PME-NA XXVIII

November 9 to 12, 2006
Mérida, Yucatán

Proceedings of the Twenty Eighth Annual Meeting of the

North American Chapter of the International Group for the

Psychology of

Mathematics Education

Mérida, Yucatán, México

November 9 – 12, 2006

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Cite as:

Alatorre, S., Cortina, J.L., Sáiz, M., & Méndez, A. (Eds.). (2006). *Proceedings of the Twenty Eighth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Mérida, Mexico: Universidad Pedagógica Nacional.

ISBN

970 – 702 – 202 – 7

History of PME

The International Group for the Psychology of Mathematics Education came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsruhe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

Goals of PME-NA

The major goals of the North American Chapter of the International Group for the Psychology of Mathematics Education are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians and mathematics teachers.

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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Preface

Organizing the twenty-eighth edition of PME-NA has been an honor, a responsibility and also a great pleasure. The vast panorama of excellent academic work that will be displayed at the conference and that is contained in these pages makes us immensely proud to be part of the project.

The Mérida Conference proposed the theme *Focus on learners, focus on teachers,* which comprises an extensive amount of possibilities in the juxtaposition of both ways of focusing mathematics education: Focus on learners *and* focus on teachers, Focus on learners *or* focus on teachers, Focus on learners *vs.* focus on teachers, Focus on learners *through* focus on teachers, etc. Thus, we proposed to emphasize the duality of the roles of learners and teachers in the educational process, and this call received many different and interesting reactions.

Our three plenary speakers are prominent figures from the three countries of North America. Luis Radford from Ontario's Université Laurentienne will use a semiotic point of view to study the algebraic thinking. Marta Civil of the University of Arizona will tackle the issue of equity by focusing not only on learners and teachers but also on parents. Tenoch Cedillo of the Mexican Universidad Pedagógica Nacional will propose a way in which teachers can –and do– learn from students. Not less notorious are the three personalities that have been selected to react to these plenary presentations: Carolyn Kieran from the Université du Québec à Montreal, Arthur Powell from Rutgers University, and Sharon Senk from Michigan State University. (We have been fortunate enough to have Carolyn Kieran's reaction paper in time for publication; the other two are unfortunately not included in the proceedings but will undoubtedly also cast a lucid light upon the topics that will be approached).

We will have six of the Working Groups that have been productively working for the last years, and engaging in topics from the complexity of learning to reason probabilistically to gender and mathematics, from in-service teacher education and teaching assistant preparation to the mathematics classroom discourse, and of course the classic WG on models and modeling. Two new Discussion Groups are proposed this year, which will certainly add to the interest and quality of the reunion: one on the lesson study and one on transnational issues in mathematics education.

After the plenary sessions and the WG and DG, which comprise the first volume of the proceedings, the second one displays the 240 Research Reports, Short Oral Reports and Posters of the traditional fifteen topics (this year the topics of Rational Numbers and Whole Numbers have been put together in one). It may be pointed out that this year we received more than 350 proposals, which unfortunately contrasted with the very limited amount of rooms available in the Conference's venue; in order to give as many people as possible the space and time for their presentations we decided on the following distribution: 28% Research Reports, 34% Short Oral Reports, and 38% Posters. As a consequence of this, we are sure that all the sessions will excel; for instance, our two sessions of posters are very promising.

This is the general overview of the papers included in these proceedings. Their richness, however, will begin to be appreciated during the Conference, in the presentations and in the following discussions. And, of course, the quality of the presented papers, the fruitfulness of the discussions and the interaction with international researchers will without doubt benefit the Mexican Mathematics Education community.

I would like to thank all the authors of proposals for contributing with such samples of good work, and also the reviewers for taking time to carefully read the proposals and give their professional opinion on them. I also wish to thank the Steering Committee for all their support, good advise, and thoughtful work throughout this year. The Local Committee has certainly been fundamental in taking this car to a good end; my gratitude to all of them. Last, but not least, the Universidad Pegagógica Nacional, host of the Conference, has given all sorts of support to make this event come true.

Silvia Alatorre
Program Chair

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VOLUME 1

PLENARY SESSIONS

ALGEBRAIC THINKING AND THE GENERALIZATION OF PATTERNS: A SEMIOTIC PERSPECTIVE

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The most important operation of the mind is that of generalization.
C. S. Peirce, *Collected Papers* 1.82.

Introduction

Several years ago, I had the opportunity to conduct longitudinal research in four Junior High-School classes about the teaching and learning of algebra. The timing was just perfect: the previous year, i.e. 1997, the Ontario Ministry of Education released a new Mathematics Curriculum based on a new type of assessment, the enlargement and reorientation of knowledge content and the rigorous description of the expected learning. To say the least, teachers were worried about the new high expectations. The time was just ripe for collaboration. There was a clear sense in the educational community that, in order to implement the new curriculum, we all had a lot to learn from each other. For me, working with three or four teachers every year for six years and following the same students in the classroom as they moved through Junior and Senior High School constituted a marvellous opportunity.

We designed a flexible teaching-researching agenda committed to meeting two main goals: First, we wanted the students to learn the algebraic concepts stipulated by the Curriculum. This was a practical concern framed by the aforementioned political educational context. Second, we wanted to deepen our understanding of the emergence and development of students' algebraic thinking, the difficulties that the students encounter as they engage in the practice of algebra and the possible ways to overcome them. The longitudinal research was characterized by a continuous loop: (1) classroom activity design → (2) classroom activity implementation → (3) data interpretation → (4) theory generation → (1) classroom activity design → (2) … Our longitudinal research was informed by the wealth of research previously conducted on the transition from arithmetic to algebra. In the early 1980s, Matz (1980) and Kaput and Sims-Knight (1983) investigated some errors associated with symbol use and Kieran (1981) pointed out different concepts associated with the equal sign. Some years later, Filloy and Rojano (1989) put into evidence some key problems that novice students face in solving equations; a bit later Sfard (1991) and Gray and Tall (1994) called attention to the students' difficulties in distinguishing between objects and processes, while Bednarz and Janvier (1996) studied the effects of word problem structure in arithmetic and algebraic reasoning. At about the same time, several researchers showed the limits of X-Y numerical tables in the generalization of patterns (Castro Martínez, 1995; MacGregor & Stacey, 1992, 1995). It was apparent that X-Y tables were emphasizing a formulaic aspect of generality based on trial and error heuristics, hence confining algebraic notations to the status of place holders bearing very limited algebraic meaning.

The research conducted in the 1980s and 1990s –the above sketch of which is obviously incomplete– led to an unavoidable and difficult question asked again and again: that of the exact nature of algebraic thinking. Commenting on the Research Agenda Conference in Algebra (Wagner & Kieran, 1989), held in March 1987 at the University of Georgia, Kieran (1989, p. 163) said: "One of the topics pointed to in the Research Agenda … as an area sorely in need of

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research attention is that of algebraic thinking." Certainly, since then, the several studies conducted by mathematics educators and historians have made an important contribution to this area (e.g. Arzarello & Robutti, 2001; Boero, 2001; Carraher, Brizuela, & Schliemann, 2000; Høyrup, 2002; Lee, 1996; Lins, 2001; Martzloff, 1997; Puig, 2004; Ursini & Trigueros, 2001). And if we still do not have a sharp and concise definition of algebraic thinking, it may very well be because of the broad scope of algebraic objects (e.g. equations, functions, patterns, ….) and processes (inverting, simplifying, …) as well the various possible ways of conceiving thinking in general.

It is clear that algebraic thinking is a particular form of reflecting mathematically. But what is it that makes algebraic thinking distinctive? Trying to come up with a working characterization to guide our research, we adopted the following non-exhaustive list of three interrelated elements. The first one deals with a sense of *indeterminacy* that is proper to basic algebraic objects such as unknowns, variables and parameters. It is indeterminacy (as opposed to numerical determinacy) that makes possible e.g. the substitution of one variable or unknown object for another; it does not make sense to substitute "3" by "3", but it may make sense to substitute one unknown for another under certain conditions. Second, indeterminate objects are handled *analytically*. This is why Vieta and other mathematicians in the 16th century referred to algebra as an *Analytic Art*. Third, that which makes thinking algebraic is also the peculiar *symbolic* mode that it has to *designate* its objects. Indeed, as the German philosopher Immanuel Kant suggested in the 18th century, while the objects of geometry can be represented ostensively, unknowns, variables and other algebraic objects can only be represented *indirectly*, through means of constructions based on signs (see Kant, 1929, p. 579). These signs may be letters, but not necessarily. *Using letters does not amount to doing algebra*. The history of mathematics clearly shows that algebra can also be practiced resorting to other semiotic systems (e.g. coloured tokens moved on a wood tablet, as used by Chinese mathematicians around the 1st century BC and geometric drawings as used by Babylonian scribes in the 17th century BC).

Drawing on the working characterization of algebraic thinking sketched above and the then-emerging Vygotskian perspective in mathematics education (Bartolini Bussi, 1995; Lerman, 1996), we formulated our research problem in semiotic terms. Starting from a broad conception of signs, we wanted to investigate the students' *use of signs* and processes of *meaning production* in algebra. Naturally, contemporary curricula favour the alphanumeric algebraic symbolism. It was our contention, however, that, ontogenetically speaking, the students' formation of the corresponding meanings and rules of sign-use were rooted in other semiotic systems that they had already mastered. Since the history of mathematics suggests that, in some cultural traditions, the evolution of some algebraic notations relied heavily on speech (Radford, 2001), we had strong reasons to look to language for the antecedents of the students' alphanumeric algebraic meanings. The results that we obtained during the first years confirmed our hypothesis, but, as we shall see in a moment, we also came to realize that language was only part of the story.

In this paper, I want to present an overview of some of our results. Although our general goal was to investigate the various aspects of students' algebraic thinking, as related to the algebraic concepts stipulated by the Ontario Curriculum, for the sake of simplicity, I will focus here on the generalization of patterns only (some results concerning equations can be found in Radford, 2002a, 2002b; Radford and Puig, in press). In Section 1, I suggest making a distinction between *generalization* and (naïve) *induction*. I will claim that, just as not all symbolization is algebraic, not all patterning activity leads to algebraic thinking. I will argue in particular that this is the case for inductive reasoning (as frequently used by the students), even if the inductive process can be expressed in symbols, such as “2n+1”. I will even go further and claim that, among the possible forms of generalization, not all are algebraic in nature (there are some pattern generalizations that are arithmetic but not algebraic, a point that I discuss later in the paper). One practical result that comes out of this is the following. In the use of patterning activities as a route to algebra, we –teachers and educators– have to remain vigilant in order not to confound algebraic generalizations with other forms of dealing with the general; we also have to be equipped with the adequate pedagogical strategies to make the students engage with patterns in an algebraic sense. In Section 2, I discuss the theoretical construct of knowledge objectification, which I use in the subsequent sections to give an account of the students’ sign use and meaning production in classical pattern activities.

1. Towards a Definition of Algebraic Generalization of Patterns

One of the introductory activities to algebraic symbolism that we proposed to Grade 8 students included the classical pattern shown in Figure A.

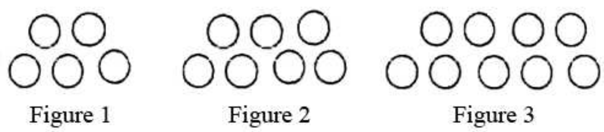

Fig. A. The sequence of figures given to the students in a Grade 8 class.

At the beginning of the activity, the students –who always worked in small groups of two to four– were required to find the number of circles in Figure 10 and in Figure 100. Their strategies fell into two main categories.

In the first one, the heuristic is based on *trial and error*: the students propose simple rules like “times 2 plus 1”, “times 2 plus 2” or “times 2 plus 3” and check their validity on a few cases. The symbolization of the rule may vary. Here is one provided by one of our small groups: “nx2(+3)”. When the students of this small group were asked to explain how they found this rule, they said: “We found it by accident.”

In the second one, the students search for a *commonality* in the given figures. Mel, for instance, wrote: “The top line always has one more circle than the number of the figure and the bottom line always has two circles more than the number of the figure”. Mel’s formula was: “(n+1) + (n+2)=”

Although both procedures lead to the use of symbolism, the heuristics are incommensurately different. The latter rests on *noticing* certain common features of the given figures and *generalizing* them to the figures that follow in the sequence. In contrast, the former rests on a rule formed by guessing. Rules formed in this way are in fact *hypotheses*. This way of reasoning works on the basis of *probable reasoning* whose conclusion goes beyond what is contained in its premises. In more precise terms, it is a type of *induction* –a type that I will qualify as *naïve* to distinguish it from other more sophisticated types of induction[1]. Thus, instead of generalizing something, when resorting to the first procedure, the students merely make an induction and not a generalization[2].

The comparison of the two aforementioned strategies emphasizes an important distinction between induction and generalization –a difference that is often overlooked and that ends up calling something generalization while in reality it is simply an induction (Peirce, CP 2. 429). At the same time, it suggests one of the traits that may constitute the core of the generalization of a pattern, namely the capability of noticing something general in the particular, a trait upon which Love (1986), Mason (1996) and others have previously insisted.

Kieran, however, claimed that this trait alone may not be sufficient to characterize the *algebraic* generalization of patterns. She argued that in addition to seeing the general in the particular, “one must also be able to express it algebraically” (Kieran, 1989, p. 165). To understand Kieran’s objection, we should bear in mind that usually, the generalization of patterns as a route to algebra rests on the idea of a natural correspondence between algebraic thinking and generalizing. Kieran took argument against the alleged natural character of this correspondence and contended that to think algebraically is more than thinking about the general. It is to think about the general or the generalized in a way that makes it distinctively algebraic in its *form of reasoning as in its expression*. As she put the matter, “a necessary component [of algebraic generalization] is the use of algebraic symbolism to reason about and to express that generalization.” (Kieran, 1989, p. 165).

I concur with Kieran’s exigency concerning the inclusion of one’s ability to *express* the general. Following a Vygotskian thread to which I shall return in the next section, what I would like to add here is that algebraic generality is made up of different layers –some deeper than others. Furthermore, the scope of the generality that we can attain within a certain layer is interwoven with the *material form* that we use to *reason* and to *express* the general (e.g. the standard alphanumeric algebraic semiotic system, natural language or something else).

In this line of thought, I want to suggest the following definition. Generalizing a pattern *algebraically* rests on the capability of *grasping* a commonality noticed on some elements of a sequence S, being aware that this commonality applies to *all* the terms of S and being able to use it to provide a direct *expression* of whatever term of S.

In other words, the algebraic generalization of a pattern rests on the noticing of a local commonality that is then *generalized* to all the terms of the sequence and that serves as a warrant to build expressions of elements of the sequence that remain beyond the perceptual field. The generalization of the commonality to all the terms is the formation of what, in Aristotelian terminology, is called a *genus*, i.e. that in virtue of which the terms are held together (see e.g. Aristotle’s *Categories*, 2a13-2a18). Direct expression of the terms of the sequence requires the elaboration of a rule –more precisely a *schema* in Kant’s terms (Radford, 2005a). I will come back to this definition later. For the time being, I want to stress two main elements involved in the definition. On the one hand, there is a *phenomenological* element related to the grasping of the generality. On the other hand, there is a *semiotic* element related to the expression through signs of what is noticed in the phenomenological realm. In the next section, I will argue that these two elements are interrelated and that they may be investigated through two theoretical constructs –knowledge objectification and the concomitant semiotic resources to achieve it.

2. Knowledge Objectification

For the novice student, noticing the underlying commonality of the terms of a pattern is not something that happens all of a sudden. On the contrary, it is a gradual process underpinned by a dynamic distinction between the same and the different. Even in a pattern as simple as the previous one (see Figure A), there are several ways to look for what may qualify as the same and the different in the given figures. Thus, talking to his two group-mates, Doug –a Grade 9 student– says: “So, we just add another thing, like that”. At exactly the moment he utters the word “another”, he starts making a sequence of six rhythmic parallel gestures (see Fig. B).

Fig. B. Excerpt of Doug’s sequence of rhythmic gestures.

Naturally, the figures all have the *same* shape, but at the same time, they are *different*: that which makes them different, Doug is suggesting, is the last two circles *diagonally* disposed at the end of each figure (see Figure C).

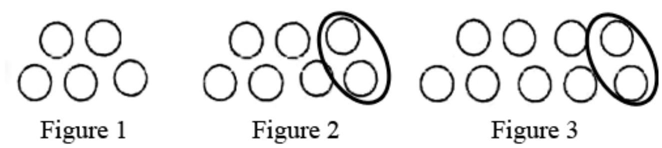

Fig. C. Doug emphasizes the last two circles in an attempt to notice a commonality in the terms of the sequence

We see hence that Doug’s grasping of the commonality is different from Mel’s (see Section 1); so too is Doug’s expression of it. While Mel saw the figures as made up of two horizontal lines and expressed generality in a verbal form, Doug saw the figures as recursively built by the addition of two circles diagonally arranged and expressed it dynamically through gestures and words.

In more general terms, what we observed in the classroom from the first day was that the perceptual act of noticing unfolds in a process mediated by a multi-semiotic activity (spoken words, gestures, drawings, formulas, etc.) in the course of which the object to be seen emerges progressively. This process of noticing I have termed a process of *objectification*.

The term objectification has its ancestor in the word *object*, whose origin derives from the Latin verb *obiectare*, meaning “to throw something in the way, to throw before”. The suffix –*tification* comes from the verb *facere* meaning “to do” or “to make”, so that in its etymology, objectification becomes related to those actions aimed at bringing or throwing something in front of somebody or at making something apparent –e.g. a certain aspect of a concrete object, like its colour, its size or a general mathematical property. Now, to make something apparent, students and teachers make recourse to signs and artefacts of different sorts (mathematical symbols, graphs, words, gestures, calculators and so on). These artefacts, gestures, signs and other semiotic resources used to objectify knowledge I call *semiotic means of objectification* (a detailed account can be found in Radford, 2003; 2002c).

In our previous example, Doug started making apparent a general mathematical structure –he started objectifying it. To accomplish this, Doug resorted to two semiotic means of objectification: words and gestures. In addition to highlighting the last two circles, the rhythmic repetition of gestures allowed Doug to achieve something notable: through this, Doug expressed the idea of something *general*, something that continues *further and further,* in space and in time.

I am not suggesting, though, that Doug's six gestures and one utterance were enough to fully disclose the mathematical structure behind the pattern. Neither am I affirming that Doug was providing a direct expression of whatever term of the sequence. What I am saying is that the objectification of the general goes through various layers of awareness. To get a better grasp of the structure behind the pattern, Doug's process of objectification had to continue. Through mediating signs, Doug continued engaging with the object of knowledge and signifying generality in more precise terms. It is obvious that the sense of generality achieved through words and gestures is not the same as the one achieved through a formula or a graph. A semiotic system provides us with specific ways to signify or to say certain things, while another semiotic system provides us with other ways of signification. The linguist Émile Benveniste referred to this situation as the principle of *nonredundancy*: "Semiotic systems", Benveniste said, "are not 'synonymous'; we are not able to say 'the same thing' with spoken words that we can with music, as they are systems with different bases." (Benveniste in Innis, 1985, p. 235). The same distinction is true of gestures and formulas.

By the same token, Benveniste's nonredundancy principle warns us against the common belief in translatability –the belief that e.g. a formula says *the same thing* as its graph, or that a formula says *the same thing* as the word-problem it "translates" (see e.g. Duval, 2002; Radford, 2002b). The nonredundancy principle does not mean, however, that what we intend or express in one semiotic system is completely independent from what we express in another one. The objectification of the mathematical structure behind a pattern that was mediated by words and gestures may be deepened by an activity mediated through other types of signs.

As previously described, the objectification of knowledge is a theoretical construct to account for the way in which the students engage with something in order to notice and make sense of it. By focusing on the students' phenomenological mathematical experience, it emphasizes the subjective dimension of knowing. But this is only half of the story. Since we are sociocultural knowers, objectification takes also account of the social and cultural dimensions of knowing. The concept of knowledge objectification rests indeed on the idea that classrooms are not merely a bunch of external conditions to which the students must adapt. Classrooms are rather seen as interactive zones of mediated activities conveying scientific, ethical, aesthetical and other culturally and historically formed values that the students objectify through reflective and active participation (Radford, in press). In these activities, embedded in cultural, historical traditions, the students relate not only to the objects of knowledge (the *subject-object plane*), but also to other students through face-to-face, virtual or potential communicative actions (the *subject-subject plane* or *plane of social interaction*).

Within the previous theoretical context, our investigation of the students' use of signs and processes of meaning production in algebra focused on a detailed study of the students' knowledge objectification as they moved along different layers of generality and awareness.

Guided by our definition of algebraic generalization and theoretical framework, some of the research questions that we tackled were the following:

1. How do the students *grasp* the commonality in a pattern?
2. What are the mechanisms (linguistic or others) through which the students *generalize* the locally observed commonality to *all* the terms of the sequence?
3. How do they *express* generality?

In the rest of this paper, I discuss these questions focusing particularly on the work done by one of the Grade 8 (13-14 years old) and also one of the Grade 9 (14-15 years old) small-groups which are representative of most of the work done by other groups.

3. *Genus* Formation: Grasping and Generalizing a Local Commonality

Roughly speaking, our classroom activities were organized along the two aforementioned subject-object and subject-subject planes as follows:

The students were presented with patterns whose complexity was commensurate to the curriculum requirements. Working in small groups, the students were invited to carry out:

1. an arithmetic investigation (often conducted by continuing the pattern on the basis of some given information, as well as answering questions about specific figures such as Figure 10, Figure 25, Figure 100);
2. the expression of generalization in natural language (in the form of a message), and
3. the use of standard algebraic symbolism to express generality.

Among the patterns that we selected, there were some classical circle and toothpick patterns (such as those shown in Figure A and Figure D) and variations of increased difficulty as the students moved through their Junior and Senior High School years (see Bardini, Radford, & Sabena, 2005).

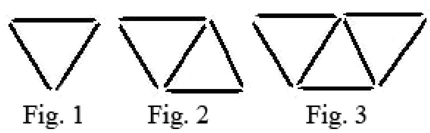

Fig. D. An example of a toothpick pattern.

As indicated in Section 1, using different techniques, the students usually succeeded in answering questions about Figure 10, Figure 25, Figure 100, etc. Let us put aside the inductive, non generalizing techniques, and focus on the generalizing strategies only.

When adolescent or younger students tackle questions about "big" figures, such as Figure 25 or 100, a frequent strategy consists in noticing a recurrent relation between consecutive figures (see e.g. Castro Martínez, 1995, and Warren, 2006, respectively). This typical strategy is illustrated in the following excerpt from a Grade 8 small-group, concerning Figure 25 of the toothpick pattern:

1. Judith: The next figure has two more than … look … […] [Figure] 6 is 13, 13 plus 2. You have to continue there […].
2. Anik: Well, you can't always go plus 2, plus 2, plus 2...
3. Judith: But of course! That's Figure 7, plus 2 equals Figure 8.
4. Josh: That will take too long!

As the dialogue implies, the students noticed that the terms of the sequence increase by two. Furthermore, the dialogue provides us with a clear indication that, for the students, this common increment applies not only to the terms that were explicitly mentioned but also to the terms that followed. One unambiguous indicator is the expression: "you have to continue there". However, up to this point, the students did not make use of the already-noticed regularity to provide an exact value for the number of toothpicks in Figure 25. Actually, as line 4 indicates, they were aware that their procedure was unpractical. According to our definition of algebraic generalization, the students have not yet stepped into the realm of algebra. They did generalize something but are still in the realm of arithmetic. What they generalized was a local commonality observed on some figures, without being able to use this information to provide an expression of whatever term of the sequence. A generalization of this kind I will call *arithmetic generalization*.

Trying to come up with another strategy, Josh proposed a more direct procedure:

1. Josh: It's always the next. Look! (*Then, pointing to the figures with the pencil he says*) 1 plus 2, 2 plus 3 […]
2. Anik: So, 25 plus 26...

Line 1 shows the moment at which Josh realized that there was a different commonality linking the number of toothpicks in a figure and the sum of the ranks of two consecutive figures. The utterance "It's *always* the next" (my emphasis) indicates Josh's awareness that this commonality applies to all the terms. Drawing on Josh's idea, Anik was then able to directly provide an expression for the value of Figure 25. Thus, the students here made an algebraic generalization –one that in a previous work (Radford, 2003) I have referred to as *factual generalization.*

The adjective *factual* stresses the idea that this generalization occurs within an elementary layer of generality –one in which the universe of discourse does not go beyond particular figures, like Figure 1000, Figure 1000000, and so on. This layer of generality is rather the layer of *action*: The *genus* of the sequence leads to the formation of a *schema* that operates on particular numbers (e.g. "1 plus 2, 2 plus 3", see Line 1). Another way to say this is that in factual generalizations, *indeterminacy* –the first characteristic of algebraic thinking mentioned in the Introduction– does not reach the level of enunciation: it is *expressed in concrete actions* (see also Vergnaud's (1996) "theorem-in-act").

Of course, the students had pragmatic reasons to remain bounded to the factual level of generality. Factual generalization was good enough to get the answers that we asked of them. This was not to be the case when the students tackled the next question. Before going there, I want to discuss another excerpt, from a Grade 9 class, dealing with the sequence shown in Figure A.

This group was formed by three students: Jay, Mimi (sitting side by side) and Rita (sitting in front of them). Prior to the excerpt that I am going to present, the students found that the number of circles in Figures 10 and 100 was 23 and 203 respectively. They perceived the given figures as formed by two horizontal rows, generalize this commonality to the other figures of the sequence and formed a factual generalization ("11 and 12", "101 and 102"; see Sabena, Radford, and Bardini, 2005). However, Mimi was intrigued by the fact that the digit 3 was at the end of the answers. In the excerpt which follows she tries to come up with another generalizing schema that would include the digit '3' and the number of the figure:

1. Mimi: Add… Add three to the number of the figure! (*pointing to the results "23" and "203" already written on the paper*).

2. Jay: No! 101 (*meaning the top row of Figure 100*), 100 (*meaning Figure 100*) and you got that, 203.

In line 1, Mimi tried to formulate a new schema. As Jay quickly noticed, the schema is faulty (line 2). Jay's utterance was followed by a long pause (5.2 seconds) during which the students silently looked at the figures. Jay became interested in Mimi's idea but, like Mimi, still did not see the link in a clear way.

Trying to come up with something, while putting his pen on Figure 1 and echoing Mimi's utterance, Jay pensively said: “Add 3”. At the same time, Mimi moved her finger to Figure 1 (close to Jay's pencil) and said: “I mean like … I mean like …” (see Figure E). Right after She intervened again and said: “You know what I mean? Like… for Figure 1 (*making a gesture; see Figure F, left*) you will add like (*making another gesture; see Figure F, right*) …”

Fig. E. Jay and Mimi pointing at Figure 1, trying to notice a commonality.

To explore the role of digit 3, Mimi made two gestures. The first one has an indexical-associative meaning: it *indicates* the first circle on the top of the first row and *associates* it to Figure 1 (Figure F, left). The second one achieves a meaningful link between digit 3 and three “remarkable” circles in the figure (Figure F, right). Although Mimi has not *mentioned* or *pointed to* the first circle on the bottom row, the circle has been *noticed*, i.e., although the first circle on the bottom has remained outside the realms of word and gesture, it has fallen into the realm of vision. Indeed, right after finishing her previous utterance, Mimi starts with a firm “OK!” that announces the recapitulation of what has been said and the opening up towards a deeper level of objectification, a level where *all* the circles of the figures will become objects of discourse, gesture and vision. She says:

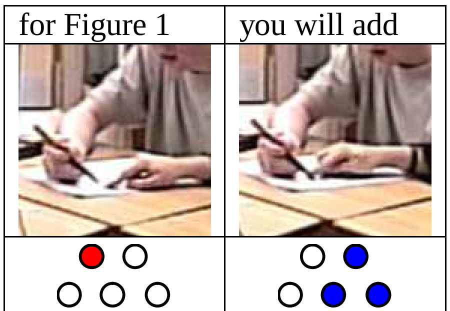

Fig. F. Perceptual objectifying effects of word and gesture on Figure 1.

> Mimi: OK! It would'be like one (indexical gesture on Figure 1; see Picture 1), one (indexical gesture on Figure 1; see Picture 2), plus three (grouping gesture; see Picture 3); this (making the same set of gestures but now on Figure 2) would'be two, two, plus three; this (making the same set of gestures but now on Figure 3) would be three, three, plus three.

Fig. G. In Pictures 1 and 2 Mimi makes an indexical gesture to indicate the first circle on the top row and the first circle on the bottom row of Figure 1; in Picture 3, she makes a “grouping gesture” to put together the last three circles of Figure 1.

Making two indexical gestures and one "grouping gesture" that surrounds the three last circles on Figure 1, Mimi rendered a specific configuration visible to herself and to her group-mates. This set of three gestures was *repeated* as she moved to Figure 2 and Figure 3. In so doing, Mimi made apparent a local commonality. Now, how does she manage to *generalize* it to all the terms of the sequence? We are here at the kernel of the generalization process. To answer this question, let us pay attention to Mimi's semiotic means of objectification.

We have already noted the crucial objectifying role of gestures. However, Mimi's gestures were accompanied by the *same* sentence structure (see Figure H). Through repetition and a coordination of gestures and words, Mimi generalized a locally perceived commonality to the other figures and moved from the particular to the general.

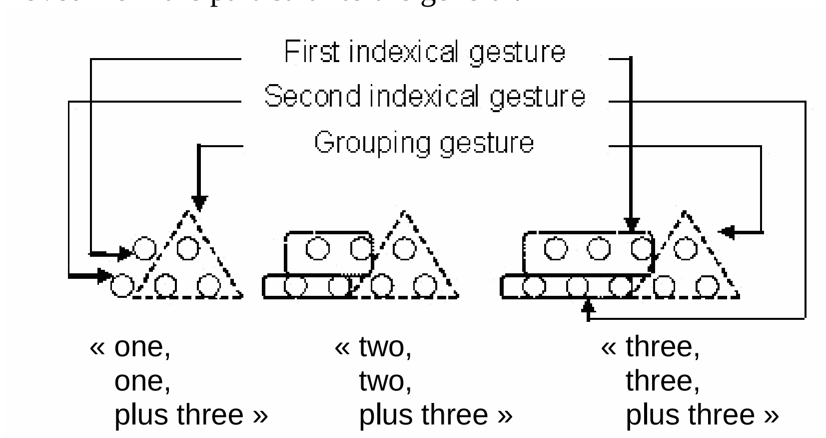

Fig. H. Mimi's objectification of a new genus of the sequence.

But in fact, in addition to gestures and words there was also *rhythm*. Rhythm was also present in Anik's utterance quoted in the first example of this section. Rhythm creates the expectation of a forthcoming event (You, 1994) and constitutes a crucial semiotic means of objectification to make apparent the feeling of an order that goes beyond the particular figures (for a detailed discussion of rhythm see Radford, Bardini and Sabena, 2006).

Mimi's generalization was hence forged with words, gestures and rhythm. Her generalization led to a schema through which the students were able to directly determine the number of circles in any *particular* figure. It is a factual generalization.

4. Showing versus Saying

Let us now discuss how students tackled the question concerning the expression of generality in natural language. The students were asked to write a message explaining how to find the number of toothpicks or circles in *any* figure to an imaginary student in another class of the same level. The level of generality that is required here is of course greater; for one thing, factual generalizations are no longer sufficient.

In Josh's group, Anik suggested a first idea:

> We can say, like, it's the number of the figure, right? Like, let's say it's 1 there. If … if … OK. You add … like, how do you say that? In order of … (*Then, implicitly referring to* Figure 2, *she says*) You add it by itself, like. You do 2 plus 2, then after this, plus 1, like. You always do this, right? […] You would do (*while she rhythmically mentions the numbers to reveal the underlying commonality, she gestures as if pointing to something*) 3

> plus 3 … plus 1, 4 plus 4 … plus 1, 5 plus 5 … plus 1. Do you know what I want to say? […] How do we say it then?

The problem, as Anik mentions, is how to express in words something general that is nonetheless easy to show through numbers and gestures. There is, in fact, a profound gap between showing and saying. The expression of the genus of the sequence (be it the first one objectified earlier by this group, based on the addition of consecutive ranks of figures, or the new one, suggested by Anik here) now has to fall in the realm of language. Indeterminacy has to be named.

After a series of unsuccessful efforts, Anik came back to their previous factual generalization:

1. Anik: Yes. Yes. OK. You add the figure plus the next figure … No. Plus the … […] (*she writes as she says*) You add the first figure…
2. Josh: (*interrupting and completing Anik's utterance says*) … [to] the second figure […]
3. Anik: So...(*inaudible*). It's not the second figure. It's not the next figure?
4. Josh: Yes, the next one [figure].
5. Judith: Uh, yes, the next [figure] […]
6. Anik: (*summing up the discussion*) You add the figure and the next figure.

To name indeterminacy in the message, the students transformed the expression "any figure" (as mentioned in the question) into "the figure" –a linguistic generic expression that does not designate a particular term of the sequence but whatever term you want to consider. The concrete actions on which the students' previous factual generalizations were based ("1 plus 2", etc.) appear now as a *single* action, as an *action in abeyance*: "You add the figure and the next figure."

The above generalization is located at a deeper layer of generality, one in which rhythm and ostensive gestures have been excluded. The students have to work here with reduced forms of expression. At the same time, to succeed at this level of generality, the students have to compensate for the reduction of semiotic resources with a *concentration of meanings* in the fewer number of signs (words) through which the generalization is now expressed. This reduction of signs and concentration of meanings constitutes a *semiotic contraction* (Radford 2002c; see also Duval 2002).

To distinguish these kinds of generalizations from factual ones, I termed them *contextual generalizations* (Radford 2003). They are contextual in that they refer to contextual, embodied objects, like "the next figure" which supposes a privileged viewpoint from where the sequence is supposedly seen, making it thereby possible to talk about *the* figure and the *next* figure.

The expression of generality beyond the level of factual generality has been investigated in the context of early algebra research. At the PME 2006 Conference, Elizabeth Warren reported a study with Grade 5 students (10 years old). Among other things, she asked the students to write in natural language the general rule for some patterns and found that between 6 and 10 students out of 27 were able to write a relationship between the position of the term and its numerical value, while between 16 and 21 students failed to do so (Warren, 2006). At the same conference, Ferdinand Rivera reported results from a research project conducted with Grade 6 students (11 years old) (Rivera, 2006). The students were presented with a slightly modified version of the sequence shown in Figure A. The terms started with one circle and increased by two circles. The students had to write a message to an imaginary Grade 6 student clearly explaining what s/he must do in order to find out how many circles there were in any given figure of the sequence.

Two answers were the following. Student 1: “You start at one and keep adding two until you get the right number of circles in all”. Student 2: “You look at the figure number and then draw the number of circles then going up you put a # less then add it all together.”

There are several interesting features in the answers. Student 2 took advantage of the geometric shape of the figures to form a *genus* of the sequence and provided a contextual generalization, the embodied dimension of which appears in the situated description of the actions as in “going up”. Student 1 formed a different genus: the common increment of two circles between figures. However, the student did not provide a direct expression for any given figure. This is hence an example of an arithmetic generalization that does not reach an algebraic character.

Let us come back to Mimi’s group. The students continued refining the factual generalization that we discussed in Section 3. Mimi said:

1. Mimi: The number of the figure like ... we’ll say that the figure is 10 (*gesture with an open hand as to indicate a row on the desk),* you’ll have ten dots (*similar gesture on the desk*) plus three (*sort of grouping gestures a bit more to the right and to the bottom, on the desk*) right? *(pause)* No…
2. Jay: (*Almost simultaneously*) No.
3. Mimi: You double the number of the figure.
4. Jay: ten plus ten *(pointing to the sheet)*
5. Mimi *(interrupting)*: So it will be twenty dots plus three *(pointing to the number 23 on the sheet)*. You double the number of the figure and you add three, right? So Figure 25 will be fifty...three. Right? That’s what it is […]
6. Jay: Figure times two plus three.

The written message was the following: “The number of the figure × 2, + 3. It gives you the amount of circles.”

The message is a mixture of mathematical symbols and terms in natural language. Undoubtedly, the comma is the most interesting element: it translates, in a written form, the spatial and temporal characteristics of one crucial distinctive event objectified in the course of the students’ mathematical experience, namely the distinction between the same and the different elements in the figures, as the students perceived them.

5. Writing Little while Saying a Lot

In Josh’s group, expressing the generalization through alphanumeric symbols –what I have called a *symbolic generalization* (Radford, 2003)– was a complex process during the course of which the students had to decide about the meaning of letters. One particular problem was to decide how to say “the next figure”. The following excerpt illustrates some of the difficulties:

1. That would be like $n + a$ or something else, $n + n$ or something else.
2. Anik: Well [no] because "*a*" could be any figure […] You can’t add your 9 plus your … like ... […] You know, whatever you want it has to be your next [figure].

When the students reached an impasse, the teacher intervened: “If the figure I have here is ‘*n*’, which one comes next?” Thinking of the letter in the alphabet that comes after *n*, Josh replied: “*o*”. In the end they ended up with the following formula: “(n+1) + n”.

The formula in Jay's group was as follows: "nx2+3". Formed out of a commonality noticed through a complex coordination of hands in space, rhythm, nouns, deictics and adverbs, the formula reached here an extremely concise expression. The "space" to be occupied by each one of its five signs (i.e. "n", "×", "2", "+", and "3") was progressively prepared by the students' previous joint mathematical experience. Thus, the symbolic letter *n* is the "semiotic contraction" of the "number of the figure" that has been so often quoted before, either directly or by means of examples. In fact, the whole formula is the crystallization of a semiotic process endowed with its situated history. It is a history in which each sign acquired a distinctive meaning and which may explain why the students do not simplify the formula into the more standard expression: $2n + 3$. The formula still hangs behind the remnants of the *narrative* side of algebra (Radford, 2002b), where signs play the role of narrating a story and where the formula has not yet reached the autonomy of a detached symbolic artifact. The letters of which a formula is made up play indeed the role of *indexes* pointing to words of the students' contextual and factual generalizations.

Obviously, some students' formulas do not correspond to the standard algebraic syntax. Thus, dealing with the sequence shown in Figure A, Samantha, one Grade 8 student, managed to produce a contextual generalization: "You must add 1 more than the figure for the top and 2 more on the bottom." Her formula was: "(n+1)+2=". Now, despite its inaccurate algebraic syntax, the formula was not written at random. A closer look at the formula indeed suggests that the formula does have a meaning. The formula was built following a syntax based on the criterion of *juxtaposition of signs*. It is a sentence structured in the manner of a *narrative* where signs become encoded as *key terms* (much as ideograms did in the written language used in Mesopotamia ca 3500 BC –where, e.g., the drawing of a foot after the drawing of a mountain in a clay tablet could mean a long walk). The formula is recounting us Samantha's mathematical experience with the general. The composed term "n+1" is telling us that, to determine the number of circles on the top row, we have to add 1 more (circle) than the (number of the) figure, and that once we have finished doing this (something scrupulously indicated by the brackets), we still have to add two (circles) to the bottom row. Now, by adding these results, we may be in a position to find the total number of circles in the figure. The inaccuracy of algebraic syntax cannot be imputed to Samantha's misunderstanding of the problem: she succeeded in finding the number of circles in Figure 10 and Figure 100. Had we asked her questions about "bigger" figures, like Figure 1000000, she would have provided the right answers. The problem lies elsewhere. It lies in the students' understanding of a cultural mathematical practice based on a specific use of signs.

6. Synthesis and Concluding Remarks

Noticing a commonality in a few particular terms of a sequence is by no means the result of a contemplative act. As Kant put it:

> I see a fir, a willow, and a linden. In firstly comparing these objects, I notice that they are different from one another in respect of trunk, branches, leaves, the like; further, however, I reflect only on what they have in common … and abstract from their size, shape, and so forth; thus I gain a concept of tree. (Kant, 1974, p. 100).

Our ability to notice differences in things is one of our basic cognitive components. Without it, we would be unable to sort the amazing amount of sensorial stimuli that we receive from the exterior and the world in front of us would be reduced to an amorphous visual, tactile and aural mass. Naturally, as many of Kant's commentators have pointed out, things are a good deal more complicated than Kant himself suggested. Noticing the differences and similarities that lead to the *genus* of a pattern in our case or to the genus of a tree in Kant's own example, occur in social activities subsumed in cultural traditions conveying ideas about the *same* and the *different* and about how these differences may be *reflected* and *abstracted*. This is why some cultures make finer or different categorizations of plants and colors than others. We certainly notice differences and similarities—not through neutral tactile, aural, visual and other sense impressions—but through our historically and culturally species-evolved senses (Gibson, 1966; Wartofsky, 1979). So, instead of a contemplative and obvious act, to notice something –anything, trivial though it may be, like the *circles* in a pattern– is already a complex cultural-cognitive process.

Now, we do not remain confined to what we materially see –perception, it is true, is always the perception of particulars. We go beyond the realm of particulars by noticing something else –something *general, conceptual*– and by trying to make sense of it. I referred to this process of concept-noticing and sense-making as a process of objectification.

The whole idea of objectification is embedded in an ontology according to which the concepts or objects of knowledge are made up of *layers of generality*. The epistemological counterpart to this ontological premise asserts that our knowledge of a certain conceptual object is concurrent with the layers of generality in which we can deal with the object. Because each one of these objects' layers is general, they cannot be fully grasped in the realm of the particular. The diaphanous or insubstantial general can only come into being through signs. This is why to objectify something is to make it come into the world of (re)*presentation*, i.e. to appear within a semiotic process.

In this line of thought, I have suggested distinguishing between the diverse strategies that the students use when they deal with the generalization of patterns. Patterning activity has been justly considered as one of the prominent routes for introducing students to algebra. However, not all patterning activity leads there. This is the case of inductive procedures based on rule formation by trial and error and other guessing strategies. These procedures do not lead to algebra because algebra is certainly neither about guessing nor about just using signs. It is rather about using signs *to think in a distinctive way*. As far as patterns are concerned, algebra is about generalizing. Now, as Kant's example intimates, to talk about generalizing is to talk about two things: (1) that which is generalized (the object of generalization), and (2) the generalized object. Drawing on Kieran (1989), Love (1986) and Mason (1996), I have suggested that the process that goes from one to the other includes two interrelated components. The first one is noticing a commonality in some given particular terms. The second one is to form a general concept –a *genus*– by generalizing the noticed commonality to all the terms of the sequence. In order for a generalization of patterns to be called algebraic, I have suggested a third component: that the genus or generalized object crystallize itself into a *schema*, i.e. a rule providing one with an expression of whatever term of the sequence (arithmetic generalizations would be those which fail to meet the third component). Next, I have discussed three layers of algebraic generality and the corresponding modes of expression: factual, contextual and symbolic (see Table 1).

Naïve Induction	**Generalization**			
Guessing	Arithmetic	Algebraic		
(Trial and Error)		Factual	Contextual	Symbolic

Table 1. Students' strategies for dealing with pattern activities and the subdivision of algebraic generalizations in accordance with their level of generality.

These layers of generality are characterized by the semiotic means of objectification to which the students resort in order to accomplish their generalizations. In factual generality, indeterminacy remains unnamed; generality rests on *actions* performed on numbers; *actions* are made up here of words, gestures and perceptual activity. In the contextual and symbolic layers of generality, the indeterminate is made linguistically explicit: it is *named*. While in contextual generality the general objects are named through an embodied and situated description of them (e.g. “the next figure”, “the top row”, etc.), in symbolic generality the general objects and the operations made with them are expressed in the alphanumeric semiotic system of algebra.

Factual generality provides the raw material that, through successive *semiotic contractions*, the students will later transform into higher forms of algebraic generality. The issue here is not just to say the same thing in a different code. It is rather about gaining access to deeper forms of consciousness. It is in this respect that the genetic link between layers of generality is most revealing. For instance, we saw the tremendous cognitive importance of words, gestures and perceptual activity in factual generality (as expressed in “1 plus 2”, etc.) and their important objectifying effects: they prepare the space where the designation of objects may occur later and where the students’ consciousness of indeterminacy may reach a deeper layer of objectification.

In this context, an important question to ask is the following: Why did the students gesture? Why did they not limit themselves to talking? Gestures helped the students to refine their awareness of the general. These gestures stood for the rows that *could not be seen*. Gestures helped the students to *visualize* (Presmeg, 2006) and hereby came to fill the gap left by impossible direct perception. Generally speaking, gestures do not merely carry out intentions or information; they are key elements of the process of knowledge objectification (Radford, 2005b)[3].

From an educational perspective, it is important to bear in mind that each one of the layers of generality presents its own challenges. As we saw in the classroom examples discussed earlier, in factual and contextual generalizations, the students often talk about “the figure” instead of “the number of the figure”; because of the embodied and metonymic mode of designation of objects, the students’ generalizations often carry some ambiguities. In symbolic generalizations, the students’ formulas often tend to simply narrate events and remain attached to the context. The understanding and proper use of algebraic symbolism entails the attainment of a disembodied cultural way of using signs and signifying through them. The disembodiment of meaning of symbolic generalizations I am talking about should nevertheless not be understood as the decline or elimination of the individual, but as a new way of engaging with, and reflecting about, the general and the particular (see Radford, 2006, p. 60; see also Roth, 2006). This attainment, I want to suggest, can only be possible through a transformation of the way in which letters signify in a formula. In addition to their indexical mode of signification, letters have to acquire a symbolic mode as well. In Peirce’s terminology, letters have to become genuine symbols. The didactic situations that may promote the transformation of the index into symbol in the students’ formulas have still to be investigated further (see Barallobres, 2005). When we invited our students to simplify formulas, some progress in the direction from index to symbol was observed, even with the youngest. Thus, several groups of Grade 8 students went from “r+r+r+1” to “rx3+1”. However, examples such as these are not enough to provide us with a clear idea of the genetic path that goes from one mode of signification to the other. My conjecture at this point is that this path is paved with subtle qualitative changes where indexicality is progressively put in the background and the letters acquire a relational meaning (see Radford and Puig, in press).

Be this as it may, I hardly believe that the didactic situations susceptible to leading our students to deeper layers of symbolic or other forms of generality can be reduced to the choice of fortuitously good mathematical problems. Powerful though it may be, the plane *subject-object* is not, epistemologically speaking, strong enough. The plane of *social interaction* must be included. The students have to learn to see the objects of knowledge from others' (teachers and students) perspectives. This is why, in the classroom, we often organized an exchange of ideas and solutions and the discussion of them between groups, followed by general class discussions (Radford & Demers, 2004). The idea, however, is not merely to 'share' solutions in order to *catalyze* the attainment of deeper layers of generality. It is rather that the objectification of knowledge presupposes the encounter with an object whose appearance in our consciousness is only possible through contrasts. Our awareness and understanding of an object of knowledge is only possible through the encounter with other individuals' understanding of it (Bakhtin, 1990; Hegel, 1977; Vygotsky, 1962). In this encounter, our understanding becomes entangled with the understandings of others and the historical intelligence embodied in cultural artifacts (e.g. language, writing) that we use to make our experience of the world possible in the first place.

Acknowledgment

This article is a result of a research program funded by The Social Sciences and Humanities Research Council of Canada (SSHRC/CRSH). I wish to thank Raymond Duval and Luis Puig for their insightful comments on a draft of this paper.

EndNotes

(1) The concept of induction has been the object of a vast number of investigations in epistemology and in education; see e.g. Peirce in Hoopes, 1991, pp. 59-61; Polya, 1945, pp. 114-121; Poincaré, 1968 p. 32 ff.). In what follows, to simplify the text, I will use induction to refer to the students' *naïve* induction described above.

(2) Because of the students' strong tendency to use inductive procedures instead of generalizing ones, we proposed some patterns with decimal numbers. One of those patters was the following: 0.82, 1.13, 1.44, 1.75, 2.06, … Here, the possible values of *a* and *b* in the rule "*an+b*" (or "*nxa+b*" as the students would write) increase considerably making trial and error a heuristic which is no longer viable. As one of the students commented after failing at several trial and error guesses, "I got more numbers in my head than ever".

(3). Currently, there is an intense interest in gestures in general, as well as in science and mathematics education. Some recent work includes Arzarello and Edwards, 2005; Goldin-Meadow, 2003; Kendon, 2004; Kita, 2003; McNeill, 2000; Robutti, in press; Roth, 2001.

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REACTION PAPER TO LUIS RADFORD'S PLENARY SESSION

A RESPONSE TO 'ALGEBRAIC THINKING AND THE GENERALIZATION OF PATTERNS'

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In this reaction to the plenary paper presented by Luis Radford, I point to selective features of Luis's theoretical framework -- a framework that offers powerful means for describing algebraic thinking within the context of generalizing activity involving patterning. Then, I highlight some examples from Luis's data using the perspectives of 'sameness-and-difference' and the 'interaction of the geometric and the numeric in geometric patterning' and suggest that these foci, which were provoked by Luis's analyses, offer additional areas for further research.

Generalization is the heart of mathematics. Because algebra is a domain of mathematics, it is a given that generalization is also at the core of algebra. In his plenary paper, Luis Radford takes us on a compelling voyage into the realm of generalization, as experienced by 8th and 9th grade students as they grapple with the challenge of developing algebraic thinking within the context of geometric patterning activity. In my response to his paper, I first point to some features of Luis's framework that pertain specifically to algebraic thinking. Then, I look more closely at particular aspects of his student data and discuss two issues that his analysis provoked within me.

Algebraic Thinking

Luis states that we still do not have a sharp and concise definition of *algebraic thinking*, suggesting that it may very well be because of the broad scope of algebraic objects (e.g., equations, functions, patterns, …) and processes (inverting, simplifying, …) as well as the various possible ways of conceiving thinking in general. I suggest that another reason can be found in the competing views of that which constitutes school algebra, with some advocates arguing for polynomial and equation-centered content and others for a primarily function-based orientation. However, by focusing on generalization as a central component of algebraic thinking, Luis finesses such cul-de-sac polemics. His perspective finds its mathematics education roots in the generalization-related research involving proving, which was carried out in the 1970s and early 1980s (e.g., Bell, 1976; Mason & Pimm, 1984). More recently, one of the pioneers of a generalization approach to algebra, John Mason, defined algebraic thinking as follows:

> Algebraic thinking is rooted in and emerges from learners' natural powers to make sense mathematically. At the very heart of algebra is the expression of generality. Exploiting algebraic thinking within arithmetic, through explicit expression of generality makes use of learners' powers to develop their algebraic thinking and hence to appreciate arithmetic more thoroughly. Algebraic symbols are a language for expressing generalities. As fluency and facility with expressions of generality develops, the expressions become more succinct, and hence manipulable. The force and desire to manipulate comes from several sources. One is from recognizing that different looking expressions sometimes purport to express the same thing [in, for example, a geometric pattern] … . Another source for desire to

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A. (Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

> manipulate algebraic expressions is from recognizing properties of numbers in arithmetic and generalizing these (hence generalized arithmetic). Another source for purposeful algebraic manipulation is from wanting to develop calculation techniques to manifest graphical properties. For example, when two graphs intersect, how can you find the coordinates of the intersection from the equations for the graphs? … In parallel with expressing generality is the use of symbols to denote as-yet-unknown numbers so that relationships can be expressed (a form of expressing generality). (Mason, 2005, p. 310)

While Mason emphasizes the role of generalization in algebra, Luis extends these ideas with an elaboration of a theoretical framework that combines semiotics and socio-cultural perspectives and applies the framework to the study of algebraic thinking, in particular, the generalization of patterns.

Definitions of algebraic thinking that are closely tied to generalizing activity lead to the question of what it is that makes algebraic thinking distinct from other kinds of mathematical thinking. Luis grapples head-on with this difficult question. In his paper, he suggests that there are three aspects that characterize algebraic thinking. The first concerns the *indeterminacy* that is proper to basic algebraic objects such as unknowns, variables, and parameters; the second, the *analytic* manner in which the indeterminate objects are handled; and the third, the *symbolic* mode used in designating its objects. With this third component, we enter the intricate world of semiotics that constitutes the essence of Luis's approach to the study of students' use of signs and processes of meaning production in algebra. Luis carefully tracks students' gestures, rhythmic motions, drawings, formulas, and spoken words as they reach deeper and deeper levels of generalization in the process of becoming aware of the commonalities in selected examples of geometric patterns. With the caveat that not all symbolization is algebraic and that not all patterning activity leads to algebraic thinking, Luis clearly distinguishes *algebraic generalizations* from other forms of dealing with the general. In other words, there are some pattern generalizations that are arithmetic in nature and which are not considered algebraic, according to Luis.

In Table 1, he synthesizes the various approaches used by students in dealing with pattern activities, two of which are deemed non-algebraic: trial-and-error guessing and arithmetic generalization.

<table>
<tr><th>Naïve Induction</th><th colspan="4">Generalization</th></tr>
<tr><td rowspan="2">Guessing
(Trial and Error)</td><td>Arithmetic</td><td colspan="3">Algebraic</td></tr>
<tr><td></td><td>Factual</td><td>Contextual</td><td>Symbolic</td></tr>
</table>

Table 1. Students' strategies for dealing with pattern activities and the subdivision of algebraic generalizations in accordance with their level of generality (from Radford)

Trial-and-error guessing leads to rules that students cannot explain, for example, "we found it by accident." Such rules, which Luis qualifies as naïve induction, are considered to be merely hypotheses and not within the realm of generalization at all. Arithmetic generalization, which is in fact generalization because the rule is applied not only to the terms that were explicitly mentioned but also to the terms that followed, is illustrated by the use of recursive rules – rules that express a local commonality but which cannot be used to provide an expression of whatever term of the sequence. We note Josh's, "That will take too long."

Factual generalizations, in contrast, offer action rules for arriving at the number of objects in any particular figure, but they do not involve the enunciation of indeterminacy, as in Anik's, "So, 25 plus 26." Contextual generalization, on the other hand, names the indeterminacy, as was for example enunciated later on by Anik; however, the generalization still refers to contextual, embodied objects: "You add the figure and the next figure." Finally, symbolic generalization expresses the generalization through alphanumeric symbols in a form that permits the calculation of values according to the position in the sequence.

The allure of Luis's categorization of algebraic generalization within the context of patterning into three ever-deepening levels of generality lies in its potential as a tool for describing students' generalizing in terms other than all-or-nothing. It also permits intermediate characterization of generalizations over the sometimes lengthy time periods that students may need in order to develop the kind of thinking that can express itself in the letter-symbolic. Furthermore, this nuancing of the generalizing process within the activity of geometric patterning can lead researchers to think about extensions to more advanced areas of algebraic thinking, as in, for example, *seeing structure and form within algebraic expressions*. In other words, algebraic symbolic generalization might itself be subcategorized into even more levels of generality when studying algebraic activity that is engaged in by more experienced algebra students. This is indeed an area where further theorizing and research could be conducted.

Two Related Issues Arising from the Given Extracts of Student Activity

Sameness-and-Difference

Luis emphasizes that generalizing a pattern *algebraically* rests on the capability of:

i) *grasping* a commonality noticed on some elements of a sequence S,
ii) being aware that this commonality applies to *all* the terms of S, and
iii) being able to use it to provide a direct *expression* of whatever term of S.

More specifically, he notes that, "for the novice student, noticing the commonality of the terms of a pattern is not something that happens all of a sudden. On the contrary, it is a gradual process underpinned by a dynamic distinction between the same and different. Even in a pattern as simple as the one below (see Figure A), there are several ways to look for what may qualify as the same and the different." Although the latter statement that, "there are several ways to look for what may qualify as the same and the different," seems quite straightforward, the student data presented by Luis suggest to me that the appropriate identification of that which is the same and that which is different within a given geometric pattern may be crucial in arriving eventually at algebraic symbolic generalization.

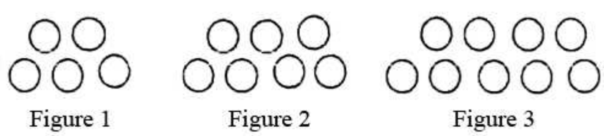

Figure A. The sequence of figures given to the students in a Grade 8 class (from Radford).

For the student Doug, what *makes the figures different is the last two circles*. Luis states that Doug saw the figures as recursively built by the addition of two circles diagonally arranged. In other words, that which was different in each figure had to do with the recursive component of the figure – in this case, the addition of two circles to each successive figure.

Focusing on the addition of two new circles for each figure, and considering that aspect as the "difference" between the figures, is a typical form of reasoning among students who use recursive thinking when analyzing growing patterns. In the mathematics education literature, such recursive thinking involving an iterative approach to generating successive values of a function is contrasted with a closed-form view that uses the figure number to arrive at the value of the function for a given figure of the pattern. However, up to now, research has not, in general, been able to pinpoint critical steps in the transition toward developing a closed-form representation in pattern generalization. Luis's example of the thinking of Mimi suggests the importance of students' being able to identify 'what is the same and what is different.'

For the pattern of circles shown in Figure A above, the three students, Jay, Mimi, and Rita perceived a structure composed of two horizontal rows, the top row having one more circle than the figure number, and the bottom row having two more. They thus arrived at the total number of circles for Figure 10 by adding 11 and 12, and for Figure 100 by adding 101 and 102. Mimi was, however, intrigued by the numerical pattern of the two results, 23 and 203 respectively -- the 3 in particular: "Add… Add three to the number of the figure! (*pointing to the results "23" and "203" already written on the paper*)."

The numerical totals for Figures 10 and 100, in combination with the arrangement of the pattern, had begun to suggest to Mimi a relationship between 3 and the figure number. Subsequent numerical analysis involving the 23 and the 203 would lead her to notice *the 3 objects that were the same in each figure* and that *what was different was really the rest of each figure* – but that this "rest" could be related to the figure number. Notice the dissimilarity between what was different for Mimi versus what was different for Doug. For Doug that which was <u>*different*</u> *was the two circles* at the end of each figure that were being added each time. For Mimi, these same *two circles, along with the one circle just to the lower left of these two, would come to constitute that which was the* <u>same</u> in each figure, while that which was *different was the basic form that changed in size for each figure number* (see Figure B).

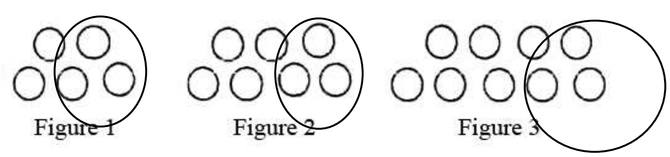

Figure B: For Mimi, the three contoured objects would come to represent the "sameness" of the pattern; the non-contoured objects would come to represent that which is different for each figure of the pattern – a difference that is related to the figure number.

Mason (2005) has said the following about *same* and *different*:

> Whenever there are two or more objects present, it can be helpful to consider what is the same and what is different, or, put another way, what is changing, what is staying invariant. To do this requires you to stress some features and consequently to ignore others, which is the basis for generalization (p. 111). … Human senses all work by detecting change. But change only makes sense if there is *something that is not changing as a background against which to detect the change*. … To appreciate sameness-and-difference requires that you discriminate features, some

> of which are shared by all objects being considered but which might not be shared by all possible objects (sameness), and some of which can be used to distinguish between the objects (difference). … When you stress certain features apparently shared by all the objects, *and ignore differences,* you are engaged in abstraction or generalization" (pp. 270-1, emphasis added).

But Mimi's comments suggest that it is not the case that one can ignore the *differences* in patterning activity. The thinking engaged in by Mimi and her fellow group members illustrates how essential it is to consider simultaneously both what is the same and what is different within the various figures of a pattern. This identification of what is the same and what is different in patterning activity can be crucial in reaching the deeper levels of generalization described by Luis.

The Interaction of the Geometric and the Numeric in Geometric Patterning

A second issue that arises from an examination of the generalizing activity of Mimi is the role played by the numeric in her thinking. When Mimi noticed the 3s in 23 and 203 for the number of circles in Figures 10 and 100, she went back to the geometric pattern to begin the process of trying to associate this numerical noticing with the geometric configuration. The coordination of the numerical with the geometric was instrumental in allowing Mimi and her group to begin to see the pattern in a new way – a much deeper way that could link the changing part of the pattern to the figure number. Eventually, this seeing was expressed as:
"$n \times 2$, $+ 3$".

Although one might argue that the 23 and 203 were artifacts of this particular pattern and the specific questions that were posed, researchers have suggested that nevertheless the two representations (i.e., the geometric and the numeric) mutually contribute to the emergence of generalization within patterning activity. More specifically, Warren (2006) has argued that, "it is the synergy between both representations that results in rich dialogues about variables, equivalent expressions, … ; we believe a continual mapping from one to the other is imperative to support these understandings."

Because numerical interpretations of pattern relations are often synthesized in table-of-values representations, Luis reminds us that, "several researchers have shown the limits of X-Y numerical tables in the generalization of patterns." He suggests further that some past studies have indicated that, "X-Y tables emphasize a formulaic aspect of generality based on trial-and-error heuristics, hence confining algebraic notations to the status of place holders bearing very limited algebraic meaning." Although it may be the case that rushing into the use of numeric tables of values can in fact curtail the richness of developing generalizations from the geometric representation, and that some students may not be able to apply the formula derived from a table of values back to the geometric pattern itself, there is no question but that at a certain moment in time the geometric perception of the physical arrangement of objects needs to be expressed numerically in order to reach deeper levels of generalization. However, a mere numeric perception of the geometric may be insufficient. Luis's data suggest that students can reach the deeper levels of symbolic generality when the passage from the geometric to the numeric is intertwined with an emerging awareness of what is the same and what is different in a geometric pattern. In this regard, the example of the thinking that emerged in the case of Mimi and her group is contrasted with what happened in Josh's group.

Mimi: OK! It would'be like **one** (indexical gesture on Figure 1; see Picture 1 in the Radford paper), **one** (indexical gesture on Figure 1; see Picture 2), **plus three** (grouping gesture; see

Picture 3); this (making the same set of gestures but now on Figure 2) would'be **two, two, plus three**; this (making the same set of gestures but now on Figure 3) would be **three, three, plus three**.

…

Mimi: The **number of the figure** like ... we'll say that the figure is 10 (*gesture with an open hand as to indicate a row on the desk),* you'll have ten dots (*similar gesture on the desk*) plus three (*sort of grouping gestures a bit more to the right and to the bottom, on the desk*) right? *(pause)* No…

Jay: (*Almost simultaneously*) No.

Mimi: **You double the number of the figure**.

Jay: ten plus ten *(pointing to the sheet)*

Mimi *(interrupting)*: So it will be twenty dots plus three *(pointing to the number 23 on the sheet)*. **You double the number of the figure and you add three**, right? So Figure 25 will be fifty...three. Right? That's what it is […]

Jay: Figure times two plus three.

This scenario culminated in the written message: "The number of the figure × 2, + 3. It gives you the amount of circles," with the comma serving to demarcate that which was different (i.e., before the comma) from that which was the same (i.e., after the comma) in the various elements of the pattern.

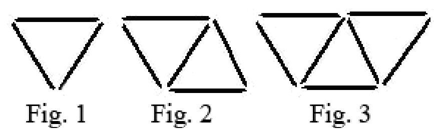

Figure C: The toothpick pattern (from Radford)

In contrast, the group of Josh, Anik, and Judith -- for the toothpick pattern (see Figure C) -- perceived a structure that, while it did not include the identification of what was the same and what was different for each element of the pattern, did involve the going back and forth between the geometric and the numeric. They developed a generalization that focused on the sum of *the* figure number and the *next* figure, and the use of this sum to determine the number of toothpicks for *the* figure:

Josh: It's always the next. Look! (*Then, pointing to the figures with the pencil he says*) **1 plus 2**, **2 plus 3** […]

Anik: So, **25 plus 26**…

[The students were asked to write a message explaining how to find the number of toothpicks or circles in *any* figure to an imaginary student in another class of the same level.]

Anik: We can say, like, it's **the number of the figure**, right? Like, let's say it's 1 there. If … if … OK. You add … like, how do you say that? In order of … (*Then, implicitly referring to* Figure 2, *she says*) **You add it by itself**, like. You do 2 plus 2, then after this, plus 1, like. You always do this, right? […] You would do (*while she rhythmically mentions the numbers to reveal the underlying commonality, she gestures as if pointing to something*) **3 plus 3 … plus 1, 4 plus 4 … plus 1, 5 plus 5 … plus 1**. Do you know what I want to say? […] How do we say it then?

…
Anik: Yes. Yes. OK. You add the figure plus the next figure … No. Plus the … […] (*she writes as she says*) You add the first figure…
Josh: (*interrupting and completing Anik's utterance says*) … [to] the second figure […]
Anik: So…(*inaudible*). It's not the second figure. It's not the next figure?
Josh: Yes, the next one [figure].
Judith: Uh, yes, the next [figure] […]
Anik: (*summing up the discussion*) **You add the figure and the next figure**.
…
Josh: That would be like $n + a$ or something else, $n + n$ or something else.
Anik: Well [no] because "*a*" could be any figure […] You can't add your 9 plus your … like … […] You know, **whatever you want it has to be your next [figure]**.

However, this group could go no further without some teacher intervention.

Although Josh, Anik, and Judith finally ended up with a correct formula, "(n + 1) + n," it is suggested that, because their processes of generalization did not follow a path that allowed them to take into account what was the same and what was different within the elements of the toothpick pattern, their formula could be considered of an *ad hoc* nature. The mysterious relation between the two figure numbers and the structure of the pattern was never made explicit. The approach that they used thus leads to further research questions regarding the generative power (or lack thereof) of the particular strategies employed in arriving at contextual, and certain symbolic, generalizations. However, there is no denying that the process engaged in by Mimi and her group touched upon structural aspects that were not uncovered in the course of action followed by Josh, Anik, and Judith – structural aspects that were grounded in the identification of sameness and difference within the pattern.

Concluding Remarks

Luis Radford, in his Synthesis and Concluding Remarks, suggests:

> The process [of generalizing] that goes from one to the other [from that which is generalized to the generalized object] includes two interrelated components. The first one is noticing a commonality in some given particular terms. The second one is to form a general concept … by generalizing the noticed commonality to all the terms of the sequence. In order for a generalization of patterns to be called algebraic, I have suggested a third component … a rule providing one with an expression of whatever term of the sequence.

However, I propose that it may not be sufficient to 'notice a commonality.' Becoming simultaneously aware of that which is the same and that which is different among the elements of a geometric pattern, while also going back and forth between the geometric and the numeric, may be crucial to students' reaching deeper levels of algebraic symbolic generalization. This particular facet of students' generalizing within patterning activity merits further research.

In this reaction, I have not centered on the semiotic tools that Luis brought to bear in his data-collection and analysis and which, in fact, permitted him to gain access to levels of student thinking that have heretofore rarely been seen in prior research on generalization within patterning activity. Let me add, as an aside, that the study of the ways in which diverse resources such as gestures, bodily movements, words, metaphors, and artefacts become interwoven during mathematical activity and, in fact, constitute a central source of meaning making in mathematics has of late been the focus of an ever-increasing body of research in mathematics education (see Kieran, 2006, for a review of this research). In conclusion, I want to emphasize that the results of the semiotic analysis conducted by Luis permit us to now have a much deeper understanding of the processes by which students arrive at generalizations within patterning activity. In the broad and diverse field of study that is collectively known as pattern-generalization research, Luis's methods of analysis serve as a model for future research.

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WORKING TOWARDS EQUITY IN MATHEMATICS EDUCATION: A FOCUS ON LEARNERS, TEACHERS, AND PARENTS

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This paper presents a reflection on my research largely grounded on my interest in students', teachers', and parents' ideas about mathematics. Starting with some considerations from a cognitive point of view, in particular preservice teachers' understanding and beliefs, I move onto sociocultural aspects. I specifically address issues related to context, valorization of knowledge, participation, and in-school and out-of-school mathematics. I draw on examples from my research in Latino, working-class communities to highlight the need (yet the complexity) to focus on all interested parties (parents, teachers, and students) and on mathematics if we are to address equity in mathematics education.

In this essay[1] I reflect on my trajectory as a researcher in mathematics education, with an eye on the theme for this conference – focus on learners, focus on teachers. My entry into the world of mathematics education research was largely focused on teachers, and more specifically on preservice elementary teachers. As a researcher, my approach was essentially cognitive—I wanted to understand their understanding and to learn about their beliefs about mathematics. I was and continue to be fascinated by how people (teachers, children, parents) make sense out of mathematics and what role their beliefs play in this process. As a teacher educator, however, I wondered about the implications of preservice teachers' understanding of mathematics and their beliefs about its teaching and learning for the children they would be teaching (Civil, 1993). I was also concerned about how preservice elementary teachers were sometimes portrayed in a negative way, focusing on their inadequate understanding of mathematics. To me, these "inadequacies" were intriguing and, while a cognitive approach was certainly very helpful, the ideas of situated cognition and social and cultural context added to my understanding of those "inadequacies." Although equity per se was not in my agenda yet, I think that some of those initial experiences opened the way towards my interest in equity in mathematics education. A concern for those who are being left out of the mathematical journey seems to guide my work. Sometimes I wonder if I have moved away from my initial cognitive-based interest in research in mathematics education to address issues that focus largely on the social and cultural context, with mathematics playing a very peripheral role. As I look over my writing from the last few years, I notice that I often raise the question "where is the mathematics?" Mathematics plays a central role in my work and recently, in our current project, I find myself pushing for the mathematics in our activities and research discussions. My interest is in equity in mathematics education, where equity to me is related to access by **all** students to opportunities to engage in rich mathematics. In this paper, my goal is to share some examples from my research throughout the years, to illustrate the role of and the need for different frameworks in mathematics education research and in particular to argue for the need to combine cognitive approaches with sociocultural ones (Brenner, 1998; Cobb & Yackel, 1996). In doing so, I also aim to emphasize the need for a serious look at what we mean by equity. The word "equity" (or references along those lines) is present in most mathematics education documents (not only in the U.S., but based on my research collaborations with a colleague in Spain and conversations with researchers

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

elsewhere in the world, it seems to be a widespread, relatively recent phenomenon), yet what do we mean by equity in mathematics education? Personally, it is hard for me now to look at any mathematics education area without an equity lens. For example, I have always had an interest in how students communicate about mathematics. Yet, my interest in this topic has considerably changed over the years. In Civil (1998) I focused on issues related to communication when students are working in small groups. My approach then was essentially cognitive, as I was primarily interested in the interplay of understanding and beliefs in small group discussions. More recently, my interest in communication relates to questions of participation (Civil & Planas, 2004): who has a voice in classrooms' discussions and whose voices are being heard. Yet, as I reflect on this more recent work and look at my current work, I am looking for how to frame the discussion of participation in such a way that mathematics becomes more central.

Research in Teacher Education: From Beliefs and Understanding to Equity

My first experience presenting at a conference was actually at PME-NA in 1989 (Civil, 1989). In that piece, a group of preservice elementary teachers were given a proportional reasoning task in which a fifth grader used incorrect reasoning (an additive approach) but the answer he obtained happened to be the correct one (at least in terms of a typical school mathematics task). The preservice teachers were to comment on this child's work. My emphasis in that paper was on questions such as "how ready are these prospective teachers to understand children's work. How are they going to handle it when one of their students comes up with a method different from theirs? What means do they have to determine the validity of a method?" (p. 292). I expressed concern for what I saw as a tendency to praise children's work without attention to the mathematics behind that thinking (Civil, 1993). Years later, I continued to express this concern, when I visited "reform" oriented classrooms in which children were encouraged to work in groups, discuss mathematics, look for different approaches to solve a problem, in short, many of the features that I value in a mathematical community of learners. But I also noticed how hard it is to listen to children's ideas about mathematics and what to do with that listening. As a result, I often heard comments along the lines of "great thinking" (was it always "great"?) and "thank you for sharing" (with no further discussion on the mathematical contribution of that sharing). Working on understanding how others (in most cases, students) make sense out of mathematics is one of the main reasons why I went into mathematics education. Whether I am working with children, preservice teachers, practicing teachers, parents, listening to their ideas about a mathematical situation fascinates me. Where is "equity" in this? When working with preservice teachers (and later on, with practicing teachers), I think I had an implicit concern for equity in that I worried about how a fragile understanding of the mathematics, and in particular of the mathematics for teaching (Adler & Davis, 2006; Ball & Bass, 2000) would affect their teaching and therefore their students' learning and enjoyment of mathematics. But that was the extent of my concern for equity at that time. In fact, I am not even aware that the term "equity" entered my conversation. In this paper I look at some of my work from those years with my current lens of equity. A typical topic in courses for prospective elementary teachers is a discussion of different algorithms for arithmetic operations. For example, one of the tasks I gave to that same group of preservice teachers to discuss was the "European" subtraction algorithm (the equal addition algorithm). I presented it to them as the way I learned how to subtract and they were to try to make sense out of it. Although my analysis of their discussion focused on cognitive aspects (e.g., assimilation to borrowing), some of their comments can certainly be looked at from a different angle:

Ann: Could you imagine if they said, "let's do math this way in American schools"?
Carol: Oh, my God!
Vicky: I don't think the kids would have as much problem with this as the teachers.
Ann: Uh, uh, you're right; that's exactly what would happen.
Carol: What's the value though? I mean, why are we doing this?

When talking about yet another algorithm for subtraction, in which the child had used negative numbers to find the answer (e.g., to do 62 – 48: 2- 8 = -6; 60 – 40 = 20; -6 + 20 = 14), Vicky said, "I do believe that you could eventually convince him that learning to carry is easier and leaves less room for error." And when talking about a left to right algorithm for subtraction, Carol said, " Wouldn't kids get confused? From left to right, wouldn't kids get confused? If I sat down with a group of kids and said, 'Ok, this is how you do it,' and showed them from left to right, I would think that when you got to the real thing, that they would get upset or that they would be confused."

Scenarios like the ones I just briefly presented can be analyzed from an understanding / cognitive approach: how do these preservice teachers understand these different algorithms? They can also be analyzed from a beliefs approach: what do they tell us about these prospective teachers beliefs about the teaching and learning of mathematics (as well as about their own beliefs about mathematics in general)? But these scenarios can also be analyzed from an equity point of view. For example, what are the implications of Ann's comment, "Could you imagine if they said, 'let's do math this way in American schools'?" or Carol's comment, "what's the value though? I mean, why are we doing this?" or Vicky's comment: "you could eventually convince him that learning to carry is easier" or Carol's comment "when you got to the real thing." What is the real thing? Is there a way (as in only one) that should be taught in "American" schools? And in the case of subtraction, is "the way" that of learning to "carry"? Is this why Carol wonders about the value of engaging in these discussions around different algorithms? My current research is located in low-income neighborhoods, with a large number of immigrant families—mostly from Mexico and Central America. Of particular concern to me is whether we are preparing teachers to address different approaches, particularly when those different approaches may be coming from low-income, immigrant children. About four years ago, I asked a class of preservice elementary teachers to write a reaction paper to an article by Perkins and Flores (2002) on the "mathematical notations and procedures of recent immigrant students." A few of the preservice teachers wrote comments indicating the need for immigrant students to learn the way arithmetic is done in the U.S. As one of them wrote, "this is nice but they need to learn to do things the U.S. way." Is it that they were concerned about their own understanding of these different ways, as one of the preservice teachers hints in the comment, "how can we be expected to know all these different ways?" Or is it related to valorization of knowledge (as in one way being better than the other) (or as in Carol's comment earlier of "what's the value though?").

With the rapidly changing demographics in the U.S., most teachers are likely to be in classrooms where children or their parents may have different approaches to doing mathematics. How do we incorporate or build on these approaches? What value do we give to the different approaches? The notion of valorization of knowledge is very present in my work, as it relates not only to my concern for equity but also to my other area of research on in-school / out-of-school mathematics. The next section explores this notion.

Valorization of Knowledge

While in the previous section my focus was on preservice teachers, here I will focus on children / school-age students and parents. For the last ten years I have been conducting research around issues of parents' views on the teaching and learning of mathematics (Bratton, Quintos, & Civil, 2004; Civil & Andrade, 2003; Civil & Bernier, 2006; Civil, Bratton, & Quintos, 2005; Civil, Planas, & Quintos, 2005). Throughout this research there is a recurrent theme that emerges in our conversations and interviews with families. This theme relates to immigrant parents' views on how their children are being taught in the U.S. versus the schooling traditions in their country of origin (which in my context is usually Mexico). As everybody else, parents bring their valorizations of knowledge to the discussion. Let me illustrate this point with an example related to different algorithms to show how this topic is of concern not only to teachers and preservice teachers.

All the parents we have talked to who learned how to divide in Mexico comment on their method being more "efficient" and "cleaner." A basic difference between the way they learned and the "traditional" approach to long division in the U.S. is that in Mexico they do not write down the subtraction, "we do it in our heads", and they only write down the result (the answer). This is what Marisol and Verónica said about the division algorithms:

> **Marisol**: When I looked at how he [her son] was dividing, he subtracted and subtracted and that he wrote all the equation complete I said, I even said, "this teacher wants to make things complicated. No, son, not that way! This way!" And he learned faster with this [Marisol's] procedure.
>
> **Verónica**: I tried to do the same with my child with divisions, that he didn't write everything, but he says, "no, no, mom, the teacher is going to think that I did it on the computer." "You don't need to write the subtraction son," I say, "you only put what is left."… "No, no, my teacher is going to think that I did it on the computer, I have to do it like that." "Ok, you think that… but I want to teach you how we learned." And I did teach him, but he still uses his method, and that way he feels safe that he is doing his homework as they told him to. The same thing with writing above what they borrow and crossing it out, I tell him, "and I remember our homework could not have any cross-outs," whereas his does.

A topic of concern for many of the families we have interviewed is their perception that the level of education is lower in the U.S., often commenting that they thought their children were behind in mathematics compared to relatives or friends in Mexico.

> **Ernesto:** I think that the educational level, in the case of my son, the schools are very basic the level in Mexico is much higher. I'm saying that because I have nieces and nephews there and here and there, I see that they have learned more things at school…No, it's that he's [one of his nephews in Mexico] in fourth grade and my son is in fourth grade too. What they're giving my son now, he (the other child) learned in second grade. So, the educational level is lower and they learn more slowly than they learn in Mexico
>
> **Bertha:** No, I'm not happy. I feel that there is repetition of a lot of things; I don't understand why the teaching is so slow, I don't like it, I don't like the system, I don't like it at all. I, when we go to México … my nieces and nephews or my husband's nieces and nephews, there are children that are more or less the same age as Jaime and I see that Jaime is behind. Here they tell me that Jaime (is) really excellent.

Researchers have made observations similar to those of our participants. McLaughlin (2002) suggests that Mexican students' mathematics background often exceeds the expectations they face when entering a school in the United States. We also have data from the children's experiences with the different educational systems. Lucinda, one of the mothers who was concerned with her daughter's schooling in the U.S. and wanted to also teach her the way she had learned in Mexico, commented that when they first arrived, her daughter was a third grader (8 years old) and was not very happy with the school in the U.S. because she said that it looked like play, "why, mijita?" asked her mother; "because they are making me do 4 +3, mom, I don't want to go this school. It's weird." And by "weird" she meant "easy."

Below is an excerpt from an interview with a sixth grader in 2001, when he had recently arrived from Mexico:

Researcher: Describe yourself as a math student

Student: I am advanced because in Mexico the schools are a year ahead. I am very fast at doing things. The teacher gives me harder work. (…)

Researcher: What is your best subject in math?

Student: Algebra

Researcher: You already know algebra?

Student: Yes

Researcher: Where did you learn algebra?

Student: The teacher [name of his current teacher] showed us. In Mexico, they had already taught me algebra. And the teacher here is barely starting to teach some algebra.

As part of our more recent work, we continue to study parents' and children's perceptions of the teaching and learning of mathematics, in particular among those who have experienced two educational systems (e.g., U.S. and Mexico), but we are also playing close attention to issues of language and how they affect students' learning of mathematics. The excerpt below is from an interview with a mother and her son (a sixth grader), about four months after they arrival to the U.S. This interview underscores the child's and mother's frustration at knowing the mathematics but not having the language (English) to participate or to fully understand the teaching:

Marta: So, I would like to know, if you can explain to me, if I went to your school in Mexico, when you lived in Mexico, what would I see in a math class? Tell me a little bit

Alberto: There, they teach things that here… there they teach you… they are ahead

Marta: They are ahead.

Alberto: And here, they teach me things too, things that they taught me there… but what they taught me there, I already know it here, it's just that here it's hard because of the English. (…)

Alberto's mother: What I feel is that yes, I notice that they teach them more things there. Now, here the difference is that you run into the language, because in this sense… That is, for them it's perfect what they are teaching them because in this way it's going to help them grasp it, to get to the level, because for them, with the lack in English that they have, and if to that we were to add, uh, what's the word? If they give them all the information, like a lot, very dense, too much teaching during this period, to tell you the truth, it would disorient them more. Right now, what he is learning, what I see is that it's things that he had already seen, but if he gets stuck, it's because of the language, but he doesn't get stuck because of lack of knowledge. (…)

Marta: So, you think that since he has already studied it in Mexico, the content, that this to a certain extent helps him

Alberto's mother: It makes it easier (…) Because he says, "ay, mom," he says, "and things that they ask and that they are really easy, and I get desperate because I want to answer, because I understood. And there are other things that I don't understand, but once I see the answer, I realize that I already knew it. … But I didn't understand the question. If I didn't understand the question, I cannot answer it, because I didn't understand them."

Alberto's mother thinks that it is a good idea that they are teaching him something that he already knows because he does not know English well yet and it would be too much for him to learn new content and English. Is this an equitable approach to the teaching of immigrant students? In Anhalt, Ondrus, & Horak (in press), the authors discuss an experience in which an instructor taught a mathematics lesson in Chinese to a group of middle school teachers. The teachers (most of whom were part of our Center CEMELA[2] and therefore, taught a large number of English Language Learners (ELLs)) realized the similarity in trying to learn in Chinese to their students' learning in English. Some of the teachers observed that because they were familiar with the mathematical content, they did not pay attention to the Chinese language and focused only on the mathematics. Teachers reported this was a powerful experience that made them think about the policy of student placement in their schools. It made them wonder about a common placement practice that places ELL students in lower level mathematics, the idea being that it will help them learn English. Teachers questioned whether through this practice students would learn neither English nor mathematics.

Are the educational needs of immigrants students being met by lowering the level of the content so that "they can learn the language"? This situation is not unique to the U.S. For the past several years, I have been collaborating with Núria Planas, a researcher in Barcelona (Spain) whose work focuses on the mathematics education of immigrant students in that city. Until 2000, immigrant students in public schools in Barcelona were placed in special classes with students with learning difficulties and physical disabilities. Currently, students with "language problems" (e.g., immigrants) are in a separate program for part of the day primarily for two subjects (mathematics and language). In that program they still work on the same adapted curriculum (as students with learning difficulties), which usually covers material two or three grades below their current grade.

In the first part of this section on valorization of knowledge I have focused mainly on parents' perceptions, and in particular immigrant parents, of the mathematics education their children are receiving in their "new" country. One could say that this is normal generational discourse—parents trying to show their children how they were taught because they feel that it was a "better" way. I argue, however, that these differences in approach take on a different light when those affected are low-income, immigrant families, whose knowledge has historically not been recognized or valued by institutions such as schools (Abreu, Cline, & Shamsi, 2002). This notion of their knowledge not being recognized or valued may even be more exacerbated if these students are given a lower level curriculum and made seem seen as "deficient" because they are not proficient in the language(s) of instruction. Planas (in press) looks into this situation by focusing on local students' perceptions of their immigrant peers' knowledge. In her research study, Planas interviewed twelve 15 and 16 year-old non-immigrant students from the same classroom in a high school in Barcelona that had a high percentage of immigrant students (60% of the students were from Morocco). In that particular classroom fourteen out of twenty-eight students were immigrants (Morocco, Dominican Republic, Pakistan, and Bangladesh). The school, as is the case with schools with high numbers of immigrants in Barcelona, is in a low-income neighborhood. Planas' research is particularly insightful in that it seeks to understand issues related to immigrant students in the mathematics classroom, not only from the point of view of these students, but also from the point of view of the "local" students (see Planas & Civil, 2002; Planas & Gorgorió, 2004). Planas' (in press) findings point to a deficit view on the part of the local students towards their immigrant peers. Attached to this deficit view is a lack of recognition and appreciation for the immigrant students' ways of doing mathematics. The "local" students point out that their peers' mathematics are different and these different forms of mathematics are not seen as useful or appropriate, as the quotes from two of these local students show:

> **Pau**: Their [immigrant students] comments help us make sense of the situations before starting solving the problems, but anyway, we cannot always start making sense of it like they do. Our maths are what they are. And theirs… they are fine, but sometimes they just don't fit in.
>
> **Maria**: We are not in the classroom to learn their mathematics but to learn ours. That's what the exams are about. (…) I am not expected to learn Murshed's way of subtracting.

It is well known that there are different algorithms for arithmetic operations. My point here is not about "the" way to divide in Mexico vs. the U.S. or "the" way to subtract in Spain vs. Morocco. My point is about whose knowledge is being valued and how these different valorizations may affect students' participation in mathematics classes. This brings me to a key concept in my research—the notion of participation.

Does Everybody have a Voice?

> I try to understand and in class, I listen and ask questions but most of the time I have absolutely no idea what is going on. And what my peers say to me sounds like a dialect of the Alaskan Eskimo. [Carol, preservice elementary teacher]
>
> There is hope yet when I can legally use my methods to solve a problem. [Vicky, preservice elementary teacher]

Carol and Vicky were students in the same section of a mathematics content course for preservice elementary teachers, for which I was the instructor. In this course I used a discussion-based approach, in which students largely worked in small groups on mathematical tasks that were often intended to create cognitive conflict (such as the proportional reasoning task alluded to earlier (Civil, 1989)). My approach (though not so clearly formulated at the time) was grounded on the idea of developing a community of learners in which students would feel comfortable questioning approaches and procedures they had taken for granted (e.g., why do we "invert and multiply" to divide fractions?). By encouraging different approaches to solving problems, I was aiming to open up the patterns of participation and, if possible, to undo the labeling that tends to classify learners as being "good at math" or "not good at math." My efforts failed with Carol, who often expressed her frustration at an approach that in her view exposed her as a failure to her peers and the instructor. She would have preferred a lecture-based approach, in which she was allowed to "remain anonymous." Vicky was also very anxious about her mathematics knowledge and kept on saying how she "couldn't do it using algebra" and would often look up to her peers who could use algebra. But Vicky became more comfortable participating in group discussions once she saw that her methods were accepted. Although she tended to label her methods as not being the "math way," the fact is that her approaches were often conceptually clearer than the more "traditional" ones her peers used. For example, for the following problem, "*If you need 1 1/3 cups of sugar and 4 cups of four to bake a cake, how many cups of sugar will you need if you want to use 7 cups of flour?"* Vicky drew the cups of sugar and flour and immediately saw the correspondence between 1/3 cup of sugar and 1 cup of flour, thus concluding that she would need 2 1/3 cups of sugar. Vicky seemed to approach the problem in what could be called a more informal way, using everyday type reasoning. The rest of her peers opted for an algebraic approach quite typical of how ratio problems are solved in school. Some of them became lost in the procedure, due to their difficulties handling mixed numbers.

This is just one illustration of the many examples that I have encountered in my work with preservice elementary teachers (and now more recently with parents), in which adult learners who often feel unsuccessful in school mathematics, bring in ways of reasoning that are clearer and more efficient than the school-based procedures. Certainly, the issue of how general are these informal / out-of-school procedures remains. Would Vicky have been able to solve the sugar-flour problem had the numbers involved been others? Or when is it appropriate for students to bring in their everyday knowledge? For example, Cooper and Dunne (2000) illustrate some of the problems that occur when students (particularly working class students) "import their everyday knowledge when it is 'inappropriate' to do so" (p. 43). But my interest in these out-of-school approaches is on their potential for the participation of more students in the learning process.

My interest in the concept of participation started with trying to understand the obstacles to participation in the sense of students not feeling confident in or not valuing their own approaches to mathematics because they were not the "school way." The large body of research on situated cognition and on out-of-school mathematics versus in-school mathematics is particularly relevant to my work (Brenner & Moschkovich, 2002; Brown, Collins, & Duguid, 1989; Lave, 1988; Nunes, Schliemann, & Carraher, 1993). Many of these studies document how successful and resourceful people are at inventing their own methods of solution to tackle tasks that they see as relevant in their everyday life. Yet, some of these studies also document a lower performance once a "similar" task is presented in a school context. To me, a key question is that asked by Hoyles (1991), "*is it possible to capture the power and motivation of informal non-school learning environments for use as a basis for school mathematics?*" (p. 149) (italics in original). This interest in bridging in-school and out-of-school mathematics and thus my interest in opening up the participation patterns moved from a somehow cognitive emphasis (as in an intellectual interest in different approaches to problems) to a more social and cultural emphasis when I started working in primarily low-income, Latino communities. I was struck by how resourceful and involved in the everyday working of the household some of the children were, while these same children were not particularly "successful" by school standards (Civil & Andrade, 2002). I was intrigued by what it would look like to try to develop learning experiences that would build on these students' (and their families) knowledge and experiences while ensuring that they advance in their learning of academic mathematics.

In Civil (in press) I discuss some of our efforts towards developing a mathematical apprenticeship in a school setting by embedding the mathematical learning in the "context of a sociocultural activity in which the pupils want to participate and in which they are able to participate given their actual abilities" (van Oers, 1996, p. 104). A construction module in a second grade class highlights my dilemmas at developing an approach to teaching and learning that emphasizes collaboration and engagement in activities that are important to the practices of the community (Lave, 1996; Rogoff, 1994), but also brings the mathematics to the foreground. The garden module in a fourth/ fifth grade classroom presents an example in which sociocultural practices are combined with cognitive approaches (e.g., enlarging a garden is followed up by a school task on exploring area of an irregular shape; task-based interviews are used to assess students' understanding of certain aspects of measurement).

Finally, there is another aspect of participation that was probably present all along in my work but has become more prominent in my recent research. It relates back to the concept of valorization of knowledge and whose knowledge is valued / recognized and when or where. We have looked at these issues in relation to the concept of norms (Yackel & Cobb, 1996), as in, for example, how immigrant students may be interpreting the norms differently from local students (Civil & Planas, 2004; Planas & Civil, 2002). In Civil (2002), I document a teaching innovation in a fifth grade class in which we tried to combine three forms of mathematics: school mathematics, mathematicians' mathematics, and everyday mathematics. Although some of the patterns of participation changed and opened up, overall the social and sociomathematical norms that were in place (and conflicted with those from our innovation) and the influence of status and students' perceptions of each other played a role in who had a voice when in the classroom. The "popular" students (which in that school often meant those in sports teams such as basketball or softball) and the students in the Gifted and Talented program had a voice in the mathematical discussions. As Lampert, Rittenhouse, and Crumbaugh (1996) write, "children do not readily separate the quality of ideas from the person expressing those ideas in judging the veracity of assertions" (p. 740).

Some Issues for Reflection

In this section I raise what I view are some issues derived from what I have presented so far in this paper. One such issue relates to the difficulties in developing mathematical learning experiences that while being true to the context (e.g., the construction or the garden modules) are also true to our mathematical agenda (Civil, in press). These difficulties, I argue, have to do largely with our values as to what we count as mathematics, as well as our own academic training that may make it harder to uncover the mathematics in everyday contexts. As a teacher in one of our study groups once asked, "if you have too much school mathematics, does it erase our practical mathematics?" In our work we have found the pedagogical transformation of community knowledge into school mathematics learning opportunities to be a non-trivial endeavor.

Students' views as to what they are willing to count as valid mathematics also play a key role in this process. Do students view the mathematics in the teaching innovations as "real mathematics"? Students may have indeed been involved in rich mathematical opportunities but if they do not see what they did as the mathematics they should be learning in school, or if the connections to what they may expect to see in the next grade are not made, are we helping these students? As I reflect on our efforts to try to bring change to the teaching of mathematics in these classrooms, I find Spradbery's (1976) work particularly relevant (even though it is 30 years old!). Spradbery describes the experiences of a group of sixteen-year-old students unsuccessful at school mathematics. Outside school, some of these students kept and raced pigeons. The author goes on to describe some of the mathematics embedded in this practice, and then writes:

> Although the mathematician may regard certain aspects of pigeon-keeping (along with many of the other daily activities of children) as being 'mathematical', such

> knowledge appears to have little value or status in the classroom. For ‘Maths’ to be ‘Maths’ (or ‘proper Maths’, as a number of children described it) it has to be separated from other everyday knowledge. (p. 237)

Spradbery (1976) describes the opposition among students—who had so far failed at school mathematics—towards an innovative curriculum that was intended to be more liberating by presenting situations for which the students were encouraged to use their own intuitions and knowledge. In our work, we also had students questioning whether what they were doing was mathematics. Currently, I often find myself wondering about innovations that try to contextualize the mathematics in situations that claim to be more relevant to the reality of the working-class Latino students with whom we work. Who decides what is relevant?

Another related issue in our work relates to the notion of norms (Yackel & Cobb, 1996). The teaching innovations we have tried to implement combine notions of communities of learners in which students engage with mathematics as mathematicians (Lampert, 1990) with notions of apprenticeship learning and other characteristics of out-of-school learning. The norms for these innovations had to be very different from the norms that were often in place (and well established) in the school where we conducted our work. Students resisted our approaches. One of the main obstacles we encountered with the fifth graders was their reluctance to engage in discussions about mathematical problems. As the teacher explained, “they [students] didn’t see the point of the discussion; they didn’t like waiting on everybody to talk. …They didn’t feel like that was work. To them, work is filling out worksheets and turning the paper in and seeing if they got it right or wrong” (Civil, 2002; in press).

A Closer Look at an Example of Bridging In-school and Out-of-school Learning

I have briefly referred to some of the teaching innovations that we tried over the years and that are described in more detail elsewhere. Here I describe a teaching experience that took place quite a few years ago and that for many will seem like a trip to the past because it is based on the use of a Logo environment. What we used (i.e., Logo) is not the point here; what I want to do is to illustrate some of the issues raised earlier in this paper and provide an example of an experience in which we tried to develop a learning environment that captured some of the characteristics of out-of-school learning while pushing for the mathematics. The choice of Logo reflects my cognitive inclination, as it provides an environment in which with very basic few commands, students can right away explore mathematics. At the same time, our approach took into account the context, hence showing my socio-cultural inclination. This was a particularly difficult fifth grade classroom in which the academic agenda of the teacher (who was new to the school) clashed with the children’s agenda. One of the researchers conducted interviews with each of the children to gain a better understanding of their social context. As she shared with me, “learning about what some of these children were going through in their daily lives made their resistance to change in the classroom quite understandable. Change for many of these children came as yet another form of upheaval in their already hectic lives.” It was under these conditions that we tried to create a change to the teaching and learning of mathematics. A key feature of the learning innovation we envisioned was the sharing of ideas in a safe and supportive environment. Establishing this feature was a major obstacle in this classroom, given that the class was dominated by a core power group of five boys, who through popularity and intimidation manipulated the classroom dynamics.

We introduced the use of Logo to the whole class and after a few sessions, we offered the option to continue working in the computer environment. A group of eight children expressed interest in doing this (but one dropped after two sessions). The rest of the class (17) stayed in the regular classroom with the teacher. With this group of seven children (who were representative of the diversity in the classroom) we set out to develop an environment that would have some of the characteristics of out-of-school learning, which we had identified from the literature (Brown, Collins, & Duguid, 1989; Hoyles, 1991; Lave, 1988, 1996; Rogoff, 1994), namely: 1) Learning by apprenticeship; 2) Working on contextualized problems; 3) Control remains largely in the hands of the person working on the task; and 4) Mathematics is often hidden; it is not the center of attention and may actually be abandoned in the solution process. Rogoff (1994) discusses three models of teaching and learning. In the transmission model, knowledge from others is passed on to the learner (adult-centered) and in the acquisition model the learner discovers the knowledge on her or his own (child-centered). In the participation model, the learner participates in a community of learners. In this model, learning takes places through collaboration and engagement in activities that are important to the practices of the community. I argue that this group of seven students learned through participation in a community of practice that emerged in the computer lab. Logo was new to all students in this class. In a sense Logo had an "equalizer effect" that may have allowed for a fresh start for all the students who stayed in the group. They were able to put aside their roles and labels. It allowed for children who had been labeled as being "less successful" in academic subjects to shine as they demonstrated their expertise and as they followed their own inquiries. Students who barely spoke to each other in the regular classroom started trading discoveries about Logo. For example, one student became an expert at using color; another became an expert at writing procedures on the flip side (this is the area where procedures are written, as opposed to immediate mode programming); another student, using his prior knowledge of the computer system, explored all the different "gadgets" included in Logo. What we soon noticed was that students were constantly sharing ideas and were well aware of who knew what. We saw "learning traveling through the lab" and students learning through interactions with their peers. It is in this sense that we believe that what took place in this Logo group had some of the characteristics of learning through apprenticeship. This idea of apprenticing is also addressed by Moll and Greenberg (1990), who write, "it is when the content of interactions is important or needed that people are motivated to establish the social contexts for the transfer or application of knowledge and other resources" (p. 326). I next give a glimpse of the dynamics in this group with a focus on the mathematical explorations.

A Mathematical Community of Practice?

While working on a task on how to draw a hexagon, two students (Daniel and Ben) found a way to make a variety of star polygons. Soon thereafter, two other students (Jennifer and Jorge) became interested in drawing star polygons. We discussed with Jennifer and Jorge some of the mathematics involved in this task. From a previous discussion on turns (grounded in part on several students' knowledge about skateboarding), these two students were familiar with the notion of 720 for two whole turns. They then found 720 ÷ 5 and tested the procedure: **repeat 5 [fd 40 rt 144]**, which produced a five-pointed star. Following this, Jorge started working on looking for divisors of 720 using, at times, the calculator available in the computer. For example, he came up with: 720 = 72 x 10. He then typed: **repeat 10 [fd 40 rt 72]**, except that this command did not produce a star polygon. On his own, Jorge continued exploring and found: **repeat 9 [fd 70 rt 80]**, which does produce a star polygon. Through a process of mathematical calculations and manipulation of the commands, Jorge found combinations that produced star polygons and recorded them on the flip side.

As Jorge and Jennifer were working on star polygons, the question of "how to draw a circle" arose. Several of the students became interested in this challenge. On the blackboard, Jennifer drew a 90º angle, and then a 40º (exterior) angle. She then said, pointing to her drawing, "it's going to have to be a very subtle turn, a subtle angle." She had the correct image of turning minutely (a subtle turn) every time, however, she did not quite know how to do it on the computer. Ben, meanwhile, did get the turtle to draw a circle. In response to Ben's accomplishment, which came as a result of step-by-step commands, our challenge to him was to draw a circle in one single command. Though the challenge was made to Ben, it was Jorge who took it up and explored with the template **repeat __ [fd 1 rt 1]**. After trying several inputs, 360 clicked all of a sudden. Sara, meanwhile, who was not sitting near this group of students, had also figured out how to make a circle and quickly shared it with Carolina and Judith who ended up incorporating a procedure to draw a circle as part of their final projects. Daniel also ended up working with circles for his final project because he became interested in creating a spiraling tunnel. To do this he needed circles of differing sizes, which led to his learning about variables in Logo. Daniel ended up with this procedure:

```
To c :n
 repeat 180  [fd :n rt 2]
end
```

In order to get the desired effect of increasingly larger circles within close proximity to each other, the need for decimals arose. We were not aware of what these children knew about decimals. We suggested that he try c .7, c .8. Daniel explored with these numbers. The first "surprise" came up when, after typing c .9, he typed c .10 and a smaller circle appeared. He then tried c 10, which gave him a much larger "circle" (actually, it did not look at all like a circle!). Following this, we drew a number line on the blackboard and asked Daniel to place .9 and 10 on a number line. He appeared to be confused by this question. Jennifer, who was working on something else but sitting near the blackboard, became interested in the conversation and helped him out. Daniel seemed unclear about what decimals were. He was not sure about what "that point," as he called it, was doing. We then switched to decimal fractions, since the students had worked with these in class. This seemed to help, but it was difficult to tell whether there was understanding or rather pattern recognition taking place. In looking at 12/10, 14/10, Daniel quickly supplied 13/10 and 11/10. His final project, however, went back to the decimal notation, this time in an increasing sequence from .1 to 1.9 (by increments of .1). In the presentation of their Logo projects to the whole class, students were intrigued by Daniel's procedure "**c**" and the various inputs. They asked Daniel to try c .50 and were surprised by the fact that it was the same as c .5. They then suggested c .47, expecting something bigger, and again were puzzled with the outcome. Looking back at Daniel's work, we think that similar scenarios could be turned into an opportunity to explore students' understanding of decimal numbers.

I would like to emphasize two salient features of the work with these seven children, at the social and cognitive levels. The social behavior governing our work in the lab with these seven children was a complete change from that present in the regular classroom. While in the lab, we did not witness any put downs or negative comments directed at each other or at us (several of these same children did not get along in the regular classroom). Students behaved in a very polite manner, informing us when they were leaving the room, and being overall very cooperative. This occurred in a natural way, since we never talked about what behavior we expected from them. We focused on their work on the computers. The students seemed relaxed, happy to be in the lab, and while quite individualistic in their work on the computers, there was considerable chitchat and sharing of ideas.

Engaging with the students in conversations about their work was difficult. The teacher had mentioned to us that many of the children in the classroom did not seem to trust adults and were not willing to converse. An added difficulty, perhaps, was that we were trying to dialogue with them on an academic topic. As we tried to have them tell us more about what they wanted to do or what they were thinking, we were often met with silence or comments we could not quite follow. But with most of them, we succeeded in engaging in a conversation on their problem solving strategies for their project. For example, with Jennifer, it was on patterns; with Daniel, decimals; with Jorge, how to make sure the football field fit on the screen and was an accurate (to scale) representation; and with Carolina, how and where to put the moon and how to obtain a visually aesthetic effect "more efficiently." The cognitive gains became clearer as the students presented their projects to the class as a whole. The Logo students were knowledgeable and comfortable with the language of the Logo environment. Each child spoke as an expert, demonstrating considerable command of the situation and clarity in the presentation, in what was for many of them a less than receptive audience (i.e., the reality of the classroom).

A Look at Our Current Work

The Logo project I just presented allowed me to experience some of the theoretical constructs such as characteristics of out-of-school learning, communities of practice, mediation, and constructivism in the context of students engaging in mathematical explorations. That work underscored the importance of understanding and paying attention to the social context. In our current work we are determined about the concept of context as we take a holistic approach to the mathematics education of working-class Latino children: parents, teachers, children are central to our work. Most of my research at the moment takes place at a school that is 90% Latino, 26% English Language Learners, and with 95% of the students eligible for free or reduced lunch (the average for the state of Arizona is 49%). At this school we have several research activities in place, including: 1) a teacher study group aimed at teachers reflecting on their practice; teachers explore mathematical content for themselves, as learners, but also reflect on students' work, and engage in discussion around language and mathematics; 2) classroom visits to not only observe the mathematics instruction but also to support the teachers and students; 3) a parental component in which parents take part in mathematics workshops (sometimes with their children and facilitated, in part, by their children's teachers); 4) an after-school mathematics club in which children are encouraged to work on contextualized mathematical projects (e.g. a garden; getting to know your community) and where both languages (English and Spanish) are used (bilingual education is severely limited in this state, thus children have few opportunities to engage in academic discourse about mathematics in Spanish during their regular school hours). All these research activities emphasize our focus on teachers, students, and parents.

A focus on teachers

The mathematics curriculum in place at this school is "reform-based" and the teachers with whom we work are working hard at implementing it. The curriculum is relatively new for most of the teachers and, as with any curriculum, it portrays a certain view of what it means to do mathematics. In some aspects, that view is not very different from what teachers at this school have been engaged in throughout their participation in professional development experiences.

For example, in these experiences they learned about group work, hands-on materials, open-ended problems, and teaching for understanding. But, as I discuss in Civil (2006), some of these professional development efforts aimed at helping teachers teach using a “reform-approach” leave me wondering whether teaching mathematics was becoming a smorgasbord of activities with no apparent road map. At this school the fact that teachers are expected to use a specific curriculum provides to a certain extent this road map that I saw lacking at other places, years back. This is not to say that it is unproblematic. I am particularly intrigued by how teachers make sense out of the curriculum and how they decide on which tasks to implement and what that implementation looks like. The work of Stein, Smith, Henningsen, and Silver (2000) on the cognitive demands of tasks is particularly helpful to my analysis of classroom teaching at this school. But in addition to the content demands of the tasks, I am also interested in what kind of support (affective and cognitive) the teacher and students create in the classroom to encourage the mathematical participation of all students. For this area, Empson’s (2003) use of participant frameworks in her analysis of two low-performing students is quite relevant as I look at interactions in the classroom. And a third area of interest is what view of what it means to do mathematics is being conveyed or co-constructed in these classrooms.

I am currently in the middle of analyzing data from several videotaped lessons from a fourth/fifth grade classroom with my focus being on mathematical discourse and in particular on students’ reasoning. These three areas of interest I just mentioned (nature and demands of tasks; support; view of mathematics) are closely intertwined in my analysis. For example, one of the lessons focused on solving word-problems on multiplication and division. The students worked in groups and for each problem they had to 1) demonstrate how they solve the word problem; 2) write an equation; 3) solve; 4) explain the process. One of the problems was “*a restaurant serves different types of sandwiches; it has four different types of meat (turkey, ham, baloney, and roast beef) and three different types of cheese (Swiss, Cheddar, and Jalapeño). How many different sandwich combinations can the restaurant sell?*” Students approached this problem in a variety of ways, using several different representations in their solutions. All the students but two interpreted the problem as “expected,” thus leading to 12 different kinds of sandwich. These two students, who were working together, tried to find different combinations, that is, with 2 kinds of meat and 1 cheese, 3 kinds of meat and 1 cheese, or 2 kinds of meat and 2 types of cheese, and so on, making it a much more demanding problem. The teacher encouraged the different groups in their work, as she walked around the room. She then asked some of the groups to present, and as they presented, she asked them questions that related to the idea of making sense “why did you choose to multiply?”, or “why did you decide on that equation?” When a group presented their work on a different problem in which they had first divided incorrectly (and they showed that incorrect way), the teacher said “you see, they did it four different ways, and three didn’t make sense to them, but they kept with it and they got it.” And later on, she said, talking in general about the different ways to solve a problem, “remember, you never give up; look at all the different ways you could do it.” It is true, however, that the teacher seemed somewhat at a loss with the two children who approached the sandwich problem by looking at combinations differently form all the other students. But although she did not really probe these two children much on the mathematics, she encouraged them to pursue their thinking and invited them to present it to the class. She certainly offered them (as well as the other students) affective support and to an extent cognitive support. Through her emphasis on making sense, looking for and at different ways, persistence, this teacher is trying to convey a view of mathematics as an area of inquiry and creativity.

Our analysis of classrooms is only a piece of this holistic approach in which we try to understand the interactions of the linguistic, cultural, social and political contexts with the teaching and learning of mathematics by paying attention to the children, their parents and their teachers. In the next two sections we focus on parents by looking at parent-child interactions around arithmetic, and we focus on students, by presenting an incipient case study of a child to illustrate how we are looking at children as learners of mathematics.

A focus on parents

In Civil, Planas and Quintos (2005), we use a Bourdieuian perspective to interpret parents' perspectives on their children's mathematics education. In that article we argue for the need to know more about students' social contexts and in particular about their parents' perceptions of their children's mathematics education as part of our efforts to gain a better understanding of students' performance in mathematics. As Marisol, a mother in a previous research project tells us, "parents and children come together." Thus, at this school we are working on this idea of parents and children coming together and, based on a prior research project in which parents and teachers worked together, we are also building on the lessons learned from that experience (Civil & Bernier, 2006) and bringing in the teachers.

A particular emphasis of this school's curriculum is the development of flexibility when working with numbers. For example, in the 4th/5th grade class, to multiply 23 by 14, a student may do (20 + 3) (10 + 4), while another student may do (5 + 5 + 4) (10 + 10 + 3)). Some of the students seem to enjoy coming up with quite complicated and, I would argue, rather inefficient ways to break the numbers. But they appear to enjoy doing this (and in some cases, when I have asked them, they are aware that there are more efficient ways to break apart the numbers). Do they do it for fun? Or do they think that that is the goal of the activity, to come up with "complicated" ways to break apart the numbers? If that is the case, what view of mathematics are they developing? How are the tasks being interpreted is a question that I find myself asking, and not only about the students but the teachers too, as some of these approaches to doing mathematics are new to them also. How tasks are being interpreted by the different parties involved is particularly important in classrooms that are trying to implement a different approach to mathematics teaching and learning, as Lerman and Zevenbergen (2004) point out:

> Bernstein (1996) is detailed in explaining how power and control are translated into different pedagogies; the implications are that if students are to be successful they need to recognise the unspoken, or invisible, aspects of some pedagogies, particularly reform ones, as we discuss later. Two important considerations need to be made; one is how tasks are framed for students—the issue of contextualisation and recontextualisation--, and the second is how they are answered by the students—the issue of recognition and realisation rules (p. 29).

In our workshops with parents and children one of the goals is to introduce the parents to these other ways to do arithmetic. So, for example, at the 2nd/3rd grade level, to do 23 + 46 +7, children may do 20 + 40, then 7 +3, and finally add the 6 to the prior result. What we are currently analyzing shows the parents trying to teach their children the way they were taught (the "traditional" algorithm) in a very procedural way, with an emphasis on how to write it "correctly." For example, in the addition 23 + 46 + 7, a father showed his daughter how to write it vertically and put a 0 in front of the 7 to keep the numbers lined up. In another case, I was working with a mother and her son; I explained to her (based on the handout the teacher had given them) a way to do 51 – 22 by first doing 30 – 22, 8, then adding the 21 we were missing from the 51. I next suggested to her that she try this strategy on 42 – 13; instead, the mother went back to 51 – 22, set it up vertically and started explaining in to her son "if you have 2, to get to 10"; she wrote "8" but then realized that that is not what she wanted. I brought up again the strategy I just showed them, and she said, "but to me, it's easier this way" [pointing to the vertical set up she had written on the paper]. She then walked her son through the "standard" subtraction procedure, "when the number on top is smaller, you ask for 1 from the one next to it, …". The mother walked him through 51-22 and then encouraged him to try 42 –13. The mother helped him make the "2" into "12" and then the child asked, "do I put a 5?" pointing to the 4 in 42. The mother said very calmly, "no, no 3." After this, I tried to explain to the mother one of the approaches to subtraction that they were using in her son's classroom, showing her how they start at 13, then they may jump to 15 (by adding 2), then to 20 (by adding 5), then maybe to 40 (by adding 20) and finally to 42 (by adding 2), and then they add all the jumps to find the answer. As I was explaining this, I remember thinking "she is probably wondering, why are we teaching this, when her method is so much quicker." We are in the preliminary analysis of these interactions but we can already see how different views about teaching and learning mathematics are in play and can potentially come into conflict, even with the parents who are coming to the workshops and therefore being exposed to how and why their children are being taught this "different" way.

A focus on learners: The case of Julián

Julián was born in the U.S. but his parents are from México and he speaks Spanish at home. By fifth grade he had attended five different schools, 2 in México and 3 in the U.S. Part of the reason for the change in schools had to do with his family moving, but another part was his not feeling comfortable and thus changing schools, "the first school I went to was, Kindergarten, and there were these kids that always called me names. It hurt me. And my teacher never understood me" [interview, February 2006]. When referring to a more recent school, from which he also moved out to come to "our" school), he said, "she [the teacher at that other school] embarrassed me. … Sometimes she was okay, but, but then when I asked questions, she said, 'why are you asking the same question over and over?' And then, that's when, ah, when she embarrassed me, and I didn't understand it. 'Well, you should, you should if you were paying attention,' and I was."

When asked about his perception as a student of mathematics in the classroom, he placed himself as third best. He takes his work very seriously: his homework is carefully written; during the scale-drawing project in the after-school mathematics club, he inquired about whether they were going to also make a three-dimensional model, which would then involve measuring the height of the walls. The facilitator said that it was up to them; Julián then told one of his peers (who did not seem interested at all in doing it), "you don't need it, I **have to**, I want to do the model." In many ways, Julián is a "school boy"; he is quite good at following the rules of school and following what the teacher tells them to do. On the problem-solving day I referred to earlier (the sandwich problem), another problem they worked on was: "*this year Mark saved $420; last year he only saved $60. How many times as much money did he save this year than last year?*" Julián and two other boys, Alberto and Leo, worked on this problem together. Alberto, who often seemed to be off task or claimed being "lost," right away said, "so, this is a division problem." Julián first said yes, then no and then said, "we need to subtract." As their first step on their paper they wrote the subtraction (420-60=360). Alberto, “we’re done”; Julián, “no, we are not… for an equation.” They seemed unsure as to what to write for an equation (this is a topic that I am currently trying to understand better, as the writing of equations seems to be very important for this teacher; it brings back the issue of how, in this case the teacher, is interpreting the task (i.e., why did she want them to write an equation for each of these problems?)). They finally settled on \$420 - \$60 = N for their equation and Julián moved to the third step of instructions on the board (1) demonstrate; 2) write an equation; 3) solve; 4) explain your process) and says:

Julián: solve it [as if reading the instructions]

Alberto: we already solved it.

Julián: we did, but how can we demonstrate it, yeah, we solved it up here (points at the subtraction) but…

Alberto: just say that (points at the subtraction)

Julián: that’s all we did

While Alberto clearly indicates that they are done with the problem, Julián looks worried as he feels that what they have done does not reflect the four steps they were asked to have; they have numbered their steps and they only have two. This episode brings up many issues: what was Leo’s role in all of this? He does not say anything, though appears to be following what his classmates are doing; Alberto had the right idea, to divide, yet Julián’s is the one that prevails, why? Alberto seems willing to let go of the instructions the teacher has given them, as he considers they have solved the problem, while Julián struggles to make sense out of the directions, in particular “demonstrate” versus “solve.” Does he want to have the four steps because that is what the teacher is asking for? Does he realize that depending on how they approach the problem, some of these steps may be unnecessary? How does Julián decide when to follow the rules and when to “challenge” them?

Julián does seem aware of the school game and the artificiality of school tasks. To support their exploration of factors, prime numbers and other elementary number theory concepts, I presented a problem that involves machines that stretch bubble gum, where machine n stretches a piece of gum to a length n times its original length. So, for example, if I wanted to stretch a one-inch bubble-gum stick to 30 inches, and the machine 30 is broken, I could use machine 6 followed by machine 5. Julián right away said to the whole class, “but it’s still the same amount of bubble gum.” He is right; the way the problem is worded, these machines stretch the lengths of the sticks of gum but do not increase the amount of actual gum. Julián understood the problem and was able to offer different combinations for the various numbers I gave them. He knew how to interpreted the task from a school point of view. I doubt that Julián in a formal assessment situation would let his everyday or common sense interpretation “interfere” with his performance (Cooper & Dunne, 2000).

To a certain extent, in the after-school mathematics club we try to develop an environment in which school and everyday mathematics are brought together along the lines of my previous research. Because we see these children as mathematics learners, both in the school setting and in the after-school, we can address some of the issues brought up by Frankenstein and Powell (1994) in relation to the separation (and maybe even opposition) between everyday and academic knowledge. We can see how a concept such as that of scale, which is often studied in school mathematics, is applied to a task that while school-based (e.g., making a scale drawing of their classroom), because it is done in a somewhat informal setting, takes on a different flavor. As mentioned earlier, Julián took the scale-drawing project very seriously. At one point in that project the facilitator (a university researcher) is working with another student using a drawing that he (the facilitator) had made of the room showing some dimensions. Julián is working on his own sketch. All of a sudden Julián looks over their work and asks about one measurement that they have on the facilitator's sheet. From there, a conversation follows in which Julián challenges the facilitator's drawing and tells him he has the wrong measurements for one of the sides, "yes [takes his pencil and starts drawing on the facilitator's sketch], from here to here, it has to go till here, you didn't draw it correctly; they have to be the same [he then starts pointing to the walls in the classroom], look they are the same." And after that, Julián just goes back to his sketch.

One of our goals in this research is to focus on Latino, low-income students as powerful thinkers and doers of mathematics, in opposition to the deficit approach that is often used to describe these students. In the scale-drawing project, we see Julián as a confident student, immersed in the task, while offering suggestions to one of his peers and engaging in a content-based conversation with the adult facilitator. Crucial to capturing the case of Julián is the affective aspect (uncovered through rapport and interviews), Julián in the classroom (and hence the teacher's role), and in the after-school setting, as a place to pursue our understanding of Julián as a learner of mathematics.

Moschkovich (1999) points out that we do not know enough about the participation structures in the home cultures of Latino students in our local contexts. Furthermore, Moschkovich (in press), in her analysis of the potential contributions of non-mathematics education studies to the study of bilingual mathematics learners, writes, "sociolinguistics also suggests that analyses of classroom communication should be informed by data on students' experiences, building profiles of students' language history, educational background, and attitudes towards bilingual communication for students, peers, teachers, and parents" (p. 29). Through our current research we hope to address the issues that Moschkovich raises. We are developing case studies that cut across different activities and people and use multiple sources of data: video-tapes and field notes of classroom and after-school mathematics club sessions; interviews (affective / perceptions and cognitive) with children, teachers, and parents; observations and video-tapes of parents' workshops. Our goal is to try to capture the experience as learners of mathematics of a few of these children by looking at them doing mathematics in 1) the regular classroom; 2) the after-school mathematics club; 3) (if possible), out-of-school activities; 4) Through their parents' eyes or with their parents. This research is allowing me to bring together my cognitive and my socio-cultural interests to hopefully gain a better understanding of the complexity of what it means to teach and learn mathematics.

Endnotes

1. Parts of this paper are adapted from Civil, 2006.

2. CEMELA (Center for the Mathematics Education of Latinos/as) is funded by the National Science Foundation under grant – ESI-0424983. The views expressed here are those of the author and do not necessarily reflect the views of the funding agency.

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LEARNING FROM STUDENTS: A STUDY WITH IN-SERVICE MIDDLE SCHOOL MATHEMATICS TEACHERS IN MEXICO

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This paper discusses the results of a professional development program aimed at in-service middle school mathematics teachers. The method we used consisted in videotaping carefully prepared mathematics classroom sessions with 8^{th} grade middle school students (13-14 year olds). These sessions focused on presenting selected curricular topics which students confronted through a series of articulated problems situations. We intentionally presented problems that did not require the use of high-level mathematical knowledge, rather the use of elementary mathematics and mathematical inquiry. Once the videos were ready, 10 monthly workshops aimed at in-service teachers were carried out in which the teachers were asked to face the same problem situations posed to students. In the closing part of each workshop, teachers watched the videos of the classroom sessions with students and were asked to discuss a set of questions aimed at encouraging them to reflect on their practice. These teacher workshops were also videotaped. Further analysis of the data suggests that the teachers experienced relevant changes both in their conceptions of teaching and learning mathematics as well as in their mathematical content knowledge.

The research carried out during the last 30 years on the learning of mathematics has provided an important knowledge base that raises the need for new educational strategies, new paradigms for teacher education, new curricula and new evaluation procedures (Kilpatrick, 1992). The results of that research have influenced the mathematics curriculum of middle school and have raised new demands for the teachers' professional task. The theoretical positions based on social constructivism have also had an impact on educational programs. Briefly summarized these theories conceive knowledge as a product of the intellectual work of communities formed by creative individuals; these ideas are reflected in courses and materials aimed at making the teacher leave his role as a deliverer of concepts, basic facts and skills in order to become a tutor of his students' development in mathematical thinking. (Cobb et al, 1990).

The new paradigms in the ways of teaching not only entail attending to basic recommendations, like asking the students to solve different types of problems or motivating them towards greater class participation; they also demand that teachers make deep changes in their mathematical knowledge and their conceptions of learning and teaching mathematics. Such theoretical positions propose that each student arrives in the classroom with his own ideas and that the teacher must bring to the classroom new experiences that induce his students to collect data to affirm or refute their conjectures.

This implies that teachers make evident in their everyday practice they are convinced that their students are not "containers expecting to be filled" and that they see their students as intellectually creative subjects able to formulate non-trivial questions, solve problems and construct theories and reasonable knowledge. This framework demands that teachers remove both themselves and the textbook from the role of intellectual authorities in the classroom, and that they deposit such authority in the rigorous arguments that they and their students produce (Thompson, 1992).

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PME studies on mathematics teaching have focused on the role played by teachers in educational improvement (Llinares & Krainer, 2006). Particularly, during the past 15 years the Mexican mathematics curriculum for middle school has introduced reforms that require teachers to develop didactic approaches based on methods of inquiry centered on student learning. These reforms also demand the implementation of a number of teachers' professional development programs. The assumption on which these programs are based is that the professional development of teachers will induce improvements in teaching and in students' learning.

This paper reports the work we have done in Mexico with in-service middle school mathematics teachers during the last few years. We begin by discussing our previous experience with teachers because it had an important influence on designing the method, instruments and teaching materials we used in the professional development program here reported. In a second section we describe and discuss the main elements of the present ongoing professional development program: the questions that guide its research component, the program goals the conditions in which it is being implemented, the profiles of the subjects participating in the project and the mathematical activities used. A third section of the paper deals with the preliminary findings we have henceforth. Finally, we put forward a set of concluding remarks regarding the work that we have done and the challenges that we believe it will be necessary to confront in the future.

Background

Our initial work with teachers grew out of a four-year-long professional development program conducted in 100 Mexican middle schools that in 1999 were equipped with symbolic calculators and materials especially designed to approach the curricular topics using this technology. As we expected, a good number of teachers enthusiastically joined the project from its outset and commenced using the equipment and materials in their classrooms; this was especially true for the youngest teachers. However, most of the more experienced teachers were rather skeptical and some overtly reluctant. Previous research work in this field indicates that effective use of these new resources requires important changes in teachers' practices (Guin & Trouche, 1999; Cedillo, 2001; Artigue 2002; Lagrange et al., 2003). From the beginning of the project our data confirmed that one of the major challenges we had to face was that of helping teachers realize that they have real reasons to change (Cobb & McClain, 2001; Cedillo & Kieran, 2003). To this end we prepared several strategies; among others, the discussion of research documents on the potential of using computer algebra systems in the classroom; the analysis of the innovative and promising approach presented in the curricular materials designed for the project; the discussion of reports informing the increasing and creative use of symbolic calculators within schools; and live classroom demonstrations with students conducted by project instructors and observed by teachers.

The data gathered during the training sessions strongly suggest that the most influencing factor in convincing the teachers to take active part in the project were the opportunities the teachers had to observe what they called "students' unexpected mathematical achievements" during the mathematics sessions conducted by the instructors. In many cases the teachers established an association between this mathematical progress and positive changes in students' attitude towards mathematics in school (Cedillo, 2006). The following excerpt from an experienced teacher's remarks during a project meeting exemplifies most teachers' reactions to this aspect:

> "In the beginning I thought it was just one more fashionable project… The regular presence of the instructors in the school, the discussions we had, and mostly what I noticed the students learned, showed me that it was worth it to try, that there were many new and interesting things that actually worked… My students are what I care for most, that is why I have already decided that I will make an effort and try it… I have to say that I am convinced that the students were showing such progress mainly due to the ways in which the instructor deals with the class… During this time I have learned a lot from my students and it has a lot to do with the didactic approach used by the instructor… I have learned from him too. I was thinking of retiring, but now I am sure that it is worth it to take part in the project … I will continue here for a while".

The teaching style adopted by project instructors was guided by the principle of *framing the classroom events according to the students' ways of reasoning as opposed to students following their teacher's ways of reasoning*. By adopting such a principle we assume that learning is an active construction process that is socially shared by the learners and the teacher. This requires the teacher to play a different role. In order to briefly describe how we conceive that teacher's role, we used the metaphor of conceiving the activity in the classroom as a multiple chess game; in such game the teacher is the expert player who simultaneously plays against 30 other players who can communicate and discuss amongst themselves before making a move. The expert makes the first move and has to be prepared to receive up to 30 different challenging responses; upon the second move on the chess board the expert player (teacher) has to give specific and challenging responses to each player, and so on. A major difference between the conventional chess game and the version we use in the metaphor is that in our chess game the teacher must manage the game in such ways that, eventually, the students legally win.

This metaphor helped make us aware of the challenge that taking on this responsibility implied for teachers. It encapsulates the stance that mathematics learning is an active confrontation between learners and mathematic challenges. It encloses the position of seeing mathematics learning as an active confrontation of the learners with challenging mathematics. Krainer (2004) suggests that to fulfill that position it is necessary to consider both the prior students' and prior teacher's knowledge: "It is unproductive to ignore students' recent understanding and fresh ideas, and it is equally unproductive to ignore the knowledge produced by generations of mathematicians. Thus, teaching mathematics is a continuous dilemma situation for teachers: on the one hand, they need to start where the students are, and on the other they aim at supporting students in developing an understanding of the mathematical concepts that are part of a socio-historically constructed body of mathematical knowledge" (Krainer, 2004, p. 87).

Lessons Learned

We learned a number of lessons from the experience briefly described above that were the basis on which we shaped the professional development program that will be discussed in the next section. As a sort of summary, we describe these lessons in the paragraphs below:

- The didactic strategy of framing the classroom events according to students' ways of reasoning proved to be a powerful resource for teachers in regards to motivating their students to engage in productive learning processes. However, adopting this strategy is a highly challenging duty for the teacher to take on. It requires that teachers have both a strong mathematical content knowledge and a strong pedagogical content knowledge. In this respect, we found that those teachers who felt inspired to adopt such a didactical strategy willingly engaged in a series of project workshops aimed at strengthening their mathematical content knowledge. The following excerpt from a teacher's remarks illustrates this:

> "… I want to thank my colleagues for the support I received from them. There were times in which I was on the verge of not participating any more because I felt embarrassed by the questions I asked, but there was always somebody who was friendly and helped me... On the other hand, my students are leaving me behind… they have been able to solve problems that I did not understand… Sometimes I did not have a way to tell them if what they had done was correct or not. My students did not realize this because they were the ones who showed me their solutions and explained them to me. Finally I decided to tell my students that all of us in the class, including me, were learning, that sometimes I was ahead of them, but other times they were in the lead… and that I wanted all of us to learn together, to provide support for each other. I know that I still have much to learn. So far I can say that I have already learned that what I will never do again is ask them to give me the solution to a problem… When something gets too complicated for me I will ask for clues… If I want to advance I must solve the problems by myself… It does not matter if it takes me a long time to do it".

- The empirical evidence we gathered suggests that teachers assign almost no value to successful teaching experiments done somewhere else; that is, in school contexts different from the ones in which they work. In contrast, the teachers react positively to teaching experiments carried out in their work places under the conditions in which they work. This data resonates with the research of Cobb & McClain (2001). In contrast, the teachers assigned an important value to the experience they had in joint meetings with colleagues from other schools. Our explanation for this apparent paradox is that in the joint meetings our discussion topic was the learning opportunities that arose for them in their classrooms.
- We found that if teachers witness someone else helping their students learn and progress, then those teachers may be more proactive in regards to changing their practices and refreshing their mathematical content knowledge. Later on they are more likely to evidence that they have changed.
- The data we collected indicates that the process of helping teachers change takes time. The first evidences of change were observed after one year of continuous work. The most evident changes took place after three years.

The Present Study

The report we are presenting here corresponds to the implementation of the Mathematics Teaching Program, which is one of three component programs constituting the Inter-American Program for Middle School Teachers and Teacher's Educators Professional Development. The other two programs address the teaching of the sciences and of the Spanish language. The whole program is funded by the Inter-American Development Bank and supported by the Mexican Ministry of Education, the Latin-American Institute for Educational Communication, and the National Pedagogical University. The program consists of two stages: the first (2003-2006) corresponds to its implementation in Mexico; in the second stage (2006-2009) the program will be extended to other Latin-American countries and include teacher's educators. Currently the program is finishing its first stage.

The major aim of the Mathematics Teaching Program is to strengthen the mathematical and pedagogical content knowledge of in-service middle school mathematics teachers and teacher's educators. Specific goals were to provide teachers with experiences that strengthen their knowledge about student thinking, and to create opportunities for teachers to explore how they might use their knowledge of students' thinking for instruction. The underlying hypothesis in this program is that an improvement in the quality of teaching might have positive effects in students' learning and competencies. We also assume that knowledge about students' mathematical thinking provides teachers with a strong basis for their instruction and also for their own continued learning (Carpenter & Fennema, 1989). The professional development program was accompanied by a research project aimed at gathering data to assess its effects.

Research Questions

The following research questions guided this study:

- When teachers and students are presented with the same set of mathematical problem situations, what are the effects of making teachers contrast their reasoning with the strategies used by 8th grade students?
- In regards to teachers' mathematical and pedagogical content knowledge, what are the effects of making teachers observe mathematics classes in which an instructor frames classroom events following the students' ways of reasoning?

Method

We attempted to achieve our goal (i.e. strengthening teachers' mathematical and pedagogical content knowledge) by adopting what proved to be the most successful strategies in the implementation of the symbolic calculators' project described above. To this end, we decided to again take the didactical approach based on framing the classroom events according to students' ways of reasoning and tested various ways to take advantage of what we observed as the positive impact of students' achievements on making teachers reflect on their practices and eventually make favorable changes in their mathematical and pedagogical content knowledge.

The ways in which we used these strategies needed to be reshaped because CAS technology is not included in the Inter-American Program and because it is aimed at middle school teachers and Teacher's Educators. To this end, we constructed the program model by adapting and refining the aforementioned didactical approach to the facilitation of classroom sessions with 8th grade students. These sessions were videotaped in order to make their use possible in as many locations as needed. Designing workshops in which the teachers would confront the same problems that students faced meant that the problem situations faced by both would be the means that allowed us take advantage of the potential impact of students' achievements. The workshops with teachers were videotaped; these videos were the main data sources for the research project and will be used when working with mathematics teachers' educators. We assumed that this strategy might provide a valuable opportunity for the teachers to learn from students and for the teachers' educators to learn from in-service teachers.

Classroom Sessions

In order to choose the mathematical content of the classroom sessions, both the research literature and the official mathematics curriculum in the Latin-American countries participating in the project needed to be reviewed. The information gathered in this review allowed us to identify those mathematical topics that are considered pertinent in the curriculums of Latin-

American countries and that have been the object of research due to their relevance, whether as difficult themes for teaching and learning or for their role as crucial antecedents for subsequent topics in higher mathematics education.

Additionally, the financial constraints and time allowed for the project were also considered. This led us to select the three curricular areas and ten themes that are described in the table below.

Arithmetic	**Algebra and Prealgebra**	**Geometry**
Multiples and divisors.	Number patterns and generalization.	Measurement and similar triangles.
Maximum common divisor	Games and algebraic regularities.	Areas and the Pythagorean Theorem.
Minimum common multiple.	First grade equations.	Measurement and trigonometric ratios.
	Reading and making Cartesian graphs.	

Two classroom sessions of 50 minutes each were assigned to tackle each topic; thus, we conducted and videotaped a total of 20 classroom sessions. The classroom sessions were carried out in two public middle schools located in Mexico City that agreed to take part in the project. The respective school boards permitted us to intervene in the school as long as the project activities did not disturb school work; this allowed us to work with different school groups within their regular school schedule. This fact represented a valuable profit for the project in terms of the freshness and spontaneity of the students' approach to the work methods required by the project's goals. Another advantage of the situation was that the students taking part in the videos were not selected in any way, neither by the school principal nor by the project staff. They were simply the students available at the time and date in which the classroom sessions were programmed; this schedule was almost always determined by the available time of the professional TV staff that videotaped the classroom sessions.

The teachers who conducted the sessions were not part of the school staff. Three university researchers who are part of the project staff were deliberately put in charge of taking the role of the teacher in the classroom sessions. This made it possible to have intensive meetings with the project academic staff to prepare the classroom sessions, pilot them, and hold the tested approaches to a continuing process of refining until we were satisfied with the results. The pilot stage was done in schools that did not participate in the main stage.

As mentioned earlier, according to the didactic approach adopted, the chosen topics for classroom sessions should be formulated in a series of articulated problem situations and the teacher's actions should be guided by the multiple chess game metaphor. This metaphor entails an instructional approach that is similar to the teaching cycle as proposed by Simon (1995) and the concept of instructional sequences put forward by Cobb & McClain (2001). Simon describes the teaching cycle with the metaphor of undertaking a long journey, such as sailing around the world. This metaphor implies that "at any point the teacher has a pedagogical agenda and thus a sense of direction. However, this agenda is itself subject to continual modification in the act of teaching… This way of acting in the classroom involves both a sense of purpose and an openness towards the possibilities offered by students' solutions to instructional activities" (Cobb &

McClain, 2001, p. 215). It is worth emphasizing that this didactical stance does not suggest in any way that the instructional activities should go on aimlessly; at any point in the classroom session the teacher has to have in mind a major instructional goal and figure out the means of achieving it.

Another methodological issue we had to attend to was the selection of problems so that they were challenging enough both for students and teachers. Since these problems had to be tied to curricular topics, the criterion we used for picking the problems was to select those that allowed us to treat the topics in unconventional ways.

In the case of arithmetic, the central theme we chose was divisibility. According to the review we made of the mathematics curricula in Latin-American countries, divisibility is approached by introducing the formal definitions of divisor, factor, multiples and prime numbers. This is followed by a series of practices to reinforce understanding of these definitions. Thus, to approach the topic of multiples and divisors we decided to use a set of questions aimed at encouraging students to engage in mathematical inquiry and to uncover relationships between these concepts; this may lead students to put forward generalizations by observing numerical regularities and finally to express and justify these generalizations by using the algebraic code. The students were allowed to work cooperatively in small groups if they wanted to do so.

Next we will describe the problem situations we used for each topic. According to the aims and extent of this paper we will describe in detail the ways in which some topics were treated and give a general description of the others.

The topic of multiples and divisors was treated on the basis of the students' responses to the following questions:

- Can you find numbers that have exactly two divisors? In the next four minutes list as many of these numbers as you can. The challenge is that your list cannot include any number not fulfilling the given condition.
- Can you find numbers that have exactly three divisors? Can you show a rule that allows us to construct many numbers having exactly three divisors? Is there more than one rule that allows us to do that?
- Can you find numbers that have exactly four divisors? Can you show a rule that allows us to construct many numbers having exactly four divisors? Is there more than one rule that allows us to do that?
- Can you find numbers that have exactly n divisors? Can you show a rule that allows us to construct many numbers having exactly n divisors? Is there more than one rule that allows us to do that?
- Can you find a natural number different from 1 that you cannot factor using exclusively prime numbers as factors?

The topic of maximum common divisor was approached using a known problem that involves three liquid containers, none of which is graduated, and whose capacity is known. The first and the second containers have capacities that may be different. The third container is larger that the other two. The problem consists in deducing a general rule that allows one to know the number of liters that can be obtained given the capacity of the first two containers. The solution relies on the concept of maximum common divisor. This problem is rarely used by teachers in most Latin-American countries, so we can assume that the students have never seen it before. We used this problem in order to allow students to recreate their notions of the maximum common divisor concept and to encourage them to find numerical regularities that eventually lead them to put forward general solutions. The problem was extended to empirically approach Diophantine equations. The specific questions and sequence in which the questions were posed to the students are described below:

- You have three jars: one has a capacity of 3 liters; the second a capacity of 5 liters. The third jar is used to hold a certain amount of liquid larger than 8 liters. Can you get 4 liters passing liquid from one jar to the other? Can you find a way to record the sequence of movements you made from one jar to the other so that you convince us that your answer is correct?
- Consider the same conditions as the problem before, but now you have one 2-liter jar and one 4-liter jar. Can you get 1 liter passing liquid from one jar to the other? Can you find a way to record the sequence of movements you made from one jar to the other so that you convince us that your answer is correct?
- Now you have one 6-liter jar and one 9-liter jar. Can you get 1 liter passing liquid from one jar to the other? Can you find a way to record the sequence of movements you made from one jar to the other so that you convince us that your answer is correct? Can you get any integer amount of liters passing liquid from the 6-liter jar to the 9-liter jar? Can you make a list with the different amount of liters you can get passing liquid from the 6-liter jar to the 9-litter jar?
- Now you have one 7-liter jar and one 10-liter jar. Can you get 1 liter passing liquid from one jar to the other? Can you find a way to record the sequence of movements you made from one jar to the other so that you convince us that your answer is correct? Can you get any given amount of liquid passing liquid from the 7-liter jar to the 10-liter jar? Can you make a list with the different amounts of liters you can get passing liquid from the 7-liter jar to the 10-litter jar?
- Look carefully at the lists you made with the different amounts of liters you can get passing liquid from one jar to another. Do you notice some regularity fulfilled by these numbers? Can you find a general strategy that allows you to know whether you can or cannot get a given amount of liquid just knowing the capacity of each jar?

The topic of minimum common multiple was approached using the gears problem. The version of the problem we posed to students involves two gears as shown in the figure below; the number of "teeth" in each gear can be different. Students were asked to find how many turns the gears have to make so that they will coincide again at the point the turning commenced. This problem was extended to include the case of more than two gears and to include word problems like "a friend of mine bought apples and oranges. For each apple she paid 5 pesos and for each orange 3 pesos. She paid the same for the apples as for the oranges. How many of each did she buy?

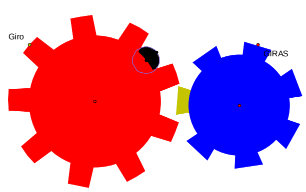

Fig. 1. The gears problem

In the case of pre-algebra and algebra we included four topics, two of them approaching the use of algebraic code to express and justify generalizations (number patterns, games and algebraic regularities). The topic of number patterns and generalization was addressed mainly on the basis of finding a function that fulfills the number relationship suggested by a sequence like the one shown in the figure below. To help the students do so, we gave them a series of questions such as the following: How many squares would the fourth figure have? How many squares would the tenth figure have? If a figure in this sequence has 225 squares, which place in the sequence does this figure have? Can you express the rule you used to answer the above questions using algebraic language?

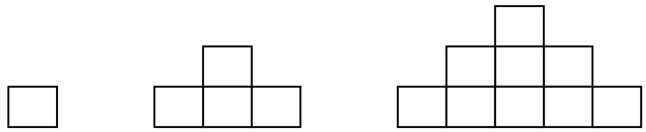

Fig. 2. The squares problem

Another problem situation used in this section was of the type "think of a number". For example: Think of an integer between 0 and 10. Add 10 to the number you thought of and keep the result. Now take away the number you thought of from 10 and keep the result. Add up the two results you kept. May I guess the final result you have? It is 20. Am I correct? Why could I do this? Find an explanation. Could I guess the number you thought of if it were greater than 10? Could I guess the number you thought if it were less than 0? Could I guess the number you thought of if it were a non-integer? Why?

The topic Games and Algebraic Regularities was treated by the Hanoi Towers problem. This problem requires students to find the rule of correspondence of an exponential function. The students were invited to play the game by using either a physical material or a piece of software that simulates the movements of the disks from one tower to another.

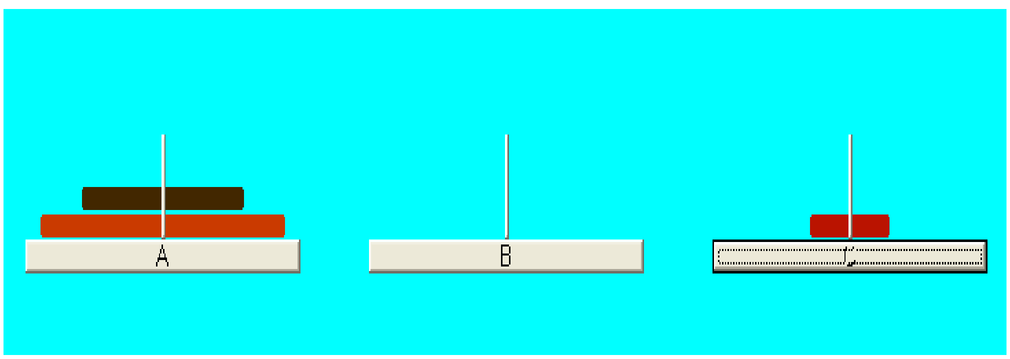

Fig. 3. The Hanoi Towers problem

The Hanoi Towers game began when students were asked to move three disks from one tower to the other. Questions such as the following were presented to students: What is the minimum number of plays you can make to move the three disks from one tower to another? If the disks were in tower A, which tower must you make the first move to? How many moves are necessary to pass the three disks from one tower to another? The game increases in complexity when more disks are used. The overall question was: If we assumed that moving one disk from one tower to another takes 1 second, how long will it take to move 64 disks from one tower to another?

The topic of first grade equations was approached by asking the students to find the missing number in a given equation. The students had not had any instruction about conventional methods for solving equations at the time this session was implemented. The activity gradually increased in complexity until equations containing brackets and division bars as grouping symbols were posed. Another type of activity posed in this session was to ask the students to create a word problem situation that can be solved by a given equation and vice versa.

The topic of reading and constructing Cartesian graphs focused on asking the students to create stories corresponding to the information shown by time-position graphs and vice versa (see the graph in the figure below). Another type of activity used in these classroom sessions was that of finding the rule of correspondence of a function from the information provided by a linear graph.

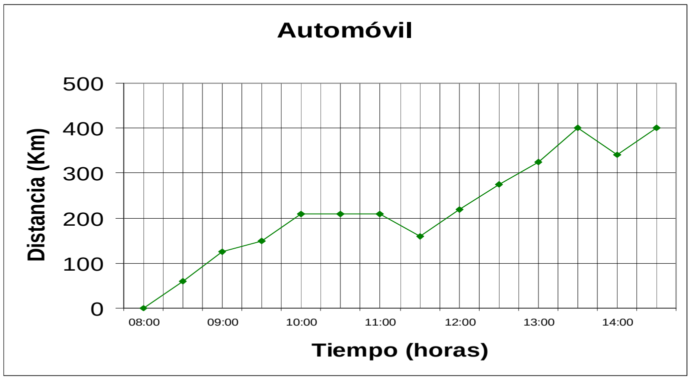

Fig. 4. The car problem

The classroom sessions on geometry focused on measurement. Triangle similarity, the Pythagorean Theorem, and trigonometric ratios in right triangles were approached as measurement tools. These topics are not included in the 8th grade syllabi, thus the students had their first encounter with them in the project's classroom sessions. The students were introduced to the topics using an inquiry strategy intended to make them encounter the central concepts involved through discovery, similar to the method used in a Sciences laboratory. To support this kind of activity the classroom was equipped with computer based geo boards, dynamic geometry software and scientific calculators. For example, to approach the topic of trigonometric ratios, the students were asked to draw a right triangle given the measurements of its interior angles. They then calculated the quotients by taking the lengths of the sides of the triangle in pairs and compared their results with the ones obtained by their fellow students. The dynamic geometry software let them do this rapidly and try with as many particular cases as they needed before drawing a set of conclusions. The use of scientific calculators allowed them to confirm their results, and they knew the mathematical names assigned to the quotients they performed. Having done this, the students were asked to provide mathematical arguments that might explain the number regularities they had observed. To close the session the students were asked to work on word problems whose solution required finding some length or angle measurements of a right triangle. This work was completed by the students in two classroom sessions of 50 minutes each. A similar approach was used to treat the topics of triangle similarity and the Pythagorean Theorem.

The students' responses to the activities described above will be interwoven when discussing the teachers' reactions in the results section.

Workshops with teachers

Ten monthly workshops of five hours each were carried out in the time period of February-November, 2005. A group of 25 in-service mathematics teachers volunteered to take part in the workshops and accepted to be videotaped. As we said before, the same activities and problem situations used with students were given to the teachers. In the workshops the teachers worked in small groups, carrying out mathematical explorations from an activity sheet and sharing discoveries.

The workshops were structured as follows: the teachers had one hour to deal with the problem situations and to discuss this experience amongst themselves and with the instructor. The next two hours were devoted to watching the videos of the two classroom sessions with students; before watching the videos the teachers were asked if they thought that 8^{th} grade students would be able to deal with the problem situations they had just faced. If yes, to what extent? How? Why? The last two hours were allocated to discussion: teachers talked about what they had done in the context of what they observed in the videos.

As an attempt to focus the teachers' attention while they watched the videos, we asked them to particularly observe and make notes about the following issues:

- Students' unexpected strategies.
- The most influencing factors in students' success in solving the problems.
- The most influencing factors leading students to become confused or to produce unclear answers.
- Students' ways of reasoning that differed from the ones s/he as a teacher would most likely use to confront the same problem situation.
- Students' ways of reasoning that were similar to the ones s/he as a teacher would most likely use to confront the same problem situation.
- Other classroom events they thought were important to discuss.

The teachers taking part in the workshops work in public middle schools in Mexico City. The most experienced teacher in this group has been teaching mathematics for 20 years, and the least experienced has 2 years teaching. Two of these teachers work in the schools where the classroom sessions with students were held. This circumstance, unplanned though it were, had a favorable influence on teachers' views of the project; in particular it added a factor of credibility.

Results

Once we analyzed the data from the videos and the worksheets completed by the teachers, we found it difficult to disaggregate the teachers' reactions and the teaching and learning episodes into separate themes. Their interventions necessarily interweave reflections on mathematical content, pedagogical and mathematical content knowledge and students' abilities. Taking into account these constraints, we attempted to organize the presentation of results in the following sections: Teachers' expectations of students' capabilities, and Teachers' reflections on their own practice.

Teachers' expectations of students' capabilities

The fact that teachers and students had faced the same problem situations provided relevant data about the teachers' expectations about students' capabilities. The data we gathered indicates that the teachers center their appraisal of students' learning and competencies on their own teaching abilities. As mentioned earlier, before viewing the video of the classroom sessions the teachers were asked if they thought that the students would be able to solve the problem situations. The first time the teachers answered this question, they emphatically said that they did not think the students could deal with the problem situations, particularly those questions that require finding a general rule to construct natural numbers with a given number of divisors. They argued that for the students to achieve such an ambitious goal it was first necessary to at least teach them divisibility rules, a method to pick up prime numbers and specific algebra instruction to deal with generalized numbers. They thought the questions requiring algebra were completely out of the students' reach because the students did not yet know enough about the topic. They said they were sure that what they had foreseen would be confirmed by the videos that they watched.

Once the teachers saw how the students managed to deal with the problems using the elementary mathematical tools they had, they began a discussion in which they looked for explanations as to why their perception of students' capabilities was so far from what the students were able to do. They found it particularly hard to believe that the students had been able to analyze the factors of a given number using algebraic code. In regards to this point, they referred to cases such as the student who went to the blackboard to explain his group's finding: "say p is a prime number… any number as p^4 has exactly five divisors because $p^4 \div 1 = p^4$; $p^4 \div p = p^3$; $p^4 \div p^2 = p^2$; $p^4 \div p^3 = p$ and $p^4 \div p^4 = 1$… We tried this with many numbers and it worked, then we tried with letters and realized it is easier with letters than with numbers".

This kind of evidence made the teachers seek explanations; they came to the conclusion that it was the ways in which the instructor guided the students' arithmetical-based reasoning that allowed them to gain self-confidence and "this encouraged the students to start producing powerful ideas by analyzing particular cases that finally led them to see possible generalizations". For example, one of teachers offered the following as a concluding remark:

> "See, the teacher (project instructor) never said no to any student, he was always patient…If a student said something wrong the teacher asked the group if they agreed with the answer that student was giving… He (the teacher) behaved this way even when a student was proposing a brilliant idea. In that way the teacher gave the students many opportunities to correct or validate their answers on their own… This teacher's attitude gives the students an opportunity to learn more… In that way students are enabled to elaborate arguments that reject or accept the solutions that they themselves propose.

The group of teachers concluded that they had thought that students would not be able to successfully face the problems presented because they had taken as their reference point their own experience when teaching these topics. They said they had been sure that the instructional sequences they used to teach were the good ones; thus, if the students had to confront the problems in the same way we posed them to the teachers, they found it almost impossible to believe that the students would be able to cope. The following excerpt of a teacher's remark illustrates this point:

> "It is the way we teach that led us to think that the students would not be able to provide correct answers to such complex questions… When I teach these topics I start by giving the students the necessary procedures and concept definitions about the content I want them to work with… Then I show a good number of examples trying to reinforce their understanding. They need the divisibility rules in order to be able to find factors, so I teach these rules to

> them… When you said the students would be given these tasks in the same way they were given to us, it was hard for us to think that the students would do well".

We expected that the teachers' view about the students' capabilities in the first workshop session would change in the next workshop, but it did not. The only change we observed was that the teachers took longer before answering what they thought would happen in the class session in which students faced the "jars problem" described earlier. Some of the teachers were not quite sure about the students' success. We pushed them to give a more precise answer: they finally agreed that they did not think the students would be able to solve the problem successfully because they knew too little "about the maximum common divisor" and the activity required not only the definition of this concept but also finding number relationships involving the concept. Once again, the achievements of the students that they observed in the videos seriously contradicted their view.

By the end of the study, 9 out of the 25 teachers taking part in the workshops shared with the group that they had begun to try a different teaching approach. One of the main obstacles mentioned was that preparing and carrying out a class similar to the sessions given by project instructors is quite time consuming; another difficulty centered around the mathematical content knowledge needed in order to properly respond to students' questions and answers in a way that his/her interventions might help students strengthen their mathematical thinking. "… With time, when we have tried the same class session several times, we can do it".

The teachers that did not try to put into practice a new teaching approach argued that although the project's teaching strategy and the students' achievements proved to be positive, a program like the one we propose is not viable for them because the level of organization in their schools is not yet sufficient to enable them to carry out a program like this one. Other teachers argued that they do not feel confident enough to explain to and discuss with their principal and school supervisor how this approach meets the topics and goals of the official school curriculum. Our data suggest that this lack of confidence is related to a weak command of mathematical and pedagogical knowledge.

The following comment illustrates the aforementioned point:

> "I am impressed by the students' achievements that I witnessed; I also see many favorable features of the way in which instructors conducted the class, how they manage the mathematical content and can adjust their responses in the moment to nicely meet the ways in which students perform. But I do not think that I can do this; the school I work in is not as well organized as the schools in which the videos were taken—I know those schools pretty well. My students do not behave like the students we saw in the videos; it is really difficult for me to keep them working and I am sure it has to do with the way in which the school's principal acts. I have said before that I liked very much several of the classes we watched in the videos; I particularly liked the class about the Hanoi Towers and tried to work the problem out with my students. That day the school principal unexpectedly came in to my class and observed it. She reported my work to the supervisor and instructed me not to do it again… I was told to constrain my teaching to the topics included in the curriculum… I am sure I will try it again later on; now I need time to better prepare myself and improve what I know of mathematics and mathematics teaching… One thing is to play the Hanoi Towers game and another very different one is to draw out the mathematics involved in

> the game so as to arrange the classroom activity sequence so that middle school students might play the game mathematically. It requires the teacher to have both good teaching skills and a good command of mathematics. The problem I see is that my workload is really heavy, as it is for most of us; anyway… I will try to organize myself in order to do this and will volunteer again to participate in this kind of program any time I have a chance."

Teachers' reflections on their practice

The teachers' comments during the workshops suggest that the students' achievements were a relevant factor in motivating them to critically reflect on their pedagogical and mathematical content knowledge. This finding confirms the central assumption we took to design the present study. As reported above, by the end of the workshop sessions (one year of work) some of the teachers engaged in trying a different teaching approach from the one they had been using before, as a direct result of the analysis and discussions of the teaching approach used in this study. Even those teachers who did not manifest a clear intention to change made reference to the ways in which observing the students' achievements made them review their teaching strategies. It seems that an important factor in encouraging them to try different teaching approaches is the attitude change they observed in students when a teacher gives them an opportunity to explore, to make mistakes, and to go back and forth feeling secure because they know the teacher is there to help them. One of the teachers explained the students' positive attitude as follows: "I think that students' favorable attitudes are related to their mathematical achievements… We saw how students gained self-esteem as the rest of the group accepted their thoughts; this was evident from the ways in which they willingly participated giving ideas and taking part in warm discussions with their fellow students and the teacher. The students we saw in the videos seemed to be happy in the mathematics classroom, now I am wondering why my students rarely show the same reaction. I am impressed by how far the students' learning can go when the teacher guides them as we saw in the videos… I would like my students to behave in that way and I am convinced that much depends on me".

The teachers' reactions indicate that their attention was primarily focused on the students' responses to the activities, but they soon turned to analyze how students' reasoning led them to elaborate their responses and finally came to discuss the ways in which the teacher conducted the class. In most of the sessions the teachers engaged in discussions to find plausible reasons to explain why the students were able to produce mathematical solutions to non-trivial problem situations. Our data shows that this kind of teachers' inquiry led them to reflect on their own teaching styles. For instance, they noticed that the teacher in the video refrained from giving the students answers and instead, responded with a new question "attempting to make the students see by themselves what their mistake is". Our analysis suggests that the teachers noticed such events because they do not act in the same way as the teacher in the video.

One of the most frequently mentioned points in teachers' discussions was their analysis of the feasibility of their putting into practice a teaching strategy similar to the one they observed in the videos. They could not help accepting that the students taking part in the videos are not so different from the students with whom they work. This type of discussion led them to conclude that it was not only the teacher's pedagogical skills that mattered in making a classroom activity develop so fruitfully; but also other important factors such as the type of materials the teacher used to support the activity, the criteria the teacher used to design the problem situation, the way in which the class was planned, and how this way of structuring the class helped the teacher anticipate what type of responses the students can produce. The following teacher's remarks illustrate this point:

> "After having discussed these topics I realized the importance of carefully preparing the class. I came to the conclusion that planning a class consists in putting together all my mathematical and pedagogical knowledge in order to help the students learn more meaningfully… I mean, planning a class requires the teacher to translate her mathematical knowledge into a teaching situation; it is not only mathematics, the crucial point is to situate that mathematical knowledge in the context of a teaching and learning situation".

Final Remarks

Our study was influenced by the work done in earlier professional development programs for teachers that were based on the assumption that reflection on solving challenging problems may have positive effects on teachers' pedagogical and mathematical content knowledge. The results found in these studies highlight the importance of these types of mathematical tasks for teachers because they can empower their practices (Zaslavky et al., 2003; Murray, Olivier & Human, 1995; Schifter, 1993). Our results also confirm previous findings about the role played by challenging mathematical tasks as means to make teachers reflect on their practices and eventually improve them. Another source of important influence for the present study was the research by Carpenter and Fennema (1989); their work, as ours, assumes that knowledge of students' mathematical thinking provides teachers with a basis for their instruction and also for their professional development.

An underlying assumption in the present study is that we think it is critical that teachers learn mathematics in the same way as they are expected to teach it. Given that our study was aimed at in-service mathematics teachers, we had to attempt an approach that permitted us to provide an opportunity for teachers to learn mathematics in the same way we expect them to teach. The challenge for us was that the subjects we worked with were already teaching. The act of putting teachers in a situation in which their knowledge confronted the students' knowledge led them to analyze their teaching within the framework of how students were being taught in the project sessions. The findings of the present study strongly suggest that teachers' mathematical background is a relevant component of their professional development as long as it is situated in their practice. The results of the present project indicate that a solid understanding of the mathematical content knowledge is needed for teachers to be able to propose mathematical activities that provide opportunities for their students to learn mathematics in a more meaningful way.

Our data provide evidence showing that affecting teacher practice through involving them in proposals that produce tangible results in student performance is a promising alternative if we want to improve the mathematical and pedagogical knowledge of in-service teachers.

The results of this study confirm that change is not an event but a process. After one year of working with in-service teachers we have evidence of incipient changes in their practices and conceptions. However, the data we have clearly indicate that the monthly meetings with teachers proved to be a favorable environment for all of us to learn from each other. This experience highlights the need to continue encouraging the creation of professional teaching communities; however, this implies an even greater challenge given that Mexican teachers usually work in relative isolation.

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WORKING GROUPS

WORKING GROUP

WORKING GROUP FOR THE COMPLEXITY OF LEARNING TO REASON PROBABILISTICALLY

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Recent foci in the Working Group have been to understand: (1) students' and teachers' reasoning when simulating probability experiments with hands-on materials and computer tools, and (2) connections between probability and statistical concepts such as inference and variability. At PME-NA 28 in Mérida, Yucatán, Mexico, the group will build on the research agenda that it began at PME-NA 26 in Toronto and expanded at PME-NA 27 in Roanoke. Group members will revisit previously posed research questions follow-up on preliminary designs for cross-national, collaborative research to be conducted in 2006. Emerging research from Working Group members will lead to a set of papers that could comprise a monograph, journal special issue, and/or joint presentations at future conferences.

Nature and topic of the working group

This Working Group was formed at PME-NA 20 (Maher, Speiser, Friel, & Konold, 1998) and has convened annually at PME-NA each of the past seven years. During the joint meeting of PME-NA 25 and PME 27 in 2003 (Hawaii, USA), at PME-NA 26 in Toronto, Canada, and PME-NA 27 in Roanoke, Virginia, we expanded our working group to include many more international researchers across 12 different countries. Through shared research, rich and engaging conversations, and analysis of instructional tasks, we continually seek to understand how students learn to reason probabilistically.

Aims of the working session

There are several critical aims that guide our work together. In particular, we are examining: (1) mathematical and psychological underpinnings that foster or hinder students' probabilistic reasoning, (2) the influence of experiments and simulations in the building of ideas by learners, particularly with emerging technology tools, (3) learners' interactions with and reasoning about data-based tasks, representations, models, socially situated arguments and generalizations, (4) the development of reasoning across grades, with learners of different cultures, ages, and social backgrounds, and (5) the interplay of statistical and probabilistic reasoning and the complex role of key concepts such as sample spaces and data distributions. Through our work, we have stimulated collaborations across universities and plan to engage in and support additional research related to the complexity of learning to reason probabilistically. Future research will seek to understand how the use of simulations can help students make sense of empirical and theoretical aspects of probability. In turn, it will help the group collaboratively conduct research that can inform the development of a conceptual framework to describe the critical aspects of students' understanding.

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Background on probabilistic reasoning

The ways in which students reason about the likelihood of an event can be considered in terms of an objective or subjective view of probability (e.g., see Batanero, Henry, & Parzysz, 2005; Borovcnik, Bentz, & Kapadia, 1991). In an objectivist perspective, probability is viewed as an inherent property of the event and can be well estimated either through a classical or frequentist approach. A repeated finite set of trials would most likely yield a different experimental estimate of the actual probability and may in fact allow one to change the estimate of the probability based on new data. In a subjectivist perspective, probability is viewed as a condition of the information known to the individual assigning the probability and not an objective property of the given event. Thus, two people may assign different probabilities to the same event based on different a priori information, even after they observe the same empirical data a posteriori trials being conducted. The law of large numbers is used to interpret empirical results in relation to theoretical probabilities and, thus supports the viability that an estimated probability from a frequentist approach will be reasonably close to the theoretical probability. This principle states that the probability of a large difference between the relative frequency of an outcome and the theoretical probability limits to zero as more trials are collected.

A frequentist approach to probability, grounded in the law of large numbers, has only recently made its way into curricular aims in schools (Jones, 2005). Teachers are encouraged to use an empirical introduction to probability by allowing students to experience repeated trials of the same event, either with concrete materials or through computer simulations (e.g., Batanero, Henry, & Parzysz, 2005; National Council of Teachers of Mathematics [NCTM], 2000; Parzysz, 2003). In these types of curricula, a theoretical model of probability based on a classical approach is not the starting point. Rather, a theoretical model is constructed based on observing that the relative frequencies of an event from a repeated random experiment stabilize as the number of trials or sets of trials (different samples) increases. However, there is general agreement that research on students' probabilistic reasoning has been lacking sufficient study of students' understanding of the connection between observations from empirical data and a theoretical model of probability (e.g., Jones, 2005; Parzysz, 2003).

Summary of activities from 2005

Eighteen researchers (faculty and graduate students) from the United States, Canada, and Mexico met During PME-NA 27 in Roanoke (see Lee, Tarr, & Powell, 2005). After analyzing a video of students' work on a computer-based simulation task (Schoolopoly task, see Tarr, Lee, & Rider, 2006), the group discussed how tasks can be interpreted differently by students and teachers. Moreover, discussions focused on issues students face in trying to generate and analyze empirical data to make inferences about an unknown probability distribution. These discussions led to different participants expressing interest in conducting various pilot research studies in 2006. Some of the ideas for follow-up research included:

- What are students' intuitions regarding whether real-world phenomena can or cannot be simulated? Are there differences between simulations and modeling tools?
- How do students (and teachers) relate technology simulation models to real-world phenomena?
- How do learners move between empirical data and theoretical models of probability? To what extent do students attend to issues of sample size, variation, sampling distributions, and data collection?

- What metaphors emerge as students engage in probability tasks, and how do these support or hinder the development of probabilistic reasoning?
- What role does agency play in the development of students' probabilistic reasoning, and how does technology influence such development?
- What are the key issues in the design of probability tasks in order to promote connections between theoretical and empirical probabilities? What issues do teachers face in implementing such tasks?

Planned activities for the 2006 meeting

At PME-NA 28 in Mérida, Yucatán, Mexico, the group will build on the research agenda that it began at PME-NA 26 in Toronto and expanded at PME-NA 27 in Roanoke. We will revisit previously posed research questions follow-up on preliminary designs for cross-national, collaborative research to be conducted in 2006. Emerging research from Working Group members will lead to a set of papers that could comprise a monograph, journal special issue, and/or joint presentations at future conferences. Clearly our proposed activities are closely aligned with Goals of PME-NA, namely "to promote international contacts and the exchange of scientific information in the psychology of mathematics education," "to promote and stimulate interdisciplinary research...," and "to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof."

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WORKING GROUP

WORKING GROUP ON GENDER AND MATHEMATICS: WOMEN AND GIRLS AS STUDENTS, TEACHERS, RESEARCHERS AND ACTIVE AGENTS IN MATHEMATICS EDUCATION

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The Gender and Mathematics Working Group has been an active participant of PME-NA since 1998. This working group's history, in brief, is included in this proceedings paper. The most recent work of the group has included a monograph project, now in revision, followed by a self-analysis of our work that has brought us to discussion and investigation of new topics. Those topics include: 1) Investigating research and teaching paradigms that develop new understandings of the relationship between gender and mathematics education; 2) Questioning the nature of school mathematics; 3) Problemetizing a (re)definition of the field of gender and mathematics; and 4) Establishing connections across technology, gender, and mathematics. A common theme emerging from our collective work on these topics is that social agency is a central element of the range of work we do, and of the decisions made in both mathematics classrooms and career decisions. Thus, the GMWG has begun and continues in this year's sessions, discussion and planning for a series of empirical and theoretical investigations around the role of social agency for women/girls who are students, teachers and researchers in mathematics education. It is this topic that will frame our work at the PME-NA XXVIII Gender and Mathematics Working Group sessions in Mérida, Yucatán, Mexico, November, 2006.

Introduction

In this year's PME-NA XXVIII meeting members of the Gender and Mathematics Working Group examine a framework for the diverse collection of works coming out of our members. The PME-NA XXVII Gender and Mathematics Working Group members in attendance in Roanoke began this discussion, and an initiative to develop a framework that employs the use of the concept of women and girls' social agency as a unifying factor in our collective work. In this paper, I review the history of the Gender and Mathematics Working Group and then outline some of the work of group members since the Roanoke sessions in the section entitled "In the Interim – Work Between Sessions." Finally, in the section entitled "The Gender and Mathematics Working Group and Its Relationship to PME-NA" I describe the relationship of our work to the PME-NA goals and to previous Gender and Mathematics Working Group endeavors; and I introduce the work under discussion at the 2006 meeting of the Gender and Mathematics Working Group.

History of the PME-NA Gender and Mathematics Working Group

The Gender and Mathematics Working Group has met annually at PME-NA since 1998 (Raleigh, NC), except for the year of the joint meeting with the International Group for the Psychology of Mathematics Education in 2003. At our first meeting, the work of the group began with reviews of gender and mathematics scholarship, and sought to identify absences from the

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A. (Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

research strands reviewed. Committing to an integration of our collective scholarship on gender and mathematics, we defined future directions for research and for the working group. An early result was a visual representation, a graphic, of our conception of the field of gender and mathematics, and the complexity of the elements with(in) which we work (Damarin & Erchick, 1999; Erchick, Condron & Appelbaum, 2000).

After the first meeting of the Gender and Mathematics Working Group, we continued to gather together at each PME-NA meeting, sharing our scholarship on gender and mathematics, redefining our direction and purpose, seeking feedback from the membership at large in PME-NA discussion groups and fine-tuning the focus of our work. Forming peer groups of individuals with common interests and related research efforts, we reviewed, critiqued, and discussed the body of scholarship we were engaged in, including research into both theory and practice. A guiding project of the working group was the creation of a gender and mathematics monograph.

At the 2004 PME-NA XXVI sessions in Toronto, the Gender and Mathematics Working Group members began moving our work into new spaces. In these sessions we explored ways in which we can more deeply examine the relationship between gender and mathematics in our work, and did so with reflection upon international perspectives and critical theory, connected work in gender and technology, and critical perspectives on pervasive, recurring questions about the place for gender work in mathematics education (Erchick, Applebaum, Becker, & Damarin, 2004). Finally, in the 2005 PME-NA XXVII sessions in Roanoke, as we discussed topics across the range of the work being completed by members of the group, we found that a unifying framework on the role and development of social agency in women and girls' experience as students, teachers, and researchers in mathematics education was emerging in our inquiry.

In the Interim – Work Between Sessions

The following work continued in the interim between the PME-NA XXVII sessions in Roanoke and the time of this proposal:

- Continuing work on an the ongoing monograph project ensued in the interim between the 2005 GMWG session's at PME-NA XXVII in Roanoke, Virginia and the 2006, PME-NA XXVIII in Mérida, Yucatán, Mexico.
- Members continued planning collaborative work, discussing the creation of proposals and individual papers on the social agency framework that emerged in the Roanoke sessions.
- Several individual members are pursing both empirical and theoretical inquiries addressing social agency and its role in women and girls' learning, teaching and researching in mathematics education. Both social and psychological perspectives will be pursued.
- Introduced at PMENA in Roanoke was the new logo of the group, available for viewing at the Gender and Mathematics Working Group of PMENA website http://www.newark.osu.edu/derchick/pmena.htm.

The Gender and Mathematics Working Group and Its Relationship to PME-NA

Our Work and the Goals of PME-NA

Since its inception, the GMWG has had a goal of impacting classroom practice in positive ways. This goal is directly related to the PME goals to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof;

and to promote and stimulate interdisciplinary research, with the cooperation of psychologists, mathematicians, and mathematics teachers. Also, technology-related issues are embedded in our work. Technology is an increasingly present and important component of the mathematics classroom. Thus, research on gender and technology informs and contributes to our work in exciting and meaningful ways.

We also are committed to another goal of PME, that of promoting international contact and exchange of scientific information in the psychology of mathematics education. In terms of further and broadening growth and exchange of ideas, inclusion of international perspectives of gender and mathematics is crucial, and participation of international colleagues is not only welcome, but essential. We address this goal in this year's GMWG agenda with the expectation of developing collegial relationships, integrating diverse perspectives into our research agendas and continuing new relationships forged around the inquiry of the role of social agency in women and girls' experiences with mathematics.

Our work is also further connected to the PMENA XXVIII conference theme of "Focus on learners, focus on teachers" with the emerging framework addressing the development of learners' and teachers' agency in mathematics teaching and learning.

A Few Words About Agency

In the Gender and Mathematics Working Group's developing work, we investigate the concept of agency as central to women's relationships with mathematics. We define agency as assertive decision-making, choices made regarding one's life and being. We recognize the agent as Phyllis Curtis-Tweed describes in discussing the work of William James, where she sees "the self as an active agent, ever experiencing, learning from and shaping experience, even though behavioural choices may result in passivity or activity" (2003, p 397). We consider agency in the case or our mathematics education work not as a moral concept, but as a cognitive one, and one that carries with it, for the purposes of this work about gender and mathematics, some basic premises.

One of those premises is that the women and girls who are subjects of discussion in our work are recognized by the scholars to be agents in their mathematical lives, to have and exhibit agency, to not need to be given agency. The perspective is akin to that of Mollie Blackburn (2004) as she writes of the agency of gay, lesbian, bisexual, transgender and questioning school students. That is, agency is not something we increase in others, either the oppressed youth in her work, or the women and girls who are the focus of our working group projects. Just like the young people in Blackburn's study, women and girls in mathematics **are** agents, "with or without us" (p. 110).

A second premise of agency as it is a part of our Gender and Mathematics Working Group work is that agency is not always explicitly visible. The choices one makes, as in the case of this year's presented works with respect to studying mathematics and developing relationships with mathematics, may not always appear as indications of cognitive agency. Choices to not participate, to resist, to speak out against or to remain silent are all choices nonetheless, and indicative of a sense of agency. Again from Blackburn, "resistance is not a failure to assert agency; rather it is a move, perhaps even an aggressive move, to assert agency for a purpose that is in conflict with the dominant person or institution" (2004, p. 109).

A third premise of agency in our work centers on power and freedom. Knowing that a dictionary definition of agency is that it is an "active force; action; power", the choices women and girls make to participate – or not – indeed are acts of power. The scholars presenting in the

Gender and Mathematics Working Group, although not necessarily explicitly claiming so, implicitly recognize as powerful the choices women and girls make with respect to their experiences with mathematics. However, we do not claim that these scholars are working for power itself but, rather, for the access to knowledge construction that mathematics educators and mathematics teacher educators can provide for women and girls. The field can and should work toward understanding and perhaps removing barriers of access to opportunities for knowledge construction. After all, "[k]nowledge construction permits considerable freedom…And knowledge has political consequences: 'truths' have implications; they do not pertain in isolation" (Code, 1993, p. 72). Why would we not want all to have that freedom?

Finally, although agency requires autonomy, we do assert that in this work the autonomy that is necessary for agency is not one likened to the autonomous human agent of philosophical and moral discourse. That agent is rational, self-conscious, individual and self-sufficient. In the Gender and Mathematics Working Group's work, agency and its autonomy are not solely about being rational. The self is recognized as wise, intuitive and emotive. How a woman or girl feels about the mathematics is important and telling in efforts to understand her choices. For us, autonomy is not about the isolation of self-consciousness, individualization, and self-sufficiency. For a number of its members, the Gender and Mathematics Working Group has become a community centered on gender and mathematics and like that environment, our work is about women and girls in cultural contexts, about decisions in classrooms and university programs, about intersections between the lives of women and girls and the worlds in which they live.

Plan for Active Engagement of Participants

As has always been the case with our Gender and Mathematics Working Group, the sessions we conduct this year are intended to be active with discussion, decision-making, and work activities. As a group we remain committed to an initiative that depends upon participant voices for direction and support. In this year's sessions, we begin with introductions, a short synthesis of the work to date, and updates on current projects and recent presentations of participants.

The major component of the GMWG sessions this year is participant feedback on ongoing projects centered on women and girls' agency in mathematics. Following are introductions to the work to be shared this year by:

- Lynda Wiest, University of Nevada, Reno who shares her work with a mathematics and technology intervention program for girls;
- Katrina Piatek-Jimenez, Central Michigan University, who solicits feedback from the group on her new work on the influences on career choice of women mathematics students;
- Abbe H. Herzig, University at Albany, State University of New York, who raises questions about the purpose of mathematics in the school curriculum;
- Markku Hannula, University of Helsinki, Finland & Tallinn University sharing findings from a review of the Finnish educational system;
- N. Kathryn Essex, Indiana University, introducing her project with 5th grade children; and
- Diana Erchick, Ohio State University at Newark, who solicits feedback from the group on a current teacher development project.

The sharing of work is brief, interactive, and introduces questions to the whole group for further discussion. Following the brief introductions of the work, the participants in attendance at the Gender and Mathematics Working Group participate in an activity to create connections across the shared work and suggest common themes. This activity provides the presenters with feedback on their individual directions, suggestion as to the relevance of agency in the collective work, and information for the presenters to use to co-construct the activities of day two's working group sessions. In the time between the two working group sessions, the presenters plan the interactive sessions of day two, focused on small group discussions around collective works *and the further development of the conceptual framework around women's agency in* mathematics and mathematics education.

Introduction of the Shared Work

Lynda Wiest, University of Nevada, Reno, shares work on "The Role of an Intervention Program in Supporting Girls in Mathematics and Computer Science." The purpose of this research was to investigate the impact and critical program elements of a mathematics and technology intervention program on middle school girls' knowledge, skills, and dispositions toward, as well as participation in, mathematics and computer science. First-time participants (N=201) in Northern Nevada's Girls Math & Technology Program across four years completed the Modified Fennema-Sherman Mathematics Attitude Scale upon entry into a one-week, residential summer camp and again three and one-half months later. Program participants and their parents also completed a survey.

Participants and their parents rated the program highly in terms of overall satisfaction. Participants showed significantly improved attitudes toward mathematics, increased interest in computers, and greater participation in both. Overall, higher scores were attained on the attitudinal measures by younger, White, and higher-SES participants. Girls of color showed greater attitudinal improvement across the two data-collection points in comparison with White participants.

On open-ended survey questions, both the girls and their parents focused mainly on the program's academic content as their most valued program features, with strong mention of computer use by the girls. The residential aspect of the summer camp was another high priority for the girls, and the social aspect was important to both the girls and their parents. Both the girls and their parents noted that the Girls Math & Technology Program offered resources, computer opportunities in particular, that the girls lacked at school. They also stated that "math camp" instructors used different teaching methods, especially more hands-on activities and group work, in comparison with teachers at school. Both the girls and their parents indicated that the girls had pursued additional or advanced mathematics (and to some degree, computer science) by having participated in this program.

Questions for discussion include:

- What is the value of an intervention program for girls in mathematics? For which type(s) of girls might this be more important?
- What might the different findings by race tell us about the needs of the girls in these groups?
- How might concepts such as mathematical voice and girls' agency play a role in these differences?
- What role might computers play in increasing females, interest and knowledge in mathematics?

Katrina Piatek-Jimenez, Central Michigan University, presents her work conducted with collaborator Sraboni Ghosh from North Georgia College and State University. The work is entitled "Influences on Career Choice: A Study of Women Mathematics Students." Piatek-

Jimenez explains that even though current statistics show that nearly half of the mathematics majors in the U.S. are women, women earn a much smaller percentage of advanced degrees in mathematics and women do not enter mathematical careers at the same rate as men. Due to this noted decline in mathematics participation after the undergraduate level, these scholars believe that it is critical to examine the factors motivating these women, who have chosen to earn degrees in mathematics, to leave the field.

During the working group session, Piatek-Jimenez describes her study being conducted with Ghosh. The study is designed to investigate why women mathematics majors choose to study mathematics at the undergraduate level and what factors influence their decision whether or not to continue with a career in this field. Piatek-Jimenez shares some preliminary results and solicits feedback from the group on this developing work.

Preliminary results raise questions about the validity of the statistics quoted above. Are women really leaving the field after earning degrees in mathematics or are these statistics a product of the way in which our society defines mathematical careers?

Questions for discussion include:

- What constitutes a "mathematical career"?
- How does the pedagogy of mathematics inform society's beliefs about the nature of "mathematical careers"?
- How would redefining what constitutes a "mathematical career" impact the research agenda of scholars interested in the field of gender and mathematics participation?

Abbe H. Herzig, University at Albany, State University of New York, examines "The purposes of Mathematics in Education." She explains that, for the past several years, she has struggled to define and understand the purposes of mathematics in education. She uses the term *mathematics in education,* instead of the more traditional *mathematics education,* in order to draw attention to the ways that the particular version of mathematics that has been constructed for educational purposes is distinct from other things that are also called mathematics.

In the work she shares with the Gender and Mathematics Working Group, Herzig explores the potentially exclusionary aspects of *mathematics itself* as it is constructed through pedagogy. This exploration is motivated by three observations about mathematics in education. The first observation is that mathematics in education is distinct from mathematics used outside of schools. Mathematics in education does not seem to prepare students to use mathematics outside of school, nor does it represent mathematics as mathematicians use it. So what purpose does mathematics in education serve?

The second observation concerns the view that mathematics in education is similar to other liberal arts studies, and that through mathematics, students develop important quantitative, critical thinking, and logical reasoning skills. However, there have been mounting concerns about whether students actually attain these skills through their school mathematics experiences.

The third, and most important, observation is that mathematics in education serves as a filter, qualifying some students for higher study in mathematics and in a host of other disciplines, and for a variety of vocations, and disqualifying many other students from those same opportunities. Further, mathematics as a filtering device has disparate outcomes for students of different races, ethnicities, genders, and social classes, making consideration of the racialized, gendered, and classed aspects of mathematics in education imperative. In order to deconstruct this filter and build a more equitable and just educational context, Herzig argues that the mathematics education community needs a clearer vision of the purposes of mathematics in education.

Questions to guide discussion are:

- Is mathematics really is an essential field of study for all students?
- *Which* mathematics is important for students to learn?
- How can the goals, curriculum, and pedagogy of mathematics in education be structured to be meaningful, inclusive, and equitable?
- How can such an exploration be brought to policy makers in education?

Markku Hannula, University of Helsinki, Finland & Tallinn University, presents for discussion his work with collaborator Kalle Juuti, University of Helsinki. He introduces their on-going project conducting "A Review of Gender Issues in Finnish Mathematics Education" and explains how Finland has a long tradition of striving towards gender equity in society and education. The Comprehensive school act from 1983 states promotion of equality between sexes as one aim of Comprehensive School. This aim has since been specified in curriculum documents, such as the Framework curriculum for the comprehensive school in 1994: "The equality of the sexes is an important part of the value basis for the school. This equality of the sexes as an educational objective means that both boys and girls are equally equipped to function with equal rights and responsibilities in family life, in working life, and in society" (NBE, 1994, p. 17).

The equity situation in Finland is good in international comparison, but the labour market is still gender segregated, and there are some gender differences in the educational outcome. In international comparative studies the gender difference in mathematics achievement among 15-year olds has disappeared in many countries, Finland being one of those countries.

However, there are still robust gender differences in students' affect towards mathematics. When attitude towards mathematics has been constructed as a single variable, studies generally have found boys to hold a more positive attitude towards mathematics. However, when different dimensions of attitude have been separated, interesting variations have been found. For example, all studies have not found gender differences in 'liking of mathematics.' Gender difference has been more clear in how difficult mathematics is seen to be and robust in students' self-confidence in mathematics. Lower self-confidence among female students has been found even on level of individual tasks, in the cases of both correct and incorrect answers. These gender differences are likely to contribute to the differential career choices as soon as mathematics is no longer compulsory.

Although the curriculum takes gender perspective seriously, most teachers seem to be gender blind and involuntarily strengthen the stereotypical attitudes among students. Hannula and Juuti argue a need for gender sensitivity in mathematics teaching and discuss some approaches towards this direction.

- Questions for discussion include:
- Why do many equity-minded teachers strengthen their students' stereotypical views?
- Does equity policy from a curriculum document influence practice in the classroom?
- Assuming equity has some influence in the classroom, either direct or indirect, what are the ways in which it does so?
- Why does equal performance not lead to equal self-confidence?

N. Kathryn Essex, Indiana University, seeks feedback and discussion on questions related to a study centered on students participating in curriculum evaluation projects. She conducted task-based interviews with 54 fifth graders from a large, urban, school district in the Midwest during December 2005. The population of this district is diverse with approximately 27% of the students being African American, 24% Latina/Latino, and 27% Caucasian. The majority of students receive free or reduced lunches. With the students' involvement in the larger curriculum evaluation project, they already have taken two written tests, one in third grade and one in fourth grade. A preliminary look at the existing data has found few, if any, gender differences in the students' achievement and in their solution strategies.

Responses and work done on tasks were analyzed to look for gender differences in correctness of responses, making sense of the tasks and methods used to solve them, errors in the strategies and procedures used, and which tasks might favor girls or boys. Few gender differences were found, and these few will be discussed. Several questions emerge from this research, and are presented for discussion in the Gender and Mathematics Working Group:

Are there really gender differences in the ways girls and boys do mathematics?

- If so, what might explain the lack of differences found in this study, compared with results found in other studies?
- In what ways might culture and Socioeconomic Status be influential?
- What might we learn from this about boys' and girls' mathematical voices?

Diana B. Erchick, Ohio State University at Newark, raises questions about how to connect conceptual frameworks, such as one grounded in agency, to the mathematics education work we do with teachers, students and schools. She embeds her discussion in preliminary data from a state-funded professional development project, the K-6 Mathematics Coaches Project. The project includes designing, delivering and evaluating a mathematics coaching professional development program in 34 low-achieving rural and urban elementary schools. Given that the percentage of elementary teachers who are women is quite high, it may not be surprising that all 34 coaches in this project are women; however, they are in leadership positions and leadership in schools has not been dominated by women.

The shared data are qualitative in nature, and are drawn from coach pre- and post- tests on content and pedagogy in year one of the project. The focus of the shared data is the qualitative change in the coaches' content and pedagogical content knowledge and discussion questions presented to the group include:

- What might we learn about the role of mathematics content knowledge in women's agency in mathematics education contexts?
- How do mathematics content knowledge and "knowing mathematics for teaching" (Ball, 2005) different and does their difference impact women's agency?
- What might we learn about the role of mathematics pedagogical content knowledge in women's agency in mathematics education contexts?
- How does mathematical voice play a role in this work?
- How can we frame the discussion of women's agency in program contexts such as the mathematics coaching program?

Closing

In pursuing inquiry around Gender and Mathematics, the PME-NA Gender and Mathematics Working Group participants have committed themselves to an interpretation of the field of gender and mathematics as complex and nonlinear. We have also chosen to investigate the absences we encounter with a respect for the reflective voices of the researchers, teachers, students, women and girls who contribute to the work. In the papers and processes of this project, we work consistently to respect the structure and voices that emerge. Original absences apparent in 1998 have grounded our work since then. Newly apparent absences now ground our new directions, and our commitment to addressing absences in the field continues. In particular, as in this year's working group, the current investigations and questionings bring to the discussion issues such as influences on girls' and women's agency in choices they make about their relationships with mathematics; the role mathematics plays and how important it is or is not; what elements of the various contexts, from elementary school to university, across the continent, in relationship with other contexts such as technological environments are contributing not-so subtle influences on women's and girls' choices and relationships with mathematics.

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WORKING GROUP

STUDYING TEACHER LEARNING:

THE WORKING GROUP ON INSERVICE TEACHER EDUCATION

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The focus for this working group is setting goals for the upcoming years in four sub-areas: studying teacher learning of mathematics; studying teacher development of PCK; studying mechanisms for teacher learning; and studying how teachers learn from their everyday classroom work. This goal-setting builds on work done by the PD discussion groups at past PME-NA meetings.

A Brief History of the Group

At the 2001 NCTM Research Presession, the facilitators of the proposed working group led a session titled "Studying Professional Development is Messy Work. What are the research issues?" Approximately 50 people attended. At the 2002 PME-NA meeting in Georgia, the same facilitators began a PME-NA-based discussion group to address continued interest in the issues surrounding research on professional development for teachers of mathematics (Arbaugh, Brown, & McGraw, 2002). Approximately 70 people attended the discussion group, which met twice during the conference.

The 2003 PME/PME-NA (Arbaugh, Brown, & McGraw, 2003) and the 2005 PME-NA (Arbaugh, 2005) discussions groups continued the work begun in Georgia in 2002. At the end of the 2005 meeting, interest remained high for continuing this group as a Working Group at the 2006 meeting in Merida.

Focus for Proposed PME-NA 28 Working Group: Setting the Agenda for Work in the Coming Years

Building on the work begun in prior discussion groups, the focus of this proposed working group is on setting the work of the Working Group over the next few years. Over the course of the meeting, we intend to set working agendas in the following areas:

1. Studying teacher learning of mathematics.
2. Studying teacher development of PCK.
3. Studying mechanisms for teacher learning (professional development models).
4. Studying how teachers learn from their everyday classroom work.

In addition to laying out an agenda for each subgroup over the next few years, participants in each subgroup will design a hypothetical study, present that study to the larger group, and receive feedback.

At the end of the PME-NA 27 working group session, we will spend time planning for future working groups. This work includes:

1. Setting goals for future meetings.
2. Generating possible products that can come from our work.
3. Committing to participation at future meetings.

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An important long-term goal for the working group will be to develop and support leadership in the area of research on teacher learning. Individuals who are beginning work in this field should benefit from engaging with a community of researchers and examining and discussing the usefulness and limitations of various frameworks and research methods. In addition, this working group will provide a much-needed arena for cross-pollination of ideas among both senior and junior researchers and encourage movement toward a coherent and conceptually rich research base in mathematics teacher learning.

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WORKING GROUP

WORKING GROUP ON MATHEMATICS CLASSROOM DISCOURSE

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This working group on mathematics classroom discourse will focus attention on the specifically mathematical characteristics of discourse in mathematics classrooms. Participants will work together in small groups to respond to various artifacts from mathematics classroom discourse. In large-group discussion, we will hear from the small groups and work together to find some common ground.

Recap of First Meeting, Roanoke, 2005

Last year's discussion group on mathematics classroom discourse (Choppin et al., 2005) was structured around three guiding questions:

1. What theoretical frameworks might be used to study classroom discourse in demographically diverse settings?
2. What are the specific mathematical characteristics of discourse, and how do our analytic techniques account for these characteristics?
3. How can the study of discourse help us understand and transform the teaching and learning of mathematics?

Participants in this discussion group began to investigate the nature and role of discourse in mathematics classrooms. The 40 participants were introduced to three theoretical frameworks as examples of a range of frameworks for analyzing the discourse. Participants analyzed and interrogated these frameworks for researching the nature and impact of discourse practices in terms of both social and mathematical aspects. Furthermore, methodological and analytical challenges were considered.

Format for Working Group, Mérida 2006

Continuing the conversation from last year's discussion group, this working group will continue to be structured by the above three guiding questions. While our discussions last year primarily focused on question one, this year's working group will focus on the second of these three questions, and will depend more heavily on the participation of the assembled group.

The sessions will be centered on the consideration of mathematics classroom artifacts. In the first session, David Pimm will use one artifact to lead the group in a discussion about the mathematics register and its implications for classroom discourse. This discussion, which relates closely to this year's focus question on mathematical characteristics of classroom discourse, will underpin small-group discussions about other artifacts.

Participants in this working group will work together in small groups to respond to mathematics classroom artifacts that may include:

- video excerpts, drawn from the TIMMS video study model lessons (RBS, 2003)

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- audio excerpts
- transcripts
- student writing
- textbooks
- technological tools (e.g. graphing calculator)
- assessment instruments

The range of theoretical and methodological perspectives that participants bring to this working group, together with the focus on unique characteristics of mathematics classrooms, will provide rich ground for small-group discussion. Groups will be given artifacts to study and will be asked to address prompts such as the following:

- What features of the discourse do you see represented in your artifact?
- Relate these features to characteristics of classroom discourse that are unique to mathematics classrooms.
- Consider alternatives to the classroom discourses you see represented in your artifact.
- Identify constraints and affordances experienced by teachers interested in implementing alternatives to these discourses.
- What is the impact of your theoretical and/or methodological perspectives on your responses to the above prompts?

After groups will have had sufficient time to work on their artifacts the larger group will be convened for the small groups to share their findings. We hope that each group will have a chance to study more than one artifact. At the end of the last session there will be some discussion about future directions for the working group and potential writing projects.

Rationale for Work on Mathematics Classroom Discourse

Theoretical Frameworks

The word *discourse* can mean various things. A range of analytical tools has been used to study mathematics classroom discourse. Each analytical tool foregrounds its own aspects of discourse. In addition to the various scholarly approaches to discourse, the term has wide currency in professional literature. For example, the NCTM *Standards* documents (1991, 2000) stress the role of discourse in the learning and teaching of mathematics, and promote particular forms of discourse in an attempt to normalize certain classroom practices.

In this context, in which various educators refer to different aspects of discourse and even use some of the same words in differing ways, there is value in bringing people with different perspectives together. We can understand our own perspectives better when we listen to others describe their perspectives. We can work together toward common goals, complementing each other's foci.

The Mathematics in "Mathematics Classroom Discourse"

Studies focusing on features of discourse that are uniquely mathematical include investigations of argumentation (e.g., Lampert, Rittenhouse, & Crumbaugh, 1996), hidden regularities in interaction patterns (e.g., Voigt, 1995), the mathematical register (Pimm, 1987), metacommenting used by mathematics teachers (Pimm, 1994), and the triadic dialog (i.e., the IRF sequence) and its relationship to forms of *habitus* (Zevenbergen, 2001).

In addition to the need for extending present scholarship relating to mathematics classroom discourse, we need to consider carefully the relationships between characteristics of mathematics and the already-identified features of mathematics classroom discourse. There is also a need to develop more analytic tools that are specifically geared toward mathematics classrooms. While we can learn much about the social order of mathematics classrooms using tools developed by discourse analysts, these tools do not often take into consideration the specific mathematical content of the conversations taking place (Steinbring et al., 1998).

Though the characteristic abstraction and generalization associated with mathematics often directs attention away from critical socio-cultural issues such as social class, gender, and race, a focus on aspects of classroom discourse that are particular to mathematics classrooms can uncover such issues. However, these issues are rarely examined in discourse studies in mathematics classrooms. Focusing discourse studies on inequities can help us understand the range of language use and interaction patterns students bring to mathematics learning and illuminate issues of authority and power (Atweh, Bleicher, & Cooper, 1998; Herbel-Eisenmann, 2003; Herbst, 1997; Zevenbergen, 2001). Though significant work toward understanding mathematics classroom discourse has been done, the research community still has far to go in its attempt to understand many aspects of discourse (Steinbring et al., 1998).

Practical Implications of this Work

There is evidence that discourse practices have not changed much in the last two decades (Spillane & Zeuli, 1999; Stigler & Hiebert, 1999) and there is little evidence of the connection between the nature of discourse practices and mathematics achievement (Steinbring et al., 1998). From a practical perspective, it has been shown that mathematics teachers' discourse patterns are quite traditional, including those of teachers who are attempting to change their classroom practices (Cohen, 1990; Herbel-Eisenmann, Lubienski, & Id Deen, 2004; Spillane & Zeuli, 1999) and a broader sample of mathematics teachers in the US (Stigler & Hiebert, 1999). This is important given that the reform movement in North American mathematics education has made some particular demands on teachers.

Most of the scant literature where teachers have been involved in examining their own classroom discourse has focused on teachers in unusual situations, for example, teacher development experiments (e.g., Cobb, Yackel, & Wood, 1993) or teachers who are considered experts in mathematics education (e.g., Lampert & Blunk, 1998). Only recently have researchers used the tools and concepts of discourse analysis with teachers as they teach in their ordinary classrooms (e.g., Rowland, 2000).

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WORKING GROUP

MATHEMATICS TEACHING ASSISTANT PREPARATION AND DEVELOPMENT RESEARCH WORKING GROUP

BASIC AND APPLIED RESEARCH INTO THE LIVES AND PRACTICES OF MATHEMATICS GRADUATE TEACHING ASSISTANTS: ADVANCING THEORY AND INFORMING PRACTICE (A WORKING GROUP REPORT)

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Teaching assistants (TAs) play vital roles in the mathematics education of undergraduates and may go on to become professors of mathematics. The K-12 literature on teacher practice delves both into professional development activities and their assessment as well as into (pre-service) teachers' beliefs about mathematics teaching and learning. Here we document an emerging body of scholarly inquiry into the graduate student TA experience and the professional development needs of TAs that is beginning to replicate, expand on, and apply ideas from these existing lines of inquiry. Topics of projects to be discussed during this session include: use of design-based research to create professional development materials, statistics TAs' knowledge of the topic of statistical sampling, and packaging research findings to inform the pedagogical content knowledge of TAs.

The working group fosters collaboration in framing and carrying out this research. Meeting time is devoted to discussion of participants' research projects at various stages of development (planning, data collection, analysis, and reporting). Participants provide feedback and discussions serve as the basis for the group's goals of building a community of researchers interested in TA issues, the analysis of similarities and differences with K-12 mathematics education, and the development of an agenda for continued work.

Interest in and awareness of mathematics graduate student teaching assistant (TA) professional development (PD) needs continues to increase in the mathematics community. The Mathematics Teaching Assistant Preparation and Development Research Working Group of PME-NA emerged in 2002 as a forum specifically for research in the area. Since that time the group has worked to be a community site where mathematics educators can connect, collaborate, receive critical feedback, and organize an agenda of relevant, common concerns. There have been significant recent developments in the field including increased interest by mathematics education graduate students, increased efforts to connect new results about TAs to the existing body of results and theory in the K-12 literature, and a general increase in the variety and richness of projects being disseminated. In this paper we note recent growth in the field and discuss developments in both "applied" and "basic" categories of work.

Activities of working group members have generated publications in addition to the PME-NA working group proceedings. For example, the organizers expanded on and published a paper from the 2002 conference proceedings in *College Teaching* (Speer, Gutmann, & Murphy, 2005). Projects that were initially presented during the working group sessions have also developed into PME-NA presentations at subsequent conferences (Speer, Strickland, & Johnson, 2005; Kung &

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Speer, 2006b). In addition, group members have given presentations on their working group-related activities in other venues, including invited talks and research conference sessions (Gutmann, 2005; Kung & Speer, 2006b; Speer 2005). An email listserv and a wiki web site (http://betterfilecabinet.com/cgi-bin/ta/index.cgi) have been developed to encourage awareness and collaboration in the field. The web site includes a "virtual research group" area where members of the working group share works-in-progress and receive feedback throughout the year.

Categorizing Research in the Field

In a broad sense, research work in the field can be categorized into two areas: applied and basic. The "applied" area is focused on the development and use of PD materials along with investigations into how existing research can inform the design of these materials and how TAs learn from planned PD experiences. The "basic" area is focused on advancing theory to understand who TAs are, how they think about what mathematics is and how it is learned, and how their communities help them define themselves and their profession.

While the research discussed here focuses on the general topic of teacher development and practice, its focus on mathematics graduate student teaching assistants means there are special issues that receive increased attention. In this world we take somewhat for granted that TAs have rich understandings of mathematics, but ask new questions about how this translates into or informs pedagogical content knowledge (Shulman, 1986) and classroom practice, topics extensively investigated in the K-12 literature. Group members bring several theoretical and methodological perspectives to their research. These include socio-cultural, deisgn-based, and cognitive lenses. This aids the group in pursuing one of its central goals: understanding how results from existing mathematics education research on K-12 mathematics teaching and learning apply to, and fail to apply to, the special world of TAs.

Applied research

In the context of this paper, applied research refers most often to the design of PD activities or materials. Here researchers question the needs of TAs in terms of knowledge about the classroom and pedagogical content knowledge. Materials are designed to meet these needs in the special context of TA professional lives that often includes being in the classroom from the start of the first semester of a graduate program, before any formal PD activities are available. Several such projects were presented at the 2005 PME-NA meetings and summarized in the 2005 proceedings paper for the working group (Gutmann, Speer, & Murphy, 2005). Hauk et al (*Video cases for novice college mathematics teacher development*) presented plans to create factual and manufactured classroom vignettes designed to help TAs confront and think through a variety of pedagogical issues. Noll (*Using a CGI professional development framework for improving statistics TAs' pedagogical content knowledge*) begins to ask specialized questions about the pedagogical content knowledge required for teaching statistics and considers how a Cognitively Guided Instruction (CGI) framework can be used to provide PD for statistics TAs. Noll's conference presentation asked participants to consider how TAs might think about mathematics and statistics differently and how this might influence their teaching practice.

Basic research

Some researchers in the field focus their investigations on fundamental issues in mathematics education that connect to and/or support the applied research described above. In an early entry, Carlson (1999) investigated a group of TAs and attempted to describe them as mathematicians and to explain the habits of mind that have made them successful. Herzig's (2002, 2004) similar line of inquiry discusses female graduate students as women and how the mathematics community is dysfunctional in its integration of them. Facilitated by the working group, four authors presented basic research at the 2005 meeting. Meel continued the theme of understanding who graduate students are and their needs in *Exploring first-year TA experiences through weekly reflective writing assignments*. Considering how graduate students acquire the knowledge needed for teaching, especially knowledge about student thinking, Kung focused on calculus TAs in *Teaching assistants learning how students think*. Extending the literature base on K-12 teacher planning Speer, Strickland, and Johnson (*Influences of college mathematics teachers' knowledge and beliefs about student understanding on their plans for instruction*) and Winter (*Lesson planning practices of graduate student instructors in mathematics*) each examined TAs' planning processes and the ways in which knowledge and beliefs about learning influence their choices of classroom activities. These titles are abstracted in the 2005 proceedings paper (Gutmann, Speer, & Murphy, 2005).

Current Working Group Projects

Members have contributed synopses of three projects to be discussed during the group meeting time. Using the categorization above, the first of these projects is applied—Belnap and Winter's design-based research approach to creating professional development experiences. The second represents basic research: Noll's inquiry into graduate students knowledge of statistics for teaching. The third (Kung and Speer) is a plan for a publication that will organize findings from research on teaching and learning in a format specifically useful for educators working with TAs and other new mathematics instructors and/or planning professional development for them.

During the group meeting researchers will share their projects as described below and solicit feedback. In each project synopsis, the researchers describe the work, indicate what "stage" of development the project is in (planning, data analysis, reporting, etc.), and set out how they intend to structure their portion of time during the working group meeting.

Mathematics teaching assistants: Video observation with peer-feedback sessions as professional development

Jason K. Belnap and Karen Winter

Extensive, nation-wide survey research by Allred & Belnap (2006a; 2006b) into the ways in which mathematics departments use TAs and the kinds of professional development provided to TAs reveals that in the United States, it has become common for departments to involve TAs in college mathematics instruction. In fact, most of these TAs are employed as instructors, having sole responsibility for their classes. Many mathematics departments provide extensive preparation for their TAs and these departments use systems of preparation involving a combination of programs that provide not only initial preparation prior to teaching responsibilities, but sustained support throughout their teaching experiences. In spite of this, most existing preparation programs are developed, not from research literature, but by word-of-mouth connections and information provided by the national mathematics organizations. Furthermore, most departments are only moderately satisfied with their preparation programs and have many questions for researchers. The topics they feel researchers need to address include: how to motivate teacher change in TAs; the benefits or impacts of preparation programs; what types of preparation are best; and what content/methodology is needed.

Researchers of TA development programs must address these needs by studying not only the TA experience, but also the impacts and design of the different types of preparation programs that exist. We need to determine how different contexts and elements in those contexts shape the impact and implementation of various preparation programs and how program designs affect TAs' teaching development.

One way that researchers can do this is through design-based research. This type of research revolves around the development and refinement of an intervention, a TA preparation program, for instance. In a cyclical process of development, a preparation program is studied and refined over time to become more effective and to better meet the desired goals. If this were it, however, these methods would add nothing more to how some departments are already producing their preparation programs.

Design-based research is more than this, though. Although the intervention is a central item and product of the research, it is not the primary goal and focus of the research. The main goal is to understand the design of the intervention, its impact on and interaction with the context. This includes identifying important aspects of both the intervention and the situation that play key roles in how the it unfolds. It provides a level of understanding that can help with study generalization and with off-site implementation.

Working group plans.

This workgroup presentation will focus on the role that design-based research can play in our research on TA preparation and development. Using our current research study as an illustration, we will present and discuss how design-based research can be incorporated into TA preparation research. Group discussion will focus both on the use of design-based research, as well as on the goals, structure, and nature of our current research project, a first attempt on our part to implement this type of research.

We would welcome feedback from both those experienced with this type of research and those new to it. Those who are experienced with these methods will be able to share their expertise and feedback both on these methods in general and on the current study design in particular. Those new to these methods can learn how it can be incorporated to further research on TA preparation and even other fields of study. We also seek feedback from those interested in research on TAs' development of social teaching networks since this is a conceptual focus of the current study.

Graduate teaching assistants' statistical knowledge for teaching in sampling situations

Jennifer Noll

During the past decade, the mathematics education community has been engaging in research on the teaching and learning of probability and statistics at both K-12 and college levels (Ben-Zvi & Garfield, 2004; National Council of Teachers of Mathematics, 2000). As graduate TAs teach the bulk of introductory statistics courses at many universities (Luzter, Maxwell, & Rodi, 2000), they have the potential to play a vital role in undergraduate statistics education and in the promotion of statistical literacy among college students. The overarching goal of my project is to broaden the developing base of research concerning TAs by initiating research on TAs' statistical knowledge for teaching. In particular, I plan to investigate the following research questions:

1. How do TAs understand the ideas of sampling? In particular,
   a. How do TAs conceptualize samples, the act of sampling and sampling distributions?
   b. How do TAs conceptualize the connections between sampling and statistical inference?
2. What statistical knowledge for teaching do TAs possess? In particular,
   a. How will TAs respond to scenarios of student responses to a variety of sampling tasks?
   b. What knowledge of student solution strategies and/or common conceptual difficulties do TAs posses?

Status

In the fall of 2006 I will collect web-based survey data from approximately 50 TAs from a convenience sample of six universities across the country. Following the surveys I will select a convenience sample of four local TAs for subsequent interviews. Survey and interview tasks will assess TAs' content knowledge of sampling, as well as their knowledge of students' statistical thinking within sampling. The selection of survey and interview tasks which address TAs' content knowledge and knowledge of students' statistical thinking are drawn from tasks used in recent research on K-12 and college students' reasoning in sampling contexts (c.f., Chance, delMas & Garfield, 2004; Reading & Shaughnessy, 2004; Watson, 2004). In order to assess TAs' knowledge of students' statistical thinking, I developed extension questions that ask TAs to respond to scenarios of student responses to the sampling tasks. I plan to use the conceptual framework developed by Shaughnessy et al. (2004) for my initial analysis of TAs' content knowledge by classifying responses into the broad categories of additive, proportional and distributional reasoning.

Working group plans

During the working group session I would like feedback on methods for analyzing and interpreting the survey and interview data. In particular, I would like feedback on my use of Shaughnessy's framework, and the ways I can refine his framework for a stronger characterization of TA content knowledge. Additionally, I would like feedback on developing a framework for TAs' knowledge of students' statistical thinking.

Transforming research findings into pedagogically useful resources for college mathematics teachers

David Kung and Natasha Speer

We have submitted a prospectus to the MAA for a volume that would make research on undergraduate mathematics learning accessible to mathematicians and mathematics graduate students. Each chapter in the volume would focus on a particular topic from undergraduate mathematics, from College Algebra through Calculus to the upper-level courses. The tentative title for the volume is *What Could They Possibly Have Been Thinking?!? Understanding Your College Mathematics Students*.

This volume is intended for use in professional development programs for graduate students and other new instructors; it would also be useful as a reference for people new to teaching at the college level or experienced college instructors who are teaching new topics for the first time. The information about the nature of student understanding and learning will be presented in a format that is accessible to people who are not knowledgeable about mathematics education research. Readers will be drawn in by beginning each topic with concrete examples of student work before moving into a discussion of how those examples illustrate general patterns of student thinking. The order of the chapters will make it easy for readers to select portions of the volume that are most relevant to their current needs and interests. Unlike other reports or syntheses of research, these chapters will provide specific examples of students' work that teachers might actually encounter on exams or during class and an analysis of the thinking behind that work based on findings from research. The examples of student work (both correct and incorrect) will illustrate the difficulties students have and the types of problematic thinking typical in that topic. This work will bridge the gap between research and practice by providing access to important information that can help these teachers understand their students' work and contribute to their improved teaching practices.

Research on K-12 teaching and professional development shows that experienced teachers have far more "pedagogical content knowledge" (Shulman, 1985) than their novice counterparts. PCK describes knowledge teachers have, distinct from knowledge of content or knowledge of general pedagogy, and includes knowledge of how student think, challenges they encounter, and strategies for helping them overcome those challenges. Thus PCK is precisely the mathematical knowledge needed to do the work of teaching. K-12 teachers have the added experience of methods courses and textbooks that contain a significant amount of PCK-related information. Teachers at the college level acquire such knowledge in a variety of ways including examining their own learning experiences, grading homework and exams, and interacting with students. What little professional development they receive is, however, unlikely to concentrate on PCK; hence the impetus for our project.

Working group plans

We have a draft chapter (about limits) and would like to get feedback from the working group members on the format of the chapter, the particular student work/thinking examples selected, and the aspects of the research literature we have chosen to illustrate. We want to ensure that the information is presented in a manner that is useful and appealing to the intended audience and want to be certain that we are representing the research findings in as comprehensive a way as possible.

Conclusion

Since the formation of the working group, the number of educators interested in and actively conducting research related to TAs has increased greatly. As this has happened, the working group has identified several important concerns that should inform a future research agenda. In keeping with the theme of this paper, they are roughly grouped into applied and basic areas.

Applied concerns

These relate to the topics PD should address and the forms of PD experiences that will be most effective.

1. We frequently assume TAs have a strong understanding of the mathematics they teach. Considering the difference among knowledge of the subject, pedagogical content knowledge (PCK), and mathematical knowledge for teaching, is this assumption warranted? On a related note, as an increasing number of students enter college needing to take courses such as College Algebra and Precalculus, the jobs of TAs are likely to change. They may be spending more and more time working with students who have weak mathematics preparation and/or significant mathematics anxiety issues. Moreover, the topics TA are responsible for helping students learn may be even further removed from their own studies than they are now. Thus, there is a need to reconsider the PCK needs of TAs and to utilize the K-12 literature about PCK specific to the content domains that new college students may be learning.
2. Since TAs' jobs involve many responsibilities, models of professional development that have been tested with school teachers (e.g., workshops, mentoring, lesson study) may need to be modified in various ways to fit into TAs' lives.
3. Many mathematics departments provide PD for their TAs, but until now there has been no real research base for them to build their programs around. The community needs to consider how research findings related to TAs can best be disseminated. Further, thought should be given to how to encourage departments to develop and update PD programs.
4. As new PD programs are created, assessment of research-based PD is becoming possible and necessary. PD programs will need to be targeted at needs identified by research and will need to be created to admit assessment of their effectiveness.

Basic concerns.

These relate most directly to understanding who TAs are, what they bring to their jobs, and how they function and learn in their social environment.

1. TAs lead complex lives in which they are professionals, yet often seen as pre-professionals. They are both students and teachers and many have families and heavy responsibilities outside the university. We need to understand more about how these people learn to value and balance their different responsibilities. Analysis should extend beyond the TA as an individual to include examination of TA communities, how these communities organize and pass on information and determine norms and values.
2. International TAs have special needs in US classrooms. Their lack of familiarity with US student cultural norms means they may feel uncomfortable responding to a variety of classroom occurrences. They may also have difficulty integrating with their TA community. Researchers should be sensitive to these special concerns and include them in their inquiry into the TA experience.
3. Not all TAs intend to complete Ph.D.s and pursue careers in academics. A greater understanding of the variety of professional motivations and goals of TAs is needed, both to understand the TA experience in general, and to make specialized recommendations about PD.

At each working group meeting the group has attempted to refine a big picture of the field and to make suggestions relevant to a research agenda for the field. An important development has been an increased attempt to apply and adapt findings in the K-12 literature to the TA experience. In 2005 and at the upcoming 2006 conference familiar domains such as pedagogical content knowledge and communities of practice represent valuable steps in this direction. The working group session promises to proceed productively on well-developed ideas.

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WORKING GROUP

MODELS AND MODELING

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The Models and Modeling Working Group at PME-NA has successfully continued its work since 1999. The purpose of this Working Group is to discuss and enrich different views in which models are used in the learning of mathematics and applied science. That is, models are considered conceptual and representational tools that allow us to better understand how students, teachers, researchers, and other educators learn, develop, and apply relevant mathematical concepts (Lesh & Doerr, 2003; Lesh, Doerr, Carmona, & Hjalmarson, 2003). To this workshop we would like to invite participants to begin or continue the development of the greatly needed communities of researchers and practitioners to expand our focus of research on the ways in which models are used in Problem Solving, Curriculum Development, Student Development, and Teacher Development. This year, a special focus will also be given to Models and Modeling as it applies to Assessment and Research Design, and its relation to Complexity Theory.

In this workshop, we will continue to reflect on *a Models and Modeling Perspective* to understand how students and teachers learn and reason about real life situations encountered in a mathematics classroom. We will discuss the idea of a model as a conceptual system that is expressed by using external representational media, and that is used to construct, describe, or explain the behaviors of other systems. We will reflect on the characteristics that are elicited, including the complexity, dynamic, and iterative features of model-development. We will consider the types of models that students, teachers, and researchers develop (explicitly) to construct, describe, or explain mathematically significant systems that they encounter in their everyday experiences, as these models are elicited through the use of model-eliciting activities (Lesh, Hoover, Hole, Kelly, & Post, 2000). During the workshop we will continue to explore these aspects of learning, teaching, and research by continuing our work in panels and smaller groups focusing in: Student Development, Teacher Development, Curriculum Development, Problem Solving, and a strong emphasis on Research and Assessment Design, and Complexity Theory.

A models and modeling perspective has proven to be a rich context for research and development. During past workshops, we have discussed and continued to work on innovative designs for research and assessment that can help answer questions involving the understanding of complex situations that are dynamic and iterative. There are several characteristics that need to be sustained by the types of research design needed. These include:

First, it is important to radically increase the relevance of research to practice, involving many levels and types of participants (students, teachers, researchers, curriculum designers, policy makers, and others) (Lesh & Kelly, 2000). Second, it is necessary to understand that the educational phenomena that are researched are complex systems, in the sense that they are dynamic, interacting, self-regulating, and continually adapting. Third, it is necessary for educational decision-makers to rely on reports that involve more than simple-minded uni-dimensional reductions of the complex systems that characterize the thinking of students,

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teachers, and researchers. Recent advances in mathematics and other scientific fields have made available the use of technologies that are capable of using graphic, dynamic, and interactive multimedia displays to generate simple (but not simple minded) descriptions of complex systems (for example, weather, systems, traffic patterns, biological systems, dynamic and rapidly evolving economic systems) (Lesh & Lamon, 1993). And fourth, research is about knowledge development; and not all knowledge is reducible to a list of tested hypotheses and answered questions. In particular, in mathematics and science education, the outcome products that are needed from our research often focus on the development of models (or other types of conceptual tools) for construction, description, or explanation of complex systems. Thus, distinctions need to be made between: (a) model development studies and model testing studies; (b) hypothesis generating studies and hypothesis testing studies; and (c) studies aimed at identifying productive questions versus those aimed at answering questions that practitioners already consider to be priorities.

From these assumptions, many participants from the Models and Modeling Working Group have been working on a research design first described by Collins (1990) and Brown (1992) called *Design Studies*. This type of research design explicitly focuses on the development of constructs and conceptual systems used by students, teachers, researchers, and other educators. Principles applying to Design Research, the types of research questions it allows to answer, appropriate methodologies involved in the design of these types of studies, and examples of Design Research Studies are some of the discussion topics that will be considered in our working sessions.

The Models and Modeling Working Group at PME-NA Mérida

The Models and Modeling Working Group at PME-NA XVIII has the following goals**:**

- To disseminate and contribute to the research on the use of models and modeling in school mathematics, with a focus on students, teachers, researchers, and policy makers.
- To create and support collaborations among researchers to build international communities of practice.
- To extend the field of mathematics education towards new directions on assessment, problem solving, research design, learning environments and complexity; as it relates to the use of models and modeling in school mathematics.

For the PME-NA XXVII Models and Modeling Working Group, several sessions will be organized throughout the Conference. In particular, there will be two main working group sessions. For each session, after a general introduction on different topics is provided, participants will be invited to select one, and smaller groups will be formed. Each sub-group will have a panel of discussants, and a discussion leader, who will approach the selected theme. In addition, participants will be encouraged to attend to other sessions that will be offered throughout the Conference, and that will further support and enrich the discussion that will take place during the two Working Group sessions.

These panels and smaller groups will be guided by topics related to models and modeling and: Student Development, Teacher Development, Curriculum Development, and Problem Solving. More particularly, for this year we would like to extend our work by placing an emphasis on Design Research Studies (Collins, 1990; Brown, 1992) as a framework for Research and Assessment Design, and Complexity (Hills, Hurford, Stroup, & Lesh, in press; Lesh & Yoon, 2004).

Participants will participate in the sessions according to their interests to discuss these issues more in depth, as well as to outline a plan of action for future collaboration for those who are interested in continuing their work through out the year.

Some accomplishments of the Models and Modeling Working Group

Some of the publications and other accomplishments of the participants of this working group. The *Handbook of Research Design in Mathematics and Science Education* (Kelly & Lesh, 2000) describes a variety of innovative research designs that have been developed by mathematics and science educators to investigate interactions among the developing knowledge and abilities of students, teachers, and others who influence activities in mathematics and science classrooms. The book *Beyond Constructivist: A Models & Modeling Perspective on Mathematics Teaching, Learning, and Problems Solving* (Lesh & Doerr, 2003) includes chapters written by many of the participants of this working group, where the authors give a fuller description of a Models and Modeling Perspective.

A special issue on *Mathematical Thinking and Learning: An International Journal* edited by Lyn English explicitly dedicated to a Models and Modeling Perspective, as a theoretical perspective (Lesh & Lehrer, 2003; Lesh, Doerr, Carmona, & Hjalmarson, 2003), and how it applies to student (Petrosino, Lehrer, & Schauble, 2003), teacher (Schorr & Koellner-Clark, 2003) and problem solving (Lesh & Harel, 2003).

A Models and Modeling perspective has proven to be rich context for research and development. Nevertheless, we have found the need to innovative research designs that can better help us answer the types of questions we are mostly interested in. A research design that has proven to be very useful for conducting research from a Models and Modeling Perspective are *design experiments* or *design research studies* (Collins, 1990; Brown, 1992). One of the works in progress of many participants of this working group is the development of a book on this type of research design, and how it can be used to conduct useful research to better understand students', teachers', researchers', and other educators' development of relevant mathematical ideas. Not only will the new book focus on design research methodologies, but it will also describe on new types of dynamic and iterative assessments that are especially useful in design research –where rapid multi-dimensional feedback is needed about the behaviors of complex, dynamic, interacting, and continually adapting systems.

Finally, a new publication is soon to be released, focusing on *Real-World Models and Modeling as a Foundation for the Future of Mathematics Education.* Some of the questions that are answered in this book, and that will also be a focus for discussion during our working group include: How can research investigate systems of interacting systems –in situations where students interact with one another, students interact with teachers and students, teachers interact within continually evolving learning communities, and the learning activities are themselves continually evolving situations? What steps can be taken to develop a research community that is more than just a community of isolated individuals?

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DISCUSSION GROUPS

DISCUSSION GROUP

LESSON STUDY AS A MODEL FOR TEACHER CHANGE IN MATHEMATICS EDUCATION

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The Lesson Study discussion group will explore current and needed research agendas for lesson study, such as the culture of schools, content knowledge of teachers, teachers' attention to students' mathematical thinking, beliefs and practices of participants, the role of outside experts and impact on student learning. An edited book is a possible outcome.

Research from cognitive science has precipitated enormous reform efforts in mathematics education in the United States focused on changing the way mathematics teachers practice their profession in K-12 classrooms to align with what we know about learning. A considerable amount of research has been conducted on the process and factors that influence teacher change. Yet many mathematics classrooms remain numbingly the same. Teacher-directed activities and lecture are frequently the primary delivery models for instruction. As a result, researchers continue to search for better understanding of the process of change and for models that support significant and lasting change in teacher behavior.

After results of the Third International Mathematics and Science Study (TIMSS, 1999) found Japanese students better prepared in mathematics than students in the United States, researchers in the U.S. turned to the Japanese educational process for answers. Of particular interest was the process for inservice teacher education that is the major form of professional learning for Japanese teachers. The process is called Lesson Study.

In a study of initial implementation of the Lesson Study process in an urban school in New Jersey, Fernandez, Cannon and Chokshi (2003) concluded that "those interested in implementing lesson study in the US cannot overlook the substantial challenges that must be overcome to make this practice purposeful and powerful . . . powerful lesson study practice will depend on what teachers bring to this activity . . . lesson study must include room for knowledgeable coaches who can stimulate the thinking of groups so they can rise above their own limitations." (181-182)

This and other research suggest that there is much to be learned about Lesson Study as a viable model for teacher change. Factors such as culture of the schools, content knowledge of the teachers, teachers' attention to their students' mathematical thinking, beliefs and practices of participants, role of the outside experts and impact on student learning are all important objects of investigation for researchers interested in studying implementation of Lesson Study.

In the past few years at PME/NA there have been an increasing number of research projects looking at Lesson Study. The researchers proposing this discussion group believe there is sufficient interest to warrant convening a discussion group to share our research, explore the possibility of collaborative research and determine if there is adequate interest in compiling an edited book that pulls together research on efforts to implement Lesson Study.

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DISCUSSION GROUP

TRANSNATIONAL AND BORDERLAND RESEARCH STUDIES IN MATHEMATICS EDUCATION

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This Discussion Group will focus on transnational and borderland research studies across sending and receiving communities in Mexico and the U.S to move transnational research agendas forward. Participants will consider multiple aspects of children's experiences with mathematics, including curriculum, classroom participation structures, mathematical reasoning and discourse (both in and out of school), and parents' perceptions and beliefs about mathematics instruction.

Focus and Aims of the Discussion Group

During this Discussion Group, researchers from several universities will present and discuss summaries of their current research projects examining mathematics curriculum and instruction in both Mexican and U.S. schools, out of school mathematical activities in both countries, and Latino parent beliefs about mathematics instruction. Through these discussions we hope to bring together researchers on both sides of the border, to foster and support an interest in pursuing these issues further, and to create a group of researchers who will work on these topics by organizing a Working Group around this theme for subsequent meetings.

For the past several decades there has been a large influx of immigrants in the United States, particularly from Asia and Latin America. These immigrants are a heterogeneous group that challenges simple generalizations. They include "highly educated, highly skilled workers... and large numbers of poorly schooled, semiskilled, or unskilled workers, many of whom are in the United States without proper documentation" (Suárez-Orozco, 2001, p. 350-1). According to the U.S. Census Bureau (March, 2000), over 50% of all immigrants in the U.S. are from a Latin American country, and the majority of these immigrants are from Mexico. Across the U.S.,

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significant numbers of immigrant children from Latin America, particularly from Mexico, are entering U.S. classrooms. Needless to say, mathematics teachers in the U.S. are struggling to understand and meet the needs of these students.

Transnational and borderland research studies across sending and receiving communities in Mexico and the U.S. are important to pursue for several reasons. First, many children experience the transition between Mexican and U.S. mathematics classrooms as disruptions in their mathematics learning trajectories. It is crucial to examine this transition in order to be able to better support recent immigrant children in learning mathematics in the U.S. (Abreu, Bishop, & Presmeg, 2002). Second, many families and children cross these borders more than once in their lives and belong to communities on both sides of the national border, so that their lived experience is not neatly separated into “here” and “there” (Suárez-Orozco & Suárez-Orozco, 2001). Therefore, it is important to examine the mathematical aspects of this population’s experiences across two countries, rather than separately in each country (Civil & Andrade, 2002). Third, it is important to understand immigrant parents’ perceptions of their children’s educational experiences in both Mexico and the U.S. As Suárez-Orozco & Suárez-Orozco (2001) write, “immigrant parents walk a tightrope; they encourage their children to develop the competencies necessary to function in the new culture, all the while maintaining the traditions and (in many cases) language of home” (p. 89). This tightrope feeling extends to the mathematics education of their children, as parents try to make sense of approaches to mathematics teaching that are often different from what they were expecting or had experienced themselves. We argue that how parents perceive and value these different approaches may affect their children’s learning opportunities (Abreu & Cline, 2005; Bratton, Quintos, & Civil, 2004; Civil, Planas, & Quintos, 2005; O’Toole & Abreu, 2005).

The discussion group will consider multiple aspects of children’s experiences with mathematics, including curriculum, classroom participation structures, mathematical reasoning and discourse (both in and out–of school), and parents’ perceptions and beliefs about mathematics instruction. The aim of this Discussion Group is to present and discuss several research projects that involve transnational and borderland comparisons in order to develop new research questions, refine data analyses, and move transnational research agendas forward. Some of the research questions that Discussion Group panelists plan to address include:

1) How do Mexican immigrant parents in the U.S. view the mathematics teaching and learning that their children are experiencing in the receiving communities in the U.S., particularly in relation to their experiences in classrooms in sending communities in Mexico?
2) How are mathematics classroom participation structures in receiving communities in the U.S. and in sending communities in Mexico alike and how do they differ?
3) How do mathematics curricula and instruction in sending communities in Mexico and receiving communities in the U.S. compare with regards to depth (over mere coverage), analytic reasoning (over mere memorization), the construction of value (over doing tasks as ends in themselves), and engagement in learning?
4) How are teachers’ views on the practice of teaching, especially with respect to opportunities to learn from each other, similar and different in Mexico and the U.S.?

Discussion Group Goals

The central goals of the Discussion Group are to:

1) Develop a shared understanding of the research questions, issues, challenges, and contributions that transnational research studies can make to research in mathematics education;
2) Develop a plan for supporting further connections among transnational projects in the future.

During the two sessions, participants will examine and discuss the design of several transnational research studies, analyze sample data collected in at least one of these studies, and discuss future plans for the Group. These activities are intended to support participants in a) clarifying research questions, b) refining research tools, methods, and analyses, c) exploring connections among different projects and studies, and d) discussing further transnational collaborations and research on learning and teaching mathematics across the U.S./Mexico border.

The planned activities will support these goals in several ways and be grounded in discussions of sample research designs, data sampling, and sample curricula. The anticipated follow-up activities for this Discussion Group include planning for a continuation of the Group as a Working Group for PME-NA 2007 and at PME-International in 2008 and ultimately organizing a collaborative writing project on this topic.

Overview of Proposed Discussion Group Sessions

Session 1:

1) Introduction and overview of the Discussion Group.
2) Brief (10 minutes each) presentations by panel members providing overviews of research projects with specific examples of how researchers have designed transnational studies. The purpose for these short presentations is to provide examples of transnational research projects and to summarize several different studies in a structured way.
3) In small groups, participants will analyze and discuss sample data from at least one of the studies presented. This will give participants an opportunity to share their own experiences in designing research studies, collecting data, and analyzing data.
4) Distribution of one or two readings for the next session (e.g., Abreu & Cline, 2005; Civil, Planas, & Quintos, 2005; Padilla & Gonzalez, 2001).

Session 2:

1) Discussion in small groups of the selected reading(s).
2) Brief (10 minutes each) presentations by the discussants that highlight key ideas using the questions listed below as a guideline.
3) Discussion in small groups in which participants have opportunities to both talk about panelists' responses to questions above and frame new questions for panelists.
4) Whole group discussion: synthesis of main ideas and future directions.

Suggested questions to be addressed by Discussion Group Presenters across the two sessions:

1) What theories and theoretical frameworks have informed the design of your research project(s)?
2) How might your work inform theory in mathematics learning and teaching? How can transnational comparisons expand our theoretical lenses?
3) What issues and challenges have you faced in designing transnational studies?

4) How have you approached defining the research questions for transnational studies?
5) How have you approached data analysis for transnational studies?
6) What specific comparisons have you focused on and why?
7) What aspects of your research do you expect will be most useful to informing practice (curriculum development, teacher professional development, work with parents, etc.)?
8) How might your work inform not only instructional practices for this population but also instructional practices for other populations?
9) Which aspects of transnational studies do you find most puzzling? Most useful? Most misunderstood?
10) How might other researchers pursue transnational research projects and what can they learn from the work done so far?

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VOLUME 2

ADVANCED MATHEMATICAL THINKING

WAYS IN WHICH PROSPECTIVE SECONDARY MATHEMATICS TEACHERS DEAL WITH MATHEMATICAL COMPLEXITY

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The current study took place in the context of a mathematics-content course designed to engage prospective secondary mathematics teachers (PSMTs) in work with the concept of function. The data are derived from a task-based interview conducted with eight junior or senior PSMTs. In the interview, the Bottle Problem was selected to engage PSMT in graphing a complex relationship (in this case involving rate of change in an applied situation). Our analyses suggest that PSMT ranged in their abilities to hold onto the meaning of the mathematical entities with which they were working, in their abilities to coordinate those entities, in their abilities to recognize the relationships between different representations of the same entity, and in their coordination of macro- and micro-perspectives. Based on these analyses, we developed characterizations of how PSMTs deal with describing and graphing complex relationships.

Dealing with complexity is essential to the success of secondary mathematics teachers (Henningsen & Stein, 1997), and one of the most complex concepts with which they deal is that of rate of change (Thompson, 1994), particularly rate of change in an applied setting. This study investigated how prospective secondary mathematics teachers dealt with complexity in the context of an applied setting whose mathematical relationships centered on the concept of rate of change.

Complexity

To deal with the complexity of a quantitative situation one needs to understand mathematical entities (e.g. function, derivative, etc.) and to be able to use that understanding in reasoning about the entities and their characteristics. When reasoning about several entities, one also needs to understand the relationships among entities, and be able to coordinate the characteristics of one entity with characteristics of the others. In the case of real-world situations, one also needs to be able to map entities, characteristics, and their relationships to the real-world situation as well as to related representations. One needs to be able to move freely between a mathematical feature of the situation and its counterpart in the real world. As one reasons about quantitative situations, one needs to hold onto the meaning of the mathematical entities, their characteristics, and relationships among them. Holding onto meaning is increasingly difficult in complex relationships because dealing with such relationships requires a coordination of all of the aforementioned components. As we studied how these individuals reasoned about a complex situation, we noted ways to think about how any or all of these essential components of understanding play out.

Complexity in a Specific Example

A specific example of a problem requiring dealing with the complexity of the interrelationship between mathematical entities (in this case, accumulation and rate of change) is the "Bottle Problem" (Shell Centre, 1999; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Our

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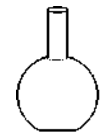
research centered on characterizing the ways that prospective secondary mathematics teachers in our study dealt with the complexity of rate of change in the context of the Bottle Problem.

The Bottle Problem

Imagine a bottle filling with water. Sketch a graph of the height of the water as a function of the amount of water that is in the bottle.



The complexity of the Bottle Problem is related to the nature of the relationships between the variables in the problem. The non-linear constantly changing relationship between accumulation and rate of change in this setting is at the heart of the complexity. The context (mention of "filling") and structure (non-traditional assignment of the independent and dependent variables) of the task also complexify the problem.

The diagram of the bottle can be used to visualize how the bottle fills with water and the resulting relationship between the height of the water and the amount of water in the bottle. The variables are given in dependent and independent roles, although the reference in the instructions to "filling" conjures up an image that is time-dependent. An implicit or explicit parameterization of height and amount as functions of time may then confound the dependent/independent relationship of the height and amount.

An additional complication in this problem is that the instantaneous rate of change of height is always changing as water "fills" the lower portion of the bottle and depends on the width of the cross-section at any specific height. Average rates of change of the height, corresponding to the cross-sectional width, can be estimated for uniform increments of change in the amount of water (Carlson, et. al, 2002, p. 357). Here, the identity of the independent and dependent variables complicates the task in that students are more likely to be asked to generate uniform increments in height than uniform increments in volume. The average rate of change of the height results in an accrual of height for uniform increments of change in the amount of water. This accrual, and the resultant accumulation, can be used to construct a graph of the covariational relationship. Adding to this difficulty of the task is the fact that uniform increments of volume are constantly changing, requiring a perspective that accounts for the limiting values of the rates of change that define the curve.

Using the Bottle Problem as a focal interview question, we were able to examine ways that prospective teachers deal with mathematical complexity in somewhat familiar settings.

The Study

Our study took place in the context of a mathematics-content course designed to engage prospective secondary mathematics teachers in work with the concept of function. The data we analyzed were taken from an interview conducted with the teachers at the beginning of the semester. The goal of the interview was to investigate students' pre-course understanding of function, a concept students would have encountered in their previous three semesters of calculus, in their discrete mathematics and matrices courses, and in additional upper-level mathematics courses. The Bottle Problem was selected to engage students in a discussion of rate of change in an applied situation that we expected would be unfamiliar to students.

Eight prospective secondary mathematics teachers took part in our study. These students (with pseudonyms of Bob, Jen, Jim, Lindsey, Maria, Ned, Tim, and Violet) were juniors and seniors in a secondary mathematics teacher certification program. A series of three task-based interviews on students' understanding of functions was conducted with each of the study participants. Interviews were transcribed, annotated, and analyzed for understanding of work on the Bottle Problem.

Data Analysis

During the analysis, video recordings and verbatim, annotated transcripts of those interviews during which participants engaged in the Bottle Problem for the first time were closely examined. We interpreted students' reasoning using line-by-line analyses of the data and a range of lenses.

The analysis process started with the use of the Carlson framework (Carlson et al., 2002) as a lens to look into participants' mental actions and reasoning. In so doing, we tried to match each student's mental actions and reasoning with levels defined in the framework. Although we found examples that fit each level of the framework, the levels did not seem to capture features of students' reasoning that seemed essential for our purposes. We recognized a complexity in what the students were attempting that the framework was not designed to capture.

There was a significant difference among students in the extent to which their thinking seemed to be operational or structural, yet those categories also fell short of capturing the complexity. Our subsequent analysis tried to characterize the ways that operational and structural perspectives came into play in how they offer different affordances and constraints to individuals as they dealt with complexity. Following such a route helped us to characterize students' thinking in general to some extent but it did not generate a detailed description and analysis of what it means to deal with complexity in a situation that embeds a plethora of ideas related to concept of rate of change and function.

As a next step, based on our observations and literature in this area (e.g., Funke, 1991) we investigated the students' handling of complexity in the following three categories: coordination of features, their identification and understanding of the nature of variables, and the connectivity they demonstrated among the variables in this complex problem. As we examined students' work in these categories, we recognized two interrelated themes that helped explain the structure of dealing with complexity: use and coordination of macro-perspective and micro-perspective; and coordination of mathematical entities and their features.

Essential understandings and mental actions required in reasoning about the Bottle Problem include the understanding of the concepts of variable, function, and rate of change, and the coordination of the representations of those concepts in this setting. It is through coordinating the complex interrelationships of these concepts that the solution to the problem emerges. In order to coordinate these concepts, one needs to hold onto the meaning of each of the concepts as well as their relationships to each other. The meaning of concepts can be held in their representations, and the extent to which students succeed in dealing with the complexity of this problem is partially a function of their ability to make solid connections among different representations of the same object. Success in dealing with the Bottle Problem also requires students to control the times at which they focus on the details of the problem and the times at which they focus on the larger structure of the problem. We have termed this type of control as their macro- /micro-perspective.

Students ranged in their abilities to hold onto the meaning of the mathematical entities with which they were working, in their abilities to coordinate those entities, in their abilities to recognize the relationships between different representations of the same entity, and in the extent to which they were in control of their macro- /micro-perspectives. In observing students working on the Bottle Problem, we developed characterizations of how they dealt with complexity. Although our observations are in the context of a few students' work on a single problem, we see our observations as a consistent explanation of issues involved in dealing with complex problems and suggest our conclusions as hypotheses about ways in which prospective secondary mathematics teachers deal with describing complex relationships that involve several variables.

Macro-Perspective and Micro-Perspective

An overarching theme that arose in our analysis is that of macro-perspective and micro-perspective. Key to the ability to deal with complexity and to think about relationships among entities and their characteristics is the ability to control the lens through which one is looking. At times, one needs to look at the overall problem (a macro-perspective) and at other times one needs to focus on the details of a smaller part of the problem (a micro-perspective). Moreover, one needs to control movement between the macro-perspective and the micro-perspective, being ever conscious of where one is in the process. The ability to move back and forth between perspectives is influenced by the conceptual tools and representations one brings to bear on the situation. It is also influenced by the ways in which one holds onto the meaning of the mathematical entities as one moves between perspectives.

Students' macro- and micro-perspectives sometimes offer affordances and sometimes present constraints to creating a graph of the covariant relationship of height and volume. The concepts of macro- and micro-perspectives seemed to apply both to students' work with the physical situation and to their work with mathematical entities. Students' perspectives are sometimes narrowly focused without a larger view, sometimes largely focused without a narrow view, sometimes combine both views, and sometimes are just inaccurate. Of course, the terms macro and micro are relative, but the context of the students' work usually made the distinction discernible.

Ned and Jim exhibit both macro- and micro-perspectives and a close connection between the two. These perspectives offer them strong affordances to completing the task. They identify critical points and regions of similar behavior to successfully analyze the covariational relationship throughout the entire bottle. (Jim does so using an inverse relationship, volume as a function of height.) In the collective work of Ned and Jim, notable points in the bottle, such as the middle of the globe and the beginning of the neck are identified. Each of three regions (lower part of globe, upper part of globe, and the entire neck) is identified as having a unique behavior. The symmetrical relationship between the lower and upper parts of the globe is also noted. These micro- and macro-perspectives of the physical situation are related to appropriate perspectives of corresponding mathematical objects. Then all of these perspectives are combined to view the physical and mathematical wholes. Ned and Jim not only coordinated the macro- and micro-perspective in both the physical and mathematical situations but also appropriately mapped the physical macro to the mathematical macro and the physical micro to the mathematical micro-perspective.

Violet has both macro- and micro-perspectives but these perspectives do not seem to be as closely related to each other as Ned's or Jim's. Ned and Jim seem able to hold both perspectives in mind simultaneously and move back and forth between them effortlessly. Violet's movement between macro- and micro-perspectives, while accurate, is slow and deliberate, making her use of the perspectives appear separate and in isolation. She is able to use her micro-perspective to determine that her linear graph cannot be correct, but only after she has actually constructed the inappropriate linear graph. Violet's macro-perspective seems less helpful than those of Ned and Jim due to the tenuous connection between it and her micro-perspective.

Lindsey exhibits a strong micro-perspective when comparing rates of change of height with respect to volume at two points in the lower portion of the bottle and at a point in the neck. Her macro-perspective is weaker. Although Lindsey notes that the lower part of the globe "could be considered at different points" (280-281) in addition to the two she has already considered, she says, "that would take a lot more effort than just considering it as two parts" (281-282), and submits the graph (see Figure 2) of a piecewise linear function (of two pieces, corresponding to the globe and the neck).

Maria exhibits a one-dimensional macro-perspective, focusing on a single attribute which she treats as global. She notes that the function is increasing, saying, "as the height increases so will the volume", and concludes that the function must be linear.

Bob's perspectives seem to constrain his problem-solving abilities. Bob uses a micro-perspective (volume is measured in cubic units) to arrive at a macro conclusion that the function of height with respect to volume should be a cubic polynomial. He then identifies micro parts of the function and finds micro parts of the physical situation to justify them.

Coordination of Entities and Their Features

A second overarching theme is that of the coordination of entities and their features. Even though the Bottle Problem was an idealized situation, some of the aspects of modeling reality were present in the problem. In the Bottle Problem, students need to identify and account for a complete set of relevant characteristics and continually check the match to the situation. Some students were able to think through a complex situation like this by identifying the relevant characteristics in conjunction with the mathematical entity and coordinating those characteristics. Other students had difficulty in coordinating features. Some of the ways this difficulty was revealed was that (1) students focused on specific features without coordinating them with other relevant features, (2) students spoke about features of an object without a clear connection between the features and the particular object, and (3) students found prototypes that captured a single feature, and then reasoned from the prototype's other features which were not reflective of the represented concept.

Coordinating an Object and its Relevant Characteristics

Ned's work is an example of an adept identification and coordination of relevant characteristics. He explained the relationship between rate and accumulation before he wrote anything on paper. Ned exhibited an easy combination of qualitative and quantitative reasoning about this problem. He identified and explained critical points on the bottle and used them to produce an appropriate graph of height as a function of volume. The leftmost two sections of Ned's graph (see Figure 1) refer to the globe of the bottle, and the rightmost linear section of the graph refers to the neck of the bottle. The second line segment (the uppermost) for the neck is a correction Ned made when he compared the rate of change of height at the neck to that of the globe.

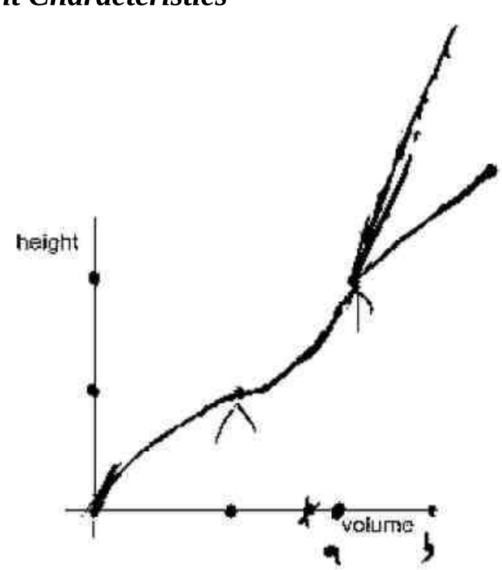


Figure 1. Ned's diagram

Focus on Essential Features without Coordinating Important Facets

Lindsey focused on essential features without coordinating important facets. She recognized the complexity of the problem and identified the fact that the rate of change of height would be different at varying heights of the bottle. But she examined only two slices of the bottle, so when she produced the graph (see Figure 2) she missed the global feature of continuously changing rate of change and produced two line segments representing the relationship between height and volume at points in her two slices of the bottle (section "a" refers to the incremental increase in height for one slice in the globe of the bottle and section "b" refers to the incremental increase in height for an equal-volume slice of the neck of the bottle). She said, "I know since the bottle is wider at this point (referring to the globe of the bottle) that the height would be slower, it would raise at a slower constant than the volume would…The height would raise faster up here (referring to the neck)…because it's smaller."

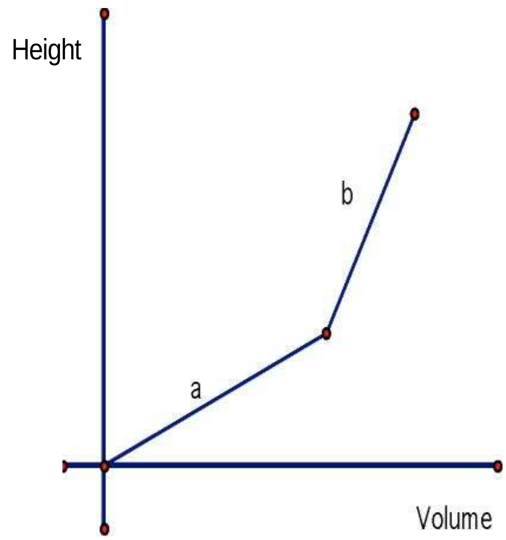


Figure 2. Lindsey's graph

Evaluating Features apart from the Object

Tim treated the features of "increasing" and "decreasing" as separate from the mathematical objects having those features. Prior to the following incident, Tim had identified "increasing" and "decreasing" as features of interest. Moreover, although he was asked to graph height as a function of volume, he seemed to be thinking of volume as a function of height and reversing the usual position of the axes for independent and dependent variables.

As shown in Figure 3, Tim floated from graphing the volume (on the horizontal axis) as increasing to graphing the rate of change in volume (also on the horizontal axes) as decreasing. In the following quote, Tim referred to the volume as increasing but then said "it" -- now referring to the rate of change in volume "is decreasing." Tim said "For every inch there is going to be a constant increasing in the volume because the bottom portion of this is getting wider as it's going up...but then…as it starts to come back into the top it's going to go back down again.

And then the neck of the bottle, once you get to there it is definitely going to be a constant increase….” At one point Tim realized the volume was going to keep increasing and said, “I’m not clear as to whether we’re charting the change in volume or whether we’re charting the overall volume of the container.”

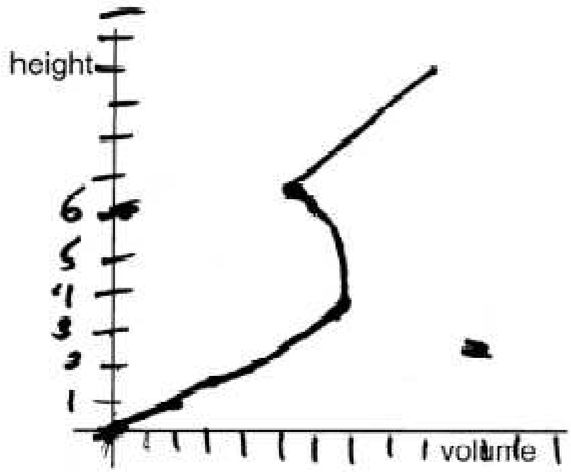


Figure 3. Tim’s graph

Finding Prototypes that Captured a Single Feature

Maria focused on the fact that both height and volume are increasing over time. She claimed that the graph of height as a function of volume would be a positively sloped line. Maria said, “as more water goes into the bottle, obviously, the volume goes up and so will…actually the height of the water will eventually raise to the top. So…I’m thinking both of them, as the height increases so will the volume. So I’m thinking of some type of linear type of graph.” She also claimed the graph would be the same regardless of the shape of the bottle–a conclusion that would be consistent with Maria’s singular observation that height increased as volume increased.

Conclusion

We have observed general ways in which the degree and nature of coordination of mathematical perspectives (micro and macro) or entities (a mathematical object and its features) affected how students deal with complexity.

We have observed how students control and use macro- and micro-perspectives as they proactively manage mathematical complexity in somewhat familiar settings. The ability to use both perspectives and to move fluently between them seems strongly related to students’ ability to graph the complex relationship in the Bottle Problem. The quality of this ability may be related to the relative strengths of students’ perspectives as well as to the specificity with which they can apply each perspective.

We also observed the strength and fluidity of the connections students have between mathematical entities and their features. Students run into difficulty when they think about features separately from the mathematical entities they are describing, when they overgeneralize the features of prototypes, and when they focus on essential features without coordinating them.

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COMPENSATION REASONING IN OPTIMIZATION PROBLEMS

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The purpose of this research report is to document and characterize a conservation phenomenon which comes up spontaneously in the solution of geometrical optimization problems: Students think that two quantities x, y, varying in opposite directions, cancel their effect over z, a quantity related to them, keeping it fixed. The findings suggest that this phenomenon affects mainly the typical optimization variables and that it's more subtle and persistent than it appears. The roots of this "logic" might be found in what Piaget (1981) calls the three basic aspects of reasoning developed by children to solve conservation problems. Using these concepts together with the mental actions from the covariational reasoning framework of Carlson et al. (2002), we propose an explanation of the conservation phenomenon and characterize it as a kind of "linear" relation between x, y, and z. We present the results of a case study with two high school students (K-11)

Introduction and theoretical framework

In several geometrical optimization problems we observed consistently and spontaneously the aforementioned conservation phenomenon, *i. e.*: The "volume" of a right circular cylinder whose diagonal remains fixed while diameter and height vary; the "area" of a rectangle whose diagonal remains fixed while base and height change; the "volume" of boxes made by cutting equal-size squares from each corner of a cardboard; and the "length" of a not straight path. Piaget (1981, pp. 11-15) describes three basic aspects of conservation reasoning that most of the children progressively develop as they reason about their world: *Identity*, *Reversibility* and *Compensation*. The last of these aspects is not found until the age of 12 or 13, when students realize that changes in one dimension can be offset by changes in another direction. We believe that high school students apply compensation reasoning when faced with the geometrical situations previously mentioned. Carlson et al. (2002) developed the notion of covariational reasoning and proposed a framework for describing the mental actions involved when interpreting and representing dynamic function events. The first three mental actions are MA1) An image of two variables changing simultaneously; MA2) A loosely coordinated image of how the variables are changing with respect to each other; and MA3) An image of an amount of change of the output variable while considering changes in fixed amounts of the function's domain. By using MA1, MA2 and MA3 we designed the investigation protocols and explain the way students mentally transform the geometrical system under study.

Methods

The data comes from two high school students (Lothorien from now on) who had successfully completed a course of Analytic Geometry (K-11). From three typical calculus optimization problems, we designed three protocols that consisted basically in a short description of a physical situation represented geometrically on a sheet of paper. Without any previous instruction Lothorien was asked to mentally transform the situation through the dragging of one geometrical element (usually a point). Confronted with this scenario Lothorien was prompted to

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draw three or four snapshots of the continuous transformation process then answer two basic questions: What changes? and What doesn't change? The second part of the protocols consisted of a simulation of the situation in Dynamic Geometry and the objective was to obtain the analytical (algebraic) function between two of the variables of the first part, through a covariational approach (see Geometrical Optimization Problems: A Covariational Approach, this volume). Collected information consisted of written answers and video-taped films. Finally, Lothorien had no time limitations and one of the researchers was always present during the sessions.

Results

First protocol: A ladder AB (see Figure 1) leaning against the wall in an upright position (a modification of a problem reported in Monk (1992)). Dragging element: The bottom B of the ladder. Optimization variables: Area and perimeter of the triangles formed by the wall, the floor and the ladder.

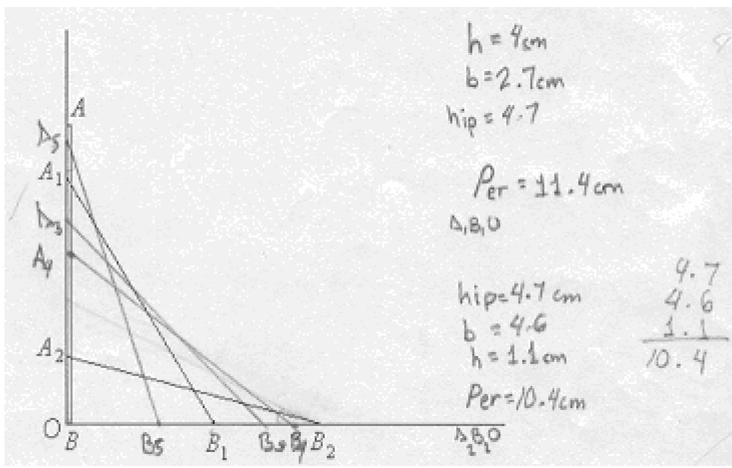

Figure 1. Sliding ladder.

While drawing the snapshots A_3B_3, A_4B_4, etc. (A_1B_1, A_2B_2 were given) the following dialog occurred between Student A and Student B (time elapsed: 2 and one-half minutes):

Student A: You know what I think?

Student B: What?

Student A: All [triangles] have the same area.

Student B: Surely.

Student A: Because the hypotenuse doesn't change.

In their answers to what changes and what doesn't change, area and perimeter don't appear as either variables or as constants. Later on, simulating the situation and watching the perimeter values in real time, one of the students exclaimed:

Student A: Ah! yes, yes, yes, yes, yes, it's changing. Why does it change? I don't understand.

Researcher: Why?

Student A: In my opinion, the perimeter wouldn't have to change because the hypotenuse is always the same one and as we drag the bottom, whatever is reduced of height, increases the base.

The researcher asked the students whether or not they trusted the values given by the program. They both answered: "no". He suggested then (resorting to external reinforcement), to work on the triangles drawn. With a ruler Student A measured the legs of two of the triangles and calculated their perimeters, obtaining different values (see evaluations in Figure 1). Then, Student A says: “Ah! It varies, but why? It did vary.” The student returns to check the measurements and when finding no errors, calculates the reduction of height and the increase of the base from one triangle to another and finds that they are not the same. Right after this, Student B chances a guess.

Student B: A_3B_3O is the biggest one. The greater perimeter is the one in the middle.

Student A: The one that has 45^0 and 45^0 on both sides [angles] … Ah! I already see where all this goes.

Once they had obtained the analytical relation of the perimeter in terms of the height, they were asked to write relevant situations of the problem. One of them was: “In this problem, although the hypotenuse doesn’t vary and despite the fact that the base increases as the height diminishes, the perimeter varies.”

Second protocol: Shot angle of a soccer player (angle formed by the position of a player and the two goal posts, see Figure 2). Dragging element: Position of the player on the side line of the field. Restriction: The player runs along the side line. Optimization variable: Shot angle.

Lothorien starts making some marks on the side line, but don’t draw the corresponding shot angles (angle with vertex P_1 was given). They go on to the basic questions and Student B writes down as first variable the position of the player and then says (time elapsed: two minutes):

Student B: Do you think the angle changes?

Student A: Let me see [draws a shot angle using as vertex one of the marks]

Student B: I don’t think it does.

Student A: Don’t you think it does?

Student B: No, because the distance between the two goal posts is always the same [points out with two fingers the two posts].

Student A: Let’s find out [draws another shot angle and measures two angles using a protractor]. One is 10^0 and the other one is much less than 10^0.

Student B: Ah! then it changes.

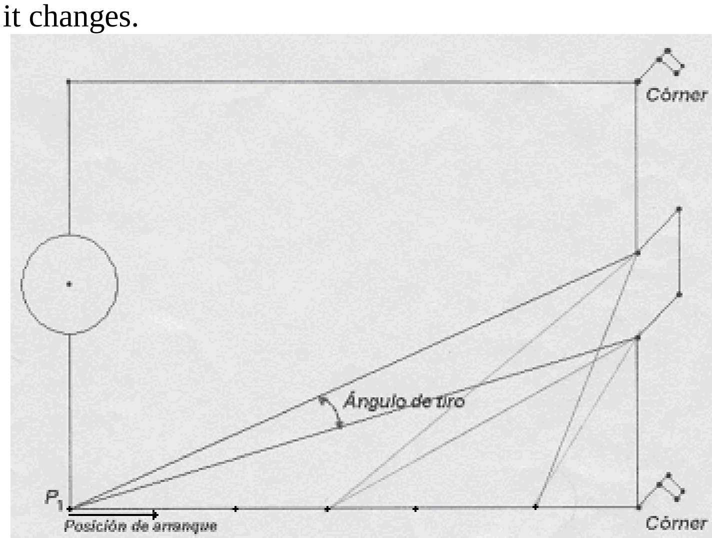

Figure 2. Shot angle.

They finish this part adding a third variable: the distance from the position of the player to the goal.

Third protocol: A polygonal path *DEF* (see Figure 3). Dragging element: Point *E*. Restrictions: Point *E* moves along the line $\overleftrightarrow{AB}$; segments $\overline{DC}$ and $\overline{FG}$ are fixed and perpendicular to $\overleftrightarrow{AB}$. Optimization variable: Path *DEF*.

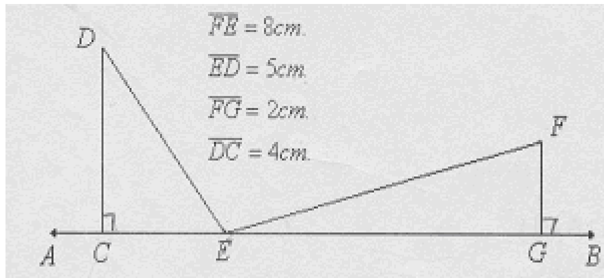

Figure 3. Polygonal path.

Lothorien is asked to visualize point *E* moving freely on line $\overleftrightarrow{AB}$ but no drawings of snapshots were given or requested. For the basic questions the students wrote down as variables "Point *E*; the angles in *E*" and four different segments, two of which are segment $\overline{DE}$ and segment $\overline{EF}$. From our previous experiences, we didn't expect Lothorien to consider path *DEF* as a variable, so there was already a question in the protocol about the nature of *DEF* (variable or parameter). The students began asking each other whether the path *DEF* is a variable or a parameter; Student A thinks it's a variable but Student B disagrees, so Student A suggests:

Student A: Do we test it?

Student B: If you want to, because as this one shortens [points out to *EF*], this one extends [points out to *DE*].

Later on, in order to describe verbally path *DEF* (one of the tasks of the second part of the protocols), Lothorien explores the values in real time and Student A becomes aware of a minimum in between point *C* and point *G*. She thought it was a decreasing variable. These stimulate them to see the values of path in point *C* and point *G*, finding out that there is a "maximum" in *C*; even more, they drag point *E* away from *C* and *G*, deciding, finally, to give a description of path *DEF*, just in [*C*, *G*].

Conclusions

Lothorien showed a misuse of compensation reasoning in protocols 1 and 3. This can be explained in terms of covariational reasoning by MA3: They coordinate the amount of change of one variable, Δx, with changes in the other variable, Δy, in such a way that, $\Delta x = -\Delta y$, resulting in a lineal model between the variables: $z = x + y$, where z stands for the optimization variable. In protocol 2, Lothorien demonstrates another form of conservation: one of the parameters, the distance between the goal posts, captures their attention impeding them from seeing the progression of the shot angle as the position of the player varies. In this case the proposed model is the constant function: $z = f(parameter)$, where z is the optimization variable and depends only on the *parameter*.

External reinforcement may help students to accept the fact that certain quantities really change but not to comprehend why (Piaget, 1981, pp. 15-16). The mere fact of accepting that a variable changes drew on some guesses about the possible optimum of the situation and made it something worthy of interest (protocol 1); however, in protocol 3 this was not enough and it was not until exploration in real time that they detected the optima. Our claim is that perception of change of a quantity and discovering their optima must not be taken for granted when solving optimization problems. To ignore or underestimate these may have the students working mechanically and without meaning. Perhaps, the solution of optimization problems could be seen also as a way to help students comprehend the changes that occur in continuous processes.

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A FRAMEWORK TO EXAMINE DEFINITION USE IN PROOF

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Mathematicians view a definition as having very specific characteristics. Besides defining mathematical objects, definitions give both structure to a proof in a global sense as well as warrants to logical implications in a local argument. Thus definitions play a central role in many proving tasks. Students are required to be able to create and interact with proofs and definitions in a variety of ways. Students are expected not only to produce proofs in homework, but also textbooks and lectures are written with the expectation that students can read and understand proofs and formal definitions. In this talk I propose a framework with which to examine students' uses of definitions and theorems in proving. The framework is a synthesis of previous literature and research results. I will also seek to illustrate the aspects of the framework using student data from a workshop in advanced calculus.

Language is a basis for several difficulties students have in proving (Dreyfus, 1999; Finlow-Bates Keir, Lerman, & Morgan, 1993; Moore, 1994; Zaslavsky & Shir, 2005). Students lack either the language skills or cultural understanding to communicate mathematics. Moore found language to be a difficulty for students along with their use of definitions and their abilities related to the specific concepts. He found students consistently exhibited the following seven difficulties:

- D1: The students did not know the definitions. That is, they were unable to state the definitions.
- D2: The students had little intuitive understanding of the concepts.
- D3: The students' concept images were inadequate for doing the proofs.
- D4: The students were unable, or unwilling, to generate and use their own examples.
- D5: The students did not know how to use definitions to obtain the overall structure of proofs.
- D6: The students were unable to understand and use mathematical language and notation.
- D7: The students did not know how to begin proofs. (Moore, 1994, p.251-252)

Notice six of his seven errors (D1-D6) are concerned in some part with the students' knowledge or use of definitions.

Berger (2004) defined two types of understanding with regards to mathematical artifacts: culturally meaningful and personally meaningful. A student is said to use an artifact, in the case of the current study a definition, in a *culturally meaningful* way when the usage is consistent with the mathematical community. A student is said to have a *personally meaningful* understanding when the student believes he understands the artifact regardless of whether this meaning coincides with the accepted cultural meaning. When a student's understanding is both culturally meaningful and personally meaningful, it indicates the learner has grasped the mathematical definition, similar to the student's concept definition and concept image coinciding as defined by Tall and Vinner (1981). This is also defined by Wertsch and Rupert (1993) as appropriating the artifact, or definition. As Weber (2001) and Hart (1994) noted, knowing a

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definition is just the first part of being able to appropriate it.

Beyond knowing a definition, the literature indicates that students need a set of heuristics which might include how to use or when to use specific definitions called *strategic knowledge* (Weber, 2001). Students need to be able to determine which theorems and definitions are important and when are they useful. They must be able to determine when and when not to use "syntactic" strategies, defined as procedural or symbolic manipulation (Hart, 1994). Edwards and Ward (2004) found that students had difficulty understanding the role that specific definitions played in a proof.

Zaslavsky and Shir (2005) recognized students used either example based reasoning or definition based reasoning when arguing about geometric and analytic concepts from given definitions. Students used their understanding about the role of definitions or their notion of the required features of a definition to justify their claims. In this paper I propose a framework with which to examine how students use definitions while proving in an advanced calculus course. The framework is consistent with a synthesis of the literature characterizing the use of definitions and was developed through the course of this study.

Methods

The data in this study comes from a semester-long one-unit workshop for mathematics majors in topics of beginning real analysis. The workshop was listed as a companion course for students concurrently enrolled in advanced calculus. The students had taken or were concurrently enrolled in a course in proof writing. Ten students, predominately juniors and seniors, met one hour each week to work on given tasks, which usually involved proving a statement. Groups of three to four students worked together on each task with some input from either a Teaching Assistant or the researcher. Students also wrote reflective e-mails weekly on their work.

Three key insights were highlighted during the data analysis which led to the development of the framework. The first of these insights was a product of the task analysis in the designing of the tasks for the workshop. The insight was into the vital role definitions played in the advanced calculus content. Definitions are the foundation and structure of many of the proof tasks in beginning analysis. The second and third insights were products of examining student errors from the data. Students who knew definitions did not correctly use them in their proofs. Likewise although one student spent significant time learning definitions, he was not successful at proving. These insights and the analysis of the tasks and student work led to the development of a framework given below which describes student uses of definitions while proving.

The Framework

In order to use a definition in a proving task, students must learn the definition, be able to know when it should be used and how to use it. These three skills form the structure of the framework. There are several sub-skills particular to the proving process (see Figure 1). These skills are not meant to be seen as a trajectory or hierarchical in anyway. Instead it is likely that students develop some of the skills concurrently and move throughout the framework. In the following sections I will define each skill and sub-skill and will illustrate an aspect of the skill with student data.

Skill 1	**Knowing**	**Skill 2**	**Knowing Which**	**Skill 3**	**Knowing How**
1.1	*Ventriloquating*	*2.1*	*Which concepts to define?*	*3.1*	*Orient the problem*
1.2	*Appropriating*	*2.2*	*Which definition to use?*	*3.2*	*Move a logical step*
		2.3	*Which aspect to use?*	*3.3*	*Structure a proof*

Figure 1. Definition Use Framework

Skill 1—Knowing a Definition

Mathematical definitions are generally provided to students as a formal definition, in the sense of Vygotsky's scientific concept (Vygotsky, 1987). Thus in order to know a definition, students must first be able to state the formal definition, in a type of ventriloquation (Bakhtin, 1986). Ventriloquation is the voicing of or use of the definition in someone else's words without a full underlying understanding of the meaning. In the seventh week of the semester a group of students were working to prove that a convergent sequence is Cauchy. They began writing on their white board with the following discussion.

Mark: *Then we're going to choose epsilon greater than zero.*
Dustin: *So we need to...*
Lynn: *Um. Am I writing it for all... do I need to?*
Mark: *Epsilon greater...choose epsilon greater than zero.*
Dustin: *Yeah for all epsilon greater than zero there exists an n such that* $|a_n - A|$ *[mumbles].*
Mark: *Positive integers, that implies hmm... How'd they get epsilon over two?*
Dustin: *For all...*
Lynn: *Writing for all not choose?*
Mark: *Okay.*
Lynn: *Or let epsilon be...*
Mark: *Oh no, we're...she's....*
Dustin: *Okay just do uh, for all epsilon greater than zero...*
Lynn: *for all?*
Dustin: *Sure. For all*
Mark: *Whatever, choose.*
Dustin: *It doesn't matter.*

In this episode, the students are looking to recall the proof as it looked in class or in the textbook. While they may be able to work with the definition of a convergent sequence in other contexts, in this context they are not using the definition on their own, but instead their use is mimicking that of the professor's. This is evidenced by their discussion about how to write the first two lines of the proof – should they write "choose epsilon" or "let epsilon" or "for all epsilon." Although they determine that it doesn't matter which phrase they write, there is a sense in the Group that this phrase is on their whiteboard without the underlying understanding of what it means, thus their use of the definition is a ventriloquation.

With further use, students should move from ventriloquation to being able to appropriate the definition (Wertsch, 1991). Appropriation is the students' ability to use the definition for their own purposes in their own problem situations. The transition from ventriloquation to appropriation is founded in the students' using their own voice. During the tenth week of the workshop, the students examined the definition of a Lipschitz function. They looked for examples, non-examples and similar concepts, like uniform continuity. The following week, Doug came to class and sketched an example of a Lipschitz function explaining it in his own words. “You know what I think a Lipschitz is? … Here. [begins drawing] Alright you’ve got some slope right? And your Lipschitz function is a function that’s gonna remain on one side beneath, it’s gonna be bounded with in some area of the graph of the real lines.” Although his initial description was slightly faulty, the sketch he drew and the example he gave were correct indicating that Doug had appropriated some of the definition.

Appropriating the definition requires students’ personally meaningful understanding to match the culturally meaningful understanding as described by Berger (2004). Through the trajectory of ventriloquation and appropriation students come to know a definition. Students who know a definition should be able to give examples of the concept, non-examples of the concept and define the term in their own words. Notice this encompasses D1-D4 of Moore’s student errors (Moore, 1994).

Skill 2 – Knowing Which Definition to Use

“Knowing which” is the skill of being able to determine which definitions and which aspects of those definitions are useful in the proving process. A student must be able to determine which particular concept definitions are applicable to the situation (Skill 2.1), which definition is most useful when there is more than one equivalent definition (Skill 2.2), and which aspect of a particular definition is useful for the proof (Skill 2.3). Determining if a definition is helpful means the student is able to figure out which concepts in the theorem should be defined as well as knowing which definitions not stated might be useful.

Consider the statement “A bounded and monotone sequence is convergent.” All three sub-skills from Skill 2 can be described by examining this statement. In the proving process there are several concepts which are apparent in the statement. There are also related concepts which are not apparent in the statement. Some of the definitions related to these concepts are vital to the proving process, while others are not necessary. In the proof of the given statement it is necessary to use the definition of a least upper bound. Since this term does not occur in the statement itself, it is not readily apparent to the students that it is necessary. Skill 2.1 is the students’ ability to recognize the need for the least upper bound.

Once it is determined which concepts should be defined, it must be determined which definitions of those concepts are most useful. There are often multiple equivalent definitions for a single concept. In some instances, all of the definitions are formal equivalent definitions. In other cases, students hold an intuitive notion or informal definition along with the formal definition. Each definition has particular affordances or purposes. Skill 2.2 identifies the students’ ability to choose the best definition from a set of equivalent definitions. In the example statement, the students had access to two equivalent definitions of the least upper bound. The students had to choose which definition was most useful for their purposes.

Once the choice of definition has been made, the particular aspect of the definition must be used. This choice may be in conjunction with the choice of which definition to use from Skill 2.2. In the following example the students had decided to use the least upper bound for their proof, but Dustin was working backwards from the definition of convergence in order to find the key to put the proof together. The following conversation ensued.

Dustin: *Right. Okay, but didn't we prove that if we take the s – ε that's like a_n or that it's like between... there has to be some element of the sequence between s – ε and s and s is the least upper bound. Did we re....*

Lynn: *Say that again?*

Dustin: *Did we say that there has to be an element between s – ε and s, and s is the least upper bound?*

Dustin was searching for a particular aspect of the definition of least upper bound. Once the group determined this aspect was part of one of the definitions they had, then they used this definition and finished their proof. Notice that in order to facilitate knowing which definition will be most helpful (Skill 2.2) and knowing which aspect of the definition to attend (Skill 2.3) it is vital to have some knowledge about the proof, goals, or purposes in the use of the definition.

The essence of Skill 2 is the determination of which definitions will be most helpful in the proving process. Knowing which concepts need to be defined allows the prover to bring to the table the necessary definitions, and avoid cluttering the table with unnecessary definitions.

Skill 3 – Knowing How to Use a Definition

Weber (2001) indicated students need strategic knowledge to be successful at proving. With respect to the use of definitions, this strategic knowledge is knowing how to use the definition. Definitions are used in many different ways, in proving they are predominately used to orient oneself to the theorem statement (Skill 3.1), move one logical step in the proof (Skill 3.2), or structure the proof (Skill 3.3). Knowing how to use a definition encompasses all three of these skills. I will illustrate the first two of these sub-skills from an episode in the seventh week of class, where the students were proving the statement, "A sequence is convergent iff it is Cauchy."

The first sub-skill, using a definition to orient oneself to the problem, might be exhibited in several instances. The students in this study drew pictures, or generated examples based on the definition. They also looked for non-examples. When proving the example statement, some of the students had not yet seen Cauchy sequences. Doug and Molly used the definition of Cauchy to help explain the concept to Ben and Jane. In this way they were orienting the Group as a whole to the concept of Cauchy convergence in the proving process.

Doug: *Alright. Definition of a Cauchy sequence is basically a sequence, oh okay. Basically you have two sequences, a_n, a_m Uh... Let a quantity be greater than…*

Molly: *Well they're both different elements of the same sequence.*

Doug: *Yeah. Let $\varepsilon > 0$, such that....*

Molly: *There exists a $n \geq N_{\varepsilon}$ based on your epsilon such that a to the... the $|a_n - a_m| < \varepsilon$.*

Ben: *So it's the idea that sub sequence is convergent?*

Molly: $\forall n, m > N_{\varepsilon}$*, right?*

Jane: *Well it's just that as the sequence elements get closer and closer together.*

Molly: *Yeah.*

While each of the students' comments contained an error, the overall idea conveyed to the group concerning the concept of Cauchy convergence was correct. Thus the students had used the different aspects of the definition of Cauchy to orient their groupmates to the statement of the theorem illustrating Skill 3.1.

Due to the nature of logical proofs, each line of the proof must follow in a logical progression from the last. In many cases the mathematical tool which allows one to move from one line to the next in a proof is a definition (Skill 3.2). In the following example, the students were proving the given statement. The first two lines the Group wrote on their white board each depended on the definition of convergence (see Figure 2)

Let $\{a_n\}_{n=0}^{\infty}$ be convergent to A. $\forall \varepsilon > 0 \ \exists N : n \geq N \rightarrow |a_n - A| < \varepsilon$
Let $n, m \geq N \ \ |a_n - A| < \frac{\varepsilon}{2}$ and $|a_m - A| < \frac{\varepsilon}{2}$.

Figure 2: Group 1's initial white board notes

Although the Group had not explicitly written that each line on their white board was dependent on the definition of convergence; the statements they made are only true because the sequence they have defined is convergent. The Group did recognize this was the case as evidenced by the statement made by Dustin later in the discussion. Dustin explained, "Well yeah the whole reason that this is true [pointing to the two lines on the white board] is because we know that a_n converges." Thus the group exhibited Skill 3.2.

In Skill 3.3, students need to know how to use a definition to structure a proof. This skill is akin to the definition based reasoning reported by Zaslavsky and Shir (2005). This skill is evident in many analysis proving tasks, including the existence of a limit, the continuity of a function, or the convergence of a sequence. As we will see this was a source of student error. In the eighth week of the semester, a group was determining the existence of the limit at x=0, of the piecewise function defined as $s(x) = x+1$, if x is rational; 1, if x is irrational. They correctly began their discussion looking for a delta, but turned to define an epsilon.

Jane — *It seems to me that you could always find an epsilon where because this thing is just going back and forth [makes motions with her hand] like that, you could find an epsilon that's – Where you could --*

Molly — *If we choose $\varepsilon = \delta$, we've got it.*

Kelly — *What?*

TA — *We need some explanation. That came out of nowhere.*

Molly — *Okay if f(x)=1, if x is not an element of the rational numbers. Okay so we take the |f(x)-1|=|1-1|. Cause we are assuming the limit is one by the diagram. So 0<ε, for all epsilon. Okay so f(x)=x+1, when x is an element of the rational numbers. So |f(x)-1|=|x+1-1|. So we get |x|<ε. But if we're taking 0<|x-a|<δ. So if we pick our $\varepsilon = \delta$ then it works.*

This group does not realize the definition of limit, "given ε> 0 there exists a delta…" requires their proof define a delta to prove the limit exists: this is the issue for Skill 3.3. Both groups working on this task made this mistake. Although both groups were able to produce a correct proof by the end of the session; neither group initially considered the definition of limit correctly. Understanding how students need to use definitions in proof writing gives insight to the skills concerning definitions which must be developed by students in proof writing courses.

Conclusion

As defined, Skill 1 is the compilation of the issues involved in learning the definition described by Vygotsky (1987), Bakhtin (1986), Wertsch (1991), Moore (1994) and Berger (2004). Skill 2 is knowing when to use a definition, and this skill is a part of the strategic knowledge defined by Weber (2001). It is also categorized by Moore as D5 in student errors (Moore, 1994). In some sense, Skill 3 is the culmination of the first two skills. In order to use the definition, one must know the definition. As students appropriate a deeper understanding of the definition, they are likely to use it in many more ways. Likewise, knowing which definition to use and which aspect of that definition to use are vital to the use of the definition. If students were unable to evoke a definition, then they would be unable to use it. Thus, Skill 3 is dependent on students learning the first two.

The proposed framework is not meant to be viewed as a learning trajectory; students can be engaged in learning these different skills concurrently. The framework does offer educators an overview of different skills which are needed by students in a proof writing setting. It appeared that many of the student errors in proving were related to each of the skills described. Therefore, the framework offers insight into particular errors students make when using definitions. This framework might also be useful for students as a tool to open the discussion concerning the different roles of definitions in proofs. In this way it might help to address the deficiency described by Edwards and Ward (2004). It may also illuminate areas of difficulty students are likely to encounter as they learn to use definitions as tools for proving.

Acknowledgements

This research was supported in part by the National Science Foundation under Grant No. 0093494. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the NSF.

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AN ANALYSIS OF STUDENTS' IDEAS ABOUT TRANSFORMATIONS OF FUNCTIONS

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This study intends to contribute to better understand students' difficulties with transformations of functions. Students were interviewed while solving problems involving transformations of functions. Results were analyzed using APOS theory (Asiala, et al., 1996). They show that few students can work confidently with these problems because they do not seem to have interiorized the processes involved in transformations, or encapsulated those processes into objects.

The study of families of functions and transformations on them (such as translations, reflections and stretches) has become important in pre-calculus courses at universities. Transformations give students new opportunities to use and reflect on the concept of function, and they can become a useful tool in more advanced topics of mathematics.

Research on students' understanding of transformations of functions is important not only because it is a topic in many pre-calculus courses, but also because it provides an opportunity to analyze students' ideas on functions and variables. It also permits to study students' flexibility with the use of different representations of function. All this information can be used as a guide in the design of teaching materials and class strategies with the purpose of trying to foster students' understanding of both functions and transformations of functions.

Antecedents

Some researchers have worked on the concept of transformations of functions (Baker, et al., 2001a; Baker, et al.; Bloch, 2000; Cuoco, 1994; Eisenberg & Dreyfuss, 1994; Quiroz, 1990; Goldenberg, 1988; Zazkis, 2003). They have found specific difficulties students have when working with a few particular problems where transformations are involved. Research focusing on a wider variety of situations is needed to further analyze how students work with situations involving transformations of functions and how their understanding of this concept relates to their concept of function.

This study focuses on the concept of transformations, using APOS theory (Asiala, et al., 1996) as a theoretical framework. A genetic decomposition for the concept of transformations of functions developed in previous research, was refined to analyze students' work and to find possible causes for their difficulties. (Baker, et al. 2001a).

The purposes were to investigate how students use transformations of functions once they have finished a pre-calculus course, to find out what their main difficulties when using them are, and determining the conditions that identify those tasks where they are able to succeed.

Research Questions

This study intends to respond the following research questions:

- Are students able to indentify when a given function can be described in terms of a basic transformed function?

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- What are the specific difficulties that students face when they work with transformations of functions? Which of those difficulties are related with the use of different representational contexts?
- What can be said about students' understanding of the concept of function from their work with transformations of functions?

Theoretical Framework

The theoretical framework used for this study is APOS (action, process, object, schema) theory (Asiala et al., 1996). The genetic decomposition of the concept of transformation of functions used was a refinement of that developed by Baker and her collaborators (Baker, et al., 2001a) with the purpose of analyzing the data obtained in this study. The genetic decomposition used for the analysis of the data follows:

Students who act at an action conception of transformation of functions can perform operations on functions and variables step by step, and these operations can be applied either in the analytical or graphical representation context; rely on memorized facts or external signs, as for example the exponents in the expressions or the apparent form of the graph; recognize differences between a function and its transformations only in terms of the syntax of the rule that defines the function, and recognize similarities between a function and its transformations, or between transformations, only in terms of some global property of the graph. When these actions are repeated on the analytical or graphical representation of a function, and students reflect upon them, they interiorize the actions into **a process**.

Students who act at a process level are able to describe changes in the basic functions as a consequence of the application of the transformation without the need to perform each step of the transformation or move the graph of a function step by step. They are able to look at the graph of the transformed function and describe the changes that result from the transformation. These students are also able to reverse the process to identify the function on which a set of transformations was applied. Students at this level show, however, difficulties in coordinating the information obtained from different representational contexts, and in flexibly translating information from one representational context to another.

When students reflect on all of these processes, and are able to think of them as a whole, in any representational context, working flexibly in different representational contexts as well, it is considered that they have encapsulated the process of applying a transformation to any function into an object. It is considered that students have an object conception of transformation, if they are able to apply actions on transformed functions and coordinate their properties in terms of possible changes in the original function, At this level, students are able to de-encapsulate any transformed function object into the process involved in its construction, and they are able to identify the basic function on which it is based and compare different transformed functions in terms of their properties in any representational context.

Schema conceptions are formed by the interconnection of several actions, processes, objects and other previously constructed schema, and the relationship between them. One possible example for a transformation of functions schema would include the schema for function, the transformation of functions object, and actions and processes on transformed functions to determine their properties or to classify them, for example into rigid or non-rigid transformations. Since students in this project had only taken one course related to the notion of transformation, it was decided by the researchers to use solely the concepts of action, process, and object from APOS in the analysis of students' responses and work.

Methodology

The genetic decomposition outlined above was used to design and analyze the actions processes and objects the researchers considered were required in the responses to the questions from an instrument designed to diagnose students' understanding of the concept of transformations of functions. This instrument was used as a questionnaire that was solved by 158 students registered in pre-calculus courses and calculus courses at a small private university in México City. The questionnaire included eleven questions where students had to work with transformations of functions using both, analytical and geometrical representations. Students' work was then analyzed comparing the constructions they showed in their answers, with those predicted by the genetic decomposition. From the analysis of students' answers to the questionnaire, 16 students were selected to be interviewed. Half of them had finished a pre-calculus course based on transformations of functions and the rest had finished a calculus course. The selection was based on students' responses to the questionnaire and the overall tendency found in their responses, in order to end up with an even number of students classified at each level according to APOS theory.

The same instrument was then used during the semi structured interviews with the purpose of conducting a more in depth investigation of their understanding of transformations of functions and to uncover the strategies they used while working with different issues related to this concept. The interview included an extra question which could only be solved in a geometrical context, and where students were expected to apply their knowledge about transformations of functions to solve an inequality. Interviews were audio taped, transcribed and analyzed qualitatively, using the criteria derived from the genetic decomposition of the concept. The analysis was done separately by two researchers. Discrepancies in the analysis were negotiated after another round of analysis by the same two researchers.

The questions of the interview were classified in four groups, according to their main global purpose. These four groups were: Identification of transformations, graphing transformed functions, using transformations of functions to perform some other actions or processes, and finding the domain and range of transformed functions. Students' difficulties were associated with these four groups in order to determine those characteristics of problems that appeared to make them troublesome for students. On the other hand, students were grouped according to their difficulties and strategies of solution, and to the actions, processes and objects they showed a tendency to use while working on all the questions. It is important to note that these terms are only used here for classification purposes, since it is acknowledged that the same student can perform at an action level in some question, and at a process or object level on others, depending on both, his or her cognitive constructions and the requirements of the question.

Results

The classification of students resulted in 7 students working at an action level, 5 students working at a process level and 4 students at an object level.

The data obtained for the group of questions related to the **identification of transformations** show that when a problem involves a function which can be the result of the application of a set of transformations on a basic function, students are generally able to identify some of the transformations that have been applied. Students who had difficulties with this type of tasks found troublesome to associate a function represented graphically with its corresponding analytical representation. They showed a tendency to use memorized facts or to make a table of data in order to succeed in these tasks. For example to a question showing the graph of a parabola which asked, a) find the values of a and b in $f(x)=(x-b)^2 + a$ and, b) what would happen for different values of a and b, a student at an action level responded "*I know a moves it up and down… the other, when it is inside… I cannot remember…I would have to make a table with values and see…*"and he proceeded to make a table, but had difficulties to infer the values from it. A smaller group of students showed a tendency to generalize the conservation of distance property, which is valid for rigid transformations, to non-rigid transformations, demonstrating that they apply actions or processes to the graph of the function as if it was a rigid object, with no reflection on what the result of those actions would be for each particular point on the domain of the function. The following is a response given by a student working at an action level: *"…I know that multiplying by 3 makes the graph narrower, it moves this way, closer to the Y axis…so* $f(x) = 3\sqrt{x-2}$ *the graph will be like…more stretched than the original…"*. In questions related to rigid transformations, students had more difficulties recognizing a horizontal translation than a vertical translation. This seems to be due to the fact that students memorize the rules for transformations and their corresponding effect on the function. When remembering this information students often associate the incorrect direction to the translation. The most difficult question in this group was related to the identification of similarities and differences between functions where the same set of transformations had been applied. This question was taken from Baker et al. (2001a). A student working at a process level showed that she had interiorized the result of applying different transformations to a well known function, but she did not recognize the similarities in terms of the transformations applied to both curves: "*I can see this is a parabola that has been moved 3 units up and 5 to the right, and it is stretched by a factor of 2…, the differences,… the other is a hyperbola not a parabola and has asymptotes …they don't touch."*. Students who worked at an action level on this group of tasks based their explanations on the differences they perceived in the algebraic formula for the function. Students who worked at a process level were able to work with functions in different representational contexts, and referred, in their explanations, to the process involved in the transformation. Students working at an object level were able to identify families of functions in any representational context, could relate graphs with their analytic representation and vice versa, could explain which characteristics of a function are conserved, and which are not, when a set of transformations is applied, and identified similarities and differences when comparing transformed functions.

Questions related to **graphing transformations** included questions where students had to graph a given function and questions where transformations had to be applied to a function in a graphical context. Students were, in general, able to apply or graph transformed functions only when one transformation had been applied, or when the basic function was very familiar to them. But, again, as in the case of identification of transformations, they had more difficulties when the transformation involved was a horizontal translation. When a stretch transformation was applied, many students referred to the graph of the function as being narrower or wider, depending of the factor of the stretch, but they were not able to explain changes to the original function beyond those they remembered from memorized rules. These students worked at an action level and were not able to realize that the rate of change of the function changed due to the transformation applied. They seemed to consider the curve as a "wire" that can be bended without changing the relative distance between its different points. A typical response given by some students working at an action or process level when asked to graph and describe the function f(x)= [2/(x-3)] – 5 was *"…as it is multiplied by 2, it is narrower. I: What do you mean by narrower? S: its graph is not as spread out as without the 2. I: is that true for any point in the domain of the function? S:…mmm, yes, it always does the same"*. Most students found it difficult to reflect the graph of a function over the Y axis, and even more when the transformation applied contained an absolute value, as was the case for a function given in graphical representation, where it was needed to find *f(-x)*. Even students working at a process level, for example, interpreted the transformation as a reflection over the X axis, instead of reflecting over the Y axis: "*I: how do you know it goes that way? S: because this minus 1 means that all the points have to be moved to the other side, down here...it is minus x"*. In this last question, the "form" of the curve changes under the transformation, but only students who showed to be working at an object level could explain the result of the transformation. For students working at an action level, applying transformations to functions presented in a graphical representation was almost impossible. They showed strong difficulties with the concept of function itself. They tried to find a "formula" for the curves and then tried to plot that function point by point. They rarely referred to transformations of functions; they did so only when the examples were familiar to them, and they had probably memorized the effects of each transformation. Students working at a process level could graph transformed functions if the transformations that were applied to the original function were rigid. They struggled with other transformations, but consistently explained using transformations the changes of the basic function. Students working at an object level were able to respond correctly and to explain most of the questions in this group. They could graph the transformed functions explaining which transformations could be used, and they were able to recognize the function on which the transformations were applied. They were also able to identify properties of the families of transformed functions.

Questions related to the **use of transformations** of functions were those where transformations of functions are a basic tool to solve a particular problem. For example, students had to relate a graph from an unfamiliar function to its analytical representation they also had to explain how to transform one function into another when both graphical representations were provided, and finally, they had to solve a difficult inequality using transformations. Those students who were found to work at an action level could not solve this group of questions. Students working at a process level demonstrated they had interiorized the actions involved in transforming a function but presented difficulties when working with trigonometric functions or with functions they could not recognize. Students working at an object level could solve these questions even though they struggled and had to reconsider their work in several occasions. On the following excerpts we present examples of answers provided by three different students each working at a different level. These examples show the different approaches they used when they were asked to solve $\frac{1}{x-1}+4\leq-\frac{1}{2}(x-2)^2+6$.

Carlos, an action level student said "*I have to solve first the squared part, and develop the expressions to leave x on one side and numbers at the other…*" he tried unsuccessfully to achieve this, then the interviewer suggested "*Can you use graphs of both functions to help you solve the question? S: Well…I would need to…make a table here…but, not really, I don't think I can graph them, this is a parabola, it is squared here, but I don't know… I am not too good at graphing difficult functions…*"

Tania, a process level student, said "…*The way I would solve that is by finding the graphs of the functions…I first thought to use algebra, but then I realized it is easier to graph… the first part, the function is a hyperbola the other a parabola….they would be like that… then…I need to…find when the first is less or equal than the second… and this happens here, from about 0,5 to infinity*". This student could use the transformations to graph both functions but her drawing was not completely correct because she considered that multiplying the parabola by ½ made it narrower, and then did not interpret correctly the intervals. Lucía, a student classified at an object level, replied: "*I will try to do it graphically…graph the function 1/x-1 + 4, and also –1/2(x-2)*^{2}*+ 6…two functions… the first is not defined at x=1, it is moved 4 units up, it is like 1/x, a hyperbola, it is also moved 1 to the right… the other is squared, it is a parabola,… it opens down, its vertex is at (2,6), it is wide because of this ½, it grows more slowly than x*^{2} *…now looking at this graph I want f(x) to be less than or equal to g(x)…so…the solution is x from more or less –1 to 1 union from 1.5 to 4 more or less…*". This student was able to use transformations in the solution of a new problem. She was able to identify the type of transformations, graph the transformed functions, and interpret the result in terms of the situation she was trying to solve.

Students' strategies to transform a function and to explain how it could be transformed reflected the same mentioned difficulties. We can conclude from their work that students worked with ease only when vertical translations were applied. When the information was provided in a graphical context, students showed difficulties to identify each transformation, or to predict its effect. Finally, students could use transformations with more ease when they were already familiar with the basic function. The last group of questions involved those where students needed to determine some properties of a family of functions, such as **domain and range**, or simply predict changes in the domain and range when a function goes through a set of transformations. Students working at an action level could determine some domains based on memorized facts, but were in general unable to determine the range of functions. They could only determine domain and range with ease for graphical representations of frequently used functions. Students working at a process level could find domains of almost all the functions that were presented to them. However, they had more difficulties when they had to find the range of those same functions. They often relied on drawing the graph of the function and, as we have already mentioned, they showed some problems to draw them correctly. Students working at an object level were able to determine domain and range of all the functions.

Discussion

Results of this study are consistent and also complement those found in previous research. They show that students' difficulties with the concept of transformation of functions are strongly related to their understanding of the concept of function. It was found that students who were classified at an action level show a weak understanding of the concept of function even though, in some cases, they have already taken a Calculus course. In particular, these students showed conceptual problems when discussing the graphical representation of functions and when looking for their domain and range. Although transformations of functions can be used in the solution of many problems, some students in this sample were not able to use them even in the case of problems that were similar to those studied in the pre-calculus course they had already taken. This situation is probably due to their point by point strategies to graph functions and their reliance on memorized facts. These students' knowledge about functions was not enough to interpret and work with transformations. Only a few of the students of this sample were capable to flexibly use those functions that were presented in a graphical context. Students showed a little more fluency when the function was presented in an analytical representation, that is, when it was presented in terms of a formula for the function; but, even in these cases, students showed lots of difficulties when they were asked about the properties of functions and to predict the properties of functions that change when a transformation is applied.

Even though we would expect that students who had taken a Calculus course would have a better understanding of the concept of transformation of functions, this was not found to be true. There were almost the same number of pre-calculus and calculus students working at the levels of action and process. However, it was found that all the students but one at the object level had already finished a Calculus course.

Transformations of functions can be classified according to the difficulty they present to students. Rigid transformations were easier for students. Students' solution strategies when applying transformations focused on what happened to the function as a whole and not on what happened to each of the points in the domain of the function. Dynamical transformations, that is, transformations where functions are "deformed" presented more problems to students. Their work demonstrated that they had not interiorized the effects of transformations on functions when it was needed to think in terms of co-variation of the dependent and independent variables of the function. Students had troubles when they had to identify which transformation had been applied to a particular basic function. When a transformation was given, they had problems finding its properties. All these difficulties were more apparent when the representation used in the question was graphical.

The results of this study demonstrate that when teaching transformations it is important to consider a wider variety of functions and to explicitly demonstrate the result of applying a transformation both at the level of what happens to the function in general, and what happens to different points in its domain. Many research studies have stressed the need to attain flexibility with the use of different representations in order to understand a variety of concepts. Results of this study show that this is also the case when teaching transformations of functions.

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GEOMETRICAL OPTIMIZATION PROBLEMS: A COVARIATIONAL APPROACH

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One mathematical task is getting the algebraic function in maxima and minima problems. The authors use a covariational approach, based on the first three mental actions (MA) of a covariational frame (Carlson et al. 2002). These are MA1) An image of two variables changing simultaneously; MA2) A loosely coordinated image of how the variables are changing with respect to each other; and MA3) An image of an amount of change of the output variable while considering changes in fixed amounts of the function's domain. The authors claim, "Our approach seems to help students create mental images of a physical situation and transform the physical objects (represented geometrically) so they can obtain an appropriate functional model". Data comes from a case study with two high school students (K-11).

Theoretical background

Several studies report student difficulties in representing and modeling a system that involves two quantities that change in tandem (Carlson, 2002; Monk, 1992; Monk & Nemirovsky, 1994; Thompson, 1994a; Kaput, 1994). Such difficulties are, in part, related to the concept of function generally applied: the correspondence concept or Dirichlet- Bourbaki definition of function, which usually leads to a strong dependence on algebraic representations ($y = f(x)$) and to a static idea of the concept (Kaput, 1994). A more intuitive approach which rests on dynamic aspects of functions is the covariational approach (Confrey & Smith, 1994, 1995; Thompson, 1994 b). Central to this concept is the coordination between two varying quantities and the development of images of covariation (Saldanha & Thompson, 1998; Thompson, 1994b). Carlson et al. (2002) developed the notion of covariational reasoning and proposed a framework for describing the mental actions involved when interpreting and representing dynamic function events. Using the first three mental actions described above, the authors designed a methodology to deal with optimization problems.

Methodology

The data comes from a case study with two high school students who had successfully completed a course of Analytic Geometry (K-11). From typical Calculus problems and using their methodology, the authors developed the protocols applied to the students. Such protocols consisted of two interdependent parts: *perception of change* and *covariation*.

Perception of change. It starts with a brief description of a physical or a geometrical system, represented in static means (usually a sheet of paper.) Next, the students are asked to mentally transform the system by "dragging" one of the geometrical elements (regularly a point), and to draw some snapshots. Answering two basic questions: What changes? and What doesn't change?, it would provide them a set of constants and variables and likely to enhance problem understanding.

Covariation. This part uses a Dynamic Geometry program. The objective is the representation and interpretation of the system through two of the quantities changing in tandem (selected by the students from the first part). The activities were planned to go progressively to

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more abstract representational contexts: from verbal descriptions to graphical, tabular, and finally to formulae representation.

There was no previous instruction or curricular interventions; furthermore, the students were not aware of the type of problems they were working on and had no time limitations. Collected information consisted of written answers and of video-taped films.

Results

We present some excerpts of three of the four protocols applied.

First protocol. The students were given a point *A* and an oblique line *l* (see Figure 1); then they were asked to put a point *P* in *l* and to join *P* and *A* with a segment of line. Next they would visualize *P* moving freely in *l* and draw some snapshots.

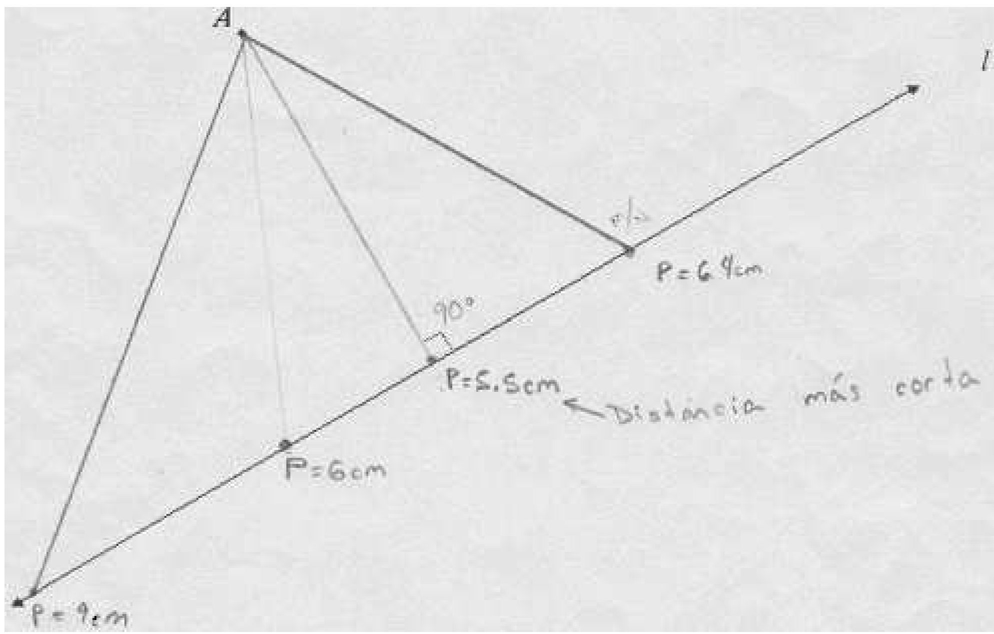

Figure 1.

When answering the basic questions they just gave as *variables*: "The distance from *A* to *P* and the location of point *P*." The *constants* were "There's going to be angles formed all the time; point *A*; it will be always the distance from *A* to *P*; and the distance will never be less than 5.5 cm [shortest distance]." Description of constants and variables is rather colloquial; let's see the following dialog in the middle of the task:

Student A: Could it be considered as constant "never"?

Student B: Yes, of course.

Covariation. After simulating and exploring the situation through several activities, the students are asked to incorporate a Cartesian system (using one of the options of the program) and obtain the equation of line *l* by themselves (not by the program). This was followed by the task of getting a formula or function to calculate the distance from point *A* to ***any point*** *P* in line *l*. The answer was elaborated in the following way.

Student A: Let's see: Get a formula or function [reads aloud the instruction] …

Student B: But, which one is point *P*?

Student A: Anyone.

Student B: Which one we take? This one for example [marks one in line *l*.] It would be necessary to calculate from here to here [places a finger in point *A* and another finger in the point marked in *l*].

They look up in their notes and write down the formula: $dist = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. Student B wants to make the calculations [replace in the formula the coordinates of point *A* and the coordinates of the alleged point "*P*"]. Student A reads to Student B the instructions, emphasizing the words ***any point***. Student B replies "So which one?" Student A says that point *P* is *any one* and that it can be regarded as *x*, *y*; that they can only substitute the coordinates of point *A* in the formula and that *P* is an unknown. The students keep discussing for several minutes and end up writing $dist = \sqrt{(-4 - x)^2 + (3 - y)^2}$. Later on, the students fulfilled the task ending with a function of just one variable.

At the end of the protocols the students are invited to reflect about their work through some questions. One of them was: Do you see any difference between the formula for the distance between two points and the formula from point *A* and a point *P* in line *l*? Student A, replies: "There's of course a difference, because in one of them there are two fixed points and on the other one there's a movable point … I would say there's a difference because in one, the formula for the distance between two points, you don't have any unknown. In the other one you have an unknown; since they are fixed you can express with a number, with a constant, and in this one, as the point is movable you don't know where it is." Their written answer was: "It changes, because in the formula between two points, we are going to get constants (numerical values) and in the other formula we are going to get constants (from the point *A*) and variables (movable point *P*)."

Third protocol. This time, the students have a drawing of a "soccer field" (see Figure 2) and a written description: "A player at P_1 gets the ball and starts to run along the side line towards the goal." The shot angle at P_1 is shown.

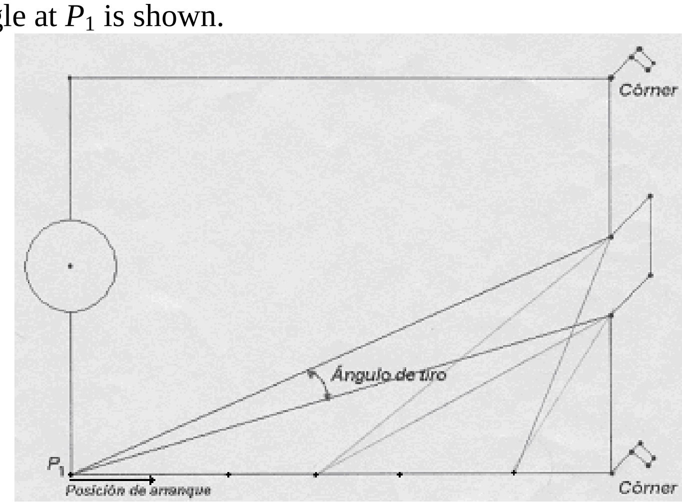

Figure 2.

Perception of change. After visualizing the player running and drawing some snapshots, the students decide to analyze two of their variables changing in tandem: shot angle and position of the player.

Covariation. One of the main activities of this part was based on tables; hence, in this protocol, the students are instructed to make a table (using the software), suggesting them that *one* of the variables should be regularly spaced. On a third column, they should calculate the differences of the non regularly spaced variable. The result is Table 1 (look at the order of the variables).

Table 1. Differences Table.

Angle[a]	P[b]	Differences
0	0	
9.66	2	9.66
11.52	4	1.86
10.49	6	-1.03
9.05	8	1.44
7.78	10	1.27
6.75	12	1.03
5.96	14	0.79
11.53	3.75	
11.53	3.79	0

[a]In degrees. [b]P: Position in cm.

Next, the students have to analyze the differences based on sign and an alleged value of zero. This assignment prompted the students to explore in real time and to notice that "positive differences indicated an increasing or opening Angle, while negative differences indicated a decreasing or closing Angle." When facing a zero value, they assume continuity and think "there must be a zero difference between 1.86 and -1.03." Furthermore, they guess such a point should be a transition point "the angle stops opening to start decreasing." Afterwards, they go on and drag P carefully, coordinating the values of P and Angle, finding out a "small" interval for P where Angle doesn't change (see last two rows in Table 1). Consequently they report that "a zero difference means the *Angle* remains the same."

The following task was to mentally visualize a horizontal line, moving up and down over their previous graphic describing the behavior of variables P and Angle. The questions related were: What's the meaning of the intersection points? The students visualize, do some hand movements and draw some snapshots, writing as answers: "Two crossing points mean they have the same Angle but in different positions of P and uniqueness of P for a single crossing point."

Fourth protocol: Given Figure 3 below, the students were prompted to drag point *E*.

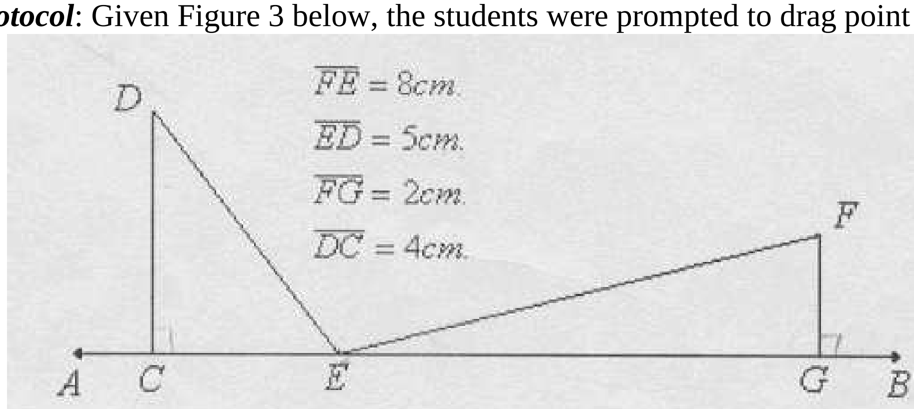

Figure 3.

Perception of change: From the very beginning there is an intense use of dynamic imagery to transform the system (no software program yet). The students drag point *E* on line $\overleftrightarrow{AB}$, particularly on the interval [*C*, *G*] and for "curiosity" outside of it. Contrary to the previous problems, they move point *E* in the neighborhoods of *C* and *G* and discuss the possibilities and consequences of putting it on *C* and *G*: According to Student B "the triangles *DCE* and *FGE* disappear and so the angle *FEG* can be very close to 90^0 but not exactly 90^0 [not even when *E* is on *G*, answering back to student A, who thinks it is 90^0]." We can see here *MA1* and *MA2* between several sets of variables changing in tandem and, something crucial to the second part, the visualization of a continuum of right triangles. The variables reported were: "Point *E*; angles of *E* [*FEB* and *DEA*]; distances from *D* to *E* and from *E* to *F* (but not path *DEF*); distances from *C* to *E* and from *E* to *G*." The constants were: "The points *D*, *C*, *F*, *G*; distances *DC* and *FG*; *DC* and *FG* are perpendiculars [to *AB*]." The students showed certain compensation reasoning in two previous protocols about the optimization variables (see Compensation Reasoning in Optimization Problems, this volume). In this case it was expected to happen in the path *DEF* so there was already a question in the protocol about the nature of path *DEF* (variable or parameter). Student B thinks it's a parameter "because as this one shortens [points to *EF*], this one extends [points to DE]." Student A thinks it's a variable so they agree to calculate, by the Pythagorean theorem, the hypotenuses of another particular case, resulting in 12.5cm, quite close to the 13 cm (see Figure 3). On arguing her point Student A moves point *E* to several positions (with two fingers Student A moves a virtual point right and left), convincing Student B. They now have their problem: an intriguing variable, path *DEF*, and of course the position of point *E*.

Covariation: For the algebraic function Student A writes "*path* =" and then goes to their set of variables and constants, which must undergo a conceptual and symbolic process of transformations, *i.e.*: point *E* (the independent variable) is substituted for variables *CE* and *EG* (they have in mind two dynamic triangles) and so they continue writing "$path = \sqrt{16 + CE^2} + \sqrt{4 + EG^2}$". In a further step, using the fact that *C* and *G* are fixed (constants) they express *CE* in terms of *EG*. Then they label segment *CE as x* (the geometric element is converted into an algebraic sign) getting the algebraic function. (It was normal for the students to cheerfully celebrate new findings, but this time was special).

Conclusions

Analysis of the protocols showed reported student's misconceptions about variables, constants and formulas (Sierpinska, 1992; Carlson et al., 2002). The students also displayed weak covariational reasoning abilities, even in *MA1* (they don't distinguish the order of variables in labeling the axes or in making tables), and *MA2* (describing direction of change, taking, let's say *x*, from left to right and vice versa). Through the 4 protocols, their verbal descriptions of two variables changing in tandem always followed the same pattern: As one of the geometrical variables increases, the other variable goes up (down) until a certain "value" is reached, before then going down (up). The pattern didn't change even after exploring with a program and observing in real time the values of the variables. The corresponding graphics also showed certain symmetry around the optimal points, reflecting a "***parabolic equation***" model. The *MA3* planned activities let the students discover and express growth and decrease of *y* in terms of the sign of the differences and the decrease (in absolute value) of the differences in the neighborhoods of the optimal points. However, this was of no use when comparing the graphic of the algebraic function and their initial model. Nevertheless, positive shifts occurred progressively. The students gradually transformed their "static" ideas of variable, constant and formula through the four problems: From given and unknown, they go effectively to constants and variables; and from geometric or algebraic formulas, where the elements on the right side are given values and unknown quantities which must be found (both) and substituted to get values for the left side, they go to dynamic functions where they get to distinguish variables on both sides of the expressions. Mental action *MA3* must be reinforced, for example, with second order differences and work concavity, and changes of concavity and point of inflection aspects. This would let the students shape their original graphics and lead them to see the variations of change. Of course, activities based on *MA5* (images of continuous changes in one of the variables for the entire domain of the other variable) would probably help to fulfill this.

Finally, it's the authors claim that optimization problems can be set out in Precalculus courses. A covariational approach to them would give chance to those students who won't take calculus courses to see a way to deal with problems of change, to model them in several representational contexts (including the algebraic one); and discuss intuitively, continuity and limit ideas. For those students who will take calculus, perhaps this approach would let them to appreciate the power and simplification of the derivative.

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STUDENTS' PROOFS FOR THE SHAPES OF GRAPHS OF SOLUTIONS IN THE PHASE PLANE

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This paper reports on student proofs for the shapes of graphs of solutions in the phase plane for systems of two linear homogenous differential equations with constant coefficients. We define proof as a convincing argument that answers the question why. By an argument, we draw on the work of Krummheuer (1995), who defines argumentation as the "interactions in the observed classroom that have to do with the intentional explication of the reasoning of a solution during its development, or after it" (p. 231). Findings from our research contribute to contemporary characterizations of proof as co-constructed and offer teachers useful insights into the types of student generated algebraic and geometric arguments they find convincing and the corresponding conceptual issues students encounter.

This paper reports on student proofs for the shapes of graphs of solutions in the phase plane for systems of two linear homogenous differential equations with constant coefficients. Although such systems might appear to be overly constrained, these systems are building blocks for more complicated nonlinear systems and therefore it is mathematically significant to investigate student reasoning with linear systems. Moreover, beginning with Poincare's pioneering geometric view of dynamical systems, phase planes have become an indispensable tool for interpreting and justifying the evolution of solutions to systems of differential equations. Little research, however, has been conducted in this important content area, and even less research documents student's proofs in which the phase plane figures prominently.

Following Henderson (2001) we define proof as a convincing argument that answers the question why. By an argument, we draw on the work of Krummheuer (1995), who defines argumentation as the "interactions in the observed classroom that have to do with the intentional explication of the reasoning of a solution during its development, or after it" (p. 231). Findings from our research contribute to contemporary characterizations of proof as a co-constructed process and offer teachers useful insights into the types of student generated algebraic and geometric justifications and the corresponding conceptual issues that students encounter.

Characterization of students' justifications leads to the second purpose this study, namely, what challenges and difficulties do students encounter as they attempt to justify the shape of solution graphs in the phase plane. In identifying salient difficulties among students, we also posit possible accounts for these difficulties. Taken together, the analysis offers a comprehensive account of student learning that will inform future instruction and curriculum design.

Prior research in the domain of systems of differential equations (e.g., Trigueros, 2000; Whitehead & Rasmussen, 2003) characterized students' mental schemes and offer important insights into student conceptions of rate and parametric equations, for example. Our research complements this more cognitively oriented work by foregrounding students' proofs within a mathematical community of inquiry. We view student learning as participation in mathematical activity (Lave & Wenger, 1991; Krummheuer, 1995). While engaging in whole-class and small group discussions, students frequently develop conjectures for the shape of graphs in the phase

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plane and justifications in support or in refutation of these conjectures. As such, we view learning as proceeding by engaging in argumentation (Yackel & Hanna, 2003).

Background

The instructional materials utilized in the classroom under study were, in part, inspired by the instructional design theory of Realistic Mathematics Education (RME) (Freudenthal, 1973; Gravemeijer, 1994). Key heuristics of RME are guided reinvention and emergent models. Corresponding instructional material provides opportunities for students to organize informal or intuitive notions into more conventional mathematical statements and findings (e.g., see Rasmussen & Keynes, 2003). In this process students gain ownership of the mathematical material, which facilitates their ability to manipulate symbols, perform algorithms, and create and use definitions. Students' mathematizing activity is facilitated by the teacher, who is proactive in supporting students' reinvention of ideas and methods for solving problems (Rasmussen & Marrongelle, in press).

In addition to the implementation of innovative teaching materials, special attention was also focused on the classroom learning environment. In order to promote productive small group and whole class discussions, the teacher continually fostered particular social and socio-mathematical norms regarding argumentation (Yackel & Cobb, 1996). Social norms are what characterize patterns of participation that are interactively constituted in the classroom by its participants, both students and teacher. Examples of social norms from this class include: students' routinely give explanations, indications of agreement or disagreement with other students' explanations, and students' explanations of other students' arguments. One should note that such norms might also characterize a history class, or an English class, for example. Socio-mathematical norms, on the other hand, are specific to mathematics, and are criteria that characterize acceptable, different, sophisticated, or elegant mathematical arguments. Thus, paying careful attention to the negotiation of social and socio-mathematical norms throughout the semester, coupled with the use of RME inspired instructional materials, the teacher created a classroom environment where students routinely offered mathematical arguments. The classroom data therefore provided a rich source of information about student reasoning regarding the shapes of graphs of solutions in the phase plane.

Methods

Data for this analysis comes from four class sessions of an eight-week classroom teaching experiment (see Cobb, 2000 for additional details of the teaching experiment methodology) conducted in an undergraduate differential equations course in a large southwestern university. Data sources consisted of video recordings of whole class and small group discussions, researcher field notes, and copies of student work. The classroom teaching experiment was conducted as part of a larger research program aimed at developing an inquiry-oriented, research-based instructional approach in undergraduate mathematics.

We began the data analysis by transcribing the four classroom sessions in which students developed argumentations for the shape of graphs in the phase plane. A coding scheme was then developed as we observed video and simultaneously highlighted arguments in the transcripts. We used problematic or especially interesting episodes to sharpen and refine the coding scheme. This collaborative coding process provided multiple occasions to share and defend interpretations of the video and corresponding transcripts, thereby minimizing individual bias by each researcher and eliminating interpretations not grounded in the video (Jordan & Henderson, 1995).

Because student argumentation was our primary research goal, we utilized Toulmin's (1969) definition of the anatomy of the core an argument as an analytic lens for interpreting and classifying students' proofs. According to Toulmin there are three parts to the core of an argument: data, claim and warrant. In this characterization, the data is what supports the claim or conclusion that is made. As it is not always explicitly apparent how the data might lead to a given claim, one might provide a warrant to clarify how the claim follows from the given data. Coding was a collaborative effort that involved multiple iterations of a coding scheme. All coding was organized and compiled in an excel spreadsheet in order to facilitate subsequent analysis. In this way, the table could easily be sorted so that one may view what types of claims were made for specific data, and conversely, what types of data were used to make a specific claim. Also, one may easily obtain the frequency of the occurrences of each claim made, and each type of data used.

Results

In the four class sessions analyzed for this report we coded 68 mathematical arguments, 61 of which were student generated, as shown in Table 1. The high ratio of student to teacher arguments reflects a classroom learning environment with social norms characterized by routine student involvement in explanation and justification.

Date	# of Student Arguments	# of Teacher Arguments	Total # of Arguments
4-18	11	1	12
4-20	6	0	6
4-22	15	2	17
4-25	29	4	33
Total	61	7	68

Table 1. Number of arguments

Further analysis revealed that students' proofs relied on six sources (data) for the basis of their claims. The six data sources that students used to make deductive claims were (1) previously proven results, (2) equilibrium solution type, (3) vectors, (4) an assumed characteristic property of the straight-line solutions, (5) the general analytic solution for the given system, and (6) the ratio of y(t)/x(t), where x(t) and y(t) are components of the general solution. Each of these six sources of data for students' arguments then had from two to four subcategories. In this report we elaborate the most prominent data sources used by students. In particular, we illustrate data sources of previously proven result, vectors, the ratio of y(t)/x(t), and the general analytic solution. Further elaboration of all data sources and subcategories can be found in Rhodehamel (2006).

Data Source – Previously Proven Result

In a previous class students had created an algebraic method for finding the x(t) and y(t) equations for the straight line solutions to the system dx/dt = y, and dy/dt = 2x-3y. That is, students had essentially reinvented the eigensolutions to this system (Rasmussen & Keynes, 2003). As a move toward creating the general solution, students were next asked to determine the shape of the phase plane graph for an initial condition between the two straight lines. Based on visual inspection of the vector field one might conclude that this graph is also a straight line. However, several students proved that this conclusion is incorrect using an argument by contradiction. The following argument by Sadie (all names are pseudonyms) exemplifies this proof.

Sadie: When we tried to find the slope of the straight line solutions we only found two slopes. So in order for there to be another straight line, we would have to find one of two slopes.

In other words, Sadie argues that because they had previously proved that this system had only two straight line solutions, the graph of the solution with initial conditions between the two straight line solutions could not also be on a straight line as this would contradict their earlier finding.

Data Source - Vector

In the following argument the data source is vector, and the claim is that solution graphs will curve towards the straight line solution y = -x heading towards the origin. In this data category students use an electronically generated vector field to make claims.

Emile: I just want to see where it goes.

Mario: Where what goes?

Emile: Where the solutions to any point goes. My thought was that they all come to y = -x [one of the two locations of straight line solutions].

Mario: They're going to zero anyway.

Emile: Yeah, they come to y = -x and then come to zero. Or infinitely close to zero, whichever it may be. Because just testing points inside like our initial point (-4,6) and (-3,5) plotting them on the vector field, they all come down to y = -x. And then you can see that any vector field below y = -x comes up and then goes to y = -x such that the same would happen for ones outside y = -2x. They'd be close to it but they would come down through the origin and they would sneak up right behind them, y = -x. I want to see if you can do something with these equations or these solutions to prove that. To prove that the graphical conjectures actually do go to y = -x.

In Emile's argument we can see the power of the vector field, but also the limitations of it. Emile began by plotting the vector for the initial condition (-4,6), and then continued to plot other vectors in order to make broader claims about the entire family of solution functions. Thus, the vector field allowed the students to make powerful visual observations about the entire solution space, but as Emile stated, "I want to see if you can do something with these equations or these solutions to prove that. To prove that the graphical conjectures actually do go to y = -x.", these empirical observations lack the formality that analytic reasoning may provide. This statement by Emile also suggests that the sociomathematical norm of what counts as a legitimate mathematical proof was progressively evolving and coming to fruition.

Data Source – Ratio of y(t)/x(t)

The following example illustrates a student's use of the data source, the limit of the ratio y(t)/x(t), where x(t) and y(t) are components of the general solution, to arrive at a mathematically incorrect conclusion. The problem was such that there were two positive, distinct eigenvalues with eigenvectors (1, 1) and (1, -1/2). The direction and location of the corresponding straight line solutions is shown in Figure 1. The student had previously determined the correct general solution to be $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = k_1 e^{t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, correctly graphed the straight line solutions, and had correctly found that the limit of y/x as t goes to infinity to be 1 for the solution with initial condition (1,0). However, he interprets this limiting value to mean that the graph of the solution with initial condition (1,0) would, in his words, "spear into the line y = x," as illustrated in Figure 1. Other students had similar claims.

This example, in addition to illustrating the use of the limit of y/x as data for an argument, illustrates one of the salient conceptual difficulties that students encountered when justifying the shapes of graphs in the phase plane, namely the incorrect interpretation of a limiting slope. In this particular example the student is able to correctly find the limit of y/x as time approaches infinity, but he is unable to correctly interpret what this means graphically. The correct interpretation of the limit would yield that the graph should ultimately achieve a slope of 1 as time approaches infinity, whereas in this case, the student has interpreted the limiting slope of 1 as signifying that the graph should physically approach the line y = x.

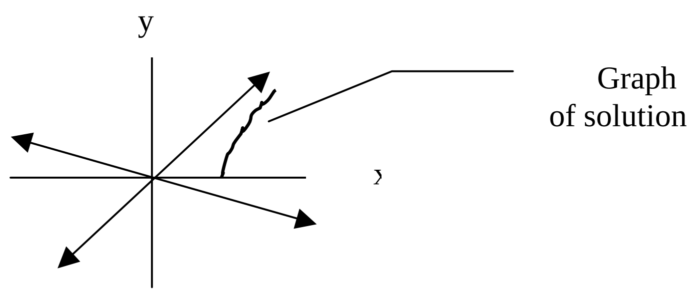

Figure 1. Student sketch of graph in the phase plane

We conjecture that students' tendencies to interpret the limiting slope in this manner are grounded in their experience with limits in calculus. For example, when taking the limit of f(x) = 1/x as x approaches infinity, the correct solution yields f(x) tending to 0. In this context, one then concludes that the graph is going to physically approach the line f(x) = 0, rather than the slope of the line approaching zero as it does in the limiting slope. This stands in contrast to the way in which one needs to interpret the result of the limit of y(t)/x(t).

Data Source – General Solution

The following argument illustrates students' use of the data source, general solution, and also illustrates another conceptual difficulty that students often encounter. As we argued in the previous section, some students were inclined to incorrectly assert that solution graphs will "spear in" towards the straight line solution with greater eigenvalue. Some students also arrived at this same conclusion using the general solution as a data source. In the example that follows, we refer to the way in which students use the general solution as general solution with subcategory quantitative, because of their focus on the magnitude of the two components of the general solution.

Brent: Okay, I just looked at what happened when t got really big. This one goes to -2t and this one is just –t [referring to the components of the general solution], so this one goes to zero a lot faster [pointing to e-2t], so as your t increases, this one [e-2t] starts to go away [that is, go to zero] and your left with only this one [e-t], that's the -1 line [the straight line solution with slope -1, y = -x]. So, as you increase t, it starts to look more like that one [point to the straight line solution along y = -x in the graph of the phase plane], so I said it went down towards that one [the straight line solution along y = -x].

Anna: [recasts Brent's argument] I guess it [the straight line solution along y = -x] would pull.

In this argument Brent realizes that the exponent, -2t, will decrease much faster than –t as time values increase. As a result, he concludes that the straight line solution that corresponds to the component e-2t will have less of an influence on the solution graph. After Brent finished giving his argument the teacher asked if other students could revoice his argument. Anna volunteered, closely following the wording of Brent's argument, but at the end she mentioned, "I guess it would pull," referring to the straight line solution. This language of "pulling" was first introduced in a previous class session when Ray [incorrectly] suggested that straight line solutions might act as attractors, similar to the way in which equilibrium solutions for first order differential equations behaved. The intuitive appealof the the notion that a straight line solution might act as an attractor, and corresponding language of "pulling" was present in other general solution arguments as well as in ratio arguments.

On the last day of discussion, in which determining and offering justifications for the shapes of graphs in the phase plane culminated, we begin to see a shift in the way in which the students reasoned with the general solution. In the following argument Anna begins by stating why she initially thought that the graph would spear-in towards the straight line solution with greater eigenvalue.

Anna: My initial thought was that this [graph similar to that in Figure] is probably correct. I was thinking in a way well, since this function [e4t] grows faster with a t increasing then um, I was saying that this function would pull our solution to itself.

When Anna makes the claim that the graph will be "pulled" into the straight line solution she notes that the magnitude of the exponent corresponding to the straight line solution that lies along y = x is much larger, so she only attends to the "pull" of that component and the "pull" of the other component (the component corresponding to the straight line solution y = -1/2x) fades into the background. In this sense, her reasoning might be characterized as univariate. That is, she focuses on one of the two quantities (or components) of the general solution to the exclusion of the other quantity. Anna goes on, however, to explain why she now sees this as incorrect and gives a justification for the correct shape of the graph.

Anna: Then the question, or, then was said, well, we still have this function [the component of the general solution with smaller eigenvalue] as well. And we concluded that this more, less curvey, more smooth curve I guess, would be the correct answer because there is two reason to it. The first one is that even though um, this grows faster, we still have this [the "pull" of the straight line solution along the line y = -1/2x], so it kind of like pulls this function, so it kind of like pulls this function towards itself as well. Um, although the slope of this graph looks more like this [see Figure 2].

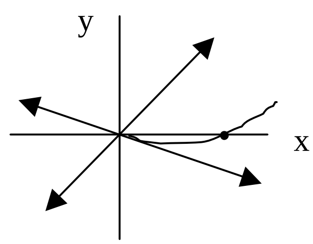

Figure 2. Anna's revised graph

There is an important difference to note between the two arguments that Anna generated in the last example. In the second argument, Anna correctly attends to the pull of both quantities, recognizing that however miniscule the contribution of a component, it will still exert some force on the solution graph. We characterize this latter type of reasoning as bivariate because she attends to both components (or quantities) in the general solution for all time, and we have found that such reasoning represents a significant conceptual shift for students. Indeed, reasoning that coordinates multiple quantities at once is known to be a challenge for many K-12 students. Our research indicates that this complexity continues to be a challenge for advanced undergraduates.

Conclusion

The goal of this report was to document students' arguments and the associated conceptual issues that arose for students. In doing so, we extend the currently limited amount of literature pertaining to students' understanding and difficulties in working with systems of differential equations. The significance of this report is two fold. First, documentation of students' arguments will allow other teachers of differential equations valuable access to the types of arguments that students might provide, thereby expanding the teachers' pedagogical content knowledge (Shulman, 1986) in this particular domain. Second, we contribute to the characterization of proof (a convincing argument that answers the question why) as a co-constructed, social process. As Krummheuer's definition of argumentation suggests, participation in argumentation means participation in classroom activity. We therefore fore grounded the notion that learning is participation in increasingly sophisticated mathematical argumentation. The example we tendered of Anna's shift in univariate argumentation to bivariate argumentation exemplifies productive advances in student participation in justification.

Consistent with the perspective that proof is a co-constructed process, the work reported here included analysis of both students and the teacher. However, we intentionally choose to highlight the contributions of the students in order to better understand the conceptual issues that enabled and constrained student arguments. In ongoing work we are reanalyzing this same data, highlighting the role of the teacher in the argumentation process.

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DEFINITE INTEGRALS, RIEMANN SUMS, AND AREA UNDER A CURVE: WHAT IS NECESSARY AND SUFFICIENT?

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A teaching experiment was conducted in a calculus class to determine what it means to understand definite integrals. One interesting result was based on students' use of area under a curve as a tool for computing definite integrals. Results show that in the problems presented in this study, students' use of area under a curve was helpful in problem solving only when a deeper understanding of the structure behind the definite integral was present.

The purpose of this research was to examine student understanding of Riemann sums and definite integrals. These concepts are imperative for students to understand for three main reasons. First, many real world applications involve functions that do not have an antiderivative that can be expressed in terms of elementary functions. For example, the antiderivative of the function $f(x) = e^{x^2}$ cannot be expressed in terms of elementary functions. Thus, the Fundamental Theorem of Calculus could not be applied, and other methods for evaluating the definite integral, such as Riemann sums would be needed.

This leads to the second reason that students need to have an understanding of the structure of Riemann sums. While Riemann sums may not be the most efficient method for approximating a definite integral, other methods, such as the trapezoid rule, midpoint rule, or Simpson's method are based on the structure of the Riemann sum. Thus, an understanding of the structure of Riemann sums will help students to understand these other methods as well.

Finally, I hypothesize that an understanding of Riemann sums is needed even when a function has an antiderivative that *can* be expressed in terms of elementary functions. Setting up the appropriate definite integral requires the student to know what to integrate, and an understanding of the structure of the Riemann sum will give the student the tools he/she needs. In all cases, it is possible to imagine the definite integral being represented by the area under a curve. This research begins to examine what is necessary for students to be able to use area under a curve as a powerful tool for solving problems that involve definite integrals.

Background

There are several pieces of literature that focus on mathematical topics that build the definite integral: $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$. Multiplication, rate of change, sequences and series, limits, and functions are all incorporated into the definite integral, and several research studies have been done to understand these topics. In addition, there are two pieces of literature that focus on the concept of integration. Orton (1983) mainly focuses on methods of evaluating definite integrals. As is common in many calculus classes, many of the definite integral problems in Orton's study involve finding the area under the curve. He discusses the structural and calculational/executive errors that students made when finding the area under the curve in several situations. Artigue (1991) discussed Orton's studies of calculus students' understanding of differentiation and integration. The study found that many students could perform routine procedures for finding

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the area under a curve, but the students rarely could explain their procedures, and some even admitted that they "didn't really know why they were doing it" (Artigue, 1991). Orton's study does not attend to understanding *why* area under a curve is equal to the definite integral of a function. My research will provide data that shows that when solving real world problems, students need to understand why this relationship between area and the definite integral holds. "For students to see "area under a curve" as representing a quantity other than area [i.e. velocity], it is imperative that they understand how the quantities being accumulated are created" (P. Thompson & Silverman, 2006).

Thompson's (1994) study focuses on student understanding of the Fundamental Theorem of Calculus. As part of his teaching experiment, he developed and implemented a module to help students understand Riemann sums in a way that develops the Fundamental Theorem. Thompson noted a distinction between accumula*tion* and accumulat*ing*, and stressed the idea of quantities accumulat*ing* for his work. Thompson also describes a younger student, Sue, who is able to construct a Riemann sum to approximate distance traveled. Sue was also able to explain that she could get better approximations by using smaller time intervals.

For Orton's study, students were given a definite integral and asked to evaluate it or were given a graph and asked to determine the function's definite integral by finding the area under the curve. In Thompson's study, students were asked to examine aspects of the definite integral that related to the Fundamental Theorem of Calculus. My study focuses on a different aspect of the definite integral. Specifically, my students were asked to look at problems where they either needed to set up a definite integral or use Riemann sums to approximate a total accumulation

Theoretical Perspective

The theoretical perspective that I have used to analyze the data is taken from the work of Piaget (1970, 1975). The basic idea is a type of constructivism with the premise that we construct not at free will, but within certain constraints. The system in which we construct is subject to certain laws, specifically reversibility, wholeness, transformation, and self-regulation (Piaget, 1970).

Possibly the most important aspect of Piaget's constructivism, structuralism, is the concept of reflective abstraction. Abstraction "in the ordinary sense of the word" refers to something being "′drawn out' from things which have that property" (Piaget, 1970). For example, a child learns what "red" is by seeing lots of objects that are red. The child may be shown a red ball, a red crayon, a red shirt, and a red block and the child eventually learns the meaning of "red". Reflective abstraction is a type of abstraction that comes from "acting on things" and ways in which we coordinate actions. Reflective abstraction deals with the elements *and* the operations we perform on them. Specifically with the definite integral, students cannot understand definite integrals simply by looking at a lot of them. Instead, the students need to *do* something with the components of definite integrals to be able to reflectively abstract and understand the structure of the definite integral.

Piaget's structuralism is a response to both Platonic and atomistic views. Within the Platonic view, knowledge is something that already exists (Piaget, 1975), and learners simply "remember" or "acquire" the information. Per Sfard's (1998) acquisition metaphor, students do not actively participate in developing the structure, but acquire the knowledge of the structure instead. A common example is based on the Gestalt perspective. When we see a person, we do not need to look at the eyes, and then the ears, and then nose, and so on before we can recognize the person. Instead, we can simply look at the face as a whole to identify the person. The idea of ungenerated wholes is central to the Gestalt perspective.

One portion of the definite integral that could be considered a Gestalt aspect is seeing the definite integral as the area under a curve, without constructing it from the structure of the limit of Riemann sums. Thus, the definite integral would not be a well-developed object, but instead would be only a pseudo-object (Sfard, 1991). This will be discussed in more detail in the data analysis section. Viewing the definite integral as the area under a curve is certainly something that we want our students to be able to do, but we also want them to be able to generate the structure of a Riemann sum in order to have a better conceptual understanding of the definite integral.

On the opposite side of the spectrum is the atomistic view that sees only a collection of individual elements which Piaget calls aggregates (1970). Within this view, the student does not see any of the relationships between the individual elements, nor does the student consider any of the operations that are performed on the elements. An example of this from Riemann sums could be a student looking at a collection of rectangles, without considering the area of the rectangles, or an example could be a student who can see the area of *n* rectangles, but cannot imagine the number of rectangles increasing infinitely to form the definite integral.

Clearly, there are ideas about the definite integral that fit into the Gestalt view or the atomistic view, but neither of these views allows us to have a fully developed concept of all that is involved in conceptualizing a definite integral. Piaget claims that structuralism is the solution (1970). Within structuralism, Piaget claims that students do construct knowledge, but the construction of knowledge takes place within a system that has its own laws (Piaget, 1970). A structure consists not only of elements or aggregates, but the structure also consists of the operations on these elements and the relationships between these elements. Specifically, structures are self-regulating and are subject to the laws of reversibility, transformation, and wholeness.

Methods/Subjects

This research was designed using a teaching experiment methodology (Simon, 1995). Thus, a hypothetical learning trajectory was created. Participants in the study were students in a calculus workshop in which the author was one of two research assistants. Students enrolled in the calculus workshop were concurrently enrolled in a traditional first semester college calculus class, and students generally reported one of two reasons for enrolling in the extra workshop. Approximately half of the students claimed they were not good at math and wanted extra help with their calculus class, and the other half reported that they loved math, and simply wanted to take another math class.

Students were videotaped as they worked in groups on activities relating to definite integrals (see Table 1), although the phrases “definite integral” and “Riemann sum” were not used by the instructors until after the activities were completed, or until the students introduced the terms themselves. All students were very familiar with Oehrtman’s (2004) approximation framework and had worked in several contexts where they were required to find approximations, both overestimates and underestimates, determine a bound for their error, and find approximations that were accurate to within a predetermined bound, epsilon. The problem solving sessions extended over two and a half one-hour class periods.

Water Problem Group 1A	A uniform pressure *P* applied across a surface area *A* creates a total force of *F*=*PA*. The density of water is 62 lb per cubic foot, so that under water the pressure varies according to depth, *d*, as *P*=62*d*. a) Draw and label a large picture of a dam 100 feet wide and extending 50 feet under water. b) Approximate the total force of the water exerted on this dam. c) Find an approximation accurate to within 1000 pounds. d) Write a formula indicating how to find an approximation with any pre-determined accuracy, ε.
Spring Problem Group 1B	For a constant force F to move an object a distance d requires an amount of energy equal to $E = Fd$. Hooke's Law says that the force exerted by a spring displaced by a distance *x* from its resting length is equal to $F = kx$, where *k* is a constant that depends on the particular spring. a) Draw and label a large picture of a spring initially displaced 5 cm from its natural length then stretched to a displacement of 10 cm. b) Approximate the energy required to do this if the spring constant is *k* = .155 N/cm. c) Find an approximation accurate to within 1000 ergs (1 erg = 10^{-5} N·cm). d) Write a formula indicating how to find an approximation with any pre-determined accuracy, ε.

Table 1

Data Analysis

Near the beginning of the problem solving session, both groups attempted to set up a definite integral, but were unable to do so. When the students in group 1A initially wanted to use an integral, they were discouraged to do so by the professor. Instead they were asked to use the ideas of the approximation framework. Thus, they approximated the force using the average pressure on the dam and the entire area. After computing this approximation, the students decided to set up an integral to check their approximation. However, they were unsure of how to set up the appropriate integral. At first they set up the integral $\int_0^{50} 5000 \cdot 62d$. They did not include the *dd* (or "*dx*"), and did not discuss this. Much more importantly, they use 5000 as the area, which is the area of the entire dam. Instead, they needed to have the area of one strip with a width of 100 and an infinitely small height.

1A Student A:	*[writes* $\int_0^{50}$ *]. Let's do this real quick. So P times A is just…it's just this [points to* $A_{contact} = 100 \cdot 50 = 5000\,ft^2$ *on whiteboard] times 62d, right? A constant…times d?*
1A Student B:	*62 d A, isn't it?*
1A Student A:	*But A is…oh yeah.*
1A Student B:	*Oh yeah.*
1A Student A:	*Isn't A constant?*
1A Student B:	*A is constant. So it's 5000 times 62 d.*

1A Student A: *[continues writing. Now has* $\int_0^{50} 5000 \cdot 62d$ *]*

Since they had difficulty constructing an appropriate integral, it seems likely that they chose this method because the problem was similar to those they solved in their calculus classes using integrals, and not because they understood (at that point) the structure of accumulation and definite integrals. These students recognized that the solution included a product of two terms, but one of the terms in their product was incorrect. The students compared their answer with the approximation they found earlier using average force and realized that one of the two solutions must have been incorrect.

1A Student B: *Ok, that's nothing close to what—*
1A Student A: *we have written down. That's sad.*
1A Student B: *That's the answer.*
1A Student A: *Maybe we did it wrong. Maybe we set up the integral wrong*
1A Student B: *yeah, that's a possibility*
1A Student A: *Maybe the area. Cause we're (inaudible) it at each level [draws a thin horizontal strip on whiteboard] and the area eventually goes to zero.*
1A Student B: *It's very upsetting that we're wrong*
1A Student A: *(laughs) It's very sad.*

Notice in the above excerpt, student A begins to discuss a necessary part of the Riemann sum, but at this point, her thoughts are not well developed enough, indicating gaps in her understanding of the structure of the definite integral. Eventually, the group computed an approximation based on the structure of a Riemann sum, allowing the students to approach the problem from a more conceptual basis, describing the underlying product structure of the definite integral to determine the appropriate integrand. The important thing to note here is that the students could only set up the integral *after* they had explored the problem using Riemann sums, and developed an understanding of the underlying structure. The following paragraphs discuss their actions when computing an approximation based on the structure of a Riemann sum.

1A Student B: *couldn't you do a summation?*
1A Student A: *yeah, we could do a summation of them.*
1A Student B: *Like do like 10 intervals and do a summation of them.*

Next, the students determined which terms to use for the product. Also, note that the idea of breaking the dam into pieces and then adding the force on each piece does not seem to be a conceptual obstacle for the students in any way. The students computed two approximations using 50 subintervals. One approximation was an overestimate (using the pressure at the bottom of each slice) and the other approximation was an underestimate (using the pressure at the top of each slice). The students recognized that both an overestimate and an underestimate would be needed in order to be able to bound the error. Next, the students needed to find a way to make their approximation more accurate and quickly decided to use more subintervals on the dam, making their Δd smaller. Although the students did not mention the word limit, the students discussed that their approximation could be accurate to within any predetermined accuracy if they used small enough intervals.

Group 1B also attempted to initially set up an integral to solve the problem, but they were also unsuccessful. In this case, the students were unsure if they function they should integrate should be the formula for force, $F = kx$, or the formula for energy, $E = Fd$.

1B Student A: *That's what it's supposed to be [pointing to E = Fd]. Or is it this one? [pointing to F = kx] Is it this one [pointing to energy formula], or this one [pointing to force formula]?*
1B Student B: *it's of energy*
1B Student A: *It's of energy, so it's the integral of force times distance.*
RA 2: *But you're not using integrals.*
1B Student A: *yeah, I know. I'm just trying to remember which one that it is that you use.*
1B Student B: *You're trying to find energy.*
1B Student A: *But in the equation…*
1B Student C: *You need the force.*

Student B thought the formula should be the one for energy, but student C thought it should be force. Although student C's method would have led to a correct answer, he was never able to justify why this was the correct method. Instead, he often said, "that's just what it is".

The primary difference between groups 1A and 1B was the aspect of the problem on which each group focused. Group 1A focused mainly on the problem within its context (water pressure on a dam), while group 1B rarely discussed their context (force of a spring). Instead, pilot group 1B drew a graph and talked mainly of area under the curve. When they abandoned their efforts to set up an integral, they moved on to area. The students graphed the force, $F = .155x$ and discussed the energy in terms of area under this curve.

1B Student C: *That's what we were figuring out, it's force with respect to distance and the area under this is energy.*

When I asked the students to explain to me *why* area under the curve was equal to energy, they could not explain, and were never confident that they were correct in graphing force, instead of energy. Their only justification was that they had gotten confirmation from one of the research assistants that this was an acceptable method. When I pushed them to explain *why* this was an appropriate method, they were unable to do so.

I hypothesize that one of the reasons the students struggled with explaining area is because they did not understand the structure of the Riemann sum. Several times throughout the video, the students in this group incorrectly said that the "summation of forces equals energy". It is not just the summation of *forces* that equals energy, but it is the summation of the *products* of force and distance that equals energy. The students were attending to the summation layer of the definite integral, but did not include the product layer. The following excerpt indicates that the statement made by the students is not a case of metonymy, but in fact a conceptual error. The excerpt below also illustrates Oehrtman's (2002) collapse metaphor. The student seems to be visualizing one dimensional lines as the area under the curve, instead of two dimensional rectangles.

1B Student E: *That the summation of the forces equal energy.*
1B Student B: *There, the answer is true.*
1B Student E: *That's how I found that. I don't know if it was right or not. All I know is that's what I found.*
RA 2: *Is it the sum of just the force?*
1B Student E: *Yeah, 'cause if you sum all the forces up underneath the graph… like the integral… you should get the energy.*

As discussed earlier, group 1A focused on the context of the problem (water pressure on a dam) and did not graph a function or consider area under a curve. One reason for this may be the nature of their activity. The first question asked the students to draw and label a large picture of a dam. This picture seems to be more helpful for reasoning about the problem than in the context of the spring. To solve the problem, the students broke the dam into horizontal slices, and calculated the area of each section and an approximate pressure on each section. The picture of the dam is a nice representation of the area of each strip. Also, since the pressure depends on depth, the picture was helpful in determining the pressure on each strip.

Conclusion

Although the title "Riemann sums" was not stressed by the professor or any of the research assistants in the class, the students in group 1A seemed to have a good understanding of the concepts involved. The understanding of Riemann sums held by group 1B is questionable. Since they only referred to their problem in terms of area under a curve, it is unclear of their knowledge of Riemann sums. Of course, area under a curve and Riemann sums are mathematically equivalent, but it is appears that the students in pilot group 1B had only a pseudo-structural understanding. They may be proficient in dealing with area under a curve, but may not be able to solve other accumulation problems without thinking about area under a curve, or may not be able to relate the area under a curve to the structure of a Riemann sum.

In particular, context 1A (the water problem) proves much more difficult when trying to use area under a curve as a tool for solving the problem. The formulas shown in the description of the problem are $P=62d$ and $F=PA$, but the function that would need to be graphed in order to apply area under a curve is $f(d)=62\cdot 100d$. In contrast, the formula that would need to be graphed for problem 1B (the spring problem) in order to correctly use area under a curve as a tool is $F=kx$, which is the formula for force that is given in the statement of the problem.

I do not in any way claim that area under a curve is a bad representation or that it should not be taught. Instead, I claim that area under a curve is not *sufficient* for understanding the definite integral. It can be a powerful tool when the underlying structure of the definite integral is present, but the above example of group 1B illustrates what can happen when this structure is missing.

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STUDENTS' PROOF SCHEMES: A CLOSER LOOK AT WHAT CHARACTERIZES STUDENTS' PROOF CONCEPTIONS

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The guiding theoretical principle in this study is the notion that the process of producing a mathematical proof can be viewed as similar to the process of solving a mathematical problem. Undergraduate students with different proof schemes were interviewed while attempting to construct proofs and their protocols were analyzed to identify patterns in problem solving and possible relationships among students' proof schemes and these patterns. Our findings suggest that students within each proof scheme do follow some similar patterns in their attempts to construct proofs and these patterns impact their proof performance.

Proof as a logical argument that one makes to justify a claim and to convince oneself and others assumes a central role in mathematics. As such there has been a growing research and policy effort to make proof central to school mathematics as well. To inform this interest in the teaching and learning of proof and mathematics, there is a substantial research base, mostly focusing on the difficulties students face when attempting to read and write proofs. (e.g., Balacheff, 1988; Chazan, 1993; Coe & Ruthven, 1994; Knuth, et al., 2002; Porteous, 1986). Several researchers have attempted to understand students' approach to mathematical proof by classifying these approaches along several dimensions – an approach currently proving fruitful in understanding students' difficulties (e.g., Balacheff, 1988; Harel & Sowder, 1998; van Dormolen, 1977). The study reported here was designed to add to this body of research on our understanding of students' approaches to proof. Our main goal was to explore further the patterns of problem-solving in building mathematical "proofs" by students at different levels in their ability to construct proofs.

Theoretical Perspective

1. Proof Schemes

Harel and Sowder (1998) argue that proving or justifying a mathematical conjecture involves ascertaining (convincing oneself) and persuading (convincing others). An individual's "proof scheme" consists of what constitutes ascertaining and persuading for that person. Harel and Sowder made the first step towards mapping students' cognitive schemes of mathematical proof and providing a framework for evaluating students' justifications. In particular, they proposed three levels of student proof schemes: (1) externally based proof schemes; (2) empirical proof schemes; and (3) analytical proof schemes.

External conviction proof schemes are ones in which students build arguments or accept the validity of an argument based on (a) the ritual or the form of the appearance of the argument, not its content – the ritual proof scheme, (b) the word of an authority, such as a textbook or a teacher – the authoritarian proof scheme, and (c) some symbolic manipulation often without reference to the symbols' meaning – the symbolic proof scheme. In all three cases, students convince themselves and others by referring to external sources.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Empirical proof schemes can be either inductive or perceptual. When a student attempts to remove doubt about the truth of a conjecture by using quantitative evaluations (using examples or specific cases) he/she is considered to have an inductive proof scheme. In a perceptual proof scheme, a conjecture is validated via rudimentary mental images, that is, “images that consist of perceptions and a coordination of perceptions but lack the ability to transform or to anticipate the results of a transformation” (Harel & Sowder, 1998, p. 255).

A proof scheme is characterized as analytical when the validation of conjectures is obtained via the use of logical deduction. Analytical proof schemes can be either transformational or axiomatic. A transformational proof scheme involves goal-oriented operations on objects. The student operates with a deductive process in which she considers generality aspects, applies goal-oriented and anticipatory mental operations, and transforms images. An axiomatic proof scheme goes beyond a transformational one, in that the student also recognizes that mathematical systems rest on (possibly arbitrary) statements that are accepted without proof.

2. Problem solving

Educators and psychologists have attempted to gain insights into student reasoning by analyzing students’ utterances and arguments during problem solving. In his seminal work on problem solving, Schoenfeld (1985) suggests that due to the complex nature of this activity, it is necessary to study and analyze in detail the verbal protocols of students engaging in problem solving. He further suggests that protocols could be partitioned into segments relating to different aspects of the problem-solving process, offering a way to examine qualitative differences and similarities in the reasoning of individuals at different levels of performance. Elsewhere, Chi and her colleagues (1989) studied students’ arguments while solving physics problems as a way to gain insights into the mechanisms underlying successful problem solving. They concluded that successful problem-solvers were the ones that were able to draw conclusions and make inferences from the given information, as well as provide explanations underlying the actions they were taking. Overall, a careful analysis of students’ verbal protocols and arguments while solving problems has the potential for revealing some of the underlying reasons for their actions and choices they make in their solution paths.

3. Proof schemes and problem solving

The notion that the process of producing a mathematical proof can be viewed as similar to the process of solving a mathematical problem constitutes the guiding theoretical principle in the study we present here on the development of undergraduate students' understanding of mathematical proof. Indeed, Harel (in press) suggests that proving and problem solving are both mental acts that characterize an individual’s “way of thinking” and, hence, the study of the two can inform one another. We suggest that using students’ interview protocols during their attempts to write proofs may help us gain further insights into students’ proof conceptions.

The baseline in our examination of students’ understanding of proof was each student’s identified proof scheme. However, within each proof scheme we sought to identify patterns in problem solving, and to identify possible relationships among students’ proof schemes and these patterns.

In particular, in the current study we examined the following questions:

- What proof schemes are exhibited by undergraduate students during their early coursework in mathematics?

- What patterns of problem solving can be observed among the proof schemes expressed by these students?
- What are the relationships between the proof schemes and these patterns?

Methods

The 34 participants for the study were undergraduate mathematics students enrolled in a first-year discrete mathematics course emphasizing mathematical argumentation and proof. At the beginning of the semester, all students in the course were invited to participate in the study. Each student was asked to solve 3 mathematics problems (Table 1) during a 30-minute clinical interview. Interviews were audiotaped and protocols were transcribed.

What happens when you add an even number and odd number? Prove that your conjecture will always hold.
Prove that for every integer n, n^2+n is even
Prove that for all irrational numbers x, $x-8$ is irrational.

Table 1: Interview tasks

Each argument was first coded with respect to student's proof scheme (Harel & Sowder, 1998). Subsequently, we analyzed students' work within each of the three levels of proof schemes (external, empirical and analytical) using a qualitative methodology. Hence, the verbal protocols were analyzed using a problem-solving perspective (Schoenfeld, 1985). Each protocol was partitioned into segments (macroscopic chunks of consistent behavior) including reading, analysis, exploration, planning, implementation, and verification with transitions between segments (ibid). Subsequently, the method of analysis involved inductively deriving the descriptions and explanations of how students proceeded in their proofs. For this analysis, each subject's think-aloud protocols were first coded to determine actions. We then classified each protocol line within a topic segment (i.e., as either analysis, exploration, and so forth), eliminating those lines representing either a reading of a problem or conversation carried on with the interviewer that did not refer to the subject matter (e.g., "So, so, OK, I read it. Do you want me to solve it?" or "Can I use a graphing calculator?"). We then wrote descriptions of the students' actions and categorized the structure of these actions. These descriptions formed the findings of the study described here.

Results

Examining the proof schemes exhibited by students was the first goal of our analysis. We used the framework proposed by Harel and Sowder (1998) to identify evidence as to which of three main proof schemes (external, empirical, and analytical) could be used to classify each of the student solutions. Both the written work and the transcribed oral remarks that students made while solving each problem were used towards this classification. As shown on Table 2, the majority of the study participants' solutions were classified under the empirical proof scheme. Hence, we chose to use only a subset of the students who exhibited empirical proof schemes for subsequent analysis. Additionally, about one third of the study participants were unable to respond to the third task.

	External	Empirical	Analytical	Total
Task 1	12	18	4	34
Task 2	7	27	0	34
Task 3	7	7	7	21

Table 2: Students' proof schemes

The verbal protocols were further analyzed using Schoenfeld's (1985) problem solving perspective. That is, each protocol was portioned into macroscopic segments and our goal was to examine each of these segments as a way to gain insights into students' approach to proof. Our guiding question was with respect to the nature of students' actions (e.g., use of representations, definitions, justifications, or explorations of existing objects). What actions do students of different proof schemes take during the various phases of their proof process and how do these actions impact their proof process?

Following the reading of each problem, all subjects attempted to construct a proof while going through steps of analysis and exploration. Yet, despite this surface similarity, the qualitative analysis of students' protocols revealed differences with respect to the content and intent of these segments. Descriptions of the problem-solving paths followed by subjects with different proof schemes are shown in Figure 1.

External Schemes

Students with external proof schemes often followed a pattern of introducing a segment of analysis following the reading of the problem statement. That is, students made an attempt to understand the problem and to select an appropriate perspective for approaching it. During this time, students tended to introduce a definition of the terms included in the problem (e.g., even number, irrational number), as they remembered reading it in a textbook or from past classroom experiences. There seemed to be no further explanation or discussion of this definition and how it fit either the current problem situation, or how they understood this definition. Hence, students did not attempt to infer any additional information or implications that could be used in the problem and appeared to be reaching an impasse. No new information was introduced and students often appeared to be expecting that the proof should follow from this definition without any further exploration or inquiry. Similarly, students who introduced a symbolic representation of the problem situation or the definition, were not able to use this representation as a way to gain more meaning.

The exploration segment involved, primarily, a ritual manipulation of the symbols, that, once again, did not lead to a proof. In a few cases, students introduced a numerical example of the problem. The main characteristic of this action was the use of this example as a confirmation of the given statement and not as an exploration tool towards an inductive conjecture, or as a way to gain further meaning into the problem. Little attempt was made to link these examples to either the definitions or symbolic representations that were introduced during the analysis segment.

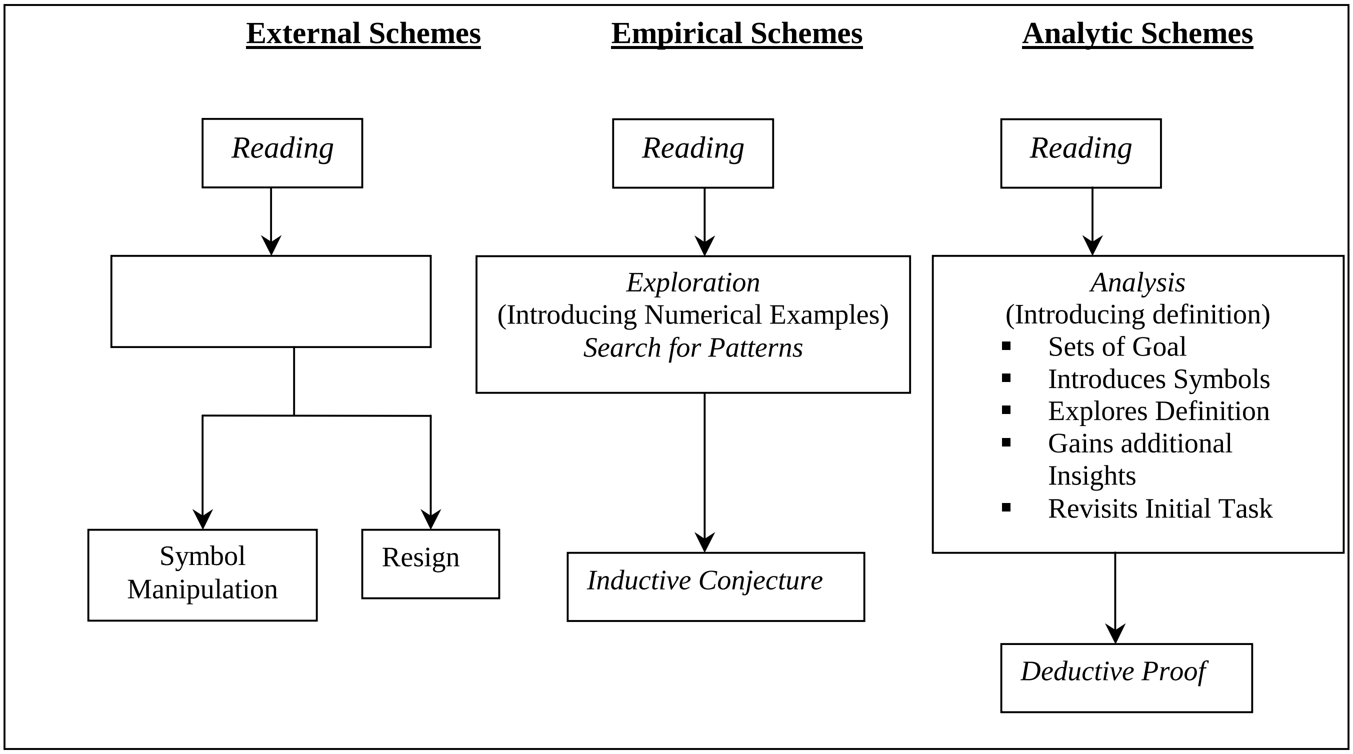

Figure 1: Problem-solving paths

Empirical Schemes

Students with empirical proof schemes differed markedly in their problem solving behavior from students with external conviction proof schemes. Following the reading of the problem, students with empirical proof schemes tended to introduce an exploration segment. During this time they tried several numerical examples until they convinced themselves (and hoped that these would also convince the interviewer) that they had a pattern that would always hold true. Their examples varied, from a structured exploration to random use of numbers, but all searched for a convincing pattern which they used as a basis for their conclusion. Rarely did students in this group introduce definitions or symbolic representations of the problem.

Analytical Schemes

Similar to students with external proof schemes, students with analytical proof schemes tended to introduce a segment of analysis following the reading of the problem statement. During this time, students introduced a relevant definition. Our subsequent analysis, though, suggested that the similarities among the three groups stopped here. Students with analytical proof schemes often produced longer segments of analysis, during which they engaged in four types of activities: (a) set a goal for their subsequent activities (often attempting to decide how they would proceed to use the definition), (b) symbolize the definition, (c) explore the definition and attempt to gain additional insights into the problem situation, and, (d) link the new information to the initial problem.

The important difference we observed between students in this group and students with external conviction schemes is the attempt made by the students with analytical proof schemes to explore their definitions and gain further information that would advance their proof. Further, throughout their proving processes, students with analytical proof schemes tended to keep goals and monitor their actions to ensure that they remained within their goals.

Discussion

To date, very little research has been done to investigate possible relationships among students' proof schemes and their problem solving strategies. In our study, we have identified some patterns that existed among our students. These led to some interesting questions for further investigation: For example, would instruction on problem solving help students advance in their proof schemes (or vice versa)? Do problem-solving strategies inhibit student progress in proving? Investigating these questions would be helpful both for researchers and teachers.

Endnotes

The research reported here was supported in part by the National Science Foundation under Grant #REC-0337703. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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A UNIFIED REPRESENTATION OF FUNCTION IN COLLEGE ALGEBRA: GRAPHS

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The graphing component of the unified representation of function will be presented which utilizes a computational arrow in all three representations of function to emphasize the transformational nature of functions. Examined through the lens of APOS theory, results suggest this approach helps students develop a more accurate action conception which should ease the transition to a process conception of function.

The function concept is often introduced to students in elementary school but still presents a conceptual challenge for some university students at the College Algebra level. This study presents an innovation in teaching functions that unifies the algebraic, graphical, and tabular representations. The graphical aspect of this approach will be presented in this report.

In the terminology of action-process-object, (based on APOS theory), the researchers believe that many students face difficulties when moving from an action concept to a process concept because these students have constructed incorrect or ineffective action concepts of function. The researchers used a theoretical model of the mental constructions used to develop the concept of function in designing this new approach. Since the same notation and techniques are used in all three representations, the arrow provides a unifying tool to aid students' understanding.

The conceptual arrow is used as a computational tool throughout the graphical, algebraic, and tabular representations and emphasizes the transformational nature of functions. By making students utilize this notation directly in working with graphs, tables, and symbols, students are more likely to interiorize the concept of function as a connection between input and output. By helping students develop a more accurate action concept of function, the transition to a process concept should be eased.

Using the traditional families of functions approach found in many texts, students have difficulty interpreting information both when working with a specific representation and when attempting to translate information from one representation to another. Based on a review of the literature and the researchers' experiences with teaching students at this level, the new approach was developed to help bridge the gap between the three traditional representations. The study presents the results from a small pilot study and two semesters of class testing of the materials developed using this new approach to teach the concept of function.

This report will focus on the use of the unified representation of function in graphing. Rather than considering how a function graph undergoing one or more transformations "moves" and/or "changes shape" on the Cartesian plane, this approach focuses on how the axes and their orientation are shifted by the transformation(s). The paper will include student interview responses, and some quantitative data supporting the effectiveness of this work to foster student understanding of function. The initial results from this study are encouraging, and detailed analysis is ongoing of the interview transcripts.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

MANIFOLD NATURE OF LOGARITHMS: NUMBERS, OPERATIONS AND FUNCTIONS

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This study addresses the understanding of logarithms and common difficulties students encounter as they study this topic. The study focuses on different tasks: some standard and others non-standard that involve logarithmic expressions or require the use of logarithms in a solution or explanations. Results indicate students' disposition towards a procedural approach and reliance on rules, rather than on the meaning of concepts.

Objectives

The miraculous powers of modern calculations are due to three inventions: Arabic Notation, Decimal Fractions and Logarithms (Cajori, 1919). However, while the first two of these inventions have been investigated in great detail by researchers in mathematics education, logarithms have received almost no attention. The mathematical concept of a logarithm plays a crucial role in advanced mathematics courses, including calculus, differential equations, number theory, and complex analysis.

This study addresses the understanding of logarithms and common difficulties which high school students encounter as they study this topic. The study focuses on the different tasks: some standard and others non-standard, that involve logarithmic expressions or require the use of logarithms in a solution or explanation. For the purpose of analysis we have modified the interpretive frameworks developed by Confrey in her study of exponents and exponential expressions, to the study of logarithms. The results indicate students' disposition towards a procedural approach and reliance on rules, rather than on the meaning of concepts. The paper concludes with pedagogical considerations.

Theoretical Frameworks

Understanding the concept of a logarithm builds on the relationship between the additive and multiplicative structure of numbers. The importance of logarithms lies in converting a product into a sum and thereby translating a multiplication problem into an addition problem; that is, the computational power of logarithms relies on the relationship between multiplication and addition. In the mathematics education literature, there is abundant research on students' learning and understanding of arithmetic operations of addition and multiplication. However, little attention has been devoted to the connection between the two.

Related to logarithms, Confrey and Smith (1994, 1995) conducted analyses of the concept of exponential function, noting the consistency in the development of students' actions while they learn about this function. To explain students' actions, Smith and Conferey (1994) investigated the historical development of logarithms, as the development of logarithmic function followed the development of exponential function. They suggested that strengthening students' knowledge of multiplicative structures may facilitate their understanding of exponents and logarithms. However, their work didn't follow its natural extension to the investigation of students' understanding of logarithms and logarithmic functions. That is where the present study is

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focused.

Two general theoretical ideas guide presented investigation: mathematical understanding and obstacles (cognitive obstacles and epistemological obstacles). When examine several theories of understanding, the study focuses on the notion explored by Sierspinska (1994), where to understand something means to overcome an obstacle. More specifically, the interpretive frameworks presented by Confrey and Smith (1994, 1995) in their exploration of students' understanding of exponential function were refined and adjusted appropriately for investigation of students' understanding of logarithms. As a result, in this research the following system of three frameworks was developed and used:

- *Framework A: Logarithms and Logarithmic Expressions as Numbers*
- *Framework B: Operational Meaning of Logarithms*
- *Framework C: Logarithms as Functions.*

Methods or Modes of Inquiry / Data Sources or evidence

Participants in this research were students enrolled in the Principles of Mathematics 12 course. Data collection relied on two main sources: written questionnaires and in-class discussions. The written questionnaires were administered to the participants during and upon completion of their study of the unit on logarithms. Data selected for the analysis consisted of a subset of the following six tasks:

1. Simplify the following expression: $\log_3 54 - \log_3 8 + \log_3 4$.
2. Solve: $\log_{12}(3-x) + \log_{12}(2-x) = 1$.
3. Which number is larger 25^{625} or 26^{620}? Explain.
4. Give the domain of the relation $\log_x(y-2) = \log_x(4-x)$.
5. In short essay format, the students were asked to explain to younger schoolmates "What is a Logarithm?"
6. In-class discussion of the question: find the exact value of $5^{\log_3 9}$.

The in-class discussions were intended to probe further ideas and beliefs that participants express in their written responses.

Results, Conclusions

The study provides a system of three interpretive frameworks, which were used to model the students' understanding of logarithms. The results present a description of students' difficulties with logarithms, and also suggest possible explanations of the sources of these difficulties. As such, this study lays a foundation for future research on this topic. It presents students' difficulties as they can be attributed to the conceptual-epistemological obstacle. Among the few implications for teaching practice that were developed in this study, the focus here is on the initial introduction of the concept. In the traditional curriculum the logarithm is introduced and defined as an exponent. However, historically logarithms were developed completely independently from exponents. Emphasis on the historical development could be a beneficial teaching practice. Further research will investigate the feasibility and the benefits of this approach.

Relationship of paper to the goals of PME-NA

Our research provides a better understanding of students' difficulties involved in acquiring an important mathematical concept of a logarithm. It is a novel study on a concept that has not yet received significant attention in mathematics education research. It paves the path to future research and development of pedagogy. In such, it is closely related to the goals of PME-NA.

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FOCUSING ON LEARNERS WITH MATH POTENTIAL AND THEIR TEACHERS THROUGH CURRICULUM, DISCOURSE, AND PROFESSIONAL DEVELOPMENT

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Some would argue that gifted mathematics students are the most neglected group in terms of not meeting their full potential. Project M3: Mentoring Mathematical Minds is a series being developed to encourage gifted students to engage with in-depth and advanced math content and in high-level discourse. Teachers piloting the series received extensive professional development. This session provides an overview of Project M3 and grade 4 student research results.

Overview of Project M3

Project M3, currently in its fifth and final year, is a $3,000,000 national curriculum and research study funded by the U.S. Department of Education Jacob K. Javits Gifted and Talented Students Education Act with the aim of nurturing math talent in grade 3-5 students. Ten urban and suburban schools in Connecticut and Kentucky are participating in the project. Some of the project goals include creating challenging and motivational curriculum units; increasing math achievement; and providing on-going professional development for teachers.

The NCTM (2000) Standards served as a guide for developing the Project M3 series. Each unit focuses on a content standard: algebra, data analysis and probability, geometry or measurement, and number and operations. The process standards, including communication, connections, problem solving, reasoning and proof, and representation, also are emphasized in all units. Exemplary gifted and talented practices that include investigating core concepts, studying concepts in depth, engaging in the complexity of the field, and personifying the characteristics of practicing professionals further reinforce the mathematical content and processes.

The communication standard is a highlight of the series. Teachers facilitate the verbal discourse using talk moves (Chapin, O'Connor, & Anderson, 2003) that include revoice (the teacher restates a student's idea), repeat/rephrase (a student restates another student's idea), agree/disagree and why (judging the mathematical validity of an idea), adding on (extending an idea), and wait time (waiting to call on students and giving a student who has been called on time to respond). Students also write responses to questions from each lesson focused on a core mathematical concept, which requires them to reason and justify their thoughts.

Teachers participated in a two-week in-depth summer training session and five training sessions prior to teaching each unit. Individualized professional development was provided every week to address the mathematics, lesson planning, discourse, and student learning.

Methodology

Students

Multiple means of assessment, including a non-verbal test, a mathematics ability scale, and teacher feedback, were utilized to identify all participants. The intervention group participated in the grade 4 number, algebra, and geometry Project M3 units taught in 2004-05.

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Data Collection

The intervention group completed pre and post Project M3 unit tests. Pre- and post-project data were collected in the fall and spring of 2004-05 from the intervention group and in the spring from the comparison group. The math subsections of the Iowa Tests of Basic Skills (ITBS) and open-response questions based on items from the National Assessment of Educational Progress and the Trends in International Mathematics and Science Study were used.

Data Analysis

Correlated t-tests were performed to determine pre and post changes for the intervention group on Project M3 units, the ITBS, and open-response questions. Independent sample t-tests were performed to determine differences between the intervention and comparison groups.

Research Findings

The research findings indicated highly significant (p < .01) differences between pre- and post-Project M3 unit scores (see Table 1), pre- and post-ITBS scores (see Table 2), and all questions on the pre- and post-open-response tests for the intervention group. As indicated in Table 3, highly significant differences were found on the Concepts and Estimation ITBS subtest and significant differences (p < 0.5) were found on the Problem Solving ITBS subtest between the comparison and intervention groups. No significance was found between scores comparing the comparison and intervention groups on the Computation ITBS subtest. Highly significant differences also were found between both groups on all questions of the open-response test.

Table 1 Comparison of Pre to Post Project M3 Unit Scores for the Intervention Group

Unit	Pre mean	Post mean	Mean difference	t-value	Df	*p*
Number[2]	2.77	10.47	7.70	27.05	185	**
Algebra[b]	2.74	13.71	10.97	38.83	181	**
Geometry[c]	3.56	14.53	10.97	42.43	178	**

[a]n =186. [b]n = 185. [c]n = 179.

**$p < .01$

Table 2 Comparison of Pre to Post ITBS Scores for the Intervention Group (n = 179)

Subtest	Pre mean (SD)	Post mean (SD)	t-value	df	*p*
Concepts and Estimation	210 (20)	226 (20)	14.09	177	**
Problem Solving	215 (23)	232 (26)	11.32	177	**
Computation	188 (19)	202 (18)	9.40	177	**

**$p < .01$

Table 3 Comparison of ITBS Scores Between the Interventiona and Comparisonb Groups

Test	Comparison mean (SD)	Intervention mean (SD)	t-value	df	*P*
ITBS-Concepts and Estimation	214 (21)	226 (21)	5.53	356	**
ITBS-Problem Solving	225 (24)	232 (27)	2.60	353	*
ITBS-Computation	201 (19)	202 (18)	0.67	356	NS

[a]$n = 179$. [b]$n = 180$.
*$p < .05$. **$p < .01$. NS = Not statistically significant.

Discussion and Conclusions

Results indicated that students participating in Project M3 appear to be benefiting from their participation in the study. Even though Project M3 does not focus on computation, there were no statistical differences between the intervention and comparison groups on the ITBS Computation subtest (see Table 3). Further investigation of whether students with math potential engaged with curriculum not focused on operations for most of the year should be addressed.

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CALCULUS STUDENTS' PERCEPTIONS OF THE RELATIONSHIP AMONG THE CONCEPTS OF FUNCTION TRANSFORMATION, FUNCTION COMPOSITION AND FUNCTION INVERSE

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Researchers investigating students' understanding of functions have found that many students have a limited understanding of functions. While much research has been conducted on students' understanding of functions, little attention has been paid to students' understanding of function transformations, function inverse, function composition and how these three concepts are related. In this study, we investigated eight calculus students' flexibility among these three concepts through task-based interviews. We used an object/process view of a function and a flexibility model to analyze the data. The data showed that the participants used varied yet limited approaches to respond to tasks involving function transformations, function inverse, function transformation and the relationship among the three concepts.

Introduction and Background

The concept of a function is fundamental in the learning of mathematics where a good understanding equips a student with more ways of problem solving. In the last twenty years, many researchers have investigated students' understanding (and misunderstanding) of the concept of a function (Leinhardt et al., 1990). These studies have found that many students have a limited understanding of functions (Schroeder et al., 2002). Many students still hold primitive understandings of functions, and firmly rooted misconceptions that have ties to historic views of functions, which described functions as formulaic rules composed of variables. They [students] understand a function to be a formula and thus connect it with actions of substitution (Meel, 2003). Students with this limited view of a function are likely to struggle specifically with processes that involve acting on a function such as transformation of functions, composition of functions, and inverting functions.

These three concepts—transformation of functions, function composition, and function inverse—are related ways of acting on a function. A transformation is any rule that takes every point of the original set and maps it to another point in the same space e.g. any function from R_1 to R_1 is a transformation (e.g., $f(x) = x + 3$). A transformation could also be a mapping from say R_2 to R_3 (e.g., $F(x, y) = (x + y, x - y, 2y)$. Using this definition, any function is a transformation. A composition of two (or more) functions is a function and hence a transformation. Conversely, any function is a composition (at least with the identity function). Thus, transformations and compositions can be used interchangeably. An inverse function is a linear transformation [recall linear transformations have the general form $F(x, y) = (ax + by, cx + dy)$]. If $a = 0, b = 1, c = 1$ and $d = 0$ then the linear transformation defines the inverse of a function. Thus, inverse functions form a subset of transformations/compositions.

In this study, we examined calculus students' flexibility in moving among the concepts of function transformation, function composition, and function inverse. More specifically, we sought to answer the following research questions:

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1. Do calculus students display versatility with regard to moving among the concepts of function transformation, function composition, and function inverse?
2. Do calculus students display adaptability with regard to moving among the concepts of function transformation, function composition, and function inverse?

We only focused on transformations from R_2 to R_2, since that is what calculus students know as transformations.

Methodology and Data Collection

Eight calculus students in beginning calculus courses (Calculus 1 or Engineering Calculus 1) at a two-year college in the northeastern part of the United States were asked to complete a set of tasks during individual task-based interviews, which helped reveal their understandings of function transformation, function composition and function inverse, and their ability to show flexibility amongst these concepts. We audio taped the interviews and transcribed them for analysis. The task-based interviews were conducted to allow us gain insight into each respondent's thinking, versatility and adaptability. We analyzed the participants' responses inductively using a framework of flexibility. We followed Schroeder et al.'s (2002) format in using first a lens of a process/object view of function, and then using the definition of flexibility.

Discussion and Conclusion

The results from this study illustrate that the calculus students in the study used varied, yet limited approaches to respond to tasks involving function transformations, function composition and the inverse of a function. Four major themes emerged in the process of coding: procedural vs. conceptual understanding, flexibility, algebra sophistication/struggle, and ways of convincing self/me. The study did not yield strong evidence indicating that participants have flexibility in their forms of function representation, or function view. On the contrary, the findings pointed to a univalent dependence on the equation representation of a function with some cases in which the participants would not continue without knowing the equation of the given function.

The participants exhibited a strong dependence on a graphing calculator to respond to questions involving function transformations. They mostly used a guess and check (using a graphing calculator) approach. We cannot help to wonder how the study results would differ if the participants were not allowed to use a graphing calculator. The response of a question such as: "Are you convinced?" would take a whole different meaning. Two participants used a transformation argument (reflection over the line $y = x$), to talk about the inverse relationship of two given functions without our initiation. There was no evidence that the participants thought of using a transformations approach to another task in which we asked them to sketch the graph of $(f \circ g)(x)$, We interpreted their ability to correctly identify the transformation that emerged from the composition (and the ability of Russell to identify that the problem would have been "a whole heck easier" if he had used a transformation approach) as an indication that they had some degree of flexibility in their view of function compositions and the transformation of a function. They however showed no versatility in their approach (also evidenced by their dependence on equation representation of a function). It is not surprising that the participants did not think of using a transformations approach since the three concepts of transformation, composition and inverse are often covered independently in pre-calculus textbooks.

Some questions for further study are: How much does the pre-calculus curricula prepare students to have flexibility among the three concepts? What would a review of commonly used textbooks reveal about opportunities given to pre-calculus students to develop their flexibility? What does flexibility among the three concepts add for a student?

STUDENT INTENTIONALITY IN HIGH SCHOOL MATHEMATICS

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This study explored the decisions students make when they select a grade 10 mathematics course and in how they choose to succeed within a mathematics course. Quantitative data was analyzed from a population of 400 students, while interviews conducted with a subset yielded to phenomenographic interpretation. Identity formation was central to students' choice of academic or non-academic mathematics courses, while keeping the credentialing of courses in view. Student intentions within a course generally fell into two categories: studenting and learning. In conversation with students, we discovered a need to distinguish between learning mathematics and learning to learn mathematics.

Because mathematics achievement is used as a critical filter for further learning opportunities, efforts to reform high school mathematics must address the wide variation in achievement among students receiving similar instruction. Attributions of the variation to ability differences or differences in prior learning fail to provide strong starting points for affecting that variation in achievement, because neither can be changed by high school students or their teachers. This paper explores the dimension of student intentionality as an explanatory construct and as a focus of attention for those interested in influencing student success in high school mathematics.

A focus on student intentions in mathematics relates to other aspects of student motivation. The classic distinction between intrinsic and extrinsic motivation (Hidi & Harackiewicz, 2000) offers a starting point, suggesting that the reasons for student actions matter. Bereiter's distinction (1990) between intentional-learning and schoolwork modules suggests that students may engage similarly in assigned tasks, but be motivated differently – to learn or simply to complete the assignment. Skemp's distinction (1987) of relational and instrumental learning addresses differences in students' sense of what it means to learn mathematics. Similarly, Flewelling and Vernay's constructs (2005) of sense-making game and knowledge game, suggests that some students want to know information (facts and skills to answer mathematics questions), while others want to know how and why the facts and skills work and fit together. If there are differences of this kind in high school students' intentions in mathematics, then those differences could help explain the variation in student success. Better still, it could be that students' intentions could be renegotiated in ways that could affect student outcomes.

Research Context and Method

The study occurred in a large urban high school in Manitoba, Canada. Student intentions were pursued in relation to two kinds of decisions: the selection of a grade 10 mathematics course, and the choices the students made to succeed within their courses. School records for all 400 students (demographics and achievement) provided data for whole-population quantitative analysis. As well, a subset of 25 students participated in three on-line surveys and interviews during their grade 10 mathematics course. Students were invited to describe how and why they made the choices they did, not only of which mathematics course they selected but how and why they engaged with the learning opportunities provided by the course they selected.

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Phenomenographic interpretation analyzed the full range of student intentionality portrayed by the data. Students in the school typically take one of five different mathematics courses in grade 10 (after a single mathematics course for all students in grade 9). More than three-quarters of the students take an academic course (in order of increasing value as an academic credential: Applied, Pre-Calculus, Honors). The two non-academic courses are Consumer Mathematics and Grade 9 Repeat Mathematics.

Intentionality of Students in Mathematics

As students made their choices of mathematics course, they were clearly concerned with the power of the different courses as credentials for further study, more than they were with the differences in content or approach among the courses. Yet few students had a focused image of what they were going to do four years later when they finished high school or of what they wanted to do as adults. As a consequence, students did not bother with facts about which mathematics courses were sufficient for which particular courses of post-secondary study. The students clearly saw the choice of courses more as a decision point in their personal decisions about who they were becoming. Students had to decide by making the choice of mathematics course how much they wanted school to be central to their life as adolescents – how they wanted to prioritize among social, recreational, and credentialing goals. They wanted to achieve a fit between their current capabilities and the mathematics course they would take. It was a present-tense matter, not the future-tense credentialing matter that the adults presumed it to be. When viewed in this way, it is possible to make sense of students' reliance on the opinions of friends and the recent experiences of relatives. It also makes sense of many students to discount their limited achievement in grade 9 mathematics in their course selection, to the extent that they saw themselves as being able to develop more mature approaches to mathematics when necessary.

The study made clear that the intentions of students in grade 10 mathematics varied in more ways than just the kind of motivation. The variations extended beyond how much they were willing to work, to reasons for participating in classroom activity, for doing the work assigned, and for preparing for tests. It is clear that some students wanted to understand the mathematics, while others wanted only to know how to do the questions in the practice work that would reappear on the tests. As well, many students expressed their intentions not in terms of learning, knowing or understanding, but in terms of their willingness to do what was expected of them.

For student intentionality to become more central to what mathematics students and teachers address, some working constructs are needed. We found that to make sense of the students' intentions in this study, we must separate intentions for studenting (succeeding as participants in school at achievement, behavior, submission) from learning (intentions for succeeding as learners). As well, to help students perceive both their current approaches and their current skills as learners as something that can be developed, we need to distinguish learning mathematics from learning to learn mathematics. We believe that both the salient details of these distinctions should develop within the discourse of teachers and students, rather than be adopted from the theoretical literature that has guided this work. We anticipate that attending to these distinctions around student intentions can help achieve more meaningful success for all students in high school mathematics.

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THE ε-STRIP ACTIVITY AS AN INSTRUCTIONAL TOOL IN LEARNING LIMITS OF SEQUENCES

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This research explored development of college students' understanding of limits of sequence through a specially designed activity, named the ε-strip activity. This study addresses how the ε-strip activity plays roles in development of students' understanding of limits of sequences.

One of the difficulties broadly experienced in understanding the definition of limits of sequences is caused by conceptualizing the idea of limits by the same order as reading the limit symbol. To be precise, students who start to learn the limit of a sequence are taught the limit of a sequence as a certain value which, as the index goes to infinity, each term of the sequence is approaching or getting close to. It should be noted that, in the ε–N definition of limit, an index number N is properly chosen after the error bound ε is determined. In contrast, such students first choose an index number and next try to determine how close the term corresponding to the index is to the limit value (Courant & Robbins, 1963; Fischbein, 1994; Kidron & Zehavi, 2002; Pinto & Tall, 2002; Roh, 2005). Comparing the process of students' thinking (for a given index N, see what is the corresponding error ε) with that of the ε-N definition (for a given error ε, find the corresponding index N), one see that the order of finding the error bound and the index is reversed. In line with viewpoint, in this study, the process of thinking implied in the ε–N definition of limits is called the *reverse thinking process*; in addition, *reversibility* means the ability to understand such a relation between ε and N.

The research design of this study is in the category of a Soviet-style teaching experiment (Kruteskii, 1969), in which the investigator engages students in instructional activities that also serves tasks to gauge their conceptual understanding. Eleven students in calculus courses at a Midwestern university with a fairly diverse department of mathematics completed a series of 1-hour semi-structured, task-based interviews once a week for 5 weeks. Monotone bounded, unbounded, constant, oscillating convergent, and oscillating divergent sequences were suggested in the interview. The task-based interviews included the ε-strip activity. This study addresses how students' reversibility could be developed through an activity, named the ε-strip activity, described as follows.

The ε-strip Activity

The ε–strip activity was specially designed to foster an environment for students to develop their understanding of the relation between ε and N appeared in the definition of limits of sequences. Each ε–strip was made of translucent paper so that students could observe the graph of a sequence through the ε–strip. In addition, each ε–strip had constant width, and its center was marked with a red line so as to examine limits of sequences with a graphical version of ε–N definition. The following list summarizes the ε–strip activity:

(1) Represent the sequence numerically and then determine its convergence/divergence.
(2) Represent the sequence graphically and then determine its convergence/divergence.
(3) Observe how many points are inside and outside the given ε–strip. Describe distribution of points inside and outside the ε–strip as the value of ε is getting smaller and smaller.

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(4) Evaluate the validity of the following ε–strip definitions A and B:

ε–strip definition A: A certain value L is a limit of a sequence when infinitely many points on the graph of the sequence are covered by any ε–strip as long as the ε–strip covers L.

ε–strip definition B: A certain value L is a limit of a sequence when only finitely many points on the graph of the sequence are NOT covered by any ε–strip as long as the ε–strip covers L.

(5) Compare ε–strip definitions with your own understanding of limits of sequences.

Results

It was found that the ε–strip activity played important roles of an effective learning environment for students to build up their reversibility, which is compatible with the ε–N definition of limits of sequences. First of all, throughout the ε–strip activity, students could not only verbalize such confusion but also gradually modify their conception of limits of sequences. Indeed, various types of sequences used in this study, some of which might be unfamiliar to students, caused their cognitive dissonance, especially when students found that the result from their own conception of limits was different from that of the ε–strip definitions. While students proceeded with the ε–strip activity, they could clearly recognize circumstance of such cognitive dissonance.

Furthermore, the ε–strip activity provided a tool for students to bear in mind appropriate pictorial images to the definition of limits of sequences. Actually, before carrying out the ε–strip activity, students experienced difficulty in explaining why and how a given sequence was convergent. In view of this difficulty, the ε–strip activity was effective for internalization of the concept of limit by transferring it from one mode of representation to another. In particular, by conceptualizing the reverse relation between ε and N in the iconic mode of the ε–strip activity, students could more easily grasp the reverse relation between ε and N in the symbolic mode of the ε–N definition, which is the final mode of representation suggested by Bruner for internalization of mathematical concepts (Bruner, 1960).

It should also be noted that there was improvement in students' reversibility through the ε–strip activities even though there was no procedure for indicating students' errors, correcting their misconceptions about limit, or confirming the propriety of the ε–strip definitions during the activities. In this point of view, the ε–strip activity can be regarded as an effective instructional method in teaching the limit of a sequence.

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WRITING USE AND ITS EFFECTIVENESS ON COLLEGE STUDENTS' MATHEMATICS PERFORMANCE

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The use of student writing activities as a part of mathematical learning has been the focus of much research. Although there are some studies indicating that writing in mathematics has a positive effect on understanding (Countryman, 1992), there are also studies showed that journal writing has no effect on learning mathematics (Croxton and Berger, 2003).

Individual differences have been seen to play an important role in students' successes and failures. Because of that, before implementing a new technique or activity, it would be better to find out the learning styles, which are the individual's characteristic way of processing information, feeling and behaving in learning situations, of students (Andrew, Green, Holley and Pheiffer, 2002).

The main purpose of this study was to answer the following research questions:

Is there a significant difference in the performance scores of students on integral that can be attributed to: (i) treatment, (ii) learning style, and (iii) interaction of treatment and learning style? What are the students' opinions about the journal writing activities, grading and feedback on them?

The study was carried out with 87 first year engineering students in a private university from three classes. Two groups were assigned as experimental groups (EG1 and EG2) and one group was assigned as the control group (CG). Students in all groups received the same instruction on integral. Experimental groups (EG1 and EG2) also engaged in journal writing activities besides lectures. Fourteen journal writings were developed to allow students to communicate their knowledge about mathematics, their thoughts and feelings about the components of the mathematics classroom and their difficulties related with integral.

Two open-ended achievement tests on integral were developed to be used as pre-test and post-test. In addition to thosetests, Kolb's Learning Style Inventory was also administered as pre-test. Follow-up interviews were conducted with ten students from EG1 and EG2 in order to investigate the students' opinions about journal writing activities, grading journal writings and giving feedback. Additionally, classrooms were observed during the treatment.

Although the results of the ANCOVA suggest that neither the groups' achievement nor the achievement of the students having different learning styles in each group differ significantly on integral, there was slightly better improvement in experimental groups according to descriptive statistics. Moreover, the results of the interviews showed that students found journal writing activities as an effective teaching method and wanted to be engaged in the activity for the future.

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LOOKING FOR TOOLS TO FACILITATE ESTIMATIONS: THE LOGARITHMS CASE

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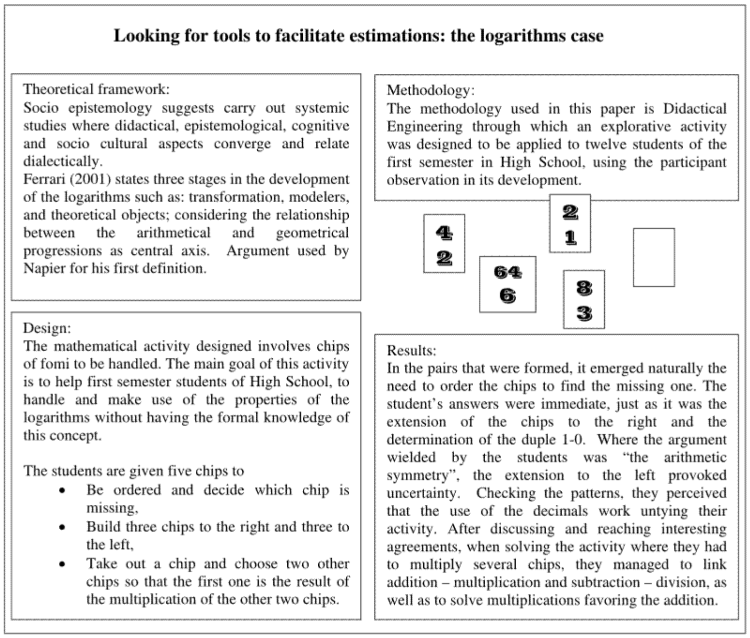

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ONE PROBLEM, TWO CONTEXTS

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When students approach a mathematics problem, in any context, they rely on their past experiences and intuitions in order to build meaning of the problem situation that might help them solve the problem. These past experiences and intuitions could form what we call a context for the problem. Langer (1989, p. 37) describes a context as, "a mindset, a premature cognitive commitment." Thus, a mindset might provide affordances or set up obstacles for thinking about a particular problem. Freudenthal (1991) draws the distinction between what he identifies as rich and poor (abstract) mathematical structures. He also discusses (1973) how it is that students come to understand abstract mathematics. He describes this as "[doing] away with the brackets protecting pure mathematics" (p. 153). Speiser and Walter (2006) suggest that "the construction of one or more presentations of the problem situation, [lead to] solutions … through reasoning based on the way the given presentations have been structured."

The purpose of this project is to analyze the decisions, choices, and reasoning made by a group of students in both a rich context and a poor context. In the rich context, the problem is stated as follows:

> At a party with five married couples, no person shakes hands with his or her spouse. Of the nine people other than the host, no two shake hands with the same number of people. With how many people does the hostess shake hands?

In this context their intuitions, based on significant experiences, allowed them to focus on how they might work toward a solution. In the poorer context, given a few months after the rich problem, the same problem was given but cast in pure set theoretic terms that included cardinalities, cross products, symmetry and reflexivity. In this version of the problem, students had minimal experience with the notation used to describe the problem and hence had very little, if any, intuition about how to proceed toward a solution.

The analysis of the student work in these two contexts shows that the students made use of various presentations (representations) from which they were able to build and reason from structures for, what for them, were two problems. Although the structure of their presentations was very similar in both cases, the students' purposes in building the two structures were very different. In the first context (rich), the conditions of the problem and the question to be answered were seen as clear. Therefore, their work was mainly focused on developing structures that might lead directly to a solution to the problem. For example, when they built structures for presenting the conditions of the problem, they attended to what handshakes might be possible. In contrast, in the second context their guiding purposes were first and foremost to clarify the conditions of the problem. Because they were not familiar with this context, their work mainly focused on the need to construct a context in which the conditions of the problem would have meaning. For example, they chose to build a structure to present the conditions by focusing on what possibilities those conditions eliminated rather than included.

In the end, however, these students recognized that they were working on two versions of the same problem. The analysis shows that even though the structures that they built were essentially the same in both contexts, it was not until the students saw how a solution was emerging from

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the second context that they recognized an identical pattern of thinking to the first context and then established a detailed isomorphism between the two versions.

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THE ROLE OF THE FACILITATOR IN A COMMUNITY OF MATHEMATICAL INQUIRY

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As a discursive form, community of inquiry (CI) pedagogy is distinguished from traditional practice by its multilogical as opposed to monological style and character. Since everyone in the system of CI can exercise control to some degree, and every characteristic of the system--whether social, psychological, logical, conceptual, linguistic or some other—can influence every other, the system undergoes a continual dialectical process of deconstruction and reconstruction (Kennedy, 2005). This identifies it as an open, emergent system, which in turn describes it as a system in continual transition, over which no one can exercise anything but what Lushyn and Kennedy (2000) call "ambiguous control." Thus construed, the process of teaching/learning in a community of inquiry is implicitly understood as a developmental and a dialectical process often marked by uncertainty and lack of clarity, and is associated with the emergence of new forms of knowledge—which, in turn, implies a greater-than-usual degree of predictive uncertainty about the system and the role of the facilitator in this system.

The role of a facilitator in such a system is ambiguous, since she has, if necessary, to encourage the scaffolding process without providing direct answers or authoritative perspectives, —that is, through provocative questioning, reformulation, and the offering of counter-examples and counter-perspectives. The facilitator's role itself may be described as paradoxical, in that she is expected to help direct the inquiry by not actually directing it. On a practical level alone the role of the facilitator in a community of mathematical community of inquiry is far more complex than the traditional teacher's, requiring as it does sensitivity, flexibility and creativity in the organization and planning of content and activities, the courage to take risks and to endure suspense in the facilitation and scaffolding of the inquiry.

Since the ultimate goal of a community of inquiry is in fact increased levels of cognitive, behavioural and motivational self-regulation, the facilitator's long-term objective is the distribution of her own function among the members of the inquiring system, a prospect that tends to subvert the traditional construction of pedagogical power. Her short term goal is to initiate students into the learning task, and to provide the opportunity for the sharing of ownership of the activity, on the assumption that ownership is a sine qua non of success or even survival of the inquiring system. Between the short and the long term goals is a continual process of both incisive intervention and sensitive adaptation. Community of inquiry demands, in brief, a form of practice and reflection, which unites theory and practice, philosophy and application, argumentation and calculation in the concrete, problem-based context of the classroom. Given both the nature of the discipline and the pedagogical traditions which still dominate it, the application of this pedagogical model to mathematics education poses a profound challenge to the teacher in her new role as facilitator of a teaching and learning process which is an emergent and a group-specific one and thus marked by uniqueness and insecurity. But it also offers the promise of the transformation of mathematics teaching and learning from a rigid, transmissional model to one which is student-centered, self-regulatory, and inquiry-driven.

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THE LEARNING OF LINEAR ALGEBRA FROM AN APOS PERSPECTIVE*

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Introduction

This project reflects the development and implementation of an innovative course in Linear Algebra. Dual parallel courses were developed: a Topics in Linear Algebra course and a Learning of Linear Algebra course. The courses were designed to complement one another and are particularly aimed at college students preparing to be high school teachers of mathematics.

Motivation for the Study and Theoretical Framework

Research at the National Center for Research in Teacher Education found that teachers who majored in the subject they taught often were not able to explain fundamental concepts in their discipline more clearly than other teachers. (McDiarmid & Wilson, 1991, p.i). Further research shows that in addition to knowledge of advanced math, effective teachers need math knowledge organized for teaching-deep understanding of the subject; awareness of conceptual barriers to learning; and knowledge of the historical, cultural, and scientific roots of math ideas and techniques (Ma, 1999). The parallel courses developed in this study aim to bridge this gap between the technical “know how” of the mathematics and “know why” need as a foundation for the building of pedagogical content knowledge.

The *Learning of Linear Algebra* course is a constructivist-based education course which employs the Action-Process-Object-Schema (APOS) theory of learning collegiate math. APOS theory represents an extension of Piagetian theories on children’s reflective learning to the realm of higher level abstract mathematics. Analysing mathematics from the APOS standpoint allows for the development of ways of thinking about how abstract mathematics can be assimilated and learned and, therefore, provides a powerful tool for the students in the course to think about what it means to learn, and how that knowledge can inform teaching approaches and strategies. The *Topics in Linear Algebra* course is designed to highlight connections between collegiate linear algebra and secondary math from an advanced standpoint.

Design of the Study, Methods, and Results

The primary data for the study consists of the course materials and syllabi developed before and during the first semester of teaching the parallel courses. Supplemental data includes student work from the courses such as concept maps of key ideas such as vector constructed ay various stages and reflective papers written by the students about their growing understanding of how people learn.

The initial findings of the project are that students benefit from the linking of learning theory with their own learning of algebra. They also show an increased sophistication in their ability to link concepts together and their ability to articulate their own experiences of learning.

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* The research reported in this paper was partially supported by National Science Foundation grant, DUE CCLI 0442574.

ALGEBRAIC THINKING

MULTIPLE VS. NUMERIC APPROACHES TO DEVELOPING FUNCTIONAL UNDERSTANDING THROUGH PATTERNS –AFFORDANCES AND LIMITATIONS FOR GRADE 4 STUDENTS

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In this paper we present different approaches and outcomes of two grade 4 students who were participants in an intervention study for the development of an understanding of functions through patterns. We contrast one student who used multiple representations (visual, narrative, graphic and numeric) to another who relied exclusively on numeric tables in order to reveal the affordances and limitations of these approaches.

Theoretical Framework and Research Project

Patterning problems are presented in a variety of contexts - geometric, tabular, and narrative with the idea that students will gain an understanding of covariational functional relationships (Schliemann et al, 2001; Warren, 2000). However, in practice, most instruction prioritizes the numeric aspect of patterning (Noss et al., 1996; 1997), which Bednarz, Kieran and Lee (1996) note reduces these lessons to data-driven, pattern-spotting activities in which tables of numeric data are constructed and a closed form formula is extracted and checked with only one or two examples. The context and meaning of the variables thus become obscured, which severely limits students' ability to conceptualize the functional relationship between variables, explain and justify the rules that they find (Stacey, 1989), and use the rules in a meaningful way for problem solving. This is a missed opportunity for learning as a number of researchers note that when visual representations are prioritized and students are able to make connections between visual and numeric patterns, they are also more able to find, express and justify functional rules (Healy & Hoyles, 1999).

For the last three years we have been working on a research project with elementary students (e.g. Moss et al., 2005) to address the issues noted in the literature regarding students work with patterns. Based on our theoretical views we have been implementing and assessing experimental interventions in which we provide opportunities for students to integrate their understanding of numeric and visual growing patterns in the context of generalizing problems. (Moss 2005; 1996; Healy & Hoyles, 1999; Moss & Case, 1999).

Research Design

The design of our overall study is modeled on principles of design research (Lesh, 2002) in that the results of each iteration inform future studies. The data sources and general research methods are similar across all studies: Pretests and posttests are conducted, classroom lessons are videotaped and transcribed and artifacts are collected. Finally, detailed interviews are conducted at 3 time periods with six students of high-, medium- and low mathematics achievement. Thus, we are able to capture not only overall changes in student learning, but also to look closely at students' strategies and to track the development of individual students learning.

The impetus for the present study came from trends that were revealed in students' strategies based on written tests and also on interviews. Specifically, we noticed that there were students in

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our studies whose primary approach to problem solving was to reason using numeric strategies regardless of the kind of challenges they were attempting. This numeric orientation was in contrast to that of other students in our experimental classrooms who used the multiple representations that were offered as a context to understand and find rules for generalizing problems. Our instructional approach was specifically designed to focus on connections among multiple representations, with an emphasis on visual representations, in contrast to a more numeric approach, which typifies traditional instruction (Kalchman & Fuson, 2001). Thus we wanted to learn more about the differences in these types of problem solvers.

Methods

Participant

The data for this study come from our Year 2 study of Grade 4 students, and the follow-up Year 3 study of Grade 5 students. The former study conducted over 3 month period in 2005 involved 51 Grade 4 students from 3 classrooms; 2 classrooms in an inner-city public school and a third from a university lab school. The Year 3 study in 2006 involved 57 Grade 5 students from 4 intact classrooms. Of note is the fact that 31 of our 51 participants in Year 2 took part in the Grade 5 study in Year 3.

Instruction

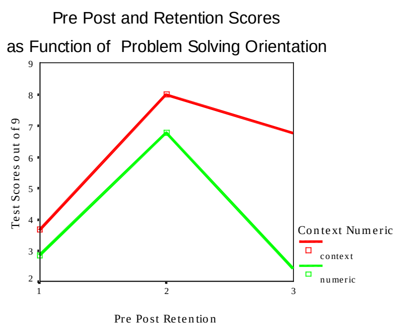
The instructional sequence in all classrooms began with “Guess My Rule” activities (e.g. Willouby, 1997; Carraher and Earnest, 2003) used to introduce students to rule finding for composite linear functions. The next set of activities involved students in building geometric growing patterns using position cards and pattern blocks. Incorporating the position cards served to help students understand the functional relationship between one data set or independent variable (i.e. the position number represented by number cards) and another data set or dependent variable (i.e. the number of blocks used in that position). In these lessons students were asked to think beyond “what comes next” to “what is the [functional] rule for this pattern”. The final component of the lesson sequence involved students in working on a series of word problems, designed to allow students an opportunity to further contextualize their understanding of the two components of a composite function (coefficient and constant)**.** The Grade 5 curriculum also incorporated a graphing component. Students built composite geometric patterns and graphed them in order to gain an understanding of slope (coefficient) and intercept (constant). Activities included creating graphs from patterns, building patterns from graphs, and interpreting graphs in “real world” contexts through story problems.

Data collection and analyses

The data collection and analyses relevant to the present study included a pre and posttest as well as a retention test given 7 months after the posttest. Six students from each class representing different levels of mathematical achievement participated in videotaped interviews that we transcribed and coded. All data were analyzed as a function of students’ demonstrated level of mathematical achievement (based on teacher’s ratings of level of exhibited math achievement and report card marks in math). In addition students were assigned to one of two groups that characterized their approach to problem solving: *Context*, those that used the representational context as the site for problem solving, and *Numeric*, students who consistently created an ordered table of values, identified differences in the dependent variable data set, then used that to determine the composite linear function.

Results

Statistical analyses of pre/post test score means of the experimental group as a whole revealed that students made significant gains. When repeated measure ANOVAs were conducted it was revealed that when the students were divided in three math achievement levels, H=14, M=24, L=18, that there was no interaction of group by time, indicating that the intervention had been successful for students in each of the ability levels. As noted above we were also interested in performance of students identified as *Numeric* or *Context*. When we analyzed gains from pre to post as a function of orientation we discovered that, while there was a slight difference in favor of the context group at both pre and post, these were not significant. It was only when we conducted an additional analysis (including a retention test) with the 31 students who were still available to be part of the research in grade 5 that significant differences appeared. Of these 31 students, 16 were designated as *Context* and 15 *Numeric*. While a multivariate repeated measures ANOVA with three time intervals as repeated factor, (Time 1 Pretest, and 2 and 3 as Posttest and Retention test respectively) confirmed that there was no effect of problem orientation between pre and posttest, this new analysis revealed a strong main effect of context in favour of the *Context* group when the retention scores were added. Figure 1 shows this interaction. It must be emphasized that there were similar proportions of high medium and low achieving students in both the *Context* and *Numeric* groups.



To understand more about these two orientations to problem solving we conducted extensive qualitative analyses to define differentiating profiles of students in each of these two groups. In this paper we elaborate on the different approaches of students in our research program—students who developed the ability to move between visual, narrative, numeric and graphic representations and those who relied almost exclusively on generating numeric function tables—by presenting 2 case studies of students, one from each group.

The Case study of SH and MK

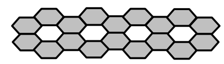
SH (*Numeric*) and MK (*Context*), were students in the same Grade 4 classroom of an inner-city public school. We selected these students for our study as both were rated as high in mathematical ability by their teacher, were competent and keen math students, and took part in Years 2 and 3 of our study. Furthermore, quantitative analyses at the end of the Grade 4 study indicated that both students showed equivalent gain scores on a pre/post test of functional understanding. However, when they were tested 7 months later, there was a significant difference in their retention scores - MK scored 100% correct and SH scored 69%. While the results of the posttest indicated that both students had developed the ability to determine functional rules of problems presented in visual, narrative and numeric contexts, the retention test results indicated that this understanding was less robust for SH, who was unable to answer a number of questions even though the problems on the retention test were similar to the types of problems she had encountered during the previous year's intervention.

SH and the Moonbat Problem Grade 4 lesson 7

As part of the lesson sequence of our study, students were presented with function problems embedded in narrative contexts, which they enjoyed and found appealing. In one problem, the "Moonbat Problem", students were asked to consider the functional relationship between the age and height of fictional moonbats. In this problem, the students were given an unordered series of ages and heights and challenged to discover the underlying functional rule height=age x6+2. SH solved the problem using a "guess and check" strategy. "I think the rule is times six plus two, and I made a t-table to figure out to the moonbat at age 100. So the moonbat would be 602cm." While this procedure allowed her to find and apply the rule, when asked two extension questions, "How tall is a moonbat when it is first born?" and, "How many cm does a moonbat grow each year?", her incorrect answers revealed a limited understanding. "The moonbat is times six plus two when it is born. The moonbat grows times six plus two every year." While SH was able to use the operations of the rule to calculate independent variable values, she did not understand the components of what the relationship meant in terms of the two data sets given. This numeric focus is further revealed in the next example that we present that was part of an interview that she participated in towards the end of the Grade 4 experimental lessons.

Hexagon flowers interview

As mentioned, six targeted students were selected to participate in a series of interviews during the course of the intervention. Below we present excerpts from an interview that one of the authors conducted. As will be seen in the verbatim protocol that follows SH has refined her method of finding a rule but still appeared to have a limited understanding of functions. The interview began with the researcher presenting a card with this picture.



SH: I'm gonna make a t-table. It's easier. [SH constructed an ordered t-table with input numbers from 1 to 4]. I know the rule. It's times four plus two.

I: How do you know?

SH: I counted them – so for 1 (flower) there's 6 (hexagons), and for 2 there's 10, for 3 there's 14, for 4 there's 18, and if you count the differences [pointing to the output column], they're all 4. So, the rule has something to do with the number 4. And if you do 1 times 4 it's 4, but then you have to add 2, so it's times four plus two.

I: Good job. What part of the pattern shows the "plus two" ?

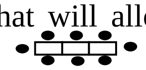
SH: [pointing to successive pairs of hexagons] "There's two, and then two and then two. There's three sets of two, so six altogether. Six or eight, depends if there's another one here."

For SH the constant was not constant, but increased in number for each successive visual representation. Thus we can see that in these first two examples SH has found a useful procedure to find rules, however we can also see that she has difficulty contextualizing her conjectures of rules in terms of the representation in which it is embedded, whether narrative or visual, and has a limited understanding of the nature of composite linear functional relationships.

In the example that follows, from a retention test administered 7 months after the intervention was completed, we can see that the procedures that she relied on were forgotten.

Tables and chairs: Grade 5 pretest (retention test item)

One of the problems that we included on the retention test, the "tables and chairs problem" is well known in the literature on generalizing problems. Students are presented with the first three positions of a pattern and asked to find a functional rule that will allow them to predict the number of chairs that will fit around any number of tables.



Although SH had done many problems of this kind during the Grade 4 intervention, and indeed could solve a more difficult variation of this problem 7 months earlier on the posttest of the Grade 4 intervention, the difficulty she had in attempting this problem in the retention test revealed how tenuous her initial understanding had been. First she attempted to use her procedure of generating an ordered table of values to determine the recursive relationship of the chairs ("plus 2 each time"), but rather than finding the constant through a second differencing strategy, which she had typically done in Grade 4, she instead looked for a numeric pattern and expressed this relationship as "input number plus next input number plus 1." When she attempted to answer the question of why this rule worked, SH was unable to relate her rule to the problem. For the extension questions, SH was unable to use her rule to predict how many chairs could fit around 100 tables. And, when asked to use her rule to figure out the number of tables for a given number of chairs, she was unable to understand the underlying reversal of thinking necessary to derive the independent variable from a given dependent variable.

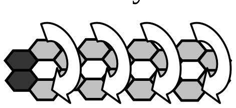
MK a visual contextual reasoner:

SH's numeric approach will be familiar to teachers or researchers who have worked with students on these kinds of generalizing activities. The students whose orientation was more contextual approached these problems differently, as exemplified by the reasoning of MK. In our view, as MK participated in the experimental lessons he appeared to develop a more robust understanding of functions, which was manifest in his fluent use of different representations both for problem solving and as the basis for explanations and justifications for his answers. For instance, on MK's retention test he not only provided an algebraic rule for the tables and chairs problem, but also illustrated the rule by drawing a diagram of the tables and chairs, and labeling the top and bottom chairs as the "multiplicative part" (the coefficient) of the composite rule (x2), and the end chairs as the constant (+2). Furthermore, MK articulated a generalized understanding that he would be able to find the number of chairs needed for any number of tables. He was also able to reverse the operations of the rule in order to calculate the number of tables, (independent variable) when given the number of chairs (dependent variable). Certainly this kind of thinking was in direct contrast to the limited grasp of the problem demonstrated by SH.

MK's problem solving for the narrative moonbat problem (for which students were asked to find the functional relationship between the age and height of moonbats) also revealed this more robust understanding of functions. Like SH, MK was able to find the correct rule ($y = 6x+2$). However, MK was also able to explain the rule in relation to the problem, something that students with a purely numeric approach, such as SH, were unable to do. As he explained. "A moonbat is 2cm tall when it's born and grows 6 cm every year." His explanation clearly showed an understanding of the components of a composite function in a narrative context.

The flower problem and structural similarities

While the two examples above indicate MK's ability to understand functional rules in different contexts (narrative/visual and narrative) the next example of his reasoning demonstrates his ability to discern functions in a purely visual context.



"It's times four plus two. Here it goes like a C, so there was a C and a C and a C and a C [MH used his finger to point out the C shapes left to right] and then these two remaining – and then it worked going this way [MH used his finger to show the C shapes going from right to left], and I knew there were four blocks in each C so it was times four, and there were four Cs so it was the fourth position." MK was able to articulate his multiple ways of seeing the pattern, and connected this to his understanding of position number to generate a rule which he explained using the visual configuration of the pattern. Impressively MK was also able to relate his reasoning to similarly structured patterns he had worked on. "It's like the Toothpick Problem when you start with one toothpick on the end and keep adding a box made of three toothpicks, so times 3 plus 1." These kinds of connections were also typical of many of the other students in this project.

Graphing: Moving to Grade 5

For a final example of the differences in the two approaches to problem solving we present data captured when the students were interviewed at the end of the Grade 5 intervention. Graphs were introduced in the Grade 5 experimental curriculum as a new representation of functions with the anticipation that this visual representation would bring out different kinds of reasoning in the students. However, as will be seen in the following excerpts from the interviews, while MK was able to look at graphs as another representation of a function, SH viewed them as a record of input and output numbers.

In the interviews students were presented with a graph showing the function y=6x+2. Within seconds of seeing the graph MK asserted that it was "times 6 plus 2". "You find out the constant by looking at the zeroth position [y axis] so that's how I found out that the constant was 2. So then I …found out the difference which is 6, so yeah, and it grows by 6 each time so that's the multiplicative [coefficient] so its times 6 plus 2." MK was then easily able to think of a function that would create a parallel slope on the graph. "Times 6 plus 4 because in order to get a parallel line you need to have a different constant but you need to have the same multiplicative to get a parallel line. I mean all you do is start at a different point and then it just goes up by the same….by 6." His use of the terms "start out" and "goes up by" indicates that MK saw the graph not just as a series of static points, but as a representation of linear growth. This was also apparent in his explanation when he was asked to find a function that would produce a slope that intersects with the one given: "Wait, let me try times 2 plus 8. Because, in order to have an intersecting line you need to have a different multiplicative and different constant. Because it starts at a different point and it goes up by a different amount. So it crosses here."

When SH was asked to find the function the graph was showing, she did so by constructing an ordered table of values. While this strategy enabled her to find the function, she could not answer any of the extension questions. When trying to think of a function that would produce a parallel slope, it was apparent that for her the graph was a collection of static points, not a representation of linear growth. Her strategy was to try out combinations of ordered pairs that would yield a series of points that were parallel to those given. She understandably gave up in frustration. When asked how graphs related to patterns, it was clear that SH saw a graph not as a way of representing a generalized rule of linear growth as MK did, but rather saw it as an organization of specific points similar to (but less convenient than) a table of values. "On a t-table it's easier to count the numbers [going to the t-chart she created and pointing quickly across from input to output column with her pencil] like 0, 2 and 1 and 8. Here [touching the graph] you have to go like this [pointing to the last value, 10,on the x axis]. If it's 10 you have to go all the way up [tracing up from the 10 on the x axis] and you have to go like this [tracing over towards the 62 on the y axis] and you might lose your place and you have to go like that and then it's like "oh it's 62" [reaching the value on the y axis]. That takes such a long time. And here [on the t-chart] you can just say 10 [writing in the input column] equals 62 [writing in the output column]. That's a lot easier!"

Discussion

While this paper deals primarily with the reasoning of two students, our goal is to contribute to a growing literature that looks at the potential of multiple representations in function learning. For instance, in their study of children's abilities to solve generalizing problems, Steele and

Johanning (2004) found that students who based their reasoning on the physical structure—the diagrams they drew—rather than only reasoning about numeric patterns were better able to interpret the relationship between quantities in the problem and represent their thinking with symbolic algebraic generalizations. Swafford and Langrall (2000) emphasize the importance of investigating the most effective use of tables and other representations in supporting generalizations. In this study, our findings suggest that it is crucial that students in Grades 4 and 5 be given opportunities to develop an understanding of functions in multiple representations. Children in our study who understood functions in multiple contexts went beyond pattern spotting to an understanding that the rules they found were generalizations of the specific cases given, as was evidenced in their explanations. Furthermore, their explanations also revealed an understanding of how the components of the function related to the functional relationships of the variables given in the problems. An implication of understanding functions through multiple representations concerns the distinction between students perceiving rules as mathematical objects rather than simply as actions (Kieran 1979). Students who were able to use the given context of a problem to find the rule were then able to describe the underlying structure of the given pattern. For these students, the rule e.g. "x6+2" became an object which could then be reflected upon in the multiple contexts in which the children worked, and could be used as a basis for generalizations. In contrast, students who used tables of values interpreted the rule as an action (multiply by six and add two) to be performed on specific numbers in input columns to generate numbers in output columns. In our future iterations of our research we will continue to explore multi representational approached to problem solving.

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SIXTH GRADERS' FIGURAL AND NUMERICAL STRATEGIES FOR GENERALIZING PATTERNS IN ALGEBRA (1)

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This study reports on findings we obtained from pre- and post-interviews of twelve 6th grade students. We address the following questions: What abilities do they have that influence the manner in which they express and justify generalizations in algebra? How, and to what extent, are they capable of extending finite samples of objects in a larger and general context? How do they justify their generalizations? How far are they capable in developing multiple representations of the same pattern and ways to assess them for equivalence? What methods do they employ in situations that involve reverse operations?

Background, Purpose, and Research Questions

From 2000 to 2005, close to 70,000 students in the US Bay area participated in open-ended assessment that involved generalizing linear patterns. Five years of data collection and analysis of 8th grade students' work have shown that while 72% of those tested could successfully deal with particular cases of linear patterns in visual and tabular form, less than 18% of them could use algebra to express correct relationships or to generalize to an explicit, closed formula (Rivera & Becker, 2005). This result is particularly troubling for us because too many 8th grade students complete a middle school mathematics curriculum unable to fully accomplish such a basic task in algebra. That is to say, while students on the surface appear to be computationally proficient on near and far generalization tasks, a closer inspection shows an inability to perform generalization correctly and completely by the gauge of conventional mathematical practices.

Further, there is little evidence-based knowledge in the current research literature base in the USA concerning how middle school children develop their ability to generalize, including reliable mathematical knowledge for teaching generalization that can assist them to succeed in related tasks. As the RAND Mathematics Study Panel (2003) astutely points out, "because most studies have focused on algebra at the high school level, we know little about younger students' learning of algebraic ideas and skills" (p. 48). Being able to successfully generalize is the hard kernel of algebraic reasoning. It is a powerful algebraic "procept" – that is, it can be viewed as both a process and a concept – and suffice it to say is absolutely essential in mathematical modeling, is a desideratum in problem solving, and is an indispensable tool in representing quantitative relationships symbolically.

Thus, in this research report, we take to task the following basic research question: What abilities do 6th grade students have that influence the manner in which they express and justify generalizations in algebra? In particular, we are interested in the following aspects of algebraic generalization that we consider to be appropriate to ask at the middle-grades level: How, and to what extent, are 6th grade students capable of extending a particular finite sample of objects in a larger and general context by way of deriving, inducing, or inferring principles, identifying common features, and expanding domains of validity over large classes of cases? How do they justify their generalizations? Further, how far are they capable in developing multiple representations of the same pattern, including ways to assess them for equivalence? What methods do they employ in situations that involve reverse operations?

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Conceptual Framework

An Operational Theory of Knowledge With Respect to Generalization. Fundamental to our framework for understanding students' abilities to perform generalization in algebra is a theory of knowledge that we have drawn from earlier qualitative studies we conducted with different groups of individuals. In all these investigations, we demonstrated how individuals tend to exhibit two different modes for expressing generality, namely: figural and numerical. To summarize, we have argued elsewhere that:

Those students who are predominantly numerical usually employ trial-and-error and finite differences as strategies for developing closed forms or partially correct recurrence relations with hardly any sense of what the coefficient and the constant in the linear pattern represent. They see variables as mere placeholders and as generators for linear sequences of numbers. Those who are predominantly figural employ visual strategies in which the focus is on identifying invariant relationships from among the figural cues given. For them, variables move beyond their placeholder function as they are interpreted within the context of a functional relationship. (Rivera & Becker, 2005, p. 202, italics added)

A Compatible Theory of Instruction. Considering the above evidence, we then sought out a compatible instructional theory that would allow us to develop classroom teaching experiments in which our participating students could express generality in their own terms. Based on an informal analysis of the different current reform-based middle school mathematics curricula, we decided to use the three introductory algebra units from the Mathematics in Context (MiC) curriculum, namely: Operations, Expressions and Formulas, and Building Formulas. The instructional theory behind the MiC curriculum is Realistic Mathematics Education (RME), and RME foregrounds the Freudenthal standpoint that a truly authentic account of the manner in which children learn generalization begins by assisting them to obtain a model of their own informal activity that would later evolve as their model for more formal processes. RME focuses on how learners' models of their informal mathematical processes could be maximized so as to enable the shift to more formal processes. Further, general mathematical knowledge evolves from a series of horizontal and vertical mathematization activities, and that the starting point of mathematical activity is usually drawn from interesting real life situations or mathematical problems that are experientially real to students. In a MiC unit, students first explore activities that target horizontal mathematization such as schematizing, discovering relations and patterns in order to build an informal mathematical model. Then, they engage in activities that focus on vertical mathematization such as generalizing.

Method (Participants, Design, and Procedure)

We worked with 29 sixth-grade students (12 boys, 17 girls, mean age of 11) in an urban school in Northern California. The students in the class were predominantly Asians (81%), while the remaining students were Hispanics, Caucasians, and African American. At the beginning of the Fall 2005 semester, all 29 students were pre-interviewed on five algebra tasks that involve patterns (see Figure 1 for a sample task). Initial results of the pre-interview, including a departmental pretest that the school implements at the beginning of each year to all students by grade level, became the basis for redesigning topics in the algebra MiC units. We then identified twelve students at differing levels of ability in generalizing whose work we investigated in some detail. Classroom activities included whole group discussions and group work. The second author met with the classroom teacher once each week, and together they developed anticipatory thought experiments that involved envisioning how mathematical activity and classroom communication and interaction might evolve from a target activity in the MiC unit. Two consecutive sequences of classroom teaching experiments were implemented over the course of Fall 2005; each sequence lasted about six weeks and targeted an MiC unit. The class met once daily for five days per week, and each session was 55 minutes long. Every teaching experiment would

Consider the sequence of figures below.

A. How many circles would figure number 10 have in total? Explain.

B. How many circles would figure number 100 have in total? Explain.

C. You are now going to write a message to an imaginary Grade 6 student clearly explaining what s/he must do in order to find out how many circles there are in any given figure of the sequence. Message:

D. Find a formula to calculate the number of circles in the figure number "n."

Figure 1. The Circles Problem (Radford, 2003)

begin with the teacher involving her class in a story problem from the appropriate MiC unit that the class needed to think about and to solve. Group work oftentimes followed a whole-group discussion as students would usually work on additional problems. When this took place, a whole group session was then conducted for closure and to enable the construction of shared strategies among groups. By the closing of the Fall 2005 semester, 11 out of the 12 students we have chosen to study in detail participated in a post-interview of five algebra tasks analogous to the ones given in the pre-interview (see Figure 2). Note that all classroom episodes, including the pre-interviews and post-interviews, have been videotaped. All students' written work have been collected as well.

Results

The results that are reported below have been obtained from the pre- and post-interview data of the twelve students whose work were closely monitored over the course of three months. Note that the tasks in the pre- and the post-interviews were stated in decontextualized form, that is, they looked different from problems that the students have been exposed to in the MiC units. The tasks have been purposefully designed in that manner because we were primarily interested in documenting the students' ability to perform generalization at the level of vertical mathematization. Note that a separate report addresses how the instructional theory of RME assisted in the formation of algebraic generalization among the students.

Preinterview Results

Seeing Patterns Additively. The most frequent mode of establishing growing patterns for all the five problem tasks involves seeing pattern sequences (in figural, numerical, and tabular forms) additively. For example, in the task given in Figure 1, the students knew that succeeding figures after the first involves "adding 2 each time."

Handling Near Generalization Tasks Through Listing and Visualizing. In near generalization cases such as item A in Figure 1, the most common method for obtaining an answer involves listing, that is, extending the last figure number by

Consider the sequence of figures below.

Figure 1 Figure 2 Figure 3 Figure 4

A. How many circles would figure number 10 have in total? Explain.

B. How many circles would figure number 100 have in total? Explain.

C. Find a direct formula for the number of circles in figure number “n”. Explain how you obtained your answer. If the solution has been obtained numerically, respond to the following question: Is there a way to explain your formula from the figures?

D. Can you think of another way of finding a direct formula?

E. Jack’s direct formula is: C = n + (n - 1), where n means figure number and C means total number of circles. Is his formula correct? Why or why not?

Which formula is correct: Jack’s formula or the formula you obtained in (A) above? Explain.

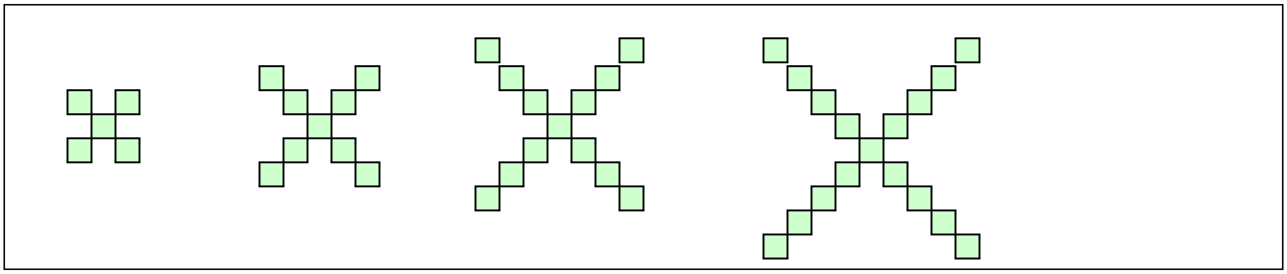
F. Elizabeth has 29 circles that she is going to use to build one of the figures. What figure number is she going to build? Explain.

Figure 2. Modified Circles Problem

listing successive cases until the desired figure number has been reached. For example, in obtaining the number of circles for figure 10 in Figure 1, James said: "Add 2. So figure 4 has 7, figure 5 has 9, figure 6 has 11, figure 7 has 13, figure 8 has 15, figure 9 has 17, and figure 10 has 19." Another near generalization strategy involves visualizing, that is, seeing a visual relationship between figure number and figural cue. For example, Mica thought that figure 8 in Figure 1 above should have 8 circles horizontally and 7 circles vertically. Hence, figure 10 should have 19 circles in all since "there should be 10 circles at the bottom and 9 circles across."

Handling Far Generalization Tasks Visually and Numerically Through Direct Proportion. Far generalization tasks such as item B in Figure 1 were difficult for most students. Those students who saw patterns visually at the beginning stage of a problem successfully obtained correct answers in comparison with those students who transformed the patterns numerically. For example, Mica thought that figure 100 in Figure 1 above should have 199 circles altogether since she saw figure 100 as consisting of 100 circles horizontally and 99 circles vertically. For more complicated growing patterns, some students found it easier to use a multiplicative relationship rather than an additive relationship. For example, in determining figure 30 from the sequence in Figure 3 below, Mica saw that each of four arms should have 30 squares. Hence, figure 30 should have "30 times 4 plus 1 square tile." Those who transformed the patterns numerically oftentimes used a direct proportion strategy. For example, Tabitha thought that if Pattern 10 had

19 circles, then Pattern 100 should have 190: "Since 10 times 10 equals 100, then 19 times 10 equals 190." This strategy was most prevalent in the task when a table of values



Picture 1 **Picture 2** **Picture 3** **Picture 4**

Figure 3. Square Tiles Pattern (Sasman, Olivier, & Linchevski, 1999)

was presented without any accompanying figural cues (see Figure 4). For example, James found that Shape number 20 needed 55 toothpicks, and to obtain the number of toothpicks for Figure 60, he reasoned as follows: "Since 3 times 20 equals 60, so 55 x 3 equals 155." Another strategy involves a difference-to-product method. Dung found the value 48 for figure 20 as follows: "20 minus 8 equals 12, so 12 times 4 equals 48."

Toothpicks are used to build shapes to form a pattern. The table below shows the number of toothpicks used to build a particular shape.

Shape number	1	3	4	5	6	20	60	n
Number of toothpicks	3	11	15	19	23			

Figure 4. Toothpick Problem

Inability to Come Up with Direct Formulas: None of the students were capable of stating a direct formula for any of the sequences.

Postinterview Results

Predisposition Towards Generating Formulas. All eleven students interviewed were successful in establishing general formulas for all the sequences involving linear patterns. In dealing with near and far generalization tasks, at least two students first obtained formulas as a way to compute particular values. In establishing formulas, students employed either a numerical method or a figural method. A numerical method involves listing several dependent values and/or setting up a table, then checking to see if there is a common difference, and finally developing a formula. For example, Tabitha initially set up a vertical table of values for the task in Figure 2 using two variables n and C. Then she observed that there was a common difference of 2, and finally wrote the equation C = n x 2 - 1. She knew the coefficient 2 pertained to the common difference and that the constant -1 was an adjustment value that she needed to add so as to make the output values match with the entries under the C column.

Ability to Justify Generalizations. All the students were capable of justifying their generalizations. For example, for James, the formula F = n x 2 - 1 for the task in Figure 2 means "doubling a row and minusing one chip." For Tabitha, it sufficed that a formula would fit the accompanying table of values.

Inability to Generate an Alternative Generalization. None of the students could come up with other ways of expressing generality for any problem tasks (see item D, Figure 2).

Assessing for and Justifying Equivalence. The most common method used in evaluating for equivalence of one or several direct formulas involves substituting particular cases and checking to see if the computed dependent values matched the original values. For example, in item E of Figure 2, James verified Jack's formula in the case when $n = 2$.

Dealing with Situations Involving Reverse Operations. Item F of Figure 2 asks students to determine an input value from an output value. At least three students extended the table and stopped as soon as the output value has been obtained. As an additional step, these students then used the corresponding formula to check for correctness. Tabitha, employed a guess-and-check strategy: First, she divided 29 by 2 and obtained 14 with a remainder of 1. Then she used Jack's formula to test two cases of n (14 and 15) and concluded the answer must be figure 15. Mario's strategy involved estimation: "How can I get something when multiplied by 2 is close to 29? I know 15 + 15 equals 30. So 15 times 2 minus 1 equals 29." A fourth strategy involves an inverse strategy. Dung said: "There is a minus 1 at the end. So 29 + 1 = 30. Then 30 divided by 2 equals 15."

Discussion

Students employed either numerical or figural strategies in establishing generalizations. In the preinterviews, at least six students operated figurally and generated more factual than contextual generalizations. None of the 12 operated at the symbolic level. In the postinterviews, 10 of the 11 students operated numerically, and all 11 of them generated correct symbolic generalizations. Further, a student who was strictly figural in both pre- and post-interviews justified his generalizations visually from the available figural cues and did not see the need to use tables. Also, 4 students who were figural in the preinterview but switched to numerical in the postinterview justified their symbolic generalizations in the postinterview figurally. Students who operated numerically in both pre- and post-interviews could not justify their symbolic generalizations beyond substitution and checking.

Those students who employed figural strategies in the pre-interview favored numerical strategies in the post-interview. The allure of numerical strategies has to deal with the fact that they appear to be very convenient and systematic. In addition, the additive relationship (for example, "adding x each time") was always stated at first in the process of generating a formula. However, the students transformed the additive relation to a multiplicative one at the symbolic level ("multiplying by x is like adding x lots of times").

We find it interesting that at least 6 of the 11 students in the postinterview preferred to set up a general formula for a problem task before dealing with near and far generalization cases. These students said that "it's easier [to do] that way [first]." Also, variable fluency at the level of symbolic generalization is necessary in order to express a general formula completely and correctly. All 11 students in the postinterview were successful in expressing their formulas using two variables.

While all 11 students were successful in assessing the equivalence of two or more generalizations for the same pattern, they were not successful in generating their own equivalent generalizations. Further, it seems that successfully justifying equivalent symbolic generalizations for the same formula depends on whether a generalization is constructive or deconstructive. Jack's formula (see item F on Figure 2) is an example of a constructive generality that involves seeing terms in an algebraic formula as representing the non-overlapping parts in a figural cue. A formula that involves deconstructive generality consists of terms that refer to the overlapping parts in a figural cue. For example, in Figure 3, the formula T = (n x 2) + 1 + (n x 2) + 1 - 1, where T represents total number of squares and n means Picture number, involves seeing two odd-numbered diagonals that share a center square. Ten out of 11 students in the postinterview could justify equivalent constructive generalities, however, none of them could explain equivalent deconstructive generalities.

Endnotes

1. This paper is based upon work supported by the National Science Foundation under Grant No. REC-0448649 awarded to the second author.

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CHARACTERIZING STUDENTS' THINKING: ALGEBRAIC, INEQUALITIES AND EQUATIONS

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This paper presents the findings of a study that explores the viability of using students' act of anticipating as a means to characterize the way students think while solving problems in algebra. Two types of anticipating acts were identified: predicting a result and foreseeing an action. These acts were characterized using Harel's framework, which involves the concepts of mental act, way of understanding, and way of thinking. Categories for characterizing acts of predicting and foreseeing were identified and developed based on thirteen 11th graders' responses to problems involving algebraic inequalities and equations. The quality of students' acts of predicting and foreseeing was found to be related to the quality of their interpretations of inequalities and equations.

Upon seeing a problem, students commonly rush into action without analyzing the problem situation. As teachers, we witness some students' inappropriate use of procedures, what Fischbein and Barash (1993) call improper application of algorithmic models. On the other hand, we also witness engagement in exploration and analysis of the problem situation among certain students. This research seeks to characterize the differences in the manner students solve problems involving algebraic inequalities and equations.

Theoretical Framework

This research combines multiple perspectives: Piaget's (1967/1971) notion of anticipation, von Glasersfeld's (1998) three general kinds of anticipation, Harel's (2001, in press) notions of way of understanding and way of thinking, and Cobb's (1985) hierarchical levels of anticipation. According to Piaget (1967/1971), anticipation is one of the two functions of knowing; the other function being conservation-of-information, an instrument of which is a scheme. The anticipation function deals with the application of a scheme to a new situation. It allows us to have foresights, strategize and plan, make predictions, formulate conjectures, engage in thought experiments, etc. Such foresights and predictions are possible because of our ability to assimilate situations into our existing scheme(s); "anticipation is nothing other than a transfer or application of the scheme … to a new situation before it actually happens" (p. 195). A scheme, as outlined by von Glasersfeld (1995), involves three components: *the perceived situation*, *the activity*, and *the expected result*. The expected result component provides the anticipatory feature of a scheme. This component constitutes the fundamental difference between a Piagetian scheme and a condition-action pair in information processing or a stimulus-response association in behaviorism.

Von Glasersfeld (1998) elaborated on Piaget's notion of anticipation by pointing to three general kinds of anticipation: (a) implicit expectations that are present in our actions, e.g., the preparation and control of our movements when we grope in the dark; (b) prediction of an outcome, e.g., predicting that it will soon rain upon noticing that the sky is being covered by dark clouds; and (c) foresight of a desired event and the means for attaining it, e.g., a child's anticipation of the capitulation of his parent if he were to throw a temper tantrum in public. In

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my attempt to adapt these three kinds of anticipation to the context of solving problems in mathematics, I was not able to infer students' implicit anticipation from their statements and actions. I therefore focused on students' prediction of a result and foresight of an action. *Predicting* is defined as the act of conceiving an expectation for the result of an event without actually performing the operations associated with the event. *Foreseeing* is defined as the act of conceiving an expectation that leads to an action, prior to performing the operations associated with the action.

Harel's (2006, in press) MA-WoU-WoT framework is suitable for analyzing students' *mental acts* (MA_s) of predicting and foreseeing. Predicting and foreseeing are among the many mental acts that one might carry out when one solves a mathematics problem. Other mental acts include interpreting, symbolizing, transforming, generalizing, justifying, inferring, etc. *Way of understanding* (WoU) refers to the product of a particular mental act and *way of thinking* (WoT) refers to a character of the mental act. For example, in the act of proving WoU refers to the proof a student produces and WoT refers to the *proof scheme* that underlies the student's act of proving. Harel and Sowder (1998) have developed a taxonomy of students' proof schemes, examples of which are *authoritative proof scheme* (one derives conviction mainly from the authority of the teacher or textbook), *empirical proof scheme* (one derives conviction from empirical evidence or visual perceptions), and *deductive proof scheme* (one derives conviction based on the application of rules of logic). Similarly, for the act of predicting (foreseeing), WoU refers to the result (action) a student actually predicts (foresees), whereas WoT characterizes the manner in which the student predicts (foresees).

Cobb (1985) identifies three hierarchical levels of anticipation: beliefs, problem-solving heuristics, and conceptual structures. At the most specific level, one's expressed conceptual structure (i.e., evoked scheme) dictates one's anticipation. An expressed conceptual structure can be viewed as a WoU associated with the mental act of interpreting. In the domain of algebraic inequalities and equations, the dependence of anticipations on conceptual structures suggests a relationship between students' ways of understanding (W_soU) inequalities and equations and their ways of thinking (W_soT) associated with the mental act of anticipating.

The research, part of which this paper reports, has three objectives: (a) to categorize students' W_soT associated with the mental acts of predicting and foreseeing, (b) to identify the relationship between these W_soT and their W_soU algebraic inequalities/equations, and (c) to explore the potential for advancing students' W_soT through a short-term instructional intervention. Figure 1 provides a schematic representation of the framework for analyzing students' act of problem-solving in terms of mental acts of predicting, foreseeing, and interpreting. Each dotted curve refers to the relationship between students' W_soT associated with anticipating (predicting or foreseeing) and their interpretations of inequalities/equations.

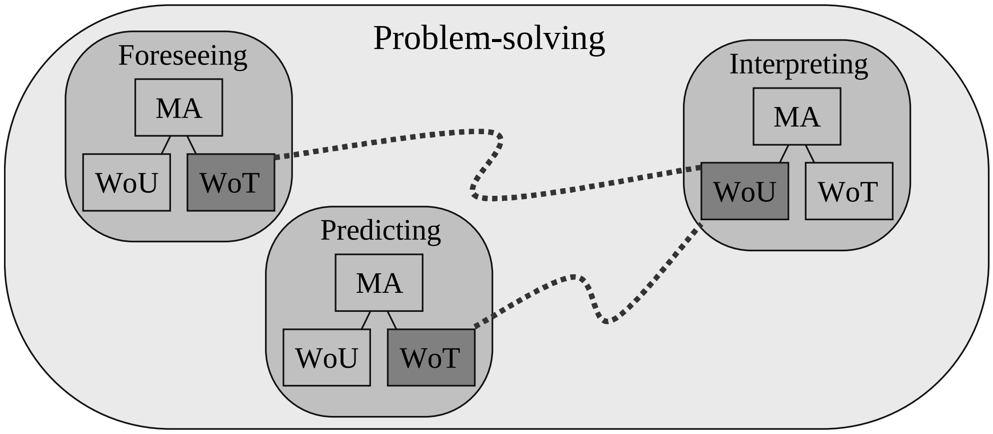

Figure 1: A schematic representation of the framework used in this research method

Fourteen 11th graders were interviewed, each for approximately 60 minutes. Four of these interviewees participated in a one-on-one teaching intervention, which was comprised of five problem-solving sessions followed by a post-interview. This research was conducted in a university-based charter school in Southern California. This school practices detracking: there is only one track for all the students; but different students at a particular grade level may be at different stages along the track. 4 interviewees were taking Algebra II, 4 were taking Pre-calculus, and 6 were taking Calculus. This distribution allowed me to observe a greater variety in students' WsoT.

The purpose of this research was to study 11th graders' thinking as they solved non-routine problems involving Algebra I concepts. Tasks used in the clinical interviews include: (a) Is there a value for x that will make $(2x - 6)(x - 3) < 0$ true? (b) Given that $5a = b + 5$, which is larger: a or b? And (c) p and q are odd integers between 20 and 50. For these values, is $5p - q > 2p + 15$ always true, sometimes true or never true?

These tasks differ from typical tasks in textbooks in that they do not direct students to perform a specific task such as "solve for x" or "simplify." This non-directive feature is found to be effective at eliciting a greater variety of anticipatory behaviors. All the tasks were phrased in the form of a question to allow students to predict the answer, if they chose to, prior to performing any actions.

All the interviews and problem-solving sessions were videotaped and transcribed. One interview was discarded because the interviewee was struggling with her arithmetic. Observation concepts (Clement, 2000) for students' WsoT associated with predicting and foreseeing and students' WsoU inequalities/equations were identified. These categories were derived from the data using a constant comparative approach (Glaser & Strauss, 1967), in which categories were constantly revised by comparing current data with previously analyzed data. The analysis involved identifying instances of the mental acts of predicting and foreseeing (inferred from student's actions and statements), generating, comparing, and refining categories for WsoU and WsoT, and consolidating and collapsing some of the categories. The consolidated categories were revised and refined in light of new information generated in subsequent phases of the analysis (e.g., analysis to account for learners' improvement).

Results and Discussion

This paper reports the research findings for the first two objectives. Excerpts of two interviewees' work are presented to illustrate the viability of using students' acts of predicting/foreseeing as a means to characterize students' thinking. Categories for ways of thinking (WsoT) associated with predicting/foreseeing will be discussed. The same excerpts are also used to highlight the relations between WsoT associated with predicting/foreseeing and ways of understanding (WsoU) inequalities/equations.

Contrasting Two Students' Work

Consider two interviewees' response to this item: "Is there a value for x that will make the following statement true? (6x – 8 – 15x) + 12 > (6x – 8 – 15x) + 6". Both interviewees, Talia and Pham, were 11th graders enrolled in Calculus.

Excerpt 1: Talia's initial response

Talia: Is there a value for x that will make the following statement true? Of course there is. Let see, umm.

Lim: Why did you say "of course, there is"?

Talia: Because, well, I figure there should be an answer to this problem, and, um, let's see, I was taught to combine like terms. I was taught this (>) is actually an equal sign.

Lim: OK.

Talia: To solve it like I would solve an equation. … (She obtained -9x + 6 = -9x and then wrote 6 > 0.) Umm, that doesn't [seem] right, because x has canceled out. What did I do wrong? … OK. Is there a value for x that will make the following statement true? Maybe there isn't.

Excerpt 2: Pham's initial response

Pham: OK. Let's see. The stuffs in the parentheses are the same. Umm, OK, first I guess I would combine all like terms. … (He got $-9x + 4 > -9x - 2$). Umm, now it's asking is there a value for x that will make the following statement true. Umm, let me see, I think 4 and -2, so you have a common term (i.e. $-9x$). OK, so it's, you have a -9, so anything [positive] that you multiply will [make it] a negative number, and this (+4) is positive. Let's see, yes, there is a value because… this, this [left] side will be greater. I guess, if it ($-9x$) was positive then, so is this side ($-9x$). So any negative number would make the statement true. … Umm, I think all numbers would make the statement true.

One difference between these two responses is that Pham arrived at the correct answer but Talia did not. Another difference is that Talia's WoU inequalities is weaker than Pham's. Talia interpreted the inequality as a signal to isolate x and treated it as an "equation," whereas Pham treated the inequality as a comparison of two algebraic expressions. A third difference is the manner in which they approach the problem. How can we characterize the thinking that underlies the actions they took to solve this problem?

Categories for Ways of Thinking Associated with Predicting/Foreseeing

Both Talia and Pham combined like terms. From a Piagetian perspective, action presupposes anticipation. So we can assume that Talia and Pham had anticipated the expediency of combining like terms. Since a WoU associated with foreseeing refers to the action one actually anticipates, both Talia and Pham are said to have the same WoU: combining like terms. Both of them were spontaneous in their foresight of combining like terms. However, the spontaneity in Talia's anticipation was characteristically different from that in Pham's. Upon seeing the problem, Talia immediately thought of what she could do to the inequality, rather than thinking about what the question was asking. Her act of anticipating had an element of impulsiveness, impulsive in the sense that she had routinized a particular WoU (i.e., combining like terms is a routine for her to solve certain inequalities/equations). I categorized her WoT associated with foreseeing as impulsive anticipation. This WoT is generally inferred when a student immediately applies a procedure without considering its appropriateness.

Pham, on the other hand, noticed that “the stuffs in the parentheses are the same” and combined like terms with the probable intent of obtaining a simpler form. He might have predicted in his mind that the left side was always larger than the right side and was confirming his prediction. He seemed to have interiorized the usefulness of combining like terms and was capitalizing on his understanding that it would be easier to reason with simpler expressions. Thus his WoT was coded as interiorized anticipation. By “interiorized”, I mean one has not only internalized (i.e. gained the ability to autonomously and spontaneously apply one’s WoU to another similar situation) a particular WoU but has also reorganized and abstracted the WoU to a higher level of understanding.

With respect to the mental act of predicting, Talia predicted “of course there is” upon seeing the problem. She seemed to have associated her having a procedure for isolating x with the inequality having a solution. Because of this, I categorized her WoT characterizing her prediction as associated-based prediction. This WoT is inferred when a student predicts by merely associating two ideas without establishing the basis for making such an association. Talia’s prediction of “maybe there isn’t” upon observing the disappearance of x from the inequality is also considered association-based because she associated the disappearance of x with the nonexistence of a value for x that would make the inequality true.

Pham, on the other hand, did not explicitly make a prediction. Instead, he reasoned with $9x + 4 > 9x - 2$. His WoT associated with foreseeing is considered analytic anticipation because he identified the goal of determining whether there is a value of x that will make the new inequality true, and foresaw the usefulness of reasoning with the common term -9x.

So far, I have introduced four WsoT associated with foreseeing/predicting: impulsive anticipation, interiorized anticipation, analytic anticipation, and association-based prediction. A total of five WsoT associated with foreseeing and three WsoT associated with predicting emerged from the data. Descriptions for these WsoT are presented in Table 1. These WsoT are elaborated in my doctoral dissertation (Lim, 2006). Relations between students’ WsoT and the quality of their solutions are also discussed in that manuscript.

Ways of Thinking	Descriptions
Impulsive anticipation	Spontaneously proceeds with an action that comes to mind without analyzing the problem situation and without considering the relevance of the anticipated action to the problem situation
Tenacious anticipation	Maintains and does not re-evaluate one’s way of understanding (prediction, problem-solving approach, claim, or conclusion) of the problem situation in light of new information
Explorative anticipation	Explores an idea to gain a better understanding of the problem situation
Analytic Anticipation	Analyzes the problem situation and establishes a goal or a criterion to guide one’s actions

Interiorized anticipation	Spontaneously proceeds with an action without having to analyze the problem situation because one has interiorized the relevance of the anticipated action to the situation at hand
Association-based prediction	Predicts by associating two ideas without establishing the basis for making such an association
Comparison-based prediction	Predicts by comparing two elements or situations in a static manner
Coordination-based prediction	Predicts by coordinating quantities or attending to relationships among quantities

Table 1. Categories for WsoT associated with foreseeing and predicting

Relations between WsoT associated with Anticipating and WsoU Inequalities/Equations

In Excerpt 1, Talia seemed to interpret the inequality as a signal to isolate x and treated it as an equation within which she could manipulate symbols. Accordingly, her WoU was coded as inequality/equation-as-a-signal-for-a-procedure interpretation. This interpretation is inferred when a student treats the inequality/equation (I/E) as an object to be worked on and does not appear to have other WsoU. Her impulsive anticipation and association-based prediction appeared to be a consequence of her interpreting the inequality as a signal to do something.

In Excerpt 2, Pham's reasoning with -9x suggests that he was interpreting it as a function whose output depends on the input variable x. Thus his WoU was coded as I/E-as-a-comparison-of-functions interpretation. His analytic anticipation of reasoning with the common term -9x was supported by this WoU.

Three additional WsoU inequalities/equations (I/E) emerged from the data: I/E-as-a-static-comparison interpretation, I/E-as-a-proposition interpretation, and I/E-as-a-constraint interpretation. In general, less sophisticated WsoU were found to be related to less desirable WsoT associated with predicting/foreseeing. For example, the I/E-as-a-signal-for-a-procedure interpretation tends to lead to impulsive anticipation. Conversely, more sophisticated WsoU are related to more desirable WsoT associated with predicting/foreseeing. For example, the I/E-as-a-constraint interpretation facilitates goal-oriented reasoning, which is an attribute of analytic anticipation. As depicted in Table 2 (the entries are based on interviewees' responses to two interview items), 9 of the 13 interviewees exhibited analytic anticipation while interpreting an inequality as a constraint. This table demonstrates that significantly more students exhibited a more desirable WoT when they showed a more sophisticated WoU, and most of those who exhibited a less desirable WoT showed a less sophisticated WoU.

		More Sophisticated WoU		**Less Sophisticated WoU**	
		I/E-as-a-Comparison-of-Functions	I/E-as-a-Constraint	I/E-as-a-Static-Comparison	I/E-as-a-Signal-for-a-Procedure
More Desirable WoT	Interiorized Anticipation	4	3		
	Analytic Anticipation	2	9	1	1
	Coordination Prediction	2	3	2	
Less Desirable WoT	Association Prediction			3	1
	Tenacious Anticipation		1	1	
	Impulsive Anticipation				2

Table 2. Number of interviewees exhibiting a particular WoT and a particular WoU

Conclusion

One objective of this research was to develop categories for ways of thinking associated with the mental acts of foreseeing and predicting. Ways of thinking associated with foreseeing provide mathematics educators with the vocabulary to communicate the way students approach a problem: whether they (a) hastily apply a procedure, (b) are tenacious in their way of understanding, (c) explore different ideas, (d) analyze the problem situation and identify a goal, and (e) spontaneously apply their ways of understanding that are pertinent to the problem situation. These descriptions correspond to impulsive anticipation, tenacious anticipation, explorative anticipation, analytic anticipation, and interiorized anticipation. An awareness of these categories can help mathematics teachers to be more explicit about their goal of advancing students from being impulsive and tenacious to being analytic and explorative.

Instruction that leads students to predict can help counteract students' tendency of rushing to apply procedures when they are assigned a problem. Having explicit terms to characterize the ways students predict allows teachers to differentiate desirable ways of thinking associated with predicting from less desirable ones. For example, coordination-based prediction is desirable because it promotes reasoning that involves change and coordination whereas association-based prediction is undesirable because it tends to foster the non-referential symbolic way of thinking. Having made these distinctions explicit, mathematics educators can design and implement instructional activities that aim to help students progress from association-based prediction to coordination-based prediction.

The relationship between the desirability of students' W$_s$oT associated with predicting/foreseeing and the sophistication in their W$_s$oU inequalities/equations suggests that we, as teachers, should attend to students' W$_s$oT associated with predicting/foreseeing while helping students to develop sophisticated W$_s$oU inequalities/equations, and vice versa. This recommendation is in keeping with Harel's (2006) call to incorporate desirable W$_s$oT and sophisticated W$_s$oU as cognitive objectives for instruction: "In designing, developing, and implementing mathematics curricula, ways of thinking and ways of understanding must be the ultimate cognitive objectives, and they must be addressed simultaneously, for each affects the other."

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RELATIONSHIPS BETWEEN ATTENTION-FOCUSING AND THE "TRANSFER" OF LEARNING ACROSS TWO INSTRUCTIONAL APPROACHES TO RATES OF CHANGE

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This paper demonstrates how differences in the nature of students' generalizations of their learning experiences are related to differences in features of the classroom environment that regularly direct students' attention toward certain mathematical properties when a variety of features compete for students' attention.

Transfer is a controversial construct, which faces a number of theoretical and methodological challenges. Numerous critiques of transfer (e.g., see Lave, 1988) have contributed to a growing acknowledgment that "there is little agreement in the scholarly community about the nature of transfer, the extent to which it occurs, and the nature of its underlying mechanisms" (Barnett & Ceci, 2002, p. 612). Several alternative models of transfer have emerged in response to critiques of the classical transfer approach (see Bransford & Schwarz, 1999; Greeno, Smith, & Moore, 1993; and Lobato, 2003). As these emergent perspectives mature, they need to move from interpretative frameworks to the development of theory, including theory about transfer mechanisms.

The classical transfer approach refers to the family of *common elements* theories that have dominated the 20th century, starting with Thorndike's (1906) "identical elements" and more currently the cognitive instantiation of Thorndike's approach (see for example, Anderson, Corbett, Koedinger, & Pelletier, 1995). In the classical approach, transfer mechanisms are typically conceived as identical elements, either common physical features of the environment for Thorndike (1906) or overlapping abstract symbolic mental representations for information-processing theorists. In classical models, transfer mechanisms are factors that can be controlled in order to produce transfer. In contrast, Lobato (2006) argued that we need a notion of mechanism that refers to an explanation of how social environments afford and constrain the generalization of learning, and thus shifts the focus from external factors that can be controlled to conceiving of transfer as a constrained socially situated phenomenon. To this end, Lobato, Ellis, and Muñoz (2003) have advanced the notion of *focusing phenomena* to link features of instructional environments with the particular ways in which individuals generalize their learning experiences.

Theoretical Framework

Focusing phenomena are features of the classroom environment that regularly direct students' attention toward certain (mathematical) properties or patterns when a variety of features compete for students' attention. Drawing upon the distinction between three types of attention (as described by Fan, McCandliss, Sommer, Raz, and Posner, 2002), focusing phenomena can be described as more than becoming *alert* or *orienting* to a task, but rather as involving *executive attention* (i.e., the biasing focus on one of several conflicting sources of information). Focusing phenomena emerge not only through the instructor's behavior but also through co-constructed

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mathematical language, features of the curricular materials, and the use of artifacts. The resulting mathematical object of focus and what students notice mathematically, are co-constituted through focusing phenomena and students' prior knowledge, experiences, and goals.

This study is grounded in the *actor-oriented transfer perspective* (Lobato, in press-b). In this approach, transfer is treated broadly as the influence of a learner's prior activities on activity in novel situations, which entails any ways in which learning generalizes. While the actor-oriented approach has focused on "similarity-making" in the generalizing process, the roles of discerning differences and modifying situations have also been analyzed (Lobato & Siebert, 2002). Taking an actor-oriented approach often reveals idiosyncratic ways in which learners generalize their learning experiences. At first these idiosyncratic forms of transfer may seem random. However, the work on focusing phenomena is demonstrating a basis by which actor-oriented transfer is constrained.

Purpose

This paper compares the focusing phenomena and associated student generalizations across two instructional environments: a high school classroom (Study 1) and a follow-up teaching experiment involving a sample of eight students from the Study 1 classroom (Study 2). The (new and unpublished) results from the analysis of Study 2 are compared with the previously published results of Study 1. The instruction in both studies dealt with the same mathematical topic of rates of change.

Methods

In both studies, data collection methods included the use of traditional transfer tasks in clinical interviews. However, once the interview data were collected, the researchers set aside their expert frame of reference and took on an actor oriented perspective in order to determine the generalizations that the students formed. Analysis of the interview data involved the application of the interpretive techniques in which the categories of meaning were induced (Strauss & Corbin, 1990). Analysis of the videotaped instruction was limited to the ways in which the instructional environment directed students' attention toward certain mathematical properties over others. Analysis drew on the constant comparative method used in the development of grounded theory (Glaser & Strauss, 1967). For more details regarding the specific adaptation of these grounded theory methods developed to identify transfer from an actor oriented perspective, see Lobato (in press-a).

Results

Study 1

In Study 1, the construct of focusing phenomena emerged from an empirical study conducted during a 5-week unit on linear functions in a high school mathematics classroom using a reform curriculum (Lobato, Ellis, & Muñoz, 2003). Qualitative evidence revealed that all seven interview participants formed generalizations about slope in which the m value in $y = \mathrm{b} + \mathrm{m}x$ was treated as a difference rather than a ratio. Classroom analysis revealed four focusing phenomena, which regularly (and unwittingly) directed students' attention to various sets of differences rather than to the coordination of quantities: (a) ambiguous "goes up by" language, (b) the use of uniformally-ordered data tables, (c) the ways in which graphing calculators were used, and (d) an emphasis on uncoordinated sequences and differences. The identification of the particular ways in which the classroom practices afforded students' generalizations suggested principled ways in which we could make design responses.

Study 2

Study 2 entailed a twenty-five hour teaching experiment with eight students drawn from the Study 1 classroom. Results demonstrate improved performance on transfer tasks, including evidence of the comprehension of co-varying quantities in rate situations rather than a focus on differences within a single quantity. Due to space limitations, a case study of a pair of 9th grade students, Carissa and Bonita, is presented in this report. Bonita struggled through many the sessions and eventually dropped out of the study. Consequently this analysis focuses on Carissa.

Analysis of the clinical interviews conducted before and after the teaching experiment, indicates a dramatic shift in Carissa's focus, from attending to differences in single quantities to attending to co-varying quantities and forming a multiplicative relationship between them. For example, in the pre-interview, Carissa was shown a table of data from a leaky faucet situation (Figure 1) and asked, "Is the faucet leaking faster at times, or is it leaking steadily the entire time?" Carissa immediately responded, "It's faster at some times." (In fact, all of the interview participants from Study 1 thought the faucet dripped faster at times.) She focused exclusively on the numbers in water column, examined the differences between numbers in the water column, and reported that the interval with the largest difference was where the faucet was dripping fastest. Her work was dominated by attention to one quantity, a focus on differences, and unitary reasoning. In another task, Carissa was asked: "Suppose you collected 16 ounces of water over a period of 24 minutes from a leaky faucet. How fast is the faucet leaking?" As with the previous task, Carissa focused on one quantity and did not appear to have formed a ratio between time and amount of water.

In contrast, her work on the same two items in the post-interview indicates a dramatically different comprehension of the leaky faucet situations. When asked whether the table of data indicated that the faucet was dripping steadily or faster at times, Carissa: (a) accounted for both amount of water dripped and time, (b) used co-variational language, (c) formed a ratio between time and water dripped, and (d) iterated and partitioned the composed unit in order to correctly solve the problem, and (e) compared dripping rates by fixing one quantity (time).

Cassandra decided to see how fast her bathtub faucet was leaking. She got a large container and put it under her faucet when she got up in the morning, and then checked periodically during the day to see how much water was in the container. She recorded the times and the amounts in the table below.

Time	**Amount of Water**
7:00 a.m.	2 ounces
8:15 a.m.	12 ounces
9:45 a.m.	24 ounces
2:30 p.m.	62 ounces
5:15 p.m.	84 ounces

Figure 1. Situation presented in pre- and post-interviews

A corresponding examination of the instructional environment of Study 2 indicates the presence of focusing phenomena of a different nature from those identified in Study 1. These focusing phenomena include (a) language of co-varying quantities; (b) data presented in pairs and compared according to an attribute; (c) measurement tasks that focused on an emerging attribute; (d) simulations to test quantitative relationships; (e) comparisons of what is the same and different across attributes; and (e) prompts to justify. The object of mathematical focus appeared to be the formation of the ratio of distance to time in order to measure the attribute of motion through space (speed). This is in contrast to the object of mathematical focus on slope as a difference that developed in Study 1.

Due to space limitations, we present only the analysis from Sessions 1 and 2 of the teaching experiment. This will provide evidence for three of the focusing phenomena identified above. The analysis began by identifying shifts in Carissa's focus during the teaching experiment sessions. Then we identified contingent instructional actions and features, for which a plausible conceptual connection to the student's ideas could be made.

In Session 1, the students were shown a *MathWorlds* computer simulation in which two characters, Clown and Frog, walked in opposite directions at different constant speeds (Roschelle & Kaput, 1996). The students were asked whether the characters walked equally fast or not and to design a method to measure how fast the clown walked.

Focusing phenomena: Nature of task. The nature of this measurement task likely served as a focusing phenomena in a very global sense, by directing students' attention to *quantities*. By quantity, I follow Thompson (1994) to refer to one's conception of *measurable attributes* of objects, such as height, distance, speed, or steepness. In other words, the focus in Sessions 1 and 2 was on sorting out the relationships among a large number of measurable attributes: elapsed time, number of steps, distance traveled, length of one step, motion through space (speed), and leg motion (how fast one's legs move). This is in stark contrast to the focus on numeric patterns in Study 1.

Shifts in focus. I claim that the mathematical focus inferred from the public displays of the two girls (i.e., their talk, representations, and actions) over the first two sessions of the teaching experiment, demonstrated a shift in focus: (a) from reasoning with one quantity alone to two quantities; (b) from the quantities of number of steps and elapsed time to distance and time; and (c) from how fast one's legs move around to how fast one moves through space.

At the beginning of Session 1, the girls incorrectly concluded that the frog was faster he "gets there first," (i.e., his time traveled is less). Correspondingly when asked to measure how fast the clown walked, both girls suggested timing the clown. This is reasonable if one considers the familiar context of racing, where the winner is indeed the one who takes the least time. Distance appeared to be implicit for both girls.

The students continued to focus on measuring a single quantity in their second suggestion of counting the number of steps. However, near the end of Session 1, their focus had shifted from one quantity to two. They used an on-line digital timer (made available once the girls expressed interest in measuring time) to measure the time that each character walked. They also counted the number of steps taken by each character. They determined that each character took 7 steps in 3 s and incorrectly concluded that the characters walked equally fast. It was not until the end of Session 2 that their focus shifted from number of steps over time to distance over time. In the process, their rationale for the steps over time method became more apparent. They talked about how fast the characters legs moved and made reference to the experience of a child walking next to a parent. In order to keep up with the parent, "the child would look like she is running." As Session 2 progressed, the students began to shift from a focus on how fast the legs moved to how fast the characters moved through space.

Focusing phenomenon: Running simulations to test quantitative relationships. Each time the girls suggested a quantity to be measured, they conducted a simulation (either by physically walking or by using the computer) in order to test their ideas. In the process of rejecting certain quantities, their focus appeared to shift to alternative quantities. Thus running simulations served to direct attention toward the consideration of new quantities and away from others, and hence served as a focusing phenomenon.

For example, after measuring the number of steps over time for each character the researcher devised a physical simulation in which the number of steps could be varied. The girls were asked to “program” the researcher to walk fast by telling her a certain number of steps and time. They asked her to walk 7 steps in 5 seconds. The researcher took 7 big steps in 5 seconds following by 7 tiny steps in 5 seconds. The girls were surprised that the researcher was faster in the first simulation. Carissa suggested that the researcher walk 7 steps in 3 seconds, in an effort to speed her up. But the researcher took baby steps and went very slowly, which produced laughter from the girls. During this simulation, a focus on the quantity of distance emerged. Carissa commented that the researcher “took small steps so it was a little bit of distance” and went on to conclude that “…it has to be like from a distance, like how far you go….you need distance because you could just walk 5 steps and not go anywhere.”

Focusing phenomenon: “Same/different” activities. During the simulations, the students sometimes attended to who pulled ahead and other times to whose legs moved faster. An activity in which the girls were asked to compare what was the same and what was different about two people (one pretending to be an adult and one pretending to be a child) appeared to allow the girls to isolate these two types of “fastness.” Specifically, in Session 2, Bonita’s cousin Andrea had been sitting in the room while the girls worked. The researcher asked her to participate in a simulation so that both Bonita and Carissa could watch. The researcher and Andrea walked together (same speed). When asked if the two people were walking equally fast, Carissa responded “yes, because your legs are going the same,” suggesting a focus on leg motion. Then Andrea pretended to be a child and bent down. The researcher and Andrea walked together (same speed) but Andrea took 2-3 steps for each of the researcher’s. The researcher asked Bonita and Carissa to compare what was the same and different about the two walkers. Carissa’s difficulty in sorting out these two attributes is evident in her struggle to find appropriate words to describe the differences and similarities: “you’re going the same way… it’s hard to say, I don’t know if it’s the same speed, but it’s the same way.” The girls noticed a key difference, namely that Andrea took more steps than the researcher, and her legs looked like she was running. They also noticed similarities, namely that the researcher and Andrea “had the same timing” (meaning they took the same number of seconds), that Andrea was trying to “keep up with” the researcher (meaning that they were equally fast moving through space), and that they both covered the same distance.

Once the attribute of “how fast one’s legs moved” seemed to be isolated from “fastness through space (keeping up with each other),” the researcher told the girls that she would use the term “same speed” to refer to what they noticed about Andrea “keeping up with” the researcher. She acknowledged that Andrea took more steps and that her feet were going around faster than her own, but she also pointed out that “how fast your feet are going isn’t speed; it’s a different rate.” In subsequent activities, the girls seemed able to focus on speed without bringing up the number of steps again. The use of this “same/different” activity appeared to play a role in this shift of focus.

Summary. In contrast to the instructional environment of Study 1, in which the initial focus in the unit was on numerical patterns and differences in *y*-values of the function, the teaching experiment appeared to focus on quantitative relationships (i.e., how measurable attributes in the situation are related). A summary of the shifts in focus and the accompanying focusing phenomena identified through the analysis are shown in Figure 2.

Conclusion and Discussion

The most important finding of this study is that qualitative differences in the nature of the individual generalizations between Studies 1 and 2 correspond to significant differences in the focusing phenomena that emerged. While all of the interview participants in Study 1 appeared to generalize their understanding of slope as a difference, the Study 2 participant treated the same situation that she had seen in the Study 1 interview in a dramatically different way after participating in the teaching experiment. Specifically, she formed a ratio of two quantities rather than attending to differences in one quantity. The significance of this exploratory comparative work lies in the ability to link particular instructional treatments (and the associated focusing phenomena) with specific student generalizations. By examining how changes in focusing phenomena are related to corresponding changes in students' actor-oriented transfer, researchers can develop a useful profile of the ways in which students are likely to generalize their learning experiences given different types of instructional experiences. This activity will provide a useful contrast of the types of foci that are related to productive student generalizations with those that unwittingly afford less powerful student generalizations.

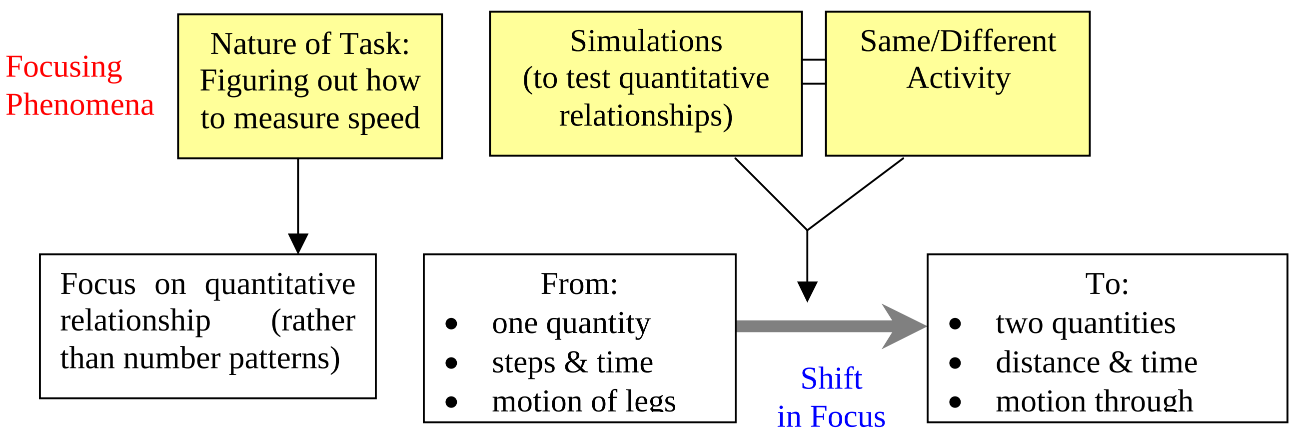

Figure 2. Shifts in Focus and Accompanying Focusing Phenomena

This work can also contribute to the ongoing transfer debate by addressing an area that has proved challenging for emerging alternative perspectives, namely that of transfer mechanisms. Because alternative perspectives emerged largely in reaction to the sole reliance on the cognitive mechanism of abstract mental schemes, tackling the issue of mechanism within these new interpretive frameworks has been challenging. Recent work on positioning or framing as a socially situated transfer mechanism is promising (Engle, in press). This research report will extend those efforts by further developing focusing phenomena as a way to account for many social aspects of generalizing activity, while also coordinating these analyses with an examination of individuals' generalizing activity and highlighting the mathematical content.

Acknowledgments

This research is supported by the National Science Foundation under grants REC-0529502 and REC-0450208. The views expressed do not necessarily reflect official positions of the Foundation.

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SUPPORTING ALGEBRAIC THINKING AND GENERALIZING ABOUT FUNCTIONAL RELATIONSHIP THROUGH PATTERNING IN A SECOND GRADE CLASSROOM

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This paper reports on a teaching study in a second grade classroom, in which functional relationship was explored through an investigation of growing patterns by explicitly integrating visual/spatial and numeric representations of pattern to promote algebraic thinking. Findings focus on three aspects of generalization: integration of representations, translation and application across representations, and generalization as also the abstraction of abstractions.

Introduction

The ability to generalize—that is, to distill from a collection of particular instances a relational abstraction transferable to new applications—has been ascribed to algebraic thinking, itself a term that Kieran (1996) explicitly broadened beyond algebra to "the use of any of a variety of representations that handle quantitative situations in a relational way". Recent research has challenged the assumed hierarchy of representations of mathematical ideas that has conventionally ranked numeric over visual/spatial (e.g.Noss & Healy, 1997; Lee, 1996; Mason, 1996; Nemirovsky, 1996). Case (e.g. Moss & Case, 1999; Case, 1998; Griffin & Case, 1997; Case, 1985) further contended that it is the integration of visual/spatial and numeric schemas within a given mathematical domain that allows children to establish what he referred to as a new central conceptual structure.

The study reported here sought to explore the notion that children's full understanding of and ability to engage in mathematical generalization may in fact rely on a critical integration of more than one form of representation of a mathematical idea. This may more specifically be described as involving children's ability to move fluidly and fluently back and forth across multiple representations in both interpreting and applying a mathematical generalization. Further, generalization may go beyond the directly experiential quantitative instances described by Kieran, to include abstractions as instances themselves where a generalization describes the relationship amongst these abstractions; this relies for illumination on Piaget's (2001[1977]) differentiation of empirical abstraction from reflecting abstraction. An exploration of these three aspects of generalization (integration of representations, translation and application across representations, and generalization as the abstraction of abstractions) will be the focus of this paper, drawn from a teaching study of algebraic thinking about functional relationship in pattern work with second grade children.

Context and Methods

This classroom teaching intervention took place in an intact second grade classroom of 22 students at a university laboratory school. This study is part of an larger ongoing international research project exploring algebraic thinking of students in second through sixth grades.

The twelve research lessons were presented during regularly scheduled math periods as part of the normal school day, three times a week over a period of four weeks. All lessons were videotaped and written transcriptions made. Digital photographs were taken of children's

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activities and constructions, and their classroom work was collected as artifacts for data interpretation. Field notes were made by the researcher, classroom teacher and research assistants.

Prior to the start of the research lessons, Number Knowledge Task (Case & Okamoto, 1996) was administered individually as an assessment of numeracy level. Pre- and post-assessments of nine patterning items in multiple representations were administered in individual interviews. Further post-interviews were conducted with pairs of students attempting two standard algebraic reasoning tasks; these interviews were video-taped and written transcriptions made.

The research lessons: Integration of representations

The lessons began with visual/spatial representations by presenting the students with a sequence of positions in a geometric growing pattern. These were made of square tiles placed in arrays that grew by a constant coefficient. The children were not taught multiplication prior to or during this study; however, they "invented" it as needed over the four weeks of research lessons (Schliemann, Carrahar & Brizuela, 2001). To introduce integration of numeric with geometric representations, an ordinal position number was placed below the geometric array that represented that position of the pattern. This helped to make clear the functional relationship between, for example, the position number 1 and one row of 3 square tiles, and the position number 2 and two rows of 3 square tiles each or 6 tiles altogether.

Numeric representations of functions were then explored using a function machine (Carrahar & Earnest, 2003; Rubenstein, 2002; Willoughby, 1997). Students took turns creating functional rules, creating non-sequential examples as clues, to challenge their classmates to "guess my rule". The children solving the challenge recorded on T-tables the input and output numbers, and their conjectures for what the rule might be. These numeric examples were non-sequential to allow a focus on the "across" (on a T-table) or functional rule rather than on the "down" pattern or "what comes next" differencing strategy identified as interfering in reaching a functional generalization in numeric patterns (Schliemann, Goodrow & Lara-Roth, 2001; Orton & Orton, 1998; Orton, Orton & Roper, 1998).

The students then integrated all aspects of the previous activities, by building non-sequential geometric pattern positions from a secret rule (composite function) on a "pattern sidewalk", a large counting line with ordinal position numbers on each section of the sidewalk. They would build, for example, positions 2, 4 and 9; other students would then guess the rule by trying to build the pattern correctly on, for instance, position 7.

Finally, the students in pairs made up their own mystery rules, and built several sequenced positions of their own patterns out of a variety of construction materials, for other students to guess the rule. These were photographed and a booklet made; students reasoned in writing about what the pattern rules might be, agreeing, disagreeing, or elaborating on one another's written conjectures. In this activity, an unexpected revelation was the spontaneous introduction by some of the students of the "zero" position of the pattern, which they explained was a "big clue" to guessing the rule because it isolated the "bump" or the constant.

Preliminary findings: Translation and application across representations

There is some debate in the field regarding whether to use correct mathematical terminology right from the start, or to rely on invented informal language with young children. In this study, because the concept of a function was first presented through geometric arrays, the informal term "bump" evolved for the constant because it appeared as an incomplete row above the array, that looked like a bump. The strong visual/spatial reference and experiential grounding that gave rise to this term supported the decision to stay with this, and see where it led.

In post assessments and interviews, all children were able to recognize and describe a reasonable general functional rule for the pattern that was presented, and had strategies for applying their rule to find extensions of the pattern, in all representations except one: skip counting (by 3s). This was despite the fact that throughout the intervention many patterns had frequently been described by the students as a "counting by [3]s pattern". This invites conjecture regarding the potential for interference of the conventional rote approach to skip counting done from Kindergarten. Other representations with which the students were more successful included arrays, drawings, and T-tables with which they were familiar, as well as narrative, two-dimensional standard algebraic reasoning task (square tables problem) and three-dimensional standard algebraic reasoning task (cube sticker problem) representations with which they were not familiar.

Within the narrative format (which describes a child with $10 saved for a scooter, who walks a neighbour's dog to earn $5 each day), all but two children showed an understanding of the rule as being a composite function with some recognition of the constant, so clearly the concept of a "bump" had transcended its geometric beginnings. One child explained, "The bumps are the extra ones that will always stay there." This understanding cut across all numeracy levels as determined by the Number Knowledge Task. However, the children's strength in applying this understanding varied. Seven children expressed clear correct generalizations, in more and less formal language, that identified both the constant and the coefficient. One of these children, who was considered highly distractible and low achieving, responded, "It's counting by 5s with a 10 bump," even though he lost interest in calculating the far transfer positions. A mid-level child responded, "Oh, I get it—it's a groups of 5 pattern with a 10 bump." A highly capable high achieving student went on to notice that the constant was larger than the coefficient (not part of the original geometric representation): "It's always the day [ordinal position number] times 5, plus 10. So there's 10 bumps and 5 normal things, more bumps than normal things—that's weird!"

A further eight students were able to apply their conceptual understanding of an implicit function rule to predict near and far positions, without being able articulate the general rule they were nonetheless expressing in working through the particular positions asked for. The remaining five included a constant at first, but "lost sight of" it as the magnitude of the numbers they were working with increased (Stacey, 1989), and incorrectly applied a "whole object" strategy in doubling the 5^{th} to get the 10^{th} position (Lannin, 2002; Orton & Orton, 1998).

The two children who did not recognize the constant at any point in this task, explicitly or implicitly, were still able to identify the correct coefficient and make the generalization that the narrative presented a "counting by 5s" pattern. They were able to apply their incomplete rule correctly to both near and far positions.

Interviews: Generalization as the abstraction of abstractions

The children were interviewed in pairs, and presented with first the square tables problem which asks if square tables are arranged in a line, with one chair at each open side of a table, how many chairs would there be for increasing numbers of tables. All pairs of students, organized by either same numeracy level or friendships (with the aim of student comfortability to promote discussion), were able to articulate a general rule, in informal or more formal language. Several pairs immediately "saw" the pattern; these students were encouraged to consider what would happen if the tables were trapezoids instead of squares (this was drawn). Responses included a surprising sophisticated consideration of multiple ways of expressing the functional rule: "It could be the number [of tables] plus 1 more, then times by 3, but then you have to take away 1," in recognition of the 2 end seats that were 1 seat shy of being the same as an extra table.

The same pairs were then presented with the cube sticker problem which asks if cubes are linked together, and a sticker applied to each cube face that was still showing, how many stickers would there be for increasing numbers of cubes. All pairs were also able to work this out, with solutions ranging from linking actual cubes and counting sides, to clear generalizations ("It's a groups of 4, and then 2 at the ends". In a very interesting leap to an abstraction of abstractions, one child recognized that these problems represented two-dimensional and three-dimensional versions of the same type of generalization: "It's [cubes] like the other one [tables], except times 4, because there's 4 sides."

Conclusion

Implications of this study for future work in understanding the role of generalizing and algebraic thinking in the mathematics learning of young children are many. Among them is the conceptual illumination of "skip counting" through a three-tiered pattern sidewalk, where the functional relationship between the position number and the number of elements in that position is made clear through the medium of geometric constructions. Further, the link between repeating and growing patterns has yet to be explicitly explored within the integrative framework of this research, where it would seem that repeating patterns can be thought of as more complex articulations of growing patterns. The rich territory of mathematical modeling, largely unexplored for elementary mathematics students (e.g. London McNab, Moss, Woodruff & Nason, 2004; Van den Heuvel-Panhuizen, 2004; Lesh & Doerr, 2000), is described in many of the same ways that help us to understand key aspects of algebraic reasoning; clearly the importance of multiple representations stands out. It seems a natural further direction to consider how these two approaches may be merged to the greater benefit of the learner.

Finally, for reasons that bear further thought, this approach supported engagement in activities that the students found "meaningful" by including what Weininger (1981) would describe as educational play. As one child explained, "I don't really feel like it's math. I think it kind of feels like it's some fun stuff. It's kind of like you're half man, half horse; it's kind of like half fun, half math. It's like you change the gear into fun!".

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GENERATING MATHEMATICAL DISCOURSE AMONG TEACHERS: THE ROLE OF PROFESSIONAL DEVELOPMENT RESOURCE

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We investigate the function of professional development resources in generating substantive mathematical discourse among teachers and providing opportunities for teachers to learn mathematics. An analysis of the implementation of a two-day sequence of tasks in an algebra content course for practicing K-12 mathematics teachers is presented. Initially teachers engaged in mathematical education discourse. We define mathematical education discourse to include making sense of students' mathematical discourse, assessing the correctness of students' responses, and making observations about classroom culture and the role of the teacher. Teachers eventually engaged in mathematical discourse, which provided opportunities for them to learn about isomorphism and closure.

Introduction

The purpose of this report is to establish evidence that practice-based professional development resources can be used to generate substantive mathematical discourse in classrooms between mathematics teacher educators and teachers and provide opportunities for teachers to learn mathematics. An analysis of the implementation of a two-day sequence of tasks in an algebra content course for practicing K-12 mathematics teachers[1] is presented. The mathematics tasks discussed in this paper were designed around videorecordings of third-grade students exploring the concepts of even and odd. Design of professional development tasks grounded in teaching practice is consistent with current thinking about what constitutes effective mathematics professional development (Ball, D.L. & Bass, H., 2000; Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). However we know little about whether and how such professional development tasks lead to substantive mathematics discourse on the part of participating teachers in such courses. This paper contributes to understanding (1) how teacher educators use resources grounded in practice (e.g., print and videocases of mathematics teaching; student work samples) in the design and implementation of mathematics related tasks to deepen teachers' mathematical content and pedagogical content knowledge and (2) the mathematical discourse that arises from the implementation of such tasks. Such research can inform the design of effective mathematics professional development experiences.

Related Research

The K-12 mathematics professional development literature suggests that content-focused professional development contributes to changes in teachers' practice, deepens teachers' content and pedagogical content knowledge, and impacts student achievement. Recent research has shown that teachers can and do learn what Ball and her colleagues refer to as 'mathematical knowledge for teaching' during professional development opportunities (c.f., Hill, Rowan, & Ball, 2005). Calls for grounding professional development experiences in the practice of teaching have been plentiful (c.f., Ball, D. L. & Bass, H., 2000; Ball & Cohen, 1999; Kazemi & Franke, 2003). However it is less clear to what extent teachers – the 'students' of professional development – engage in substantive mathematics discourse as a result of analyzing tasks grounded in practice (student work, case studies, etc.). Professional development work to date has used mathematical tasks to launch discussions about K-12 students' mathematical thinking and learning (c.f., Smith, Silver, & Stein, 2005). We are interested in exploring whether or not these tasks can help generate discourse among teachers about their own mathematical knowledge. In particular, can practice-based professional development tasks be used to generate opportunities for teachers to deepen their own knowledge of mathematics? We explore this question by examining teachers' discourse in a mathematics professional development setting. We take the perspective that learning opportunities arise when people consider each others' ideas and engage in argumentation (c.f., Yackel & Cobb, 1996) - that is, when they engage in mathematical discourse.

Setting

The research reported in this paper is part of a larger design research study (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) in an algebra course that was part of a three-week long mathematics professional development summer institute. During the algebra course, mathematical investigations were often initiated by watching video of K-12 students' mathematical activity and analyzing this activity, or by analyzing K-12 curricular materials. These activities led participating teachers' (heretofore referred to as PTs) into advanced explorations of fundamental algebraic concepts such as operations, functions, and the commutative and associative properties.

Method

As stated earlier, this research is part of a larger, design research study in an algebra course for practicing teachers. Multiple forms of data were collected as part of the design research study. These data include: records of task design, daily class scripts, daily videorecordings of the algebra class, and digital copies of PTs' work. Daily class scripts – numbered sequences of intended class activities including questions and discourse protocols – were written daily prior to class and stored electronically. The daily class scripts serve as a record of the intended actions and questions for each class period. Videorecordings of each class session were captured with two cameras and were digitized daily. Finally, digital photographs of PTs work were taken and catalogued.

The specific tasks analyzed for this paper spanned two class days and involved PTs examining a transcript and video of a third-grade class in which the students discussed even and odd numbers (Ball, 1993). The third-grade classroom discussion was focused on a student, Sean's, suggestion that some numbers (like six) were both even and odd because they were comprised of an odd number of twos – these numbers were later referred to as Sean numbers. After analyzing the discussion presented by Ball (1993), PTs were engaged in exploring whether the set of Sean numbers was closed under various arithmetic operations.

We began data analysis by reviewing the class scripts and identifying the mathematics teacher educators' questions and actions intended to prompt the PTs to investigate mathematical ideas in the transcript of the third-grade classroom. Next we reviewed the videotapes that corresponded with the questions and actions identified on the class scripts. Videotapes of the implementation of the class scripts were reviewed to determine to what extent the implementation of the mathematical tasks was successful at eliciting discussion about mathematics among the PTs. A mathematical discourse observation protocol (Weaver, Dick, & Rigelman, 2005) was used as a basis for assigning codes to instances of mathematical discourse in the classroom videotapes. The discourse observation protocol is a tool for documenting the quantity and quality of mathematical discourse that transpires during mathematics lessons. We were interested in documenting evidence of mathematical discourse that engaged PTs in thinking about substantive mathematical ideas. Two points warrant further elaboration: First, we were looking for evidence of mathematical thinking among PTs and thus coded only the PTs' discourse (not utterances of the mathematics teacher educators). Second, we made the stipulation that the discourse must center on *mathematical* ideas or procedures. Thus, discussions of pedagogical approaches or strategies were not considered part of mathematical discourse. A sample of the discourse coding scheme is shown in Table 1.

Code	Definition	Explanation
S	Stating/Sharing	PT makes a mathematical statement or assertion without an explanation of how or why.
E	Explaining	PT explains a mathematical idea or procedure by stating a description of what he or she did, but the explanation does not provide any justification of the validity of the idea or procedure.
C	Conjecturing	PT makes a conjecture based on her/his understanding of the mathematics behind the problem.
J	Justifying	PT provides a justification for the validity of a mathematical idea or procedure.
G	Generalizing	PT makes a statement that is evidence of a shift from a specific example to a general case.

Table 1. Example mathematical discourse codes

Results

Mathematical Education Discourse

PTs initially focused on the actions of the students and teacher in the transcript, even though they were given instructions to reflect and make notes on the mathematical content of the discussion. Our coding of the three minute whole class discussion (in which PTs shared summaries of their small-group discussions of the transcript) netted only four instances of mathematical discourse and only one discourse code (S: Sharing).

While very little of the small-group and whole class discussion of the transcript initially involved mathematical discourse, we noticed that the PT's discourse focused on understanding Sean's thinking, assessment of Sean's difficulties, and evaluating Sean's and other students' mathematical thinking. In other words, the PTs engaged in discourse about mathematical *education*. Although we were not able to code this mathematical education discourse using the mathematical discourse coding scheme described above, we noted three distinct ways in which PTs engaged in mathematical education discourse: (1) PTs attempted to make sense of students' mathematical discourse; (2) PTs evaluated students' responses as correct or incorrect and/or assessed students' difficulties; (3) PTs did not specifically attend to students' mathematical discourse, but commented on the role of the teacher or the culture of the classroom. We juxtapose a sample of discourse from Paula and Sue Ann, whose comments provide examples of (1) and (2), respectively. We remind the reader that Sean called numbers 'even and odd' if an even number had an odd number of pairs.

Paula: Basically they're talking about really interesting concepts. They went ahead and said 'I have three groups of two, so I have an odd number', so they had taken it an additional step. So that's kind of how I read it. So the underlying concept was odd and even.

Sue Ann: I thought it was interesting that Mei [another student in Sean's class] starts asking questions to start Sean thinking. And so she uses the example of 10, that it can be an odd, and it just reinforces the relationship. And I was thinking that sort-of happens a lot in the classroom, um, that he memorized that three and five were odd numbers and so he can't get beyond that understanding…But you get caught up in those misunderstandings, you memorize one little piece and can't get past it.

Paula did not point out shortcoming in Sean's thinking; rather she attempted to make sense of his idea. She restated Sean's explanation, "I have three groups of two so I have an odd number" and recognized that he made a nonstandard generalization ("they had taken it an additional step").

In contrast, Sue Ann did not try to figure out why Sean's idea made sense to him, rather she evaluated his answer as right or wrong and offered conjectures about the nature of his difficulties. In particular, Sue Ann hypothesized that Sean memorized three and five were odd numbers and that he was inappropriately applying this memorized fact.

Just as there are different types of mathematical discourse (e.g., justifying or sharing), we propose that there are different types of mathematical education discourse. In the example of Paula and Sue Ann's responses, we found that one PT (Paula) attempted to make sense of Sean's ideas from his point of view. On the other hand, Sue Ann was concerned with assessing Sean's difficulties and evaluating if his response was correct or incorrect.

We find these differences in mathematical education discourse significant for at least three reasons: First, teachers who make sense of students' ideas (rather than evaluating them as correct or incorrect) use mathematical content knowledge in a different (and perhaps deeper) way than teachers who only evaluate student responses as right or wrong. Second, teachers may learn to think about mathematical ideas in different ways (ways suggested by their students), thus deepening their own mathematical understanding. Third, as teachers try to figure out thinking, they may have the opportunity to make connections to mathematics history for themselves and for their students. For example, Euclid also found Sean's numbers interesting; they appear in Book IX (proposition 33) of the *Elements* (Heath, 1956). Thus, while the idea of mathematical education discourse is under development, we have found it a powerful way to characterize the ways teachers engage with student thinking and suspect that such characterizations have implications for the development of teachers' mathematics content knowledge.

Opportunities for teachers to deepen their mathematical content knowledge

Following this initial discussion, the PTs engaged in a series of mathematical explorations about the mathematical issues raised in the third-grade class. First, PTs were asked to characterize the numbers that Sean had in mind. Next, the PTs were invited to investigate the results of combining two "Sean numbers". The goal of these investigations was to deepen the PT's knowledge of generalization and justification and support the development of a number of algebraic ideas, including closure and the isomorphism[2] between the even integers under addition and the integers under addition. Analyses of the class discussion of these two tasks showed that teachers engaged in mathematical discourse at the conjecturing, justifying, and generalizing levels. Thus, these two activities served to move teachers to engage in substantive mathematical discourse and provided opportunities for teachers to deepen their knowledge of algebra.

Beginning with the task of characterizing the numbers that Sean had in mind, the PTs were engaged in substantive mathematical discourse that created a number of opportunities for them to deepen their understanding of mathematics. In particular, a number of mathematical issues came to the fore as the PTs considered what happens when two Sean numbers are combined. In each small group, the PTs determined that the result of adding two Sean numbers is an even number that is not a Sean number (called an even/even by the teachers). After this small-group exploration, the PTs were shown a video of a fourth grade student, Alison, proving that the sum of two odd numbers is an even number. The video was intended to function as a catalyst for discussion about justification and communication. After watching this video, the PTs shared their results in a whole class discussion. Michelle presented her group's poster (shown in figure 1), which was meant to justify that the sum of two Sean numbers is an even/even.

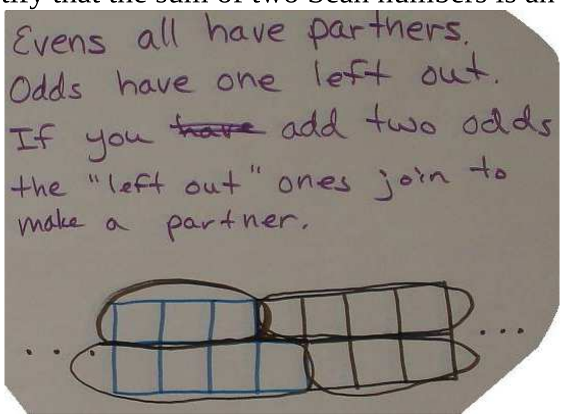

Figure 1. Michelle's poster showing how her group combined two Sean numbers

This diagram and proof does not appear to be appropriate because it does not show two Sean numbers being added together. Additionally the language suggests that this is a proof that the sum of two odds is an even, not a proof that the sum of two Sean numbers is an even/even. (There is evidence that Michelle was actually thinking of this picture as showing not 7 +7 but 6 + 8 with 8 misidentified as a Sean number with 4 as the "odd factor". However the other PTs and the course instructors did not notice this.) Michelle's poster was immediately challenged by another PT, Ann.

Ann: I don't understand how this is related to Sean's numbers

Michelle: Um, Sean had, uh, let's see, yeah, that's a good point, up here, this part is related to Sean's numbers, and this is showing how the odd factors can always have, um, the two odd factors, can always have partner, that any time you add two odds together … he had the even factors and the odd factors and whenever you put two odd factors together then you're going to have an even amount, or a partner, it's going to even out the partnership.

Jack: Does each small square there represent two, as a pair?

Michelle: So let's say, for example, I had looped these and say Sean's got, three times three, two threes, two sets of three and two sets of four, ok, so if I loop them this way, it might show it in a better way, that here's his set of three, ok, three, let's think of it like, um, if his number was fourteen. [Michelle is grouping to show that the two threes form an even number and the two fours form an even number.]

Instructor 1: Maybe this is something that everybody can continue thinking on

Instructor 2: So here's the question, how can you take this diagram and make it look directly like putting two Sean numbers together?

As a result of Michelle's presentation, a number of mathematical issues became salient. Perhaps the most significant (aside from numerous issues related to justification) was the isomorphism between the even integers under addition and the integers under addition.

The following day, the PTs were given a chance to read a transcript of the video they had watched of Alison proving that the sum of two odd numbers is an even number. This activity lead to a discussion of how Michelle's proof from the previous day (which looked very much like Allison's proof) could be improved. In particular, the PTs were asked if they could modify Michelle's diagram so that it showed two Sean numbers being added together. During the whole-class discussion that followed Michael shared a way to modify the diagram by changing the odd numbers into Sean numbers (implicitly using the isomorphism between the integers and the even integers that is provided by doubling). Sandra and Alice both made observations that suggest they came to see Sean's numbers have much the same structure as the odd numbers. (Note that under the isomorphism $f(x) = 2x$ from the integers to the even integers, the odd integers are mapped to the Sean numbers.)

Michael: I was thinking that the definition is that it's any number with an odd amount of pairs so the key is to show that these boxes aren't one number that they're not just one they're a pair of numbers and so first started just putting little dashes in here so here's a Sean number 6 added to 6 would be 12 so it's no longer a Sean number and maybe we thought it would be easier for maybe some people to understand for kids to understand if these boxes were divided into two pieces so made a little slash through them but it's basically the same.

…

Sandra: My paper's really messy, but I was thinking and scratching I was thinking that Sean and Alison are thinking the same way because Alison was telling us about odd numbers and that a single is a one and you know an odd number has a pair and a single right now and Sean is kinda thinking the same way if you take the assumption that two is the one and four is the two [he] just doubled it so Sean's doing the same pattern but making but you know six ten fourteen on and on and on but [he's] thinking that $2 + 4n$ is the [Sean] numbers…

Instructor 1: So you're seeing a connection between the formula that you showed and in the very beginning for odd numbers being $2n + 1$ and the $4n + 2$.

…

Alice: I kind of did the opposite because I wrote this is really small I broke the pairs down into ones so these this is six and this is six and they're and they have an odd number of pairs so when I squished those together I got an Even/Even because it has an even number of sets of two it has six sets of two but then I decided that I would look at it as blocks kinda like Michael did and then I saw that it did resemble Alison's tally marks because of those two in the middle that will pair up so you know that I was thinking of it at the end more like five sets plus five sets is ten sets so an odd plus an odd is an even.

As the PTs engaged in substantive mathematical discourse that was situated in these two video cases from elementary classrooms, they had a number of opportunities to deepen their own content knowledge. The excerpts above illustrate one of these opportunities as the PTs began to have a deeper understanding of the number theoretic structure of the integers. This understanding included an implicit understanding (made explicit later in the course) of the isomorphism between the even integers under addition and the integers under addition. This implicit understanding was leveraged as the PTs were able to use their prior understandings of even and odd to make sense of a class of integers that was new to them. Other learning opportunities that grew out of these practice-based professional development materials involved 1) important mathematical processes including defining, generalizing and proving, 2) connections between geometry and algebra – as they learned to use diagrams appropriately to prove algebraic statements, and 3) other mathematical concepts including the closure of sets under operations.

Conclusion

PTs initially engaged in mathematical education discourse, that is, discourse focused on students' thinking and the actions of students and the teacher in the video transcript. They moved to developing and justifying conjectures about the special set of numbers that Sean was thinking about; i.e. they engaged in mathematical discourse. In turn, these activities played a significant role in promoting opportunities for the development of teachers' knowledge of algebra concepts such as closure and isomorphism. This investigation raised a number of questions for further inquiry: What are the consequences of different types of mathematical education discourse in terms of providing motivation and opportunities for teachers to learn mathematics? For instance, can teachers learn mathematics by engaging in mathematics education discourse that involves making sense of students' mathematical thinking? What types of mathematics education discourse provide the greatest challenge (or opportunity) for transitioning to discourse about mathematics?

Endnotes

1. The research reported here is funded, in part, by the National Science Foundation (NSF-HER-0412553). The views expressed in this paper do not necessarily reflect the views of the National Science Foundation.

2. The integers form a group under addition and the even integers form a group under addition. The function $f(x) = 2x$ is an isomorphism from the group of integers to the group of even integers.

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LEARNERS' DIFFICULTIES WITH DEFINING AND COORDINATING QUANTITATIVE UNITS IN ALGEBRAIC WORD PROBLEMS AND THE TEACHER'S INTERPRETATION OF THOSE DIFFICULTIES

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This study examines 8th grade students' coordination of quantitative units arising from word problems that can be solved via a set of equations that are reducible to a single equation with a single unknown. Quantitative unit conservation also emerges as a necessary construct in dealing with such problems. We introduce a theoretical framework that encompasses these two constructs. Our data consist of videotaped classroom lessons, student interviews and teacher interviews. We generated a thematic analysis by undertaking a retrospective analysis, using constant comparison methodology. Our first result is about students' coordination of pairs of units (e.g. dime standing for the name of the coin and/or the number of dimes, the value of a dime being the second unit). Our second result is about students' attempts to balance the two sides of an equation by conserving units.

Theoretical Background

This study is part of Project CoSTAR (Coordinating Students' and Teachers' Algebraic Reasoning)[1] that has as its main purpose the coordination of research on students' understandings and teachers' practices and interpretations of students' actions relative to algebraic reasoning. This particular study is informed by recent research that coordinates analyses of collective classroom mathematical practices and individual cognition (Lobato, Ellis, and Muñoz, 2003), research on knowledge that teachers use as they engage their practice (Ball, Lubienski and Mewborn, 2001), and research on students' understanding of algebraic symbols (Kieran and Sfard, 1999), and their construction and coordination of quantitative units (Olive, 1999; Steffe, 1988, 2002).

The ability to coordinate different units in a quantitative situation is an important skill for students to develop in order to be successful in both representing and solving algebraic word problems. Whether this be coordinating different levels of units in a whole number multiplicative situation (e.g. Steffe, 1988) or in a fraction situation (e.g. Olive, 1999; Steffe, 2002) or in dealing with intensive (e.g. miles per hour) as well as extensive (e.g. number of hours) quantities (Schwartz, 1988; Kaput, Schwartz and Poholsky, 1985) the crucial point is to understand what is being done with the varying quantities in these situations and how the units involved can be related.

The names of quantities involved in a word problem do not suffice to adequately reflect the nature of those quantities. We need to delve further into the nature of these quantities in order to uncover the units associated with them. Just as a point on the coordinate plane is associated with its x- and y-coordinates, that is coordinated as an ordered pair (x,y), a quantity is born only when it is correctly represented as the pair (name, unit), thus coordinated properly. For instance, the coordination (dime, number of dimes) is not the same as (dime, value of a dime) or (dimes, value per dime). Schwartz (1988) used the term *referent* in a way similar to how we are using *name* and called such quantities *adjectival quantities* (p. 41) He stated that all quantities have *referents* and that the "composing of two mathematical quantities to yield a third derived quantity can take

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either of two forms, *referent preserving composition* or *referent transforming composition."* (p. 41) The referent transforming composition, Schwartz claims, forces us to distinguish between two different kinds of quantity: *extensive* quantity and *intensive* quantity. An extensive quantity can be counted or measured directly, whereas an intensive quantity is derived from the multiplication or division of two like or unlike quantities, and is usually recognized by the use of "per" in it's referent unit (e.g. miles per hour, price per pound).

In word problems involving extensive and intensive quantities, one further step is needed, beyond coordination of each one of those quantities. We somehow would need to reconcile all these quantities, each of which can be coordinated in the form (name of quantity, unit of the quantity). In other words, we not only look at each coordinated quantity separately, but also look at all these quantities together as a whole. This coherence of the whole requires that we meaningfully combine each coordinated quantity: A coordination of coordinated quantities. We refer to this second level of coordination as "quantitative unit conservation" that covers a range of mathematical practices (e.g., taking care of priority of operations, using parentheses appropriately, and substituting literal expressions for other literal symbols) associated with solving word problems. All these mathematical practices serve one crucial idea, and that is to maintain the equality of expressions on both sides of an equation (Chazan and Yerulshalmy, 2003), while being aware of what's happening on both sides: Things we are adding or subtracting have to be like terms while those we multiply or divide do not necessarily have to be so.

In this paper we explore the units coordination arising from situations that can be represented by linear equations involving more than one unknown or variable but that can be reduced to an equation in a single variable; that is, a system of linear equations that can be solved by substitution. Through our analysis of the classroom discussions, students' explanations and responses to interview tasks, along with interviews with the classroom teacher we have come to realize that the identification and coordination of the units involved in the problem situation are critical aspects of the teaching-learning process.

Context and Methodology

This study took place in an 8th-grade classroom in a rural middle school in the southeastern United States. The 24 students were between 13 and 14 years old and had been placed in the algebra class based on their success in 7th-grade mathematics. All eight class lessons on a unit that focused on writing and solving algebraic equations from word problems were videotaped using two cameras, one focused on the teacher and the other on the students. Four students were interviewed twice in pairs (a pair of girls and a pair of boys) during the three weeks of the study. Ms. Jennings, the classroom teacher was also interviewed twice during the three weeks. All interviews were videotaped. Excerpts from the classroom videotapes were used during both student and teacher interviews to provoke discussion of the learning that was taking place in the classroom. Excerpts from the videotapes of student interviews were also used in the teacher interviews.

This paper focuses on the two class lessons and subsequent student and teacher interviews that dealt with the following word problem from Unit 4 of College Preparatory Mathematics (CPM) Algebra 1 (2002):

> *Ms. Speedi keeps coins for paying the toll crossing on her commute to and from work. She presently has three more dimes than nickels and two fewer quarters than nickels. The total value of the coins is $5.40. Find the number of each type of coin she has. (CP-16, p.10)*

Students were challenged to write an equation to represent the problem.

In this problem situation, the monetary values of specific coins are intensive quantities (they are the values per coin) when trying to calculate the total value of all coins and the numbers of each coin are extensive quantities. Distinguishing these two different types of quantities surfaced as a problem during the classroom discussions. Associating appropriate units with the different quantities and combining unknown quantities emerged as further problems during the student interviews.

Analysis Process

Each day the classroom video data from the two cameras were viewed and digitally mixed using a picture-in-picture technology. A written summary of the lesson with time-stamps for video reference was created from the mixed video. This written summary also contained comments about any significant events and screen shots from the video when needed for clarification or highlight. These written "lesson graphs" were then used to select excerpts from the classroom video to be used in the student or teacher interviews, and to plan questions and related problems to pose to the interviewees in an effort to understand how the students (and teacher) had interpreted the problem and the classroom discussions that followed from different students' attempts to address the problem.

After the end of the three weeks of data collection, the corpus of classroom video data were reviewed, along with their lesson graphs to generate possible themes for a more detailed analysis. All student and teacher interviews were transcribed from audio files created from the videotapes of the interviews. A chart of relationships among class lessons, student interviews and teacher interviews was then created. A retrospective analysis, using constant comparison methodology, was then undertaken during which the classroom video, related student interviews and teacher interviews were revisited many times in order to generate a thematic analysis from which the following results emerged.

Results

A major confusion arose during the first class lesson on 10/27/04 in naming the quantities in the situation. Students had chosen the letter N to represent the nickels in the problem, however, it became apparent from the discussion that, while N stood for the *number* of nickels for the teacher and for some of the students, for others it either represented the value of the nickels or just stood for the coin. When Ms. Jennings asked the students "What are we gonna call dimes?" (right after writing "n=nickels" on the classroom board), some students answered "two N", and this could be a corroboration that those students saw N as the value, and not the number of the coin under consideration. The following dialogue between Ms. Jennings and a student, Cathy, taken from the classroom video illustrates the confusion:

Protocol I: Student's confusion about naming coins (from classroom video on 10/27/04)

Ms. Jennings*: We are just naming our variables right now. We haven't begun to make an equation yet. We have to know what we are naming, before we put in an equation.*

Cathy*: So why can't we just put them all with their first letter? Like n equals nickels, just keep doing, d dimes, q quarters.*

Ms. Jennings*: Let me ask you this question and see if you can solve it: "n plus d plus q equals 5 dollars and forty cents. How many of each one do I have?"*

By challenging Cathy with the statement "N plus D plus Q equals five dollars and forty cents. How many of each one do I have?" in the above dialogue, Ms. Jennings may have added to the confusion (over what the letters the students had decided to use represented in the situation). Ms. Jennings actually wrote on the white-board during the lesson: “n=nickels, d=dimes, q=quarters” following Cathy’s suggestion.

In her interview a few days later, Ms. Jennings commented that students name a coin by its first letter to make it easy to identify in the equation but that later confuses them. The source of this confusion partly comes from what Ms. Jennings had written on the board: “n=nickels, d=dimes, q=quarters”. Ms. Jennings pedagogical approach in the classroom is to accept students’ suggestions without evaluation from her, with the intent of having her students evaluate and discuss what is said during the lesson. This approach leads to rich discussions and productive arguments among the students, but we believe, can also leave some students confused as to what is mathematically acceptable and what is not. Ms. Jennings's introductory question “What are we gonna call nickels, dimes, quarters?” could have been misleading (as she did not specifically say *number of* nickels, dimes and quarters).

In the interview with students Pam and Maria on the morning following the classroom lesson, the interviewer showed the classroom video episode from Protocol I above:

Protocol II: Students' interpretation of Cathy's remark (from student interview on 10/28/04)

Interviewer: *Okay. What do you think Cathy means by N for nickels, D for dimes, and Q for quarters?*

Pam: *That represents how much you have, that’s what she’s talking about.*

Interviewer: *How much?*

Maria: *No, she was thinking about the value of each one.*

Interviewer: *Oh, rather than what?*

Maria: *The number of coins.*

Later in the same interview, Maria distinguished the differences among three different types of quantities: the value of a coin, the number of that coin and the total value of all the coins of that type. She was then able to combine her total values for each type of coin to produce the total of all coins ($5.40).

Protocol III: Unitizing quantities (from student interview on 10/28/04)

Maria*: Yes. Okay. Okay, this is the value of the nickel, so it would be…*

Interviewer*: What is “this?”*

Maria*: .05. And, in any number, let’s say 5, so it’ll be .05 times 5 will give the amount of nickels*

Interviewer*: The amount of nickels?*

Maria*: No, the value of the whole nickels that you have.*

Interviewer*: Does that make sense?*

Maria*: Yeah. And, then you do the same for D and Q and it comes out to $5.40.*

Interviewer*: Do the same for D and Q for me.*

Maria*: Okay, let’s say, 10 dimes. So, it’ll be 10 times .1 will give you the value and the same for Q. If you do times any number, so Q… the letters mean any number you can think of.*

Interviewer*: Well, what in the terms of the problem what do those letters stand for?*

Maria*: The number of coins you need to get $5.40.*

Maria’s statement “the letters mean any number you can think of” is evidence that she knows she is dealing with the quantity “number of a coin”. Moreover, she separately calculates the total value for each coin, and this could be seen as her coordination of units before adding them together. In fact, during this interview, by explaining this unit coordination, Maria makes sure that the “terms” she is adding are like terms, and then she concludes the addition and writes the first equation 0.05n+0.1d+0.25q=5.40; and after substitution, the second equation 0.05n+0.1(n+3)+0.25(n-2)=5.40.

During the process of obtaining her equivalent form 0.05n+0.1(n+3)+0.25(n-2)=5.40, Maria performed several notable mathematical practices: First, her correct substitution of expressions for literal symbols, as in this case, n+3 for d, and n-2 for q; second, her placing of parentheses around those expressions appropriately. In this way, each product on the left hand side represented a composed quantity, and had to possess a unit inherent in its structure. Moreover, each product, having the same unit *value*, was connected meaningfully via the addition operation. This was when she identified these products as *monetary values*. In this whole process of obtaining the equivalent form, there is another meaningful mathematical practice, which we call *quantitative unit conservation*: Not only did each product on the left hand side of the equation have the same unit as the quantity on the right hand side, but their combination in the form of a sum – they could be combined because they were like terms – had the same unit as the quantity on the right hand side of the equation. In this way, there is this notion of coherence between each term on the left with the term on the right, as well the coherence of the combined expression on the left with the expression on the right.

In contrast, the two boys who were interviewed, while able to make the unit coordination to produce the first equation similarly to Maria, were not able to produce the second equation through substitution. The interviewer asked Greg to write down what D equals in terms of "that number of nickels" (pointing to the N). Greg wrote the expression: (.05n+3). Commenting on what Greg wrote, Ben said that it represents the value of the dimes. Ben's interpretation suggests that he was aware of the different types of quantities involved (value and number of coins) but may have had problems coordinating the quantitative units meaningfully. Ben was aware that 0.05n and 3 both must have the same unit, *value*, in order to be added. There is also the possibility that Ben interpreted the statement in the problem "three more dimes than nickels" to mean that the *value* of the dimes was three more than the value of the nickels. This would explain the acceptance of 0.05 (5 cents) as the unit value used to find the value of the dimes.

Upon the interviewer's suggestion, students went through a little experiment to test the truth of the expression (.05n+3) and realized that their conjecture must be false. The interviewer then shifted the focus back to what D was in terms of N. Protocol IV begins at this point in the interview:

Protocol IV: Creating Expressions for D and Q (from student interview on 10/29/04)

Interviewer: *But, I don't want the value, I want D. What's D in terms of N?*

Greg: *N plus 3.*

Interviewer: *Can you write that down? Put D equals.* [Greg writes: D=N+3] *Now, what do you know about Q?*

Ben: *Q is 2 less then N.*

Interviewer: *Yeah. So, write down the equation for me.*

Ben: *N minus 2.* [Greg writes Q=N-2]

Interviewer: *Okay. So, now can you rewrite this equation* [pointing to the first equation: (0.05 n) + (0.1 d) + (0.25 q) = 5.40] *just using N? Okay, can you do that for me?*

Greg: *To get the value?*

Greg started by writing .05n, then put a plus sign, and then put n+3 and stopped. He hesitated for a while in this step, and he asked himself "*How can you get the value?*". He erased the n+3, and replaced it with .1 n + 3, without parentheses. He did the same thing for the last expression and wrote .25 n – 2 . His complete expression for the second equation was 0.05n+0.1n+3+0.25n–2=5.40. Upon the interviewer's question whether he agrees, Ben said that 0.05n was correct. They both hesitated for the remaining terms on the left hand side. They tried to compare this expression, namely their second equation with their first equation (0.05 n) + (0.1 d) + (0.25 q) = 5.40 (note that Ben had placed parentheses around each of the expressions that indicated the value of each set of coins in this first equation).

Greg eventually realized that he needed to add parentheses to produce the second equation: .05n+.10(n+3)+.25(n-2)=5.40. The lack of appropriate parentheses in Greg's first expression for D (.05n+3) and in his first attempt at the second equation led to Ben and Greg's difficulties in coordinating units within their quantities and seeing the coherence of the whole equation, namely that each product on the left hand side must be consistent in units with the term on the right hand side, and that they could then be added because they were in terms of the same unit: monetary value.

The teacher interview on 11/03/04 started with asking Ms. Jennings her perception of what was going on with the students. Her initial comments are worth noting:

> *Well, my first perception is probably that I jumped it too fast and that they weren't ready to think about three different variables in terms of one. And, also, the fact that with coins we have decimals to play with didn't make the problem any easier for them to think about. Even though they know what value is represented by coins, to multiply that value to a variable that they don't understand yet was a leap.*

Ms. Jennings recognizes the complexity in the coins problem but focuses on there being more than one "variable" and the use of decimals rather than the difficulty we perceived as distinguishing between names of coins and the different quantitative units. Throughout the class lesson Ms. Jennings used the name of the coin, (nickel, dime, quarter) to stand for the number of coins. There are several instances from the interview where this issue becomes apparent. For instance, her comments *"I guess my job is to re-route them into naming a dime in terms of a nickel by the information from the problem."* could imply that for the teacher, a dime is a variable that can be expressed in terms of the other variable, nickel. She probably used the names of coins assuming that number of the coins was to be understood.

Ms. Jennings' commented about the boys' construction and confusion over the expression (0.05n+3); she said that Ben really knew what he meant, but he was having trouble with how to write it as an expression. She thought that if he knew the number of nickels and multiplied by the 5 cents and then added a 3, he would see his mistake. Ms. Jennings commented, however, that Ben would not realize his error without knowing a specific number of coins. In other words, when focused on the nature of these quantities, Ms. Jennings realized that the coin must be associated with the units inherent in its structure. When questioned about Ben's understanding of the role of n in (0.05n+3), she said she does not think Ben sees n + 3 as one number. She added *"I think he only sees the n as a number and whatever it is, you're gonna add 3 to it"*. Ms. Jennings is aware of students' difficulties with interpreting algebraic expressions as a "process" to be carried out rather than as an "object" on which to operate (Tall et al., 2001). In another instance from her interview, Ms. Jennings was asked to compare the behavior of the boys with that of Maria. She commented that Maria understood that the n + 3 was an expression meaning d and that Maria had an understanding of what an expression means: just to rename another thing by another name. The boys, however, did not really have that understanding. According to Ms. Jennings, Ben and Greg understood how to name the new variable, but they didn't know how to use it to describe an amount of money with an amount of coins. In other words, they could not make the "name-unit" coordination.

Summary and Conclusions

With this study, we tried to bring about a theoretical framework that encompasses a way to look at word problems that can be solved via a set of equations that are reducible to a single equation with a single unknown. We claim that through this lens, one could interpret all the quantities arising from such word problems as not simply ordinary quantities named in the problem, but as mathematical referents with associated units. The Coins Problem helped us focus on each coin in the problem as an object with a name and the unit associated with the specific coin. A dime is not a quantity: Dime is only the name of the coin stated in the problem. Once a dime is associated with its value, for instance, that's when a quantity is born. Therefore, with this quantitative unit coordination, “value of a dime” or “value per dime” becomes an intensive quantity that must be combined with another (extensive) quantity (number of dimes) to find the total value of the dimes.

We relied on evidence of students' judgments concerning this coordination of the quantitative referents and their units. Maria’s successful coordination of the quantitative referent and its unit, and the difficulties that Ben and Greg experienced in interpreting the various quantitative referents and combining them with appropriate units to produce quantities that *conserved* units within an expression, support our theoretical conjectures. We also saw that solving such problems requires students to conserve quantitative units. Thus we hypothesize that quantitative unit coordination necessitates quantitative unit conservation.

One other conclusion we draw is that the use of parentheses in these expressions indicates a student’s ability to unitize a quantity and see the relation between the different quantities and their units. Also, a teacher’s use of her students’ convenient but ambiguous naming of quantities (without clarification) can contribute to students’ confusions.

Endnotes

(1) CoSTAR is supported by a grant from the National Science Foundation, grant # REC 0231879. The opinions expressed in this paper are those of the authors and do not necessarily reflect the views of NSF.

(2) All names are pseudonyms.

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SPEECH AND GESTURE IN PATTERN GENERALIZATION TASKS INVOLVING GRAPHS: EVIDENCE THAT PERCEPTIONS INFLUENCE CONCEPTIONS

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This paper presents a study aimed at describing middle school students' views of graphical representations. From interviews of middle school students solving pattern generalization tasks using graphs, both speech and gesture responses were analyzed to determine students' beliefs about graphical representations. Results indicate that a bounded *view of graphs, as evident in gestures, may influence performance on pattern generalization tasks.*

Background

There is limited empirically based research in the literature about student reasoning in the context of early algebra graphical representations. Zacks and Tversky (1999), using a sample of Stanford undergraduates, found that students view graphs as inherently limited by their form. Stevens and Hall (1998) showed how a student, involved in a tutoring session using Cartesian coordinate graphs, was influenced by the graph's spatial relation to the grid edges as he predicted how an entire function would change in appearance when an equation was transformed. Videotaped interview data allows for the collection of multiple modes of response that can be analyzed to provide a more accurate and complete account of middle school student thinking. Our gesture analysis methodology draws from the established theoretical work by Goldin-Meadow (2003) and Alibali and Goldin-Meadow (1993).

Methods

Thirteen students, in grades 6-8, participated in videotaped interviews. All students were from a large urban middle school with a high percentage of non-Caucasian students (91.7%) and students in the free/reduced lunch program (86%). Problem 1 asked students to interpret a two-dimensional, Cartesian coordinate graph with four points plotted along a linear function, $y = 3x + 1$. In Problem 1a, a low-complexity FP (far prediction) task, students were asked to determine the cost to make 10 copies of a CD using the information shown in the picture. Problem 1b, a high-complexity FP task, asked students to determine the cost to make 31 copies of a CD, and thereby extrapolate beyond the bounds shown in both the x- and y-axes.

Results

Students in this study articulated both *bounded* and *unbounded* beliefs about the graph. *Bounded* beliefs about a graph are articulated when a student's speech or gesture response demonstrates that the perceived information is constrained by the graph's physical boundaries. *Unbounded* beliefs about a graph manifest when a student's speech or gesture response indicates reasoning beyond the physical boundaries of the graph. Since FP tasks are tasks with solutions outside of the numerical range shown on one or more axes of the graph, only unbounded verbal responses are considered to reveal the novel information about students' interpretations presented through their gestures. Figures 1 and 2 show examples of bounded and unbounded verbal and gesture responses from the study:

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TIFF (LZW) decompressor
are needed to see this picture.

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are needed to see this picture.

Figure 1. Bounded Responses **Figure 2. Unbounded Responses**

For Problem 1a, 46% of the students (N=13) were successful at the low complexity FP task. By definition, none of those who expressed a verbally bounded view of the graph were successful. Nine out of the 13 students (69%) provided an unbounded verbal response, and therefore had some opportunity for success. Of these, 66% were successful, even though 55% exhibited a bounded view through their gestures. On the low complexity problem, the nature of the boundedness indicated by their gestures did not predict success once students exhibited an unbounded view in a verbal form. In the high complexity FP task (Problem 1b), 23% of the sample was successful, confirming that this was indeed more difficult for students than Problem 1a. Of the 13 students, 10 students (77%) gave unbounded verbal responses but only 30% of those were successful. For this item, the boundedness of the gestures that accompanied students' speech does appear to be predictive of their FP problem-solving performance. Even with an unbounded verbal response, when gestures indicated a bounded view, students were four times more likely to miss the FP task. When gestures were unbounded, students were twice as likely to produce the correct FP.

Conclusions

The evidence suggests that there may be a relationship between gestural unboundedness and correct performance on both one-dimensional and two-dimensional far prediction tasks, especially for the high-complexity FP task (Problem 1b). The bounded gestures of students may indicate a prevailing belief about the bounded nature of the graph. For students exhibiting a bounded view, their reasoning may be perceptually constrained by the limits of the graph, thus prohibiting them from answering correctly on FP tasks. Further, this belief may be influencing their performance on the task. In the low-complexity FP task (Problem 1a), students with bounded gesture performed better than the students with unbounded gesture. Preliminary analysis of strategy choices reveals that adopting a strategy using a computational method rather than utilizing a spatial strategy may help students with a bounded view of the graph obtain a correct answer.

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USE OF THE CAS (COMPUTER ALGEBRA SYSTEMS) AS SYMBOLIC MANIPULATOR IN THE RESOLUTION OF ARITHMETIC-ALGEBRAIC WORD PROBLEMS

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In this presentation, I will describe the observations of using the CAS like symbolic manipulator CAS (Computer Algebra Systems) to solve arithmetic-algebraic word problems. The CAS is used as syntactic resolutor (symbolic manipulator) it makes that the student become concentrated in the semantic part of the arithmetic-algebraic word problem and meditate about he has in the screen of the calculator with CAS, and what brings in it again (formulas, algorithms, etc.) to generate and to use algebraic expressions with the purpose of exploring, to discover and to manipulate things that he ignored. The observation of the symbolic manipulator's relationship (CAS) with the analysis-synthesis process when solving the arithmetic-algebraic word problems, and of return, when giving sense to the results that provides the CAS, it corresponds to the study in the manner of the students of third grade of secondary school in Mexico, between 13 and 15 years and with algebra knowledge, they solve such problems.

Theoretical Model

The Theoretical Model that was used to approach the investigation was that of "The Local Theoretical Models" (Filloy, 1999; Filloy y Rojano, 2001). This Theoretical Model allowed analysing to the observations of the phenomena around the resolution of arithmetic-algebraic verbal problems (Filloy, Rojano y Rubio.2001; Rojano, 2002) and it allowed knowing to senses that the students gave to the use of the CAS like symbolic manipulator with relationship to such problems.

The Observation Moment

The election of the observation moment consists on finding some point of the curriculum of mathematics, in the one, which the students, starting from their school antecedents, require the knowledge to teach for, extend their learning of the algebra. In the curriculum of mathematics in Mexico, this moment is located after studying the first one and second grades of secondary school, after studying pre-algebra and elementary algebra.

Population

The study, in which this document is based, was made in a group of 28 students of third grade of secondary public school in Mexico, in function of diagnostic questionnaires to verify the school antecedents of the students, allowing to classify them in the axes: syntactic, semantic and of basic contents (Rojano, 1985; Rubio, 1994) after the work of a school year in Mexico, to carried out the clinical interview.

Observations

The following ones are some of the observations of the resolution of arithmetic-algebraic verbal problems with the use of the CAS like symbolic manipulator:

1. It exists a to go and to come from the students, of the symbolic manipulator (CAS) to the problem and vice versa (the machine guides in the diverse results that finds in the

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synthesis phase, returning to the analysis with new ideas or ways of looking at the problem, that is to say, power the analysis when the student is made a more competent user of the CAS) what generates a bigger understanding of the problem, of his confirmation or rectification, by means of a more appropriate use of the CAS.

2. With use of the CAS like symbolic manipulator, the students tend to admit the possibility to make inferences on something that they ignored in a verbal problem.
3. When using the CAS in the resolution of arithmetic-algebraic verbal problems, the following procedure is observed: analysis of the problem, synthesis of an equation, resolution of this with the CAS and confirmation of one or more results.
4. The quick exploration that allows the CAS to validate numerically the equality among algebraic expressions, allows to achieve senses so that they are used in a competent way the meanings of the errors that have been able to make when solving an arithmetic-algebraic verbal problem. For example, if the obtained results, in fact, they verify the relationships among the well-known or unknown quantities that it proposes the one enunciated of the problem.
5. The resolution of arithmetic-algebraic verbal problems with the CAS like symbolic manipulator, strengthens the relationship between the logical sketch and the analysis of the problem since it allows that you can unchain a resolution strategy. In fact the first phase of the use of the CAS consists on the reading and understanding of the text of the problem, this is carried out by means of a logical sketch of the situation; this involves among other things, a representation logical-mental of the problem in which is integrated the fundamental information of the problematic situation and where the relationships are identified that are central to be able to unchain some resolution strategy.

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INTER-STUDENT QUESTIONING IN STUDENTS' INVESTIGATIONS INTO ALGEBRA: A DIALOGUE BETWEEN KIANJA AND JEREL

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My study investigates mathematical behaviors students engage to decide on the validity of their classmates' ideas and reasoning. I examine a video clip focusing on Kianja and Jerel, who argue about the equations corresponding to points plotted on a Cartesian plane as well as to tables of coordinate pairs. The following research question frames my investigation: What are the questions that students use to understand each other's viewpoints and the types of evidence they consider to prove or disprove each other's mathematical statements? In general, the analysis of the data shows how students can use questions and mathematical representations to reason about their ideas, and to engage in a mathematical discussion.

Statement of Problem and Purpose of Study

I investigate the mathematical behaviors that students engage to decide on the validity of their fellow students' ideas and reasoning. I analyze an 11-minute video clip of five students, focusing on Kianja and Jerel, who argue about the equations corresponding to points plotted on a Cartesian plane as well as to tables of coordinate pairs. These are seventh grade students attending a middle school in an economically depressed city in New Jersey. The data is from an Algebra strand used within a larger three-year study supported by the National Science Foundation. This study, which occurs in an after-school mathematics program, investigates the development of the mathematical ideas and reasoning of students in algebra, combinatorics, and probability.

Research Questions Addressed

What are the questions that students use to understand each other's viewpoints and the types of evidence they consider to prove or disprove each other's mathematical statements? How does Kianja decide the validity of Jerel's method of finding an equation that corresponds to a set of points? How does Kianja convince Jerel that he is wrong?

Methodology

I use a modified version of the methodology outlined in Powell, Francisco, and Maher (2003) for studying the development of mathematical ideas using video data, the steps of which are (1) Viewing attentively the video data, (2) Observing and wondering, (3) Identifying critical events, (4) Transcribing critical events, (5) Coding, (6) Constructing storyline, and (7) Composing narrative.

The codes that I found emerged from the data and not from any source. The method for developing codes is based on grounded theory, which is found in Charmaz and Mitchell (2001). From the data, two categories of codes emerged: questions and evidence. Within the questions category, two subcategories surfaced: surface-level questioning and mathematical questioning. Under the surface-level questioning subcategory are two codes: (1) QP: questions used to clarify certain points, and (2) QR: rhetorical questions. Under the mathematical questioning subcategory are three codes: (1) QF: questions brought about from confusion about another's viewpoint, (2) QT: questions used to test understanding, and (3) QM: questions used to elicit certain responses that inform the answerer of his/her mistake. Within the evidence category, four codes were identified: (1) R: rules, (2) G: graphs, (3) T: tables, and (4) PAS: previous statements said by students to explicate their ideas to one another.

Results or Anticipated Findings and Implications

Kianja uses various questions and evidence to invalidate Jerel's conjectured method for finding an equation that corresponds to a set of points. Kianja uses an equation that Jerel has presented to point out a contradiction between his equation and his stated method. Throughout this video clip, Kianja and Jerel rely upon evidence provided by rules, graphs, tables, and their own previous explications directed at one other to make their presentations and to disprove one other's statements.

Jerel presents two overhead slides that show how his thinking about equations progresses. On his first slide, Jerel infers from a table of coordinate pairs that multiplying the change in *y* by *x*, and then adding on another number produces the corresponding equation. Jerel notes that the change in *y* is 2 and writes the rule $x \times 2 + 10 = y$. On the second slide, Jerel further deduces from some points plotted in the Cartesian plane that multiplying the change in *y* by *x* and adding this to the change in *x* results in the corresponding equation. Since his difference in *y* is 3 and his difference in *x* is 1, Jerel develops the equation $x \times 3 + 1 = y$, which is a special case of his conjectured method.

Kianja takes the lead in trying to show that Jerel's above method is invalid. Her technique of doing so is to take the graph of his first rule, $x \times 2 + 10 = y$, to show that if his method were true, the coordinates would have to go up two (change in *x*) and over ten (change in *y*), when in fact, they actually go up two and over one.

Kianja uses rhetorical questioning to move the conversation along and to express skepticism. Kianja expresses skepticism when she asks Jerel after his presentation "Is that all I need to know?", and follows that with "If so, can we move on now?" It seems here that Kianja is reluctant to discuss Jerel's method, but when prompted by the researcher, Kianja immediately states that Jerel's method will not always work.

In launching into an explanation of why Jerel's method is not valid, Kianja quotes Jerel's previous statements to validate her own assertions. Kianja says, "I'm gonna repeat what you said. You said, you went over one and went up three, that's how you got times three plus one, okay? That's what you said, right?" Using Jerel's statements would seem to strengthen Kianja's arguments. Since Kianja uses Jerel's statements and places them in a new context, it would seem that Kianja's argument would reverberate more strongly with Jerel, more so than if she had not done so.

Kianja also asks questions based on her understanding of Jerel's conjectured method to get Jerel to see her viewpoint and his own mistakes. To relate the other rule, $x \times 2 + 10 = y$ to Jerel's conjectured method, Kianja asks Jerel, "What ten do you see on that paper?" while pointing to a graph of plotted points, inquiring of Jerel what difference of ten he can see on the paper. Presumably, Kianja does this to get Jerel to see the disconnect between his claim that in the rule $x \times 3 + 1 = y$, 1 is the change in *x*, and the fact that 10 is *not* the change in *x* in the rule $x \times 2 + 10 = y$.

Kianja uses other questions to engage Jerel in their mathematical conversation. She questions Jerel to determine whether he understands what she is saying. In general, the analysis of the data shows how students can use questions and mathematical representations to reason about their ideas, and to engage in a mathematical discussion.

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THE IMPACT OF A UNITS-COORDINATING SCHEME ON CONCEPTUAL UNDERSTANDING OF AN IMPROPER FRACTION

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This paper investigates how a units-coordinating scheme affects a seventh grader's understanding of improper fractions. The study shows that extending the scheme to three levels of units may be critical in developing conceptual understanding of improper fractions. This conclusion is based on three observations: a student's difficulty in finding an improper fraction equivalent to a mixed number, implicit view of a composite unit through a unit of one, and difficulty in making a whole when given an improper fraction.

This paper concerns how a units-coordinating scheme, which is one of the concepts of multiplication in whole number contexts (Steffe, 1994), affects a seventh grader in understanding improper fractions. A units-coordinating scheme coordinates two composite units, say a unit of five and a unit of seven, by inserting the unit of five into each of the seven units of one to produce a composite unit containing seven units of five. Coordinating units in continuous contexts is distinguished from doing so in discrete contexts in that partitioning is involved in continuous contexts. Starting with a continuous unpartitioned unit, implementing a coordination of, say, five and seven means to partition the unit into five parts and then each part into seven parts.

Mike, a seventh grader, provided the data for this study. The data used in this paper is based on the examination of 16-videoteaped sessions that took place over one semester period. In each session, 20-30 minutes long, Mike was asked to work with computer software called Tools for Interactive Mathematical Activity (TIMA) (Olive, 2002). TIMA allows students to make rectangular regions called bars and partition the bars into parts, the parts into subparts, etc.

Close examination of Mike's progress over the one-semester period provides three major observations that may answer how a units-coordinating scheme influences the development of understanding of improper fractions. The first observation is that Mike had difficulty in finding an improper fraction equivalent to a mixed number. When asked to make a bar so that a three-inch bar is twice as long as the bar, Mike first made a bar and partitioned it into three parts to represent the three-inch bar. He then partitioned each of three parts into two parts, pulled one of them out and repeated it three times. Mike correctly identified that the result is one and a half inches. However, when he was asked to convert the result into a fraction, his answer was three sixths with a reasoning that each part in the result indicated one sixth. Even though he was able to think the three parts comprising the result was one and a half inches long, he referred to the whole of the six-part bar to name the three parts in inches using a fraction. That is, he conflated the meaning of the three parts as its length in inches with its meaning as three out of six parts. This conflation may result from his inability to maintain a second level of a unit, here, the unit of one, in his activity of coordinating a unit of two with a unit of three; he was unaware of the fact that he was dealing with a unit of one, not three, as a unit for partitioning to get a half of a unit of three. When asked to make a bar so that a five-inch bar is three times longer than the bar, he repeated the above procedure. In response to the teacher's question as to the length of each of the five parts in final bar, he said it was one third of one inch. However, it took a long time for him

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to see that the five parts he just produced indicated five thirds of one inch.

The second major observation is that Mike implicitly regarded a composite unit in terms of a unit of one but wasn't explicitly aware of that. For the question of making a share for one person when seven candy bars are shared among nine people, Mike partitioned each of seven bars into nine parts and pulled out one part from each partitioned bar. Responding to what fraction a share for one person was, he answered one ninth of all the candy bars. He elaborated that each part is one ninth of each candy bar, so if he lined them up he would get one ninth of all the candy bars. He also said that one share is seven sixty thirds of all the candy bars but never answered on the basis of one candy bar such as seven ninths of one candy bar. This indicates that he just considered two levels of units and didn’t yet extend to three levels; he thought one part in one candy bar (one ninth) and nine parts in seven candy bars (seven sixty thirds), but didn’t think of seven parts in one candy bar (seven ninths). It is apparent that Mike came to be able to maintain second level of units while coordinating two given units: to consider nine equal shares from seven bars, he considered one ninth of seven candy bars through seven one-ninths of one candy bar. However, it is not clear whether he could consider seven parts comprising the result in terms of third level based on the maintained second level such as seven units of one ninth of a bar for one ninth of a unit of seven bars.

The third observation is that Mike had difficulty in making a bar so that a given bar is an improper fraction of the bar. Given the question of making a new bar so that a given bar is seven fifths of the new bar, Mike struggled with even restating the question. He rephrased the question saying that the new bar is seven fifths of the given bar and then made it by dividing the given into five parts and repeating one of them seven times. By coloring the parts in the new bar corresponding to the given bar, he realized the given bar is five sevenths, rather than seven fifths. He then made a bar with five of the parts resulting from dividing the given bar into seven parts. However, it was not easy for him to conceptually produce a mixed number, one and two fifths, equivalent to seven fifths; it was evident that his concept of seven fifths remained in two levels of units, disregarding the unit of one. It took a long time for him to realize that there is one five-part and two remaining parts in the seven-part bar, so seven fifths is equal to one and two fifths. Based on the understanding of improper fractions through converting them into mixed numbers, he was able to make a bar so that a given bar is thirteen sevenths of the bar. When asked to make a twenty-five sevenths bar from thirteen sevenths bar, he first made a whole bar using a seventh and then a twenty-five sevenths bar by repeating the whole bar three times and adding four sevenths to the three wholes. One seventh seems considered as an iterable fractional unit (Steffe, 2002) but it was not, in that he used one seventh only in order to make a whole and recognize the remaining parts with respect to the whole, rather than generate a fraction based on one seventh. Such a way of making a fraction reveals that his fractional reasoning was additive rather than multiplicative.

The three observations lead to a conclusion that a seventh grader’s difficulty in working with improper fractions is related to his inability in extending a units-coordinating scheme to three levels of units. This result indicates that conceptual understanding of improper fractions can be better achieved if a student is able to deal with a unit of one comprising a given unit as an iterable unit (when given two units), to reconsider another unit in terms of the iterable unit of one, and to construct a unit on the basis of the reconsideration.

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MATHEMATICAL FLUENCY MEASURED WITH THE FOUR PARAMETERS OF FOREIGN LANGUAGE LEARNING: APPLICATIONS OF THE INTEGRAL

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The concept of mathematical fluency was developed, and the four parameters used in foreign language learning: reading comprehension, writing, speaking and listening comprehension, were employed to measure it. Local fluencies, with mathematical fluency as a global amalgam of these, were classified. Interviews, action research and observations of students learning how to generate solids of revolution, and to use integral calculus to calculate their volumes, were carried out. In depth analysis employing the four parameters of foreign language learning offers a methodology for studying student learning and understanding, that can be generalized to other mathematical areas, and adapted to quantitative as well as qualitative methods.

Introduction

Mathematical fluency, as is defined in this study, includes traditional aspects such as accuracy, efficiency and flexibility, but is motivated by fluency as a goal when learning a foreign language. To ask "why" fluency is important in mathematics learning is similar to asking "why" fluency is important in foreign language learning. If fluency isn't achieved, learning, when it occurs, is fragmented, partial and weak, and almost always ephemeral. How is fluency to be measured? This question was what triggered the idea of similarity with fluency in foreign language learning. Now, of course, there is no isomorphism between mathematics and language. To begin with, no-one is a native mathematics speaker. However, when analyzing the four parameters of foreign language learning: speaking, writing, understanding (listening comprehension, reading comprehension) it was felt that they could be useful in measuring mathematical fluency.

Mathematical fluency is linked not only to performance, but also to conceptual understanding, and employs the four parameters used to evaluate foreign language learning, that is, *reading comprehension*, *listening comprehension*, *speaking* and *writing* as indicators. One of the main objectives of this study was to measure what happens when taking into account these four parameters, and relating what is called *local fluency* to the assimilation of new concepts, in particular the concepts which are needed to carry out and understand applications of the integral. *Global fluency* is an amalgam of local fluencies. A research design that incorporated the four parameters of foreign language learning as a measure of mathematical fluency was developed, and the concept of mathematical fluency itself had to be adequately defined. The design was developed for qualitative research, but can be extended to large scale quantitative research as well.

Theoretical Framework

The elements of the theoretical framework provided the language and concepts which permitted the detection of mathematical fluency as defined in the present study, and as measured by the four parameters of foreign language learning. In particular, the concept of *procepts*, as developed by Tall and Gray (1991) helped to analyze and pinpoint difficulties that the

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unsuccessful student might show, as the integral symbol is a prototype of the symbols that simultaneously represent a process (integration) and a concept (accumulation).

Extra-mathematical and *structural metaphors* (Pimm, 1987), the first referring to metaphors outside the realm of mathematics (for example, an *imaginary* number), and the second referring to metaphors within the actual language of mathematics (the complex number as a *vector*), can be helpful within a particular context and misleading when transferred to a different one. These metaphors were also detected in this research, through the fluency parameters. The presence of *cognitive obstacles* (Brousseau, 1997) was also explored, in particular the ones defined as *didactical*.

Mathematical fluency was defined by the researcher of this study as *accuracy*, *efficiency* and *flexibility*. Although this definition apparently coincides with the definition of fluency seen in other works, the researcher marks certain differences or generalizations when defining the components, not accounted for, to her knowledge, in previous studies. The researcher's definition was motivated by fluency as a goal when learning a foreign language. Mathematical fluency, as is detected in the context of this study, has the four components of foreign language learning: speaking (with coherence and logic, that is, correctly, according to standardized norms), listening and reading with understanding (comprehension) and writing. It is important to emphasize that the four parameters are considered *measures* of mathematical fluency. For example, *speaking* mathematics is not considered, in this study, as a method to foment mathematical fluency, but an indicator itself of mathematical fluency.

Methods and Modes of Inquiry

The research questions were formulated with respect to mathematical fluency, and with respect to the specific mathematical content. The preliminary study, as well as the main study, was carried out with a standard second calculus class at a public community college in the northeastern United States. The studies were a combination of *action research* and *interviews* (case studies).

Results and Implications

The results of this study showed a definite relation between mathematical fluency as measured by the parameters of foreign language learning, and performance in the particular aspect of calculus that was researched. The implications for future investigation are multi-fold; the use of the four parameters of foreign language learning is not by any means limited to research in calculus learning, and can be a tool in doing research at all levels. Testing for fluency in foreign language is often on a massive level (the Test of English as a Foreign Language, TOEFL, for example), but the parameters of reading comprehension, writing, listening comprehension, and recently, speaking, are part of the evaluation process. This means that testing for fluency in mathematics, defined with the components of accuracy, efficiency and flexibility, can also be designed on a large scale level in which quantitative techniques, if necessary, can be used. These designs can be for purposes of research, or also in terms of placement or qualifying exams.

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INFINITY OF NUMBERS: A COMPLEX CONCEPT TO BE LEARNT?

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The paper is based on survey results, and will focus on the development of students' understanding of infinity. The same tasks were answered by different age groups of students: grades 5, 7, 11, and teacher students. The results show that most of the students did not have a proper view of infinity, even not at the teacher education program.

Infinity awakes curiosity in children already before they enter school: "Preschool and young elementary school children show intuitions of infinity" (Wheeler, 1987). However, this early interest is not often met by school mathematics curriculum and not discussed in school, and infinity remains mysterious for most students throughout school years.

Actual and potential infinity

Consider the sequence of natural numbers 1, 2, 3, … and think of continuing it on and on. There is no limit to the process of counting; it has no endpoint. Such ongoing processes without an end are usually the first examples of infinity for children; such processes are called potentially infinite. In mathematics, such unlimited processes are quite common. However, the interesting cases in mathematics are, when infinity is conceptualised as a realised "thing" – the so-called actual infinity. It requires us to conceptualise the potentially infinite process as if it was somehow finished (Lakoff & Núñez, 2000). The transition from potential to actual infinity includes a transition from (an irreversible) process to a mathematical object. In the history of mathematics, the exact definition of and dealing with actual infinity is something more than one hundred years old (e.g. Boyer, 1985; Moreno & Waldegg, 1991). Infinity has been an inspiring, but difficult concept for mathematicians. It is no wonder, that also students have had difficulties with it – especially with actual infinity and density. Previous research has identified typical problems and constructive teaching approaches to cardinality of infinite sets (e.g. Tsamir & Dreyfus, 2002). Fishbein, Tirosh and Hess (1979) inquired students' view of infinite partitioning through using successive halvings of a number segment. They concluded that students on grades 5–9 seem to have a finitist rather than a nonfinitist or an infinitist point of view in questions of infinity. Even at the university level, the concept of infinity of real numbers is not clear for all students (cf. Merenluoto & Pehkonen 2002). For example, Wheeler (1987) points out that university students distinguished between 0.999… and 1, because "the three dots tell you the first number is an infinite decimal". In this paper we want to find out what is the level of students' understanding on infinity in Finnish schools, and how this understanding develops on different levels: grade 5, grade 7, grade 11, and at elementary teacher education.

Methods

The paper combines some partial results of two research projects implemented in Finland (Hannula et al, in press). In grades 5 and 7 of the Finnish comprehensive school, the representative random sample of Finnish students consisted of 1154 fifth-graders (11 to 12 years of age) and 1902 seventh-graders (13 to 14 years of age). In the sample of elementary teacher students, we had all first-year students (altogether 269) from three Finnish universities (Helsinki,

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Turku, Lapland). A reference sample from school (grade 11) was selected at random (N=1200). In both research projects, there was a questionnaire inquiring students' mathematical understanding, and a part of the tasks singled out students' conceptions and skills in infinity. We focus here on the two following infinity tasks.

- Task A: Write the largest number that exists. How do you know that it is the largest?
- Task B: How many numbers are there between numbers 0.8 and 1.1?

About results

To each question, we can find answers that remain on the level of finite numbers, answers that describe processes that do not end (potential infinity) as well as some answers that indicate that the student has an understanding of the final state of the infinite process (actual infinity). In the fifth grade, 20 percent of the students have some understanding of the infinity of natural numbers, but only few have any understanding of density of rational numbers. The situation improves, as the students get older (and selected). Infinity of natural numbers is understood earlier than density of rational numbers, and potential infinity is understood earlier than actual infinity. It is somewhat worrying that even in the selected group of 11th graders barely half of the students understand density of numbers. As students get older, the potential infinity becomes less frequent; it seems as if it was an intermediate stage that leads to an understanding of actual infinity (at least in these contexts).

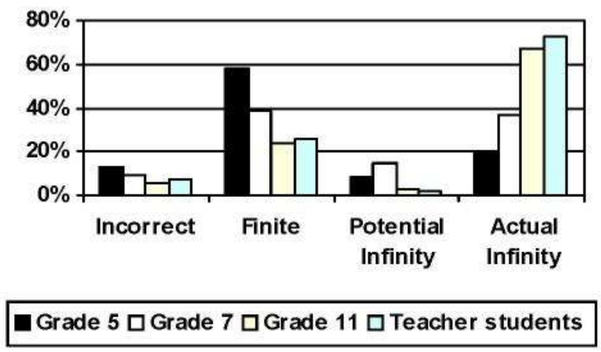

Fig. 1: Answers to task A.

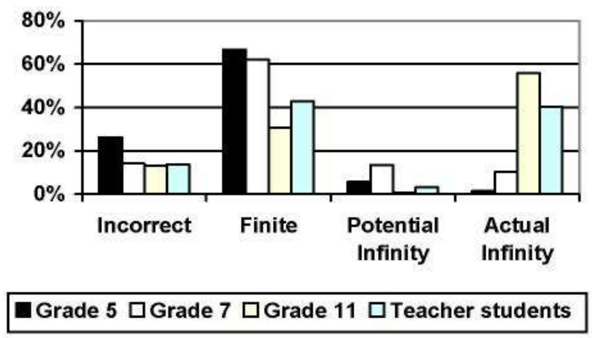

Fig. 2: Answers to task B.

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ACCOUNTING FOR SIXTH GRADERS' GENERALIZATION STRATEGIES IN ALGEBRA (1).

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This qualitative study reports on findings we obtained from pre- and post-interviews of twelve 6th grade students. We address the basic question: What abilities do they have that influence the manner in which they express and justify generalizations in algebra? Results indicate that the students established generality figurally and numerically, and that they were are capable of symbolic generalization towards the end of three sequences of teaching experiments.

Objective

Twenty-nine 6th grade students (12 males, 17 females, mean age of 11) participated in a classroom teaching experiment involving the formation and generalization of linear patterns. Our basic objective in this research investigation is to provide a descriptive account of elementary structures of thinking relevant to expressing generalizations of patterns of figural and/or numerical objects in algebra. Such an account has been empirically justified on the basis of findings from clinical pre- and post-interviews, with the teaching experiment on generalization as providing some level of instructional intervention over the course of five weeks.

Theoretical Framework

Our theoretical framework has evolved out of our previous research investigations in the area of generalization in algebra at different levels (Rivera & Becker, 2005, 2003; Becker & Rivera, 2005). We consistently observed the predominance of either figural or numerical modes of generalizing that individual interviewees seem to have consistently demonstrated over several different problem situations. Those who generalized numerically primarily established their formulas from the available numerical cues. They were not consistently capable of justifying their generalizations non-inductively or in some other valid way. They also frequently employed trial and error as a numerical strategy with no sense of what the parameters in particular formulas represent. Some of their numerical methods contained fallacies and contradictions, and they were object-oriented in the sense that the formulas they developed were justified solely in terms of how well the formulas fit the limited information they examined. Those who generalized figurally were capable of justifying their generalizations non-inductively and in other valid ways due, in part, to the manner in which they were able to connect their symbols and variables to the patterns that generate the figures. They were relation-oriented in the sense that they saw sequences of figural cues as possessing invariant structures and thus, were necessarily constructed in particular ways. We also note that, while the students who were predominantly figural generalizers did not see the need to set up a table of values in order to establish a general formula, those who were predominantly numerical generalizers were predisposed to initially setting up a table in order to perform a numerical strategy with little regard to how the dependent values may be perceived otherwise (for example, figurally). There were a number of numerical generalizers who viewed variables as mere placeholders with no meaning except as a generator for certain sequences of numbers. With figural generalizers, they saw variables not as mere placeholders but within the context of a functional relationship and, thus, were more likely

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

capable of generalizing to an explicit, closed formula. There were cases, too, in which some learners manifested pragmatic modes for expressing generality, that is, their generalization reflected a capacity for employing both numerical and figural strategies.

Results

Figure 1 presents what we refer to as a theory of generalizing types involving figural and/or numerical patterns in algebra. The theory is a further refinement of our evolving theoretical framework. Drawing on results of our work with the sixth-grade class, we further classified figural and numerical modes of generalizing as either additive or multiplicative in character. Also, some students have been documented to be using analogical strategies that they would then analyze figurally or numerically. In the model, we identify two phases of the generalizing process: protorepresentational and representational. The representational level is the domain of variable use and fluency. We have found that students with no knowledge of variables were still capable of generalizing partially or in a situated manner. We have also found that students with some knowledge of variables produced either partial generalizations or full algebraic generalizations depending on their competence in understanding variables in a functional context. Thus, understanding of variables leads to differing levels of function use (early, situated, symbolic) that consequently influence the performance of generalization.

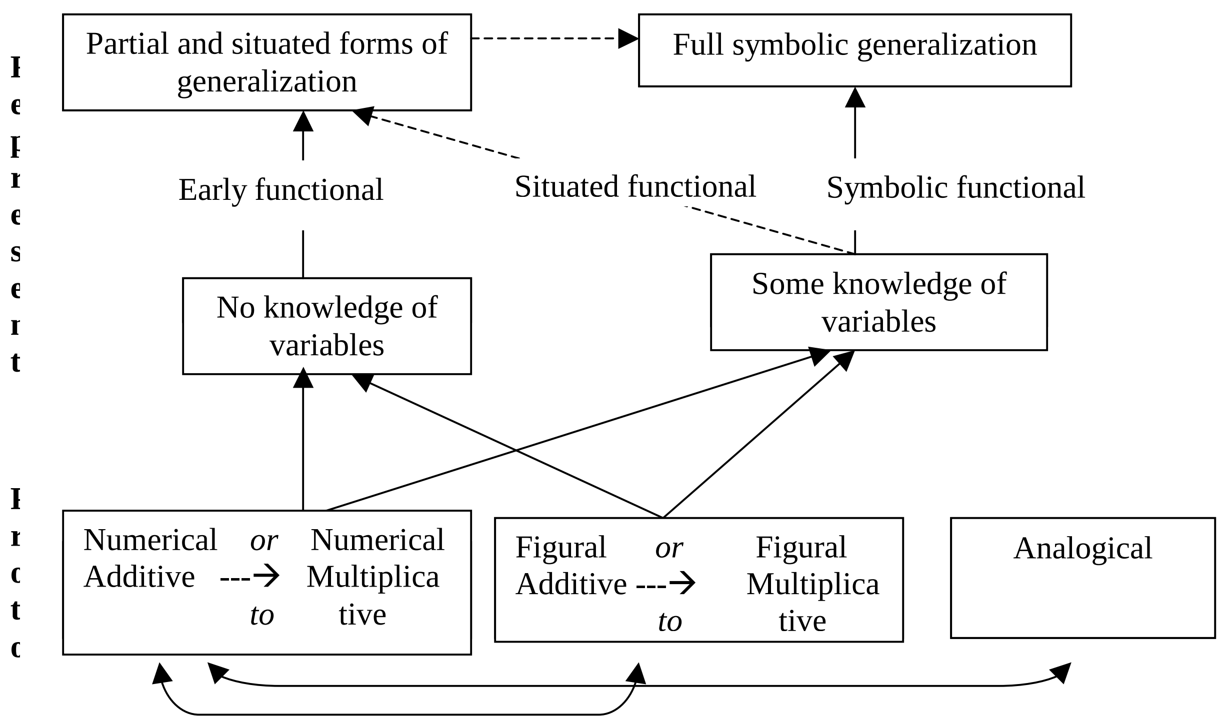

Figure 1. A Theory of Generalizing Types Involving Figural and Numerical Patterns

Endnotes

1. This paper is based upon work supported by the National Science Foundation under Grant No. REC-0448649 awarded to the first author.

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CONSTRUCTION OF MEANINGS TO THE MATHEMATICAL OBJECTS OF VARIABLE AND FUNCTION THROUGH PROBLEMS

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Research provides evidence regarding the genesis of the meanings of variable and function concepts as well as to the signification process occurring during the sense production of the tabular and algebraic representations for a linear function. Empirical evidences were obtained through a study case, where 15-year-old junior high school students were faced to verbal problems of continuous variation. It was proved that a rate of change given as datum and a solution implying a set of values in problems are central starting points for a student to unchain a numerical operational process through which he/she can be detached from the meaning of a literal as an unknown value in an equation and pass on to the meaning of a literal as a variable in a linear functional relationship expressed as an equation in two variables.

The study comprised two stages: the stage of teaching and the experimental stage. A teaching model based on a numerical approach (see, e.g. Rubio, 2002; Rubio&Valle, 2004) was used allowing students to: a) develop their analytical ability; b) give sense to relationships between the unknown and known elements of a problem; c) construction of meanings to the algebraic representations of the problem (such as 1st grade, 2nd grade equations and systems of equations.). The other stage consisted in a case study with four students from three different knowledge stratums, selected from a previously stratified school group according to their performance in a diagnostic questionnaire and in the teaching stage. Clinical interviews of the students-case, which were videotaped, proved that: i) the cognitive tendencies (Filloy, 1991) to use numerical quantities to unchain the analysis of problems new to them, where rate of change as datum occurs and which solution involves a set of values; ii) students already had an intuitive knowledge on the rate of change notion, which they used during the search process of the solution by facing the first problems; iii) during the meaning construction process of the algebraic representation of a function expressed as an equation in two variables, students spontaneously used numbers, but going different ways: some of them achieved it almost directly as of the numerical relationships established by them and others after having organized such relationships in a table; iv) once giving “sense” (Filloy, 1991) to a literal as a variable in a linear functional relationship is achieved, the student is able to give meanings to parameters “m” and “b” of the linear functional relationship represented by a general equation in two variables of the type: y = mx+b.

Reference Setting

Works having used a numerical approach to solve problems and make possible the transition of the arithmetic thought to the algebraic one (see, e.g., Bernardz & Janvier, 1994; Kieran, Boileau & Garançon, 1996; Rojano & Sutherland, 1993; Rubio, 2002; Rubio & Valle, 2004) are part of the reference setting of this research. Studies confirming that the learning of the variable

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concept is a difficult and slow process (Ursini-Trigueros, 1997). Modeling works having as their main aspect the construction of the variable meaning and as a crucial point the establishment of a relationship among variables (see, e.g., Janvier, 1996). Researches, such as those by Blanton and Kaput, 2004 regarding the origin of the "functional thought" in children and, works on semiotics (Eco, 1995) and their relationship with the mathematical education (Puig, 1997; Filloy,1999).

Conclusions

The research evidenced that the use of numbers (numerical quantities) instead of literals acts as a semantic mediator or bridge between an arithmetic use of the unknown or variable and an algebraic use of the same. From interviews can be observed that the use of numerical quantities together with the sense they give to the rate of change fosters an operational process and a co-variation thought (Blanton & Kaput, 2004), where relationships between the variable magnitudes of a problem, primarily established with numbers, acquire a meaning for the student, thus allowing to produce "senses" for the algebraic representation of the functional relationship between variables as of a generalization process of operations carried out with numbers.

The study showed that the teaching model and the families of word problems used in both research stages allowed the students to generate signification processes, which make possible a systematic advance in the competence of the mathematical conceptos of the unknown, equation, variable and linear function, in more general uses every time.

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A PSYCHOLINGUISTIC APPROACH TO TEACHING/LEARNING MATHEMATICS BASED ON BRAIN PATTERN PROCESSING OPERATIONS

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The poster presents an emerging framework on teaching/learning mathematics, grounded in a 10-year project conducted as design-based research and in education-research literature (mathematics, Schoenfeld, 1992; 2nd language acquisition, Krashen, 1995), and focusing on the mathematical domain of algebra. The project was motivated by my observations as an educational practitioner and as founder/director of a natural language school, and bears implications for teaching, learning, and design. The framework foregrounds cognitive and affective factors contributing to or hindering mathematics learning and connections between these factors (see Schoenfeld, 1992). These factors, I demonstrate, can be modeled as cohering around the construct '*pattern*' that undergirds a heuristic model of brain-pattern processing.

As mathematics is arguably the 'science of patterns' (Devlin, 2003, Schoenfeld 1992) with a language to deal with them (Esty, 1992), I discern structural, functional, and developmental continuity from simple pattern-processing perceptual activity (e.g., recognition, comparison, matching) to learning language and basic mathematical skills (Amezcua, 1999). Thus, similar pattern-processing cognitive faculties are active in learning languages and mathematics. Much of naturalistic learning is the development of 'equivalence classes', e.g., 'table.' To the extent that mathematics-learning shares with language-learning cognitive faculties, students need ample opportunity and supportive contexts to recognize and construct equivalences. Algebra, though, is particularly demanding, as equivalences, e.g., $[123=10^2+2*10+3] \equiv [x^2+2x+3]$, are stated but not initially evident or intuitive to the learner. Abstraction is based on the innate ability to recognize equivalent classes, but observation strongly suggests that at its natural level the skill is insufficient to deal with mathematics' requirements. Strengthening this ability allows fluency in the mathematical language to emerge making possible the transferring of skills between one area to another via the abstraction process.

The psycholinguistic approach to teaching/learning mathematics that emerged from my study offers tools for diagnosing learning problems and for designing strategies for their resolution. The framework deals with affective factors metacognitively. Students are lead to recognize and assess their hidden beliefs by showing them that their equivalence-class recognitions skills in daily activities are the same as those required in mathematical endeavors. Through this, one can remove learning blockages by replacing old beliefs with more effective ones. Much of my research was done from a qualitative perspective, but I am conducting new quantitative studies in rural areas in Mexico where the difficulties are especially challenging to ground the research.

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UNDERSTANDING INTEGERS: USING BALLOONS AND WEIGHTS SOFTWARE

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The purpose of this paper is to describe software we developed (*Balloons and Weights*) that supports students' integer understanding and to share results of its use in a fifth grade classroom. Integers are an integral part of the middle school curriculum (NCTM Standards, 2000) and mark a transition from arithmetic to algebra because of the abstract thinking required when working with them (Linchevski & Williams, 1999). Given the abstract nature of integers, it is not surprising that students have tremendous difficulty operating on them. Despite the importance of understanding integers and the difficulties students have understanding and operating on them, relatively little research has been conducted in this area. The development and use of software described in this study represents one attempt to address this void.

A paper-pencil version of the *Balloons and Weights* software was initially reported by Janvier (1983). In our interactive version, balloons and weights can be attached to a basket. Helium balloons represent positive integers, whereas weights represent negative integers. One balloon raises the basket one unit. One weight lowers the basket one unit. Thus, one balloon and one weight cancel each other to create a zero pair. Adding on balloons or weights represents addition of integers, whereas removing balloons or weights represents subtraction. The result of adding on or removing balloons and weights is represented on a vertical number line. Also, the model is set in a context of traveling up and down in a basket with balloons and weights attached that is experientially real to students. Because the weights in this model indicate a direction (pulling down) and result in a position, the rules associated with their actions are not as arbitrary as those for a typical horizontal number line. The software affords animation that can serve to support students' imagery when they make and test conjectures about what will happen when, for example, one begins with 4 weights and adds on 7 weights (-4 + -7) or compares beginning with 10 weights and removing 3 weights (-10 - -3) with beginning with 10 weights and adding on 3 balloons (-10 + 3).

This software was used in a 5th grade classroom to introduce addition and subtraction of integers. The students engaged in a variety of experiences in which they added and removed balloons and weights and were asked to describe the outcome of each action and relate that to the symbolic notation. By the end of the lessons most students in the class were able to solve symbolic integer problems without the explicit use of the software by drawing on the imagery supported by the software.

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A FRAMEWORK TO ANALYZE THE ALGEBRA MATHEMATICS REGISTER

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Algebra entails various types and levels of understanding that deal with concepts and procedures. From previous research on this topic I selected four categories of mathematics understanding as a *working framework* to analyze the mathematics register used to communicate algebraic ideas.

1. *Instrumental understanding*: Knowing rules of and how to apply and carry out a procedure without necessarily understanding the reasoning behind the rules (Skemp, 1982).
2. *Procedural understanding*: Knowing "what do and why" (p. 9) and the "ability to deduce specific rules or procedures from more general mathematical relationships" (p. 45)
3. *Conceptual understanding*: Acquiring "knowledge that is equated with connected networks [. . .] knowledge that is rich in relationships" (Hiebert & Carpenter, 1992, p. 78).
4. *Symbolic understanding*: "[T]he ability to connect mathematical symbolism and notation with relevant mathematical ideas and the ability to combine these ideas into chains of logical reasoning." (Skemp, 1982, p. 59).

Note that the definitions above are not clear-cut, they do not exclude each other, nor do they indicate a sequence in which students may acquire a particular type of knowledge.

The following framework defines **categories of the algebraic mathematics register** which, somehow, parallel to the categories of understanding.

1. *Instrumental Register: The **Verbs** Register***.** This mathematics register conveys and reflects *instrumental knowledge*. It is mostly formed by *verbs* to denote actions and sequence of actions. Examples: "A*dd* 2, *divide* by 4, *and plug* in the value."
2. *Procedural Register: The **Verbs** and **Logical Connectors** Register***.** This register conveys and reflects *procedural knowledge*. It contains *verbs* that denote actions or a sequence of actions. The difference with the instrumental register is that it also includes *logical connectors* or logical expressions such as: if/then and this/because. Example: "Divide by 2 on both sides of the equation because we are applying the inverse operation of multiplication."
3. *Conceptual Register: The **Nouns** and **Adjectives** Register*. This register conveys and reflects *conceptual knowledge*. It makes use of *nouns* to name the concept. A*djectives* or *adverbs* may be used to talk about properties of a concept or procedure. Example: a quadrilateral (noun/concept) is a four-sided (adjective/description/property) polygon (noun).
4. *The Formal and Symbolic Register*. Symbolism is central to algebra. There are *symbols* for concepts (x = variable, m = slope), symbols for procedures or operations (+, -, ( )), symbols for relationships (<, >, =,), and symbols or expressions to denote logical statements (for all ∀, and, or). Students are expected to perform procedures that require symbolic manipulation.
5. *The "Making Sense" Register*. Not necessarily related to the working framework. This register is used to explain reasoning and ideas when connecting mathematics to previous knowledge and experiences. Example: When asked about the best phone plan (base price and

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cost per minute) a student expressed “it depends on how much I use my phone.” In algebraic notation his statement can be represented as: “if X <= N (if I use it a little bit) X1 <= Y2 (then company 1 is cheaper than company 2); if X > N then Y1 > Y2.

This framework can be used as a research tool to explore students’ understanding and as a tool for teachers for analyzing their own use of language in the classroom.

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SHAKING HANDS, COMPLETING GRAPHS: WHAT CAN BE GAINED BY LEVERAGING EVERDAY EXPERIENCES IN THE DEVELOPMENT OF ARITHMETIC MODELS THAT SUPPORT ALGEBRAIC REASONING?

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In a study centered on a class of third-grade students and their work on a variation of the Handshake Problem, Blanton and Kaput (2005) consider how teachers can leverage their students' familiar situations and everyday experiences involving number and arithmetic operations to support the development of algebraic reasoning. Specifically, the authors consider the everyday experiences of shaking hands and knowledge of norms about appropriate handshake gestures. They maintain that the task "drew directly on students' everyday experience to make sense of the problem" (p. 226) and that "the opportunity to enact a familiar situation facilitated the development of arithmetic models" (p. 226).

The number of handshakes in a room with n people is equal to the number of edges on a complete graph on n vertices. The Complete Graph Problem asks students to determine the number of edges on complete graphs of 5, 6, 7, and 20 vertices. By comparing student work on the Handshake Problem to work on the Complete Graph Problem, this researcher looked for advantages that may be gained by leveraging familiar situations and everyday experiences in the development of arithmetic models that support algebraic reasoning.

The Handshake Problem and the Complete Graph Problem were implemented by this researcher in three classrooms of pre-service elementary teachers at a large public university in central Texas. Data in this study was collected through field notes, observations, and interviews. The data was analyzed by looking for patterns in students' experiences and in their written work. When observations or written artifacts raised questions, they were answered in interviews during and following the activity.

Student comments during work on the Handshake Problem included "you don't shake hands with yourself" and "once I count the handshake I do with you, I can't count the handshake you do with me." Comments made during work on the Complete Graph Problem included "you can't connect a dot to itself" and "you can't connect to a vertex that was already used."

Students completing the Handshake Problem recorded the number of handshakes in a room of 5, 6, and 7 people and looked for patterns among the differences in the total number of handshakes. Similarly, students completing the Complete Graph problem recorded the number of edges on complete graphs of 5, 6, and 7 vertices and looked for patterns among the differences in the total number of edges. Students used the pattern they had found and extended it to find either the number of handshakes in a room of 20 people or the number of edges on a complete graph of 20 vertices. These findings suggest no advantage in leveraging everyday experiences and familiar situation in the development of an arithmetic model to support algebraic reasoning.

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THE ROLE OF GESTURE AS A FORM OF PARTICIPATION IN NETWORKED CLASSROOMS

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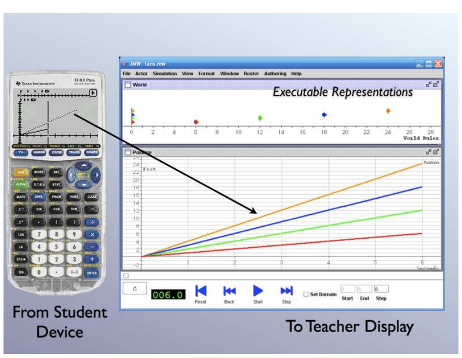
We have been studying the role of gesture as a mechanism to understand how students make sense of mathematical structures (e.g. families of functions) in networked classrooms. We have integrated SimCalc software into Algebra High School classrooms. The software works on a TI-83/84+ and in parallel on a desktop PC, in conjunction with TI-Navigator's wireless network. The software allows students to create functions algebraically or graphically (e.g. dragging hotspots) and see dynamic representations of these functions through animations of actors whose motion is driven by the defined function. We have created sets of activities that exploit these new technological affordances in mathematically meaningful ways. For example, students are in groups and are asked to create a function that moves an actor for 6 seconds at a speed equal to their group number. So groups 1, 2 and 3 create y=x, y=2x and y=3x respectively for a domain [0,6]. Students work is then aggregated into the computer software via the network, and the teacher then has control of what is shown, (e.g. the collective motion, the graphs, the algebraic expressions) to meet various pedagogical purposes. We have built a hide/show feature to allow students to collectively conjecture and make generalizations about how their contributions are contextualized within a class set of contributions. In such an activity, the important concept is slope as rate (something that underlies the mathematics of change and variation), and the family of functions vary via the parameter *m*, in y=*m*x, their group number. The parameter is identifiable and hence more meaningful for students as their independent contributions create the variation. As their collective responses emerge and they are asked what they expect to see, we have observed an interesting combination of mathematical speech and gesture as students reason and make sense of the family of functions. In this activity, students in various school settings, describe the whole set as a "fan" and have used their hands (fingers splayed out) to describe what they expect to see before the teacher shows the set of graphs (for example). Our poster will describe a set of categories of gesture that relate to our various activities we have used in classrooms and describe how gesture is an expressive form of participation and mathematical reasoning. The diagram below illustrates screenshots from the calculator to the computer, and a student gesturing a "fan".



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STUDENTS' CONCEPTIONS OF THE EQUALS SIGN AFTER ALGEBRA II

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An analytical framework was developed by comparing conceptual analyses of the equals sign (Dienes, 1960; Ferrini-Mundy et al, in press; Usiskin, 1988) with students' conceptions from the literature cited above. Four tasks were developed, and given to two high school students who were either enrolled in or had completed pre-calculus. The students were interviewed and audiotaped while working on the tasks. Transcripts of the recording were coded according to the theoretical framework, and cycles of analysis were conducted for confirming and disconfirming evidence (Strauss & Corbin, 1990).

The results suggest that students with strong operational skills in algebra have difficulty articulating different meanings for the equals sign. In addition, a focusing phenomenon (Lobato et al, 2003) was discovered: use of hybrid notation in manipulating equations that directs attention to "doing the same thing to both sides", while obscuring the use of the distributive property to combine like terms.

Beyond Algebra II, students encounter even more uses if the equals sign, e.g. for functions, limits, vectors, and linear transformations, to name but a few. Students need to develop a sense of what "the same" means in these different contexts, but I surmise that many instructors overlook this and focus on operations.

The poster session features task descriptions, the subjects' backgrounds, examples of students' work, and more detail about the analysis.

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CAN TEACHERS PROMOTE LEARNER RESPONSIBILITY THROUGH EXTERNAL ACCOUNTABILITY?

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While it is often assumed that implementing external accountability (testing, grading, disciplinary actions) promotes learners' responsibility, the authors' experiences researching urban Algebra I (1) classrooms have caused them to confront the reasonableness of this assumption, which is rooted in the behaviorist perspective that the most efficient means to develop a behavior is to provide or withhold rewards (Spadano et al, 1997). The authors argue that a focus on external accountability in fact weakens responsibility, which requires internal accountability.

The data for this poster was gathered as a part of a larger case study project studying the role of content knowledge in urban, algebra I teachers' instruction. The authors collectively observed three teachers for approximately forty, 90-minute observations; they also conducted four, 1-hour interviews per teacher. The teachers that were observed work in a large, urban high school with a FARMS (2) rate of 58%; the student body is primarily African American and Hispanic.

The district, school and teachers are confronting the accountability pressures of the NCLB (3) legislation, which emphasizes *mastery of skills* as opposed to *understanding*. The 2005-2006 freshman class is the first class which must pass the year–end Algebra I High School Assessment (HSA) in order to graduate in the state of Maryland. To attempt to manage this challenge, the district has placed ALL incoming freshmen in Algebra I and has implemented a standardized algebra curriculum, while the mathematics department has instituted common exams.

In analyzing the classroom data for the three Algebra I teachers, the work of Joseph Spadano et al's (1997) describes a "Continuum of Educational Orientations", which extends from *social behaviorists*, with their teacher-centered classrooms, to *experientialists*, with their learner-centered classrooms. We, the authors find that the teachers' practice can be located in different places along the continuum, depending on which lens is used to consider the data (e.g. a learning focus versus a student behavior focus), as well as the time of year.

These differing locations of practice also mirror the teachers' continued struggle to balance accountability with responsibility, which Spadano et al show are emphasized in teacher-centered and learner-centered classrooms respectively. As teachers manage the tensions between these two foci, they draw on their perspectives of their own and students' responsibilities, their understanding of the contextual elements of an urban setting, the constraints being imposed by the district and NCLB, their knowledge of students, and their own teacher preparation. As teachers manage this tension, increasing the focus on accountability is generally at the expense of learner responsibility.

Endnotes

1. Algebra I includes symbol manipulation, solving of equations and beginning topics in data analysis.
2. Free and Reduced Meals and Services
3. No Child Left Behind

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PROPORTIONAL VARIATION WITH DYNAMIC GEOMETRY

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This proposal is an extension of the work we have done to further the understanding of the variation and the literal as a variable (Najera, 2004). We take the dynamic geometry environments (DG) as an appropriate means to support the construction of meanings in relation to the proportional variation (PV), and, as a way to approach lineal functions.

Here, we are interested in the communication like the chain of meanings or the mediators employed by the students to make sense of the variation. According to Dorfler (2000, pp. 99-100): *If we know the meaning of relevant words or sentences, then we are able to understand. If we don't, we are not able to understand.* From a semiotic perspective, following Pierce (1960, p. 228), meaning depends not only on the sign or *representation* and its referent, the *object*, but also it depends on an *interpreter,* some kind of representation, an *idea* about the mediating reality of the relationship between referents and signs. Here, the Figures may work as iconic referents of the variation, and the comments, labels and legends, annexed to the illustration, as indexes of the variation, as antecedents of a semiotic inference, a central element to generate the symbolic understanding of the PV.

Experiment: Students (typical age: 12 ½ years) work in pairs, interact with dynamic illustrations like the ones shown below, where, in each case, the point ***p*** may be dragged generating the variation of the segments and the modification of their values. The values are changing, turning this way into indexes of the variation. The students read their results to their classmates and discuss what the most adequate way of representing the results is. In this dialogue words such as *dragging, invariant, variable, dependency, label, variation, slope,* etc. have an important role in the students' construction of meanings.

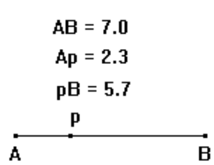

Fig.1

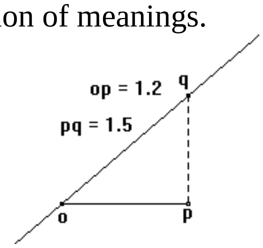

Fig.2

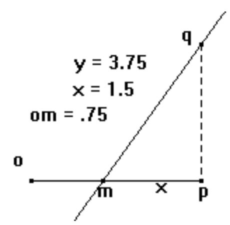

Fig. 3

We explore the potential of Fig. 2, e. g., to study the notion of constant velocity. Some students considered *op* as time and *pq* as the traveled distance, then they were able to symbolize $pq = 1.25op$ and verbally express: *When **p** is being dragged, the traveled distance depends on time, it is proportional to time… the velocity is equal to pq over op and it is invariant*; the figure 1 introduces, simultaneously, the notions of variation and invariant; Figure 3 works with the function $y = mx + b$. Accordingly, the exploration allows to consider that the capacity to symbolize, emerges from the inferences done based on perceived data, illustrations shown in the screen, and the meanings constructed in the interaction with the computer tools.

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A FOCUS ON VARIABLES AS QUANTITIES OF VARIABLE MEASURE IN COVARIATIONAL REASONING

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Recent emphasis in mathematics education has been placed a covariational conception of function, but focusing on "quantities varying," makes evident a subtlety in covariational reasoning. This paper will analyze the potential difficulties one encounters when one's understanding of involves a graph constraining the ways in which variables vary, particularly an inability to reason about why fairly complex functions behave as they do.

Introduction and Purpose.

The predominant conception of function in mathematics today can be described as functions as correspondence, or "a rule that assigns each element x in a set A exactly one element, y, called f(x), in a set B." This current definition of function persists despite the fact that many mathematicians and mathematics educators (Eisenberg, 1991, Thompson, 1994, Wilder, 1967) criticize this conception on pedagogical grounds. In response to the criticisms of the correspondence conception of a function, a number of researchers (Carlson, 1998, Carlson, Jacobs, Coe, Larsen, & Hsu, 2002, Confrey & Smith, 1995, Thompson, 1994, Thompson & Thompson, 1994) have proposed a covariational conception of a function. The covariational conception of a function is based on Euler's notion of function: "[when] some quantities depend on others in such a way that if the latter are changed the former undergoes changes themselves, then the former quantities are called functions of the latter quantities" (Kleiner, 1989).

It is true that this notion of function is consistent with "reform" mathematics, which calls for a shift in attention in the mathematics curriculum from functions as rules and formulas to functional relationships in both pure and applied settings, however there are subtleties to the current conceptualization and frameworks for understanding and classifying students' understanding of function as covariation of quantities and covariational reasoning. Two strands of research which address these issues are the work of Carlson, Jacobs, Coe, Larsen & Hughes (Carlson et al., 2002) and APOS (Action-Process-Object-Schema) Theory (Dubinsky & Harel, 1992).

Mental Action **Description of Mental Action**

Mental Action	Description of Mental Action
MA1	Coordinating the value of one variable with changes in the other
MA2	Coordinating the direction of change of one variable with changes in the other variable
MA3	Coordinating the amount of change of one variable with changes in the other variable
MA4	Coordinating the average rate-of-change with uniform increments of change in the input variable
MA5	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function.

Figure 1: Carlson, et al's (2002) Covariation Framework

Carlson, et al (2002) present a covariation framework which describes five "mental actions" and five coordinated levels of covariational reasoning ability. The covariation framework

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"contains five distinct developmental levels. … [One's] covariational reasoning ability has reached a given level of development when it supports the mental actions associated with that level and the actions associated with all lower levels" (p. 357). The mental action form of their covariation framework is shown in Figure 1.

Covariational reasoning has also been related to the work in APOS theory on functions. Thompson (1994) notes that many mathematics students tend to see a function as a "command to calculate" and that early algebra students are no more likely to see the expression x (12 (x – 5)) as representing a number as elementary students are to see that the expression 4 (12 (4 – 5)) represents anything other than something to do. Researchers (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thompson, 1996, Dubinsky & Harel, 1992) have labeled this such a conception of a mathematical concept as an action conception. A process conception of a function involves the learner automating lengthy sequences of operations into an expression that, in his or her image of it, "evaluates itself" (Thompson, 1994). When a student possesses a process conception of function, he or she can imagine the function as something that performs the sequences of operations but no longer needs to actually think about the chain of operations when envisioning the result of the evaluation. The development of covariational reasoning is related to the progression from an action to a process conception: once a person conceives of a function as the covariation of quantities, they "can begin to imagine 'running through' a continuum of numbers, letting an expression evaluate itself (very rapidly!) at each number" (Thompson, 1994, p. 26) and can therefore conceptualize the way in which the quantities covary.

It is this notion of "quantities varying," though, that makes evident a subtlety in covariational reasoning. Though the varying quantities are a conceptual precursor to a fully-developed conception of a function, these variable quantities are often not the focus of instruction or analysis of students understanding of functions. Both conceptualizations of understanding the concept of function discussed above rely on the underlying notion of an independent variable varying and a dependent variable varying accordingly. The way these two variables vary is quite different: the independent variable is free to vary, but the dependent variable is "constrained," in that as the independent variable varies, the dependent variable must vary in a particular way. As an example, consider a function representing the height of a roller coaster car at any given time during the ride. In this example, that constraint on the dependent variable makes sense – why would one be concerned with the height of the car in a vacuum? The variation of that quantity would give us no useful information about the velocity, acceleration, energy, etc. of the situation.

In this paper, we will examine a teaching experiment that placed the variability of quantities in the foreground and was designed to better understand how students develop covariational reasoning abilities that position them to analyze functional relationships. The study was grounded in Saldanha & Thompson's (1998) conceptualization of covariation as (1) an understanding of a variable is a measurable quantity (it has a magnitude) whose measure can vary; (2) the coordination of two variables, each of which can be envisioned as varying independently; (3) envisioning a graph as a collection of points; (4) envisioning the collection of points as being generated by keeping track, simultaneously, of two quantities whose values vary; and (5) envisioning that every point in a graph represents, at once, simultaneous values of two quantities.

Methods and Data Sources.

The data for this study was gathered as part of a constructivist teaching experiment (Steffe & Thompson, 2000). The analytic tools employed fall under the heading of "grounded theory" (Glaser & Strauss, 1967). Analysis involved the development and refinement of hypotheses through a process of continual review, constant comparison, and revision. The analysis consisted of multiple iterations of the generation and refinement of hypotheses, first from a global perspective (reviewing the entire data corpus to identify segments of theoretical importance) and second from a local perspective (line-by-line coding of the segments identified and continual development and refinement of hypotheses).

Participants for this study were a group of 11 undergraduate mathematics and secondary education dual majors at a large, private university in the Southern United States. For this study, we focus primarily on four class sessions that took place in the first few weeks of the Fall semester. Each class session was videotaped and immediately following the class, was transcribed and annotated. All class artifacts are also analyzed as part of this study.

Results and Conclusion.

The results to be described in this paper focus on students initial inability to reason covariationally about the straightforward, but quite complex question: Explain why the graph of f(x) = sin(nx) behaves as it does. Though their explanations would qualify as covariational reasoning (and at a minimum) MA1, analysis of students' initial explanations of the behavior of these functions indicate their reliance on graphs as the primary image underlying their explanations. Further analysis indicates that their understandings and explanations were grounded a more advanced conception of function, but something falling shy of covariation: their focus was not on how the quantities covaried but how the graph behaved. For one to explain the behavior of these functions, they must keep track of (at least) three quantities: x, nx, and sin(x), and only two of these quantities appear in the graph. A conceptual explanation of this problem involves an understanding of how the quantities x, nx, and sin(nx) each vary independently, an understanding of how x and nx covary, how nx and sin(nx) covary, and finally how x and sin(nx) covary. An explanation of this sort enables one to understand why the family of functions behaves as it does. The proposed paper will discuss the students' developing understandings as they took part in a course that focused on appropriate images of variables and functions that would support covariational reasoning. The affordances of this developing understanding, both in the students' mathematical development and in their pedagogical conceptualizations (all were future teachers) will be discussed.

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ASSESSMENT

METHODS FOR CONTROLLING FOR OPPORTUNITY-TO-LEARN

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One of the most critical variables in determining students' achievement is opportunity to learn (OTL). In this paper we describe low cost methods for studying OTL that enable researchers to understand variation in both the implemented and the attained curriculum. Throughout, we draw upon data collected during field-studies of curriculum materials developed by the University of Chicago School Mathematics Project.

Results from international comparisons of mathematics achievement have highlighted the importance of both curriculum and teaching in relation to students' learning (McKnight et al., 1987; OECD, 2004; Stigler & Hiebert, 1999; Valverde, Bianchi, Wolfe, Schmidt, & Houang, 2002). In particular, both curriculum materials and teachers' implementation of the materials influence pupils' opportunity to learn (OTL) mathematics. In fact, OTL is among the most reliable predictors of students' achievement (Burstein, 1993; Hiebert, 1999; Shavelson, McDonnell, & Oakes, 1989; Valverde et al., 2002). Thus, one methodological question for researchers is "How can one gather evidence in reliable, cost-effective ways about the OTL provided by teachers' use of curriculum materials?"

In the United States, the report, *On evaluating curricular effectiveness* (National Research Council, 2004), recommends that comparative studies of curricula be conducted to designate a curriculum as providing scientific evidence of its effectiveness. However, "How do we compare curricula that are aiming at different learning goals? Do we, for example, assess only the goals they have in common, thereby failing to address new goals the project has adopted? Or do we assess all goals, common or not, thereby asking students to respond to assessment tasks for which they have had little or no preparation?" (Kilpatrick, 2003, p. 478). Thus, a second methodological question for researchers conducting comparative studies is "How can students' achievement be compared in a manner that is fair to both curricula given the differences in OTL in both the intended and implemented curricula?"

We believe that conceptualization of the construct of OTL and methods for measuring it are critical issues for mathematics educators. In this research report, we address the two methodological questions by drawing upon data from field-study evaluations of the University of Chicago School Mathematics Project (UCSMP) curriculum materials. We use these data to illustrate methods for documenting OTL in relation to the implemented curriculum in classrooms. We also describe methods for data analysis that control for OTL in relation to achievement, and note how different conclusions can be reached when OTL is controlled. We hope that other researchers involved in curricular studies will find such methods promising.

OTL Measures Related to the Implemented Curriculum

In order to document OTL provided by classroom teachers, researchers sometimes use classroom observations supplemented by follow-up interviews or textbook diaries (Stigler & Hiebert, 1999; Tarr, Reys, Chavez, & Shih, 2006), possibly audio-taping or video-taping the observations. Although such methods provide rich descriptions of classroom discourse, they are

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costly in money and personnel, particularly if many classrooms over a wide geographic area are involved. Hence, they are almost never used to describe what happens in multiple classrooms over the course of an entire school year.

In the evaluation studies of the materials developed by UCSMP, researchers used a combination of teachers' self-reports on curriculum implementation, classroom observations, and interviews to document opportunities to learn. During the 2005-2006 school year, field-studies were undertaken for *Transition Mathematics* (Third Edition) and *Algebra* (Third Edition), courses primarily for 7th and 8th/9th grade students, respectively. For each textbook chapter, teachers field-testing the UCSMP materials completed a chapter evaluation form on which they indicated if a given lesson was taught and identified the problems assigned from that lesson. They also rated lessons and problems, documented when and how they supplemented the textbook, and answered specific questions about sequence or approaches used in the chapter. The chapter evaluation forms were sent electronically to each teacher; the teachers could return them either electronically or via paper copy.

For each UCSMP teacher, the number of lessons that were taught, including end-of-chapter summary materials that are an integral part of the text, was noted. In addition, because the UCSMP texts are written with the expectation that teachers will assign almost all the problems in each lesson, the number of problems assigned in each chapter was also compiled; in this case, the review and test problems at the end of the chapter were not included because teachers are expected to use these problems based on the needs of students rather than regularly assigning all of them. Finally, to understand how the nature of curriculum implementation varied across the school year, the percent of lessons taught and the percent of problems assigned were compiled for chunks of 4–5 chapters in each book.

As shown in Tables 1 and 2, for each teacher using the UCSMP Third Edition materials, two measures of OTL in relation to implementation of the curriculum are obtained: the percent of lessons taught; and the percent of problems assigned.

Table 1. Opportunity to Learn UCSMP *Transition Mathematics* (3rd Ed) by Teacher

Teacher	Percent of Lessons Taught				Percent of Problems Assigned			
	Ch 1-4 (*n*=44)	Ch 5-8 (*n*=55)	Ch 9-12 (*n*=42)	Overall (*n*=141)	Ch 1-4 (*n*=903)	Ch 5-8 (*n*=1028)	Ch 9-12 (*n*=706)	Overall (*n*=2637)
B	82	75	62	73	73	65	47	63
C1	98	89	0	65	79	84	0	60
D	100	91	0	67	73	70	0	52
E	95	100	40	81	71	73	28	60
F	100	98	67	89	98	93	55	85
Overall	95	91	34	75	79	77	26	64

Note: Based on Thompson & Senk (2006b).

Table 2. Opportunity to Learn UCSMP *Algebra* (3rd Ed) by Teacher

Teacher	Percent of Lessons Taught				Percent of Problems Assigned			
	Ch 1-4 (*n*=39)	Ch 5-8 (*n*=51)	Ch 9-13 (*n*=60)	Overall (*n*=150)	Ch 1-4 (*n*=707)	Ch 5-8 (*n*=940)	Ch 9-13 (*n*=984)	Overall (*n*=2631)
A	97	98	55	81	95	97	52	80
C	92	88	37	69	81	79	27	61
D	97	92	82	89	84	64	62	69
E	97	88	47	74	96	81	43	71
F	95	67	3	49	25	24	2	16
Overall	96	87	45	72	76	69	37	59

Note: Based on Thompson & Senk (2006a).

From the data in Table 1, we can conclude that these five teachers are roughly similar in their implementation of the first two-thirds of the text, although teacher F typically assigns a higher percentage of problems than the other four teachers. However, differences appear in the final third of the text; in particular, two of the five teachers taught none of the content in the last four chapters. Thus, although students in the classes in all five of these schools ostensibly have studied the curriculum of *Transition Mathematics*, there are significant differences in the opportunities that students have had to learn the course content.

Among the five *Algebra* teachers (see Table 2), there is again dissimilarity in the percent of lessons taught in the final third of the book. Further, it is clear that Teacher F assigns a much smaller percentage of the problems than the other teachers. The fact that her students did so many fewer problems than their peers in other schools raises questions as to whether their opportunity to learn is really similar to that of their peers, even when the teacher taught the same lessons.

Across both studies most teachers taught the lessons in order, with the number of lessons taught based on how they paced the course. However, Teacher B in the *Transition Mathematics* study was an anomaly, consistently skipping lessons throughout the school year. With respect to problems assigned, Teacher F in the *Algebra* study is an outlier, consistently assigning far fewer problems than any of the other teachers in the two studies. Thus, with relatively little expense researchers have accurate measures of what aspects of the intended curriculum were actually covered; this information can supplement OTL measures related to the assessed curriculum and achievement.[1]

Limited classroom observations and interviews provide additional data about teachers and the classroom context so the researchers can describe factors that influence pacing, selection of lessons taught, and percent of problems assigned. These limited observations confirm the reliability of the teachers' self-reports on implementation for the entire school year.

OTL Measures Related to the Assessed Curriculum

Comparative studies of curricula often examine issues that arise and outcomes that are attained when students use one of two sets of instructional materials, say curriculum A and curriculum B. As noted by Kilpatrick (2003, p. 485) and the National Research Council (2004, pp. 112-113) the class, not the student, is the most appropriate unit of analysis in such comparative studies, and classes should be matched on critical variables as much as possible.

All of the UCSMP field studies for *Transition Mathematics*, *Algebra*, *Geometry*, and *Advanced Algebra* have used matched-pair designs, with classes in the same school matched on the basis of pretests of relevant prerequisite knowledge. Thus, each field study is a set of small studies replicated for each pair of classes. When possible, the pretest has been a standardized measure; when a content-specific standardized measure has not been found, UCSMP researchers have created a test of knowledge that students entering the course should have regardless of the prior curriculum studied, based on recommendations of professional organizations and existing research.

To examine the effects of the curriculum and teaching on students' achievement and to account for the different goals that curricula and teachers may have, another measure of OTL was determined for each teacher. This measure was based on a question adapted from one that had originally been used in the Second International Mathematics Study (SIMS) (Schmidt, Wolfe, & Kifer, 1992). For each posttest item the teacher was asked, "During this school year, did you teach or review the mathematics needed for your average students to answer the item correctly?" Teachers were given three choices: (1) Yes, it is part of the text I used; (2) Yes, although it is not part of the text I used; (3) No. For each teacher and each posttest, the percent of items to which the teacher responded "Yes" was calculated.

This measure of OTL based on teachers' responses permits data on students' achievement to be reported in three ways:

- Overall achievement on the Entire Test, with the OTL percentage reported by class or teacher.
- Achievement at the school level on a subset of items for which both teachers in the pair (the teacher using curriculum A or curriculum B) indicated that students had a chance to learn the content needed for the item. This test, which varies by pair but which controls for OTL at the school level, is called the Fair Test.
- Achievement for the entire study sample on a subset of items for which all teachers in the study (whether teaching curriculum A or B) indicated that students had a chance to learn the content needed for the item. This test, which controls for OTL at the study level, is called the Conservative Test (Thompson & Senk, 2001).

A repeated measures *t*-test on the mean of the pair differences (Gravetter & Wallnau, 1985) is then computed to determine if a significant difference in achievement exists between curricula on any of the three analyses.

We illustrate this methodology using results of analyses of students' achievement on a 36-item UCSMP-constructed multiple-choice test called the *Advanced Algebra Test*; these data are from the field study of the second edition of *Advanced Algebra* reported in Thompson and Senk (2001) and Thompson, Senk, Witonsky, Usiskin, & Kaeley (2001).[2] Table 3 contains the percent correct on this test for eight pairs of classes in four schools using the *Advanced Algebra* (Second Edition) UCSMP curriculum or the more traditional curriculum in use in the comparison classes. Table 4 provides the summary statistics for the repeated-measures *t*-test computed for each of these three approaches to data analysis.

In Table 3 the data in the columns labeled OTL indicate that in all schools there are large differences in teachers' reports about their students' opportunities to learn the content assessed on the *Advanced Algebra Test*. Thus, it is not surprising that when OTL is not controlled (i.e., the achievement on the entire test is considered), a significant difference in achievement between students using the UCSMP *Advanced Algebra* or comparison curriculum is observed (see Table 4), with UCSMP students scoring about 13% higher than comparison students. However, when controlling for OTL at the school level, the overall difference in achievement is also significant, with UCSMP students scoring about 9% higher on the Fair Tests (see Table 4). When OTL is controlled at the study level on the Conservative Test, the number of items for comparing achievement is less than half of the items on the instrument, and no significant difference in achievement is observed. Hence, the three analyses illustrate that conclusions about differences in achievement are strongly related to teachers' reports of OTL.

We have used such analyses of Entire Tests, Fair Tests, and Conservative Tests for various instruments in studies of four curricula developed by UCSMP. This methodology allows us to conclude when differences in achievement are likely to be an artifact of differences in OTL and when they appear to be robust regardless of OTL. In addition, we have observed that generally achievement levels are higher for the Fair Tests than for the Entire Test, not unexpected given that the Entire Test assesses content that students have not studied. Achievement on the Conservative Test often falls between achievement on the Entire Test and the school's Fair Test.

Table 3. Mean Percent (Standard Deviation) Correct by Pair for Advanced Algebra Achievement Analyzed in Relation to Opportunity-to-Learn

Pair ID	No. of students in pair	Entire Test				Fair Tests			Conservative Test	
		UC		Comparison		UC	Comp		UC	Comp
		mean (sd)	OTL	mean (sd)	OTL	mean (sd)	mean (sd)	No Items	mean (sd)	mean (sd)
J18	18, 14	60.8 (9.0)	100	55.2 (10.2)	69	63.8 (10.8)	61.7 (9.9)	25	61.5 (15.1)	65.7 (15.0)
J19	11, 15	58.8 (13.5)	100	53.7 (11.0)	69	59.6 (12.7)	60.0 (14.7)	25	58.2 (16.4)	63.6 (17.6)
K20	22, 24	63.8* (13.0)	94	45.9 (10.0)	72	66.8* (12.8)	53.5 (12.9)	26	66.1* (15.5)	54.7 (14.2)
K21	16, 23	64.8* (14.0)	94	43.0 (11.9)	72	68.3* (15.6)	51.2 (14.1)	26	65.0* (18.1)	50.7 (17.5)
L22	19, 20	57.6* (16.9)	92	38.8 (9.1)	75	61.5* (18.0)	46.5 (10.8)	26	55.8 (22.2)	48.0 (15.2)
L23	13, 15	44.7 (11.2)	92	38.3 (11.0)	75	50.9 (11.0)	45.4 (14.8)	26	45.1 (13.4)	44.9 (17.5)
M24	29, 22	58.4* (12.7)	92	37.8 (13.8)	47	59.6* (13.6)	46.0 (19.6)	17	58.2* (14.6)	46.1 (19.6)
M25	22, 23	39.6* (13.5)	92	30.8 (9.9)	47	39.6 (17.3)	36.8 (14.0)	17	38.5 (19.0)	35.1 (13.8)
Total	150,156	56.1 (15.4)		42.0 (13.1)					56.1 (19.0)	50.0 (18.4)

Notes: Because the analysis of the Fair Tests is on a different set of items for each school, no overall mean percent correct is possible for this test. *indicates a significant difference in the pair means.

Table 4. Summary Statistics for Advanced Algebra Analyses Controlling for OTL

Test	No. of Items	UCSMP - Comparison		*p*
		mean %	s.d. (in %)	
Entire Test	36	13.125	7.281	0.0014*
Fair Tests	17-26	8.825	6.833	0.009*
Conservative Test	15	4.950	7.598	0.108

* indicates significant difference in achievement among students using the two curricula on a repeated measures *t*-test to determine if the mean (UCSMP-Comparison) is different from 0.

Discussion

In this paper, we have illustrated several methods for capturing information about opportunities to learn at the classroom level, and several ways to analyze achievement data taking OTL into account. We argue that these methods for measuring OTL have much lower costs associated with them than classroom observations.

Data from a questionnaire about the number of lessons taught and number of problems assigned allow researchers to obtain profiles of the nature and extent of curriculum implementation by the teacher. Data from a questionnaire about the teacher's perception of the extent to which his or her students had the opportunity to learn the content needed to answer specific questions allow researchers to investigate the extent to which achievement differences between groups are artifacts of OTL or robust even if OTL is controlled.

Furthermore, analyzing the data from a given instrument in three different ways using different sets of items permits maximum data usage while controlling for OTL. For instance, all the items on each posttest are used when assessing achievement on the Entire Test. Such an analysis assesses achievement on some content that students have not explicitly studied; perhaps such analyses assess students' ability to transfer knowledge to new situations.

Analyses of data from the Fair Tests address achievement on the common content at the school level. The Fair Tests control for the socio-economic context of the school and reflect what both teachers in a given school value. Given the high-stakes accountability assessments that educators in the U.S. must administer as a result of the No Child Left Behind legislation (2001), the Fair Tests might also reflect differences in state frameworks for a given course. Analyses of the Fair Tests maximize data at the school level while acknowledging differences that occur between schools.

The Conservative Test is the most restrictive and uses the least amount of available data, because it is based on only those items for which all teachers participating in a study, whether at high or low performing schools, report that students have had an opportunity to learn the content. Such a test might be viewed as core content on which all can agree, regardless of philosophical or pedagogical beliefs related to the teaching and learning of mathematics.

It should also be noted that the instruments used in the studies cited in this manuscript are specific to a particular course, rather than general measures of achievement. OTL measures seem to be particularly important when using instruments that assess content specific to a given course. However, we have found that teachers' reported OTL varies even on standardized measures.

Although we believe the OTL measures we have used are an important first step in addressing this issue, there are other issues related to OTL that need to be considered by the research community. For instance, is the simple teacher question on OTL (essentially *yes* or *no* to student opportunity to learn the content) sufficient? At times, we have evidence that teachers covered the chapter in their book in which an item's content would be found, but responded *no* to the OTL question (Senk & Thompson, 2006). So, what type of OTL measures might be used to determine which aspects of the content are taught (e.g., skill vs. concept vs. application)? How might OTL measures for pedagogical approaches be developed that are also low-cost but reliable and how might such measures be integrated into interpreting achievement on the assessed curriculum?

Endnotes

1. Data on students' achievement from the evaluation studies of the third editions of *Transition Mathematics* and *Algebra* were not available when this manuscript was submitted.
2. For copies of the 36 items, see Thompson & Senk (2001) or Thompson, Senk, Witonsky, Usiskin, & Kaeley (2001).

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EXPLORING THE OPEN-RESPONSE TASK AS A TOOL FOR ASSESSING THE UNDERSTANDING OF FIFTH-GRADE STUDENTS IN THE CONTENT AREA OF FRACTIONS

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Inferring student understanding is imperative in mathematics teaching and learning. Methods for assessing understanding require constant exploration and evaluation, including open-response tasks as tools for gaining insights into student thinking. This detailed analysis of student solutions to selected open-response tasks on an exam administered to elementary students examines the role of strategy choices as indicators of mathematical understanding and proficiency.

Purpose and Framework

Inferring student understanding is at the heart of improvement in mathematics learning and teaching. Assessment provides valuable information which can be used to "promote growth, modify programs, recognize student accomplishments, and improve instruction" (NCTM, 1995, p. 27). It is imperative that the methods for assessing student understanding be constantly explored and evaluated. Our decision to research the open-response task as an assessment tool results from our perspective that students make sense of mathematics by exercising personal agency. Personal agency is the freedom and responsibility to choose to act (Walter & Gerson, 2006). Open-response tasks allow students to exercise personal agency, hence eliciting not only *what* students know, but also *how* students explore concepts. Open-response tasks require "students to explain their thinking and thus allow teachers to gain insights into . . . the 'holes' in their understanding" (Moon & Schulman, 1995, p.30). How can "holes" in student understanding be inferred from analyses of student strategies in solving open-response tasks? Does a student's strategy choice indicate a lack of proficiency or conceptual understanding that may be evidenced in alternative strategies?

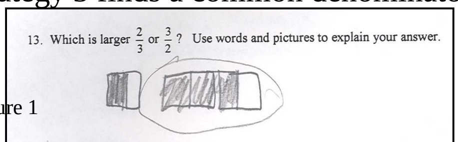
Method

Pre- and post-tests consisting of 12 multiple-choice and 12 open-response questions were administered to 774 elementary students. We used a mixed method of qualitative and quantitative research procedures to infer student understanding of probability and fractions and to analyze students' strategies on open-response tasks. A detailed analysis was conducted on 2 open-response and a combination of 13 conceptually related multiple choice and additional open-response questions on 172 post-tests given to fifth graders at two elementary schools. Each open-response task was graded individually by two research team members. Discrepancies were discussed and resolved before a final score was given. Qualitative analysis was performed by assigning strategy codes to all student solutions on open-response tasks. Related questions answered incorrectly by each student were quantified. We noted relationships between incorrectly answered questions and alternative student strategies.

Data and Analysis

The 172 fifth graders' exam scores improved significantly ($p < .01$) from their pretest to their post-test performance. We discuss one open-response question here which asks students to identify the larger of two fractions (2/3 or 3/2), and use words or pictures to explain their answer. Sixty-three students received full credit for this problem. Three strategies received full points. Strategy 1 compares 2/3 and 3/2 to one whole by using a picture or explanation (Figure 1), strategy 2 compares 2/3 and 3/2 to one whole and represents 3/2 as a mixed number (Figure 2), and strategy 3 finds a common denominator (Figures 3 and 4).

Figure 1



Figure

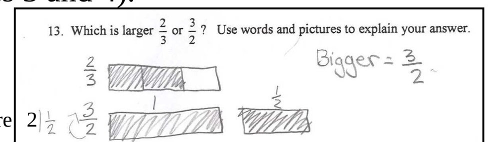


Figure 3

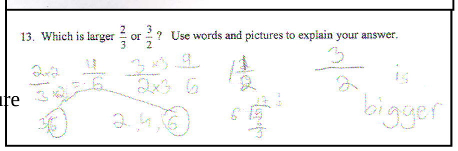
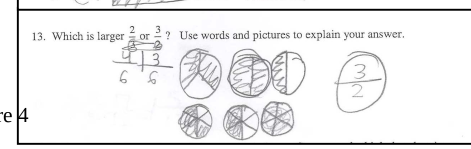


Figure 4



Thirty-five of the 63 students employed an intuitive strategy (strategy 1). Of these 35 students, 57% missed two or more related questions involving finding a common denominator and 29% of the students missed multiple-choice question number 5 dealing with representing a fraction as a mixed number (see Table 1). Twenty-two students (35%) used strategy 2. These students performed better on question number 5 than those who used strategy 1.

Seven students (11%) used a procedural strategy (strategy 3) to compare 2/3 and 3/2. However, only two of these students provided evidence that finding a common denominator is accomplished by re-unitizing each fraction so that each unit is divided into the same size of pieces (see Figure 4). Four of the 7 students who found a common denominator without interpreting its meaning missed 2 or more questions dealing with finding a common denominator. These four students might lack the intuition of knowing when to use this strategy. For example, analysis of Joshua's exam suggests that when comparing fractions he found a common denominator, but when using a common denominator would be most useful for solving certain addition or subtraction problems he added or subtracted across the numerators and denominators. Five of the seven students missed an intuitive problem. A key finding of this study is that two of the seven students who did not miss the intuitive problem and who performed well on problems involving finding a common denominator were those two students who demonstrated their intuitive and procedural understanding as shown in Figure 4.

Table 1 Strategy	Percentage of students who used this strategy	Missed 2 or more common denominator questions (procedural cd)	Missed question #5 (procedural mn)	Missed question #14 (intuitive)
Comparison, Picture, Explanation (intuitive)	56%	57%	29%	37%
Comparison, Picture, Explanation, with Mixed Number (intuitive/procedural mn)	35%	63%	4%	45%
Common Denominator (procedural cd)	11%	57%	14%	71%

*As evident in this table, percentages are greater for the number of students who missed questions assessing understanding of those concepts that are evident in alternative strategies. For example, the percentage of students that missed question #5 who did not include a mixed number in their solution (mn) was higher than those who did include a mixed number (cd = common denominator, mn = mixed number).

Conclusions

A careful review of student strategies on open-response tasks warrants a new interpretation of the open-response task as an assessment tool. These fifth grade students who worked procedurally on open-response problems to find common denominators did not provide evidence of intuitive understanding through drawings or explanations. Students who did not employ the strategy of finding a common denominator might not be proficient with that procedure. Students may be able to solve open-response problems, such as those involving finding common denominators, but when conceptually similar problems are placed in another context, students may be unsuccessful in demonstrating their content knowledge or mathematical skills. Implications for teaching include assessing whether or not students can flexibly use their understanding of fractions in multiple contexts. Open-response tasks provide opportunities for teachers to consider how students do not solve problems and encourage their students to develop understanding of those concepts addressed by alternative strategies. Through the use of open-response tasks and analysis of alternative strategies, teachers can gain greater insight into student thinking and provide opportunities for their students to learn as they exercise personal agency.

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INTRODUCING PRE-SERVICE TEACHERS TO FORMATIVE ASSESSMENT: IMPROVING ASSESSMENT DESIGN AND ACCOUNTABILITY IN SCHOOL MATHEMATICS THROUGH A NETWORK-BASED LEARNING ENVIRONMENT

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The purpose of this proposal is to contribute in three important directions. First, it will contribute to pre-service teacher preparation by exposing pre-service teachers to an experiential instructional unit on formative assessment through the use of a network-based learning environment. Second, it will contribute to the development of evaluation and assessment tools that allow for valid, reliable and feasible interpretations of students work on classroom-based (performance-based) assessments with the future goal of better linking these formative assessments to larger-scale assessments of students' mathematical knowledge. Third, it will contribute to the use of technology, like the network-based calculator system, in the design of more participatory learning environments to promote learning.

Research goals and objectives

This research study contributes in three important directions: first, to pre-service teacher preparation by exposing pre-service teachers to an experiential instructional unit on formative assessment; second, to the development of evaluation and assessment tools that allow for systematic interpretations of students' work on performance-based assessments in math; and third, to the use of technology, like network-based calculator systems, in the design of more participatory learning environments to promote learning. The research questions that guided this study are: *Can a network-based learning environment be designed to facilitate formative assessment in the classroom, by making students' mathematical understanding observable, and allowing the teacher to provide opportunistic and timely feedback of students' observed participation? What do pre-service teachers learn, as part of their teacher preparation program, after an instructional unit on formative assessment about this topic and about middle school students' knowledge as elicited on a performance-based assessment in mathematics?*

Theoretical Framework

This proposal builds on two important research findings about classroom assessment. First, that improvement in formative assessment will make a substantial contribution to the improvement of student learning (Black & Wiliam, 1998); and second, that more frequent classroom assessment of students' learning yields to better teaching (Bransford, Brown & Cocking, 1999; Ball & Cohen, 1999). Some of the difficulties associated with formative assessment are that it is time consuming, difficult to implement, and thus, teachers rarely use it in their classrooms. The current release of the Texas Instruments Navigator System (TI-Nav) provides wireless communication between students' calculators and the teacher's PC. Because students' individual input can be aggregated and displayed in a shared space, it is an ideal setting that enables real-time opportunistic feedback, self-evaluation, and supports instructional strategies (Stroup, Ares & Hurford, 2005). These are all fundamental aspects of formative assessment. A professional development unit, beginning in the pre-service teacher preparation program, needs to be created for the appropriate implementation of formative assessment and the

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use of performance-based assessments in the classrooms. This will give future teachers all the benefits of observing, assessing and evaluating students' mathematical knowledge. These skills can be applied to reflect on their own teaching and in better identifying math proficient students, including those who are thought to be unsuccessful (Wiliam, Lee, Harrison & Black, 2004).

Research Design and Antecedents

According to the National Research Council (2001), one of the problems encountered in assessment design lies in the tensions among: how students learn math, how their knowledge is elicited and documented, and the instruments used to interpret this knowledge. For this study, we draw on students' mathematical knowledge in the form of models that can be observed and documented through the implementation of performance-based assessment tasks called *model-eliciting activities* (*MEAs*) (Lesh *et al.*, 2000). MEAs are real-life complex problems that are meaningful for students, and in order to be solved, students must develop mathematical models that elicit and document their ways of thinking about relevant math ideas. Examples are provided: http://128.83.243.140/classes/knl2006fall/MEAwebsite/meas.htm.

Previous research was conducted where three different MEAs were implemented with 117 7th grade students in a public Midwestern urban middle school. These activities targeted relevant mathematical content in NCTM's Principles and Standards in School Mathematics (2000). Four math educators assessed and evaluated students' performance on these activities. In conducting formative assessment, they defined a pool of descriptors that well-described students' math knowledge as elicited in these tasks. Students' solutions to these MEAs, and experts' descriptors of students' math knowledge were used for this study.

Methodology

Participants for this study were a purposeful sample of 33 pre-service teachers, obtaining a degree in Math or Science, and also enrolled in a recognized teacher preparation program in a large higher-education institution in Texas. They were in their 2nd year in the teacher preparation program, and had had a year of teaching practicum. The setting for this study was in a university classroom. To become familiar with the task, participants were asked to solve a MEA that elicited their understanding of math ideas: variation (quadratic, linear, inverse, direct), variable identification, algebraic and pre-algebraic notation, patterns, profit, and cost.

Stage I Each participant was provided with the same 5 examples of 7th grade students' work of the MEA they previously solved. They were given a scoring rubric and asked to provide a holistic score to students' performance on this activity by using the TI-Nav.

Stage II Each participant was provided with a pool of descriptors that experts identified to potentially describe students' math knowledge as elicited in the given MEA. They were provided with a new set of 5 examples of real students' work on this MEA, and were asked to assess by deciding which descriptors best fit the student's work. Later, they were asked to individually score students' performance using the scoring rubric. All was done with TI-Nav.

The aggregate of all the descriptors (for each of the 5 examples) was presented in a graphical format in the shared space of the TI-Nav. Participants were encouraged to extend their formative assessment experience to generalize in: (1) the graphical display as a pictorial descriptor of students' math knowledge elicited on the MEA; (2) what students might know, and what they need to know; (3) how might students' knowledge be extended through instruction; (4) how this might relate to students' performance score; and, (5) how the graphical display might show inter-rater agreement on the descriptors. Data analysis was conducted focusing on participants' evaluation of middle school students' work, their description to students' work on the MEA, and their reflections on the experience of formative assessment.

Results and Conclusions

The network-based environment provides a quick and reliable way to pull forward pre-service teachers' ideas for discussion, reflecting back to the others the current (and changing) ideas about students' math knowledge. Evidence shows that pre-service teachers benefited from being exposed to real students' work as a way to improve their ability to do formative assessment. Pre-service teachers went beyond only assessing whether students obtained the correct answer or not by being more aware of the subtleties of what students' work shows them about their math knowledge and how different students have approaches to solving the same problem. The descriptors of relevant math knowledge created by experienced teaches helped these pre-service teachers scaffold their understanding of formative assessment, and its benefit to improve student math learning and monitor student progress. Nevertheless, there was little said about the use of formative assessment in informing instructional decisions. Many envisioned the use of a similar setting as a process of continuous display of ideas, reflection, and analysis in effectively implementing formative assessment in their future math and science classrooms.

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VALUES & EXAMS: MEASURING THE QUALITY OF TEACHER-GENERATED ASSESSMENT

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Values/beliefs in measurement are highly controversial and rarely explored. Yet is a vital part of classroom assessment. Based on a quantitative research study, this paper examines mathematics teacher's values in relation to the exams they create. The framework is based on Messick's four faceted model. The results help identify areas of strength and weakness in the quality of teacher assessment.

Values and beliefs play an integral role in teachers' lives. Teachers determine on a daily basis what students should learn, in which way and how to assess learning. This is all accomplished while staying within the confines of teachers' professional ethos and ministry guidelines. This paper examines the cohesion of these relationships to analyse the quality of teacher-generated assessment.

Perspective

Teachers develop final assessments that measure student's ability within a course. This is considered "assessment of learning." The Western and Northern Canadian Protocol [WNCP], (2006) state: "The purpose of assessment of learning is to measure, certify, and report the level of students' learning, so that reasonable decisions can be made about students" (p.56). Thus exams, which are developed at the end of the year for this purpose, are comprehensive and ought to provide testing results, which are generalizable from school to school.

Teachers are expected to create these exams in relation to the pre-defined set of values set out by the Ministry, yet they also have their own personal values to contend with. As a result, a struggle may exist between teachers' internal beliefs and the external values that are determined by the central authority (Katz, Earl, & Olson, 2001). Messick (1995) also states: "The consequential basis of test interpretation is appraisal of value implications of the construct label itself, of the broader ideologies that give theories their perspective and purpose…" (p. 748). Thus, a major determinant of the quality of a teacher-constructed assessment is a teacher's ability to remain in line with measuring the intended outcomes.

As part of a larger research study examining the validity of teacher-generated grade 9 mathematics exams in Ontario (MPM1D exam); this paper explores quality of assessment as the extent to which curriculum, teachers' belief/value in what is tested is coherent. The paper also investigates the extent to which exam content aligns to ministry expectations.

Method

The data for this study was collected in the fall of 2005 and analysed in early 2006. MPM1D is a mandatory course in all participating schools; students who successfully complete the course earn a credit towards graduation. MPM1D is based on four mathematical strands: Number Sense and Algebra, Relations, Analytic Geometry and Measurement and Geometry, the strands are also subdivided into expectations (Ministry of Education and Training, 2000).

Two data sets were collected from 7 teachers in 3 independent Ontario schools. Data set A

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was collected through coding a set of teacher-generated MPM1D exams based on the alignment model developed by Porter, Chester, and Schlesinger (2004). Data set B was collected through a teacher questionnaire. The instrument for Data set B was developed and field-tested with ten mathematics teachers and items are linked to the alignment model. The questionnaire asked teachers to identify the percentage of curricular expectations they believed should be on the exam by selecting one of 7 percentage intervals (0, 1-5, 5-10, 11-15, 16-20, 21-25, 26-30, 31+) for each MPM1D curriculum expectation.

Descriptive methods were used to illustrate the construction of the exams, questionnaire results and ministry expectations. Chi-square goodness of fit and inter-rater reliability analysis was used to examine the agreement on MPM1D content.

Results & Discussion

In the classroom, teachers may feel they are making informed decisions about students, the reality is that all measures are estimate of student competence and the more precise the testing instruments the better the diagnoses of student learning. The results identify the construction of all the exams appears to be heavily skewed away from the Measurement and Geometry strand, this was also mirrored in the questionnaire. This is of great concern as a balance of these strands forms the foundation of higher-level mathematics.

Ambiguity in ministry expectations made the study more complex. However, this may be purposeful as not to prescribe the curriculum. Adversely this makes examining the relationship between ministry excitations and teachers' values and/or exams unreliable.

The most fascinating finding was the relationship between the different school exams. The exams are aligned very well on the curricular expectations, however when examining the distribution of marks for the curricular expectations, alignment is very low. Suggesting that exams are comparable between schools, while the results are not, different schools place emphasis on curriculum differently.

The research conducted for this paper, explores how cohesion between curricular values, ministry expectations and teacher-generated exams contribute to the quality of assessment. The methodology employed is unique, however still needs refinement for future studies. Overall, what appears is the MPM1D exams are developed in a similar manner to other classroom assessments, making the quality of the exams questionable. The exams creation process needs to be greatly refined to increase the quality of interpretation. This is important, as these exams are methods of accreditation. Thus educators need to critically examine the purpose "assessment of learning" – more specifically in design and intent otherwise we run the risk of missing what mathematics students ought to know developmentally.

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WHAT CAN WE UNDERSTAND ABOUT ACHIEVEMENT GAPS IN MATHEMATICS BY STUDYING CLASSROOM PROCESSES?

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An alternate view of achievement gaps based upon achievement levels rather than demographic groups is proposed. Four distinct patterns of instructional effects on low and high achieving students are discussed.

"Closing the achievement gap" is a goal espoused by many — from those in the mathematics education research community to the authors of No Child Left Behind. How we define these gaps influences our approach to closing them. In this presentation, we consider an alternative definition and advocate a classroom-based approach that focuses on teacher moves and student engagement, two factors directly related to achievement.

A standard definition of achievement gap is a significant difference in standardized test scores between some aggregate of white, largely middle-class students and some aggregate of "other" students, defined as not white or not middle class. Posing the problem this way has led to solutions aimed at the "other group," including helping "them" catch up or accelerate their learning, and identifying and addressing cultural factors from institutional racism to taboos on "acting white." In practice, this kind of definition can lead to ascribing qualities of the aggregate to the individual, such as characterizing all students in the "other" group as low-achieving. This is a serious category mistake that only makes things worse.

We propose an alternative view, a shift from a view of differences between demographic groups across schools to differences between achievement groups within classrooms. Consider in each school, or classroom, the students who are low achieving and the students who are high achieving—where achievement is measured by standardized test scores or by school-based assessments. The issue of large scale achievement gaps can be framed differently, in terms of these groups. The problem is that the ratio of low-achieving to high-achieving African-American students, for example, is higher than the ratio of low to high-achieving White students. Surprisingly, evidence suggests there is a great deal of stability in differences between low and high achievers (regardless of race or ethnicity), and a positive correlation between low achievement one year and low achievement the next. These patterns may be an artifact of the psychometric properties of standardized testing, a reflection of the "accumulation of disadvantage" (Valian, 1998), or the net effect of classroom processes that reinforce prior levels of achievement. (We do not assume they reflect immutable differences in children's ability to learn.)

What is the role of classroom processes in maintaining or ameliorating gaps between achievement groups in classrooms? To answer this question, we examined prior studies of classroom instruction that reported achievement gains disaggregated by levels (e.g., low, medium, high) and concentrated on inquiry learning, where inquiry learning is an imprecise umbrella term for taking a problem solving approach, encouraging argumentation and justification, emphasizing conceptual understanding, and so on. We focus on classroom processes (teacher moves, student engagement around tasks) as the explanatory unit, not curriculum programs – whether it's Everyday Math or Singapore Math – as proxies for these

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processes. We found four patterns of effects on low and high achieving students.

Some classroom processes support higher achievement for all. This leaves a persistent gap if the increase is even across both groups (Saxe et al., 1999), but in some cases actually widens the gap by raising the ceiling on achievement, giving high achieving students even more room to excel. For example, in a case study of a highly skilled cognitively guided instruction third-grade teacher, Koehler (2004) found lowest performing students made the most dramatic gains relative to pre-established proficiency levels on a district-mandated standardized test. The highest performing students were operating at least a standard deviation above middle-schoolers. The teachers in these studies had a high level of knowledge of children's mathematical thinking, and used this knowledge to differentiate – or essentially, fine tune – tasks to maximize opportunities for student engagement.

A second group of classroom processes supports higher achievement for middle or high-achieving students but not for low-achieving students, clearly widening the gap. Some studies (Baxter, et al., 2001; Lubienski, 2000) have found higher gains for middle- and high-achieving students than for low-achieving students. One possible explanation is that tasks were, on average, too difficult for low achieving students and so those students were meaningfully engaged at a lower rate. Another possible explanation is that teacher moves differentiated among students depending on whether students were considered low or high achieving, resulting in qualitatively different opportunities for engagement.

A third category is marked by closing or narrowing the gap between low and high achievers. The idea of "narrowing" suggests that the higher achieving group is held in place while the lower achieving group catches up, or that the lower achieving group learns at a faster rate than the higher achieving group. If an achievement measure has a ceiling effect, it lends the appearance of narrowing, rather than reflecting more realistic learning gains. For example, U.S. states showing a narrowing of achievement gaps between demographic groups (not achievement groups) that was not corroborated by the National Assessment of Educational Progress (NAEP) in fact were relying upon ceiling effects (Lee, 2006).

We propose a fourth category, based upon the notion that prior achievement levels should not foreclose or substantially limit what students can learn from instruction: classroom processes that result in gains that are not predictable based on initial achievement rankings. Some gain considerably, some do not, but it is difficult to predict who will do what based on previous achievement scores. This kind of approach ruffles now rigid boundaries in classrooms between low and high achieving groups. We found no examples of classroom instruction where achievement gains fit this pattern.

With the exception of the second pattern, each of these patterns arguably supports a version of equity. However, we propose creating and studying instruction that exemplifies the fourth pattern is a critical program of research towards equity in school mathematics.

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SUPPORTING MATHEMATICS IMPROVEMENT: ANALYZING CONTRIBUTING FACTORS

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This paper concerns a project to explore factors affecting improvement in mathematics achievement at Grades 3 and 6. We believe that understanding the relationship between school practices and patterns of improvement is fundamental to understanding strategies that provide opportunities for all students to successfully learn mathematics. The concept of "learning environment" as set out by Bransford et al. (2000) informs our analysis of the data.

Since 1997, when a new curriculum was launched, student achievement in Ontario schools has been described in terms of four "achievement levels". The provincial standard is level 3 – a "high level of achievement" and a level at which parents "can be confident that their children will be prepared for work at the next grade" (The Ontario Curriculum, Grades 1-8, Mathematics,Ministry of Education and Training, 1997, p. 8).

As a research group seeking to understand the factors contributing to mathematics improvement, we are concerned that the provincial focus on achievement of the standard ignores improvement shown by students below level 3. It embodies a dubious philosophy of education in which there are winners and losers and where "reaching the standard" or "failing to reach the standard" is expected to have significant motivational effects on students, teachers and schools.

In the first phase of our work, we developed a weighted average system, whereby weights 1, 2, 3, 4 are assigned to the percentages of students with achievement at levels 1- 4 respectively and the weighted percentages are added to generate a school score between 0 and 400. We used the WA system to examine the results of all grade 3 and 6 students in two Ontario school boards and identified a number of schools in which there was marked improvement in mathematics achievement, although that improvement may not have been recognized by the provincial formula. We believe that this approach allows us to recognize improvements in achievement of the majority of learners in the majority of schools.

In phase 2 of our work we conducted case studies at 12 of the schools that had shown a strongly positive improvement on our measure, in order to develop a deep understanding of contributing factors. We carried out questionnaire surveys of teachers and principals, observed mathematics lessons, and interviewed teachers who may have had an impact on student results. For example, if a school had improved its grade 3 scores, we met with those teachers who had been involved with the kindergarten to grade 3 program at that school in the past 5 years.

Theoretical Framework

In examining the collected data we used the concept of learning environment as envisioned by Bransford, Brown, Cocking, Donovan, and Pellegrino (2000); they state that learning environments must first be learner-centred. That is, they must take into consideration the knowledge, skills, attitudes, and beliefs that learners bring to an educational setting. Because of the complexity of meaningful knowledge and skills, scaffolding must be provided to help learners carry out components of the task that they cannot yet manage on their own. Learning

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environments must also be knowledge-centred. In structuring such environments it is critical to provide effective and ongoing professional development to deepen understanding of content knowledge. Thirdly, learning environments must be assessment-centred. The assessment should focus on higher level thinking skills, not only on factual or procedural recall. Fourth, Bransford et al. state that learning environments must be community-centred. Finally, the design of learning environments must be coordinated so that all four elements function together.

Study Findings

Our results showed that the study schools had implemented changes in all four areas. They had developed particular initiatives that took into account students' individual needs and prior knowledge; their teachers were involved in a range of professional development projects and were making significant use of provincial curriculum documents and exemplars; on an ongoing basis schools were using the results of diagnostic assessments to inform program decisions, and they had developed connections with the broader community. However, there were differences.

Schools in Board A were largely autonomous. They analysed their own provincial results and developed a range of unique responses at the school level. In most cases, programs were aimed at improving school culture and/or literacy skills. For example, teachers attended literacy workshops, and schools mobilized classroom teachers, special education teachers, parents, and even local seniors to help implement a variety of reading/writing support strategies. In contrast, Board B developed mathematics initiatives at the board level. For example, workshops were designed around kits of manipulatives provided (and maintained) by the board, numeracy specialists were assigned to groups of schools to model best practices and mentor teachers, and teacher candidates from the local university were brought in as tutors. There was no common mathematics text for schools in Board A, while all schools in Board B used the same (new) text.

In Board A schools some teachers were using ideas from numeracy workshops, emphasizing problem solving and use of manipulatives, and encouraging students to share their ideas, and explain their thinking, but interviews revealed that these changes were recent and could not be related to improvements in mathematics achievement over the past five years. On the other hand, in Board B there was extensive evidence of well-established 'best practices' in mathematics teaching, an emphasis on the use of multiple approaches to solving problems, displays of mathematical work, and a pervasive attitude that "math is fun".

Teachers were asked to reflect on possible reasons for the improvement in mathematics scores. In Board A, teachers mentioned consistency in staff, school climate initiatives, attention to early identification and remediation in literacy and numeracy, focused reading programs, collaboration and communication at the division level, and goal setting. In Board B, teachers credited openness to new strategies, teachers who enjoy math and have backgrounds in it, teacher dedication, research-based mathematics curriculum materials, and "knowing what's required".

Thus, study results indicate that diverse approaches contributed to the study schools' mathematics improvement. In one board, a strong emphasis on literacy led to collateral benefits in mathematics; in the other, board-wide mathematics initiatives were the key factors. At the same time, in both boards, responses to the initial problem of low scores involved the implementation of research-based strategies to strengthen the foundations of a learning environment - learners, knowledge, assessment and community.

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DESCRIBING AND VALUING VARIATIONS IN LEARNING FROM TEACHING EXPERIMENTS USING STANDARDS-BASED AND TRADITIONAL CURRICULA

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This report describes a teaching experiment designed to document effects on students' understanding and other valued learning outcomes from an example of high-fidelity implementation of social-constructivist pedagogy and curriculum compared to an example of traditional behaviorist mathematics instruction. A mixed-methods analysis provides evidence of significant differences in important learning areas not documented by a state criterion-referenced test.

Conceptual Perspectives

Authors of *Principles and Standards for School Mathematics* (The Standards, NCTM, 2000) argue that "Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (p. 15). Clearly, achieving this goal requires major changes in curriculum, teaching, and assessment. The Standards also argue that "Teachers need to move beyond a superficial 'right or wrong' analysis of tasks to a focus on how students are thinking.... Although less straightforward than averaging scores on quizzes, assembling evidence from a variety of sources is more likely to yield an accurate picture of what each student knows and is able to do" (p. 23).

Over a decade ago, Hiebert and Carpenter (1992) argued that "one of the most pressing problems in education is the development of procedures for assessing higher-order thinking…. Significant progress… depends on good measures of understanding, so that the specific outcomes of instruction can be assessed. Progress in achieving widespread implementation of curriculum programs stressing understanding depends on being able to document the outcomes of such programs" (p. 89). In today's accountability environment, documenting students' understanding and problem solving as outcomes of particular learning experiences remains a significant challenge for teachers and other stakeholders.

This study examines the following research question: How can we gather evidence of learning that goes beyond standardized test scores and fairly represents the broader goals of the NCTM Standards, particularly the process standards through which understanding is demonstrated?

Methods and Evidence

The first author taught mathematics for one hour each school day for six years to the same group of students at one elementary school in the Western U.S. She began teaching this experimental group as a natural class of 22 first-grade students and continued to teach these students on a "pull-out" basis through sixth grade. A control group began with 26 students at a neighboring elementary school with very similar demographics. The data examined in this study comes from the first year of this six-year longitudinal study.

The experimental group received instruction in mathematics that can be characterized as social-constructivist and highly consistent with the NCTM Standards. This instruction was orchestrated and supported by the units from the *Investigations in Number, Data, and Space*

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curriculum (Russell & Tierney, 1998). These materials were used in a manner which qualified as a *high-fidelity* implementation (Reys, 2004), based on the following criteria: (1) regular and primary instructional resources, (2) significant amounts used during the year, and (3) instructional decisions influenced by the philosophies of the materials. Instruction in the control group classrooms consisted primarily of direct instruction in counting, learning routine computational procedures, and memorizing number facts typical of the traditional focus on learning basic skills.

Near the end of the first-grade year, each student was interviewed by the first author. Students' responses during these interviews were videotaped and transcribed. The problems used were created to provide evidence about specific learning goals, including the mathematical content and types of problems emphasized in the NCTM Standards and research on children's understandings of number and operations. These goals were associated with production of correct answers (CA), problem-solving strategies (PS), understanding of concepts (U), and communication (C).

The authors used a five-point (0-4) general scoring rubric as the foundation for developing topic-specific and problem-specific rubrics for scoring students' responses to each of the interview problems. These rubrics provide ways to quantify the qualitative differences among student responses, while giving value to those performances identified in the NCTM Standards and the authors' goals. These rubrics describe and value the more advanced performances as well as the typical inadequacies seen in the interview data.

The students in both groups also completed a constructed-response problem-solving activity and participated in their states' end-of-level criterion-referenced standardized testing.

Results and Discussion

A multivariate analysis of variance (MANOVA) test of the significance of the differences between the experimental group and control group (simultaneously considering composite percentages for the interview, the constructed-response task, and the state criterion-referenced test) indicated significant evidence of a treatment difference (p-value <0.00001). Investigating the effect of each of these data sources individually shows there is strong evidence of a treatment difference for the interview and constructed-response task. The mean percentage score on the interview for the experimental group was about 17 percentage points higher than the control group at a 95% confidence level. However, there is no evidence of a treatment difference between the experimental group and the control group for the state mathematics core test.

Conclusions

The methodology applied in this study provided clear evidence of significant differences in learning outcomes between two groups of students that were not documented by their state criterion-referenced testing program. Important learning goals for students expressed in the NCTM Standards were documented, described, and valued using rubrics for analyzing students' responses to problem-solving tasks conducted during videotaped clinical interviews.

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THE IMPACT OF A NON-HIGH STAKES STATEWIDE TEST ON TEACHERS' EXPECTATIONS FOR STUDENT PERFORMANCE

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This research examined the impact of an end-of-course assessment on high school teachers' expectations for students in Algebra I classrooms. Data sources included official state documents, interview transcripts, and test results. Results showed that although the test implicitly communicates expectations for students to engage in mathematical processes—problem solving, etc.—the expectations are not explicitly communicated to teachers and students.

To graduate from high school, all students in Indiana must take Algebra I. Aligned with *Indiana's Academic Standards for Mathematics*, the Algebra I End-of-Course Assessment (Algebra I ECA) is a final exam measuring what students know and are able to do upon completion of the course. As part of Indiana's school accountability system, the Algebra I ECA is designed to ensure the quality, consistency, and rigor of Algebra I across the state. Passing the Algebra I ECA is not currently a requirement for graduation. A guiding question for this research was, "Why do students perform so poorly on the Algebra I ECA?" Additional questions included, "What are the relationships among the intended curriculum, the implemented curriculum, and the achieved curriculum in Algebra I in Indiana schools?" "What expectations for teachers and students are communicated through the intended curriculum?" "What expectations for students are communicated through the implemented curriculum?"

A particular focus in this investigation was the role of mathematical *processes*—problem solving, reasoning, communication, representation, and connections—in the various Algebra I curricula. These processes are important ways of acquiring and using mathematical content knowledge (NCTM, 2000), and the ability to engage in these processes is an important part of mathematical competence (Hiebert, 2003).

To examine the intended Algebra I curriculum the researchers studied various documents posted on the Indiana Department of Education (IDOE) website. The documents describe graduation requirements, standards, textbooks, and details regarding the Algebra I ECA itself. To gain insight into the implemented curriculum, the researchers interviewed 12 teachers from 8 high schools and examined various printed materials – including a departmental curriculum guide, a list of "power standards," a final exam study guide, and several final exams – from the participating schools. Evidence for the achieved curriculum was found in students' performance on the Algebra I ECA.

Results

The mathematical content of the intended Algebra I curriculum is clearly communicated through documents posted on the IDOE website. The role of mathematical processes, however, is not as prominent in these materials.

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Of the eight schools in the study, only three suggested they made changes in their curriculum because of the test. The five that did not attribute the test as a factor in their implemented curriculum suggested new textbooks or state standards as influential. Schools least influenced by the test indicated the lack of consequences related to test scores as a reason to de-emphasize the test at this point. Although some schools do not value the test at this point, they do recognize expectations that are being communicated through the test. Teachers' comments revealed that some Algebra I teachers do not expect their students to develop long-term mathematical understanding beyond recall of facts and procedures. Those schools more influenced by the Algebra I ECA cited a stronger focus on word problems, a stronger focus on "thinking," a stronger focus on graphing, earlier coverage of topics, and more review as changes made in their curricula.

Evidence of the achieved curriculum appeared in the results of the Algebra I ECA. The number of students tested was 69,198 with 24.3% passing. The minimum attainted scaled score was 200 and the maximum attained scaled score was 800. The scaled score cut was 579 and the state average scaled score was 510.7. The results for the various categories of test items will be shown in a table (not included here for lack of space).

Discussion

The results of the Algebra I ECA suggest the need for change. Etchberger and Shaw (1992) note that "perturbation" is the first step in a process of change in teachers. They describe perturbation as, "a dissatisfaction or uneasiness with the way things are, e.g., teachers may not be happy with their present teaching methods or satisfied with their students' understanding" (p. 412). The Algebra I ECA has created such a reaction among teachers, even when they claim they are not making changes in their curriculum because of the test. The teachers who do make changes do so only after approving the new expectations as beneficial to students. Some teachers respond to the perturbation by changing their implemented curriculum. The aspects of these changes include adjustments in mathematical content, expectations for student engagement in mathematical processes, and opportunities for more review. Other teachers, though, do not make changes because: (1) they agree with the ideas but they have low expectations for their students, or (2) they do not agree with the ideas because their own vision of mathematics (algebra) is different.

It has long been known that large-scale assessments such as the Algebra I ECA influence teaching and learning. "Tests are more than a simple instrument of measuring achievement. They are interactive with the learning environment since they communicate to teachers and students society's values about what students should learn" (Webb, 1992, p. 679). The Algebra I ECA communicates expectations for knowledge of mathematical content and engagement in mathematical processes, but this communication is merely implicit, not clearly articulated. The authors will offer suggestions to address the problem.

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EVIDENCE-BASED PROFESSIONAL DEVELOPMENT PARTNERSHIP WITH MIDDLE SCHOOLS TO IMPROVE STUDENT TEST ACHIEVEMENT

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Effective professional development (PD) is grounded on best practices (Ball & Cohen 1999, Clarke 1994) and evidence (Holcombe 2004, Love 2002) and has student learning and achievement as the ultimate goal (Guskey & Sparks 2004, Smith 2001). Under a 2005-06 Texas Education Agency grant, we developed evidence-based PD using item analysis of data from the state's high-stakes mathematics test (TAKS: Texas Assessment of Knowledge and Skills) for middle school students' achievement in mathematics. The 23 participating teachers are from high-need (based on % of students not passing mathematics TAKS) and low-SES (based on percentage of students participating in free or reduced-price lunch) schools. These schools' student bodies are about 80-90% Latino/Hispanic.

Our 14 PD workshops engaged teachers in analyzing student error patterns and adapting pedagogy. Each session was launched by teacher reflections upon low-performing items from 1 of the 6 TAKS objectives. Teacher exploration of items went beyond teaching-to-the-test to unpacking big conceptual ideas and strategies (e.g., multiple representations) to help improve achievement on a much larger collection of items, and situate this understanding in a larger set of curriculum objectives and in the K-12 continuum.

We used these measures: (1) TAKS is administered in mathematics at Grades 3-11, and satisfactory performance on grade 11 is required for HS diploma. www.tea.state.tx.us/student.assessment/; (2) Teacher Observation Protocol (teacher lesson observations were accompanied by journal reflections and observer instruments informed by issues addressed during the workshops); (3) Teacher Knowledge Survey (TKS) of TAKS-like problems for teachers to solve, identify big ideas, and explain how they would teach them; (4) Teacher Reflections after each workshop session, focusing on any changes in teaching.

Our results include solid improvement in student TAKS scores. The TKS revealed lowest performance on the "patterns, relationships, and algebraic reasoning" and "measurement" objectives, which are the lowest performing two TAKS objectives for MS students!

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DEVELOPMENTAL UNDERSTANDING OF MATHEMATICS WITH ELEMENTARY SCHOOL STUDENTS

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Introduction

This research describes the process and findings for the development and validation of developmental continua in elementary school mathematics. A major goal was to provide a framework for identifying the phases at which students are operating in mathematics. This framework is based on key concepts and skills matched with phases of development that students pass through to understand mathematics.

Developmental continua describing the phases that students pass through as they acquire the skills and understanding of concepts associated with given subject are of great use to teachers since they link curriculum, assessment, and instruction.

In addition to the need to organize mathematics curriculum guidelines in ways that facilitate teaching and learning, educators are also concerned with understanding how students think about mathematics. One prevailing view is that an improved understanding of how students think mathematically will lead to improved student achievement (Ross, McDougall & Hogaboam-Gray, 2002).

Methods and Data Sources

Data collection to validate the developmental maps occurred during two stages of testing in 2003 and 2004. Students in Kindergarten to Grade 3 answered sets of questions in oral interviews; students in Grades 4 to 6 completed sets of written questions. Correlations among responses to questions in the same developmental phase and across developmental phases were examined to establish that the items were, in fact, empirically related, as had been indicated on the relevant map (using a mean >0.5, $\underline{r}$ =.40, and $p<0.05$). Questions yielding low mean scores and questions that were negatively correlated with other items in the same phase or with other items in adjacent phases were further examined and rewritten.

Findings

Field-test research has validated much of the researchers' original hypotheses regarding the phases of development for the five math strands. For the purposes of this paper, we will only describe the findings relating to Number and Operations. There were five phases in the Number and Operations developmental map, with five concepts and three skills.

Research for Number indicates that very few students, even at Grade 6, reach Phase 5 (the flexible phase), and many do not even reach Phase 4. This suggests that deeper understanding of some of the topics presented to students may occur later than we think. Another reason might be that the educational system does not regularly provide sufficient conceptual underpinning for students to allow students to reach this phase.

Research for Operations resulted in a similar finding: very few students, even at Grade 6, reach Phase 5 in the Operations strand because relatively few students are able to work with decimal operations. This is particularly the case in the area of multiplication and division. This

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

also indicates that relatively few students consider alternatives and make explicit choices about how to calculate in particular situations

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GEOMETRY

PRODUCING A VIABLE STORY OF GEOMETRY INSTRUCTION: WHAT KIND OF REPRESENTATION CALLS FORTH TEACHERS' PRACTICAL RATIONALITY?

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We report on the development of representations of teaching based on sequential-art sketches of classroom stories. We demonstrate with focus group data that these resources can help sketch compelling classroom stories and elicit the practical rationality of mathematics teaching.

For quite some time policy makers have looked for levers for instructional improvement – whether increasing teacher knowledge, upgrading instructional resources, or raising standards for student achievement. But instruction, like many other human activities that take time and recur over time, is organized by a rationality, a way of doing the activity that makes sense to participants and that tends to keep the activity stable and viable. We posit therefore that in order to design and promote improvements that are feasible and sustainable reformers also need to know about the mathematical work that teachers and students customarily do as they interact in classrooms. We have borrowed from Bourdieu (1998) the notion that a practical rationality, tacit and shared, undergirds the decisions and actions of the mathematics teacher in specific instructional situations. In the present paper we describe and illustrate a novel technique that project ThEMaT (Thought Experiments in Mathematics Teaching) has developed in order to study this practical rationality empirically, in selected situations in secondary algebra and geometry.

The project is based on the hypothesis that practitioners' instructional actions respond to obligations to the discipline, the students, and the school institution, but are neither determined by those obligations nor chosen at will through individual management of personal resources. Rather, courses of instructional action are constructed as viable, tactical plays of a game that pursues curricular and other stakes through the collective production of work over time. We conceive of the practical rationality invested in the teaching of algebra and geometry as composed of a system of dispositions that serves the purpose of warranting a range of possible tactical plays that a teacher of a given school subject might consider viable to do. We conceive of this system of dispositions as including the categories of perception and appreciation that actors of a practice can draw upon to relate to (possible or real) events and things in that practice. By categories of perception we mean the categories available in a practice with which a teacher can identify and describe events or things. By categories of appreciation we mean the categories available in a practice with which a teacher can have an attitude toward, or allocate value to, events or things.

In this paper we report on our conceptualization of a novel resource for eliciting the practical rationality of mathematics teaching based on sequential-art sketches of classroom stories deployed in three media forms: animation, slide show, and comic book. These stories are designed to engage practitioners in thought experiments about instruction, thought experiments that, we argue, can elicit practitioners' practical rationality. We conceptualize the use of this media against the background of our prior use of video for similar purposes and illustrate the kind of data that we have been able to collect with it.

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Practical Rationality and Instructional Situations

Complementary to the notion of practical rationality is the hypothesis that joint work in any mathematics classroom is framed by instructional situations (Herbst, 2006). Instructional situation refers to the system of norms that regulate the exchange or trade between, on the one hand, the work that teacher and students do together to sustain their classroom relationship and, on the other hand, the claims that they can make about the knowledge at stake. Specifically we hypothesize that classroom life is organized in segments of interaction whose goal is to produce mathematical work and exchange it for claims on what is at stake. An example is the instructional situation called "doing proofs" (Herbst & Brach, 2006) which is a normative way of organizing some exchanges in high school geometry classes in the US.

We model instructional situations by proposing systems of norms that express hypotheses about what is at stake in the exchange, what is the division of labor (who is supposed to do what, and how), and what is the organization of time for work (when are things to be done, and for how long). By a norm we mean a central tendency around which actions in instances of a situation tend to distribute. Since, these actions are performed by humans at a scale in which they could willingly act at variance, we expect that norms exist to mark central tendencies but that these are not necessarily estimated by the most frequent set of actions enacted in instances of that situation. Thus, instructional situations are sets of norms that identify similarly regulated phenomena but not necessarily regularities in behavioral manifestation; teachers operating under the same norms may produce different actions. As we will illustrate below, the notion of practical rationality helps us make sense of that apparent contradiction.

Practical rationality is the system that helps practitioners notice and justify (or else denounce) departures between actual actions and (implicit) norms. The empirical datum that justifies the existence of a norm is the frequency of observations of similar ways of noticing and valuing different behaviors. Thus the operational definition of practical rationality calls for uncovering the categories of perception and appreciation that actors of a given practice invest in noticing and negotiating the status of specific, problematic actions.

The mathematics education community has in the past drawn in similar ways on the notion of belief system, whereby a belief system is what warrants and puts a value on actions. Practical rationality is akin to belief system in that both attempt to explain what regulates action. But our use of practical rationality (and of dispositions) is not just another way to say belief system since beliefs have usually been attributed to individuals. Instead practical rationality is proposed as regulatory of specific situations and to be common to the individuals who play a similar role in those situations; it is what accounts for the continuities in instructional practice that persist as the actors change. Practical rationality points to a collectively held system of warrants for specific actions in a situation (e.g., there is a practical rationality for engaging students in proving). Conceptually, practical rationality is the system of dispositions that allows actors in a situation handle the presumption that they should or should not abide by a norm.

To illustrate the ideas described above and get into the substance of this paper, we'll elaborate on the instructional situation of "doing proofs" in the high school geometry class which is one of four situations our project is studying. We hypothesize that several norms regulate division of labor and organization of time in this situation. In particular, there are norms that apply to who can do what to diagrams and when. Three of those hypothetical norms are:

1. the diagram to be used in a proof contains all the objects that students will need in order to do the proof and no more than those; and no (generally) false properties are represented by way of objects included in a diagram

2. the diagram to be used in a proof is drawn in its entirety (including labels) before the writing of the proof starts, with the only exception of markings (hash marks, arcs for angles, etc.) which might be added while the proof is being done
3. the teacher is in charge of drawing the diagram (including all the geometric objects) that will be used in a proof, whereas the student may be responsible for marking the diagram with information given or discovered while the proof is being done

We submit that these norms express a set of default regulations regarding interaction with diagrams while proving. They are neither immutable laws nor explicit preferences of actors; rather, they are hypotheses made by an observer that help explain classroom observations and help sketch classroom stories that are marginal to customary practice. The ThEMaT project constructed three stories that show departures of these norms around properties of the intersections of angle bisectors in a kite, a square, and a parallelogram. In "The Kite" a student is allowed to prove that in a kite the angle bisectors meet at a point (a true property) starting from a diagram that she has drawn and that shows an apparent (non-rhombus) kite and its diagonals (one which bisects two of the angles of the figure, the other which does not). In "The Parallelogram" the teacher leads the class in proving that the angle bisectors of a parallelogram meet at a point (which is false), using a diagram that he has drawn of a parallelogram whose sides look congruent and whose angles between diagonals and sides are marked with equal arcs. In "The Square" a student who is sharing from his seat a proof that the angle bisectors of a square meet at a point requests the teacher to erase one of the diagonals of the square on the board (which contains two diagonals). We created these stories to purposefully contravene the norms listed above since our theory led us to predict that instances not abiding by those norms would elicit ad hoc remarks by participants, remarks that point to, describe, or put a value to the departures from the norm. The practical rationality of "doing proofs" includes the categories on which those remarks build; no matter whether they indict or praise the actions done.

Note, by the way, that these sample norms demonstrate why practical rationality is defined as dependent on an instructional situation, in that it is obtained empirically from the reactions to breaches of norms that characterize a specific instructional situation (e.g., "doing proofs" in the American geometry class). Evidently, it is possible that the empirical study of practical rationality in different situations might yield as a result the conjecture that some elements of practical rationality will be common across situations and might be attributable to the teaching of a particular course (such as geometry or algebra).

A Resource for the Study of Practical Rationality

We have developed a technique that permits us to study practical rationality empirically. In Herbst & Chazan (2003), when we introduced practical rationality, we described how we had gotten a glimpse of it as we examined conversations among practitioners looking at an edited video episode of a lesson where the teacher had made a decision at variance from what (we hypothesized) a teacher would do when engaging students in proving. We have used video because of the effectiveness of records of practice (Lampert & Ball, 1998) and video cases (Jacobs & Morita, 2002) in getting teachers to talk about teaching. We argued then that video episodes have the potential to elicit practical rationality because video episodes are not just records of events and cases of a kind of teaching, but also artifacts (i.e., reconstructions of events with a recording protocol that transforms the events), and probes into teaching (i.e., catalytic of a normative response from those who create practice, like Rorschach's blots are).

Embedded in the notion that a record of practice could operate like a probe into teaching, by compelling practitioners to remark upon deviations from a norm, is the notion that practitioners' feel for their practice is summoned by vicarious experiences with instances of that practice. All practices consist of actions that happen in time—they take time, but also they can be timely or not; they depend on what happened before and constrain what can happen after. In this sense, video episodes are very different than other cases such as written cases; video episodes allow one to probe the temporal dimensions of instructional situations both in the sense of duration of events and in the sense of order (and timeliness) of events. In spite of that potential of video, we also realized that systematic studies of the practical rationality invested in a specific instructional situation could not depend on the possibility to obtain video records of events that illustrated deviations of specific hypothetical norms. This and other considerations led us to explore whether other temporal media forms alternative to video (such as animations, comic books, and slide shows), could also be leveraged to be records, artifacts, cases, and probes.

The second consideration that led us to investigate other media forms is related to the argument that video recorded episodes are cases of teaching, which builds on the notion that the craft knowledge of teaching is stored and communicated in the form of stories (Carter, 1993). The same reasons that recommend videocases as useful for teacher educators to provide prospective teachers an 'anticipatory socialization' into the problems of teaching were the ones that discredited video as a resource for us to prompt expert practitioners to relate stories of what they might do instead. As Richardson & Kile (1999) noted, videocases provide "a moving picture of a classroom context" (p. 122) and "are probably the best representation [of reality]" (p. 133). This we took to be a disadvantage, in that video can too forcefully narrate one story, address one context, and thus obliterate any need to rely alternative stories that should have happened instead or could have happened in another context. Whereas videocases might help viewers study a specific case and to explore what happened in that instance, our research on practical rationality required sketchier representations that invited all geometry teachers to project themselves and their students into the case. If participants in a conversation could do that, we expected the conversation might not just address the general issues of the case represented, or call forth the alternatives that individuals might undertake in their class instead, but also elicit the common categories of perception and appreciation with which colleagues would relate to the events represented and the alternatives elicited.

Following that line of argument, we understood that in addition to describing the media itself along the lines of their qualities of records, artifacts, cases, and probes, we needed to focus on the properties of the interaction between the media and the target viewers, experienced teachers of geometry in the case under consideration. We conjectured that our sequential-art representations of teaching needed to draw on various modalities so as to create the sense that stories were on the one hand conceivable but also sufficiently sketchy so as to encourage individuals to make inferences and imagine alternatives. They needed to be encoded in a sufficiently general visual language so as to encourage dialogue across those inferences and alternatives. According to those considerations, and against the background of the use of video to represent teaching, we inquired into whether conversations prompted by other representations could showcase the properties of temporality and reflectiveness usually enabled by videocases, as well as those of projectiveness, alternativity, generality, and normativity. By projectiveness and alternativity we mean that the representations should allow geometry teachers to project into the media the particulars of their contexts and imagine alternative storylines that could have happened instead. By generality and normativity we mean that the representations should provide material for participants to address general issues of an instructional situation and that the conversation should include normative statements about such instructional situation. We illustrate below the extent to which our representations have demonstrated those properties after we describe our characters.

We investigated the use of various forms of sequential art (Mc Cloud, 1994) to construct representations of teaching. For the three forms investigated—comic book, slide show, and animation—we used settings and characters that deflect attention from themselves onto the story they sketch. The project has developed a number of character sets. The character set used in the episodes described below, the ThExpians (Figure 1), consists of two-dimensional figural, but schematic drawings in two different sizes (teacher and student). They have heads, limbs, and bodies made of simple geometric figures. ThExpians do not display the kind of immanent characteristics that identify a given person (e.g., affiliations, handicaps, race) but they have available a finite number of facial expressions and gestures (encoded with smaller, simple geometric figures) which are used to nonverbally do their share in sustaining human interaction over time, showing a range of emotions (e.g., anger, puzzlement). Each character is named after a Greek letter in each story whereas the teacher is always referred to by his or her role.

The rationale to investigate the qualities of comic books, slide shows, and animations in prompting conversations, hinged in the particular affordances that they provided regarding temporality. Across all of these modalities, we conjectured that participants' sense of the passing of time would be supported by representations that displayed interactions at the timescale of the utterance (Lemke, 2000). We also conjectured that, for those representations to be projective, they would need to omit the many other elements that enhance the sense of uniqueness and reality of video, including elements of action that take place at smaller timescales (e.g., fine motor movement, phonetic variation, ambience noise) and elements that take place at larger timescales (e.g., what the previous day's lesson was about, the time of the year this lesson takes place). Thus representations sketch conceivable stories but the viable stories are produced in the collective thought experiments.

The media forms studied (animation, comic book, slide and show) were chosen because they embodied different manifestations of the passing of time: time as bidirectional, discrete sequence of events of arbitrary duration; time as unidirectional, discrete sequence of events of standardized duration; and time as unidirectional, continuous sequence of events of uncontrollable duration. The verbal content was represented through written speech bubbles, oral single narrator, and multiple-speaker voice over. Three different storylines were represented in such a way as to support participants' production of comparable geometry classroom thought experiments about proving properties of geometric figures (kite, square, and parallelogram), each of those storylines in two different media forms. The three storylines were different but in all cases they represented deviations from the hypothetical norms listed above about interaction with diagrams when engaging students in proving.

Figure 1. A ThExpian presents a conjecture

Conversations among Teachers about these Representations

We convened a group of nine experienced mathematics teachers. Most of them had taught or were teaching geometry in high school but a couple of them had taught geometry in college and three had moved from classroom teaching to part time or full time teacher education. During a four-hour meeting participants were given the opportunity to look at the six representations described above, one at the time and engaged in a collective dialogue about the story. The session was videotaped and transcribed.

The themes in the conversation among participants were examined in search for indications of the properties targeted (temporality, reflectiveness, generality, projectivenness, normativity, and alternativity). First of all we were interested on whether the media would support conversations about strategic and tactical aspects of the teaching involved. Apropos of the “given” in the slide show of The Parallelogram story that diagonals of a parallelogram are angle bisectors, Brent offered the following reflection, which illustrates how the stories could be seized and turned into objects of study and discussion like cases are:

I think some of the students in the class might find it difficult to accept things as hypothetical givens. I mean if a teacher leaves that given as unchallenged, then that's accepted I believe in the minds of many of the students.

But beyond those general reflections, conversations also addressed more specific issues of temporality in the stories. For instance, Karin made some comments about how later moments in the comic book version of The Kite story seemed to depict too fast a story to be believable:

In frame 88, the teacher says ‘look at this diagram, if the angle bisectors did meet at three points, what would we know that has to be true about all these triangles?’ I, imagining my students working on this, there would be a lot of discussion about what triangles to attend to, or a pregnant pause where everybody wasn't really sure what that question was asking. And so you don't get like the length of pauses in this representation, but there doesn't seem to be the kind of bickering about the difficulty of the problem that we saw earlier.

Conversations also addressed issues that correspond to the distinction between context and situation. The media compelled participants to project their own selves and students into the stories, as the following quote from Melvin apropos of the student who wanted to erase the diagonal in the animated version of The Square demonstrates:

this other student, [Lambda], who I'll call Alex, because I know Alex, has an idea in his head and the teacher, which is me, doesn't totally understand what he's getting at, but that happens a lot. So at some point in time, you have confidence that he knows where he's going, but I don't know how to help him to interpret to the rest of the class.

In addition to allowing participants to identify their contexts with the stories, conversations also addressed general issues of instructional situations that were depicted through the media without any of the richness of a particular context. In regard to The Parallelogram story, Ben indicated:

I would have to understand why you would spend a lot of time trying to prove something that wasn't true in regards to maybe the more important part of properties of the rhombus, where you could show…

And as expected, the conversations also addressed issues that pertain to the distinction between possible and normative (or expected). On the one hand, participants shared alternatives that they thought they might possibly do instead. About the comic book on The Kite, Anthony said:

I would immediately challenge the fact that the short diagonal is a bisector, as opposed to letting the students start proving it, and run into trouble.

On the other hand, participants did point to normative aspects of the instructional situation of engaging students in proving. Referring to The Parallelogram, Anthony said if you're going to prove that the diagonals intersect at a point… that the bisectors meet at a point, then your givens should be this line and that line are the bisectors, not that they are the diagonals.

Conclusions

The focus group meeting gave credence to our conjecture that these representations of teaching could engage practitioners in thought experiments in mathematics teaching. Participants viewed representations of teaching practice in unusual media forms but could focus on the stories of mathematics teaching sketched out in these media. They took these stories seriously as stories about teaching; for example, they critiqued the moves of the teacher on many occasions. Participants' behavior while looking at the representations, and their comments afterwards, also give evidence that the way in which representations portray time makes a difference. Participants did browse back and forth in the comic books. They made comments more like those of an observer than those of an actor. In contrast, animations elicited reactions to more timely events such as the frustration evidenced by the teacher in The Square story when one of the students could not grasp how the problem was about the relationship between diagonals and angle bisectors. Whereas the slide shows were similar to the animations in relation to the kind of discussions that they elicited, participants indicated some sense of irritation at the way the slide shows boosted the importance of otherwise unremarkable events by inducing so many pauses. Overall we found evidence that the six properties of temporality, reflectiveness, generality, projectiveness, alternativity, and normativity can be helpful in contrasting conversations about video from conversations about sequential-art based representations of teaching, especially helping us identify the potential value added of the latter.

Endnotes

1. The research reported in this article is supported by NSF, grant ESI-0353285. Opinions expressed here are the sole responsibility of the authors and do not reflect the views of the Foundation. The character set ThExpians was made for ThEMaT by Jack Zaloga, under direction of Patricio Herbst and with the assistance of Gloriana González. The stories referred to in this paper were created by ThEMaT at the University of Michigan.

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A CONCEPTUAL-BASED CURRICULAR ANALYSIS OF THE CONCEPT OF SIMILARITY

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As they engage with activities in mathematics textbooks, students have a variety of opportunities to make sense of the concept of similarity. The nature and sequence of these activities have an impact on the development of concept images that support students as they make sense of the terms "similar figures" or "scale drawings" and the properties they hold. In this analysis of the treatment of similarity in three middle grade textbook series, the authors share their analysis of the concept definitions and concept images supported by these texts.

The term "curriculum" has different meanings in different contexts. According to the Center for the Study of Mathematics Curriculum, the most familiar terms include the ideal curriculum, the intended curriculum, the enacted curriculum, the achieved curriculum and the assessed curriculum. The focus of the present study was on the intended curriculum, which typically includes teacher's manuals, student books, and additional resources such as technology, assessment, etc.

On the basis of goals, prior curriculum content analyses can be divided into two major categories. One goal is to evaluate the alignment or effectiveness of a given curriculum against a set of pre-determined criteria. Typically, a score or a grade will be assigned as the results of such analysis. Examples of this type of study include analysis done by Project 2061 of the American Association for the Advancement of Science on algebra and middle school curricula. The other goal of content analyses is to understand the characteristics of different curricula, for example, their philosophy, content coverage, sequence, and instructional approaches. The current study shares the latter goal and our unit of analysis is a particular concept, the concept of similarity. Similar studies have been done by Cai and his colleagues on the textbook treatments of arithmetic average and early algebraic concepts (Cai, Lo & Watanabe, 2002; Cai, 2004). We draw upon theoretical constructs of concept definitions and concept images to help us analyze the potential of each curriculum to promote conceptual understanding of similarity.

Many events in daily life provide us experience with similar figures, for example, sun shadows, mirrors, photos, and copying machines. The applications of similarity include map and model making as well as surveying. Similarity also serves as a building block for more advanced study in trigonometry and calculus. Because curriculum plays a central role in school mathematics learning, it is important to ask what curricula have done (or failed to do) to address the conceptual difficulties inherent in the teaching and learning of the similarity concept. This paper will discuss some key issues that emerged from an analysis of middle-grades textbooks with illustrations taken from various curricular materials.

Concept Definitions and Concept Images

According to Tall and Vinner (1981), concept definition is a form of words used to specify a concept. Concept images include all the mental pictures and associated properties and processes built through experiences over the years. The learners may or may not be aware of the connections among these components. Influenced by an individual's experiences both inside and

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outside of school, concept images can vary in terms of the degree of richness and connectedness. It is also possible for an individual to hold contradictory concept images without perceiving the contradiction if they are associated with different contexts and not being called for simultaneously. Limited and fragmented concept images will be of little use to subsequent learning or practical uses in real life contexts.

For any given concept, a concept definition is given either explicitly or implicitly to the students. Contrary to common belief that mathematics is built upon precise definitions and axioms, mathematical definitions can vary greatly depending on the contexts. A group of researchers, when analyzing the classification of quadrilaterals in various textbooks, found five different definitions for isosceles triangles (Usiskin, Griffin and Witnosky, 2004). Similarly, the curricula may contribute to the development of concept images very differently depending on the nature and sequence of activities they use to engage students.

Prior Studies on the Concept of Similarity

The majority of findings on the concepts of similarity are embedded in studies of proportional reasoning, which has been extensively studied. Much has been learned about students' errors and difficulties in solving proportion tasks (Hart, 1984; Lamon, 1993) as well as task variables that affect students' choice of strategies and performance (Kaput & West, 1994). Lamon (1993) identified four semantic problem types. The last type, stretchers and shrinkers, includes tasks based on the concept of similarity. Lamon found the stretchers and shrinkers were the most challenging type for the six-grade students in her study to understand.

Research studies focused specifically on the concept of similarity report some seemingly contradictory findings. On one hand, researchers documented that children at early elementary grades are able to recognize similar figures visually (Swoboda & Tocki, 2002) and some were able to then judge the reasonableness of scaled images (Brink & Streefland, 1979). On the other hand, when working on missing value tasks involving similar figures, close to 40% of the 15 year olds still focus on the additive change rather than the multiplicative change of the given values when solving for the missing side length (Hart, 1998).

Some researchers have attempted to identify the nature of the difficulty with the similarity concept. Chazen (1987) identified three such aspects: 1) notions of similarity, 2) proportional reasoning, and 3) dimensional growth as the most difficult similarity-related concepts for students. He pointed out that the use of the term “similar” might mislead students who have strong associated images with the term set in non-mathematical contexts. For example, some students might think all rectangles are similar because they are generally alike and all have four right angles. Both Lehrer, Strom & Confrey (2002) and Swoboda & Tocki (2002) suggest that one way to clarify this situation with students is to treat “similarity” as a special way of classifying shape. In our analysis, we will examine if and how each curriculum attempts to address the conceptual difficulties raised by prior research studies.

Methodology

The primary data source for our curricular analysis includes units of similarity from two Standards-Based curricula: *Connected Mathematics 2* and *MathScape*. Even though the development of these curricula were supported by the National Science Foundation and share some common characteristics, such as attempting to incorporate situations from natural and social science as contexts for mathematics, there are significant differences among them. To broaden our perspective, we also examined lessons from a Japanese textbook series: *Study Together*. While *Connected Mathematics 2* treats the concept of similarity in one cohesive unit entitled *Stretching and Shrinking*, *MathScape* revisits the concept during four distinct units over three grade-levels of study. While *Gulliver's Travels* (Grade 6) includes the most concentrated treatment of the topic, additional lessons from the units *From the Ground Up* (Grade 7), *Getting in Shape* (Grade 7) and *Roads and Ramps* (Grade 8) were also examined. From *Study Together*, we included only the unit on scale drawing at the grade 6 level. It should be noted that the concept of similarity is closely related to the concepts of ratio and proportion. However in our analysis, we included only the lessons that focus primarily on the development of similarity concepts, rather than on the concepts of ratio and proportion in general.

For each curriculum listed above, we first identified the stated concept definition for similarity. We then developed a concept map, in the form of a flowchart, to illustrate the potential concept images that could be developed through the suggested activities. Finally, we determined if and how each curriculum attempts to address the conceptual difficulties inherent in the notion of similarity.

Findings

Both *Connected Mathematics 2* and *MathScape* devote over 20 lessons to the topic of similarity; whereas *Study Together* provides less than half of this amount on this topic. This difference contributes directly to the types of activities as well as the learning experiences each curriculum provides for students. Nevertheless, all these curricula share the same goal of helping students build a conceptual understanding of the properties of similarity, and be able to use these properties to solve real life problems. Our analysis centers around these main foci, diverging in the case of unique features provided by each curricula.

Concept Definitions

Both *Study Together* and *MathScape* introduce the concept of similarity in the context of scaling. *Study Together* presents to students four houses on grid paper and asks students which of the three remaining houses has the same shape as the first. In *Gulliver's Worlds*, *MathScape* has students read passages from Gulliver's journals to identify clues about the sizes of various objects in the Brobdingnag. Using these clues and the measurements of the same objects in our land, students are asked to find the scale factor that relates the sizes of the two lands. Neither curriculum offers explicit definitions of "same shape." *Study Together* assumes students are capable of determining whether two shapes are similar by looking at them, and *MathScape* assumes that students understand that scaling will preserve the shape.

As a contrast, *Connected Mathematics 2* provides two different definitions of similar figures in their student book, one during the beginning of the unit, one in the glossary toward the end of the book.

- *… for two figures to be similar, there must be the following correspondence between the figures.*
- *the side lengths of one figure are multiplied by the same number to get the corresponding side lengths in the second figure.*
- *the corresponding angles are the same size.*

The number that the side lengths of one figure can be multiplied by to give the corresponding side length s of the other figure is called the scale factor. (p. 25)

Similar figures have corresponding angles of equal measure and the ratios of each pair of corresponding sides are equivalent. (p. 103)

Notice that the first definition provides a sufficient condition for “similar figure” while the second provides a necessary condition. Merging the necessary and sufficient conditions provides the traditional if and only if form of mathematical definitions for similar figures.

Concept Images

All curricula examined in this study share two related goals, that is, help students 1) develop the notions of similarity, and 2) apply the properties of similarity to solve problems. In the following section, we will discuss these two aspects.

Develop the notions of similarity. Our analysis has identified three major types of activities that are used by these curricula to develop the notions of similarity, *differentiating, measuring, and constructing*. In the differentiating activity, students are either asked to determine if a given pair of figures are of the same shape, or to identify those figures with the same shape among a set of given figures. The assumed basis for this determination is either the intuitive notion of same shape, or the properties of similarity. The second type of activity is the measuring activity. Students are asked to measure a variety of attributes either directly or through estimation and use those measurements to explore certain patterns and relationships. The third type of activity is the constructing activity. The curricula may provide students with specific tools (e.g. grid paper, ruler & protractor) and/or step-by-step instructions (e.g. rubber band stretcher) when carrying out this type of activity. The activity may include a specific scale factor or leave it open while providing other information such as measures of side lengths or angles. In the latter case, students’ choices are typically bounded by the limitations of the materials or space afforded to them.

These different types of activities, used separately or in conjunction provide students with different opportunities to develop concept images. Differentiating activities prompt students to examine the notion of “same shape” more critically. Measuring activities affirm that the measures of the corresponding angles will be the same and the ratio of the corresponding sides will be equal among all figures that are similar. Constructing activities helps students see how the above properties can be used to create same shape figures. Together, they help to establish the necessary and sufficient conditions that are needed to mathematize the intuitive notion of same shape. Even though all three curricula include these three types of activities, there are significant differences in terms of the set up, the sequence and the purpose for using these activities among these curricula.

MathScape provides extensive opportunities for students to construct and measure similar figures, however does not provide any experiences with differentiating beyond assessing the quality and status of student constructions. In this assessment, students are taught to use estimation strategies to account for real-life margins of error as they have been working with complex images and complex scaling strategies.

Unlike *MathScape, Study Together* starts its introduction of the concept of similarity with a differentiating activity. It then asks students to perform various measurements to identify the properties of “scale drawings” and to use the properties as the basis to reject the additive pattern. *Study Together* then engages students in the constructions of scale drawings. The curricula suggests four different methods: using grid papers with the same size grid, using grid papers with different size grid, using a ruler and protractor, using the idea of central dilation as shown in Figure 1 (p. 72), and using copy machines where the scale factor is represented using percentage.

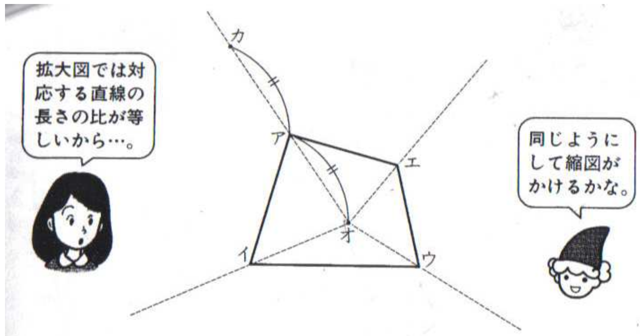

Figure 3. (1)

The unit from *Connected Mathematics 2* provides the most extensive experience with the concept of similarity for students in two dimensions, and interweaves experiences with the three types of activities. For example, it introduces three different ways to construct similar figures: rubber band stretcher, algebraic transformation $(x, y) \rightarrow (k_1x+a, k_2x+b)$ and rep-tile. It provides students opportunities to examine the conditions under which a mere subset of the properties will be sufficient to preserve the shape. Furthermore, the ratio formed by comparing the adjacent side lengths within a figure is explored as another necessary and sufficient condition for similar rectangles and triangles.

Apply the concepts of similarity. Our analysis of the three examined curricula identified four main types of activities asking students to apply the concepts and the properties of similarity. Direct applications such as finding the scale factor or a missing measure of two or more given figures are the first of these types. One main characteristic of this type of task is that the problem statement makes it explicit that the concept of similarity is involved. This can happen when the students are told explicitly that the figures are similar or that they are scale drawings of each other.

The second type of application tasks also asks students to identify the scale factor or a missing measure, but instead of working with given figures, students need to either work with the concept of corresponding parts directly without visual support, or to make their own scale drawings from which to work. All curricula include tasks of these two varieties that demonstrate to students the usefulness of scale drawings in solving problems.

Eliminating explicit clues in the problem statement that the concept of similarity is involved yields a third type of activity only found in the *Connected Mathematics 2* curriculum. To be able to solve this type of task successfully, students not only need to recognize that the concept of similarity is involved or embedded in the situation, they also need to make the correct associations between corresponding parts. Figure 2 illustrates one such task taken from *Connected Mathematics 2* (Stretching and Shrinking, p. 85).

4. Stacia stands 8 feet from a mirror on the ground. She can see the top of a 100-foot radio tower in the center of the mirror. Her eyes are 5 feet from the ground. How far is the mirror from the base of the tower?

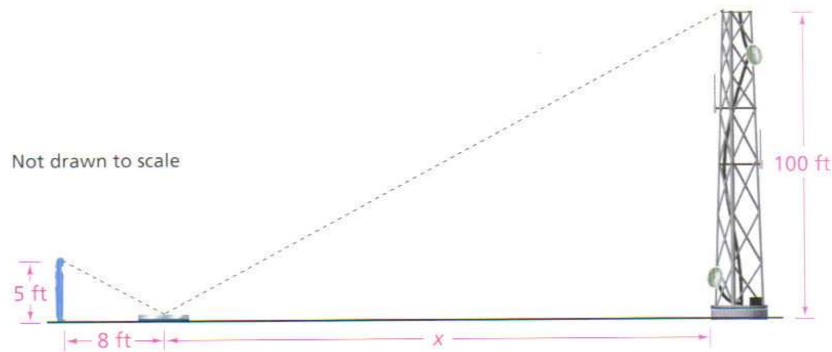

Figure 4

The fourth type of task extends the basic one-dimensional properties of similar figures to higher dimensions. Both *Connected Mathematics 2* and *MathScape* provide students with opportunities to examine the relationship between the scale factor and the corresponding area growth. *MathScape* goes a step further to engage students in building three-dimensional scale models to get a sense of volume growth. This process helps students build strong images that, numerically speaking, area grows much more quickly than length and volume grows even faster. Later, the exact mathematical relationships (K^2 for area growth and K^3 for volume growth, K: scale factor) are discovered by investigating area and volume built using unit squares and cubes.

Our analysis so far captures all but one type of activity which we believe is also very important to the development of concept images: communicating mathematical ideas. *Study Together* engages students in extensive small group and whole class discussions in conjecturing and debating. *Connected Mathematics 2* includes "mathematical reflections" at the end of each sub-unit for students to write about what they have learned. *MathScape* ends each unit with an extensive final project that requires students to pull together the main ideas and skills they have learned. For example, the final project from *Gulliver's Worlds* asks students to create three dimensional models of collected objects from Gulliver's travels and to describe the measurements of the object on display and to compare them to the same objects in our land. The experiences of constructing models and examining models constructed by others provide ample opportunity to build a strong images of scale drawings/models for the students.

Discussion

One major finding of our analysis is the focus on scaling in *Study Together* and *MathScape*. Even though scale drawing is an important application that depends on similar figures, this context alone does not convey the full range of the concept of similarity. Scale drawings are a very special case of similar figures, where correspondence is taken for granted. It is natural to think that the roof of an original drawing should correspond to the roof of the image. However, as see in the mirror task above, there are applications of the concept of similarity that are not based on scale drawings. We conjecture that the lack of experience with identifying the corresponding parts may contribute to the low success rate among high school students in seeing the similar triangles ABC and DAC in Figure 3 (Chazen, 1988).

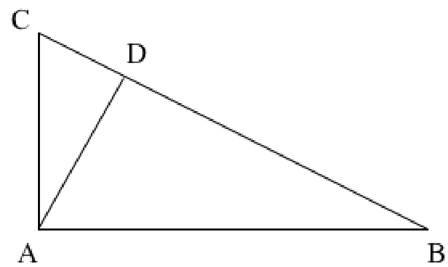

Figure 5

Another important finding of this study is the importance of providing students with different types of activities in order to help them establish the necessary and sufficient conditions for concept definitions. The measuring activity helps students to see the existence of the properties of corresponding parts between figures with the same shape; the constructing activity provides students with experience of creating images of the same shape by using the properties of corresponding parts, while the differentiating activity draws upon both students' intuition of same shape and their understanding of the properties of corresponding parts. The strength of these connections will help students to avoid overly relying on the visual clues. After all, all skinny rectangles may be considered "similar" because they all look alike as opposed to fat rectangles or squares. To be able to reach the conclusion that a 3 by 4 rectangle is "shaped" differently from a 4 by 5 rectangle, it is necessary to have both an intuition about the general shape, and recognition of that the unequal multiplicative relationships between the corresponding side lengths will distort the shape.

The myriad of differences that we have noted amongst the three curricula may be attributed to the intentions and emphases placed on the concept by individual curriculum developers. None of the three curricula treats in quite the same way, choosing to highlight different aspects, related concepts, and contexts. *MathScape* seems to consider scale drawings and models to be a fruitful context to highlight other related concepts such as estimation, measurement, and geometry. This is quite different from the focus of *Study Together* on developing multiple construction methods and rich imagery to lay the groundwork for geometric proof in a future course. Also, while *MathScape* chooses to develop both two and three dimensional scale models, *Connected Mathematics 2* focuses exclusively on similarity in two-dimensions yet goes far beyond the context of scaling in the development of the concept, including forays into informal proof. Individual treatments of similarity highlight the mathematical complexity of this concept. Future research studies are needed to examine the effect of these different curricular treatments on the development of concept definitions and concept images.

Authors' notes

This research is supported by Center for the Study of Mathematics Curriculum.

Endnotes

1. Translation of text in figure: The teacher says: "In an enlarged diagram, the ratios of corresponding segments are equal..." The boy says: "Can we draw a shrunken diagram in the same way?"

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DISCOURSE, METAPHOR, AND MULTI-REPRESENTATIONAL INSTRUCTION IN THE TEACHING AND LEARNING OF NON-EUCLIDEAN GEOMETRY

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The purpose of this study was to investigate the discourse elicited by a multi-representational view of non-Euclidean surfaces, the artifacts used to model these surfaces, and the metaphorical discourse used to construct mathematical understanding. A multi-tiered teaching experiment was conducted in a 15-week undergraduate course in non-Euclidean geometry. The results suggest that the careful use of metaphor helped provide an intuitive base for a more conceptual understanding of geometric concepts.

Introduction

Given the communicative nature of mathematical learning, semiotics as a theoretical perspective has become a viable framework for research in mathematics education. While the term semiotics in its root form refers to the study of signs in communication, the breadth of the body of mathematics education research spans a variety of issues related to semiotic theory. Some semiotic issues investigated are the role of representations (Doerfler, 2000; Presmeg, 1992, 2002; Sfard, 2000b), discourse (Presmeg, 1997, 1998; Sfard, 2000a, 2001), and cultural artifacts (Hoyos, 2002) in the semiotic mediation of the construction of mathematical knowledge.

Representations can be thought of as the form, perceptual or cognitive, which the mathematical concept takes. Furthermore, mathematical representations may take the form of a literal symbol (Sfard, 2000) or exist in the mind of the learner (Doerfler, 2000: Presmeg, 1992). Cultural artifacts refer to physical objects that are external to the cognizing being, and mediate the internal construction of psychological constructs (Mariotti, 2000). Cultural artifacts may be technologically advanced or complex (e.g. - dynamic-geometry software, computer algebra systems, or applets), or technologically simple (e.g. – pencil, paper, Lenart Spheres, or everyday objects). In an instructional setting, the teacher constructs an activity utilizing the artifact in order to promote the construction of a mathematical concept. At the same time, the student uses the artifact to accomplish the given activity. The artifact serves as a semiotic mediator to elicit meaningful mathematical discourse on a representational level; in other words, artifacts are tools that help us talk and write about mathematical objects. Furthermore, the relationship between representations, discourse and cultural artifacts is reflexive in which there is an active interaction between the three strands in the construction of one's mathematical understanding.

Within this three-fold framework or representations, discourse, and cultural artifacts is intuitive process that the learner utilizes to make sense of the mathematical investigation and discourse. While the term intuition has many different connotations, Fischbein, Tirosch, and Melamed (1981) characterize intuition as a direct acceptance without the necessary support of an explicit detailed justification. Furthermore, there is an immediacy to this form of knowledge. In this study, the lessons and instruction were designed to enable an intuitive thought process to develop a deep conceptual understanding of non-Euclidean geometry.

To help guide this intuitive process, the literary notion of metaphor (Presmeg, 1992, 1997, 1998) served as an important construct to help foster the students' mathematical intuition, and to better understand the discursive activity in the meaning making process in this study. The use of

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metaphor allows a person to use one construct to stand for another. However, with a metaphor, meaning is mediated by the connection between one domain of experience with another seemingly unrelated domain (Presmeg, 1992). For example, in this study to better understand the nature of a geodesic, or line, on various non-Euclidean surfaces, students were encouraged to imagine the experience of a bug walking along that geodesic. So on a sphere, as a bug walks along a great circle, its physical experience would be the same as walking on a straight line on the Euclidean plane. As a discursive tool, one can see how the power of metaphor lies in making meaning for a new concept through a previously constructed conception (Presmeg, 1998). Furthermore, the metaphor creates an extended context in which the learner may intuitively reason about the nature of geodesics on a non-Euclidean surface.

Sfard (2000), Doerfler (2000), and Presmeg (1992, 1998) suggest that meaning making in a mathematics environment is mediated by both the representations of the mathematical concepts that students are given and the discourse that ensues as a result of the representation. The purpose of this study was to investigate the discourse elicited by a multi-representational view of different non-Euclidean surfaces, the artifacts used to model these surfaces, the students' intuition, and the metaphorical discourse used to construct mathematical understanding. In particular, the research questions that this paper will address are in what ways did the use of metaphor and intuition help to make sense of the multiple representations of the non-Euclidean surfaces and to mediate the discursive activity in the construction of mathematical meaning.

Methodology

To investigate these phenomena, a naturalistic (Moschkovich and Brenner, 2000), multi-tiered teaching experiment (Lesh and Kelly, 2000) was conducted in an upper level undergraduate course in non-Euclidean geometry at a medium sized Midwestern university. Three researchers participated in this study to provide three tiers, or perspectives, consisting of the researcher level, the teacher level, and the learner level. One researcher assumed the role of traditional researcher by observing and videotaping every class session from the back of the room. The second researcher assumed the role of teacher-researcher (Ball, 2000). This researcher was the normal classroom instructor for this course and was responsible for the planning and teaching of all the lessons throughout the semester. The third researcher assumed a role that we identify as learner-researcher. In this unique role, the third researcher assumed a position of being a naïve-learner by attending all classes throughout the semester and participating in all the in-class activities in discussion. Since this researcher had never taken a course in non-Euclidean geometry of this nature, he was truly naïve in his limited understanding of the geometric concepts being taught, thus providing a perspective of how a student thinks during the activities and discussions.

The student participants in this study consisted of three male and three female undergraduate mathematics majors. The six participants were randomly selected from those in the class that volunteered to be a part of the study. The six students were put into two groups, one group of three students and a second group made up of the remaining three students and the researcher-learner. Each 75-minute class period was taught using a variety of instructional styles: lecture, small-group activity, and large-group discussions. Furthermore, a multitude of artifacts were used including Lenart Spheres, a dynamic spherical-geometric system, a dynamic hyperbolic-geometric system (Poincare model), Java applets, and a multitude of everyday objects. Throughout the semester over two dozen class sessions, in their entirety, were videotaped and transcribed forming the primary data source for this study. Other data sources include videotaped pre-study interviews, videotaped post-study interviews, and copies of participants 'written homework, and copies of participants' formal written assessments. The data was analyzed using iterative refinement cycles for video analysis (Lesh and Lehrer, 2000) in which each data was reviewed by all three researchers. The goal was to have the multiple perspectives from the researcher, teacher and learner converge on individual episodes or written artifacts to identify points of agreement and disagreement.

Results

Using Presmeg's (1992, 1997, 1998) concept of metaphor to create mathematical meaning provided a helpful theoretical lens to interpret the data. In particular, two themes arose. The first is the role of teacher-generated metaphors to facilitate discourse to help students make sense of mathematical concepts (Figure 1). As a teacher tries to convey a certain mathematical concept, they choose an appropriate metaphor that reflects the mathematical concept. The metaphor is shared with the learner. The familiarity of the metaphor then helps the learner make sense of the unfamiliar mathematical concept.

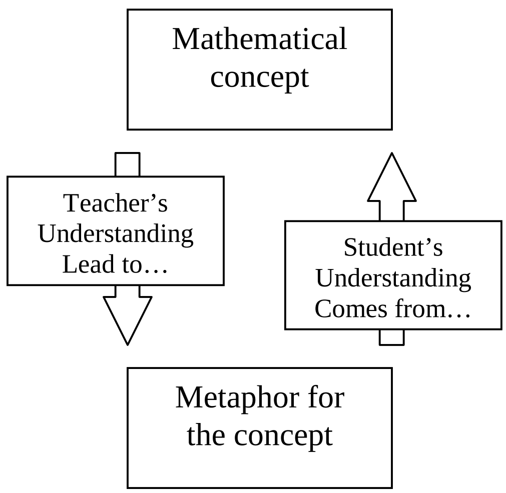

Figure 1

For example, to foster an intuitive understanding of geodesics on non-Euclidean surfaces to support more mathematically rigorous notions of geodesics, the professor introduced a variety of metaphors. One metaphor was the idea of imagining the experience of a bug as it walks along a given surface (Henderson and Taimina, 2005). As the bug walks in a straight path, from its perspective, the bug is walking along a geodesic of that surface. On a plane, as a bug walks in a straight path, from its perspective, it is walking along a geodesic, or line in a traditional Euclidean sense. However, on a sphere, as the bug walks straight, from its perspective, it is actually traveling along a great circle, which is a geodesic. Conceptually, recognizing both a Euclidean line and spherical great circle as being lines is a strange notion when viewed from an extrinsic global, or bird's eye, perspective. However, from the bugs intrinsic perspective, the local experience of walking straight on a plane and on a sphere are identical, thus providing the rational that the Euclidean line and the great circle are indeed both lines for their respective surfaces. This bug metaphor proved to be a powerful metaphor as students considered geodesics on other non-Euclidean surfaces (saddle, torus, cones, and cylinders)

The second theme is the role of student-generated metaphors and the resultant discourse to help make sense of mathematical activities and concepts. In some cases, students would generate their own metaphors, or extend teacher-generated metaphors in order to make sense of the mathematical investigations and discussion. The following case story is taken from a classroom discussion on the concept of planar curvature. In this episode, the teacher was conducting a class discussion based on the students' homework assignment in which they had to calculate the curvature of two different circles with different radii. The purpose of the activity was to show that for any given circle, the planar curvature is constant for any circle with radius, r, but dependent on the given radius.

Case story – curvature of circles

After the students put their answers to the homework problems on the board, the professor answers students' questions about the algebraic representations. After explaining the algebraic manipulation for the planar curvature of a circle, he asks, "Does that help you? Does that make sense?"

In a moment of honesty, one student responds, "I would not have done that on my own…it (the algebra) got really ugly, really quickly."

At this point, the instructor does an algebraic generalization with the students to show that the planar curvature of a circle with radius r is, as one student put it, "negative one over r."

The instructor summarizes the activity and discussion saying, "It tells you that the curvature at any point is the same and is related to one over R (radius)…does this make sense to you guys?"

Referring back to a discussion from the last class the instructor asked, "Ok, does this agree with your intuition that we developed last time…that the curvature of a circle should be constant?" In the previous class, the curvature of a circle was developed intuitively through the experience of one's physical experience of driving a car around a small circular track, at a constant speed. Although one would feel a centrifugal force, the force would be constant. However, while driving at the same speed on a larger circular track, the force would be constant, but not a great as driving on the smaller circular track.

There is a brief momentary silence over the room as the students jot down notes. It seems like many of the students are passively waiting to move on. Thinking the students are ready to move on the professor asks, "Are there any questions on the first three (problems) here?" One student raises her hand, "Mia?"

Not yet ready to move on to a new question Mia responds, "I have a question…I understand why it (curvature of a circle) is constant…but…but…in my intuition…the circle all (of them) should have the same curvature."

The professor clarifies, "all circles should have the same curvature?"

"Yes, its like we talked about how when you drive a car around a curve, it would depend on how big your car is," referring to the previous class discussion.

"Well, I want you to try to… let me try to appeal to your intuition, if we…suppose we drive in a circle in your car," pretending to drive a car the professor turns an imaginary steering wheel hand over hand as if making a hard fast turn, "we turn the wheel and hold it there, right?"

"Right."

"And you're going to go…well, close to a circle…so if you turn the wheel all the way over and you drove it 50 miles and hour…you got it up to 50 miles and hour… would you have a different experience than if you turned it a little, say a quarter turn and you were still going in a circle. Would you feel something different?" asked the professor.

Mia responds, “Yea, but if your car was really small and your were a really small person…like a bug, and you ride…very fast…in a very small circle, you would feel the same way when you ride in a big circle and ride in a big car…”

“Ok, so you’re getting into more what a person would feel depending on the size of the car and the size of the cir... if I understand you right, if you were a really small person moving on a really small circle, you would feel the same thing if you were a large person moving on a large circle…” said the professor trying to understand and clarify what Mia was saying.

“…right.”

The professor continues, “Ok, so we have to have a fair comparison, we have to have the same…you want to try to think about this as…it’s the same small person doing both or the same large person doing both, you’re not allowed to change the size of the car or the person doing it.”

“Ok.”

Bringing the discussion to a close the professor concludes, “And that’s one way of thinking about it, again, we have the mathematics behind this, which tells us that it (the curvature of circles with different radii) is different, but again I want to appeal to your intuition to try to explain why that might make sense.”

Case story discussion

This story is an example that shows both the strength and potential limitations of metaphor and intuition. The strength of the metaphor lies in its ability for students to conceptualize and bring clarity to a concept like curvature that may seem arbitrary. For example, one could think that since all circles are similar, then they should have the same curvature. On the other hand, a small circle, like a wedding ring, seems to bend more than a large circle, like in a racecar track. From an intuitive perspective, both notions seem reasonable. To bring clarity to a seemingly arbitrary situation, the professor chose a metaphorical experience, that of driving a car in a circle, to help guide the students’ intuitive process, and to give the students another “way of thinking” about the curvature of a circle. This was meaningful and useful discursive tool to help students make sense of planar curvature.

However, since a metaphor is a concept that exists outside the domain of study, over extending or misapplying the metaphor may result in a reasonable but incorrect conclusion. In this case, the professor had used a formal algebraic proof to show that the curvature of a circle is dependent on the radius, but constant for any given circle. While at first, its seemed as if all the students were willing to accept this algebraic proof, Mia shared her thought that the proof, while mathematically correct, seems counterintuitive to her notion of curvature. To Mia, her intuition seemed to have a relatively high degree of obviousness and confidence (Fischbein, Tirosh, & Melamed, 1981) yet conflicted with the algebraic representation presented by the professor and another student. The conflicting issue did not necessarily arise out of incorrect intuition, but rather an extension of the metaphor that led to an incorrect conclusion. She extended the metaphor so that the person driving a car along the smaller circle is proportionately smaller than the larger person driving on the larger circle. This is an example of an unintended consequence of a metaphor. Unlike the teacher who chooses a metaphor to reflect a concept, students must use a metaphor to make sense of a concept. In this case story, Mia incorrectly extends the metaphor. In order to bring clarity and correctness to Mia’s counter-intuition, the professor added a necessary constraint to the metaphor to make it consistent with the algebraic representation. In this episode, the instructor’s constraint on the metaphor, that the car and bug size remain constant was necessary to ensure the intended mathematical understanding was reached.

Conclusion

Although one class out of the semester was discussed in this report, the themes discussed in this paper were consistent across other classroom episodes. In general, the professor's use of metaphor and artifacts to cultivate students' intuition provided a meaningful way for students to gain a better conceptual understanding of non-Euclidean objects. While unintended consequences of the students' intuition and use metaphor had initially led misconceptions, the resulting discourse between students and professor was mathematically rich as they negotiated the intended meaning and the student constructed meaning of the mathematical concepts.

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GOING BEYOND THE RULES: MAKING SENSE OF PROOF

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Teachers of geometry claim to want their students to think logically and organize coherent arguments. Our research shows that most students are only attaining these goals on a superficial level. Students focus on learning rules of the game but are unable to resolve inconsistencies between the rules of formal proof and their non-mathematical reasoning experiences.

Objectives

When asked why teaching and learning proof continues to be an important component of most high school geometry classes, teachers of geometry often claim that learning proof helps students learn to think logically and to organize coherent arguments that explain why something is true. With these as goals for geometry students in proof-based courses, to what extent do students attain these goals? Much research over the past couple of decades has explored methods for teaching formal proof as well as student understanding of proof (Harel & Sowder, 1998; McCrone & Martin, 2004, 2005; Senk, 1985; Weber, 2001). Our current study also focuses on student understanding of formal proof. However, we investigate student participation in the discourse of proof in various settings as one measure of their understanding of proof.

In particular, we address the following two questions:

- Can students learn how to write, analyze and communicate about proofs?
- How is their ability to participate in the discourse of proof linked to their understanding of proof?

These questions are addressed by analyzing student responses to a research questionnaire and to interview questions, as well as by examining transcripts of videotaped classroom discussions.

Theoretical Perspective

Student participation in the discourse of the classroom is one measure of student understanding (Cazden, 1988; Sfard, 2000) In fact, Sfard characterized communication as thinking and learning as gaining access to a certain discourse. In order to elucidate these ideas, she described object-level rules, rules governing content of a discipline, and meta-discursive rules, rules governing the flow of the exchange of information within a discipline. In particular, Sfard claimed that student understanding may be assessed by determining the extent to which students follow these discursive rules. Because Sfard's ideas fit well with our own experiences related to student learning in classroom situations, we investigated implicit and explicit rules that governed student discourse about proof and the nature of students' participation in proof discourse to assess student understanding of formal proof in the high school geometry classroom.

Results and Conclusions

Based on analyses of classroom transcripts, interview transcripts, and other data sources, we were able to construct a portrait of students' participation in formal discourse related to proof. We detected a conflict between the "rules of the game" and students' apparent convictions about what constitutes proof. In this section, we examine student participation in the discourse of proof as it is related to two aspects of proof highlighted by McCrone and Martin (2004).

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Validity of Proof

Although there are many ways to assess a proof's validity, in this section we focus on the nature of the reasoning (inductive vs. deductive) and the logical ordering of a proof's components. One theme that arose within this category was that students preferred deductive arguments for constructing a "valid" proof because that was what was expected within the culture of the classroom. The students seemed to imply that by using this generalized form of argument they were able to demonstrate their understanding of geometric content and the proof "algorithm" they were supposed to be learning in class. Even so, most students agreed that empirical arguments were acceptable ways of justifying geometric statements.

Purpose of Proof

Our focus in this category was on proof as a tool to convince and explain to oneself or others that a mathematical statement is true. Some students saw the purpose of proof as being very local, such as to prove a specific geometric relationship, perhaps congruency, in relation to a unique diagram. Additionally, many students who were able to echo the notion that proof can be used to convince a reader of the validity of a statement, indicated that proofs are only convincing to teachers and others who were informed about the particular formats and rituals of proof used in their classroom.

Conclusions

We found that most students attempted to or were successful at following the meta-discursive rules of the classroom in relation to the general nature of a proof and in analyzing a proof. However, we have shown that, perhaps because these rules are so different from their everyday language and ways of forming convincing arguments, many students developed ways to rationalize apparent contradictions between their intuitive sense-making and the artificial world of the geometry classroom. Their rationalizations are exactly that: rational explanations for rules that may appear irrational. Perhaps teachers at this level need to be more explicit in helping students understand how and why mathematical language and practices are different from those in a loosely structured everyday culture.

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INVESTIGATING PROPERTIES OF ISOSCELES TRAPEZOIDS WITH THE GSP: THE CASE STUDY OF TWO PRE-SERVICE TEACHERS

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A four-month pilot teaching experiment on the learning of geometry was conducted with two pre-service teachers and two in-service teachers. The purpose of the study was to understand how learners with some knowledge of geometry are able to reorganize it and develop a better sense for geometric objects and their properties. The tasks of the study used semi-structured constructions in the GSP environment and open-ended questions to give learners freedom to explore, to make conjectures, to investigate them (inductive thinking), and to prove them (deductive thinking). One of the tasks of the study investigated isosceles trapezoids and their properties. The two pre-service teachers not only expressed these properties, but also they proved them using their previous knowledge of similar and congruent triangles.

Theoretical rationale

Several studies in the teaching and learning of geometry (Choi-Koh, S. S., 1999; Mariotti, 2000; Jiang, Z., 2002; Christou, Mousoulides, Pittalis, & Pitta-Pantazi 2004; De Villiers, M., 2004, among others) have used the GSP environment to mediate the interaction between teacher and students. The GSP not only fosters the learners' constructions and ways of thinking but it also mediates and makes tangible the learners' dialogues with themselves and their constructions.

Methodology

Teaching experiment. The teaching experiment methodology consists of long term interactions between researchers and learners to focus on their conceptual constructions and cognitive manifestations. This pilot study lasted four months and consisted of one-to-one task-based interviews with two pre-service and two in-service teachers. One researcher was the interviewer and the other was the participant observer. The tasks of the study used semi-structured constructions in the GSP environment and open-ended questions. The teachers were interviewed weekly throughout the academic semester; each interview lasted 90 minutes. Here we analyze a task solved by the two pre-service, Susan and Michael.

Task. Figure 1 was given to them in the GSP environment. By dragging points and segments, they first explored and conceptualized the cases of congruence and similarity between the triangles ΔOAB and ΔOCD. Then, using appropriate open-ended questions and open hints, they constructed trapezoids and isosceles trapezoids. Both teachers used the congruence of the non-parallel sides and of the two pairs of base angles in isosceles trapezoids to prove new found properties.

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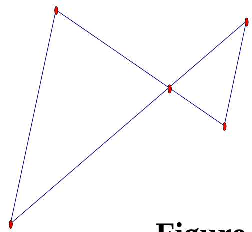

Figure 1

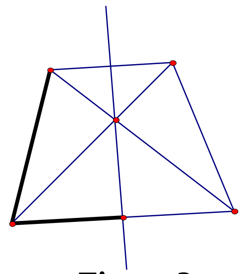

Figure 2

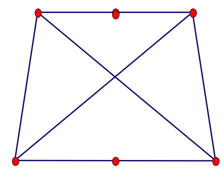

Figure 3

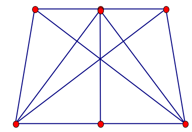

Figure 4

Analysis

Michael used the congruence of the angles in a base to construct an isosceles trapezoid connecting this property with the similar property of isosceles triangles. In contrast, Susan started from two line segments $\overline{PR}$ and $\overline{RM}$ (Fig. 2) and constructing a perpendicular line on $\overline{RM}$ at the point M she mirrored these two segments over this perpendicular line. Finally, she joined the points P and S (Fig. 2) to construct the isosceles trapezoid.

Each teacher proved different properties in different order reasoning in different ways. Susan first conjectured that “if a trapezoid is isosceles then its diagonals are congruent”. She used Figure 2 to prove the congruence of triangles ΔPRQ and ΔSQR by SAS concluding that the diagonals $\overline{PQ}$ and $\overline{SR}$ are congruent. Similarly, Michael, using the isosceles trapezoid he constructed (Fig. 3), proved the congruence of the triangles ΔDGF and ΔEFG by SAS. So, they gave the same proof.

On the other hand, they approached differently the property of the perpendicular bisector to the bases of isosceles trapezoids. Susan conjectured that “if a trapezoid is isosceles then the intersection of its diagonals lies on the perpendicular bisector of its two bases”. Using the Figure 2, she proved the congruence of the triangles ΔTRP and ΔTQS by ASA concluding that $\overline{TR}$ is congruent to $\overline{TQ}$ and $\overline{TP}$ is congruent to $\overline{TS}$. Hence, she proved her conjecture. Michael, on the other hand, conjectured that “the line that connects the midpoints of the bases in an isosceles trapezoid is perpendicular bisector to both bases”. He first constructed Figure 4 and proved the congruence of the triangles ΔUTC and ΔVWC by SAS concluding that $\overline{CU}$ is congruent to $\overline{CV}$. He completed the proof proving the congruence of triangles ΔCDU and ΔCDV by SAS. Finally, Michael wrote all the properties of isosceles trapezoids.

Susan conjectured and completed the proofs of the properties of isosceles trapezoids faster than Michael. Then, we asked her to try to prove the following proposition: “if the diagonals in a trapezoid are congruent then the trapezoid is isosceles”. She struggled to complete the proof. First she constructed congruent diagonals but not in a trapezoid. Second, using the Figure 2, she compared triangles but in all cases an element was missing to be congruent and she got stuck. Then, the interviewer using appropriate open hints helped her to realize the need of right angles. She thought to construct the perpendicular line to both bases through the intersection of the diagonals of the trapezoid but she realized that this perpendicular was not convenient. So, she figured out that the appropriate lines were the perpendicular lines from the points R and Q to the bases of the trapezoid and she constructed Figure 5 creating a rectangle. She proved the congruence of triangles ΔRVS and ΔQUP by hypotenuse and leg and the congruence of triangles ΔRVP and ΔQUS by two legs. Finally, she proved that the trapezoid was isosceles because sides $\overline{PR}$ and $\overline{SQ}$ were congruent.

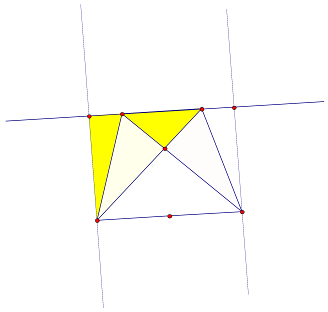

Figure 5

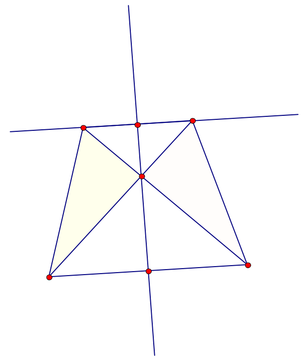

Figure 6

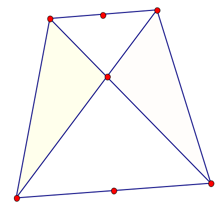

Figure 7

Then, she continued proving the proposition "If the intersection of the diagonals in any trapezoid lies on the perpendicular bisector of one base, then the trapezoid is isosceles". Susan constructed Figure 6 and she proved the congruence of triangles ΔGEB and ΔGED by SSS and of the triangles ΔHGA and ΔHGC by ASA concluding that the trapezoid was isosceles. After this proof the interviewer asked Susan to rethink and try to give a different proof. So, using the property of the point G being equidistant from the points B, D she concluded that the triangle ΔGBD was an isosceles triangle. She also proved that the triangle ΔGAC is an isosceles triangle using the property of parallel bases. So she proved that the trapezoid was isosceles as its diagonals were congruent. Using the Figure 7, she also proved the congruence of triangles ΔGAB and ΔGCD by SAS that gives the congruence of sides $\overline{AB}$ and $\overline{CD}$ making the trapezoid isosceles. In conclusion, Susan was able to prove more propositions than Michael did, but Michael was more explicit in his proofs and wrote them in detail.

Summary

The analysis indicates that, step by step, the two pre-service teachers achieved a holistic sense of the properties of isosceles trapezoids and were able to prove them. The GSP environment mediated their explorations, their inferential thinking (inductive and deductive), and their dialogues with themselves and their constructions as they acted and reacted upon them.

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REVEALING STUDENTS' CONCEPTIONS OF CONGRUENCY THROUGH THE USE OF DYNAMIC GEOMETRY: AFFORDANCES AND CONSTRAINTS OF ARTIFACTS IN A GEOMETRY CLASS

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In this work I ask what conceptions of congruency emerged discursively as students interacted with dynamic diagrams in a geometry class? The main data sources are videos of an "instructional experiment" (Herbst, 2006) in two sections of honors geometry. In the first two days of a 12-day unit on quadrilaterals, students worked to fill a table of properties relating characteristics of a quadrilateral and the quadrilateral formed by connecting successive midpoints of the original one (m-quad). The teacher asked, "What quadrilateral would you need to start from in order to get an interesting m-quad?" Students drew sketches by hand and assumed that sides and angles were congruent if they looked so. These claims relied on what I propose to call a visual-perception conception of congruency (PERC), which uses spatial and graphical features of the diagram such as the orientation, size, or position. The 3rd and 4th day, the teacher kept the same problem but changed the task (Herbst, 2006) by giving students access to the table of properties and Cabri-Geometry. Students' use and interpretation of numerical values that resulted from measuring and dragging showed a change in their conception of congruency, namely a measure-preserving conception of congruency (MEAP), which takes two objects as congruent if they have the same measure.

Measuring, which is usually perceived as an illegal or undesirable activity in the high school geometry class, can allow students to relate geometry objects by their properties rather than their shapes. The use of dragging and measuring in the geometry class contrasts with teachers' usual reluctance to measuring and their support of a transformation conception of congruency (TRANS), which establishes that two objects are congruent as long as there is a geometric transformation that maps one segment onto the other one. While measuring still does not promote the TRANS endorsed in classroom mathematics, it could be the seed for students to discover new mathematical ideas and go beyond visual perception.

Endnote

The data used for this research has been collected by GRIP under grant NSF REC 0133619 (Reasoning in Geometry, PI P. Herbst). Opinions expressed here are the sole responsibility of the author and do not reflect the views of the Foundation.

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"INSTALLING" A THEOREM IN HIGH SCHOOL GEOMETRY: HOW AND WHEN CAN A TEACHER EXPECT STUDENTS TO USE A THEOREM?

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Efforts to increase students' share of labor in the development of new knowledge require us to understand how new knowledge is customarily developed in classrooms. Like other classroom activities, the introduction of new knowledge in the mathematics class is done jointly (even in the extreme case, when one of the registers of interaction is nonverbal): the teacher does some things in some way, the students do other things in other ways. Furthermore, the introduction of new knowledge is done over time, some actions happen earlier and others later, and all of them take time as well as occupy places in time. Whereas accomplishing this joint work over time is contingent on many factors, what is to be accomplished, the element of mathematical knowledge at stake, somehow preexists that interaction. The claim that a given class has come to know something requires a judgment call over an exchange between work done and what that work could mean for an (mathematically educated) observer. The teacher may not necessarily be responsible to tell, sanction, or produce new knowledge, but she is, by virtue of the title she has, responsible to manage the place where students come to know. To understand whether and how students might take responsibility for developing, recognizing, inscribing, and remembering new knowledge we need to first understand what the customary exchanges leading to the claim that a class knows something, what the customary division of labor and organization of time are like.

In project ThEMaT (1) (Thought Experiments in Mathematics Teaching), we have been studying a case of this phenomenon in the context of the US high school geometry class: how theorems are installed. The expression "installing a theorem" designates the activity whose goal is for the teacher to be able to hold students accountable for knowing a theorem that she could not have held them accountable to know before. We expect that this activity might include actions as apparent as stating a declarative proposition and sanctioning it as theorem, but also subtler things such as translating a statement about concepts into a statement about objects. What are all those actions? How are they done, and by whom? When are those actions done in relation to each other and how long can they take? Our poster shows a model that describes the installation of theorems in geometry classes as a system of norms; where by norm we mean a central tendency around which actions tend to distribute, or a default that is applied whenever nothing ad hoc or special is done in its stead. We have been studying the norms associated with the installation of theorems by way of a novel experimental method that builds on Bourdieu's (1998) notion of practical reason: Confronting groups of practitioners to representations of the installation of a theorem that deviate from norms we hypothesize, and observing whether they say something and what they say to mark the deviation perceived (see also Herbst & Chazan, 2003).

Endnotes

1. The research reported in this article is supported by NSF, grant ESI-0353285. Opinions expressed here are the sole responsibility of the authors and do not reflect the views of the Foundation.

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PROBABILITY AND STATISTICS

SOME CONCEPTIONS AND DIFFICULTIES OF UNIVERSITY STUDENTS ABOUT VARIABILITY

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Some conceptions and difficulties are reported in this paper which in a group of 11 university students were showed by them around the topic of variability. The results show that some students consider that variability is a product of the quantity of the data as well as the irregularity in its distribution. In homework assignments with a probabilistic context they had difficulties in recognizing the sample variability and they choose for the most probable results, meanwhile in homework assignments with statistical context, they tend for the explanation the variability by ignoring the sampling and having support in deterministic reasonings.

Background and purposes

The concept of variability and the reasoning process involved in it, hold a central place in the study of statistics. Several researchers (e.g., Moore, 1990; Wild & Pfannkuch, 1999) consider it as a basic component of the reasoning process and statistical thinking. Nevertheless its importance, for a long time variability has stayed relegated into the curriculum and as a topic of research in statistics education. The most frequent reference about variability it was usually made –and still keeps this influence in no few curricula-, inside the topic of dispersion or variability measurements, with a stressed emphasis in formula and calculation process, such as standard deviation and variance, without any conceptual analysis about its meaning and also without a real purpose of developing the statistical thinking of the students.

With the difference of the topic of variability, the study of central tendency measurements had held much more the attention and the efforts of many statistical educators until some few years ago. As a results, nowadays different results of research we have around the concepts and beliefs students have about averages (e.g., Mokros & Russell, 1995; Pollatsek et. al. 1981, Konold & Pollatsek, 2004), but just a little is known about the conceptions and beliefs over the topic of variability or dispersion. And one reason about it lies in the lack of research on the topic (Shaughnessy, 1997).

Among the causes of this lack of attention to variability is that averages are often used to estimate or predict what will happen in the future, or just to compare two different groups or treatments. However, the knowledge of variability can not be disregard in order to these estimations and comparisons have sense in statistical inference. As a result we notice that a conceptual gap is present in most students when the topic of variability is discussed, which needs to be boarded by educational research (Shaughnessy, 1997).

This tendency has started to revert in recent years as well as some investigations had to emerge around the thinking of the students have about variability. Some examples of this, are the research works of Reading & Shaughnessy (2000), Torok & Watson (2000) and Watson & Kelly (2002), who have studied the variability in different contexts with students of elementary and middle levels.

At the university level is even more limited such investigation about variability, even when students need to understand it in order to start the study of statistical inference besides of making the correct interpretation on such findings. Our teacher experience and the reflections of other

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statistical educators around the topic, show us that students learn to calculate dispersion measurements such as standard deviation and variance, but without understanding its meaning.

The present research work represents just a part of a greater study in which the purpose has been the research of meanings in university students about the sampling distributions and concepts around it, among in them, variability but particularly, sample variability holds a very important place. We have specifically established and questioned the following: What are the conceptions and difficulties that university students have about variability in a previous level to the study of statistical inference?

Methodology

The study is explorative and responds to a qualitative methodology. The main instruments of data collection were the questionnaire and interviews with some students. They were 11 voluntary students (19-21 years old) who were taking course of statistical inference at the National Polytechnic Institute in Mexico City.

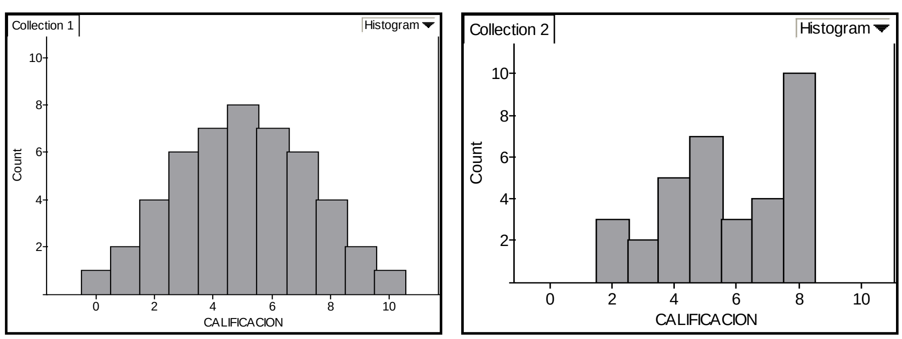
Those features of variability that were taken into account in the study were:

1. Variability of a set of data from a graphic point of view.
   a) Given two data distributions identify the distribution of greater variability
2. Sample variability
   a) Predict the possible results of the samples taking into consideration the parameters of the population.
   b) Identify the sample variability as the cause of the difference between the results of a sample and the parameters of a population.

Results and discussion

Let's firstly analyze some items related to variability of the data from a graphic point of view:

1. *Mark with an X the distribution which has more variability and explain with details the reasons of your election.*



Three students answered correctly this item, but only one of them establishes and founds well his answer, as we see immediately:

Student	Response
Omar	Because the graphic occupies a bigger range.
Denis	For me the first one has more variability considering the fact that it has a greater number of data and is from 0-10.

Libnia	I chose it because it has more bars in the histogram.

In the case of Omar and Denis, they relate correctly the variability with the range of the distribution, however, this latter student adds that the variability is related with the number of data, which is incorrect. Libnia doesn't involve the range in her answer and considers that the variability depends on the number of bars. The rest of the students considered that the second graphic has greater variability than the first one. Their explanations are focused in the difference between the heights of the bars, that is, in the irregularity of the histogram. Let's see some other cases:

Student	Response
Ana Lilia	For showing the results in different levels.
Edgar	Because there is more difference between the heights of the bars.
Gerardo	The second one has more variability since the bars of the histogram increase and decrease in different proportion in comparison to the first one.
Jorge	If variability is represented as the bars of the histogram, the one that has more variability is the distribution 1, but if variability is represented by the size or the difference that other bars have, so the histogram 2 has more variability

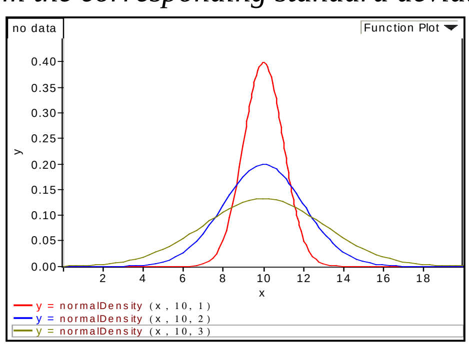
These students consider that the variability of a distribution of data has a relation with the irregularity of its shape. They never relate the variability with the range of the distribution, as Omar and Denis did.

The Jorge's case deserves special attention, since it shows confusion about the criterium for the evaluation of the variability of the distributions and besides that in both cases it exhibits a misconception.

According to the latter mentioned, we notice two misconceptions from students about variability of a set of data from the graphic point of view.

a) Variability depends on the quantity of data.
b) Variability depends also on the irregularity of the distribution.

2. *In the following graphic three populations distributions are presented in which the mean is* $\mu = 10$ *and its standard deviations are* $\sigma = 1$, $\sigma = 2$ *and* $\sigma = 3$ *respectively. Place over each one of them the corresponding standard deviation.*



The purpose of this item was investigate whether students are relating in a correctly way the standard deviation with the variability of a distribution, since we make the conjecture that some students learn to make the calculations but they don't know the meaning, nor the implications in a data distribution. Only six students answer and base their answers in a correct way. Some of them are showed in the following table:

Student	Response
Monica	Because the standard deviation is found closer to the mean.
Gerardo	The smaller the deviation, the higher the curve.
Ana Lilia	The greater the standard deviation, the more flattened the curve is.
Donovan	Because the greater the deviation the curve is on the way of widening.

As we can see, these students relate in a correct way some properties of the distributions, such as the height and the extension, just for assigning the values of the standard deviations. However, if we analyze their responses and if we compare them with the results of the first item, we notice that the students have an isolated and unconnected comprehension about the variability in a distribution and its quantification through the standard deviation, since they weren't able to relate it with the variability of the distributions in the first item. They have also learnt to make the calculations of the standard deviation and to establish some relation with properties of the theoretical distributions, but they don't understand its meaning when they apply it in different contexts, such as that of the empirical distributions.

The latter mentioned, constitute a representative example of the prevailed teaching process about the variability in the university level, where more emphasis has been put on the calculation of measures of variability, but without the searching of the understanding of its meaning in a data distribution and its relevant properties, what shows a conceptual gap as Shaughnessy (1997) says.

In corresponding terms to the topic of sample variability, the questionnaire included several items. Let's analyze the following (with probabilistic context), which pretended that the students identify some possible sample results, having known the population parameter.

3. *If a well manufactured coin is thrown a great quantity of times, the proportion of "heads" that are going to appear will be very close to 0.5. Let's suppose that you take 5 samples of 10 throws each one. Write down how many "heads" would you expect that they would appear in each of the 5 samples.*

Number of heads in the sample 1: _____
Number of heads in the sample 2: _____
Number of heads in the sample 3: _____
Number of heads in the sample 4: _____
Number of heads in the sample 5: _____

Six students appreciated the sample variability in a correct way and they considered results around of 5 heads, even though solid arguments were missed which make well justifications to the responses. By their own, those who answered incorrectly they wrote responses such as 5, 5, 5, 5, 5 in most of the cases. Those latter arguments contain elements of probability, such it is shown in the two following cases:

Student	Response
Jorge	Because the proportion of "heads" and "tails" is equalprobable in 0.5, so in each the same probability is expected.
Omar	Because the probability doesn't change

In these students responses we notice that their knowledge about probability represents an obstacle in order to appreciate the sample variability. Even then, their response considers the result with greater probability to occur, they aren't conscious that combinations of other results closer to 5,5,5,5,5 that could occur with greater frequency. Similar results were obtained by Shaughnessy & Ciancetta (2000), in a study with secondary students.

In other items of the questionnaire (with statistical context) it was searched that the students could explain the difference between the sample results and the population as a product of the sample variability. The following item is an example of them:

4. *The M&M chocolate company says that 30% of their chocolates that come in a yellow bag presentation, are in color brown. A sample was taken with 10 bags and the result was that the proportion of brown chocolate was of about 25% ¿How do you explain the previous result?*

Five out the eleven students point out that the difference is because the samples vary, other four say that the difference is because the sample is too small. These students have into consideration adequate elements in their explanation about the difference of results. The first ones referring to sample variability and the second ones, in pointing out in an implicit way a representative sample, when they refer to the size. By their own, the other two students, they are not conscious of these properties when pointing out that the company was wrong.

5. *A 50 data sample is selected from a population of temperatures presenting a sample mean of 20 degrees centigrades. ¿Which one would be your best estimation of the population mean ?*

a)Would it be exactly 20 degrees?

b) Would it be close to 20 degrees?

c) Wouldn't it be possible making an estimation, since μ is an unknown parameter and that the information is only of a single sample.

d) Another response.

The purpose of this item was to investigate if students are conscious that the sample variability generates a sample error at the moment of making an estimation. Only five students point out that the sample mean would be close to the parameter, what shows an adequate reasoning about variability, other five consider that it is not possible to make an estimation with the base of the results of a single sample and one student considers that the mean would be exactly the same parameter. As it can be seen, more than the half of the students have reasoned in an inadequate way about variability as the main cause of the sample error.

With the same target than the item before, we have following one:

6. *Under similar conditions, three survey companies have done a survey in order to determine the citizen opinion over the performance of the president of the country. ¿Would you expect that they get the same result? Explain your answer.*

In this item, seven students point out that they wouldn't expect the same result, however, in some arguments is ignored the sample as a source of variability. Such it is the case of Denis and Monica:

Student	Response
Denis	No, because we all have different kinds of point of view or opinion
Monica	No, because they aren't the same citizens and everyone thinks differently

By her own, Libnia and Jorge, give credits to the sample process the variability in the results.

Student	Response
Libnia	No, because it wouldn't be the same section or proportion of the population
Jorge	No, because the sample means have a variation in comparison to the population mean.

As we can notice, most of students consider that the results of the surveys, which are based in different samples vary among each other. Nevertheless, most part of them, in their arguments ignore the variability owed to the sample process and adduce the reasoning with deterministic disposition in order to explain the variability.

Getting to resume:

Among the misconceptions that showed the students about variability we find the following:

a) From the graphic point of view, most of the students considered that the variability depends on the quantity of data or better upon the irregularity of a distribution and they don't take into consideration the length of that distribution.
b) They have a superficial understanding about the meaning of the standard deviation as a measure of variability and they also don't relate it correctly with the properties of a distribution, specially in empirical distributions.
c) In probabilistic context situations, such as hazard games, many students choose the most probable result and they do not consider that a mixture of other results can occur more frequently.
d) In statistic context situation, such as samples and surveys, they tend to ignore the sample process as a cause or source of the variability itself and try to explain the results through deterministic arguments which don't have any relation with the uncertainty generated by the sample process.

Conclusions

The results of this research, even at an exploratory level, point out that the students showed different misconceptions and difficulties in the understanding of the variability, as well as a superficial understanding of the standard deviation as a measure of the variability, in spite of being university students who had taken at least a statistics course in previous levels and at the moment of the research they were taking a statistical inference course. The latter without any doubt represents an obstacle for the understanding of more advanced topics such as those of sampling distributions and inference methods, which require insight and deeper knowledge much more for knowing how to calculate a measure of variability.

One suggested cause is that the teaching students received during the statistics courses was focused into the calculation of variability measures, such as could be the standard deviation and the variance, and not explicitly in the development of a thinking which involves variability and the understanding of statistical concepts. This is also a proof about the conceptual gap developed by Shaughnessy (1997).

Without any doubt, the results show that it is necessary a lot of research with students at this level, in order to know and discover at a greater detail their conceptions and difficulties of understanding about variability in different situations, that could permit the generation of teaching strategies in order to develop the statistical thinking in the students besides the fact of an adequate understanding about variability.

Relationship of paper to the goals of PME-NA

The present work of research has a close relation with the major goals of the North American Chapter of PME, considering that it reports results from a research made in order to understand the conceptions and difficulties that university students have and face around the issue or topic of variability, a very important and basic concept for the teaching and learning process of statistics.

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THE CONCEPTUAL CHANGE TEACHING METHOD & PRESERVICE TEACHERS' UNDERSTANDING OF PROBABILITY

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Many students have intuitive biases that are contradictory to probabilistic reasoning (Fischbein & Gazit, 1984; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993), which makes the teaching of probability complex and difficult. Preservice teachers need teacher education programs that address this issue. Castro (1998) introduced a teaching method called the conceptual change method. This paper is a literature-based argument for the use of this method within teacher education programs to effectively address preservice teachers' understanding of probability.

Several researchers have established the importance of teaching and learning probability (Batanero, Henry, & Parzysz, 2005; NCTM, 2000; Shaughnessy 1992). However, one of the difficulties in teaching probability is due to the fact that many students have intuitive biases that are contradictory to probabilistic reasoning (Fischbein & Gazit, 1984; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993). They rely on these biases when making estimates of the likelihood of events as opposed to using more formal probabilistic reasoning.

This problem is found not only in K-12 students but in preservice teachers as well. In his study of preservice teachers' probabilistic reasoning, Koirala (2003) noticed both informal and formal reasoning used when solving probability problems. Formal probability is often based on university courses and informal probability is based on everyday intuitions and experiences. Tversky and Kahneman (1974) define a judgmental heuristic as a strategy that relies on a natural assessment to produce estimation or a prediction, which is based on students' perception of events. In light of these issues, Shaughnessy (1977) poses the question, "is there an effective way of teaching elementary probability so that students would learn to rely upon probability theory … rather than relying upon heuristic principles which may bias probability estimates?" (p. 298).

Researchers claim that the most effective way to address misconceptions is to bring about a conceptual change (Castro, 1998; Fischbein & Gazit, 1984; Konold et al., 1993; Stohl, 2005). In a study done by Castro (1998), an instructional model based on conceptual change was compared to a more traditional model. Castro uses Piaget's theory of learning to frame the study: there are two different ways of acquiring knowledge. In the first way students have adequate intuitive cognitions to solve a problem. The new knowledge is assimilated into their existing cognitive structure. In the second way students have inadequate or incorrect intuitive notions and a radical change is needed for students to reorganize their concepts. This process is called accommodation.

There are four criteria needed for an accommodation to occur: there must be dissatisfaction with existing conceptions, the new conception must be intelligible, the new conception must be initially plausible, and the new concept must suggest a new 'research program' to open up new problems. (Castro, 1998). Once students are exposed to a situation where their existing conceptions are inadequate or incorrect, the symbols and words of a new conception must be understandable to the student. In addition, this new conception must be consistent with other knowledge and aid in solving problems generated by the former conception. This process is not

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

immediate; it is usually a process of stages of gradual adjustment. Thus the teacher's role is to guide the student through this process of adjustment starting from their initial conceptions.

One of the key proponents of the conceptual change method of teaching is the building on students' prior knowledge and experience. When dealing with questions of uncertainty and chance, students rely on past experiences and intuitions to find answers. Many of these conceptions are inconsistent with formal probabilistic reasoning and thus require change. Konold et al. (1999) believe that one way to produce conceptual change is to create situations that produce cognitive conflict when the answers are based on a particular incorrect intuition. Therefore within the teaching of probability there should be opportunities for students to experiment and test their conjectures. Preservice teachers should be exposed to models of teaching for conceptual change within their teacher education programs.

An area that is missing in the literature regarding the teaching of probability is that of effective teacher education. The conceptual change model proposed by Castro should be used as a method for teacher educators to address preservice teachers' incorrect intuitive probabilistic reasoning. This in turn will allow those teachers to effectively teach probability to their students because they will have a greater awareness of students' misconceptions as well as their own.

One of the key aspects that sets conceptual change teaching apart from traditional teaching is the fact that it is student-centered, not teacher-centered. Rather than the teacher being a transmitter of information, the teacher is more of a moderator and supervisor. They provide guidance while students decide if their own previous conceptions are adequate or need to be modified. The dual nature of probability, both empirical and theoretical, requires experimental learning situations that are designed to explore this connection. Through experimentation students can test their conjectures and form convincing arguments based on their findings.

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COMMUNICATING ASPECTS OF DATA: AN EXAMINATION OF ELEMENTARY PRESERVICE TEACHERS' CONCEPTUALIZATION OF DISTRIBUTION

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Little is known about how elementary preservice teachers understand distribution, for example how to best describe, represent, and compare distributions. This study was investigated the development of understandings of distribution as expressed in the measures and representations used to communicate aspects of a given distribution.

Theoretical Perspective

Several studies provide insights into concepts considered integral components of distribution. Research suggests that visually representing a distribution has the potential to draw attention to particular aspects of data. A study carried out by Hammerman and Rubin (2004) suggests that access to a visual representation of a distribution may influence the value(s) that one chooses to represent that distribution. Similarly, Makar & Confrey (2005) found that preservice teachers are more likely to use measures of variability to represent a distribution if the data are presented graphically. Uncovering prospective secondary math and science teachers' understanding of variation was facilitated by attending to their nonstandard statistical language, revealing a 'strong relationships between expressions of variation and expressions of distribution' (Makar & Confrey, 2005, p. 27).

Other research focuses on preservice teachers' understandings of measures used to index distributions of data (Canada, 2004; Gfeller, Niess, & Lederman, 1999; Heaton & Michelson, 2002; Leavy & O'Loughlin, 2006; Makar & Confrey, 2002) and many of these studies converge on the same finding – preservice teachers' understanding of measures of center tends to be procedurally rather than conceptually-based. Leavy & O'Loughlin (2006) found that elementary preservice teachers demonstrated fluency in using and manipulating the mean algorithm but demonstrated gaps in conceptual understanding of the mean. The prevalence of procedural understandings is further supported by Gfeller, Niess & Lederman's (1999) finding that computational algorithms were the most prevalent method used by preservice teachers for solving problems related to the mean.

Participants and methodology

The 23 participants were enrolled in a math methods course as part of a one-year master's degree program leading to elementary certification. Instruction on data analysis centered on two statistical investigations and the data modeling aspects of statistical inquiry (see Lehrer & Romberg, 1995), with a focus on distributions and the ways to represent and compare them. Instruction was anchored in the statistical inquiry, addressed the statistical skills required in the inquiry, and concentrated on statistical questions surfacing from the inquiry.

This one group pretest posttest design incorporates aspects of teaching experiment methodology (Steffe & Thompson, 2000) and teacher development experiment (Simon, 2000). Two or more of the four research team members were present in the classroom during every teaching session and were involved in the daily organization and evaluation of classroom mathematical practices.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Findings and Conclusions

Initially, participants did not represent distributions in ways that highlighted structural features, rather, the focus was on calculating descriptive statistics, in particular the mean, as a way to describe data. Participants held fundamental misconceptions about the mean that limited the ways in which the mean was used to represent the distributions. Misconceptions ranged from believing that the mean could not be used to compare data sets of unequal size, to difficulties calculating weighted means, to a lack of understanding of the limitation of the mean in representing skewed data, to the belief that data values having a magnitude of zero values need not be included in the mean. In addition, none of the participants used graphical representations in a way that revealed aspects of the data. Increasing preservice teachers' awareness of and attention to the concept of distribution is possible. Over half the participants moved from using graphical representations to display group means to a focus on graphical representations as a way to depict a distribution. These participants used graphs as tools to maximize insight into a distribution, as a way to facilitate the observation of variables and the detection of outliers, while at the same time supporting the identification of trends and patterns in the data. This improved appreciation for the power of graphical representations was realized only for those participants who started out with an appreciation for graphical representations. For others the resilience of the mean algorithm was prevailing.

Supporting the development of understanding of distribution requires that preservice teachers have their attention drawn to the notion of distribution. The use of statistical inquiry supported the evolution of a distributional perspective due to the emphasis drawn by the context on the variation of data values along a scale of measurement (i.e. how height of the beans varied within the range of possible heights). A focus on data set as an entity was also essential as it provided a meaning and context for the construction of representative values (see Mokros & Russell, 1995) – measures that were initially applied without an underlying rationale. The act of comparing data sets forced the entity view in that the act of comparison required the search for comparison values, each of which needed to be representative of the body of data. Finally, once the notion of distribution was established and the concept of data set as an entity developed, understandings of distribution were further nurtured though exposure to the range of measures and representations that supported the continued effort of describing, analyzing and comparing distributions.

Gains in understanding were influenced by having access to strategies used by peers when engaging in data description and comparison. While classroom teaching experiences supported the development of skills and conceptual understanding, what they seemed not to do was convince participants of the utility of such measures when engaged in data analysis. Such experiences provide opportunities for participants to learn in practice, to develop communities of learners that engage in authentic statistical inquiry, and who continuously seek to find more efficiently and statistically justifiable ways of thinking about distribution.

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USING RESEARCH ON STUDENT THINKING IN SAMPLING CONTEXTS TO CREATE ENGAGING DISCOURSE IN STATISTICS CLASSES: AN ILLUSTRATIVE EXAMPLE

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This study is part of a larger NSF funded grant investigating students' conceptions of variability. The research presented here focuses on one student's thinking in a series of sampling tasks that asked him to make predictions for single outcomes and repeated samples. In reflecting on hypothetical student responses – presented in a semi-structured interview, the student revised his initial thinking. This paper illustrates possibilities for using student thinking coupled with dynamic tasks as a way to promote engaging statistical discourse.

Reading and Shaughnessy (2004) point out that the K-12 statistics curriculum has overemphasized measures of center at the expense of measures of spread, and that there exists a gap in the statistics education literature in the area of student reasoning about variability. However, to fully appreciate the complexity of statistical enquiry one must consider the critical role that variation plays in statistical thinking (Pfannkuch & Wild, 2004; Reading & Shaughnessy, 2004). The research presented here is part of a larger NSF funded grant involved in an effort to fill in some of the gaps in research on students' statistical reasoning with the primary goal of investigating students' understanding of variability.

Research Questions

Two overarching research questions for this NSF research grant are: 1) How do students acknowledge, describe, and record variability when they are asked to represent or to make decisions about sampling data, repeated measurements data, and multivariate data in tables and graphs? and 2) In what ways do students' conceptions of variability evolve from their initial conceptions as they experience classroom-based investigations involving sampling/re-sampling, comparing data sets, repeated measurements, and investigate large multivariate data sets?

Background, Methods, and Procedures

Middle and high school students from six schools (two urban, three suburban, one rural) participated in this research project. Three week-long classroom 'teaching episodes' were conducted over a two-year period: the first focused on sampling distributions, the second on repeated measurements, and the third on univariate and multivariate data sets in the contexts of fast food restaurant information. Surveys and task-based interviews preceded the first two teaching episodes, and a task-based interview preceded the third teaching episode. In all six research classes, four students (two males, two females) were chosen for interviews.

The tasks and student responses discussed in this report are excerpts from the first and third interviews and are focused on students' understanding of variability within sampling contexts. The first interview specifically addressed sampling contexts, and two tasks on the third interview revisited the topic of sampling distributions in order to investigate growth and change in student thinking over the course of the project. In particular, the following sub-research questions were investigated: Would students be overly focused on an expected value? Would students recognize

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a reasonable amount of spread around the center value? Would student reasoning be consistent with their predictions? Would there be evidence of student growth of understanding from the first to the third interview? After presenting an example of the interview tasks, I discuss the thinking of an 11th grade student, Jeremy, that serves as an illustration of how a framework for student thinking emerged and how these tasks could be used in classrooms to engage students in meaningful statistics activities.

Task

1. Suppose you have a container with 100 candies in it. 60 are red, and 40 are yellow. The candies are all mixed up in the container. You pull out a handful of 10 candies.
   a. How many reds do you expect to get? Why?
   b. Suppose you did this several times. Do you think this many reds would come out every time? Why do you think this?
   c. How many reds would surprise you? Why would that surprise you?

After this initial task, students were asked to make predictions for six handfuls of 10 candies and then 50 handfuls of 10 candies. Students were also shown hypothetical examples of other students' predictions for six handfuls and asked to pick out ones they thought were likely.

Discussion and Results

Jeremy's responses are described with reference to a five-tiered framework for statistical reasoning that emerged in the analysis of the larger research study (see Shaughnessy et al., 2005). Jeremy is characterized as a proportional reasoner since he predominantly attends to the ratio of reds to yellows in the jar. What is particularly interesting is the role these tasks could potentially play in promoting statistically rich classroom discourse and in facilitating student transitions to more sophisticated stages of statistical reasoning. Jeremy's survey responses yielded little information about his reasons for predictions, but the interview tasks engaged Jeremy in statistical discussions that provided insight into his thinking, as well as ways in which his thinking changed as he reflected on particular interview tasks. For example, Jeremy initially expected an equal number of handfuls above and below six red candies, the expected value. However, his thinking changed significantly when he examined the hypothetical responses of other students. Jeremy decided that he liked one of the hypothetical responses more than his original response because it had more outcomes above the center than below it. From this point on, including during the third interview, Jeremy believed that there was a higher probability of having more handfuls above the expected value. He also exhibited inconsistencies in his reasoning about the number of reds above and below the center compared to his predictions for 50 handfuls. His prediction of more outcomes above the center was not consistent with this belief. Jeremy's inconsistencies reveal that although he has some sound statistical ideas he has not fully integrated these ideas with prior knowledge. Jeremy's engagement with the sampling tasks in the interviews provides a promising illustration of how to develop meaningful statistical discourse in the classroom.

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CONDITIONAL, MARGINAL AND JOINT PROBABILITIES VS. TIME AXIS

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This paper shows the differences observed in students' answers when faced with problems involving composite experiments and asked about conditional, marginal and joint probabilities, problems that despite being equivalent can be distinguished from one another by whether or not there is simultaneity in the actions that make up the experiments.

Introduction

The notions that students have and the difficulties they are confronted with concerning conditional probability have been studied in detail for a long time. (Falk, 1979 ; Pollatsek et al. 1987; Gras and Totohasina, 1995).

A deeply-rooted conception in the students is the chronological one, by which the occurrence of the conditioned event is thought to be impossible after the conditioning one (Falk, 1979).

Sánchez & Hernández (2005) studied the effect of time on the use of the product rule for independent events. In their work, the authors devised a series of questions that consider so-called synchronic and diachronic situations: "we will call synchronic situations, those which are made up of two or more actions taking place simultaneously, and diachronic situations, those formed by two or more actions taking place successively one after another" (Sánchez & Hernández, 2005, p. 300). Their results, although not very conclusive, show that students use the product rule more often in the synchronic case than in the diachronic one.

With this in mind, we have conducted a research that sheds some light on the effect of the time variable on calculations of direct and inverse probabilities as well as on the use of the product rule for independent events and the computation of marginal probabilities. In order to compare answers we assume, unlike Sánchez & Hernández (2005), equivalent problems, one with diachronic situations, the other with synchronic ones. We designed a series of questions about two problems in the urn context. In the first problem, used by Falk's in his study, two draws are made one after another (diachronic), whereas in the second problem the two draws take place at the same time (synchronic). The full text of the problems can be found in the Appendix.

Description of the students

The questions were asked to 30 third-year students of the Teaching-oriented Undergraduate Mathematics Program of the Universidad Industrial de Santander. They had already taken at least one course on basic probability and statistics and were familiar with combinatorial processes, classical probability, tree diagrams, conditional probability, independent and dependent events, the Total Probability Theorem and Bayes' Theorem.

Results and Discussion.

The first question, about direct conditional probability in diachronic version, was correctly answered by 27 students, whereas in its synchronic version only 8 answered correctly. In this case it is clear that the diachronic version is a lot more intuitive.

To the second question, about inverse probability, the answer ½ was given by 19 students

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when it was formulated in diachronic version; the same answer was given by only 9 to the synchronic version of the question, favouring again the chronological conception of conditional probability. The correct answer, 1/3, was given by seven students when the question was in diachronic form and by 14 when it was in synchronic form. This shows that the chronological conception loses strength when the draws are simultaneous.

The third question regards the product rule. It is observed that the time interval between the two actions induces better answers from the students. Indeed, 12 students, under the diachronic version, answered correctly 1/6, while only three answered likewise under the synchronic version. This behavior is contrary to that reported by Sánchez & Hernández (2005).

To answer the fourth question, 15 students argued that it "depends on the outcome of the first draw" under the diachronic version, while only one student stated similarly under the other version. The correct answer, ½, was given by only 4 students when the question was in diachronic form, but the synchronic one made 24 students answer correctly.

Conclusions

The results demonstrate the influence of the time axis on the resolution of problems dealing with conditional probability, product rule, Total Probability Theorem and Bayes' Theorem. The synchronic situation mitigates the chronological notion of conditional probability and the 'dependence argument' to compute marginal probabilities, but it also complicates the computation of direct conditional probability and the product rule.

In didactical terms, the obtained results stress the importance of presenting equivalent forms of identical problems, distinguished by the variation of variables irrelevant for the problem, the time variable in particular. This assortment of random phenomena all equivalent would allow the student to perceive what is essential in the experiments and dutifully overlook the 'confusion variables'.

Appendix

Problem 1. An urn contains two white balls and two black balls. A ball is drawn, put aside, and then another one is drawn. Answer.

What is the probability that the second ball drawn be white, if the first was black?

What is the probability that the first ball drawn was black if the second one is white?

What is the probability that the two balls be white?

What is the probability that the second ball drawn be white?

Problem 2. An urn contains two white balls and two black balls. Two balls are simultaneously pulled out, one with the left hand, the other one with the right hand. The questions are the same as in the former problem, only substituting first draw by left-hand draw, and second draw by right-hand draw.

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USING SIMULATIONS IN AP STATISTICS: THE EFFECT OF INSTRUCTIONAL CHANGE ON STUDENT LEARNING

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Students who have been classically educated in statistical inference techniques such as hypothesis testing and confidence intervals may perform well in statistics classes but can they use their statistical reasoning when faced with making an inference based on empirical data that has been collected? There is currently a lack of research on students' understanding of the connections between empirical data collection and theoretical probability using simulation tools (Jones, in press). This study attempts to build on previous research on the statistical reasoning of US high school Advanced Placement statistics students when using such tools (Rider & Baker, 2005).

The first research study conducted by the authors utilized a task designed for sixth grade students (Stohl & Tarr, 2002) and determined that AP Statistics students, although successful in their traditional course, lacked a conceptual understanding of theoretical probability within a simulation environment. With this realization, the AP Statistics teacher incorporated more simulation activities in everyday instruction. To examine the effect of this instruction, this current study considers how students who have been exposed to probability through the use of simulation tools approach the same task. Our analysis focuses on how students approached the task, the size of the samples they chose to collect, and whether they applied statistical inference techniques to provide compelling evidence to support their estimates of the theoretical probability. We are also examining the students' interpretation and use of the law of large numbers and the central limit theorem. We are using case-based methods to study pairs of students' work as they operate the computer "microworld" (Probability Explorer, Stohl, 2002) to estimate the outcomes from randomly generated die. The purpose is to examine the difference between classically instructed students and students who have utilized simulation activities on a regular basis.

Data for the current study is under analysis, however data from the original study showed that although students could apply inference techniques appropriately, their use of samples of size 30, based on their misunderstanding of the central limit theorem and the law of large numbers, hindered their ability to arrive at appropriate estimates for the theoretical probability. Our poster will contain a description of the task the students completed, a comparative analysis of the students' statistical reasoning and success in estimating theoretical probability, and examples of students' work.

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THE RELATIONSHIP BETWEEN ARRANGEMENT STRATEGIES AND REPRESENTATIONAL TYPES IN DATA TASKS OF GRADE 4 – 7 LEARNERS

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The National Curriculum Statement in South Africa emphasises the critical role of representation in data handling, referring to the way in which different representations can either highlight or hide features of a situation. The curriculum recommends that when the data are collected or generated, special attention must be given to the representation thereof and learners should be guided to understand how to organise the data in a manner that allows them to conduct the proper data analysis to answer the question posed in the beginning.

A key idea of learning in statistics is to form and change data representations to arrive at a better understanding, a process that is called transnumeration (Wild & Pfannkuch 1999). Genuine understanding will most probably emerge and become evident to others when transnumeration takes place, that is, when a learner is able to summarise and represent a data set in a number of different ways and is capable of translating between these different representations (Chick 2003). Data arrangement plays an important role in this transformation of data during the phases of transnumeration (Mooney 2002; Chick 2003).

In the research project described here the purpose was to elicit Gr 4-7 students' spontaneous representations in two data tasks to determine the arrangement types, representational types and level of statistical thinking evident from these representations. One of the questions concerned the relationship between representation and arrangement. A sample of 144 mixed ability students in Grade 4-7 in a suburban government school in South Africa took part in the study. In the analysis eight different representational types were found, namely pictures, lists, tables, pictograms and frequency tables, bar graphs, pie charts, line graphs and anomalous representations. Four main categories, namely clustered, sequential, summative and regrouped summative arrangement strategies were found in seven different combinations. For each of the six appropriate representational types, two or three different arrangement types were used. Summative arrangement was a popular arrangement strategy used for all appropriate representational types. No pictograms and pie charts were categorised as inappropriate representations, while inappropriate representations were found for all other representational types.

Several arrangement types were found for each representational type. No direct relationship between arrangement strategies and representation types were found. The refined frameworks for categorising representational and arrangement types that were designed to accommodate all the different combinations of the broad categories, however proved adequate tools to shed light on learners' strategies to deal with data representation.

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COMPENSATING FOR IMPERFECT CONTENT KNOWLEDGE: USING CONTEXTS AS A MEANS TO SUPPORT URBAN STUDENTS IN ENGAGING WITH AND MAKING SENSE OF DATA

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Teaching is a complex activity. While we have some idea of the role teacher knowledge plays in teaching mathematics (Ma, 1999; Shulman, 1986), we are less clear about the types of knowledge and understandings that well-regarded teachers draw upon when teaching in urban settings. Our study examines the types of knowledge three well regarded urban Algebra I teachers drew upon in their teaching of data analysis and statistics. In particular, we pay attention to their use of context-rich data problems.

The high school is located in an urban setting in the Eastern United States and has a large minority population and high mobility. Three African American female teachers (T1, T2, T3) were studied. All were well regarded teachers and were nominated for participation by their school principal. The Algebra I/Data Analysis course taught was high stakes in that all students are required to pass a state mandated exam pertaining to this content in order to graduate. The teachers varied along a number of dimensions: teaching experience (2, 3.5 and 15 years respectively), mathematical content knowledge (all have undergraduate degrees in mathematics, T1/T2 have stronger preparation in data analysis and statistics), and formal academic experiences focusing on pedagogy (T1/T2 are certified, T3 has not gone through a standard teacher preparation program and was enrolled in an alternative certification program).

Our analytic methods drew on the constant comparative method used in the development of grounded theory (Glaser & Strauss,1967) in which data are constantly compared as they are analyzed against current conjectures. Eight observations of classroom teaching, a test of statistical content knowledge, two interviews, and pre and post teaching conversations were conducted throughout the school year. Analysis of the ways in which each teacher used real life examples and familiar domains revealed a stark contrast in the teachers' use of context. T2 and T3 attempt to use context in their data analysis sections to strengthen student interest and understanding whereas T1 embraced a purely mathematical and context-free style. Our analyses yield the following conclusions: (1) The use of real life contexts can scaffold teacher knowledge and create opportunities for learning for students that go beyond what it seems the teacher knows, (2) Contexts are not all alike, the use of context rich examples is not a sufficient condition to ensure student engagement, contexts must be familiar and compelling for students, and (3) School textbooks and curricular materials presented less compelling examples, successful instances of incorporating context required substantial investment of time and effort in identifying data sets that resonated with urban students. In conclusion, while mathematical content knowledge is a critical component of mathematics teaching, we observed that other characteristics such as selection of compelling statistical contexts and tasks augmented the teacher's ability to engage students in making sense of fundamental statistical ideas.

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SCHOOL MATHEMATICS STUDENTS' REASONING ABOUT VARIABILITY IN SCATTERPLOTS

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Research on students' reasoning about scatter plots has mostly concentrated on the reading, translating, and representing processes (Moritz, 2004). Our research placed students in the position of raising their own questions and conjectures about the types of variability they notice in a scatterplot, with subsequent probes for possible causes of that variability.

Research Questions

- Will students reason just about particular points, or employ data reduction strategies and reason about clumps in describing variability?
- How will students explain the variability within and across categories of variables?
- Will students reason from absolute numerical values, or will they address the proportions or percentages, thus combining information from several variables?

Discussion

Less than half of the students (36%) were able to articulate the relationship between the two variables and to use of that information to reason about the scatterplot. Relating the two variables requires an understanding of proportions, and the ability to focus on both axes when reading scatterplots.

Conclusion

The students' responses demonstrated a wide spectrum of possible thinking about bivariate information. Students made predictions and comparisons: I) Focusing only on outliers or particular values; II) Appealing to clumps or clusters of the data; III) Creating their own hypothetical cut-off lines; IV) Reasoning only from frequencies (purely additive reasoning); V) Transforming the initial data by using proportions or percentages (proportional reasoning); VI) Explicitly referring to both centers and spreads when making comparisons across the restaurants (distributional reasoning). The rich spectrum of responses to this task suggests that students pass through a variety of levels of thinking about bivariate graphical information. Students' responses to such tasks can provide researchers with solid clues about the developmental paths of student thinking about graphs, and can give teachers an opportunity to tap student responses to promote classroom discourse and shared thinking.

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PROBLEM SOLVING

A FRAMEWORK TO EVALUATE UNIVERSITY STUDENTS' USE OF DERIVE SOFTWARE TO COMPREHEND AND APPLY THE CONCEPT OF DEFINITE INTEGRAL

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This study documents the type of mathematical competence that first year university students develop as a result of working on series of problems that involve definite integral concepts. During the problem solving sessions students had the opportunity of using DERIVE software to solve the tasks, that were specially designed (Camacho, Depool and Santos, 2004), to comprehend and apply definite integral concepts. In this report, we sketch a framework to evaluate the students' mathematical competences that emerge during the sessions and value the importance, for instructors, to think of the curriculum contents in terms of problems or activities that guide the students' learning trajectories within CAS (Computer Algebra Systems) environment.

Introduction

The definite integral is a relevant content that engineering students need to comprehend and use in diverse contexts during their learning experiences. Thus, it becomes important to characterize initially, mathematical features, resources and strategies associated with the concept that are important for students to reflect in order to develop a robust understanding of that concept. In addition, the use of Computer Algebra Systems (CAS) to generate and operate distinct representations of the concept or problem seems to help instructors identify and evaluate potential learning trajectories that students may follow during instruction. Research questions that were useful to orient the development of the study include: What main mathematical concepts are involved in the process of understanding the definite integral? How the concept of area under a curve is related to the definite integral concept? What does it mean for a function to be nonnegative and continuous on one given interval? What does it mean the limit of the sum of the areas of the inscribed rectangles? How is the concept of definite integral defined in terms of the limit of the Riemman sums? How is the limit process represented geometrically? And, what type of reasoning do students develop about the concept as a result of using Derive software? In particular, important data used to discuss the research questions came from the implementation of instructional activities where students had opportunity of reconstructing the definite integral concept in accordance to its epistemological development. That is, guiding the students to think of the concept, in terms of finding areas of bound regions by using the idea of approximation and limit visually, dealing with numeric and algebraic representations, and construing geometric representations of the concept.

The use of CAS, in this case Derive, seems to offer students the possibility of approaching the concept graphically by representing a variety of functions and drawing simple figures (rectangles, trapezoids, or parabolic trapezoids) to approximate the area under those curves. What does it happen visually to the area, when the number of the drawn simple figures increase? What does the sum of those areas represent when the number of simple figures increases? Is there any relationship between the primitives associated to those functions and the limit of the

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

sums of the area of the inscribed figures? These types of questions helped students discuss strengths and limitations that appear while using the software to deal with problems or situations that involve the use of the concept of definite integral.

Conceptual Framework

Principles that helped organize and structure the study involve the recognition that students develop resources, strategies and ways of reasoning within a learning community that favors their active participation in problem solving activities (Schoenfeld, 1992). The use of CAS offers students the opportunity to represent information and relationships to solve problems in terms of visual, numeric and algebraic approaches. In this context, it is important to examine de process that students show in transforming the use of the artifact (Derive software) in a problem solving tool to explain, in terms of properties and meaning, connections and relationships among the distinct representations and operations that appear while solving the problems. In this transformation process, students need to develop skills and strategies to use software' in order to generate proper representations of the problem that lead them to pose and discuss pertinent questions to examine and deal with the problem. In this context, the use of the tool becomes important for students to explore the problem from perspectives that include numerical, algebraic, and graphic approaches. Thus, students learn mathematics as a result of reflecting on distinct ways that concepts can be represented and used in problem solving activities. In particular, when students use the software to approximate areas of limited regions, they need to think of ways to refine partitions of the interval to draw rectangles, trapezoids, etc to determine the area through summing areas of simple figures. The concept of limit becomes relevant to interpret the refining process of the interval partitions. Students can also use the software to solve some definite integrals directly; however, in order to introduce the expression into the program, students need to analyze the behavior of the function to determine the integration limits and to visualize graphically the meaning associated with solving the integral.

How are aspects of mathematical practice that involve the use of techniques or algorithms and concepts reconciled with the use of CAS? Artigue (2002) argues that "[techniques] have also an *epistemic value*, as they contribute to the understanding of the objects they involve, and thus techniques are a source of questions about mathematical knowledge (p.248). Thus, analyzing and transforming results produced with the use of CAS becomes important for students to understand the meaning of operations and concepts. That is, "technique –whether mediated by technology or not- fulfils not only a pragmatic function in accomplishing mathematical tasks, but an epistemic function in building mathematical concepts" (Ruthven, 2002, p. 283). Heid (2002) suggests that results from CAS studies challenge the assumption that students' abilities to perform procedures must precede the development of conceptual understanding. "…[CAS studies] have provided evidence that, prior to developing related by-hand routines, students can learn at a greater depth than in a traditional skills-before-concepts curriculum" (p.98).

To document the students' process of transforming an artifact into an instrument, that is making the use of computer algebra system (CAS) functional to comprehend or solve mathematical problems, we recognize that students need to develop cognitive schemes to transform a physical device or material into a mathematical tool. Artigue (2002) states that: "for a given individual, the artifact at the outset does not have an instrument value. It becomes an instrument through a process, called instrumental genesis, involving the construction of personal schemes or, more generally, the appropriation of social pre-existing schemes" (p. 250). This instrumental genesis can be explained in terms of constrains and potentialities of the artifact and their relation to the cognitive schemes that students develop as a result of progressively using the tool in problem solving activities.

In order to understand and promote instrumental genesis for learners, it is necessary to identify the constrains induced by the instrument; “command constrains” and “organizational constrains”. It is also necessary, of course, to identify the new potentials offered by instrumental work, (Artigue, 2002, p. 250).

The students’ process of transforming an artifact, in this case the DERIVE software, into a problem solving instrument shapes not only their ways to use the tool, but also their sense and conceptualization of mathematics and problem solving activities. That is, tools’ use influences directly students’ ways to deal with mathematical activities. The transformation process is linked to the tool characteristics (potentialities and constrains) and to the subject’s activity that include his/her knowledge and ways to use it (Trouche, 2005). With the use of CAS, students build a conceptual system that guides their mathematical behaviors. Ruthven (2002) points out that “…building a coherent conceptual system and an overarching concept of framing involves the progressive coordination of many other specific schemes (p. 279).

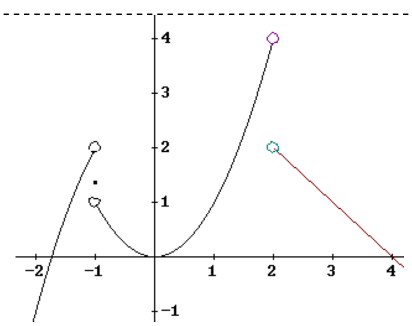
Design, methods and procedures

Forty engineering students who were taking a first year calculus course participated in the study. The course included a combination of regular classes where the instructor mainly presented the content to the students, and series of computer lab sessions in which students had the opportunity to work on sequences of tasks with the help of DERIVE software. During the development of the laboratory session, students worked on the tasks in pairs and the instructor monitored the students’ work. All the sessions were videotaped and each pair handed in its work (electronic file). In addition, all the pairs were videotaped during the development of the sessions. For the study, we focused on the work shown by 6 students during 9 sessions of computer laboratory. In particular, we are interested in analyzing information that comes from students working, in pairs, on two problems. ¿How do students make sense of the relevant information embedded in each problem? What type of representation do they use to identify and deal with concepts and mathematical properties attached to the problem? To what extent does the use of the software helped them to visualize relationships and apply corresponding procedures to solve the problem? These types of questions were used to guide and structure the students’ approaches to each problem

Problem 1:

Given the function

$$f(x)=\begin{cases}-x^2+3 & x<-1\\ 1.36 & x=-1\\ x^2 & -1<x<2\\ -x+4 & x>2\end{cases}$$

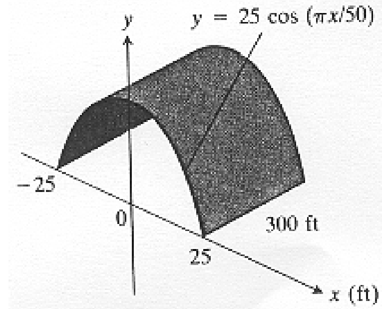


a. Calculate, if it is possible, the bound area of the curve and the interval [-2,3]

If it is possible, find the value of the definite integral on the interval [-2,3]. It is not possible then you must explain your reasoning.

Problem 2:

An engineering company will construct a tunnel under a mountain. The tunnel will have 300 feet of length and 50 feet width. The shape of the tunnel is an arc whose equation is $y = 25Cos(\pi x/50)$. The up part of the tunnel will be sealed with a paint that cost $ 1.75 dollars per squared foot. How much will it cost to apply the painting (Thomas y Finney, 1996, p.399)



Preliminary Results

To structure and present main result of the study we focus, firstly, on identifying features of the mathematical interpretation that students gave to the content involved in each task, secondly, the effects produced by the software facilities (syntax, command use, etc.) in students ways to represent and deal with the task, and thirdly, the extent to which the students' experience in using the software influences the type of mathematical reflection that they get involved during their interaction with the problems.

For the first task. We are interested in discussing the extent to which students utilized the software to make sense of the task statement, identify relevant information to translate the problem into mathematical operations, to apply the tool to solve the problem, and to make sense of the results. For example, do the students identify visually the region limited by the curve and the given interval? Do students pay attention to the intersection points of the function with x-axis? How do students calculate the area of the bound region? Do they use different methods to determine that area? How do they compare the results? How do students interpret the result?

While students worked on the first task, there is evidence that students in general visualized on the graph the region limited by the function and the x-axis. This was clear when they used the Utility File to approximate the corresponding area. However, some students utilized the software to determine the area under the curve, by introducing the function's expression and taking the extremes of the interval as the inferior and superior integration limits without analyzing the behavior of the function under such interval. That is, students seem to have acquired a procedure to calculate definite integral, with the software, by associating the initial and end point of interval as the corresponding integration limits and introducing the function into the software to get some results. These students never explained what that result meant in terms of the given problem. Thus, to illustrate the work shown by the students in the first problem, we reproduced part of the interview carried out with a pair of students:

PAIR 2

62) R (Researcher): What did you do?

63) EJ: First, we represented the function, then we are asked to calculate the area, so we calculate the integral for each subinterval from -2 to -1; from -1 to 2; and from 2 to 3 (pointing to the graph)

64) R: ¿Would that give you the area?

65) EJ: No. It will give us the measure of each one.

66) R: and the area?

67) EJ: Then, we add them up. Because, in case that one part of the graph lied below the X-axis, then we World take the absolute value; but we can observe that the entire graph is located above the X-axis..

68) R: ¿Is the graph of the function above the X-axis?

69)EJ: Ajaa. The integrals' values are positive, so we added them up to get a value. Later we calculate the integral, with the whole matrix form, and we got the same results. Here again, we added all the matrix row value and we got similar results.

$$\int_{-2}^{3} \begin{bmatrix} \mathrm{IF}(x < -1,\ -x^2 + 3) \\ \mathrm{IF}(x = -1,\ 1.36) \\ \mathrm{IF}(-1 < x < 2,\ x^2) \\ \mathrm{IF}(x > 2,\ -x + 4) \end{bmatrix} dx \qquad \begin{bmatrix} 0.6666777639 \\ 0 \\ 3.000132202 \\ 1.545712809 \end{bmatrix}$$

The area is 5.212522774

$$0.6666777639 + 3.000132202 + 1.545712809$$

$$5.212522774$$

70) R: Write down you conclusions

71) EJ: Ajaa. *They wrote* " to calculate the definite integral between -2 and 3, the result is expressed as a matrix, since we are dealing with a function defined by parts which gave us the area value for each part. The total area would be then the sumo f the absolute values of each part"

".

72) EJ: Now using Barrow's formula, we calculate the definite [integral] (pointing to the integral $\int (-x^2 + 3)\ dx$) and evaluate in the integration limits, we are going to add all to see what we get..

Other students chose the integration limits of the region based on identifying the discontinuity points of the function. Thus, they solved the definite integral by selecting integration limits associated with pieces or parts of the function in accordance with their domains. There is evidence that students often use the software efficiently to carry out series of operations, but they experience difficulties in interpreting in mathematical terms what they get through the software.

For the second task: Do students visualize the area to be sealed? Do they identify relevant information to determine the area? How do they choose the formula to calculate the arc length? How do they determine the integration limits? What units do they use to express the area? What response or solution do they get? Do they interpret the solution?

It was observed that some students experienced difficulties in interpreting the problem in terms of using the definite integral concepts. Specially when they need to apply the concept in situations that do not involve areas. For example, a pair of students (DR and PV) interpreted that the arc length can be calculated directly by finding the integral of the given function. It seems that the use of the software helped them carry out difficult operations or calculations; but they lack resources and strategies to make sense of the meaning of the results in terms of integral within the length context.

PAIR 2

42) DR: we are asked to find the total cost to seal the tunnel (pointing to the figure); but in order to find the tunnel's area, we have the arc tunnel function. Thus, we find the areas (pointing to the figure).

43) R: The area?

44) DR: The integral (PV) we get the integral of the function.

45) R: with the integral, what are you going to calculate?

46) DR: wit the integral of the function between -25 y 25.

47) R: What do you get with that?

48) PV: This is something like the width.

49) R: the width?

50) PV: the measure of this *(pointing to the border of the figure).*

51) R: what do you call that?

52) DR: Arc.

53) R: And?

54) DR: That result is multiplied by 300, which is the depth of the tunnel to get the area.

55) R: ¿What is the area?

56) DR: $2.387324146.10^5$

57) R: Yes?

58) DR: Ajaa. Then we are told that the squared feet costs 1.75 dollars, then this amount ($2.387324146.10^5$) is multiplied by 1,75 to get the total cost, which is $4.177817255 \cdot 10^5$.

59) R: isn't that amount a lot?

60) PV: that is what we got.

61) R: What was the integral value?

62) DR: *looking for the commands*

63) R: where is the function?

64) DR: *(pointing to the expression)*

$$F(x) := 25 \cdot \cos\left(\frac{\pi \cdot x}{50}\right) \qquad \int_{-25}^{25} 25 \cdot \cos\left(\frac{\pi \cdot x}{50}\right) dx$$

65) R: I believe that you are not right, this is not the formula.

66) PV: it isn't the formula?

67) I No68) DR: why are you saying that ?

69) R: What do you think?

70) R: What formula do you get?

71) PV: This is the formula (*pointing to* 

)

72) R: when you solve the integral, are you sure that you get the length of the arc?

73) DR: We, by using analogy, thought that if we integrate the function, the function is the arc of the tunnel (looking at the figure).

74) R: Do you think that the length of the arc can be calculated by solving the integral of that function?

75) DR: The arc is not the given function and the integral (PV) is the measure of what we got here, 25…but the total area is the product of that number and 300…

There is evidence that for students to approach the problems they need to develop a robust understanding of the involved mathematical concepts. It is clear that the pair of students, who worked on the second task, lacked understanding of what the definite integral concept involves. In particular, it seems that students do not recognize that the integral expression can also represent other type of situations that are different from areas. It was also observed that students tend to use the tool (CAS) mechanically to solve integrals but without reflecting on the meaning of what they do or obtain.

Concluding Remarks

What type of experience do students need in order to use the software efficiently? To what extent the students' experiences in using the tool (domain of the syntax, commands, limitations, etc.) influence the way they represent and deal with mathematical problems? These were central questions that guided the development of the study. A central finding is that students' learning trajectories should involve both attention to the students development of resources and strategies that allow them to examine mathematical concepts from different angles or perspectives, and ways to select and use proper syntax and software commands to represent and explore mathematical tasks and problems. In particular, the software should be taken as a vehicle to represent and explore properties attached to those problems. Thus, students constantly should use the software to formulate and examine set of questions that lead them to comprehend and solve the problems.

Endnote

This research is supported in part by DGI (Dirección General de Investigación I+D+I) under grant No. SEJ2005-08499, and by CIMAC (Centro de Investigación Matemático de Canarias) Foundation.

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DOCUMENTING CHANGES IN PRE-SERVICE ELEMENTARY SCHOOL TEACHERS BELIEFS: ATTENDING TO DIFFERENT ASPECTS

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Changing the beliefs of preservice teachers is a prominent topic in mathematics education. In this paper we present the results of a study that focuses on the changes in preservice elementary school teachers' beliefs during a method course. Their evolving beliefs are documented in reflective journals. These journal entries are analyzed according to established categories describing mathematical beliefs. Results indicate that through their own experiences with mathematics in a non-traditional setting the participants' beliefs about mathematics and the teaching and learning of mathematics are positively affected to include beliefs commensurate with reform teaching practices.

Introduction

Even before undertaking their first education course, prospective teachers have developed a wide range of beliefs about mathematics as well as about the teaching and learning of mathematics (Ball, 1988; Feiman-Nemser et al., 1987). As often stated, these beliefs are quite stable and robust and therefore difficult to change. Schommer-Aikins (2004) points out that beliefs are "like possessions. They are like old clothes; once acquired and worn for awhile, they become comfortable. It does not make any difference if the clothes are out of style or ragged. Letting go is painful and new clothes require adjustment" (p. 22). Nevertheless, one of the roles of the teacher education programs is to reshape teacher beliefs and to correct misconceptions that could impede effective teaching of mathematics (Green, 1971).

This study uses reflective journals to examine in what ways the beliefs of a group of preservice elementary school teachers did and did not evolve during their enrolment in a mathematics method course that was designed and taught with the implicit goal of changing their beliefs.

Theoretical framework

In this paper we deal with beliefs about mathematics and its learning and teaching. In general, such beliefs can be referred to as "messy constructs" (Furinghetti and Pehkonen, 2002; Pajares, 1992). Some of this 'messiness' can be reduced, however, if we focus on the composition of these beliefs. Törner and Grigutsch (1994) suggest that beliefs are composed of three basic components called the toolbox aspect, system aspect and process aspect. In the "toolbox aspect", mathematics is seen as a set of rules, formulae, skills and procedures, while mathematical activity means calculating as well as using rules, procedures and formulae. In the "system aspect", mathematics is characterized by logic, rigorous proofs, exact definitions and a precise mathematical language, and doing mathematics consists of accurate proofs as well as of the use of a precise and rigorous language. In the "process aspect", mathematics is considered as a constructive process where relations between different notions and sentences play an important role. Here the mathematical activity involves creative steps, such as generating rules and formulae, thereby inventing or re-inventing the mathematics. Besides these standard perspectives on mathematical beliefs, a further important component is the usefulness, or utility, of

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

mathematics (Grigutsch, Raatz & Törner, 1997).

Robust beliefs are difficult to change. However, an abundance of research purports to produce changes in preservice teachers of mathematics. Prominent in this research is an approach by which preservice teachers' beliefs are challenged (Feiman-Nemser et al., 1987). Another prominent method for evoking change in preservice teachers is by involving them as learners of mathematics (and mathematics pedagogy), usually submersed in a constructivist environment (Ball, 1988; Feiman-Nemser & Featherstone, 1992). A third method for producing changes in belief structures has emerged out of the work of one of the authors in which it has been shown that preservice teachers' experiences with mathematical discovery has a profound, and immediate, transformative effect on the beliefs regarding the nature of mathematics, as well as their beliefs regarding the teaching and learning of mathematics (Liljedahl, 2005; Smith, Williams, & Smith, 2005). All three of these approaches are combined in the design and teaching of the aforementioned mathematics methods course.

Methodology

Participants in this study are 39 preservice elementary school teachers enrolled in a *Designs for Learning Elementary Mathematics* course for which the first author was the instructor. During the course the participants were immersed into a problem solving environment. That is, problems were used as a way to introduce concepts in mathematics, mathematics teaching, and mathematics learning. This designed for the course emerged out of the literature on producing changes in preservice teachers' mathematical beliefs. This included, for example challenging their beliefs (Feiman-Nemser et al., 1987), involving them as learners of mathematics (Ball 1988; Feiman-Nemser & Featherstone, 1992), or occasioning experiences with mathematical discovery (Liljedahl, 2005; Smith, Williams, & Smith, 2005).

Throughout the course the participants kept a reflective journal (Mewborn, 1999) in which they responded to assigned prompts. These prompts varied from invitations to think about assessment to instructions to comment on curriculum. One set of prompts, in particular, were used to assess each participant's beliefs about mathematics, and the teaching and learning of mathematics (*What is mathematics? What does it mean to learn mathematics? What does it mean to teach mathematics?*). These prompts were assigned in the first and final week of the course. The data for this proposal comes from the journal entries responding to these prompts.

The three authors independently coded the data according to each of the four aforementioned components of mathematical beliefs: toolbox, utility, system, and process. Discrepancies in coding were resolved as part of a recursive process of discussion-coding-discussion that the three authors engaged in. This recursive process not only led to a more stringent treatment of the data, but also led to a greater and shared understanding of the interpretive framework at hand. We use two excerpts from the participants' journals to exemplify our shared understanding of some aspect of beliefs with respect to mathematics as well as the teaching and learning of mathematics (for further examples see Rolka, Rösken & Liljedahl, 2006).

Beliefs about mathematics – toolbox aspect:

- *"My first impression is that math is numbers, quantities, units. In math there is always one right answer. […] Math is about [...] memorizing formulas that yield the right answer."* (Stephanie)
- *"When first pondering the question "What is mathematics?" I initially thought that mathematics is about numbers and rules. It is something that you just do and will do well*

as long as you follow the rules or principles that were created by some magical man thousands of years ago." (David)

Beliefs about learning and teaching mathematics – process aspect:

- *"The other thing that stands out is the difference between formally teaching students, and actually facilitating learning. By being a facilitator of the learning process, we are able to choose situations, activities and problems for the students to work on either individually or in groups, and through this approach, students are able to […] try different ideas, and develop strategies."* (Robyn)
- *"I think to teach mathematics you need to let the thinking be put in your students' hands. You need to give them ownership of ideas and let them feel safe and free within the classroom."* (Michelle)

Results and Discussion

The coded data were treated in two distinct, but related ways. First, the coded data were aggregated to produce a holistic picture of the evolving beliefs of the class as a whole (Rolka, Rösken, & Liljedahl, 2006). The results of this analysis are summarized in *Figures 1-4*.

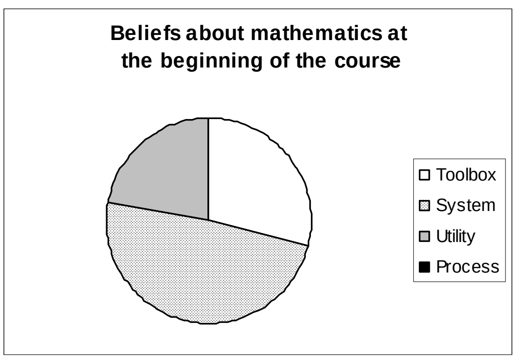


Figure 1

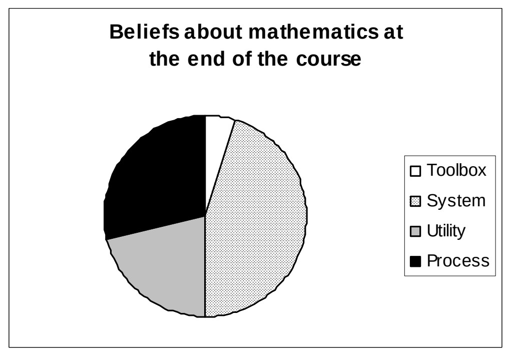


Figure 2

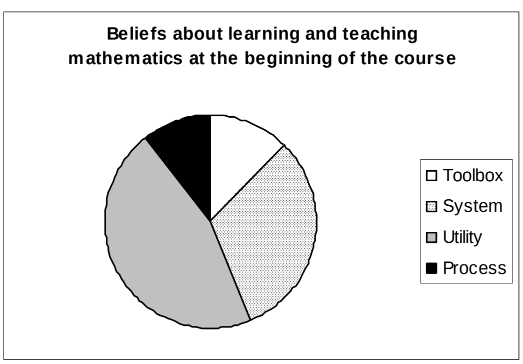


Figure 3

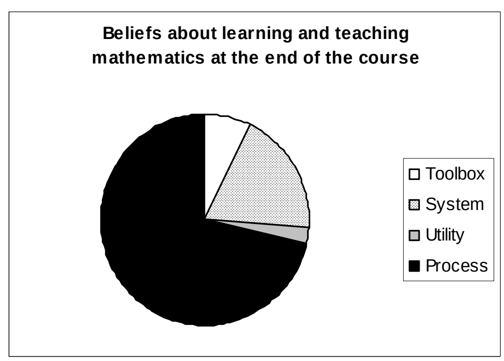


Figure 4

The most obvious change is the degree to which a process aspect of mathematics and the teaching and learning of mathematics has been introduced into the collective beliefs of the class. To see what gave way to this change a further analysis of the data was performed.

As such, the data were analyzed with a focus on how individual participants' mathematical beliefs changed. For each participant the data was examined to see how their beliefs at the beginning of the course compared to their beliefs at the end of the course. For example, in her first entry Becky's beliefs about mathematics correspond to the toolbox and the utility aspects.

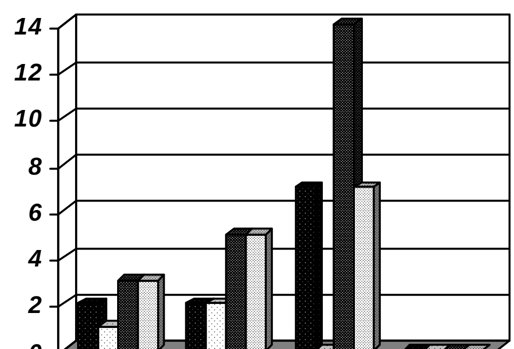
> *So all I can think of when asked this question [what is mathematics?] is numbers, the study of numbers and perhaps how math exists in this world all around us, how it encompasses us, from everything from our 10 fingers and toes to the products we bought for groceries to the angle of the sun or the curvature of the earth.*

In Becky's last entry, however, her beliefs about mathematics correspond to the process aspect.

> *I was focused on seeing math as a tool; that its benefit was in its uses. I put the stress on the applications [...] and equated meaning to its usefulness as a tool. Throughout the course, this has been the biggest evolution in my thinking, as I have moved from this definition of math towards one that focuses more on math as the thought processes or reasoning that goes on inside of us to make sense and give meaning to number relationships and patterns. [...] For me, math has truly transformed from being a skill or procedure that can be used merely for efficiency to being imbedded within a process of meaning-making that goes on inside the individual, a construction of understanding that we make up.*

The changes in Becky's beliefs are coded as [toolbox, process] and [utility, process], where [x, y] is an ordered pair denoting [initial belief, final belief]. For most of the participants (n=36) more than one belief was expressed in their responses to at least one of the journal prompts. This necessitated the above coding method in which more than one ordered pair was assigned for a participant.

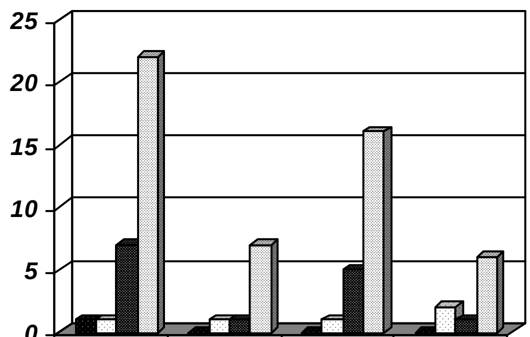
Due to space restrictions, we limit ourselves to the presentation of one student's change in her beliefs about mathematics. However, we do present the results of our coding of the data. *Figure 5* presents the changes in beliefs about mathematics across all of the participants.



	Utility Before (n=9)	Toolbox Before (n=14)	System Before (n=28)	Process Before (n=0)
Becomes Utility	2	2	7	0
Becomes Toolbox	1	2	0	0
Becomes System	3	5	14	0
Becomes Process	3	5	7	0

Figure 5: Changes in beliefs about mathematics

Figure 6 presents the changes in beliefs about the teaching and learning of mathematics across all of the participants.



	Utility Before (n=31)	Toolbox Before (n=9)	System Before (n=22)	Process Before (n=9)
Becomes Utility	1	0	0	0
Becomes Toolbox	1	1	1	2
Becomes System	7	1	5	1
Becomes Process	22	7	16	6

Figure 6: Changes in beliefs about the teaching and learning of mathematics

Each of the tables in *Figures 5 and 6* are to be read vertically. For example, in *Figure 5* there were nine participants who initially expressed a belief of mathematics that was coded as *utility* [Utility Before]. Two of these participants maintained a belief of mathematics as *utility* at the end of the course, one adopted a belief that mathematics was a *toolbox*, and so on.

From these figures a number of results emerge: results that are also supported within the fine grain analysis of individual participants' journal entries. One of these has already been alluded to above and pertains to the predominant shift towards a process belief about the teaching and learning of mathematics. In all, there were 45 instances where initial beliefs were either replaced by, or complemented with, a process belief. This can be seen in the first three columns of *Figure 6*. Not seen in *Figure 6*, however, is the result that this adoption of a process belief was embodied within 24 of the 30 (80%) participants who did not initially profess a process belief.

A second result that emerges from this data is the robustness of the systems belief about mathematics. 14 out of 28 participants who had a demonstrated systems belief of mathematics at the beginning of the course retained that belief at the end of the course. This is by far the greatest retention of any one belief aspect within the study and speaks, at least in part, to the resilience of the characterization of mathematics as logic, rigorous proofs, exact definitions and a precise mathematical language. This resilience is further accentuated by the fact that7 of the 14 participants who retained a systems belief of mathematics at the same time shifted from a systems belief about the teaching and learning of mathematics to a process belief [1]. We are still trying to make sense of this bifurcated shift in beliefs. Our conjecture is that the difference may lie within the changes to their belief structures as opposed to just their beliefs. That is, beliefs are not discrete entities – they cluster together to form belief structures (Green, 1971). We think that the bifurcated nature of the changes in beliefs is due to the fact that these participants' beliefs about mathematics reside within a different belief structure from their beliefs about the teaching and learning of mathematics. Ongoing research with these data as well as with these participants is currently under way in order to further explore this conjecture.

Conclusion

There are two main conclusions from this study. The conclusion most relevant to the results presented here pertains to the effectiveness that the problem solving environment had on the recasting of these preservice teachers' beliefs about what it means to teach and learn mathematics. Through their own experiences with mathematics in a non-traditional setting most of the participants come to see, and furthermore to believe, in the value of teaching and learning mathematics in the sense of the *process* aspect. This is an important shift in that it most closely aligns their beliefs with contemporary theories of learning as well as contemporary ideas about what constitutes effective practice (NCTM, 2000). There is an important subtlety here, though, that is not to be overlooked. Beliefs are not practice. Although there is strong evidence to suggest that beliefs govern practice (Chapman, 2002) we are not to be fooled into thinking that beliefs automatically translate into practice. Research has repeatedly shown that the adversity faced in the early years of teaching can have profoundly detrimental effects on novice teachers' best intentions of practice. Further research is needed, and is ongoing, to determine the robustness of these beliefs in the face of such adversity and to closer examine the transition from intentions of practice to actual practice.

A further result pertains to the robustness of the systems belief about mathematics. For many of the participants in this study it seemed as though these beliefs were impervious to their recent experiences with mathematics within the context of the course that the study was situated. Robust beliefs are to be expected. What was unique within this study, however, was that these beliefs remained unchanged while other (closely related) beliefs did not. Robust beliefs within such a context were not expected. We conjecture that this robustness is, in fact, due to the resilience of the belief structure that these individual beliefs reside within. Again, further research is needed, and is ongoing, to closer examine this conjecture.

Endnote

1. The other 7 of the 14 participants who retained a systems belief of mathematics also retained their systems belief about the teaching and learning of mathematics.

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MANNERS OF UNDERSTANDING AND SOLVING MATHEMATIC PROBLEMS OF HIGH SCHOOL STUDENTS

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In this article we report on the work realized by high school students when confronted with a combination of problems that involve different methods of solution in a scenario of instruction based on problem solving. During the process of implementation, the students had the opportunity to work in small groups, to present and defend their ideas to the whole class, and to constantly revise their work as a result of the criticisms and opinions they were given during their presentations and discussions in class. In this context, the students exhibited different cycles of understanding that permitted them to comprehend the fundamental ideas associated with the solution, and eventually, they resolved the tasks.

Recent proposals for a mathematics curriculum (NCTM, 2000: Balanced Assessment Package for the Mathematics Curriculum, 1999; 2000) suggest the organization of the teaching and learning of mathematics around the resolution of problems. In these it is recognized that the experiences of the students are enriched when they work with attractive problems or tasks that are posed in real contexts wherein they have the opportunity to apply and extend the basic mathematical relationships. The convenience is also recognized of implementing a manner of working in the classroom in which collective work in the whole class and in small groups is combined with individual work, and wherein the students can present and defend their ideas before the others. This also permits them to invigorate their comprehension of the mathematical contents and fortalice their abilities in the resolution of problems. In this study we were interested in documenting the strategies, representations and resources that the students used when confronted with a combination of problems that were interesting for them and were therefore easy to understand, that contained fundamental contents of the curriculum and that, because of their design, permitted the recuperation of the processes utilized in the attempts at solution. Some of the questions that guided the development of the study are: What manners of comprehension and methods for solution appear during the processes of problem resolution? What is the role of the professor during the development of the sessions of application? What signifies that the students are learning mathematics?

Conceptual Framework

Mathematics learning involves the development of a disposition on the part of the students to: explore and investigate mathematical relationships, employ distinct representations in order to analyze particular phenomenon, to use distinct types of arguments and to communicate results (NCTM 2000). This disposition permits them to better their initial attempts because the students exhibit cycles of understanding in the distinct phases of problem resolution (Lesh et al. 2000) that permits them to constantly refine their models of solution and advance their mathematic comprehension.

Furthermore, it is recognized that learning mathematics is a continual process that is favored in an atmosphere of problem resolution (Schoenfeld 1998) wherein the students have the

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opportunity to develop manners of thinking that are consistent with the routine work of the discipline. In this context, the students conceptualize mathematics in terms of problems that they must examine, explore and resolve through the use of distinct mathematical strategies and resources (Hiebert et al. 1997).

To bring about teaching and successful, significant learning, the Balanced Assessment Package for Mathematics Curriculum (1999; 2000) group proposes the utilization of tasks that are designed in a manner which is easy to understand and interesting for the students. These involve fundamental curriculum concepts and ideas presented in a manner in which the work of the students can be analyzed and documented. Furthermore, the package also emphasizes the implementation of an instruction that combines group and intellectual work.

Participants, Methods of Investigation and Procedures

The present report forms part of a study undertaken in Mexico in various public schools at the high school level in the city of Morelia, Michoacan with reference to a combination of tasks wherein the participation of the students was promoted in processes of the resolution of problems. Eighteen students of the third year of high school participated in sessions of two classroom hours wherein they were given a combination of problems with the following phases of instruction (Sepúlveda and Santos, 2004): i) Prior Activity. The teacher gave a brief introduction to the task, emphasizing the importance of the students' participation. ii) Group work. The students began the task, in groups of three, for a period of approximately 30 minutes; their actions were registered in the written report of each team, and this was turned in to the teacher, and the activities were audio taped. iii) Group presentations. Each team presented their solution to the whole class, receiving opinions and criticisms from the others. iv) Collective discussion. The teacher promoted discussion of the expositions, leading and guiding the participation of the students and when necessary, systemizing the methods of solution presented. v) Individual work. The students had the opportunity to work individually on the task and incorporate their reflections and understandings that were generated during the interaction.

The analysis phase consisted of three stages: In the first the attempts at solution and modes that arose when the students worked in teams were identified; in the second, it was verified whether there were variations in the different models after the presentations and collective discussion; and finally, in the third, the transcriptions of the ideas given by the students were analyzed. In this manner, the written reports of the students, the audio recordings and the observations of the teacher made up the database for our analysis.

Presentation of Results

To illustrate the type of analysis and results that emerged in the study, we present the work realized in one of the tasks. In this, what was of interest was to see whether the students could recognize that in increase in the lineal dimension did not produce a similar increase in area. This was represented in a problem that involved a builder who had to explain how to cover the skylight of a building with a mosaic that was square, similar to a given drawing. The task was presented on a grid of a mosaic with small rectangular pieces of glasses (as in Figure 1) of three different colors; blue 144, amber 144, red 288. The measurement of the side did not include the border that is formed by the 40 rectangular pieces of black glass.

The students were asked to answer the following:

- How many pieces of glass of each color are necessary to cover the skylight of a building of 121.92 cm. x 121.92 cm.? (Do not forget that the border is black).

- Write a description to explain how to obtain the number of pieces of glass of each color necessary to cover a similar drawing of any size.
- Suppose that there were only 6,000 pieces of red glass that that there were extras of the other colors. What is the size of the largest drawing, as in the Figure 1, that can be made?

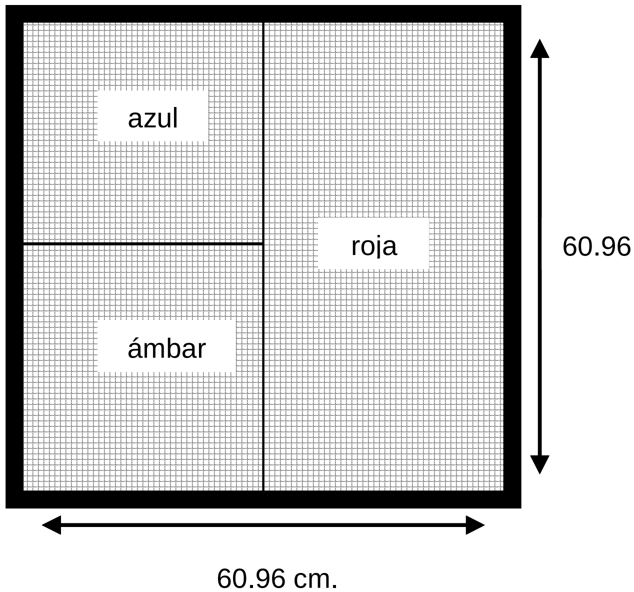


Figure 1. Illustration that accompanies the task "Evaluate a drawing"

When they started the work in teams, some of the students rushed to say that if the side of the square were doubled, then the number of pieces of glass on the interior must be doubled; but they changed their opinion after hearing the arguments of their classmates.

The written reports of the work of the small groups demonstrates that five of the groups coincided in the answers to Question 1; however, three distinct approaches were distinguished:

- Teams A and B sketched squares with sides 1 and 2, calculated areas and parameters, and established the proportions $\frac{1}{4} = \frac{144}{x}$ for the interior pieces of glass, and $\frac{1}{2} = \frac{40}{x}$ for the black pieces. They argued that the figures are similar and that "the number of black pieces of glass is doubled because of having the relation of 1 to 2. The interior pieces are quadrupled because of the square of the areas is 1 to 4" (Figure 2).
- Teams C and F had an geometric approach; the drew a square such as that of the given Figure and another that contained twice as many black pieces of glass on each side. Team C concluded that the given drawing fit four times in the new one and required 80 black pieces of glass (double), 576 blue pieces of glass, 576 amber pieces of glass, and 1,156 red pieces of glass (quadruple); team F established proportions and resolved the problem (Figure 2).

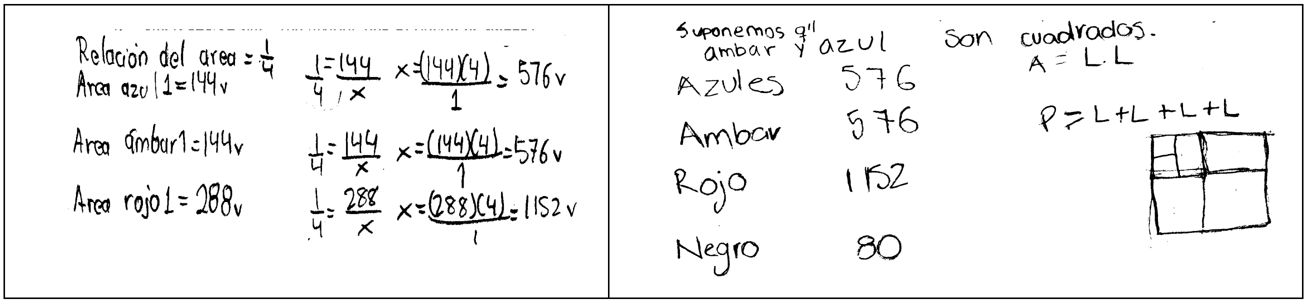


Figure 2. Approaches of teams A and C in Question 1

- Team D obtained the square of the areas $A_1 = 60.96^2$ and $A_2 = 121.96^2$; $\frac{A_2}{A_1} = 4$; multiplied 4 times 288 and obtained the number of red pieces of glass, etc. Initially, the tape recordings show that some of the students rushed to say that if the side was double, the interior pieces of glass would be double, but changed their opinion after hearing the first arguments.

In Question 2 was evaluated as the step which generally represented a problem for the majority of the students. Only the written report of team F contains a concrete answer. Their approach consisted in applying proportionality according to the lineal dimension (black pieces of glass) or of area (interior pieces of glass). The other teams' work contains verbal expressions or phrases that do not accord with what was asked in the instructions.

Teams D and F answered Question 3. Team F drew a square with side x, and applied proportionality through the rule of three: $\sqrt{576}$ is to 60.96 as $\sqrt{12000}$ is to x. Team D obtained the reasoning $x = \frac{6000}{288} = 20.83$, called A1 the area of the given square, calculated the area of which they were going to construct: $A_2 = A_1 x = 77419.2$; therefore $l_2 = \sqrt{77419.2} = 278.26cm.$

During the group presentations, when team A went to the front to explain their solution, they had a strange form of accommodating the black pieces of glass. The following is a part of the discussion between Abraham [A] (team A), Sarahi [S] (team F), and the teacher [T];

- A: If there are 40 black pieces of glass in the given drawing, and each side requires 10, how will they be placed? If you put 10 above and 10 below then the other 20 don't fit, or at the most eight and eight fit, …taking out the two that go in each corner.
- T: Why don't you do a Figure e, try to draw the placement of the black pieces of glass.
- A: … I did but it didn't work, there were empty places…
- S: Make your drawing bigger and start marking for the black pieces of glass from the extreme left. That way you have 10 frames, and then start to count the 10 pieces of glass on the side that follows, but from the width of the exterior of those that you have already placed.

  Abraham was not convinced in spite of the intervention of the teacher to support Sarahi's proposal. The teacher made an appointment with Abraham for later (here is included a part):
- T: Draw a large square with the corresponding shape of the black pieces of glass. Remember the recommendation that Sarahi made.

- A: Okay. I start to count from here and count 10, but … I keep missing spaces to fit in the 40, don't I? … (Persists in his error).
- T: What would happen if the black pieces of glass were square? How would you place them?
- A: (Abraham makes various attempts to place them) … Oh yes, if they are square they go exactly to the corner. Their width and length are equal and where the first one ends or starts the last one coincides with the interior pieces of glass; but …
- T: Make a new drawing and place the glasses in the corners (Abraham does the drawing and marks the squares of the corners). How many pieces of glass have you used and how many are left to place?
- A: … 36 divided by 4 is … 9. Yes, there are 9 left to place on each side.
- T: Draw them. What do you see?
- A: Ah, from the end of glass 10 starts the first piece of glass of the 10 that go on this side (points to the design that resolved the problem).
- T: Now do it supposing that they are not square. (Abraham draws a large square with a thick outline and places the 40 black pieces of glass).

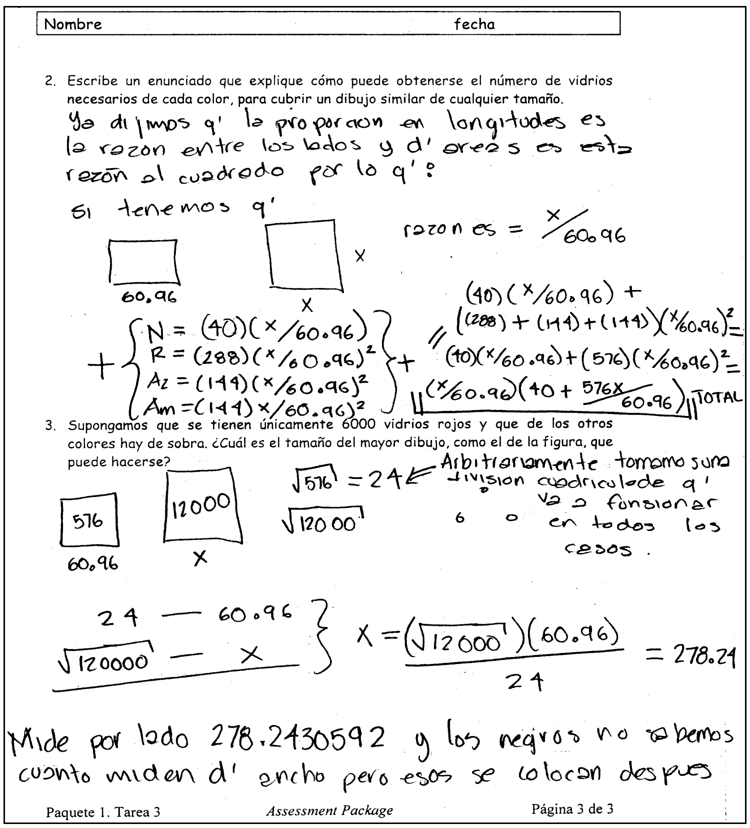
Another aspect that motivates the discussion is related to the form and measurements of the interior pieces of glass, and if the black pieces of glass are of the same measurements as those of the interior. During the presentation of team C, in which Francisco was a member, the following dialogue occurred (Francisco [F], Teacher [T], Miguel [M] (team D), Sarahi [S]).

- F: Since you tell me that there are 144 blue pieces of glass, this means that the side of the square where they go has 12 pieces of glass, which means that all of the interior pieces of glass are square.
- T: Let's see. What do you think? (Asks the entire group).
- M: Yes, they are square.
  There is doubt whether to argue or agree with the argument of Francisco until Sarahi intervenes.
- S: The interior pieces of glass are not necessarily square. You have based that on the fact that the square root of 144 is 12, but you can be sure that there are other measurements of rectangular glass that give 144. You could show this with 24 and 6, for example.
- F: … but it would work if they were square.
- S: Yes it would, but it would also work with pieces of glass that are rectangular, not necessarily square. (She points to a piece that makes this apparent).

The presentations of teams D and F motivated the collective discussion with their contributions and various students from the other teams commented that they had understood the solutions to Questions 2 and 3. Particularly in Question 3 there were advances in individual solution on the part of the students; however, after the collective discussion some of the students tried to give the answer to Question 2 but the answers were not complete, except for the individual answer of Sarahi who, it seems, had sufficiently developed the skill of making generalizations. It should be mentioned that Sarahi, being the most advanced student of the group, had a manner of implementing the activity that helped the others to understand her and respond correctly. Some of her first interpretations at the start of the work in small groups were mistaken, and after the interaction with her classmates, with other teams and with the teacher she said "she had not understood the problem well". Afterwards, during the presentations of the teams, she actively participated in the discussions and contributed with ideas so that the others understood the solution of the questions. Figure 3 shows the individual work of Sarahi.

Discussion

Practically, the presentations initiated the collective discussion that was converted into a platform for the discussion of issues related with the understanding of the problem, the use of distinct representations, mathematical relationships, and the solution of the problem. In the first approach fragmented knowledge appeared, as well as incomplete or incorrect ideas; however, when the students had the opportunity of discussing and exploring their ideas with the other students, they improved their initial approaches and proposed more "robust and sophisticated" manners of resolving the task. The fundamental ideas that emerged in the work of the students involved the use of figures, calculation of parameters, areas, and the application of proportionality to justify the type of variation between a lineal dimension and one of area.



Nombre fecha

2. Escribe un enunciado que explique cómo puede obtenerse el número de vidrios necesarios de cada color, para cubrir un dibujo similar de cualquier tamaño.

Ya dijimos q' la proporcion en longitudes es la razon entre los lados y d' areas es esta razon al cuadrado por lo q':

si tenemos q'

60.96 x x

razon es = $x/60.96$

$$\left\{\begin{array}{l} N = (40)(x/60.96) \\ R = (288)(x/60.96)^2 \\ Az = (144)(x/60.96)^2 \\ Am = (144)(x/60.96)^2 \end{array}\right.$$

$(40)(x/60.96) + [(288) + (144) + (144)](x/60.96)^2 =$

$(40)(x/60.96) + (576)(x/60.96)^2 =$

$(x/60.96)(40 + \frac{576x}{60.96})$ TOTAL

3. Supongamos que se tienen únicamente 6000 vidrios rojos y que de los otros colores hay de sobra. ¿Cuál es el tamaño del mayor dibujo, como el de la figura, que puede hacerse?

576 60.96 12000 x

$\sqrt{576} = 24$ ← Arbitrariamente tomamos una division cuadriculada q' va a funcionar en todos los casos.

$\sqrt{12000}$

$$\left.\begin{array}{l} 24 — 60.96 \\ \sqrt{120000} — x \end{array}\right\} \quad x = \frac{(\sqrt{12000})(60.96)}{24} = 278.24$$

Mide por lado 278.2430592 y los negros no sabemos cuanto miden d' ancho pero esos se colocan despues

Paquete 1. Tarea 3 *Assessment Package* Página 3 de 3

Figure 3. Individual work of Sarahi

The form of work in the classroom and the meaning of the arguments derived in the form of interaction all implemented the activity. They were organized in a plan completely distinct from the traditional; on one hand, the arguments of validation of the result could have come from one or various students at the precise moment in which they spoke, whether it be during the group work or in the presentations of the teams, such as occurred in various issues that resulted in being problematical: the placement of the pieces of black glass or whether the interior pieces of glass were square. On the other hand, this form of task in the classroom permitted the approximation of the possibility in which the student was able to construct knowledge in the measure of how their understanding was evolved, which definitely occurred when Francisco accepted that the pieces of glass were not necessarily square because of a piece that showed that 4 square pieces of glass had the same area as 4 rectangular ones of double the longitude and half the width. Or during the interview with Abraham, when he understood about placing the black pieces of glass, which was derived from the suggestion that he supposed that those pieces of glass were square. That is to say, this form of instruction converted the task itself converted into a learning tool.

The task resulted in being useful for the students with a distinct level of development to advance in their understanding level, as in the case of Sarahi, who was able to write the general expressions involved in the notion of proportionality in order to determine the number of pieces of glass of each color in Question 2, and of Daniel, who argued correctly his answer to Question 1 (Figure 4). In general, the answers corresponding to the individual work are more complete than the answers contained in the report elaborated in the group work. Some of the students had serious difficulty when they wished to calculate the area units of each of the interior pieces of glass and to apply proportionality in order to answer Questions 2 and 3 because this required a careful management of the small quantities that they obtained. In Figure 4 it can been seen how a variety in the use of resources permitted Juan Carlos to answer Question 3. Only the group E demonstrated inattention in the group dynamic.

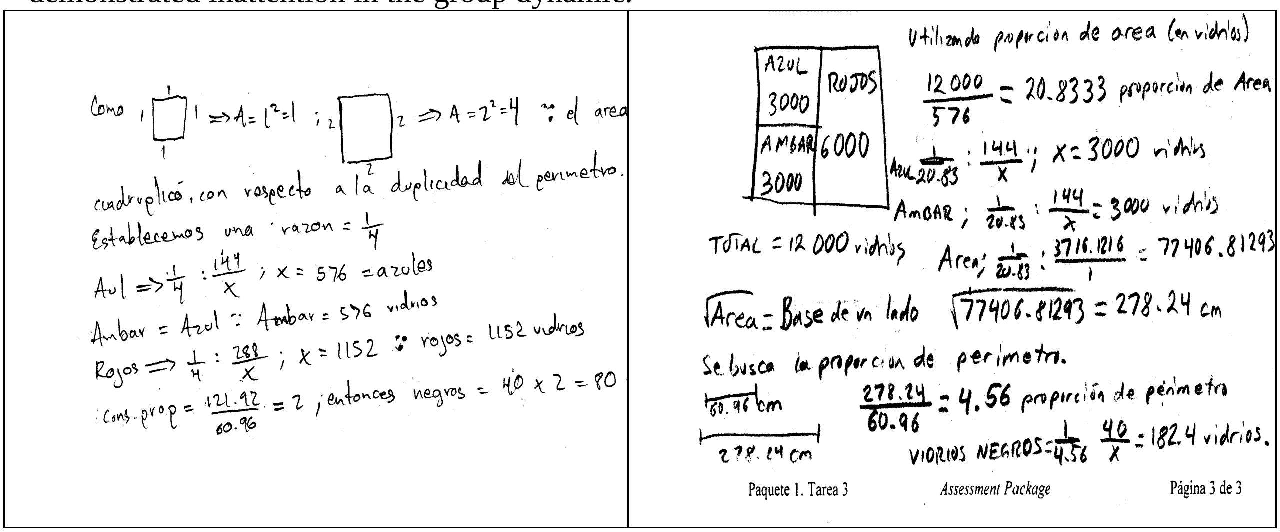


Figure 4. Individual answers. David (team A) and Juan Carlos (team D).

Remarks

We comment two important aspects over this work: i) The importance of designing or reformulating activities in which the students have the opportunity to utilize previously studied mathematic resources and the process of solution demands from them the extension or consideration of new resources or concepts for the solution of problems. Here one must identify the mathematical potential of the activity before using it in the classroom. ii) The implementation of the activity in the instruction must consider the active participation of the students in the distinct phases of solution. In particular we recommend that initially the students work in small groups of three; afterwards each group should present their attempts at solution to the whole class. In such a way the group that is making the presentation has the opportunity to defend their methods of solution and the other students, along with the teacher can formulate questions and ask for explanations that help them understand and justify what they have presented. The public presentations were a forum for discussing points related to the use of certain relationships and the necessity to justify the work in each of the groups. In general, during the work the students on this group of problems they experienced difficulties as much in the use of the language as in the use of the resources to pose and communicate their ideas, but the form of instruction permitted a refinement of their ideas in their approximations to the problems, which permitted them to get ever closer to the solution.

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FOCUSING ON TEACHER LEARNING: REVISITING THE ISSUE OF HAVING STUDENTS CONSIDER MULTIPLE SOLUTIONS FOR MATHEMATICS PROBLEMS

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Few teachers would doubt the value of providing students with opportunities to consider multiple solutions for mathematics problems; yet, this practice is rarely seen in studies of U.S. mathematics teaching. To consider the complexities and challenges entailed in this seemingly straightforward pedagogical practice, we provide an analysis of data collected in the BIFOCAL project. Using these data, we examine how teachers' thinking about this issue changed over time in relation to a sequence of professional development sessions.

During the past three years, in the context of the BIFOCAL (Beyond Implementation: Focusing on Challenging and Learning) project, we have worked with mathematics teachers in the middle grades (grades 6-8) and offered them opportunities to (re)consider many issues pertaining to teaching practices that affect students' opportunities to learn mathematics. One issue that arose early and resurfaced often in our work together was the practice of having students consider multiple solution approaches for mathematics problems. In a previous paper (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005), we reported on our work during the first year of the project with 12 teachers, who were experienced users of a so-called standards-based mathematics curriculum, and we examined some of their concerns that appeared to serve as obstacles to displaying multiple solutions in their classrooms. We speculated that other teachers would likely perceive many of these obstacles in considering the issue of multiple solutions. Tracing the emergence of this issue during participants' year-long professional development experience, we also identified two major transformations that seemed to account for what they learned. First, they shifted from treating multiple solutions as a slogan associated with reform-oriented teaching, to seeing the use of multiple solution approaches as a complex, nuanced aspect of instructional practice. Second, they refined their understanding of how the display of multiple solutions could help advance their mathematics instructional goals. Based on what participants said and wrote during the year, we argued that they gradually became better prepared to manage this complex pedagogical practice; yet we lacked data to directly support this assertion. The current paper builds on and extends this earlier analysis.

The current paper extends the earlier work in three ways. First, we analyze data from the second year of the project, particularly performance data on a task that was constructed and administered to elicit subtle changes in teachers' thinking regarding the use of multiple solutions and their ideas about enacting this aspect of teaching. Second, because we worked with a larger sample of teachers with more diverse backgrounds in the second year of the project, we are able to examine whether this broader sample of teachers faced obstacles like those identified in our previous work. Finally, we consider what aspects of their professional development experience might have contributed to any perceived changes. In particular, in this paper we seek answers to the following two questions: (a) To what extent and in what ways did teachers' thinking regarding the issue of multiple solutions change over a sequence of professional development sessions? and (b) To what extent do perceived changes appear to be related to professional

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

development experiences?

Context and Research Methods

In this section, we provide a brief description of *BIFOCAL* and the project participants. We also describe the data we collected and the process we used to analyze them.

The BIFOCAL Project

During its second year, *BIFOCAL* enrolled 60 mathematics teachers from elementary, middle, and high schools in the Detroit Metropolitan area that had been using standards-based curricula. The teachers met roughly once every other month in a series of six day-long workshops. These sessions were designed to assist teachers to improve their instructional practice by engaging in two main sets of activities: (1) case analysis and discussion and (2) modified lesson study. (For more details about the project, see Silver et al., 2005.) We used cases that were designed to stimulate reflection, analysis, and inquiry, given that they illuminate many of the challenges faced by teachers working with cognitively challenging tasks in the middle-grade mathematics classrooms (Smith, Silver & Stein, 2005a, 2005b). Before examining a case, participants were asked to solve a mathematical task drawn from the case, or a very similar task that would allow examination of the mathematical ideas encountered in relation to the task. A whole-group discussion followed, during which the participants presented and discussed different solutions to the mathematical task. During the sessions teachers also completed several cycles of a modified lesson study process, in which they planned a target lesson with colleagues, taught the lesson, and reflected on their instructional moves in relation to evidence of students' thinking and understanding. Professional development sessions were planned and facilitated by the BIFOCAL project team. An iterative process was used to plan the sequence of sessions, with a tentative plan developed for a series of sessions and then adjusted as needed based on observed teacher participation and their comments and reflections following each session.

Data sources and analysis

Several different kinds of data were systematically collected during *BIFOCAL*. In this paper, we mainly draw on responses to a task that was administered to teachers at the beginning and end of year two (hereafter called the "One-Less Task" or *OLT*). Initially developed and used at the University of Pittsburgh, the *OLT* explored teachers' stance toward the issue of having students share multiple solutions, as well as their approaches to implementing this aspect of teaching in their own practice. The *OLT* consisted of a brief narrative case (from Barnett, Goldenstein & Jackson, 1994) and a set of follow-up questions. The case describes a classroom lesson in which a teacher asks her students to solve the following problem: "*A secondhand store trades four of their comic books with five of yours. How many of their comic books will they trade for 35 of yours?*" The students propose several different solutions, one of which is incorrectly based on an additive rather than a multiplicative rationale (i.e., Geraldo argues that the trading results in the number of your books being one less than their books, and so he concludes that trading 35 of your books will yield 34 in return). After reading the case, participants answer some questions (2 on the pre-test and 3 on the post-test; see Figure 1).

Teachers' responses were coded using a scheme developed at the University of Pittsburgh (for question 2) and additional schemes developed by our research group (for questions 1 and 3). Three persons independently coded the responses, and they achieved a high degree of inter-coder agreement for all three schemes (the lowest level was 82% agreement); any disagreements were discussed and resolved to reach a consensus coding. Given the focus of this paper on documenting changes in teachers' thinking and searching for plausible reasons for these changes, we focus here on the qualitative analysis of the task responses, but when they are relevant we refer to findings from the quantitative analysis of the task responses. We also draw on other sources of data collected in the *BIFOCAL* project (e.g., teachers' responses to a survey asking participants about their beliefs and instructional practices, and participants' post-session reflections).

Q1: The teacher in the case says: "*I consciously foster the notion that there are many ways to solve problems by having my students share and discuss their methods. The variety of solutions never ceases to intrigue me.*" In contrast, a colleague of this teacher expresses a very different view: "*I consciously avoid having my students share and discuss their methods for solving a problem. Although sometimes students present really good solutions, at other times they might present wrong solutions on the board. I worry about confusing the class.*" Where do you stand in relation to these two different positions?

Q2: Please (i) describe the "one less" error and the thinking that might have led Geraldo and other students to make this mistake, and (ii) suggest a few things the teacher could do to help these students correct the error and develop a deeper understanding of the mathematical ideas involved.

Q3: Think back to the student solutions presented in the case. Suppose these solutions were posed by students in your classroom. Please describe how you might design the "share" or "summary" portion of the lesson, including the reasons for your decisions (appeared only in the post-test).

Figure 1: The questions used in the One-Less Task

Findings

This section is organized in four parts. The first three address the first research question, and the last one addresses the second research question.

Unpacking the complexities and subtleties of the issue of sharing multiple solutions

Responses to Q1 suggest that teachers initially accepted the idea of multiple solutions as a good idea, at least in theory. Teachers' pre-test responses to Q1 showed that, at the beginning of the year, a majority of participants (about 70%) agreed with the first teacher. The rest of the teachers supported the idea of sharing multiple solutions, but raised concerns about time constraints, the feasibility of using this approach in classes with "low-ability" students, and the likelihood of creating more confusion rather than helping students develop understanding, especially if the incorrect solutions were to be presented. These concerns resonate with those reported in our prior study of first-year participants. On the post-test, the number of participants who agreed with the first teacher in Q1 increased to about 85%, and there was a corresponding decline in the number of teachers who voiced concerns about time issues or the issue of confusing students. Thus, there appeared to be a shift in the direction of stronger support for having students share and discuss multiple solutions when solving a mathematics problem. Perhaps more importantly, however, teachers' responses at the end of the year suggested that they were more aware of some of the complexities and subtleties of this practice.

Consider, for example, one teacher's pre-test response to Q1:"*My preference is absolutely the first choice given: supporting multiple solution strategies for any problem. I believe this* *"policy" supports the fostering of critical thinking in students and that their learning experience is more meaningful if they've figured out the solution on their own."* The teacher subscribes to the idea of using multiple solutions, arguing that this approach fosters critical thinking and helps students see meaning in their work. But, as our earlier study showed, supporting the idea of enacting multiple solutions in teaching is not sufficient to ensure enactment in the classroom. This teacher's post-response illustrates a deeper appreciation of the ways in which this approach could advance her mathematics teaching agenda, well beyond merely "honoring" student work: "*I prefer to always accept all solutions that any students try, no mater how different or confusing; but I believe it matters* <u>how</u> *you use them. It's very important in being a good teacher to be able to anticipate students' responses in carrying out a lesson. You can address misconceptions in many ways: leave them as food for thought to revisit later, compare them to work already presented to help students see the error right then. I prefer to start with a solution strategy that is visual and will assist students who struggle, to draw them in before I lose them completely […] I feel that if a student is taught only one solution, they walk away feeling that their approach is probably wrong, and they lose confidence*" (emphasis in the original). The teacher values having students share multiple solutions, even the incorrect ones, and argues that this approach fosters students' confidence and understanding. Yet, now she moves into the particularities of this approach: She underscores the importance of being able to orchestrate the sharing of multiple solutions well; she stresses that the teacher should anticipate some of students' responses and misconceptions well in advance in order to build her lesson in ways that help students develop understanding; she even considers multiple ways of capitalizing on students' errors; finally, she has developed an idiosyncratic way of sharing multiple solutions and she explicates what makes her approach legitimate.

To document change in thinking, it may be even more informative to consider one teacher whose post-test response to Q1 appeared to suggest more rather than less concern about sharing multiple solutions when compared with her pre-test response. The teacher's pre-test response was: "*I lean much closer to the first view point* [support sharing multiple solutions, even the incorrect ones]. *I also try to give a particular version as a way to try remembering if none of the versions are making sense.*" In her post-test response, the teacher notes that she is "*Somewhere in between. If you are skilled at drawing connections and fostering students' ability to do the same, then multiple approaches can help students solve more types of mathematical problems. If you are not very skilled, then the students might get confused even more*." This teacher appears to acknowledge the challenges inherent in enacting this practice. She suggests the importance of a teacher developing proficiency in helping students share, compare, and connect different approaches. Other aspects of her extended response on the post-test suggest that she also recognizes the challenge of helping students learn to consider multiple solutions and build connections among different approaches. Her post-test response suggests that she has deepened her understanding of the entailments of this teaching practice and the requirements it imposes on the teacher.

The participants' deepening appreciation of the complexities of this aspect of pedagogy was also evident in their responses to Q3. A majority of teachers claimed that if responses like those in the *OLT* case were to occur in their classrooms, they would examine students' different solution approaches and select which solutions to have students share (about 68%). About 40% of participants referred to the "mechanics" of enacting this approach in practice, specifying how students would be expected to work (e.g., in groups, individually), and what particular equipment they would use (e.g., sharing multiple solutions using the overhead projector, presenting multiple solutions on papers posted on the walls). But their consideration of classroom practice went well beyond mechanics. About half of the teachers pointed out the importance of purposefully sequencing the sharing of students' solutions and asking students to justify their answers. Going even further, one in three teachers asserted that they would ask students to draw connections among the different solutions displayed. These elaborations appear to suggest that the teachers were gradually becoming more prepared to manage this issue in their practice.

Treating the incorrect solutions

One of the complexities entailed in having students share multiple solutions is the issue of treating incorrect approaches. We examined project participants' stance toward this issue drawing on their responses to Q2. At the beginning of the year about one-third of the teachers stated that they would try to help students develop understanding by having them act out the situation ("I would ask students to trade books"). One in five teachers stated that they would have students use manipulative materials to help them understand their error. A comparison of teachers' responses to Q2 at the beginning and end of the year suggests a shift in the way teachers would treat the one-less error and structure their lessons to assist their students. In general, teachers' post-test responses were more nuanced. More teachers referred to using multiple representations and employing different mathematical ideas in the post-test than in the pre-test (about 57% and 31%, correspondingly). Additionally, more teachers indicated that they would require students to justify their answers and make connections among different solutions and representations (about 23% in the post test compared to 8% in the pre-test).

These general trends can be seen in the responses offered by particular teachers. Consider, for instance, one teacher's pre and post responses: *"Geraldo and other students need more visual and hands on activities. I would have the class do the problem, then use real comic books as a mini experiment. I would post the variety of correct answers as well (pre-test).* "[I would] *have "one less" people explain their way; have other correct- students explain their way especially using scale factor; show problem visually; show examples of when you would use "one less" and how it is different than* [the] *current problem" (post-test).* At the beginning of the year, this teacher saw the use of manipulative materials and acting out the situation as the only viable approaches. In contrast, at the end of the year he suggested more intellectually demanding approaches, such as: having students *explain* their answers and *compare* problems in which the additive approach works to problems in which this approach leads to an incorrect answer. In this way, the teacher appears to have realized that is possible to capitalize on a student's mistake as a way to have students consider both additive and multiplicative situations. Also, we note that the post-test response targets the underlying mathematical idea of the problem (i.e., the scale factor). A similar shift in thinking is evident in the following pair of answers for another participant: "*The teacher could ask Geraldo to actually "trade" books until the store totaled 35 books. Have the students draw a picture, then make a table"* (pre-test); and *"I would ask* <u>*all*</u> *the students to explain how they got their solutions, listening for correct mathematical language and reasoning. Then, I would ask clarifying questions, restate the students' process and/or see if the class has any questions. The "minus 1" students need to see the error of their thinking, especially after discussion about the other solutions. "What amount of comic books would you trade for 4 of the store comic books? 8 of the store? etc Is this ratio for one time only or each group of 4?"* (post-test). Like the teacher in the first example, at the beginning of the year, this teacher talks about treating the one-less error in very general terms (i.e., draw a picture, make a table, and do the trading). Her answer at the end of the year reveals a shift in emphasis. First, she attends to students' reasoning and use of correct mathematical language. Second, she expects that the students inclined to make the error would develop understanding through participation in a discussion of several different approaches. Finally, she refers to a central mathematical idea that she wants students to consider in solving the problem (i.e., the ratio).

Reconceptualizing the roles of teacher and student in the process of sharing multiple solutions

Our analysis of participants' pre and post responses to the *OLT* also suggests that there was a shift in their conceptualization of the roles of the teacher and students during the process of sharing and discussing multiple solutions. Our quantitative analysis of teachers' pre-test and post-test responses indicates a shift in perspective. At the beginning of the year, the participants were more than seven times more likely to indicate that they would try help students correct the one-less error by directing them through the process (i.e., modeling the trading or guiding students to do the trading), essentially having the teacher assume responsibility for all of the thinking.

An example of this shift in conceptualization of teacher and student roles is evident in the following pre and post responses to Q2 from one participant: "[I would] *have students look at each transaction of 4 to 5 separately. For example, you may have 35 comic books to turn in, but you can only trade 5 at a time. While explaining this method,* [I would] *use the other students' models to visually reinforce this idea.* (pre-test) *"I would first ask the students to prove their answer, hopefully leading to some type of chart, or ask them what would happen if I didn't trade 35 in once, but traded 5 at a time over the course of a couple of weeks. This might lead them to take one less from every exchange and not just at the end"* (post-test). Notice that the teacher's role in the pre-test response is conceptualized as dominant: the teacher guides students to do the trading; she also does the explanation, using several students' responses. In the post-test response the teacher is seen as assisting students to arrive at the correct answer, but her role is conceptualized differently. Instead of directly explaining how children could do the trading and why it works, she would ask students to explain the process and underlying rationale. The following pair of responses is suggestive of a similar transformation in another participant's thinking: "*The teacher could have actually had the students model the trading to show that it was actually 1 less on 7 different occasions"* (pre-test) *"The teacher could ask other students to explain in their words- sometimes hearing a student explain rather than a teacher is more beneficial"* (post-test). Similar to her colleague, in her pre-test response this teacher reveals an inclination to direct her students to the right answer. In contrast, in her post-test response, she appears to shift responsibility for problem solving and reasoning more to her students, having students develop and present their own solutions and then try to convince their classmates that their approach works.

What might have caused the changes in teachers' thinking?

The preceding analysis documents some changes in teachers' thinking about having students consider multiple solutions in mathematics classes. It also provides evidence that, at the end of the year, participants appeared to have a deeper, more nuanced view of this complex pedagogical practice and also to be better prepared to manage it in their classrooms. A consideration of our findings in relation to the experiences of project participants suggests some plausible reasons to account for these changes.

First, in the BIFOCAL project sessions teachers were offered *recurrent* opportunities to experience the benefits of sharing multiple solutions as *learners* themselves. One participant mentions in an End-of-Session Reflection (ESR): "*There is more than one way to approach a* *problem. I was immediately going to the algebraic method and solve. I had a difficult time seeing things visual. I have never been a visual learner*" (February, 2005). Another participant made a similar observation in her ESR: "*Seeing the group presentations really makes me think how diverse my students must be in their approach to a problem. I need to exploit those differences for the good of the class and the class discussion"* (February, 2005). The sessions also helped teachers realize that in addition to fostering students' confidence, sharing multiple solutions could create significant learning opportunities for their students: "*I was impressed at the multiple ways to represent a problem, and how simple problems can serve as building blocks for algebraic thought"* (March 2005). The more opportunities teachers were offered to reflect on this aspect of their thinking, the more nuanced their thinking around it became.

Second, the analysis and discussion of teaching cases in BIFOCAL project sessions helped teachers (re)consider their teaching practice and become aware of issues that might impede their use of multiple solutions in practice. For instance, many teachers were confronted with the dilemma of challenging students' thinking while at the same time ensuring that students would be engaged and remain on-task: "*I am concerned about how to be effective when it comes to balancing student interest and student frustration. Need to develop strategies that equip students to engage in the problem with interest and without being overwhelmed by frustration*" (September, 2004). Surfacing and discussing these concerns appears to be a critical step in shifting from a general endorsement of sharing multiple solutions to actualizing this in practice.

Third, the discussions of this issue appear to have caused some *disequilibrium* for teachers. In fact, participants often reported leaving a session with more questions than they had at the beginning. For example, one teacher wondered in her ESR: "*How many different representations will help students better understand the problem? When do you stop*?" (March 2005) and another one pondered issues of managing the sharing of multiple solutions in her ESR: "*How will I manage students and time during the sharing of their solutions*?" (February 2005). On the other hand, reflecting on this issue and its entailments often offered them ideas with which they could experiment: "*Sometimes I do not take the time in class to look at multiple strategies. I think I vary from day to day, but I need to take more time every day*" (March 2005).

Finally, it appears that changes in teachers' thinking were also accompanied by changes in their beliefs about teaching and learning mathematics. In particular, the analysis of project participants' answers to the survey questions revealed that they were less reluctant at the end of the year to consider allowing their students to experience disequilibrium and struggle as they learned mathematical ideas and solved complex problems. For instance, participants expressed less support at the end of the year for survey statements indicating that teachers should arrange their instruction so that students avoid frustration and uncertainty or that teachers should always explain clearly and completely how a problem should be solved. That is, teachers appeared to have become less inclined to do all the thinking for their students.

Conclusion

Our analysis of data collected in the second year of the BIFOCAL project appears to support many of the observations made in our earlier research based on a smaller group of participants in the first year. In particular, many of the perceived obstacles to using multiple solutions appeared to be operative for the larger group of teachers. But our analysis also suggests that providing teachers with systematic and recurrent opportunities to consider and reconsider their mathematics teaching in relation to their students' opportunities to learn can allow them to develop a deeper, more nuanced understanding of the desired practice as well as the inherent challenges. In so doing, they become better prepared to actualize innovative practices in their classrooms.

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MATHEMATICS IN THE MAKING: MAPPING THE DISCOURSE IN PÓLYA'S "LET US TEACH GUESSING" LESSON

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This paper describes a detailed analysis of verbal discourse within an exemplary mathematics lesson—that is, George Pólya teaching in the Mathematics Association of America [MAA] video classic, "Let Us Teach Guessing" (1966). The results of the analysis reveal an inductive model of teaching that represents recursive cycles rather than linear steps. The lesson begins with a frame of reference and builds meaning cyclically/recursively through inductive processes—that is, moving from specific cases, through recursive cycles, toward more general hypotheses and rules. Additionally, connections to univocal (conveying meaning) and dialogic (new meaning through dialogue) discourse are made.

"First, guess; then prove… Finished mathematics consists of proofs, but mathematics in the making consists of guesses" (Pólya, 1966).

Introduction

The National Council of Teachers of Mathematics [NCTM] has consistently recognized communication and problem solving as essential components of reform-oriented mathematics education (NCTM, 1989, 1991, 2000). Recent research demonstrates that even among teachers who report agreement with reform ideas, instructional practices emphasize routine procedures, giving learners little opportunity to investigate, conjecture, reason, and justify (Spillane & Zeuli, 1999). Talk alone is not sufficient; the quality and type of discourse affect its potential for promoting conceptual understanding (Kazemi & Stipek, 2001). To learn how to orchestrate meaningful discourse, mathematics teachers need evidence drawn from exemplary practice. This paper describes a detailed analysis of discourse within an exemplary mathematics lesson—that is, George Pólya teaching in the Mathematics Association of America [MAA] video classic, "Let Us Teach Guessing"1 (1966). Additionally, the results of the analysis are connected to models of teaching that have been previously reported (Truxaw & DeFranco, 2004, 2005), thus providing a focus on teachers and learners of mathematics.

Background

Decades before current mathematics reform documents were published, similar themes were espoused by George Pólya, noted mathematician, mathematics educator, and problem-solving expert, in works such as "How to Solve It" (1945/1985), "Mathematical Discovery" (1958/1962) and his "Mathematics and Plausible Reasoning" volumes (1954). In 1965, Pólya was filmed teaching a mathematics lesson to a group of university students (1966). He began the lesson with a rich problem that was unfamiliar to the students—that is, into how many parts is space divided by 5 planes?2 During the lesson, Pólya used discourse to guide the students through cycles of evidence-based guesses (i.e., plausible reasoning), investigations, and explanations that result in mathematical sense-making—about the problem itself, about generalizations of the problem, and about strategies for approaching problem-solving. This lesson and Pólya's teaching have stood the test of time as exemplars. In fact, other researchers have investigated the lesson; in particular,

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Leinhardt and Schwarz noted, "The advantage of examining this particular episode of teaching is that the mathematical and epistemological knowledge carried by the teacher in this lesson can justifiably be considered exemplary" (Leinhardt & Schwarz, 1997, p. 397). While Leinhardt and Schwarz focused on "the instructional explanation of guessing as a heuristic for solving the Five Planes Problem" (1997, p. 305), this paper will examine the verbal discourse and its implications for teaching and learning associated with "mathematics in the making".

While this research draws from divergent theoretical viewpoints, sociocultural theory, with its contention that higher mental functions derive from social interaction, provides a meaningful framework for analysis and discussion of discourse as a mediating tool in the teaching-learning process (Vygotsky, 1978, 2002). When considering language as a mediator of meaning, it is useful to take into account the two main intentions of communication—that is, dialogic (i.e., constructing meaning through give-and-take communication) and univocal (i.e., one-way transmission of knowledge) (Lotman, 2000; Wertsch & Toma, 1995). Structures associated specifically with classroom discourse are also relevant. For example, the basic structures of classroom discourse include the following: moves, exchanges, sequences, and episodes (Lemke, 1990; Mehan, 1985). The move, exemplified by a question or an answer from one speaker, is recognized as the "smallest building block" (Wells, 1999, p. 236). An exchange is made up of two or three moves and occurs between speakers (typically including initiation, response, and evaluation or follow-up moves: I-R-E or I-R-F). A sequence contains a single nuclear exchange and any exchanges that are bound to it. The episode is the level above sequence and represents "all the talk that occurs in the performance of an activity" (p. 237).

In addition to structures, identified forms of talk and verbal assessment include the following: *monologic talk* (i.e., involves one speaker—usually the teacher—with no expectation of verbal response), *leading talk* (i.e., occurs when the teacher controls the verbal exchanges, leading students toward the teacher's point of view), *exploratory talk* (i.e., speaking without answers fully intact, analogous to preliminary drafts in writing) (Cazden, 2001), *accountable talk* (i.e., talk that requires accountability to accurate and appropriate knowledge, to rigorous standards of reasoning, and to the learning community) (Michaels, O'Connor, Hall, & Resnick, 2002), *inert assessment* (IA is verbal assessment that does not incorporate students' understanding into subsequent moves, but rather, guides instruction by keeping the flow and function relatively constant), and *generative assessment* (GA is verbal assessment that mediates discourse to promote students' active monitoring and regulation of thinking).

Previously reported research (Truxaw & DeFranco 2004, 2005) demonstrated that graphic maps of discourse (called sequence maps) could be developed to represent the flow of the talk and verbal assessment, as well as the overall function of the discourse (i.e., tending toward univocal or dialogic). Additionally, the sequence maps, when combined with multi-level analysis of transcripts and other evidentiary data (e.g., interview transcripts and field notes), could be used to develop associated models of teaching. Three models of teaching were reported: a deductive model (associated with univocal discourse), an inductive model (associated with dialogic discourse), and a mixed model (a hybrid of the other two). This paper describes how these strategies and constructs were applied to an investigation of the talk, verbal assessment, discursive functions, and associated teaching of a lesson taught by mathematics education expert, George Pólya.

Methods and Procedures

The data derived from the video, "Let Us Teach Guessing" (Pólya, 1966). The dialogue from the video was transcribed, coded, and analyzed using strategies that had been developed in previously reported research (Truxaw & DeFranco, 2004, 2005). In particular, the transcripts were formatted into tables and numbered based on “utterances” (i.e., speaker’s turns—from here-on-out to be called “lines”) (Bakhtin, Holquist & Emerson, 1986). Moves within each line of text were coded using strategies adapted from Wells (1999) and Nassaji and Wells (2000). Next, the coded text was parsed into 19 sequences. Individual sequence maps (i.e., diagrams representing the flow of forms of talk and verbal assessment within a sequence) were developed by applying the coded data to a previously developed graphic template of classroom discourse (Truxaw & DeFranco, 2004). After that, sequences were deconstructed into sub-units that included data from sequence maps, transcribed text, and evidentiary data from Pólya’s published works. The sub-units were then reconstructed within the context of instructional episodes and mapped onto a model of teaching.

Results and Discussion

Not surprisingly, initial viewing of the lesson showed expert teaching derived from deep understanding of both content and pedagogy. Further, the analysis of the discourse uncovered details that may help teachers move toward more reform-oriented practices. For example, the coding of the transcripts showed the use of triadic exchanges (I-R-F) to facilitate the discourse within the lesson. Although triadic exchange structure has been associated with “illusory understanding” (Lemke, 1990), Pólya demonstrated its use in conjunction with a rich mathematical problem and with discourse aimed at building (rather than simply conveying) students’ understanding of mathematics. An important question is: *how* did he do this?

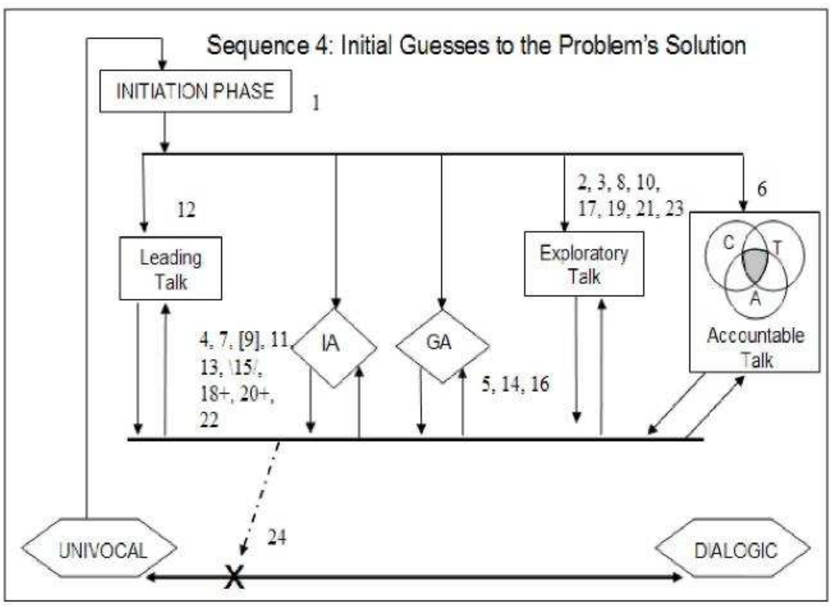

Figure 1. Example of sequence map. lesson continued, the sequence

The analysis of this lesson yielded 19 sequence maps. The sequence maps allowed us not only to count each form of talk and verbal assessment, but, more importantly, to indicate *when* and *how* each was used in the dialogue. The sequence maps showed the flow of the moves within each sequence as well as relationships of sequences to each other. For example, sequence maps 1-3 showed monologic talk that was univocal in nature. In contrast, sequence 4 showed leading, exploratory, and accountable talk and both IA and GA that, although univocal overall, had slight tendencies toward dialogic function (see Figure 1). As the maps shifted back and forth between exploratory stances (e.g. guessing) and talk that tended more toward conveying meaning (e.g., the introduction of problem-solving strategies).

Increasing instances of accountable talk were documented as strategies and evidence were built.

GA was infused at critical junctures—typically when students had gathered sufficient evidence to question previous conjectures.

Inductive Teaching

Three models of teaching based on analysis of middle grades mathematics teachers were previously reported: deductive, inductive, and a mixed models (Truxaw & DeFranco, 2005). The model of teaching developed from Pólya's lesson most closely resembled the inductive model—that is, moving from specific cases, through recursive cycles, toward more general hypotheses and rules (see Figure 2). The discourse that mapped onto this model will be described next

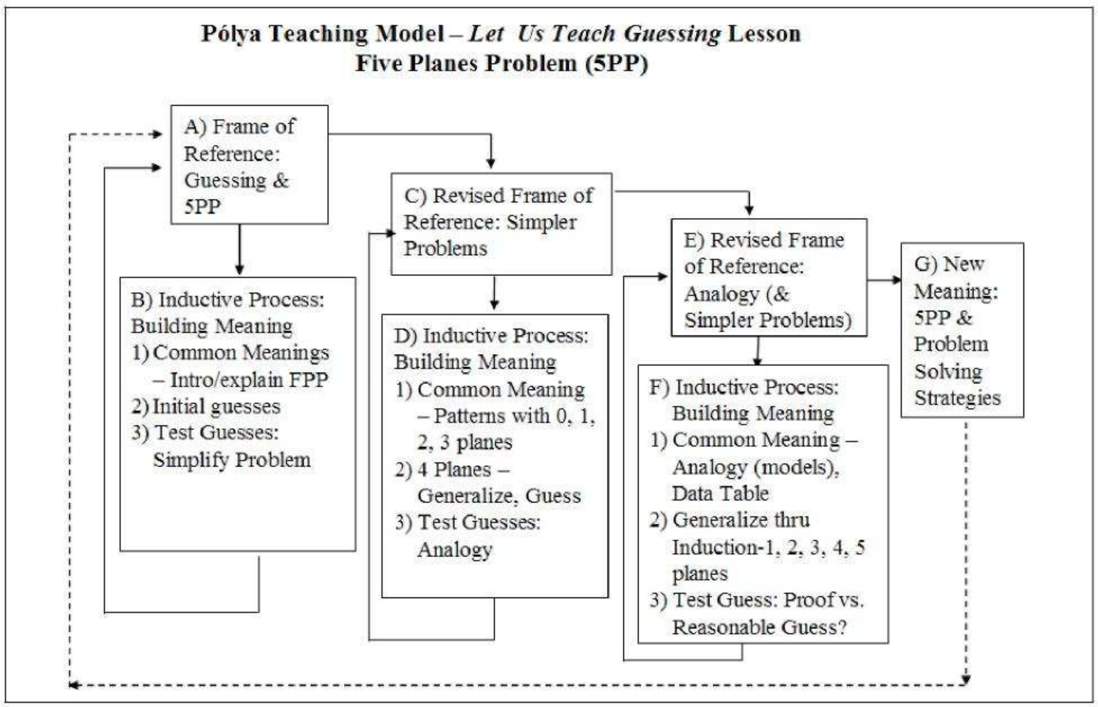

Figure 2. Inductive model of Polya's lesson.

The first three sequences established a ***frame of reference*** (see Figure 2-A) (i.e., the five planes problem [5PP] and the theme of guessing) and communicated ***common meaning*** related to the problem and procedures (see Figure 2-B-1). In sequence 1, Pólya introduced the theme of the lesson—that is, guessing:

Pólya: …Mathematics when it is finished, complete, all done, then it consists of proofs. But, when it is discovered, it always starts with a guess…

In sequence 2, he shared rules related to guessing:

Pólya: …If you know already my problem, don't answer my question. That would be unfair, if you know already the answer … it wouldn't be guessing, and you would spoil the fun of all of us…

In sequence 3, Pólya described the five planes problem [5PP]—that is, *into how many parts is space divided by 5 planes?*

In sequence 4 (see Figure 1 for sequence map), students made ***initial guesses*** (see Figure 2-B-2). Pólya facilitated exploratory talk using both IA and GA. An excerpt follows.

Pólya: Who is ready to say? Don't be bashful. Go ahead. Yes. Say something.

S1: Um, 25.

Pólya: 25. How did you get it? *[Writes it on the chalkboard.]*

S1: I looked at 5 times five.

Pólya: Five times five. That's an idea. There is an idea. Good. Anybody ready for another guess? Yes, please?

S2: 32

Pólya: Thirty-two. Oh, oh. There's something behind your… Oh, 32. Interesting…

In sequence 5, Pólya introduced the strategy, "solve a simpler problem" in order to begin to ***test the guesses*** (see Figure 2-B-3). This ***revised the frame of reference*** (see Figure 2-C). In sequences 6-10, Pólya facilitated investigation of space cut by one, two, and three planes, thus, establishing ***common meaning*** about these simpler cases (see Figure 2-D-1). Additionally, extreme cases, such as parallel planes and zero planes were discussed. After the students had the opportunity to investigate patterns from these simpler examples, in sequence 11, ***guessing*** (see Figure 2-D-2) was again infused.

Pólya: Now, let me come to the next case. We have four dividing planes. Try to guess it. Four dividing planes. How many parts? The consensus of the group, based on the observed patterns, was that four planes would divide space into 16 parts.

Pólya: So we got really the 16 in a reasonable way of guessing. We observed. We found the pattern. And we said, and so on. It will go on like this. We made a generalization. That's very important, you see.

In sequences 12-15, Pólya facilitated ***testing the guess*** (See figure 2-D-3) using another strategy—that is, *analogy*. Pólya suggested that the students consider *lines divided by points* and *planes divided by lines* as analogies to *space divided by planes*. This again ***revised the frame of reference*** (see Figure 2-E). As Pólya and the students worked through the analogous problems (using all four forms of talk and both IA and GA), responses were discussed and recorded on the blackboard. This helped to further build ***common meaning*** (see Figure 2-F-1). Pólya used both IA and GA to encourage the students to look for patterns and to generalize beyond the individual cases. The students generated new meaning about space divided by four planes—that is, that *15 parts would be formed, rather than the originally conjectured 16 parts*. This discourse was coded as dialogic because new meaning was constructed through give-and-take communication.

In sequence 16, the 5PP was revisited. Pólya used GA to promote students' active monitoring and regulation of thinking. Students made connections among the strategies, patterns, and common meanings that had been built throughout the lesson. Students hypothesized and defended their conjectures, using ***inductive processes to build generalizations*** (see Figure 2-F-2). The group came to consensus that s*pace divided by five planes would form 26 parts.* This new meaning represented another example of dialogic discourse within this lesson.

In sequences 17 and 18, Pólya facilitated a discussion about whether the answer (26 parts) has been ***proven or, rather, was based on a reasonable guess*** (see Figure 2-F-3). The discussion led to reminders that it is important to ***test guesses***—although plausible reasoning pointed to 26 as the answer, they had not yet proved it. The strategies, themes, and ***new understandings about***

the 5PP and the problem solving strategies were summarized (see Figure 2-G). Finally, in sequence 19, Pólya told a short story that reinforced the themes of guessing and proof.

The analysis revealed that the discourse in Pólya's lesson followed patterns that were compatible with those represented in a previously reported *inductive model of teaching*. Pólya's lesson represented recursive cycles, rather than linear series of steps. The lesson began with a frame of reference (i.e., the 5PP) and built meaning cyclically/recursively through inductive processes—that is, establishing common meaning related to the problem and to problem-solving strategies, conjecturing (i.e., guessing), investigating, and revising conjectures based on additional evidence. An answer based on plausible reasoning was developed through dialogic discourse, but the answer was not *proved*. The frame of reference of the lesson was the 5PP, but the outcomes of the lesson moved *beyond the solution* to the problem to include strategies and ways of thinking that could be applied to other mathematical problems—thus, mathematical meaning was developed.

Final Remarks

Within Pólya's lesson, monologic talk, leading talk, and IA—often associated with univocal discourse—were evident (see Table 1); however, they were used as *part of a larger cyclic process* that *also included* exploratory talk, accountable talk, and GA. The percentages of verbal assessment and talk moves shown in Table 1 are informative, but *when* and *how* each was used further explicate Pólya's teaching practices. For example, while IA was the predominant form of verbal assessment used throughout the lesson, it should be noted that *IA was rarely used in an evaluative way* (I-R-E); rather, it was used as follow-up (I-R-F) to keep the discourse moving—often to encourage exploratory talk. GA appeared to be used strategically; typically, it was infused after common understanding of key ideas had been established. Further, the analysis indicated that GA was particularly productive in facilitating dialogic discourse when it was used in later repetitions of the cyclic process. In sum, dialogic discourse (and accompanying evidence of new meaning) seemed to be built through recursive cycles of establishing common understanding, guessing (i.e., plausible reasoning), and infusing GA to test and move beyond the guesses. In the end, the inductive teaching facilitated by Pólya helped the students to build new meaning about the problem and about mathematical problem solving. While we would not suppose to suggest that an analysis of the discourse in Pólya's lesson can uncover the nature of his expertise or art, the hope is that some clues related to facilitating *mathematics in the making* have been revealed. In this way we hope that this research contributes to a focus on learners through a focus on this exemplary teacher and scholar.

Table 1. Percentages of Verbal Assessment Moves and Talk Moves in Pólya's lesson

Verbal Assessment		Talk			
IA	GA	Mono	Lead	Expl	Acct
75.8%	24.2%	23.0%	32.4%	20.3%	24.3%

Endnotes

1 The authors would like to thank the Mathematical Association of America for granting permission to use data drawn from the video, "Let Us Teach Guessing." The video is copyrighted by the MAA and all rights are reserved.

2 This problem is described in detail in volume 1 of Pólya's *Mathematics and Plausible* Reasoning (1954) text.

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ORAL RETELLINGS: SOLUTION STRATEGY FOR COMPARING WORD PROBLEMS?

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This study explored oral retellings used as a problem solving strategy for compare word problems in mathematics. The research involved 29 sixth graders (14 boys and 15 girls) from two classes who took a pretest, participated in four 30-minute instructional periods, and completed a posttest and delayed posttest. Results showed no statistically significant change in overall success in problem solving or in the number of computational errors students made. However, students made fewer operational errors following instruction in oral retellings. This finding suggests that students may more accurately select operations based on their retelling of the word problem, thus benefiting from instruction in oral retellings as a strategy for solving compare word problems.

The ability to solve problems is "a hallmark of mathematical activity and a major means of developing mathematical knowledge" (National Council of Teachers of Mathematics, 2000, p. 116). Word problems stand "front and center" as problem solving contexts for developing analytic thinking and cognitive abilities (Latterell & Copes, 2003). Compare word problems have been the focus of abundant word problem research (Lester, 1994) and have been deemed the most difficult type of word problem to solve (Carpenter, Fennema, Franke, Levi, & Empson, 1999). Researchers have begun to look at the potential of oral retellings in measuring students' comprehension of compare word problems (Verschaffel, 1994). In this study, sixth grade students were instructed in oral retelling as a problem solving strategy. To deepen understanding of the psychological aspects and implications of instruction using oral retellings, the researchers examined the effect of instruction on the following: (a) overall success in solving compare word problems, (b) the number of computational errors made, and (c) the number of operational errors made.

Method

Two classes of sixth graders from an elementary school in a suburban city in the western United States participated in the study. The participants consisted of a stratified random sample of 29 students (14 boys and 15 girls) who were fluent English speakers. All 29 students participated in four 30-minute class sessions in which they practiced retelling narrative and expository text selections and then practiced retelling mathematical word problems. The students also wrote original word problems and retold them with partners.

The instruments employed in the study were administered by the classroom teacher and included a pretest, posttest, and delayed posttest period, each following a similar, free-response, paper and pencil format. Each 10-item test consisted of a warm-up question, four fillers, and six target word problems. Only the target problems were scored and analyzed. The target items were three compare word problems worded consistently and three compare word problems worded inconsistently. The posttest was administered on the day immediately following the completion of instruction in oral retellings. The delayed posttest, administered two weeks after treatment,

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provided insight into the enhanced, constant, or diminished effects of the treatment over time and the effects of instruction on near and far transfer.

Results

Pretest, posttest, and delayed posttest target items were scored as “correct” or “incorrect”; incorrect solutions were analyzed to determine the type of error made. Arithmetic or counting errors were grouped as computational errors. Incorrect operation selection was considered an operational error. Errors that did not appear to fall into either category, approximately 2% of the data, were omitted from the analysis.

The first research question examined overall success in problem solving. An analysis of variance for repeated measures yielded no evidence of a statistically significant difference in the pretest, posttest, and delayed posttest scores ($F = 2.688$; $p = 0.077$). Instruction in oral retellings did not significantly affect students’ overall success in solving word problems.

The second and third research questions investigated the effects of instruction on the number of computational and operational errors. Results of an analysis of variance for repeated measures for computational error rates yielded no evidence of a statistically significant difference ($F = 0.177$; $p = 0.839$) between the pretest, posttest, and delayed posttest. Instruction in oral retellings did not significantly affect the computational error rates of students. Operational error rates were also compared using an analysis of variance for repeated measures. This analysis yielded a statistically significant difference in the operational error rates between the pretest, posttest, and delayed posttest ($F = 4.534$; $p = 0.015$). Instruction in oral retellings appeared to have a significant effect on students’ selection of an arithmetic operation when solving compare word problems.

Discussion and Recommendations for Future Research

This study showed that students continued to struggle with compare word problems even after instruction in oral retellings. The results of this study provide further evidence of the need to focus on students’ comprehension of these problems and offer students a variety of problem-solving strategies that are based upon a clear understanding and representation of the problem. Further research into oral retellings of compare word problems should be conducted to clarify the utility and role of retellings in the problem solving process. Suggestions for future research include using oral retellings with younger populations and among English as Second Language populations.

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STUDENT COGNITIVE PROCESSES IN SITUATIONAL ALGEBRA TASKS

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Clinical interview observations of secondary mathematics students engaged in novel situational algebra tasks are reported. Discrete word tasks were constructed to both perturb and elicit demonstrations of subjects' abilities to employ personal problem solving templates, exhibit quantitative reasoning skills, and to discern proficiencies with symbolic representations. Coding and analysis focus is on two subjects and their processes of identifying and examining recognized patterns of relationship in order to express a solution as a closed form symbolic generalization of the situation. Guiding research questions are: To what extent did the students see the task in terms of situational modeling and analysis of covariational relationship, proportion, or functional dependency? How do they access and employ qualitative problem solving strategies? When faced with perturbations, how did they work their way around cognitive obstacles in their solution path? Were they able to express this relationship in precise verbal and symbolic statement or proposition?

Motivations

Current pedagogic arguments emphasize relational development of qualitative and symbolic reasoning skills as essential for the eventual success of mathematics students in reifications to higher understandings. Skemp coined the ideas of instrumental learning versus relational learning – "learning an increasing number of fixed plans [versus] building up a conceptual structure from which it's possessor can develop any number of plans..." (Skemp, 1976, p. 14). Usiskin (1998) identifies common conceptions and misconceptions regarding school algebra and symbol usage, in that Algebra can be viewed as generalizations of arithmetic, a study of solution method, a study of relationship, or as a study of structure. Variables thus take on meanings that are conception-dependent. Sfard and Linchevski (1994) argue that reasoning skills and conceptual development is a series of reifications from operational/instrumental to structural/relational understandings. Arcavi (1994) addresses symbol sense within an aesthetic awareness of roles that symbols play in mathematics.

Herscovics (1989) has categorized cognitive obstacles encountered in school algebra curriculae. Work within the cognitive obstacle tradition has included that of Philipp and Chappelle (1999), who report difficulties faced by students and instructors when viewing algebra as a generalization of arithmetic; students are not readily able to integrate syntactic and semantic understandings. MacGregor (1998) has indicated that secondary mathematics students have exhibited significant difficulties with translations from intuitive verbal understandings of task into taught procedures for algebraic form. Zaslavsky, et al, are involved in very interesting work in the study of the default assumptions that students and instructors bring into mathematical task, and in reactions to designed task-induced perturbations (Zaslavsky, et al, 2002). These intriguing viewpoints motivate a great interest in the default assumptions and the varieties of reasoning that students, and instructors, bring into engagement with mathematical task, and in the ways that they deal with any cognitive perturbations that they come across in task. The cognitive obstacles that learners encounter in engagement of mathematical task include, but are not limited to, student difficulties with syntactic/semantic integration, struggles with recognition of applicable solution templates, vague assumptions regarding variable choices and delimitation which lead to

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imprecise engineering of situational algebraic representations, and internal debate regarding cognitive "costs" discerned for achievement of solution.

Presmeg and Balderas-Cañas (2001) validate that word task attentions and interpretations demand an emotional attachment. They make a case that problem solvers manifest a metacognitive awareness of their own mathematical cognition and affect as they solve problems. Izsák (2003 & 2004) suggests a theoretical cognitive pathway for algebraic modeling where students generate representations of situations by drawing on their understandings of physical and numerical patterns, conceptual schemata, and symbolic templates. They use patterns that can be modeled by their familiar algebraic representations.

Methods

Word, or story, tasks contain inherent cognitive obstacles in that they must be interpreted from a syntactic form into a semantic conceptualization, and then again into a symbolic language in algebraic generalization of the situation. Qualitative protocols were designed around a task bank of offered tasks that were chosen and contextualized from classic discrete mathematical tasks. All represent a relative degree of relational shadings, including co-variational linearity, recursion, proportionality, and all can be approached from a variety of directions mathematically.

Clinical interview series were conducted with AP mathematics students from the same secondary community. Subjects were interviewed independently in undisturbed classroom settings. Two students, Phillip and Chris -- both AP Pre-Calculus students with records of high achievement -- were chosen for juxtaposition analysis. Each interview began with a presentation of a card-stock page containing five word-form discrete tasks. The subjects were asked to choose only one of the tasks and then make attempt at solution. All interviews were recorded, transcribed, and coded versus investigation criteria. Thematic analyses were drawn from these transcriptions (Strauss & Corbin, 1990).

Results and Analyses

Reported results for interviews contain comments and notable consequences that emerged within the interview processes. Interviewees exhibited varying degrees of algebraic reasoning in their approaches to task. Below are examples of observations actions from the two interviews:

Phillip - *The Golden Apples*

- Found interesting sense of co-variational linearity within task.
- He had interesting conventions for variable usage.
- Phillip experienced obstacles by attacking the task in a syntactic manner.
- Phillip saw experience as an exercise and expressed enjoyment during engagement in task.

Notable consequences:

Phillip's approach to the task was strictly algebraic and symbolic in nature. Most obstacles encountered were from arithmetic error due to haste. Phillip Suggested an inductive proof for verification of universality of his solution, but could not accurately describe induction nor engineer proof.

Chris - *Cutting the Pizza*

- Initially attempted centrically intersecting cuts; experienced confusion when method challenged by off-center cut.
- Chris overcame confusion and stated observed recursive pattern easily but could not engineer algebraic/symbolic statements of situation well. Exhibited significant syntactic/semantic disintegrations, tendencies towards quitting task.
- Chris saw task as a problem to be endured and expressed a marked degree of discomfort with the process.

Notable consequences:

Chris' solutions focused upon use of t-tabulation methods. He did not use a diagram until it was suggested for clarification purposes.

Subjects showed an intriguing variation in their abilities to initially verbalize situational relationships. Significant differences were observed in display of subject understandings of variable usage and in transiting from syntactic to semantic when engineering symbolic representations of situation. Indications from the investigation support claims discussed as framework motivations: the use of qualitative reasoning and a strong symbol sense are essential to success in situational algebra word task engagement, and that there are significant cognitive obstacles encountered by students in transit from the syntactic to semantic.

Pedagogical implications entail employment of co-variational approaches that embrace relationship emphasis and directions or scaffolding that emphasizes and nurtures this type of reasoning. Recommendations for further investigation are warranted in that word modeling to symbolic representation integration is a significant part of the instructional methods for algebra, and other mathematics disciplines, at every level, in that many textbooks and exam formats rely upon this type of task.

COMPARING MAPS DEVELOPED BY PRACTICING ELEMENTARY SCHOOL TEACHERS AND HIGH SCHOOL STUDENTS THROUGH TASK-BASED MATHEMATICS

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We present a qualitative comparative analysis of the mathematical concepts elementary school teachers, participating in a content-based professional development course, and high school students, in a pre-calculus course, developed as they worked collaboratively on a conceptually challenging task. These two groups developed some similar and some very different mathematics concepts while working on the same task.

Metaphor

We introduce a useful metaphor, topographical explorations, for learners' activities while working on open-ended mathematics tasks. Through the prism of the metaphor, students take on the role of an explorer while working on open-ended mathematics tasks and figuratively draw a 'topographical map' that traces the development of their mathematics based on the way-points (points between major points on a route) of mathematical concepts they developed. Student explorers may create different way-points on their maps or developed understanding of some of the same features of the topography, although the route may have been explored at different times or from different perspectives, resulting in different emphases on core mathematical ideas. The focus, here, is on the way-points drawn by the students instead of the end location or completeness of a map which might indicate mastery of all related mathematics concepts.

Related Literature

There has been substantial research into the positive effects that task-based instruction has in the classroom. Dewey's (1916/1944) principle of continuity suggests that students gain richer understanding by connecting to previous knowledge and experience. He also suggests that learning by experience is a forward and backward connection between what we do and what occurs as a result. Task-based instruction typifies Dewey because the students have the opportunity to personally construct mathematical concepts (Fraivillig, 1999), which allows them the ability to make choices which is an essential component in learning (Walter & Gerson, in press).

Research Question

How does the map drawn (mathematics concepts explored) by elementary school teachers compare to the map drawn (mathematics concepts explored) by high school students while working on the same conceptually challenging task?

Method

In this qualitative comparative study, we use a grounded theory approach to analyze the mathematics and interactions of teachers and students as they work on open-ended tasks. Grounded theory is particularly suited to our study because the tenets of grounded theory support an inductive approach to interpreting the meanings of the interactions of people involved in problem-solving situations (Strauss & Corbin, 1998; Zaslavsky & Leikin, 2004).

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Participants and Setting

Twenty-five practicing elementary teachers from thirteen schools in one district participated in an experimental professional development program. Teachers met for more than three hours each week for five semesters in evening class sessions, explicitly designed to foster mathematics learning in teachers as they collaboratively explored challenging tasks. Episodes presented here, from the third-semester course, trigonometry, focus on three teachers in one group. Twenty-five high school students in a pre-calculus course participated in a three day exploration of a conceptually challenging task. These students had limited previous experience with collaborative learning. Episodes presented here focus on four students in one group.

Tasks

A detailed comparative analysis will center on teachers' and students' collaborative work in representing a spiral shell. The *Placenticeras* Task was designed to challenge university honors calculus students (Speiser & Walter, 2004). Teachers in the professional development program and the high school students were given a photocopy of an ammonite shell and were asked to collect data and then describe the spiral of the shell.

Data Collection

All class sessions were videotaped, research team members took field notes, and participants' written work was collected. Video descriptions and transcriptions were linked to video time codes in hours:minutes:seconds to afford fine-grained analysis of the development of teachers' and students' connected and invented mathematics. Selected episodes were identified in which the focus group of teachers and the focus group of students built connections between the mathematics tasks they each previously explored and the mathematics they were each inventing.

Selected Data and Preliminary Analysis

An overview is presented of the way-points (mathematics concepts) in the order they were explored by each group.

Elementary School Teachers

- Common ratio (25-02:26:01, 1-01:26:24)
- Sine curve (25-02:52:29)
- Damped sine curve (25-02:54:29)
- Right triangle trigonometry (1-01:06:53)
- Trigonometric inverses (1-01:10:55)
- Exponential functions (25-03:19:24, 1-01:39:04)

High School Students

- What is a function (00:08:12)
- Exponential function (00:23:15)
- Common ratio (00:55:10)
- Graph transformations (01:02:05)
- Transformations between polar and rectangular coordinates (01:38:52)

Discussion

The maps drawn by these two groups are not the same. Common ratio and exponential functions are common way-points, but there are many way-points which are unique to each group. Despite differences, both groups explored fundamentally important mathematics. Which concepts emerged depended on the background of the participants and the interpretation of the task. Although the mathematics explored in each group was different, one topographical map is not inherently better than the other. Participants in both groups deepened their understanding of mathematics and have foundations from which to continue to draw and complete their individual maps.

Implication

Task-based instruction is a powerful tool in teaching. This comparative analysis between elementary school teachers and high school students working on the same task demonstrates how a single task can illicit understanding of different mathematics concepts. Thus, tasks not only allow students the opportunity to choose how to develop their understanding of mathematics concepts, but are rich in the number of mathematics concepts available to explore.

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COMPUTER SIMULATIONS FOR WHOLE NUMBER ARITHMETIC—ARITHMETIC CARTOONS

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The proposed project will develop and test arithmetic cartoons meant for grades K-2. These cartoons are acted out situations, on a computer screen, that occur in real life. The solutions that students will produce while working on the activities created around the cartoons will involve "models" (Lesh & Doerr, 2002) for describing and solving problems regarding whole number arithmetic. These cartoons will not just focus on problems in which the problematic aspect is that students must make meaning out of symbolically described situations (usual textbook problems) but they will focus mainly on problems in which the problematic aspect is that students must make symbolic description of meaningful situations (Ibid).

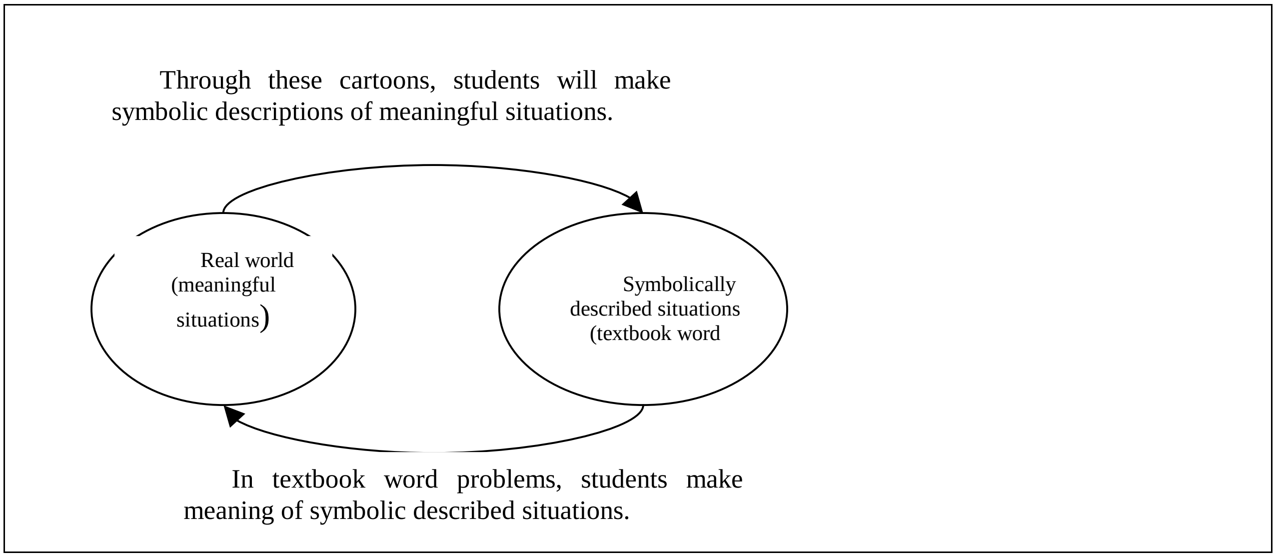

Figure 1: Textbook word problems vs. Arithmetic cartoons

Models and Modeling

Previous research (Fuson, Carpenter, and Steffe) that has been done in the area of whole number addition and subtraction talks about the models that researchers have developed to explain children's learning of whole number arithmetic. For example, those models have either been task specific (join, separate, combine etc… problems by Carpenter et. al.), researcher's interpretation of how children solve the word problems (count all, count on etc… by Fuson et. al.) and researcher's *model* of different stages children go through while learning about whole number arithmetic (perceptual, figurative, etc… stages by Steffe et. al.). The models that we are talking about are the conceptual systems that students will develop while solving the problems presented through the cartoons. They test and revise these conceptual systems during problem solving and express them using various representational media (Lesh & Doerr, 2002).

The models discussed in this project are both internal (conceptual systems that students develop and use to solve real life problem solving situations) and external (conceptual systems

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that students express using variety of representational media). Previous research (Rational Number Project, 1980's – 1990's) has shown that representational fluency (shown in figure 2—Lesh's translation model) is a very important part of understanding a given conceptual system. The arithmetic cartoons project assumes that children will develop a better representational fluency while working on the activities created around the cartoons and eventually develop a better conceptual system of whole number arithmetic.

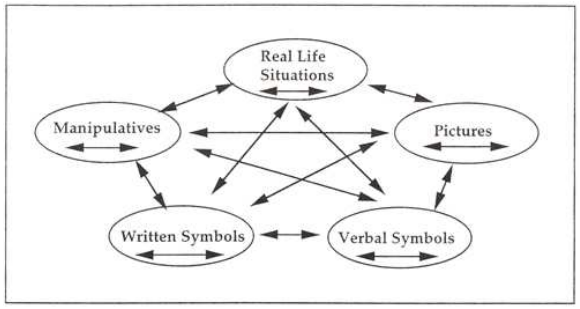


Figure 2: Lesh's translation model

What kinds of situations do we want students to describe using whole number arithmetic?

As stated in the very first paragraph that in usual textbook word problems the students are required to make meaning out of symbolically described situations whereas, through our arithmetic cartoons, we want students to make symbolic descriptions out of meaningful situations (Lesh and Doerr, 2002). The cartoons will be simulations of "real life" problem solving situations in which the big idea of whole number arithmetic (addition, subtraction and to some extent multiplication and division) will be involved. Previous research (Lesh et. al., 2000) has shown that the understandings that are emphasized in textbook problems represent shallow and narrow aspect of problem solving relevant in real life. Real life problems often involve concepts and abilities that seldom fall into only one topic area (for example, most of the situations may involve both addition and subtraction simultaneously rather than only addition or subtraction), they often involve several interacting representational media (symbols, pictures, stories, words, concrete materials) instead of only symbols or only words, and different embodiments (or constructs) of numbers (number line, counts, directed quantities etc…).

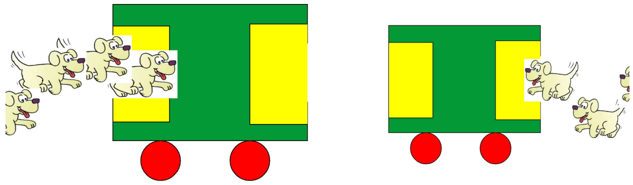
A short clip of the Flash movie and the story that goes along with the cartoons is shown below:

The Runaway Puppy Rescue Center

My friends Harry and Gary run a rescue operation; they rescue run away puppies. They use a wagon to transport the puppies back to The Runaway Puppy Rescue Center.

To catch the runaway puppies, Harry makes puppy treats. These are special treats because they smell just like a Happy Hamburger sold only at Charlie's Hamburger Place next to Jungle Jack's Skateboard Park. Humans cannot resist the smell and dogs are completely helpless with hunger from the delicious aroma. Treats that smell like Happy Hamburgers made rescuing puppies easy, Gary can even rescue them.

Harry liked to put his puppies on the left side and Gary put his rescued puppies on the right of the wagon. *One day they rescued 8 puppies all together*. <u>Harry put the five puppies he rescued into the right side of the wagon and Gary placed his three puppies in the right side of the wagon.</u>



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IMPACT OF INSTRUCTIONAL SETTING, PROBLEM POSING, AND LANGUAGE ON ELEMENTARY SCHOOL STUDENTS' PERFORMANCE IN SOLVING MATHEMATICAL WORD PROBLEMS

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Solving story or word problems is a challenging task in elementary mathematics classrooms. Research shows that students' poor performance in word problem solving could be a result of their miscomprehension of the problem (Cummins et al., 1988). Among factors influencing miscomprehensions are: difficulties in perception of mathematical language (Kane, 1967), insufficient subject matter knowledge (Mayer, 1992), problem posing (Butts, 1980), language deficiencies (Mestre, 1988), ineffective text processing (Nathan, Knitsch, & Young, 1992), and lack of effective reading strategies in problem solving (Shuard & Rothery, 1988). We examine the following research question: what impact a problem posing along with other variables such as instructional setting and language have on students' performance in solving word problems?

Research Design

The research sample consisted of 141 second graders from an urban southwestern public school with predominantly Hispanic student population. Data collection conducted through the use of a short and extended version of a 'trick' word problem (the problem that doesn't contain a solution). The trick problem was selected for the study purposefully; it is a non-routine problem that requires students to comprehend the problem before solving it. The short version was presented as a mathematical word problem and the teacher (graduate student) introduced herself as a math teacher. The short version used in the monolingual class was: "There are 125 sheep and 5 dogs in a flock. How old is the shepherd?" Students in the bilingual class had an option to approach the problem in English or Spanish (the problem was translated into Spanish).

The main difference between the short and extended version was that the extended version was introduced to the students as a reading assignment and the teacher didn't introduce herself as a math teacher. The extended version for the monolingual class was presented in the following way: "Once there was a shepherd who had 125 sheep and 5 dogs. He would take his sheep up to the hill to eat. They would stay there all day and in the late afternoon the shepherd would walk them back down. The dogs would help him keep the sheep together so they wouldn't get lost. At the end of the day the shepherd would be tired, but happy because he still had all his sheep. Help us to answer the following questions: (a) how many sheep did the shepherd have? (b) where did the shepherd take his sheep to eat? (c) how old is the shepherd? (d) who would help the shepherd keep the sheep together?". The extended version was also translated into Spanish for the bilingual class. There were two sub-groups for each version of the word problem posing: without and with a direction which consisted of only one statement: "Read everything carefully!" In all the groups the teacher provided an individualized assistance helping students understand what each unfamiliar word (if any) meant.

Results

Data collection and analysis were performed using the following 4 categories based on variations of students' responses. Results of the study on instructional setting, problem posing,

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and language in solving word problems are presented in Table 2.

	130 (+) 120 (-)	Wrong variations (5, 105, 129, 505, …)	"Reasonable" answers (15-75)	Correct responses
Short version (N=70)				
Without Direction S. (33)				
a) Monolingual	6 (35.3%)	6 (35.3%)	5 (29.4%)	0 (0.0%)
(17)	5 (31.3%)	8 (50.0%)	3 (18.8%)	0 (0.0%)
b) Bilingual	11 (33.3%)	14 (42.4%)	8 (24.2%)	0 (0.0%)
(16)				
	9 (60.0%)	3 (20.0%)	3 (20.0%)	0 (0.0%)
With Direction S. (37)	13 (59.1%)	6 (27.3%)	3 (13.3%)	0 (0.0%)
a) Monolingual	22 (59.5%)	9 (24.3%)	6 (16.2%)	0 (0.0%)
(15)				
b) Bilingual				
(22)				
Total - short version	33 (47.1%)	23 (32.9%)	14 (20%)	0 (0.0%)
Extended version (N=71)				
Without Direction S. (35)				
a) Monolingual	0 (0.0%)	3 (15.0%)	9 (45.0%)	8 (40.0%)
(20)	0 (0.0%)	1 (6.7%) + 1	10 (66.7%)	3 (20.0%)
b) Bilingual	0 (0.0%)	(6.7%)*	19 (54.3%)	11 (31.4%)
(15)		4 (11.4%)		
	0 (0.0%)		4 (20.0%)	15 (75.0%)
With Direction S. (36)	0 (0.0%)	0 (0.0%) + 1	2 (12.5%)	12 (75.5%)
a) Monolingual	0 (0.0%)	(5.0%)*	6 (16.7%)	27 (75.0%)
(20)		2 (12.5%)		
b) Bilingual		2 (5.6%)		
(16)				
Total - extended version	0 (0.0%)	6 (8.5%)	24 (33.8%)	41 (57.7%)

Table 2. Results of the study

Discussion and Conclusion

Considering limitations of the study (small sample size, non-randomized selection of the research sample, absence of follow up cognitive interviews with students, etc.), we feel safe to open a discussion on the following observations and interpretations from the data we collected:

1. There is a strong impact of the problem posing variable on students' performance. The most impressive result was shown by both monolingual and bilingual groups that used the extended version with the direction statement "Read everything carefully!"
2. There is a mixed impact of the direction statement (an instructional setting variable) on the students' performance depending on two other variables. Data tells us that the direction statement played a "negative" role in the short version setting: more students performed arithmetic operations without understanding the problem. On the contrary, the students in the 'extended version' groups didn't even try to use meaningless computations at all!
3. The study also showed a small impact the language variable had on students' performance: in the 'short version' groups monolingual students were capable to come up with more 'reasonable' answers than bilingual students.

4. Observations of student behavior showed that the students in the ‘extended version’ groups were more engaged and open minded while approaching and solving the problem; they asked more question for clarification purposes compare to the students in the ‘short version’ groups who were mostly silent and less active.

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STUDENTS' DIFFICULTIES IN UNDERSTANDING FRACTIONS AS MEASURES

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This study investigated students' understanding of fractions as measures using number lines. Fifty-six seventh graders were asked to locate ¾ on two number lines: one was 1-unit long, and the other 5-units long. Results showed only 9 students correctly located ¾ on the both number lines. Analysis suggests that the students' difficulties stem from an overgeneralization of part-whole partitioning strategies in measurement contexts.

Several reports of national and international assessments of students' understanding of rational numbers have documented students' difficulties conceptualizing fractions as measures (Jakwerth, 1996). Our observations of students at a middle school mirrored those difficulties. In this paper, we report on the students' responses to tasks that asked students to identify fractions on number lines. We use the students' responses as a window into their conceptual understanding and struggles in making sense of fractions as measures.

Theoretical Background

Several researchers identified different aspects of rational numbers, two of which are part-whole and measurement (e.g., Kieren, 1993). Traditionally students have been introduced to rational numbers as part-whole representations with two-dimensional shapes partitioned into multiple equal parts. In this study, we focused on children's understanding of fractions as measures and used a number line representation to exemplify such a measurement understanding. A measurement understanding is defined as one that is able to see and use a given unit to measure any distance from the origin (Lamon, 1999). The distance is then some amount of partitioned units from a point of origin, and the collection of units on the number line consecutive, continuous measurements from that origin.

Although partitioning of the unit occurs in both part-whole and measurement contexts, there are not the same. Part-whole partitioning involves comparing the number of equal parts to the total number of equal parts, while in measurement "the number of equal parts in the unit can vary, and what you name your fractional amount depends on how many times you are willing to keep up the partitioning process" (Lamon, 1999, p. 113). Moreover, students' partitioning experiences in part-whole contexts are often limited to dealing with one unit, such as partitioning a pizza into two 1/2s. In contrast, in measurement contexts, students often deal with measures of multiple units. Due to a similar language, similar sets of symbols, and similar representations, we hypothesize that students will overgeneralize part-whole partitioning strategies in measurement contexts.

Methods

Fifty-six students from three 7th grade classrooms in Pheonix, Arizona participated in this study. The school mainly served minority students (85%). Data was gathered through written assessments. In this paper, we focus on students' responses to 2 of the 8 problems to highlight their understanding of fractions as measurement--the first problem asked students to locate ¾ on a number line from zero to one, and the second ¾ on a number line from zero to five. All the

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students were taught by one math teacher using Mathematics in Context (MiC).

Results

Nine of the 56 students in 7th grade (16%) answered both problems correctly. Jack's strategies exemplify students' valid implementation of unitization and partitioning strategies in linear measure contexts. In Figure 1, Jack correctly interpreted the unit on a number line as zero to one. He found a half of one and correctly estimated the location of ¾. In Figure 2, he first estimated the length of each one-unit on the number line between 0 and 5. He then halved the first unit to find ½ and correctly located ¾ at the midpoint of the line segment from ½ to one.

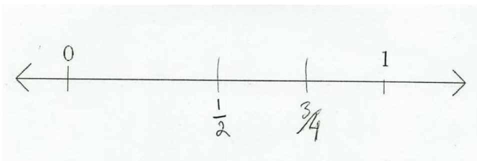

Figure 1. Jack's strategy for locating ¾ on a number line from zero to one.

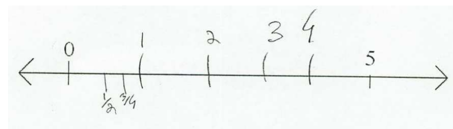

Figure 2. Jack's strategy for locating ¾ on a number line from zero to five.

Thirty-six students (64%) identified ¾ on the number line from zero to one, but not on the number line from zero to five. For example, Jessica found ¾ correctly on a number line from zero to one, by partitioning the number line into four equal pieces and estimating ¾ appropriately (Figure 3).

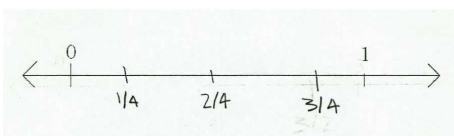

Figure 3. Jessica's strategy for locating ¾ on a number line from zero to one.

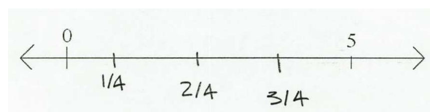

Figure 4. Jessica's strategy for locating ¾ on a number line from zero to five.

Jessica, however, overgeneralized her first strategy when using a number line from zero to five, providing her with an inappropriate location of ¾ (Figure 4).

Conclusion

Among students' difficulties in transitioning from part-whole to measurement, this paper focused on two issues. First, many students seemed to use a number line as if it was a fraction bar, partitioning it as if the entire visual number line was a unit of one. Second, it seemed that students were unable to see a number line as a continuous collection of iterated units. While part-whole aspects of students' strategies were valid for the first problem, many students overgeneralized the strategies and failed to identify an appropriate unit or measure 3/4 of one unit on the second problem. The results suggest that we need to consider the over-emphasis of part-whole aspects of fractions and limited representations in rational number instruction.

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THE IMPACT OF WRITING ON STUDENTS' PERFORMANCES ON THE POST TEST

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Scholars studying writing to learn strategies within classrooms argue that writing can be used as an effective learning tool. One of the functions of writing is to augment understanding through cognitive and metacognitive actions that it demands. In addition, during writing, one negotiates meaning; and in negotiation, one is generating knowledge and augmenting learning (Powell & Lopez, 1989).

The recent research emphasis has been on incorporating writing into mathematics classrooms to develop students' problem solving abilities. According to Polya (1981), problem solving is a process of making conjectures among the data (known), the unknowns, and the conditions of the problem. Kenyon (1989) argued that writing can be used effectively for promoting the problem solving process because it allows students to gather information, organize their thoughts, and make connections between their mathematical concepts, problem solving strategies, and the conditions of the problem.

The mathematics reasoning heuristic (MRH) was developed to scaffold students' problem solving skills with writing tasks embedded into mathematics activities. The MRH is a pedagogical tool that combines teacher's mathematical understanding, students' understanding of mathematics, negotiation of ideas, and writing.

Method

The purpose of this study was to look at high school students' performances on the post-test. The main data source was students' pre- and post-test scores obtained from the teacher. This study was conducted with a ninth grade algebra teacher who had two classrooms. A classroom with 16 students was chosen as the control group and the other was chosen as the treatment group with 24 students.

Results

An analysis of covariate (ANCOVA) model was estimated to look at the group difference. The results showed that students in the MRH (treatment) classroom significantly outperformed students in the control group on the post-test when their pre-test scores were controlled ($F(1,37) = 5.688$, $p = 0.022$). The effect size for this group difference was 0.55 **SD**, which is a medium effect.

Discussion and Conclusion

The significant results indicated that the treatment group students improved more than the control group students. This difference can be attributed to the intervention that was practiced by the teacher. Along with writing tasks, students engaged in classroom discussion where they could have opportunity to negotiate common meaning and individual understanding of the topic.

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ENHANCING PROBLEM SOLVING THROUGH WRITING

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Problem solving and writing processes resemble in many ways. The authors exploring writing process suggest that the writer engages in a series of complex cognitive actions to transform his experience through the specific symbol system of language into an icon (the graphic product) (Emig, 1977). Similarly, a problem solver constructs internal and external representations of the object and develops strategies to make the connections between the information extracted from the problem text by means of language and the conditions and goals of the problem (Mayer, 1982).

Method

The purpose of this paper was to investigate the characteristics of problem solving and students' problem solving skills within writing tasks over a longitudinal course of study (three chapters in a math textbook); and to explore their explanation of mathematical concepts (e.g., word-choice) to younger audiences. Students were asked to explain what they had learned, in a text format using mathematical examples, to different audiences so that they could understand.

Results

Problem Solving Skills

The analysis of students' second writing samples in chapter 7 revealed that students put more emphasis on understanding the problem by extracting the relevant information from the problem text to set their solving strategies.

Using Mathematical Language

Students' use of mathematics language showed variation, and improvement, across the writing samples depending on the target audience. For example, when students wrote a letter for a fifth grade student about ratios, proportions, and percents, they explained the mathematical terms either using simpler everyday words or drawing pictures. For example, a student started with asking her audience whether he/she heard of ratio, proportion, and percent. She then gave a mathematical definition for ratio, "A ratio is a way to compare numbers." She further provided real life examples, "if you have 8 brownies and 12 cookies, you could set the ratio up like this 8:12."

Discussion and Conclusion

Improving students' problem-solving and reasoning skills is one of the major goals in mathematics education. Writing is one of the tools that teachers can use in their classrooms to help students enhance their abilities to solve mathematical problems. Writing gives students the chance to reflect on, and react to their thoughts and ideas (Kenyon, 1989) and helps students develop reasoning and a specialized mathematical language through the language they more

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USE OF EVERYDAY LANGUAGE AND WORD PROBLEMS

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Solving arithmetic word problems has been studied from several perspectives: considering a schematic approach (Carpenter y Moser, 1982), attending the semantic structure (Nesher, 1982) and considering as well the place of the unknown (Vergnaud y Durand, 1986).

Nevertheless, an element that has not been considered in depth is the language proficiency the student has and its relation to solving addition word problems.

A qualitative research was carried out in order to see what role reading comprehension and use of everyday language had when solving addition word problems. Word problems were classified according to Puig and Cerdán's (1988) semantic classification: 'combine', 'change', 'compare' and 'to make equal' type of problems.

Using two instruments during individual interviews, it was possible to assign each of the 17 students a language level and an ability level when solving addition word problems. Student's age was 10-11 years old; all of them had already studied in previous years the four types of verbal problems. Bearing in mind that language was the key factor of the research, seven children with hearing problems (moderate to severe) took part in the study.

In order to assign the language level, Ortega and Garza's (1982) instrument was applied. The attainment level when solving word problems was determined with 16 problems: four for each semantic category, each given orally as well as written down. All problems included small numbers which addition did not surpass 20, with the purpose of focusing on the language and the semantic categories used. Children that could not answer correctly one of the verbal problems, a theatrical presentation of the problem was used to identify if the difficulty was in their language ability or in understanding the semantic structure of the problem.

Use of language and word problem solving was classified into three levels: high, medium and low.

With regard to problem solving, nine out of the 17 students obtained a high level in problem solving, of which seven had a high level in use of language, one medium and the other low. One student obtained a medium level in problem solving with a high level in use of language. The rest, seven students, got a low level in problem solving, of which four had a medium level in use of language and three a low one. The hearing problem factor had no influence on these results.

Considering the use of language, students with a medium or a low level had mayor difficulties with the aspect of "auditory memory" and "following orders" comprehension. Nine students had problems with one or the two aspects. Seven of these nine students where the only ones that got a low level in problem solving, regardless if they had hearing problems or not.

In conclusion, it was found that the use of language plays an important role in solving addition word problems, being the "auditory memory" and "following orders" aspects that children need to master in order to be good problem solvers. Thus, a high level in use of language leads to a high level in problem solving; a low level use of language results in a low level in problem solving. All the same, a low level in problem solving does not imply that the student has a low level of use of language.

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PROBLEMS THAT MAKE A DIFFERENCE TO KANGAROOS

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The « Kangourou des Mathématiques » competition aims at showing its participants the beauty and entertainment of mathematics by problem solving. On the basis of the data of over 100.000 participants' answer sheets of the German version of this multiple choice competition, a choice of a couple of problems is examined which predicts best the outcome of the competition. These prove not to be simply the most difficult ones; they cover rather a wide range of mathematical areas. The samples of those questions indicate that problems connecting different areas of school mathematics are likely to decide about the prize winners.

The making of the multiple choice problems for the kangaroo competition

The kangaroo competition is international in organisation and tradition. It was founded in France in 1991 as « Kangourou des Mathématiques » following the example of the "Australian Mathematics Competition" – a fact its name alludes to. First and above all the Kangaroo project aims at showing the beauty of mathematics by problem solving. Each year, an international board selects suitable problems for the different age-groups from a pool of problems contributed by the 14 participating countries. The organisers of each country select approximately 75 % of the problems for their national test from the catalogue. The remaining part can be chosen freely or modified from original problems. The problems are not designed for a diagnostic purpose; conceptually, they are rather set to reflect a certain spirit of mathematics and mathematical problem solving in its whole diversity.

Objects of research in view of the kangaroo design

The merits of problem solving are widely acknowledged (Pehkonen, 1991). Working on problems in mathematics is widely believed to be a key of mathematics education at all levels (NCTM 1999). In the well known competitions in mathematics, problems are designed to determine the most talented problem solvers. These are used and designed for the diagnosis of mathematical abilities (Kalmann 2002). A canon of solution strategies is established, which is claimed, again, to be universal for mathematical contents (Engel 1998).

This kangaroo project questions the tradition of problem solving, which does not mean it would not respect it highly. The relation of mathematical content and problem solving strategies is not well understood yet; some selection of problems stress, for instance, the combinatorical component. It seems also interesting to us to find out whether the Kangaroo competition, as an event for all, gives a broader range of problems than Mathematics olympics, which tend to be an event for the gifted. The project involves the following parts:

- Identifying the choice of those problems which reflect best the achievement in the whole test,
- Analysing this shortened form of the test from the point of view of problem solving in order to find out in which regard these problems are special,
- Working out criteria for the quality of problems and to understand more about the relation between mathematical contents of the problems and heuristic strategies.

Data analysis method: finding wallabies

In Germany, questionnaires for five groups of age are offered; the total number of participants refers to the year 2001. The results are being discussed here:

Grades	3 and 4	5 and 6	7 and 8	9 and 10	11 to 13
Total of participants	17,690	37,059	27,218	15,370	6,457
Prize winners	814 4.6 %	1,808 4.9 %	1,396 5.1 %	815 5.3 %	348 4.5 %

Table 1: Participation in Germany in 2001

The approach of the discriminant analysis is not described here in detail because the studies on the wallaby-problems will be more interesting for the readers. The approach involves a kernel for the ordinal three parameters: correct answer, no answer, and wrong answer. As already mentioned, no distinction is made between the various wrong answers. The data is divided into two subsets: a training set for the estimation and its complement for checking the quality of the decision rule in the second step.

Wallabies for grades 3 and 4 as well as for grades 11 to 13

In this section, the results on the year 2001 are given. The older results are taken on purpose because in this way accompanying studies with students today rule out the possibility that they have worked on these problems at the time because they are now in a different age-group.

The answers in the group of grade 11 to grade 13 are included here to give an impression of how diverse the statistics to the problems is distributed. The original problems can be found at the website www.mathe-kaenguru.de in the archive “Chronik” and the folder “Aufgaben”, subfolder “2001”.

The discriminant analysis singles out three problems as the wallaby which predict best the outcome of the test. The group of prize winners agrees with those who answer all these problems up to an estimated error rate of 4.37 %. One can add a fourth question to the wallaby which increases the error rate only marginally. In the following table, the wallaby questions are written in italics.

Problem	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Correct (%)	52 45	47 41	42 35	35 32	34 34	23 16	48 48	6 6	39 29	33 35	7 9	35 35	23 23	34 32	57 59
Wrong (%)	28 38	47 50	21 31	46 55	38 41	45 57	28 29	35 39	37 49	33 37	75 77	35 37	57 57	24 30	33 32
n. a. (%)	20 17	6 9	37 34	19 13	28 25	32 27	24 23	59 55	24 22	34 28	14 14	30 28	20 20	42 38	10 9
Problem	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Correct (%)	45 41	20 15	9 8	42 36	33 33	36 38	25 17	20 14	39 42	21 18	7 6	45 43	14 14	20 19	7 7
Wrong (%)	31 38	26 54	41 48	36 33	46 48	34 41	31 47	30 42	31 33	33 44	66 59	29 37	49 57	47 47	40 42
n. a. (%)	24 21	36 31	49 44	32 31	21 19	30 21	34 36	50 44	30 25	36 38	27 25	26 20	37 29	33 34	53 51

Table 2: Percentages of correct, wrong and no answers of prize winners and others in grades 11 to 13

For the grades 3 and 4, the questions number 8, 11 and 21 (in the year 2001) were identified as wallaby problems.

Discussion

The results of the test for the grades 3 and 4 and that from grade 11 were chosen because they reflect common aspects to all group-ages and contain also some particularities.

Older students become more and more careful with giving an answer. Although it can be doubted whether abstaining from an answer is a good strategy, they seem to want to be quite sure before ticking the appropriate box. Thus, there is a striking difference in ages for multiple choice questionnaires that penalize wrong answers compared to abstaining from giving answers. For the wallabies this means that these are more reliable for older students because the option of giving no answer at all is almost neglected by the younger children.

The classical solution strategies, which appear in the wallabies, cannot always be clearly distinguished. For sure, in the examples presented here, we find a problem aiming at the invariance principle (grades 3 and 4, problem 8) as well as a combinatorical problem (grades 11 to 13, problem 23) which some students approached with a strategy of systematic trying. Combinatorical problems appear frequently in wallabies, but they are not dominant.

It can also be observed that some problems link different areas of mathematics. Question 20 (grade 11 to 13) links geometry and algebra by the way the choice of possible answers is given. And question 25 (grade 11 to 13) involves very different aspects within the area of analysis.

Following the results, it could be worthwhile examing whether the solution strategies for problem solving have been stressed too much in the tradition of mathematics education. They seem to indicate that problem solving competence involves significantly both the ability to work on problems of different mathematical areas and to connect different mathematical areas in the solving process of a single problem.

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SECOND GRADERS' SOLUTION STRATEGIES AND UNDERSTANDING OF A COMBINATION PROBLEM

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Many researchers emphasize that in order to understand students as learners of mathematics and to build on students' thinking, teachers need opportunities to examine and understand children's strategies while solving mathematics problems (Carpenter, Fennema, Franke, Levi, & Empson, 1999; NCTM, 2000). Carpenter et al. (1999) identified student strategies for solving multiplication and division problems, but avoided analysis of student strategies in combinatorics. Maher and Martino (1992) characterized the solution strategies of two students working on a combination problem presented to students by the researchers.

In this poster presentation, we will report preliminary findings on the strengths of children's thinking and the different solution strategies they develop while working on a combinatorics task presented by their classroom teacher. This qualitative research is part of a professional development project with twenty-eight elementary teachers. The video analyzed here is from one session in a classroom of twenty-two second grade students taught during the last month of school by Melanie, a second-year teacher. Melanie introduced a combinatorics task to her students by asking them to explore how many different outfits could be made if there are three shirts and two pairs of pants.

Students used various strategies to solve the shirts and pants problem. Tanner stated, "It [the problem] says there's three pairs of shirts and two pair of pants. If there's one more pants you could make three." Tanner believed he needed the same number of pants as shirts to make three outfits. Tiffani used colored squares of paper to represent the shirts and pants and arranged them in such a way that she held the shirts constant and varied the pants. If we denote the number of shirts as 3S and the number of pants as 2P, then a mathematical representation for Tiffani's strategy could be $2P(S + S + S) = 2PS + 2PS + 2PS$. Caleb used colored squares of paper to represent the shirts and pants and solved the problem in two parts. First, he held one pair of pants constant and varied the shirts. A mathematical representation for the first part of Caleb's strategy could be $(S + S + S)P = SP + SP + SP$. Second, Caleb held the other pair of pants constant and varied the shirts, which could be represented as SP+SP+SP. Shelby used a connecting-line strategy. She said, "Since there's three shirts and two pairs of pants, well, you could take two shirts and each put them with a pair of pants. Then switch the ones. And then like if you left out the red shirt, put the red shirt on that one [pants] and then on that one." A mathematical representation for Shelby's strategy could be $(2S + S)2P = 4SP + 2SP$.

Additional data and implications for building on student thinking will be discussed.

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PSM–METHOD: A TOOL FOR PROBLEM SOLVING

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Problem solving map (PSM) –method

The main idea of the PSM is that pupils will learn to collect and write down notes that will help to solve the problem. The PSM emphasizes metacognitive thinking. The application of metacognitive techniques has two important mathematical purposes: It allows pupils to keep track of what they have done and it allows pupils to make connections between their problem solving work and their knowledge of subject matter and mathematical procedures (Finkel 1996; Schoenfeld 1985).

Method

This study was carried out in the 6th grade of a small Finnish school using a quasi-experimental design. The experimental group (N = 17) was taught problem solving over six weeks in 30 lessons integrated in to their regular school days, mainly in mathematics but also in mother tongue, science, art and craft. In teaching problem solving Schroeder's and Lester's (1989) ideas on the three components: about problem solving, for problem solving and through problem solving were used. The control group (N = 35) studied mathematics and other school subjects in their normal way. The PSM -method played a central role in the course. Pupils' problem solving performance was measured in pre- and post-tests and 1.5 years later in a delayed test.

Results

The differences between the control and the experimental group in three tests could be compared if the results of the control group were standardized to 100% in all the three tests (table 1).

Test	Experimental group	Conrol group
Pre-test	97%	100%
Post-test	126%	100%
Delayed test	115%	100%

Table 1. Standardized results in the pre-test, in the post-test and in the delayed test

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NO WAY THAT'S TWO HUNDRED AND FIFTY INCHES DEEP! MATHEMATICAL UNDERSTANDING IN AN APPRENTICESHIP CLASSROOM

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The larger study from which this poster comes is concerned with exploring how adults in workplace training learn, understand and use mathematics as they engage with it in the context of their specific trade.1 This poster will focus on the growing mathematical understanding of three apprentices, known here as Joe, Andy and Mike, who are in the second year of an apprenticeship training program to become credentialed ironworkers. The taught program is based at an Institute of Technology in Vancouver, BC, and involves classroom and practical sessions. The apprentices were in a larger class of about twenty students and Joe, Andy and Mike worked closely together, for about one hour, at a desk, where they were video and audio recorded. In this session they have been posed the task of establishing the size of choker sling required to lift an assembled structure of four large iron beams into an upright position, and later of determining where the crane should be positioned to accomplish this. The structure consists of two upright beams, one top crosspiece, and one middle beam. This is lifted into position using two chokers in a sling arrangement around the top beam. It is the size of these chokers that the apprentices have been asked to calculate, something that is dependant on the total weight of the structure to be lifted. In analysing the data, and in seeking to describe and account for the way in which the students work mathematically whilst solving this problem, we employed elements of the Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1994). However, we also characterise mathematical thinking as particularly complex in the workplace, involving the drawing on and working with three different forms of mathematical understandings: understandings of the task as posed; understandings of the mathematical relations required by the task; and understandings of the task as an actual job to carry out. We contend that it is the way that these apprentices are able to build an understanding that shifts, and builds connections, across these three dimensions, that leads to their success with the task.

Endnote

1. The research reported in this poster is supported by the Social Science and Humanities Research Council of Canada, (SSHRC) through Grants #831-2002-0005 and #501-2002-0002.

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EFFECTS OF INTER-GROUP AND INTRA-GROUP SHARING OF MATHEMATICAL PRACTICES, PROCESSES, AND PRODUCTS IN A MIDDLE SCHOOL MATHEMATICS CLASSROOM

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Introduction

This study supports the claim that the mathematical knowledge of both a classroom community and individual students in the community is advanced through the significant influence of shared student communications, interactions, and products. In the past, some attention has been given to the sharing of such practices among students in collaborative working groups (Noddings, 1989), but little to no attention has been given to the sharing of such practices between working groups. In many mathematics classrooms, students are introduced to ideas through the teacher's presentation of them to the entire class. This study investigates two middle school mathematics classrooms where students are often introduced to practices, processes, and products through the spread of student-initiated ideas. The spread of these ideas occur both within students' collaborative working groups and between the collaborative groups. Characterizing the effects of sharing students' practices, processes, and products both in and between working groups as they work collaboratively on thought revealing mathematics problem solving tasks forms the basis of this study.

Collaborative learning environments that involve students working on thought revealing mathematical tasks encourage the development, exchange, and sharing of knowledge (Lesh, Hoover, Hole, Kelly, & Post, 2000), where mathematical knowledge is a broad term referring to mathematical ideas, facts, concepts, problem solving strategies, tool related practices, etc. The notions of development, exchange, and sharing does not suggest imitation but instead a process of interpretation, adaptation, and evolution. When students work on complex problem solving activities, there are often discussions involving negotiations, clarifications, explanations, and justifications among students. During this process of negotiating meanings of goals, ideas, possible strategies, progress and other issues, students share and spread new ideas. One of the purposes of characterizing the effects of such sharing is to clarify means for engaging students in processes of knowledge advancement. The challenge is not simply to provide opportunities for collaboration but to design the classroom environment so that knowledge is shared.

Data Collection and Analysis

A random sample of six groups of students from two different mathematics classes was studied. Data collected include (i) audiotapes and videotapes of student group work and group presentations, (ii) student work (i.e. group product, individual homework assignment), (iii) informal interview with teachers and students, and (iv) field notes. Multiple sources of data were analyzed in relation to one another to check for congruency and to enhance the credibility of their findings. Discussions and interactions between students were captured on audiotape and videotape. Data from videotapes were primarily used to visually capture whole class interactions and support audiotape data. All audiotape data were transcribed or summarized. Data were coded and categorized according to the pattern coding method (Miles & Huberman, 1994) to specifically trace the development and diffusion of ideas among students.

Results

The findings of this research indicate interactions occurring both within and between working groups foster the development and sharing of knowledge. Data also show that varying views and ideas should be accessible and valued as a resource for the entire classroom community. Interactions occurring within students' working groups and those occurring between groups of learners are both relevant with respect to encouraging the development and exchange of new ideas. Although groups of students often work on the same task, each group may reconstruct the problem differently. Within their own working group, students naturally negotiate and interchange ideas.

In inter-group interactions, non-group members have access to alternative ideas. These alternative ideas should be accessible to all students throughout the implementation of a problem solving task. Through connecting ideas from their own group with ideas from neighboring groups, students are able to connect multiple ideas to form one new idea. During intra-group collaborations, students focus on developing and structuring their designs and models. The inter-group environment leads students to further discussions of the quality of their models, functions of their designs, and justifications for their reasoning. Both types of interactions elicit and foster the development, sharing, and appropriation of knowledge through opportunities of peer dialogue, submission of contrasting ideas, and active seeking of information outside the group.

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MIDDLE SCHOOL MATHEMATICS THROUGH MURALING

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The author designed and taught two semesters of math-through-muraling classes under a nationally-funded after-school program. Over 90% of students at this middle school qualify for free or reduced cost lunch. The author was also a participant-observer in four daytime math classrooms, for one of which a mural activity was implemented at the end of the year.

The fall class used concepts of polygon and circle geometry to construct a large mural based on familiar regional weaving motifs. Goals were assigned such as "record an aspect of our project which you think future viewers may wish to know about," but execution, including, for the above example, choosing how to collect, record, summarize and present data was entrusted to the students as a whole. Students painted their statistical and geometric analyses into the mural. Vigorous mathematical debate was often observed during this after-school class. Of note was the voluntary attendance and extended, concentrated mathematical work of students regarded as behavior problems in their daytime math classes.

During spring semester, muraling as a model-eliciting activity (Lesh, Hoover, Hole, Kelly, & Post, 2000) was explored. Using an image of a Fibonacci spiral for mural design, the activity was only successful once structured to physically require collaboration. Thereupon students quickly progressed through multiple cycles of increasing mathematical perception as they struggled to make their sections of a model for the mural fit with other groups' sections.

Last, to address challenges observed in an 8th grade classroom, a mural activity was designed for the class in which students used algebraic notation to communicate with other teams about color proportions. The goal was to establish a situation wherein:

- interacting with and generating data about meaningful phenomena form the foundation for translating experience into algebraic representation (Moses, 2001),
- communication in algebraic language could be appropriated by students, and
- strongly disengaged students could be valued and re-engaged.

In each turn at the mural, a team read the latest equation, decided how to translate the relationships between colors encoded therein into an appropriate total volume of paint for the section, painted, saw the result, decided what direction they wanted the mural to take from there in terms of color, represented this in a new equation, and wrote this equation onto the next section. Equations penciled onto the wall were the only mode of communication between teams in this relay, in which each team had several turns over two weeks.

Project goals were met, as measured by positive feedback from classroom teachers, students quickly adopting mathematical language in dialog with each other, and high challenge and engagement for most students: the quietest in-class students now persistent with their questions and the most disruptive in-class students taking the mural seriously. Students applied in-class material to the solution of mural equations and responded effectively when mural equations were referenced as aids in understanding in-class material. Opportunities for muraling to act as formative assessment (Black & Wiliam, 1998) were also realized.

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MATHEMATICAL IDEAS ELICITED WHEN STUDENTS SOLVE MATHEMATICALLY RICH TASKS IN TWO LANGUAGES

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Is there a difference in the mathematical ideas elicited when solving the same meaningful problem in two different languages? Do students whose first language is Spanish have difficulty doing mathematics because of the math, or because of their lack of confidence in their second language (Khisty, 1997)? According to Moschkovich (2002, p.190), “Students are now expected to communicate mathematically, both orally and in writing, and participate in mathematical practices, such as explaining solution processes, describing conjectures, proving conclusions, and presenting arguments.”

Model-eliciting activities are based on real-life situations where students, working in small groups, present a mathematical model as a solution to a client’s need (Lesh, Hoover, Hole, Kelly, & Post, 2000). Students must develop a mathematical model which the client can comprehend, find useful, and apply to other situations. These models reveal “important aspects about how students are interpreting the problem solving situations” (Lesh and Doerr, 2003, p.9). These activities are thought-revealing, allowing students to elicit their knowledge in a variety of ways: orally, as students communicate among each other when solving the problem and present their solution to the whole group; and in writing, when they respond to the client in a letter format.

This study will look at two groups of 8th grade students. Both groups will have a combination of students whose second language is English, and whose first language is Spanish; and students whose first language is English. One group will solve a model-eliciting activity in English and the other group will solve the same model-eliciting activity in Spanish. The focus of this study is to look at the extent at which students’ mathematical ideas are elicited, and if language places a difference in these mathematical ideas. Are the same mathematical ideas elicited no matter what language the activity is in? Is student performance on this task related to the language of the activity?

In order to answer these questions, we will conduct classroom observations as students in both groups solve the activity. These observations will focus on the language that students use in communicating with other peers, the mathematical ideas elicited while students solve the problem and present their solutions. We will also collect students’ solutions to the model-eliciting activities and use a scoring rubric to assess their performance. Later, we will relate students’ performance on this task, with the mathematical ideas used, and the language used, both, by the students and in the task; we will find how these variables are related to each other.

During the poster presentation we will discuss which mathematical ideas were elicited from the two groups. We will also discuss how each group went about solving the problem, what ideas were elicited, and their solutions to the problem. We will compare the mathematical ideas embedded in each group’s solutions and discuss why they did/did not differ.

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STUDENTS' SOLUTION STRATEGIES TO DIFFERENTIAL EQUATIONS PROBLEMS IN MATHEMATICAL AND NON-MATHEMATICAL CONTEXTS

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This study focused on probing undergraduate students' understanding of two concepts of differential equations, that of slope fields and equilibrium solutions, as they solved complex problems in mathematical and non-mathematical contexts. The term complex problem refers to a problem that requires the solver to consider the concepts of slope fields and equilibrium solutions together and as non-isolated facts. Problems in a mathematical context are those expressed purely in mathematical terms. By contrast, problems in a non-mathematical context are framed in real-word applications settings.

Also assessed were participants' abilities to solve problems that evaluated different types of understandings of the concepts of slope fields and equilibrium solutions separately. These latter problems are referred to as *simple problems*, all of which were presented in mathematical contexts only. The specific questions guiding the research were: 1) Does performance on complex problems vary by context (mathematical, non-mathematical)? 2) When considering a complex problem in a mathematical and a non-mathematical context, are participants who answer the problem in one context correctly more likely to answer the corresponding problem in the other context correctly? 3) Does performance on simple problems predict performance on complex problems?

In order to investigate the three research questions, a written test was designed to consist of four complex problems and six simple problems, three pertaining to slope fields and three pertaining to equilibrium solutions. Two of the complex problems were in mathematical contexts and for each of these, there was a corresponding problem in a non-mathematical context designed to be identical in terms of its solution and mathematical requirements. This written instrument, named the Differential Equations Concept Assessment (DECA) was administered to 91 participants drawn from three introductory differential equations courses. Of those participants, 13 were interviewed to provide detail for interpreting performance on DECA.

The data obtained from DECA and the interviews showed that participants performed significantly better on complex problems in non-mathematical contexts than on complex problems in mathematical contexts. There was a significant relationship found between performance on a problem in a mathematical context and performance on the isomorphic problem in the context of population growth, but a significant relationship was not found between a different pair of isomorphic problems, one in a mathematical context and the other in the context of learning. However, for all the complex problems, participants illustrated a preference for algebraic rather than geometric methods, even when a geometric approach was a more efficient method of solution. Although performance on simple problems was not found to be a strong predictor of performance on complex problems, the simple problems proved to elicit difficulties participants had with aspects of slope fields and equilibrium solutions. For example, participants were found to overgeneralize the notion of equilibrium solution as being any straight line and as existing at all values where a differential equation equals zero. Participants were also found to identify slope fields as determining only equilibrium solutions.

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RATIONAL AND WHOLE NUMBERS

TEACHERS' IMPLEMENTATION OF STANDARDS-BASED ELEMENTARY WHOLE NUMBER LESSONS[1]

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Recent calls for research investigating the impact of the Principles and Standards for School Mathematics have indicated a need to better understand the relationship between teachers and Standards-based curricula. This study examines teachers' implementation of a Standards-based lesson strand in terms of the level of fidelity to two curricular forms: the literal and the intended curricula. Results indicate that level of fidelity to the literal curriculum is not indicative of the level of fidelity to the intended and vice versa. It is also shown that level of fidelity to the intended curriculum varies within sites despite professional and contextual similarities.

Three years after the release of the NCTM *Principles and Standards for School Mathematics* [*Standards*] (NCTM, 2000), the Board of Directors of NCTM organized a conference focused on the impact of the *Standards*. One outcome of this conference was a call for research that examines the role and influence of the *Standards* on K-12 mathematics education, including research on the interaction between teachers and instructional materials. Coinciding with the NCTM Research Catalyst Conference was the initiation of a multifaceted investigation of an NSF-funded, comprehensive *Standards*-based[2] elementary curriculum, *Math Trailblazers*. One component of this large-scale investigation was the Whole Number Study, a study aimed at documenting teachers' implementation of whole number lessons, student learning of whole number concepts, and the relationship between the two. In this paper, we share the design of the Whole Number Study and then focus in on a particular data set – the videotaped classroom observations - presenting both our analytical approach and findings on teachers' implementation of *Math Trailblazers* whole number lessons.

Perspectives on Curricula

Curricula can be thought of as having many forms or levels. Cuban (1992), for instance, distinguishes between the intended, the taught, and the learned curriculum. From his perspective the intended curriculum "is written" and includes "that body of content contained in state frameworks, district courses of study, listings of courses taught in a school and syllabi" (Cuban, 1992). The taught curriculum is "what teachers do (lecture, ask questions, listen, organize classes into groups, etc.) and use (chalk, texts, worksheets, machines, etc.) to present content, ideas, skills, and attitudes" (Cuban, 1992). The Instructional Materials and Curriculum Working Group also described various forms or levels, making distinctions between the ideal, intended, enacted, assessed, and achieved curriculum (Reys & Roseman, 2003). Here the idea of a taught curriculum is replaced with the more holistic notion of an enacted curriculum. "While teachers rely heavily on textbooks and district curriculum guides, they also often make major alterations to textbook lessons resulting in an *enacted curriculum* that looks very different from the intended curriculum. … the enacted curriculum represents the opportunities students have to study and learn specific areas of mathematics (Reys & Roseman, 2003, p. 134). The notion of an enacted curriculum is more holistic than Cuban's taught curriculum in the sense that it goes beyond a list of actions and materials and considers how teachers and students interact with curricula when lessons are brought to life in classrooms.

Alternatives to Cuban's definition of the intended curriculum have also been proposed. For example, Reys and Roseman (2003) state: "To be effective, general curriculum standards must be translated into specific grade level learning expectations and materials that guide the day-to-day decisions of teachers and help them focus on the important mathematical learning goals in significant ways. Teachers and publishers utilize these expectations and materials to build lessons to implement the *intended curriculum*" (p. 134). Viewed in this way the intended curriculum is more than a written curriculum or a set of standards, for it encompasses the expectations that guided the curriculum's development.

In this paper, we examine teachers' use of three curricular forms: literal, enacted and intended. We use the term *literal curriculum* to refer to the instructional materials provided to teachers, i.e., textbooks, workbooks, implementation guides, software, etc. We view the enacted curriculum as comprised of the "opportunities to learn" mathematics that arise as teachers and students engage in lessons (Reys & Roseman, 2003). We take the distinction between Cuban's and Reys and Roseman's definitions one step farther and define the *intended curriculum* as the curriculum discerned from applying the stated philosophical approach to the mathematical content as articulated in the instructional materials. This applying, we posit, fosters an image of the potential opportunities to learn one can create by engaging in the curricular activities in specific ways.

Research Questions and Data Collection

The purpose of the Whole Number Study was three-fold. First, we hoped to gain a deeper understanding of how a *Standards*-based lesson strand is used in schools. Second, we wished to explore the relationship between implementation and students' learning of a key content area. Third, we aimed to produce research-based recommendations that would inform revisions to the curriculum. To achieve these goals, we asked: Which components of the curriculum do teachers use, omit, modify, or supplement; which factors influence teachers' use of whole number lessons; to what extent are students developing the whole number concepts and operations that are the foci of the whole number strand; and, how are whole number lessons implemented in classrooms? To address these questions, teachers completed level-of-use surveys, Lesson Reviews, and extensive interviews. Also, researchers interviewed 2-3 students and videotaped 2-3 lessons per participating classroom. Data collection for grades K-2 primarily took place during the 2003-04 academic year with a total of 19 classrooms. Data collection for grades 3-5 primarily took place during the 2004-05 academic year with a total of 20 classrooms. We recruited 1-2 classrooms per grade level per site. Due to space limitations, we restrict our discussion to the grade 1-2 classroom observation data. These classrooms were recruited from 9 schools with varying demographics (SES, size, and location) and differences in terms of length of use of the curriculum.

Classroom Observation Analytic Approach

Initial attempts to analyze the observation data employed the Horizon Research, Inc. Protocol (HRI, 2000) for evaluating reform classrooms and then the Reformed Teaching Observation Protocol **(**Pilburn, et al., 2000**).** However, we found that analyzing the implementation of whole number lessons required a more detailed protocol in terms of teachers' use and enactment of whole number curricular materials. Thus, we developed a two-component protocol for analyzing classroom observations.

The first component of the protocol involves comparing the videotaped classroom observation to the *literal lesson*: the instructional materials provided to teachers for the lesson, in particular, the explicit written suggestions and recommendations. We created a description of the literal lesson by outlining the recommendations and suggestions in the instructional materials and then coded each recommendation and suggeston as either *implemented*, *partially implemented*, or *not implemented*. Having compared the observation to the literal lesson, we then produced a summary rating of the extent (high, moderate, low) to which the literal lesson was implemented in the classroom. This rating - *the level of fidelity to the literal lesson* - reflects the level of alignment between the literal lesson and the classroom observation.

The second component of the protocol involves comparing the intended and enacted lesson. In order to identify the intended curriculum, we generated lists of the specific "opportunities to learn" described within the whole number strand instructional materials.[3] The perspective taken when identifying opportunities to learn was that these are the opportunities teachers and students can create by engaging in the lessons in specific ways. The identified opportunities were then grouped into two categories: *opportunities to reason* and *opportunities to communicate*. Opportunities to reason included students' opportunities to: (A1) reason to solve problems, reason about mathematical concepts; (A2) use or apply concepts, strategies or operations, or to refine strategies so that they become more efficient; (A3) select from multiple tools, representations or strategies; (A4) compare and make connections across tools, representations, or strategies; and (A5) validate strategies or solutions, reason from errors, or inquire into the reasonableness of a solution. Opportunities to communicate included students' opportunities to: (B1) describe ways of reasoning about tools, representations, strategies, operations or communicate mathematical ideas; (B2) interpret another student's ways of reasoning about tools, representations, strategies or operations; (B3) clarify or justify reasoning or explanations; and (B4) characterize mathematical operations. Within the two categories, we defined each code. Due to space limitations, however, we will only provide the definition for A3.

> A3: *Select from multiple tools, representations, or strategies*: Situations in which students may consider a variety of tools, representational approaches, or strategies in an effort to make appropriate choices based on the problem context. This code includes situations in which students may spontaneously select and include tools, representations, or strategies while problem solving.

Following the classification of specific opportunities, we identified the intended and the enacted lesson for each observation. The *intended lesson* is defined as the potential "opportunities to learn" that are expected to occur when the stated philosophical approach is applied to the mathematical content as articulated in the lesson. The *enacted lesson* is defined as the opportunities to learn mathematics that occurred as students and teachers engaged in the lesson. To identify the enacted lesson we reviewed videotapes and transcripts of the classroom observations. The transcripts were coded in terms of the opportunities to learn that arose, arose in a limited manner or were missed and then listed in an enactment record. We then compared the intended lesson to the enactment record to determine the extent to which the intended lesson was implemented in the classroom. This rating - *the level of fidelity to the intended lesson* – reflects the level of alignment between the enacted lesson (the observed opportunities to learn) and the intended lesson (the potential opportunities to learn).

Results

In this paper we focus on two findings that emerged from our investigation of teachers' implementation of *Standards*-based whole number lessons. The first concerns variability among teachers within sites. The second concerns variability between the two levels of fidelity (LOF) – LOF to the intended lesson and LOF to the literal lesson. With respect to variability among teachers within sites, our data indicate that teachers may implement curricula with varying degrees of fidelity despite having similar classroom settings, professional development, and levels of experience teaching the curriculum. In regard to variability between the two levels of fidelity, we found that an observation's level of fidelity to the literal lesson was not indicative of the level of fidelity to the intended lesson, and vice-versa. Below, we discuss and provide examples to illustrate our findings.

Variability among teachers within sites, our first finding, was most evident among the new-user cohort at School 7, an elementary school located in a mid-sized city. At the time of data collection, School 7 had a predominately-white, middle-class student population of over 500 students. The cohort of grade 1-2 teachers from School 7 had jointly participated in professional develop during the summer prior to the year when data was collected and teachers began using the curriculum for the first time. Several teachers in this cohort reported in their exit interviews that teachers had been asked by the district to "follow the textbook" for a year before attempting to alter the curriculum. That the cohort of grade 1-2 teachers took this request seriously is evident from the LOF to the literal lesson ratings for their observations, with the grade 1-2 observations consistently indicating a high or moderate LOF to the literal lesson. Thus, the cohort consisted of teachers who had similar classroom settings, professional development, levels of experience teaching the curriculum and whose observations indicated a high or moderate LOF to the literal lesson. The enactment records for these observations, however, indicate that teachers at this site enacted the observed lessons with varying LOF to the intended lesson. Variability in the LOF to the intended lesson indicates variability in the degree to which opportunities to learn are created when the lesson is enacted in classrooms.

Two grade 1 teachers at School 7, for example, showed markedly different enactments of the lesson *Arrow Dynamics*. *Arrow Dynamics* is one in a series of lessons in which students: (a) use the 100 Chart to identify numbers and explore relationships between numbers; and (b) write addition and subtraction number sentences in various contexts. A 100 Chart is a ten-by-ten grid with the numbers 1 to 100 sequentially listed (Figure 1).

31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60

Figure 1. Three rows of a 100 Chart

During the lesson, students play a game using a 100 Chart and a spinner (Figure 2) that can indicate one of four options: ↑ (-10), ↓ (+10), → (+1), ← (-1). Students put their markers on the number 45 and then, in pairs, take turns spinning and then moving on the 100 Chart until one player reaches 100. After each student completes a turn, the student writes a number sentence to represent the move.

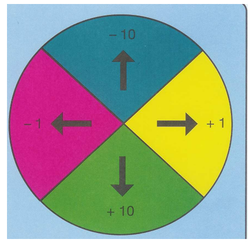

Figure 2. Arrow Dynamics Spinner

Teacher 113's enactment record for Arrow Dynamics indicates that the lesson was implemented with a low LOF to the intended lesson. A low LOF to the intended lesson indicates that opportunities to learn were consistently missed or limited during the lesson. For example, opportunities to reason to solve problems (A1) were either missed or arose in a limited manner in all but one instance during the observation. Analysis of the rationale for the A1 codes indicates that when interacting with students the teacher repeatedly directed students on where to place their markers and stated the appropriate number sentence. Below, we provide an excerpt of a typical interaction between the teacher and a student during the lesson.[4] This way of interacting with students, depending on the context, resulted in a series of either missed or limited opportunities for students to reason about the appropriate placement of their marker for the given spin and the numbers and operations to include when representing their move with a number sentence.

Teacher: Okay. This should say sixty-three. Erase fifty-five. Right here. [Jeff erases error and fixes it as teacher helps.] Now put in sixty-three. Now was it a plus ten? Alright. So you're at sixty-three, move it down ten more. Right here. What would it be?

Jeff: Seventy-three

Teacher: Seventy-three. So sixty-three plus ten makes seventy-three. [Jeff writes on record sheet.] So you need to write seventy-three here. Erase… no, no no. Right here. Right here. The sixty-three plus ten is seventy-three right? Seventy-three. Good. [Jeff nods.] Now, you take that big number, and you put it here. Let's just erase all this, 'cause this is all wrong. [Teacher erases Jeff's page.] Okay. Now, let's spin it. Alright, let's spin it. See what you get. [Teacher puts the spinner back on the page.] Let's go back and see if your… [Jeff spins spinner.] What do you have to do? [Student moves his marker.]

Jeff: Minus ten.

Teacher: Minus ten, so write that down. Take ten away from seventy-three.

We also observed trends in the opportunity to communicate code rationale, which indicate a general scarcity of opportunities for students to express their interpretations of moves and operations indicated on the spinner or their understandings of a spin in terms of actions on the 100 Chart. Thus, the low LOF to the intended lesson rating was the result of a series of auxiliary instructions and specific classroom ways of acting that limited or removed students' opportunities to engage in the mathematics the lesson was intended to address.

Teacher 112's enactment record for *Arrow Dynamics* indicates that the lesson was implemented with a high LOF to the intended lesson. A high LOF to the intended lesson generally indicates that appropriate opportunities to learn consistently arose during the lesson. We say "appropriate" because, for example, one would not expect to observe opportunities to select representations, tools, or strategies (A3) in a lesson that calls for specific representations, tools, or strategies. Analysis of the rationale for the A1 codes indicates that opportunities to reason to solve problems consistently arose in Teacher 112's classroom.

Allen: [Spins the spinner] Oh, minus ten. [He begins to move the green counter.]
Teacher: Okay. Which one are you?
Sophia: Forty-five
Allen: [He moves to forty-five.] Oh, darn it. That was where we started.
Teacher: So what are you going to write? Allen, let me hear you thinking.
Allen, let me hear your thinking. Would you talk out loud so I can hear you? What do you have?
Allen: Fifty-five
Sophia: Minus ten
Teacher: No, that's your thinking Sophia. I want to hear Allen.
Allen: Minus ten… equals… forty-five. [Allen records number sentence as he talks.]

Students in Teacher 112's classroom also had several opportunities to interpret another students' ways of reasoning (B2), whereas not one instance of a B2 code was identified in Teacher 113's implementation. Thus, interactions among Teacher 112's students and between the teacher and students provided opportunities for students to reason through open-ended questions and mathematical operations, to generate number sentences that represent moves on the 100 Chart, and to communicate their thinking, i.e., students had multiple opportunities to engage in the mathematics the lesson was intended to address. As these examples demonstrate, Teacher 112's enactment of the lesson *Arrow Dynamics* was markedly different from Teacher 113's enactment.

School 7 was one of several sites for which similarities across factors, such as classroom setting, professional development, and level of experience teaching with the curriculum, were not indicative of similarities in the LOF of the intended curriculum. Thus, the classroom observation data supports the claim that variation in the LOF to the intended curriculum will occur at a classroom level despite similarities across factors, such as those listed above.

Variability between the two identified levels of fidelity for an observation, our second finding, refers to general trends in the fidelity ratings. These trends indicate that LOF to the literal curriculum is not indicative of LOF to the intended curriculum, and vice versa. This finding is demonstrated below with the Fidelity Table for grade 1 (Figure 2).[5]

Literal Lesson Rating

Intended Lesson Rating

LOF	Low	Moderate	High
Low	118	113	113
Moderate		117	112, 118, 117
High		100	112, 112, 114,114

Figure 2. Grade 1 Fidelity Table

Consider, for example, the lessons with a high LOF to the literal lesson. In this set, high LOF to the literal lesson was not indicative of a high LOF to the intended lesson, for observations were rated as having high, moderate and low LOF to the intended lesson. Recall, the LOF to the literal lesson indicates the level of alignment between the classroom observation and the literal lesson: the instructional materials provided to teachers for the lesson, in particular, the explicit written suggestions and recommendations. Thus, a high LOF to the literal lesson indicates that the lesson guide recommendations and suggestions were followed. Variation in the LOF to the intended lesson, however, suggests that even when the recommendation and suggestions are followed, teachers may struggle to foster the intended opportunities to learn.

Thought of from a different perspective, a low LOF to the intended lesson is not indicative of a low LOF to the literal lesson, for observations with a low LOF to the intended lesson were rated as having high, moderate and low LOF to the literal lesson. Recall that the LOF to the intended lesson indicates the level of alignment between the intended lesson (the potential opportunities to learn) and the enacted lesson (the observed opportunities to learn). A lesson with a low LOF to the intended lesson is a lesson where the enactment record indicated that opportunities to learn arose in a limited way or failed to arise throughout much of the lesson. Thus, the observations with a low LOF to the intended lesson demonstrate, in another way, that the degree to which the intentions of the curriculum are realized may not be indicative of the degree to which the teacher implemented the various recommendations and suggestions in the lesson.

Concluding Remarks

Our findings support Schoenfeld's claim that " … data gathering, coding, and analysis must try to indicate the character of the implementation and its fidelity to intended practice" (Schoendfeld, 2006, p. 17). Had our analysis of teachers' implementation of whole number lessons been limited to teachers' implementation of the literal curriculum, we would have lost critical information regarding how an implementation may shift, add to or limit the intended curriculum. As researchers examine and evaluate *Standards*-based curricula the limitations of simple measures of curriculum implementation must be taken into consideration.

Endnotes

1. This material is based upon work supported by the National Science Foundation under Grant No. 0242704.

2. "Standards-based" refers to "curriculum materials developed in response to the NCTM's Standards documents (e.g., NCTM 1989)" (Remillard & Bryans, 2004).

3. In this paper, the phrase "opportunities to learn" refers specifically to opportunities for students to learn as opposed to opportunities for teachers to learn.

4. All names are pseudonyms.

5. Multiple listings of a teacher's numeric code indicate multiple classroom observations.

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SIXTH GRADERS' CONSTRUCTION OF QUANTITATIVE REASONING AS A FOUNDATION FOR ALGEBRAIC REASONING

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In a year-long constructivist teaching experiment with four 6th grade students, their quantitative reasoning with fractions was found to form an important basis for their construction of algebraic reasoning. Two of the four students constructed anticipatory schemes for solving problems that could be solved by an equation such as ax = b. In doing so, these students operated on the structure of their schemes. In solving similar problems, the other two students could not foresee the results of their schemes in thought—they had to carry out activity and then check afterwards to determine whether their activity had solved the problems. Operating on the structure of one's schemes is argued to be fundamentally algebraic, and a hypothesis is proposed that algebraic reasoning can be constructed as a reorganization of quantitative operations students use to construct fractional schemes.

In this paper, I demonstrate how 6th grade students' quantitative reasoning with fractions can form an important basis for their construction of multiplicative algebraic reasoning. In particular, during a year-long constructivist teaching experiment with four 6th grade students, the coordination of fractional schemes and whole-number multiplying schemes was seen as pivotal in the students' work toward solving basic linear equations of the form ax = b.

Since ax = b is essentially a statement of division, considering its construction and solution requires understanding how students produce division, which entails understanding students' multiplying schemes and multiplicative reasoning. Furthermore, any statement of division inherently involves reasoning with fractions: While fractions may be implicit or disguised in solving equations like 4x = 28, they soon become explicit in solving equations like 3x = 7. So in contrast to recent studies on elementary school students' early algebraic reasoning that are largely additive in nature (e.g., Blanton & Kaput, 2005; Carpenter, Franke, & Levi, 2003; Carraher, Schliemann, Brizuela, & Earnest, 2006), I was specifically interested in investigating students' construction of multiplicative algebraic reasoning.

Two of the four students in the teaching experiment constructed anticipatory schemes for solving problems that could be solved by an equation such as ax = b. The anticipatory nature of these schemes meant that the students did not have to carry out activity in a computer microworld and then check afterwards to determine whether their activity had solved the problem. Instead, these two students could foresee, in thought, aspects of the results of implementing their schemes (1). In doing so, these two students operated on the structure of their schemes. The larger purposes of this paper are (1) to argue that operating on the structure of one's schemes is fundamentally algebraic and (2) to propose a hypothesis that algebraic reasoning can be constructed as a reorganization of quantitative operations students use to construct fractional schemes.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Quantitative and Algebraic Reasoning

Fractions as Quantities

I use the phrase quantitative reasoning with fractions to refer to the purposeful functioning of a person's fractional schemes and operations in the context of quantities and quantitative relationships. In my study, this approach to fractions meant that problem situations often involved quantities whose values were fractional amounts identified with standard units of measurement, such as 3/4 of a yard of ribbon. More generally, fractions were measures of quantities (often lengths) made in relation to an identified (but non-standard) unit quantity. So, conceiving of 4/5, for example, meant being able to draw 4/5 of a previously identified unit length. Taking 1/3 of 4/5 of a unit length invariably meant being able to draw this amount and determine the resulting length in relation to the unit length. This approach to fractions was facilitated by the use of JavaBars (Biddlecomb & Olive, 2000), a computer program in which students can draw rectangles (bars) of various dimensions and operate on those bars by partitioning them into parts, further partitioning parts, disembedding parts, iterating parts, etc.

Early Multiplicative Algebraic Reasoning

In my study, I did not want to take the equation ax = b as a given. Instead, I wanted to understand more about what was required for students to generate that equation out of their reasoning and what schemes students might construct to solve it, even in the case where a and b are fractions. This focus meant that I was interested in how students operated on both known and unknown quantities that stand in multiplicative relationship to each other: For example, if 3/5 of a length is 7 inches, how do students come to know that 3/5 multiplied by the unknown length is a length exactly 7 inches long? How do they come to know that 5/3 of the 7 inches produces the unknown length? From this point of view, in generating the equation ax = b, operating on both knowns and unknowns is involved. At the very least, conceiving of (3/5)x requires conceiving of 3/5 operating multiplicatively on an unknown quantity represented by x.

Some researchers have proposed explanations for students' difficulties in operating on unknowns when solving linear equations (e.g., Filloy & Rojano, 1989; Herscovics & Linchevski, 1994), while others have contested such difficulties (e.g., Brizuela & Schliemann, 2004). Often researchers have not considered a operating multiplicatively on x as "operating on the unknown," opting instead to examine students' solutions of more complex equations with "unknowns on both sides," such as ax + b = cx + d. The reasons for exploring such equations have varied but sometimes have resulted in students' solutions of ax = b being dismissed as not yet algebraic—as merely involving the reversal of arithmetic operations in order to be solved. Yet students in Filloy and Rojano's study did not correctly solve equations of the form ax = b in the case of 102x = 51, which indicates that reversing one's thinking to solve ax = b is not trivial and can be considered a component of early algebraic reasoning (cf. Sfard and Linchevski, 1994).

Abstracting Conceptual Structures

More broadly, many researchers characterize algebraic reasoning as generalizing mathematical activity into structural ways of thinking (e.g., Carpenter, et al., 2003; Sfard & Linchevski, 1994). Consistent with this view, in my study a central distinction between students' quantitative and algebraic reasoning was the extent to which students had abstracted a conceptual structure. A concept, according to von Glasersfeld, is "a kind of place-holder or variable for some of the properties in the sensory complex we have abstracted from our experiences of particular things" (1991, p. 49), where the place-holder is often linguistic.

By abstracting a mathematical conceptual structure, I mean abstracting a “program of operations” from the experiences of using particular schemes that includes an awareness of how the schemes are composed (their structure) and an ability to operate with this awareness. So attributing a conceptual structure to a student means inferring the student is operating on or with the structure of his schemes. I use this notion of abstracting conceptual structures to account for differences in the schemes constructed by the four students in the study. In particular, students’ multiplicative structures were central explanatory constructs. I conceive of students’ multiplicative structure as the units coordinations that they have abstracted from the activity and results of their multiplying schemes and can take as given. Taking a units coordination as given means a student can project the coordination into a situation and operate further with it. Such units coordinations are said to be interiorized.

Methods of Inquiry

To investigate the purposes I have described, I conducted a year-long constructivist teaching experiment (Steffe & Thompson, 2000) in which I taught two pairs of sixth-grade students at a rural middle school in north Georgia from October 30, 2003 to May 12, 2004. The four students were invited to participate after individual selection interviews conducted during September and October of 2003. In the interviews I used fraction tasks to select students who were reasoning multiplicatively (see Hackenberg, 2005, for full details).

The pairs and I met twice weekly in 30-minute episodes for two to three weeks, followed by a week off. Most sessions included the use of JavaBars, and all sessions were videotaped with two cameras for on-going and retrospective analysis. One camera captured the interaction between the pair of students and myself, and the other camera recorded the students’ written or computer work. Two witness-researchers were present at all sessions to assist in videotaping and to provide other perspectives during all three phases of the experiment: the actual teaching episodes, the on-going analysis between episodes during the experiment, and retrospective analysis of the videotapes.

In retrospective analysis, I aimed to construct second-order models (Steffe & Thompson, 2000) of the students’ ways and means of operating and changes in those ways of operating. A second-order model is a researcher’s constellation of constructs formulated to describe and account for another person’s activity. In my study, I used scheme theory (Piaget, 1968; von Glasersfeld, 1995) as a central tool toward this end, and thus I viewed mathematical learning as a process in which a person makes accommodations in her schemes and operations in response to perturbations (disturbances) brought about by her current schemes and operations in on-going interaction within her experiential reality.

Data Excerpts

All four students coordinated their fractional schemes with their whole-number multiplying schemes, at least to some extent. However, only two of the four students, Michael and Deborah, embedded their whole number multiplying schemes into their fractional schemes to construct powerful anticipatory schemes for solving problems that can be solved by basic linear equations of the form $ax = b$. I call such problems reversible multiplicative reasoning (RMR) problems.

For example, on February 18 Michael solved the following RMR problem by using his whole number multiplying scheme in service of his reversible fractional scheme:

The Candy Bar Problem. This collection of 7 inch-long candy bars is 3/5 of another collection. Could you make the other collection of bars and find its total length?(2)

Michael quickly formed a goal to divide 7 inches into three equal parts. However, he had no immediate way of operating to achieve that goal—seven seemed to be a perturbing element. His activity in some previous episodes in January, in which he solved problems such as making a 2/2-bar into a 3/3-bar without erasing the half mark,(3) was an important basis for his elimination of this perturbation. That is, Michael partitioned each of the 1-inch bars into three equal parts and determined he had 21 parts, which he knew could be divided by 3. He then announced, prior to carrying out further activity, "I know how many there are now." He made the new collection by making five of the three equal parts of the 7 inches.

In this solution, Michael used the activity of his multiplying scheme to convert the composite unit of 7 inches into a composite unit consisting of number of parts (21) that he could split into three equal parts. I infer that he could operate in this way because he could view the 7 inches as a unit of seven units, into each of which he could insert more units (three units) to produce a number of units (21) that he could reorganize, in thought, as a unit of three units each containing seven units. So Michael inserted the coordination of units (the activity of his multiplying scheme) into the activity of his reversible fractional scheme—thereby operating with the structure of his multiplying scheme on the structure of this other scheme. This way of operating was novel for Michael, and during the rest of the study he used this new scheme, a reversible multiplying scheme with fractions, to solve other RMR problems like the Candy Bar Problem.

Problems like the Candy Bar Problem proved to be quite a challenge for Michael's partner Carlos. As a result, by mid-March I began to pose "basic" RMR problems where both the known quantity and the quantitative relationship were whole numbers:

Basic RMR Problem. That 2-foot candy bar is three times longer than your candy bar. Make your candy bar and tell how long it is. (No erasing the foot-mark on the bar.)

Carlos struggled to solve this problem over two teaching episodes, and even after several tries he wanted to erase the foot-mark and then split the unmarked bar into three equal parts. However, on his third try on March 29, he partitioned each of the two feet into three equal parts for a total of six parts, and pulled away two of the six pieces, explaining "I multiplied two by three and got six." So like Michael, Carlos used the activity of his multiplying scheme to convert a composite unit he could not split into three parts (the 2-unit bar) into a composite unit that he could split (the 6-unit bar). Yet because he had seen his partner operate in similar ways many times before, and because he may have been aiming for a particular visual image (he had created a bar that was 1/3 of 2 feet prior to this last solution), it is difficult to judge to what extent this was a significant or permanent modification for him.

Carlos did use this way of operating in subsequent episodes, at least when the goal was to make a 2-part bar into three equal parts. However, when working with bigger numbers, such as making a 5-part bar into four equal parts, his tendency was to partition each part of the 5-part bar into some number of small parts, select a number of those small parts that he estimated would constitute 1/4 of the whole bar, and iterate that selection four times to check. If he was not successful, he would adjust his estimate and try again. So he did not appear to make a modification in his activity at the same level of generality as his partner's modification. My main explanation for this difference between the boys was that Carlos had not yet interiorized the coordination of three levels of units (Hackenberg, 2005). That is, Carlos could use his multiplying scheme to determine that a 5-part bar with each part partitioned into 4 equal parts would produce a 20-part bar. But he could not take that structure as a given while mentally reorganizing the 20-part bar into a different number of units of units—i.e., as a unit of four units each containing five. I call his scheme for solving basic RMR problems an enactive reversible multiplying scheme (without fractions).

On March 29, Bridget also demonstrated that she could solve basic RMR problems by solving this problem:

Chocolate Bar Problem. Four yards of chocolate is three times what Phillip has. Can you make Phillip's chocolate bar? How long is his bar?

Bridget partitioned each of the four yards into three equal parts, disembedded one of those parts, and iterated it to produce a 4-part bar. In explanation she said, "you can't divide three into four, so you go, three times four is twelve. And twelve divided by three is four. You have to divide it by three because you said three times."

To test out the nature of Bridget's scheme for solving basic RMR problems, I posed a series of them in the next teaching episode on March 31. Bridget began to solve them by partitioning each unit of the known quantity into the number of parts given in the relationship between known and unknown. However, then she said things like "Hmm. I don't know what else to do," and "Why am I doing that?" Although she generally completed solutions of the problems, she did not appear to be certain about her activity unless it was confirmed in some way by me or by her partner Deborah (who seemed certain). During the episode I challenged Deborah to solve a basic RMR problem without enacting the physical coordinations in the microworld, and she did. When I then posed another such problem to Bridget, she exclaimed "I can't do that in my head!" Thus I can conclude that like Carlos, Bridget had constructed an enactive reversible multiplying scheme to solve basic RMR problems. She needed to carry out the activity of the scheme in the microworld in order to solve these problems, and thus a three-levels-of-units structure was something she could make in activity. Like Carlos, she did not seem to be able to take such a structure as given so as to reorganize the known quantity into a different units-of-units structure, in order to anticipate the result of the scheme.

Meanwhile, Deborah began to construct an anticipatory scheme for solving the most complex type of RMR problem she encountered. On May 12, using the context of a homemade racecar contest between two teams, I posed this problem to Deborah:

Race Car Problem. The Lizards' car goes 1/2 of a meter. That's 3/4 of how far the Cobras' car went. Can you make how far the Cobras' car went and tell how far it went?

Deborah partitioned her 1/2-meter bar into three equal parts, disembedded one of the parts, and iterated it to produce a 4-part bar which she called 4/6 of a meter. When I asked her what she had to take times the Lizards' distance to make the Cobras' distance, she promptly said, "three-fourths—I mean four-thirds." In explanation, she pointed to the Lizards' distance and stated that it was 3/4 of the Cobras' distance.

I then posed the same problem except the Lizards' car went 2/3 of a meter. Deborah partitioned each third of her 2/3-meter bar into six equal parts, disembedded one of the parts, and iterated it to produce a 16-part bar. In explanation, she said, "I knew each third is four pieces. So four times four, because you need four thirds for this one," pointing to the Cobras' car's distance. A short time later, she also referred to the 16-part bar as "four-fourths."

At this point in the teaching experiment, Deborah had constructed a reversible multiplying scheme with fractions similar to Michael's. However, as shown here, a characteristic of her solutions was her use of reciprocal relationships between the two quantities. In fact, she was the only student of the four to state these relationships swiftly and to use them in operating. Her initial justification for knowing that the Cobras' distance was 4/3 of the Lizard's distance indicates that she had constructed a multiplicative relationship that did not rely on reference to material parts. In other words, she did not explain that the Cobras' distance was 4/3 of the Lizards' distance because the Cobras' distance was four equal parts and the Lizard's three. Instead, she relied on the given multiplicative relationship, that the Lizards' distance was 3/4 of the Cobras' distance. Her explanation leads me to infer that she knew the following: If Bar A was 3/4 of Bar B, then Bar B was 4/3 of Bar A. When solving the second Race Car Problem and other variations, she used this kind of reasoning. Yet she did not compress her reasoning so much as to, for example, take 4/3 times the known distance to solve the Race Car Problems—i.e., she did not appear to view these problems as problems that could be solved by fraction multiplication.

Discussion

So, what was algebraic about Michael's and Deborah's ways of operating in contrast with their partners? Both Michael and Deborah operated on the structure of their reversible fractional schemes with the structure of their multiplying schemes. That is, they both seemed to view the known quantity as a unit of so many units where the goal was to divide it into some number of parts—the number being determined, in the case of a fractional relationship, by the number of units of fractional size that the known quantity was of the unknown quantity. Once they used the activity of their multiplying schemes to accomplish this goal, each part they had created was a unit fractional part of the unknown and could be used in iteration to make the unknown. This aspect of their scheme was one that they could foresee and one reason for calling their schemes anticipatory. In contrast, Carlos and Bridget needed to carry out activity materially and check the result in retrospect—the results were not "contained" in the implementation.

Thus anticipatory schemes have a general, structural quality to them. They seem general in the sense that they apply to a wide range of situations, and this generality is related to their structural nature. For example, to both Michael and Deborah I can attribute a conceptual structure for splitting any known whole-number quantity into any number of parts required to solve a particular RMR problem. In contrast, I cannot be sure how general Carlos's or Bridget's reversible multiplying schemes were, nor can I claim they were operating with an awareness of the structure of their schemes.

My main explanation for the difference between the constructions of anticipatory versus enactive schemes involves the students' multiplicative structures. Being able to take the coordination of three levels of units as given prior to activity meant that both Michael and Deborah could project two different units of units structures into the known quantity, and flexibly switch between them, while also keeping track of the "larger" unit structure between known and unknown. Not having interiorized three levels of units was a significant constraint in Carlos's and Bridget's activity. However, being able to coordinate three levels of units in activity seemed to facilitate their construction of enactive schemes.

Although powerful, this explanation does not account for the main difference between Michael and Deborah: the use of reciprocal relationships in solving RMR problems. My current explanation is that Deborah had abstracted fractions as programs of operations that included all the ways to make as well as "unmake" the fractions. So, for example, 1/4 meant not only to disembed one part from a bar partitioned into 4 parts; it also meant that 1/4 taken four times, or 4*1/4, would yield the original bar. Thus if one bar was 3/4 of another bar, it was 3 times 1/4 of the other bar, and the other bar was 4 times 1/4 of itself, which was 4 times 1/3 of the original bar. Abstracting fractional concepts seemed to allow her to flexibly switch between viewing either quantity as the unit to which the other quantity could be compared. The reason Deborah made this abstraction while Michael did not is not yet clear.

Of course, in one sense the students' activity was clearly not algebraic: No students operated on standard algebraic notation in lieu of carrying out their own mental operations. In addition, with the exception of Deborah's use of reciprocal relationships, the students did not explicitly operate on unknowns (or knowns) multiplicatively with fractions (they did do so with whole numbers). However, this research supports and extends the notion put forth by early algebra researchers that arithmetic (including "arithmetic" with fractions) has an algebraic character (Blanton & Kaput, 2005; Carpenter, et al., 2003; Carraher, et al., 2006).

In fact, although early algebra researchers have indicated that reconceiving of arithmetic can bring out its algebraic qualities, they have not yet proposed a reorganization hypothesis for algebraic reasoning similar to the hypothesis proposed by Steffe (2002) for students' construction of fractions. Steffe's hypothesis is that students' operations on discrete quantities are reorganized to produce operations on continuous quantities—that students use the operations they have used to construct whole numbers in the construction of fractions.

I propose that such a hypothesis might be fruitful for understanding how students can construct multiplicative algebraic reasoning out of their previous quantitative reasoning with fractions. The main question in proposing such a hypothesis is: What quantitative operations are reorganized? One possible response has its origins in the research presented here: If students use their interiorized coordinations of three levels of units to operate with multiple units structures at once, it could be that students further abstract these interiorized units coordinations so that they can "apply" not just to numerical units but to schemes themselves. That is, algebraic reasoning may involve taking schemes as units that can be inserted into other schemes in at least 2-level and perhaps 3-level structures. In turn, this units-coordinating activity likely requires abstracting this "mega-structure" of embedded schemes as a program of operations.

Conclusion

The usefulness of this research, and of the proposed reorganization hypothesis for algebraic reasoning, will be measured by the degree to which it helps researchers understand more about the nature and value of early algebraic activity for elementary and middle school students. Ultimately, research that investigates and refines this reorganization hypothesis will contribute to testing Kaput's (1998) proposal that algebraic activity integrated across K-12 schooling will lend coherence, depth, and power to K-12 students' mathematical educations.

Endnotes

1. Still, activity in a computer microworld often accompanied the implementation of their schemes.
2. In solving RMR problems, the students usually determined the length of the unknown. Due to space limitations in this paper, I will restrict myself to discussing only how they made the unknown quantity.
3. To solve this problem, Michael partitioned each of the halves into six equal parts

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ASPECTS OF MATHEMATICAL KNOWLEDGE FOR TEACHING FRACTION MULTIPLICATION

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I report on mathematical knowledge that two U.S. 6th-grade teachers used while teaching fraction multiplication for the first time with lengths and rectangular areas. Both teachers were using versions of the Bits and Pieces II unit from Connected Mathematics. Data came from videotaped lessons, teacher interviews, and student interviews. To explain where each teacher did, and did not, adapt to her students' explanations and drawn representations, I examined the unit structures that each teacher evidenced and the purposes for which they used drawn representations. The results highlight the importance for teachers of reasoning flexibly with three levels of units when responding to students' representations of fractional quantities.

Context and Objectives

Research on teachers' knowledge has expanded from studies of subject matter knowledge of various content areas to the organization of knowledge for teaching particular topics (e.g., Ball, 1991; Ball, Lubienski, & Mewborn, 2001; Borko & Putnam, 1996; Ma, 1999; Shulman, 1986). As part of this development, current discussions of teacher knowledge are often framed in terms of subject matter, pedagogical, and pedagogical content knowledge (e.g., Borko & Putnam, 1996). When introducing the notion of pedagogical content knowledge, Shulman (1986) emphasized knowledge of students' thinking about particular topics, typical difficulties that students have, and representations that make mathematical ideas accessible to students. Building on the notion of pedagogical content knowledge, Ball et al. (2001) emphasized the importance of examining how teachers use mathematical knowledge in the course of their work and argued that, with respect to research methods, this implies working backwards from practice to infer mathematical knowledge that supports both routine and non-routine aspects of practice. Describing the mathematical knowledge that teachers need to teach particular topics remains a central challenge for the field.

The present study examines instruction in two U.S. sixth-grade classrooms to uncover aspects of mathematical knowledge for teaching fraction multiplication with drawn representations. Both teachers were teaching fraction arithmetic with reform-oriented materials for the first time and were using drafts of the revised *Bits and Pieces II* unit that has since been published as part of *Connected Mathematics 2* (CMP; Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006). The CMP materials use lengths and rectangular areas as representations of fractional quantities and ask students to solve problems about situations in which fractions are embedded. I focus on the use of drawn representations in teaching not only because discussions of pedagogical content knowledge and mathematical knowledge for teaching refer to representations, but also because reform-oriented curricula in the United States often place new demands on teachers and students to interpret and reason with a variety of representations.

Background and Theoretical Framework

Research on teachers' knowledge has examined fraction division and decimal multiplication

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

more closely than fraction multiplication. Research on fraction division has reported that teachers can confuse situations that call for dividing by a fraction with ones that call for dividing by a whole number or multiplying by a fraction (Ball, 1990; Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Ma, 1999). Research on decimal multiplication has built upon the notion that people have intuitive models for arithmetic operations (Fischbein, Deri, Nello, & Marino, 1985) and that the model for multiplication is repeated addition. For instance, Graeber and colleagues (Graeber & Tirosh, 1988; Graeber, Tirosh, & Glover, 1989) administered to 129 preservice elementary teachers a written test consisting of word problems adapted from those used by Fischbein et al. They report that a higher percentage of the teachers solved multiplication word problems correctly when the multiplier was a whole number but often used division in problems that should have used a decimal less than one as the multiplier.

Conceptual units of various types have played a central role in research on children's understandings of rational numbers (Behr, Harel, Post, & Lesh, 1992; Olive & Steffe, 2001; Steffe 2001, 2003, 2004). The present study builds most directly on the research of Steffe and Olive, who have examined how elementary students solved tasks involving lengths and areas by coordinating two and three levels of units and by using disembedding, iterating, and partitioning operations. The most successful students constructed operations based on three levels of units. Steffe (2003, 2004) defined one such operation, *recursive partitioning*, to be taking a partition of a partition in the service of a non-partitioning goal. To illustrate, students might begin taking 1/3 of 1/4 by partitioning a unit into four pieces and then partitioning the first of those pieces into three further pieces. Determining the size of the resulting piece is a non-partitioning goal, and students could accomplish this in more than one way. Students might iterate the resulting piece and count to see that 12 copies fit in the original unit. This solution requires decomposing an initial unit into a unit of units (one unit containing 12 twelfths). Alternatively, students might recursively partition by subdividing each of the remaining fourths into three pieces. This solution involves decomposing an initial unit into a unit of units of units structure (one unit containing 4 fourths, each of which contains 3 twelfths). The first solution is based on *2 levels of units* and the second on *3 levels of units*.

The theoretical frame for the present study emerged from analyses of two teachers and builds on past research about conceptual units. The common parts-of-a-whole entry point into fractions emphasizes two levels of units. The extension of fractions from parts of wholes to parts of parts creates opportunities to establish three-level unit structures when relating parts of parts back to the original whole. Because the first solution to 1/3 of 1/4 discussed above illustrates that reasoning about parts of parts is not necessarily the same as reasoning with three levels of units, I will distinguish between the two throughout. To reason with three levels of units one must relate all three levels at once, not just two of the three levels at a time. The analyses below will demonstrate that *flexible* three-level structures are necessary for teachers if they are to adapt in response to the range of ways that students might assemble such structures.

In addition to levels of units, I also consider purposes for which teachers use drawn representations. One use is simply to *illustrate* solutions also arrived at using an alternate method, such as a numeric computation. A second use is to *infer* a computation method by determining solutions for various problems using drawn representations and then looking for numerical patterns (for fraction multiplication the pattern might be products of numerators are numerators of products and products of denominators are denominators of products). Each teacher emphasized one of these two purposes. A third use is to *adapt* how one represents structures of quantities in response to students' thinking by attending to the variety of ways that students might begin to assemble three-level unit structures as evidenced by their explanations and drawings. This would require, in turn, the ability to perceive and produce three-level unit structures in a variety of ways and to understand opportunities for determining general numeric methods afforded by different approaches. Neither teacher used drawn representations to adapt.

Methods and Data

Data for the present report come from a larger study of teaching and learning mathematics conducted by a team of researchers in a rural middle school in the Southeastern United States. The school adopted the CMP materials in the 2001-2002 school year and has a racially and economically diverse student body. Data for one teacher were collected in Spring 2003, and for a second teacher in Spring 2004. Both teachers began the transition to reform-oriented materials with limited professional support. The district hired a consultant to help teachers select units for the first year, and the research project provided further support starting in Spring 2003.

Members of the research team videotaped each teacher's lessons during the same class period every day for 4 to 5 weeks. Each afternoon we analyzed that morning's lesson for mathematical ideas, problem-solving strategies, and representations that, from our perspective, seemed central. We identified excerpts during which the teacher and students apparently had difficulty understanding one another and replayed such excerpts during student and teacher interviews.

I conducted weekly, semistructured interviews with pairs of students selected from the same classrooms to represent a cross-section of achievement (3 pairs in 2003 and 4 pairs in 2004). During the interviews, I had the students work on tasks similar to those in the lesson excerpts and asked questions to gain access to mathematical understandings that they used. Then, I had the students watch the lesson excerpts and asked them to comment on what they thought their teacher wanted them to learn. As the interviews progressed, I moved back and forth between tasks and excerpts to access ways that students used their understandings of the content to make sense of the lessons. Although central to the research design for the larger project, these interviews will not be in evidence during the analyses below.

I then planned with other members of the research team weekly teacher interviews that used the same lesson excerpts and related student interview excerpts as prompts. These researchers asked the teachers to summarize their preparation and enactment of the lessons, examine student work from the lessons and interviews, comment on what students understood and where they struggled, and discuss how they might address students' observed difficulties in future instruction. These interviews provided further access to understandings of the mathematics (including drawn representations) and of students that teachers used during the observed lessons.

Once the data were collected, I conducted further, more detailed analyses using a version of the constant comparative method described by Cobb and Whitenack (1996) for conducting longitudinal analyses of classroom videorecordings. These analyses used talk, gestures, and inscriptions as evidence for teachers' and students' understandings of the content and the lessons. I treated knowledge teachers evidenced in interviews as confirming evidence in cases where it appeared consistent with knowledge teachers evidenced in lessons. In cases where knowledge evidenced in interviews appeared inconsistent with knowledge evidenced in lessons, I reexamined both sets of data and tried to refine my interpretations to achieve a consistent account of what teachers said and did in both contexts. Finally, I examined the teacher's edition of the *Bits and Pieces II* unit to determine which mathematical ideas the materials emphasized and how they presented the role of drawn representations in the activities.

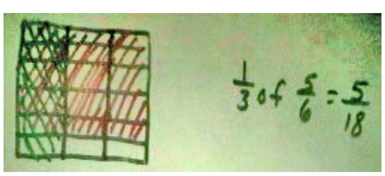
Results and Conclusions

Bits and Pieces II develops fraction arithmetic through problems about situations in which fractions are embedded. Many of the problem situations can be modeled using lengths or rectangular areas as representations of fractional quantities. The introduction to the teacher's edition states that *Bits and Pieces II* does not teach a preferred algorithm. Rather, the stated goals include developing ways to model sums, differences, products, and quotients using lengths and rectangular areas; looking for and generalizing patterns in numbers (consistent with the second purpose for drawn representations discussed above); and developing students' strategies into general algorithms (consistent with the third purpose). The main result I emphasize is that the unit structures the teachers produced shaped in fundamental ways the purposes for which they used drawn representations and the extent to which they adapted in response to their students.

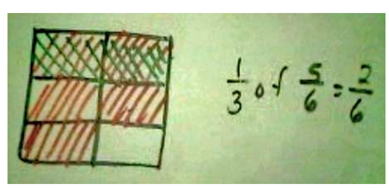
Ms. Reese

Prior to the present study, Ms. Reese (all names are pseudonyms) had taught algebra to seventh-, eighth-, and ninth-grade students for approximately ten years. She reported that her high school classes had focused on "traditional mathematics" and algorithms, had rarely used manipulatives, and had included drawn pictures only occasionally to introduce a topic or when there was confusion. Ms. Reese was in her second year of CMP implementation, but at the time of the study was teaching *Bits and Pieces II* for the first time.

Ms. Reese could produce three levels of units but struggled to adapt and respond to some students' representations of parts of parts. A central example occurred during her second day of instruction on fraction multiplication. Students were using squares to represent parts of parts of brownie pans. At one point, Ms. Reese worked with a student whose incorrect representation for 1/3 of 5/6 was similar to that shown in Figure 1b. The student explained that the answer was 2/6 because the whole was broken into three pairs of smaller rectangles. If, in producing her drawing, the student took the whole pan as a unit that was divided into thirds each of which was further divided into sixths, she would have produced three levels of units when arriving at her answer. Ms. Reese said that something was not right, questioned whether the student had shaded 5/6, and counted the shaded pieces to check. That Ms. Reese hesitated and then counted the five shaded pieces suggested that she was genuinely stuck. The student continued to explain when Ms. Reese interrupted, "Oh. I gotcha. OK. So you just didn't divide it up again. You just put like this is one third (pointed to two pieces), that's one third (pointed to another two pieces), and that's one third (pointed to the last two pieces). OK. That's fine." Apparently, Ms. Reese produced three levels of units, but she did not address the problem with the student's work.



(a)



(b)

Figure 1. (a) Ms. Reese's approach to 1/3 of 5/6. (b) Two students' approach.

Ms. Reese observed another student who had a similar incorrect solution and stopped the class so that she could compare two strategies for determining 1/3 of 5/6. Ms. Reese began by telling the class that it was alright to have different answers so long as they were equivalent fractions. She demonstrated her solution first (Figure 1a) and pointed out that she partitioned the brownie pan into sixths by drawing lines in just one direction (horizontally). She shaded five pieces and pointed out that she could not form three equal groups directly from those pieces (an instance of reasoning with three levels of units). She then partitioned her brownie pan vertically into thirds, shaded the first third of the whole, and reminded students that the bottom piece was not included. Thus, she focused on 1/3 of 5/6 but did not discuss the fact that she had taken 1/3 of *each* shaded sixth (a drawn instantiation of the distributive property). With some prompting about the size of the pieces, students said the answer was 5/18. Ms. Reese then recounted the alternate student solution, led the class to express the solution as 6/18, and emphasized that the two answers were truly different. Finally, she asked which of the two answers was consistent with the pattern emerging from previous problems (e.g., 1/2 of 2/3 is 2/6 and 3/4 of 1/2 is 3/8).

From this point forward, Ms Reese rejected solutions in which students cross-partitioned for just one fraction. For instance, later during the same lesson, she rejected a student's correct (from my perspective) drawing for 2/3 of 3/4 reproduced in Figure 2a, saying that the student would run into the same problem shown in Figure 1b. A moment later, she accepted another drawing reproduced in Figure 2b. Ms. Reese knew that difficulties could arise when cross-partitioning for the first fraction, but may not have focused on the underlying role of the distributive property. In an interview, Ms. Reese still hesitated to accept the approach shown in Figure 2a. Thus, she was not sufficiently flexible with three levels of units to consistently respond to her students' correct and incorrect representations of parts of parts or to *adapt* when using drawn representations.

In subsequent lessons, Ms. Reese and her students used lengths to solve three examples, 1/4 of 2/3, 1/3 of 1/2, and 2/5 of 1/2. In each case, Ms. Reese evidenced recursive partitioning, and hence further reasoning based on three levels of units. She also asked students if new solutions were consistent with the emerging pattern. Ms. Reese's ability to produce three levels of units supported her use of drawn representations to determine solutions to fraction multiplication problems and to *infer* a numeric pattern in which numerators were multiplied together and denominators were multiplied together, the second use of drawn representations discussed above.

(a) (b)

Figure 2. (a) Ms. Reese rejected this drawing for 2/3 of 3/4. (b) She accepted this drawing.

Ms. Archer

Ms. Archer was a first-year full-time teacher but had taught in the district as a long-term substitute, mostly in high school classrooms. She had a bachelor's degree in mathematics and, at the time of the study, was enrolled in a 2-year alternative certification program. In contrast to Ms. Reese, who used lengths and rectangular areas to determine products of fractions and inferred a computation procedure from a pattern in the results, Ms. Archer told students from the beginning that a fraction *times* a number and *of* the number meant the same thing. She almost always multiplied numerators together and denominators together before turning to drawn representations. Thus, she used drawn representations to *illustrate* solutions also arrived at through an alternate method, the first use of drawn representations discussed above.

Ms. Archer's troubles explaining drawn representations of fraction multiplication and responding to some students' questions appeared rooted in reasoning with just two levels of units. As one of several examples I found, Ms. Archer introduced the rectangular area model (interpreted as a pan of brownies) for fraction multiplication with the example 1/2 x 2/3. As shown in the teacher's edition, she first partitioned the unit square vertically into thirds and shaded two parts. She then partitioned the unit square horizontally into halves and shaded one part. In so doing, she described 2/3 as two of three equal parts of the whole pan (two levels of units) and 1/2 as one of two equal parts of the whole pan (also two levels of units). At this point she had drawn a two-by-three array with five parts shaded. She told students the answer could be found where the two fractions "overlapped" and that this demonstrated "why" 1/2 x 2/3 = 2/6. One student argued that the diagram showed the answer to be 5/6 because five of six pieces were shaded. Ms. Archer simply stated that the picture was correct and called on another student who said the fractions "mixed" in two of the six parts. Ms. Archer accepted the latter explanation but never addressed the former. When she moved on to the next part of the lesson, problems from the book asked students to show "part of the part in the brownie pan." These problems introduced the part-of-a-part language into the lessons and, as Ms. Archer worked with students, she appeared to be learning how to represent parts of parts alongside her students. For instance, she first rejected a representation of 3/4 of 1/2 in which a student represented the answer, correctly, as 6/16. The student had divided the unit square into a four-by-four array, shaded one half (eight parts), and then double shaded six parts. During an interview, Ms. Archer rejected the solution a second time but then accepted it after she reduced 6/16 to 3/8. With the additional level of units (sixteenths), she did not recognize quickly that the solution was correct.

The cases of Ms. Reese and Ms. Archer cast into relief aspects of mathematical knowledge for teaching fraction multiplication. In particular, examining places where both teachers were, and were not, able to engage and effectively respond to their students' reasoning revealed the central role that reasoning with nested levels of units can play. Reasoning with just two levels of units constrained Ms. Archer's explanations of drawn representations of fraction multiplication. Reasoning with three levels of units, Ms. Reese was able to solve fraction multiplication problems using lengths and areas as representations of fractional quantities, but she was not sufficiently flexible with three-level unit structures to consistently respond effectively to students' correct and incorrect solutions. The results demonstrate correlation between the unit structures that each teacher could produce and the purposes for which each used drawn representations. Ms. Archer's difficulties explaining 1/2 of 2/3 with the unit square suggested that she would be challenged to determine solutions to fraction multiplication problems using just lengths and rectangular areas as representations of fractional quantities. This, in turn, would preclude using drawn representations to infer or adapt. Ms. Reese could clearly use drawn representations to infer but not to adapt. I do not claim that nested levels of units and uses for drawn representations are a complete account of mathematical knowledge for teaching fraction multiplication, but I do conjecture that these tools will be useful in understanding mathematical knowledge that other teachers use when teaching with reform-oriented materials and, most importantly, places where they do, and do not, engage and build upon their students' thinking.

The research reported in this article was supported by the National Science Foundation under Grant No. REC-0231879. The opinions expressed in this paper are those of the author and do not necessarily reflect the views of NSF.

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INVENTED STRATEGIES FOR DIVISION OF FRACTIONS

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A study of computational strategies developed by sixth-grade students and prospective and practicing teachers identified two-step approaches for measurement and partitive division of fractions problems. These strategies, grounded in direct modeling and problem contexts, extend one-step strategies developed for division of whole numbers by adding a step which converts units by multiplying (unitizing). The order of the two steps, as well as the type of division performed, depends on problem type. The study also identified common computational errors, and observed the use of co-measure units in adapting strategies.

Introduction

Division of fractions is arguably the least understood of the arithmetic operations studied in school mathematics. Instruction often begins and ends with the invert-and-multiply algorithm traditional in the U.S., although even most adults are hard-pressed to explain why it works. Research has shown that children can develop their own computational strategies for operations on both whole numbers (e.g., Carpenter et al., 1998) and rational numbers (e.g., Warrington, 1997)—in fact, students who develop their own strategies before working up to traditional algorithms have been seen to make fewer errors in their later use of the traditional algorithms (Carpenter et al., 1998). However, the additional complexities involved in working with rational numbers often challenge student and teacher alike, with the result that students are typically presented with the traditional algorithm for division of fractions before having an opportunity to construct meaning for it themselves. This practice interferes with students' construction of meaning (e.g., Mack, 1990, Nagle, 1999), so that their knowledge of dividing fractions tends to be disconnected from their understanding, as well as "buggy" (making procedural errors such as inverting the wrong fraction before multiplying).

Studies of children's invented strategies for division typically distinguish two types of problems: measurement division, also called repeated subtraction, in which the group size is known and the number of groups is the unknown; and partitive division or fair sharing, in which the number of groups is known but the size of each group is unknown. Studies of invented strategies for division of whole numbers showed that students tend to approach measurement division problems by making groups of the known size, but approach partitive division problems by distributing items from the total to each group, one or a few at a time (e.g., Ambrose et al., 2003). This distinction between problem types is equally important in division with fractions (e.g., Flores, 2002; Siebert, 2002). Measurement interpretations are much more common than partitive ones for division of fractions problems, principally because it is difficult to imagine a natural context in which the number of groups is not a whole number. Ott, Snook and Gibson (1991) found that both literature and textbooks tend to ignore the partitive division of fractions. Sharp and Adams (2002) described students' conceptualizations of context-free division of fractions problems as measurement interpretations, and all the context-situated problems given in their figures are measurement division problems. Additionally, Warrington (1997) showed that children can develop their own computational strategies for dividing fractions. One interesting observation that can be made from the examples she describes is that her students' explanations

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for (context-free) problems involving the division of a whole number by a proper fraction tended to be given in terms of a measurement interpretation, whereas their explanations for a problem involving the division of a proper fraction by a whole number were given in partitive terms. It is therefore important to consider both types of division problems when examining strategies for division of fractions. This study examined the computational strategies developed for both measurement and partitive division of fractions problems by sixth-grade students, prospective teachers and experienced teachers working (separately) on the same problems. The study also examined ways in which invented strategies for division of fractions paralleled those for division of whole numbers.

Methods

The study involved 22 sixth-graders, 24 prospective teachers enrolled in an undergraduate content course, and 26 practicing middle school teachers. The sixth-graders attended a Title I elementary school and were of various ethnicities; 9 participated in October, and the other 13 in late April of the same school year. The middle school teachers were from a different district which included sixth, seventh and eighth grades in middle school. In each case, subjects were asked to solve word problems involving division of fractions. They were provided with materials for modeling, and encouraged to work in small groups to solve the problems and to describe their solutions individually using concrete models, pictures, words and numbers. All sessions lasted approximately one hour. Some sixth-graders returned for a second session to complete their work.

Data collection included subjects' written work, photos of models constructed, observation of subjects while they worked and discussed their solutions, and in some cases interviews immediately following the session to clarify solution approaches when the available artifacts did not make them clear. Analysis focused on the first four problems (see Table 1) and involved description and then classification of approaches to each problem. The problems were chosen for the following properties: Questions 1 and 4 are measurement problems with the number of groups (the quotient) a noninteger greater than 1 and the divisor a proper nonunit fraction. Questions 2 and 3 are partitive problems, but in the former case the number of groups (the divisor) is less than 1, whereas in the latter it is greater than 1.

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1. You have 2 ½ oranges. If each student serving consists of ¾ oranges, how many student servings (or parts thereof) do you have?
2. You have 1 ½ oranges. If this is enough to make 3/5 of an adult serving, how many oranges constitute 1 adult serving?
3. Sarah is making posters by hand to advertise the school play, but the posters she has designed are not the same size as a standard sheet of paper. She has 3 ½ sheets of paper left, which is enough to make 2 ⅓ posters. How many sheets of paper does each poster use?
4. If Alberto is also making posters, but his posters only use ⅔ of a sheet of paper, how many of Alberto's posters will those 3 ½ sheets of paper make?

Table 1. Problems used in the analysis

Results

Strategies

In general, subjects used either a one-step or a two-step strategy particular to problem type. The numbers involved in a problem (in particular, whether the number of groups was less than 1, or the fractional part of the known group size involved a small enough denominator) influenced the number of steps in the strategy used to solve it.

A minority of the adults (43% of the prospective teachers and 13% of the practicing teachers), and none of the sixth-graders, applied a measurement division-of-whole-numbers strategy to Question 1 (q.v.), making as many ¾-oranges as possible (see Figure 1). Sharp and Adams (2002) describe one such strategy as an “early strategy”. All of the sixth-graders and the remainder of the adults used instead a two-step process (see Figures 2 and 3):

1. Cut the dividend into quarter-oranges.
2. Make groups of three quarter-oranges.

This can be expressed more generally as first multiplying by the denominator of the dividend, which converts units (here, from oranges to quarter-oranges), and second performing a measurement division of whole numbers (here, 10÷ 3). All solutions to Question 4 used this two-step strategy. This structure can also be seen in the strategies used by students to solve measurement division of fractions problems in existing literature such as Sharp, Garofalo and Adams (2002) and Perlwitz (2004).

Subjects also solved the partitive division problems in Questions 2 and 3 primarily by two-step methods occasioned by the fractional divisor (see Figure 3). This approach was:

1. Divide the oranges into three equal parts.
2. Obtain five such parts (often phrased as “get two more” parts).

The first step corresponds to a partitive division by a whole number (the numerator of the divisor), followed in the second step by a multiplication (by the denominator of the divisor) that converts units, from fifths of a serving to servings. The only subjects who used a different approach for Question 2 were a group of sixth-graders who misinterpreted the question as measurement division and attempted to find how many 3/5-orange servings were in 2 ½ oranges. This group cut the half-orange into quarters instead of fifths, and thus ended up with a miscount in the remainder.

However, the most common approach to Question 3 was some variation of a one-step division-of-whole-numbers approach. Here the dividend is 3 ½ and the divisor is 2⅓, and all but one of the adult groups who solved this problem first subdivided the sheets of paper into smaller parts—halves, thirds, or sixths—before dealing them out into piles. They typically dealt out into 2 piles and stopped with a few parts of a sheet left in hand to determine how many to deal into the ⅓-pile. This preliminary subdivision into smaller (common-denominator) units like sixths, called co-measure units (e.g., Olive, 1993), also seen (but not identified in these terms) in the solution to a measurement division problem in Perlwitz (2004), may account for how partitive division-of-whole-numbers strategies can be adapted to division-of-fractions problems, contrary to the expectations of some researchers (e.g., Sharp, 1998). The only subjects to solve Question 3 with a two-step division-of-fractions approach (partitive division by 7, then multiplication by 3) were one group of middle school teachers; this required reconceptualizing the divisor as an improper fraction (7/3) rather than a mixed number.

Errors

Patterns also emerged in the missteps made in approaching the problems. In measurement division problems, the most common error was to divide the fractional part of the dividend into as many parts as the denominator of the divisor: for example, quartering the half-orange into eighths when solving Question 1. Half (50%) of the sixth-graders did this, and two of these groups then continued to divide the whole oranges into eighths also before self-correcting when making groups of ¾ (which they modeled as 6/8).

The other common difficulty encountered with measurement division problems was reporting a remainder in terms of the wrong (old) units: half the sixth-graders also first reported the answer to Question 1 as "3 ¼" (see Figure 3) before being prompted for units (labels) made them realize that they had 3 servings and ¼ orange. Perlwitz (2005) reported a similar error by college students on a measurement division of fractions problem; without using units in their answers, they were even unable to explain the observed discrepancy with the quotient obtained through the traditional invert-and-multiply algorithm. Remainders were never an issue, however, in solving the partitive problems.

Finally, there was also a confusion of division types. The only common error in the two partitive division problems, Questions 2 and 3, was to apply the first step of the strategy for measurement division instead, e.g., dividing into 1/5-oranges in Question 2. Half the sixth-graders discussed doing this (see Figure 4), and one group actually did so, as did one group of preservice teachers. One group of sixth-graders also applied partitive division by 3 to the leftover ¼-orange in Question 1, to place one piece with each serving.

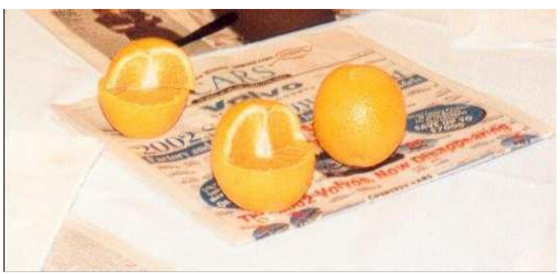

Figure 1. One-step solution to Question 1

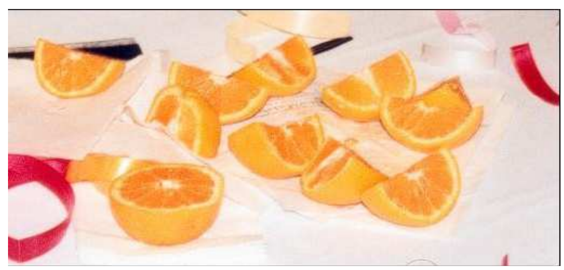

Figure 2. Two-step solution to Question 1

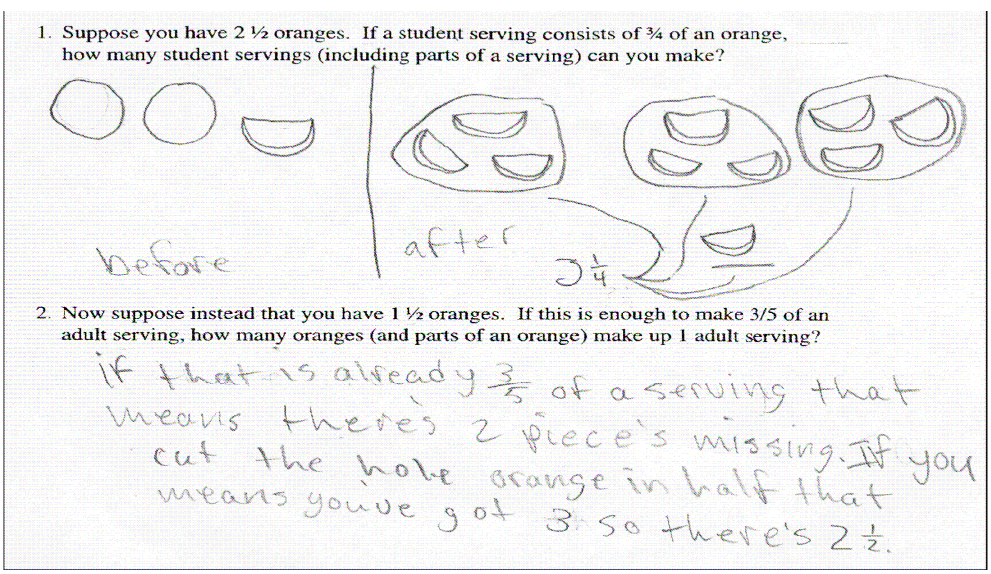

Figure 3. Student work for Questions 1 and 2

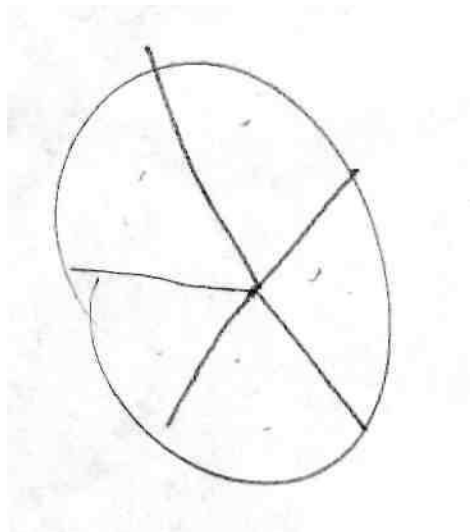

Figure 4. Student sketch proposing dividing the whole orange in Question 2 into fifths

Implications for Teaching

In addition to the distinction between strategies developed for measurement and partitive division problems, there is a sharp distinction between division-of-whole-numbers strategies, which involve only one step and can be applied to some division-of-fractions problems, and the two-step strategies particular to division of fractions. In general, the two-step strategies developed for division-of-fractions problems simply add to the one-step division-of-whole-numbers strategies an extra step which deals with the unit conversion inherent in operating with fractions and can be described mathematically as multiplying by the denominator. This process, called unitizing, is considered critical to conceptual development in mathematics. It is important to note that this extra step is the first step in measurement problems but the second step in partitive problems; also, the division step in each case is different, according to problem type. Measurement problems such as Question 1 allow both one-step and two-step approaches since the divisor (¾) is familiar and easily recognizable in physical models. Partitive problems like Question 2 do not allow one-step approaches because the number of groups is less than 1. Further study may be needed to distinguish the influences of mixed numbers, improper fractions, and unit fractions in the development of these strategies, as well as the role of co-measure units in problems where dividend and divisor have unlike denominators.

The development of these strategies by learners has several implications for teaching. Teachers must attend carefully to the sequencing of problems they use (see also Sharp and Adams (2002), p. 338). Students will approach their first division-of-fractions problems armed with division-of-whole-numbers strategies, so those first problems should admit one-step as well as two-step strategies—in particular, measurement problems should have familiar, recognizable divisors, and partitive problems should have divisors greater than 1, with fractional parts as simple as possible. Later, more advanced problems will require two-step approaches. The sixth-graders in this study solved the three problems with divisors less than 1 with uniformly two-step approaches (whereas several of the teachers and prospective teachers applied one-step approaches). It is also important to balance students' exposure to measurement and partitive division-of-fractions problems. Although the latter type are often harder to solve, and certainly harder to write well, they extend students' invented partitive division-of-whole-numbers strategies, and lead to solutions where remainders do not prove as much a sticking point as in measurement division problems. Finally, only after many experiences using these two two-step strategies may students recognize that the two steps in each strategy are the same, and can be performed in a single step, via the traditional algorithm.

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RESEARCHING TEACHERS' KNOWLEDGE FOR TEACHING MATHEMATICS *

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This report theorizes and provides empirical evidence of how researchers and educators might recognize categories of teachers' knowledge for teaching as teachers teach and discuss with peers their student's mathematical behavior and their practice. Its theoretical orientation engages work by Shulman on pedagogical content knowledge, Ball and Bass on mathematical knowledge for teaching, and Steinbring on teachers' epistemological knowledge. The empirical evidence emerges from the practice of teachers working with working class African American and Latino students in a poor, urban school district in the United States of America. The results of this investigation, part of larger, broader inquiry, suggest that the categories of teachers' knowledge implicate each other.

Introduction

Teaching mathematics is a multifaceted human endeavor, involving a complex, moment-by-moment interplay of different categories of knowledge. Teachers' mathematical knowledge, pedagogical competence, and insight into the development of students' mathematical ideas and reasoning are key to improving students' mathematical achievement. High quality standards, curriculum, instructional materials, and assessments are also important but not enough to improve students' learning of mathematics. As Ball, Hill and Bass (2005) argue, "little improvement is possible without direct attention to the practice of teaching … [h]ow well teachers know mathematics is central" (p. 14). Conceivably, this explains why recently there has been considerable discussion and research on teachers' subject-matter knowledge, pedagogical content knowledge, and mathematical knowledge for teaching (for example, Adler & Davis, 2006; Ball, 2000; Fennema et al., 1996; Hill, Rowan, & Ball, 2005; Shulman, 1986). The problem we theorize and explore empirically is "How might educators and researchers investigate and understand the development of teachers' mathematical knowledge for teaching?" Our perspective seeks descriptions of how teachers develop their mathematics knowledge for teaching in the complex, discursive interaction of actual practice as students evidence their mathematical ideas and reasoning and in the course of teachers' discussion of students' mathematical behaviors.

Theoretical Perspective

The theoretical perspective for our methodological approach has several sources. It is based on the assumption that teachers engage several categories of knowledge to enact successfully the mathematics education of their students. They clearly need knowledge of mathematics as well as knowledge of the subject that is specific to their work as teachers. In agreement with Shulman (1986) and Ball, Hill, and Bass (2005), our perspective recognizes that to teach a school subject like mathematics effectively necessitates knowledge of mathematics that "goes beyond the knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (Shulman, 1986, p. 9), or what Ball (2000) terms "mathematical knowledge for teaching. In their practice, teachers also need management and organizational knowledge that is distinct from

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

subject matter knowledge—pedagogical knowledge (Shulman, 1987). Furthermore, effective teaching requires teachers to attend to and endeavor to understand the mathematical ideas and reasoning of their students (Maher, 1998; Sowder, in press). This category of knowledge is specific and varies moment-to-moment and refers to teachers' inference into the status of students' knowledge. As Steinbring (1998) notes, a teacher "has to become aware of the specific epistemological status of the students' mathematical knowledge. … to diagnose and analyze students' constructions of mathematical knowledge and … to compare those constructions to what was intended to be learned in order to vary the learning offers accordingly" (p. 159). This category of knowledge—teachers' awareness of the epistemological status of students' mathematical understanding—enables, for instance, teachers to pose appropriate, new challenges for students to consider as they further build mathematical ideas and reasoning.

Researchers can infer teachers' mathematical knowledge for teaching by analyzing their practice in action, including interactions with students, questions they ask, issues they make salient to students, student artifacts they use, as well as post-session analyses they perform of their actions, plans, and students' work. Interaction also provides a lens through which to view mathematical knowledge, mathematical knowledge for teaching, pedagogical knowledge, and awareness of the epistemological status of students' mathematical understanding. These four categories of knowledge though conceptually different do at times, as we have observed, interact and even intersect. When they do intersect, they are essentially indistinguishable one from the other. Teachers' mathematical knowledge for teaching can be observed through their pedagogical moves; that is, by way of their pedagogical knowledge revealed in their moment-to-moment discursive interaction with students. In this paper, based on our methodological approach, we provide empirical evidence to substantiate the theoretical claim that teachers' mathematical knowledge, mathematical knowledge for teaching, pedagogical knowledge, and awareness of the epistemological status of students' mathematical understanding are in some instances mutually constitutive.

Method

This study is an adjunct of larger, ongoing analyses that emerge from a multi-prong, three-year research endeavor, "Informal Mathematics Learning Project" (IML). Two primary goals of the IML project involve investigating (1) how middle-school students (11 to 13 years old) develop mathematical ideas and reasoning over time in an informal, after-school environment and exploring relationships between agency and students' learning as well as (2) how teachers facilitate IML sessions and attend to students' ideas and reasoning. The IML research sessions occur in a middle school, after-school program in Plainfield, New Jersey, an economically depressed, urban area, whose school population is 98 percent African American and Latino students. These sessions were held after the regular school day to avoid some constraints of schools, such as time, curriculum, and testing.

For an academic year and a half, including the intervening summer, three pairs of teachers facilitated 20 sessions, 90-minute each, with a cohort of approximately 20 students, who began in their sixth grade, while graduate students from Rutgers University observed as ethnographers. This cohort explored similar mathematical tasks that had engaged an earlier cohort of students with whom researchers from Rutgers University worked, while the teachers participated as observers, taking field notes, and as co-investigators in post-session debriefings. Nonetheless, the teachers were not given a script; rather, they developed their own by selecting tasks and planning their own sessions. For about 50 minutes after each research session, the two teachers who facilitated the session, the other four teachers who observed the session, and two to three graduate students along with one Rutgers researcher discussed their observations and reflections on the tasks and on the ideas and reasoning of students. Research and debriefing sessions were videotaped. Students' inscriptions, graduate students' observation notes, teachers' planning documents were collected and stored electronically.

Through the course of IML sessions, the teachers invited students to work on strands of mathematical tasks. These tasks range across areas of mathematics that include rational numbers, combinatorics, probability and data analysis, and algebra. By design, the tasks are open-ended and well-defined, in that students were invited to determine what to investigate and how to proceed, identify patterns and search for relationships, make and investigate mathematical conjectures, develop mathematical arguments to convince themselves and others of their solutions, and evaluate their own arguments and those of others.

To understand the nature and development of mathematical knowledge for teaching, we analyzed data from the teachers' planning, implementation, debriefing sessions, as well as teachers' written reflections on the sessions they facilitated. For this report, we present an analysis of the first two IML sessions that two teachers, Lou (six years teaching experience) and Gilberto (three years teaching experience), facilitated as well as the corresponding work of students. For each session, there were between three and five video cameras, each with a boom microphone, capturing images from different student work groups and whole class discussions. Our videodata analysis follows methodological suggestions outlined by Powell, Francisco, and Maher (2003) and within this framework, we coded all data inductively and deductively. Our initial coding scheme intended to flag instances of teachers' using, commenting, and questioning about mathematics and pedagogy. Analyzing the data to understand teachers' mathematical knowledge for teaching, we noticed several instances of an intersection among teachers' awareness of the epistemological status of students' mathematical understanding and teachers' pedagogical and mathematical knowledge , some of which we present in the following section.

Results

The purpose of this paper is to theorize and explore an emergent approach for understanding the nature and development of teachers' knowledge. Above, we described a method for flagging critical events from data that provide investigators with insight on teachers' content and pedagogical knowledge as well as their awareness of the epistemological status of students' mathematical understanding . This section describes how we applied our methodology. Space only permits us to present a sequence of four critical events, occurring in one debriefing session.

The sequence of critical events concern students' presentation of ideas and teachers grappling with how understand the students' ideas and the underlining reasoning and how to orchestrate the next session based on the students' discourse that transpired in that day's after school session. These critical events provide us a window into the teachers' knowledge of pedagogy, mathematics, mathematics for teaching, and epistemological status of student learning. In the research session, students worked on the following task with Cuisenaire rods: If the light green rod has the number name two, what is the number name for the dark green rod? Three individual students each presented a different solution at an overhead projector.

Tiffany stated that since the light green rod has the number name two, then the white, the red, the purple, the yellow, and the dark green rods have respectively the number names zero, one, three, four, and five. Devon asserted that the white, the red, the light green, the purple, the yellow, and the dark green rods have the number names one, two, three, four, five, and six, respectively. With different results, both Tiffany and Devon lined-up their rods according to their heights and used their ordinal position to reason what number names to assign the rods. The third student, Sameerah, reasoned that since light green has the number name two, then the dark green has the number name four because two light green rods have the same length of one dark green rod and therefore, two plus two is four. Her reasoning is based on the additive property of length (two light green rods placed end-to-end are equivalent in length to a dark green rod) to name the dark green rod four. The session concluded with Lou and Gilberto asking the students to think about the three different solutions and announcing that the following day they will revisit them.

The first critical event occurs during the debriefing session when Alice, a university researcher, asks the teachers to assess the validity of the three student solutions described above.

Alice:	What do we think? Are they all equally valid?
Teacher1:	Yes
Teacher2:	No
Alice:	Okay, I'm hearing, some -[1] Jennifer what do you think?
Jennifer:	I guess I would respond to your question by saying "yes". They were valid to me because they were able to explain and justify their thinking behind it. It's not necessarily how I would have interpreted…[i] But that - the way the students who just lined them up in steps and explained to me, that if you call this one two and the ones below it are one step down each and the ones above it - to me that's a reasonable explanation and justification and explanation for their reasoning.
Alice:	Okay for their reasoning, now help me to understand what's going on in their reasoning
Jennifer:	Using just the attribute of length. That's all they were looking at and it made sense to me. If this one is a certain number
Alice:	Ok, if - you just said attribute of length
Jennifer:	Yes, those weren't their words
Alice:	Wait just a minute, though. What would the attribute of length be if you gave something the number name two?

The above discussion was selected as a critical event because the teachers begin to assess the validity of the students' solutions. By so doing, they discuss the students' reasoning and try to make sense of the students' understanding. They are exhibiting their mathematical knowledge for teaching and discussing their awareness of the epistemological status of students' mathematical understanding.

When Jennifer says that the students are using "just the attribute of length" to solve the task, Alice asks the teachers to discuss the mathematical meaning of assigning the number name two based on the attribute of length. The above discussion continues and is flagged as a second critical event:

Jennifer:	Because the number name two -
Alice:	What does that mean in relation to the attribute of length?
Kim:	It depends on the unit.
Jennifer:	It means that if I have one [rod] that's shorter, it's a number that going to be less than two and if it's longer then it's going to be greater than two.
Alice:	Oh, okay, well that is certainly one thing that it means. Uhm, let's really push on this because mathematically, this is what we're having to agree on. You've all have said, and agreed, that it is the attribute of length that we are interested in and according to the attribute of length, you're giving light green the number name two. What does that mean about - anything? To say that it is two in length

Gilberto: That it is one plus one
Alice: It is one what… yeah I agree, but one what?
Gilberto: One unit
Danielle: Whatever the length it is
Kim: Unit
Alice: Okay, if this is two, then it is one unit plus one unit in length

Here, a shift occurs in the conversation. Alice asks the teachers to consider what it means mathematically to give a rod the number name two. Hence, the discussion shifts from assessing the validity of the students' solutions and understanding their reasoning to a discussion of the underlying mathematical ideas of the task. While we hear a couple of the teachers use the term "unit" to answer Alice's question, we also hear responses that involve comparing lengths and applying the additive property. As a community, the teachers discuss and negotiate their mathematical knowledge as it pertains to comparing the lengths of Cuisenaire rods.

Once the teachers agree on the mathematical consequence of giving the number name two to the light green rod, they return to the assessing validity of the students' solutions. This is a shift in conversation from mathematics back to assessing the validity of students' reasoning is our third critical event.

Danielle: So based on that [Tiffany's] argument, the young lady that assigned, that said we're going to take the white off and were going to have that be zero then we're going to make red one and, and light green two and what was next, yellow three, and purple or purple then yellow and then she named it five, if we're looking at length, then its not, you can't justify that based on that answer because, you know, if light green is two then red should be half of light green, if, right, as one, if light green is two then yellow should have been exact, a double, she was naming yellow four, so yellow can't be four, however, because its not the double measure -

Jennifer: So it's not the attribute of length, it's the attribute of position.

In this discussion, Danielle notes the contradictions that would occur if yellow is given the number name four and red the number name one. In the first critical event, Jennifer assessed that the students were using the attribute of length to justify their solutions. After the discussions that occurred during the second critical event, Jennifer returns to correct her first statement and assert that the students are using the attribute of position to justify their solutions. This evidences the use of teachers' knowledge of mathematics to understand the status of the students' mathematical reasoning.

With this awareness of the epistemological status of students' knowledge and reasoning, Gilberto turns to designing an intervention for the next after school session, which we have flagged as our fourth critical event:

Gilberto: [He moves to the overhead projector and lines Cuisenaire rods in height order: red, light green, purple, yellow, dark green.] So what they are taking into account here is order - the position, first, second, third, fourth, and then this [pointing to dark green rod] is the fifth position. So then, I say, well if it [light green rod] is two, then the number name for red will be one. She is going to, she might answer that. Then I will ask her, if this is one [pointing to red] - ok if this is two [pointing to light green], then she is going to say this is one [pointing to red], and then I think we should ask her, how many ones do you need to make two? And then she will probably come with something like this [aligning two red rods end-to-end and placing them adjacent to one light

> green rod] and then we will see that if red is called one, two ones will be bigger than two.

Prior to this statement, Gilberto acknowledges that the students must take the lengths of the rods into account to successfully progress through the Cuisenaire tasks that the teachers have planned over the following five after school sessions. He uses Tiffany's solution as a starting point for designing an intervention that he hopes will lead the students to recognize a contradiction and shift their reasoning from positional to additive. Gilberto's understanding of the underlying mathematical ideas of the task, his pedagogical intervention for the upcoming sessions, and his insights into the students' reasoning evidences specific aspects of his knowledge of pedagogy, mathematics, mathematics for teaching, and awareness of the epistemological status of students' mathematical understanding.

Discussion

By analyzing teachers' practices in action, we notice that their knowledge of mathematics, mathematics for teaching, and pedagogy, as well as their awareness of student learning intertwine and intersect. As the teachers discussed the students' reasoning and tried to make sense of the students' understanding, they exhibited their mathematical knowledge for teaching and their epistemological awareness of the students' mathematical understandings. In these conversations, their discussion shifted from assessing the validity of the students' solutions to an examination of the underlying mathematical ideas of the task. Consequently, as a community, the teachers negotiated their mathematical understanding, which they applied as they returned to assessing the validity of the students' solutions and discussing the epistemological status of students' knowledge. Using the shared knowledge of the group, Gilberto designed an intervention comprised of pedagogical moves informed by the teachers' collective understanding of possible student trajectories. These pedagogical moves included his awareness of the epistemological status of students' knowledge. Furthermore, his pedagogical intervention was based on developing a proof by contradiction (if the red rod has the number name 1, then since the length of two red rods, whose combined length is 2, is longer than the length of one light green rod, whose length is also 2) evidences his mathematical knowledge. Cumulatively, we are able to infer his mathematical knowledge for teaching from his pedagogical moves. His four knowledge domains (mathematics, mathematics for teaching, pedagogy, and awareness of the epistemological status of students' mathematical understanding) interact, one influencing the other.

We have found that researchers can acquire an understanding of four types of knowledge of teachers—teachers' knowledge of pedagogy, mathematics, mathematics for teaching, and epistemological awareness of students' mathematical—understanding by studying teachers' practice and their reflections on their practice. Specifically, researchers can obtain insights into the development of teachers' knowledge by observing how teachers analyze students' mathematical behavior, grapple with the mathematical, epistemological, and pedagogical issues involved in addressing challenges they perceive in facilitating students' growth in students' mathematical ideas and reasoning, as well as by studying teachers' pedagogical moves (Powell & Hanna, 2006).

Endnotes

* This work was partially supported by a grant from the National Science Foundation, REC-0309062 (directed by Carolyn Maher, Arthur Powell, and Keith Weber). Any opinions, findings,

and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

1 In the quoted transcripts, the symbol "-" indicates pause in speech, "…" indicates inaudible speech, and bracketed words provide background information.

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BENCHMARKS AND ESTIMATION: A CRITICAL ELEMENT IN SUPPORTING STUDENTS AS THEY DEVELOP FRACTION ALGORITHMS

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This presentation will share findings from research that explored the mathematical practices a class of sixth-grade students and their teacher engaged in when learning about fractions using a problem solving or constructivist approach. In particular, the important role of benchmarks and estimation as students explore fraction operations and develop algorithms for operating with fractions will be highlighted.

In the past decade there has been an emerging focus on understanding students' invented algorithms (Carpenter & Fennema, 1992; Kamii, 1985). Part of this work has focused on understanding how to create learning opportunities where algorithm development draws upon students ideas but at the same time leads to the development of powerful and efficient methods for operating with numbers. This development of algorithms should involve a process that leads to determining the efficiency and usability of an algorithm. Lappan and Bouck (1998) argue that "[t]he invented algorithms of students are often very efficient and with a teachers' help can become powerful, generalizable methods" (p. 184). McClain, Cobb and Bowers (1998) also indicate that the teacher and the curriculum are influential parts of the algorithm development process but discussions where students justify their reasoning is critical.

Designing experiences that lead students to create and understand algorithms is a relatively new area of exploration. As the mathematics education community continues to explore the development of algorithmic thinking several questions remain. What does it look like when students engage in tasks that support them in developing algorithms? What do these tasks look like? What role does the teacher play in this process? This paper presents the findings of research that explored the mathematical practices a class of sixth-grade students and their teacher engaged in when learning about fractions using a problem-based or constructivist approach. The goal is to highlight the important role of benchmarks and estimation as students participated in a unit that developed algorithms for operating with fractions.

This work draws upon the situated nature of learning and the notion of practice where practices are common patterns of behavior that students (and their teacher) engage in. Cobb, Stephan, McClain, and Gravemeijer (2001) speak of mathematical practices or "taken-as-shared ways of reasoning, arguing, and symbolizing established when discussing particular mathematical ideas" (p. 126). Mathematical practices are emergent in classroom activity as well as particular to mathematical ideas. Regarding fractions, this research seeks to understand what mathematical ideas are common to or at the center of conversations engaged in as part of an instructional unit where students develop strategies for operating with fractions. The setting for this work is a classroom where 23 sixth-grade students and their teacher use a problem-based curriculum with an inquiry-oriented approach to learning mathematics. The fraction practices that emerged in this research were shaped by the design or intent of the curriculum as well as by classroom activity making them both intended and emergent at the same time (Wells, 2000).

Using curricular analysis in conjunction with analysis of observational field notes and

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classroom video of lessons, five interactive elements were identified to form the practice of learning to operate with fractions. These five interactive elements are: 1) problem context, 2) diagrams/visual models, 3) symbolism, 4) algorithms, and 5) benchmarks and estimation. The development of an operation occurs through problems that lead to combining, separating, partitioning, replicating, sharing and grouping quantities as well as estimating, modeling, and writing number sentences. For each operation, students work thought several contextual situations and are eventually asked to pull their ideas together and articulate an algorithm for each operation.

One example happened when discussing an activity that involved playing a game where students had to estimate fraction sums and use a number line to illustrate the reasoning leading to the solution. The class had just moved to discuss finding the estimated sum of 6/10 and 6/7.

Mrs. Kay How about 6/10 + 6/7?
TJ: Wouldn't they be 12/17 if added?
Mrs. Kay I don't know? Does it make sense? How much, about, is 12/17?
Class It is about 3/4.
Mrs. Kay About how much is 6/10 and 6/7?
[Students talk with their partners.]
Student C 6/10 is a little more than 1/2 and 6/7 is almost one whole.
Student D 6/7 is about 3/4.
Mrs. Kay Will the sum be more than 3/4?
Class Yes

In this conversation, students used estimation to find a reasonable sum and disprove TJ's suggested algorithm. In the discussion students drew upon the mathematical context of the game they were playing, the diagram of a number line [this is not apparent in this shorten version of the discussion], symbolism, benchmarks and estimation, and a potential algorithm. Across 6 of 13 tasks where students were exploring fraction operations, benchmarks and estimation stood out as an important conversational element in determining if an algorithm that was offered was indeed a legitimate algorithm. It was used by students to argue when someone's algorithmic approach did not make sense, by the teacher to help students decide on the validity of an algorithm that was proposed, and in the curriculum to help students develop operational number sense.

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EXPLORING AN INTERMEDIARY PHASE OF ARITHMETIC AND ALGEBRAIC THINKING

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Algebraic thinking is an important topic in school mathematics. In this paper, the authors will provide an overview of 7th grade students' solution methods regarding problems that elicit algebraic thinking. Data, which includes student written work on adapted tasks from the Balance Assessment Project, indicate that the students highlighted intermediary schemas of arithmetic and algebraic thinking.

Algebraic thinking is deemed an important topic in school mathematics (Chambers, 1994; Chappell, 1997). In the Principles and Standards for School Mathematics understandings of patterns, relations, functions, mathematical models and quantitative relationships are recognized as key facets of algebraic thinking (NCTM, 2000). In essence, algebraic thinking "embodies the construction and representation of patterns of regularities, deliberate generalization, and most important, active exploration and conjecture" (Chambers, 1994, p. 85). Algebraic thinking should function as a means of shifting from arithmetic to algebraic concepts (Kaput, 1999; Kieran & Chalouh, 1993). This shift has proven to be difficult for students (Herscovics & Linchevski, 1994; Lee & Wheeler, 1989; Stacey & MacGregor, 2000; Usiskin, 1988). Accordingly, it is imperative to explore students' reasoning as they approach problems that elicit facets of algebraic thinking (Nathan & Koedinger, 2000).

Methods and Data Sources

This study involved twenty-four middle school students, who were enrolled in a pre-algebra class. These students were participants of the Diversity in Mathematics Education (DiME) Project at the University of California at Berkeley. The problem set utilized in this study was adopted from the Balanced Assessment Project and adapted by members of the Diversity in Mathematics Education (DiME) Project at the University of California at Berkeley. A subset of problems is presented in below.

Figure 1. CUPS (Part I)

Tom makes a table to show the number of white cups in each stack and the height of each stack.

Number of white cups	2	4	6	8
Height of white cups in centimeters (cm)	10	14		

*Permission to reprint was granted by the Balanced Assessment Project

Fill in the missing numbers in Tom's table and find the height of a stack of 12 cups. Explain.

Students' responses were examined to determine: (1) how students approach problems that elicit algebraic thinking (e.g., did they resort to extending tables), and (2) the extent to which students present evidence of algebraic or arithmetic thinking (e.g., what features (if any) are

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highlighted).

Results

In the following sections, we will provide an overview of students' solution methods. In the interest of brevity, we will discuss a solution method that highlights an intermediary phase of arithmetic and algebraic thinking. For instance, for problem 1 in Figure 1, 21 (88%) students provided accurate heights of 18 and 22 centimeters for 6 and 8 cups, respectively. When asked to determine the height for 12 cups, only 11 (30%) students provided an accurate height of 30 centimeters. One student, Brandon, provided the following response: "The answer is 36 cm. because I added 18+18 becaue 6+6 = 12 and hight of 6 cups 18 cm so if you 18+18 you the hight of 12 cups" [sic]. Brandon concluded that the height of a stack of 12 white plastic cups would be 36 cm. He reasoned that since 6 (cups) + 6 (cups) = 12 (cups) and the height of 6 cups is 18 cm that the height of 12 cups would be 18 cm. + 18 cm. or 36 cm. Here he seems to be applying an additive part-whole proportional stacking heuristic that considers the stack as smaller parts that can be measured independently and put back together to find the whole instead of one continuous stack as called for in the problem. Moreover, Brandon's approach highlights the following proportional relationship: (6/18) = (6x2)/(18x2) = (12/36). Interestingly, his approach does highlight reasoning beyond specific quantities (e.g., "add 2 cm for each cup added") and emphasizes reasoning about relationships between quantities. Consequently, Brandon's line of reasoning may highlight an intermediary phase of arithmetic and algebraic thinking. In essence, this phase could be perceived as a way that students may begin to reason about relationships between quantities after arithmetically considering specific quantities.

Conclusions

The shift from arithmetic thinking to algebraic thinking is difficult. However, as evidenced by the reasoning presented here and additional samples of student work in this study, the shift may not be absolute, but instead involve an intermediary phase of thinking that is characterized by aspects of both reasoning about specific quantities (arithmetic) and reasoning about relationships between quantities (algebraic).

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LOST IN TRANSLATION: THE 'BEAN SNARE' AS A CASE OF THE SITUATED–SYMBOLIC DIVIDE

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The NCTM Standards (2000) recommend that instructional programs enable all students to create, use, and translate between mathematical representations. Yet, students are apt to fail in transferring between situated and symbolic notation (Martin & Schwartz, 2005). We propose that this transfer failure is due to critical shades of meaning being lost between media. Also, we explore the pedagogical value of having education researchers and practitioners analyze such semiotic breakdown.

The third author designed the *Bean Snare* (Figure 1) to spark discussion of the complexity of constructivist design, teaching, and learning, i.e., subtle interactions of content and context as well as multi-media, multi-modal, and multi-representational aspects of collaborative reasoning about a situated mathematical problem. Note how the presentation surreptitiously leads us down the garden path to a mathematically incorrect statement.

What is lost in translating between this situation (combining groups of white/black beans) and the standard mathematical notation (adding fractions)—our group concluded after a semester of lively debates—is a crucial fragment of meaning implicit within the ostensive statement ("…2 *of them* are black") and accompanying deictic gesture (indicating each *whole group*). Thus, in inscribing the combining action, the multiplicand (the cardinality of each group of beans) is inadvertently omitted (so it should be 2/3 * *3* + 3/5 * *5* = 5/8 * *8*). Alternatively, the paradox lies in shifts between 2 frames of meaning—proportionality (*a/b* or *a:b*) and sets (*a* + *b*).

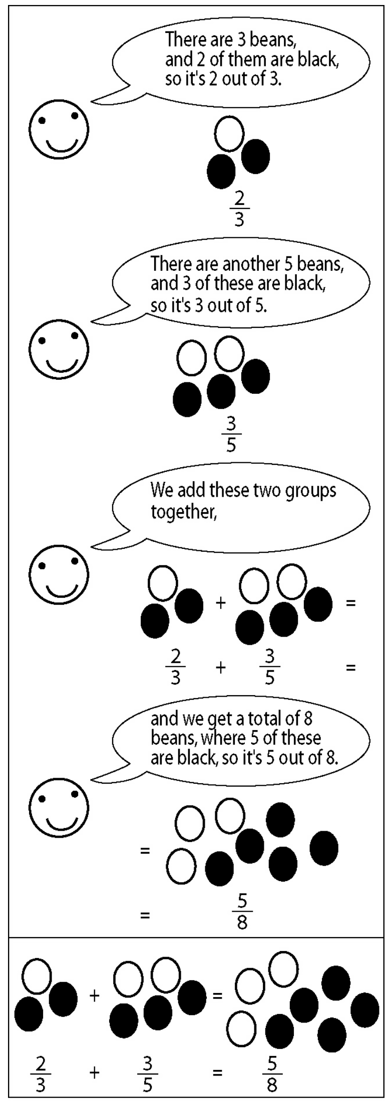

Figure 1. The Bean Snare

Situated mathematics can help ground mathematical meaning. Yet, if designers and/or teachers fail to recognize potential pitfalls inherent in mathematization, then the concrete contexts may constitute a disservice. The Bean Snare is a case of an activity that can generate insight into the intricacies of situated-mathematics curricula, i.e. the nature of mathematical reasoning and learning, challenges of pedagogy, didactics, and design, and issues of policy making around the "math wars." In future work, we will interview in-service teachers, using the bean snare, to investigate potential tension between their pedagogical beliefs (e.g., constructivist) and their formative-assessment practices (see edrl.berkeley.edu).

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EXAMINING HOW STUDENTS' LACK OF PROCEDURAL SKILLS IMPACTS THE DEVELOPMENT OF THEIR CONCEPTUAL UNDERSTANDING

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As a part of a longitudinal study involving rational number understanding of middle school students, we became interested in the question of how students' lack of procedural skills impacted the subsequent development of their conceptual knowledge in certain instances. Specifically, we explored how their lack of long division skill with decimal or fractional remainders affected the concepts in the tasks they were given. In videotaped individual interviews, the students were given word problems to solve and we looked for cases where students' exhibited a lack of procedural skills while problem solving and then identified the concept that was impacted by this procedural deficiency.

Based on our interview data, we identified two situations or categories where the students' lack of procedural skills hindered subsequent conceptual understanding. In Category 1, the lack of procedural knowledge interrupts conceptual understanding, where the students' choice of the wrong procedure or a set of procedures prevents them from progressing from Concept A to Concept B. In Category 2, students selected the correct procedure, but did not execute it properly. Basically, students lack some of the procedurals skills involved in the progression from Concept A to Concept B, leading to an impoverished understanding of Concept B.

Category 1 is illustrated by a student converting a fraction (Concept A) to a percent (Concept B). She knows that to change 9/24 (a fraction which is hard to express as a percent using part whole relationship or memorization) to a percent, she should use division, but she divides 100 ÷ 9, which is the incorrect procedure. Thus, she is lacking both the concept and procedure for fraction as division, which prevents her from changing this type of fraction to a percent.

Category 2 is illustrated with the following situation involving ratio questions. The student could only describe and understand ratio with whole numbers (Concept A). The student's impoverished conceptual knowledge of ratio was affected by her inability to divide beyond the decimal point and to divide multi-digit numbers (Concept B in two related parts), even though she knew the necessity of using division to get the ratio. She could divide 23 ÷ 4 to get 5 R3 but could not divide 1300 divided by 500, so she could not find the ratio when it included a decimal fraction.

Across both categories, the students' lack of procedural knowledge is affecting the further development of the concept in the question. We argue that this is evidence that conceptual and procedural knowledge are interrelated and should be taught in concert.

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REASONING AND PROOF

DOES PROOF PROVE?: STUDENTS' EMERGING BELIEFS ABOUT GENERALITY AND PROOF IN MIDDLE SCHOOL

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This paper presents results from a multi-year research study[1] that examined the development of middle school students' competencies in mathematical reasoning and proof. Written assessment data collected from 78 middle school students over a period of three years shows that while improvement occurs from 6th to 8th grades, results indicate that competencies needed to generate mathematical proof remain to be developed.

Proof has always been a central aspect of the discipline of mathematics and the practice of mathematicians, but it is a relatively recent aspect of mathematics education for students at all grade levels. Traditionally, students' first encounters with proof occurred during high school geometry courses, where the formal two-column proof was often the only method of proving students encountered, and explorations of proving general mathematical statements in other mathematics courses such as algebra were typically not supported. However, researchers such as Schoenfeld (1994) and Wu (1996) assert that proof cannot be separated from mathematics and it is an essential part of the process of doing and communicating mathematics in all content areas. From its inclusion in the 2000 *Principles and Standards for School Mathematics* (NCTM), reasoning and proof has gained increased attention as a central part of mathematics education for students at all grade levels. The 2000 NCTM document recommended that students be encouraged to view reasoning and proof as fundamental aspects of mathematics, know how to make and test conjectures, and evaluate and select various types of reasoning and methods of proof. Existing research, however, indicates that students' understandings of proof are weak in light of these recommendations (e.g., Balacheff, 1988; Bell, 1976; Healy & Hoyles, 2000; Porteous, 1990; Senk, 1985). Understanding the notion that a proof treats the general case is critical for students' success in evaluating and generating mathematically correct proofs. A number of researchers have documented evidence that students tend to view empirically-based arguments as sufficient justification for demonstrating the truth of a mathematical argument (see Porteous, 1990; Fischbein & Kedem, 1982; Balacheff, 1988, Healy & Hoyles, 2000). Existing research, however, has not studied students' competencies in proving over a cohesive grade band (e.g., middle school or high school). The purpose of this paper is to present results from a longitudinal study of middle-school students' conceptions of proof. We explore how students' understandings of proof change during their middle school education by exploring the following questions related to notions of generality: Do students tend to generate empirically-based arguments or proof-like arguments to justify a mathematical statement? and To what extent do students recognize that a proof treats the general case?

Proof framework

Researchers have hypothesized that the development of students' proving competencies might follow a developmental progression and, indeed, various frameworks have been proposed that reflect such a developmental progression (e.g., Balacheff, 1988; Bell, 1976; van Dormolen, 1977; Waring, 2000). Building upon these aforementioned frameworks, we used four levels of proof concept development: Level 0 – students respond either "I don't know" or give information already presented in the problem (for this paper, Level 0 also includes non-codable and no response/I don't know); Level 1 – students consider checking a few cases as sufficient; Level 2 – students are aware that checking a few cases is not sufficient and may attempt a general argument, but the argument is on the "wrong track" (i.e., would not lead to a proof) or is incomplete (would lead to a proof if completed); and Level 3 – students generate an argument that treats the general case of the statement (i.e., a proof).

Methods

Longitudinal data were collected from 78 students beginning in grade 6 and continuing throughout grades 7 and 8 (students range in age from 11 to 14 years). All students attended the same middle school that utilized *Connected Mathematics* as a curriculum in mathematics courses from sixth to eighth grades. The use of *Connected Mathematics* is relevant since the authors (Lappan et al., 2002) assert the following: "Throughout the curriculum, students are encouraged to look for patterns, make conjectures, provide evidence for their conjectures and strategies, ...Informal reasoning evolves into more deductive arguments as students proceed from grade 6 through grade 8" (p. 8). Thus, it seems reasonable to conjecture that student understanding regarding the generality of a proof may be more fully developed for this sample than that of students from schools with more traditional curricular programs (as reported in previous research).

The primary source of data was student responses to written assessment items designed to measure their proving competencies. The students completed one assessment in the fall of sixth and seventh grades, one assessment in the fall of eighth grade, and one assessment in the spring of eighth grade. Each assessment contained between 6 and 7 questions designed to measure proving competencies; each assessment had a core set of questions that remained essentially the same for each administration of the assessment.

This paper will focus on student responses to three of the core assessment questions. Two of the questions (Items 1 and 2) required students to provide a justification for a mathematical statement. The third question (Item 3) presented students with a statement introduced as a mathematical truth and asked students to choose which of two arguments justifies the truth of the given statement. One argument utilizes only examples-based reasoning, while the other argument treated the general case.

For Items 1 and 2 in which students were asked to generate justifications, their responses were coded according to the aforementioned framework. Results presented for Item 3 focus exclusively on students' choices between the examples-based justification and the proof, not on the students' explanations of why they chose one argument over the other.

Results

Due to page length limitations of the conference proceedings, a selection of results will be briefly presented and discussed here (more detail as well as additional results will be presented during the conference session).

Assessment Item 1

Item 1 asked students to respond to the following item, with parts of the context changed, such as names, for each assessment (cf. Porteous, 1990): *Jesse discovers a cool number trick. She thinks of a number between 1 and 10, and adds 4 to the number, doubles the result, and them* *she writes this answer down. She goes back to the number she first thought of, she doubles it, she adds 8 to the result, and then she writes this answer down. [The preceding text was following by an example using the number 5.](a) Will Jesse's two answers always be equal to each other for any number between 1 and 10?(b)Does your explanation in part (a) show that the two answers will be equal to each other for numbers greater than 10?*

Typical Level 1 responses to this item include affirmations of the rule as well as further examples of numbers for which the rule works:

> *(a) Yes, it would always be equal because I tried a few problems.*
> *[Student has the correct calculations for the numbers 3, 8, and 10][Student has the correct calculations for the numbers 11-13].*
>
> *(b) Yes, it would work for numbers more than 10 because I tried numbers that are higher then ten. [8th grade student]*

Level 2 reasoning on Item 1 is usually an attempt to describe why both operations yield the same number, but such descriptions, as shown in the excerpt below, often fall far short of a satisfactory proof due to their ambiguity:

> *(a) Yes her 2 answers will always be the same because there doing the same thing except reversing the order.*
>
> *(b) Yes my answer does because even though the two methods are different they are doing the same thing. 50, 50+3=53, 53x2=106. 50, 50x2=100, 100+6=106. [7th grade student]*

Typical Level 3 responses describe why the net procedures applied in each case change the original number in the same way. Interestingly, most student responses did not make use of symbolic/algebraic arguments nor mention the distributive property:

> *(a) Yes because since you add 4 then double it then that would be the same as doubling and adding 8 because 4 doubled = 8. [8th grade student]*

Consistent with previous research, a significant number of the 78 students preferred the use of examples when justifying the truth of a statement. For each assessment, fewer than half of the students' responses were coded as attempts to produce a general argument (Level 2 or 3). There is a positive trend towards more successful proof production from the 6th to 8th grade. Figure 1 illustrates the frequency of responses to Item 1 among the longitudinal sample across all four assessments.

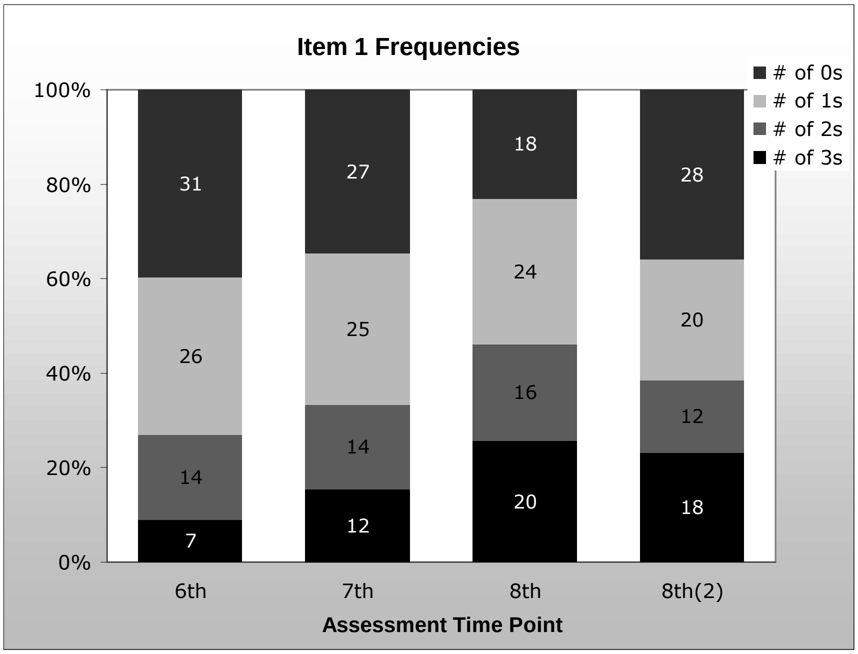

Figure 6

In particular, if we consider attempts to produce a general argument (combining Levels 2 and 3), we see an increase from 27% in 6th grade to 38% at the end of 8th grade (interestingly, we see a slight drop from the fall of 8th grade to the spring of 8th grade). In addition, the percentage of examples-based arguments (Level 1 responses) remains relatively constant throughout all four assessments, ranging from a high of 33% in 6th grade to a low of 26% in the spring of 8th grade.

A notable portion of the longitudinal sample, 9%, produced a proof for this item in 6th grade. Further, approximately 36% of the students never formulated a general argument for this assessment item across all four assessments. Overall, only 42% of students produced an adequate proof (Level 3 response) to Item 1 on at least one of the four assessments. Note that the apparent drop in proof production in the final assessment is not statistically significant.

Assessment Item 2

Similar to Item 1, Item 2 prompted students to explore a general mathematical statement and provide justification for their reasoning. Item 2 asked students to respond to the following item: *We know that an odd number added to another odd number is always an even number, and we know that an even number added to an odd number is always an odd number. (a) What happens if you add any three odd numbers together, is your answer always odd? (b) Provide an explanation that would convince your teacher that the answer is always odd.*

Typical Level 1 responses include examples of three odds that add up to an odd as well as claims that this reasoning satisfies as proof or is at least enough to convince the student of the statement's verity:

(a) I agree because I showed examples and I think the statement is true.1+3+5=9; 7+9+11=27; 13+15+17=45.

(b) All you have to do is add 3 odd numbers and find out what the answer is, like in the example above. Once you are done looking at the numbers, I hope I have convinced you to believe that if you add any 3 numbers, you will always get an odd sum. [6th grade student]

Unsuccessful general arguments, Level 2, were represented by various strategies. Some students used an analogy with the cancellation of the sign of negative numbers when multiplying, others made mention of the alternating pattern of even and odd numbers and some even used visual arguments. All Level 2 arguments made an explicit attempt to treat the general case, but they were unsuccessful in that they were not specific or well-stated enough to prove the statement. An example is given below:

(a) Yes because, you could think of it like + (even) - (odd) and if you have odd odd =even, but odd, odd, odd =odd.[student draws three - signs, and crosses out the first two] cancel out leaving you with -.[8th grade student]

For Item 2, the successful proofs looked very similar to one another, differing only in whether they used an illustrative example in addition to the proof. The most typical response provided a logical linking of the given mathematical facts to establish that three odd numbers added together yields an odd number. Similar to student arguments for Item 1, students producing Level 3 arguments did not make use of symbolic representations (i.e. even: 2n, odd: 2m-1). The following Level 3 response is representative:

(b) It will be odd because if add the first two numbers together, it will equal an even number. If you take the third number and your answer of the first 2 numbers, which is even, you take the odd number plus the even number and an odd number plus an even number equals an odd number, so your answer will be odd. Example: 3+3+3=?; 3+3=6+3=9(odd). [6th grade student]

Unlike Item 1, the results for Item 2 showed a consistent and statistically significant increase from 6th to the final 8th grade assessment in the number of general responses; again, considering both Level 2 and 3 arguments, there was an increase from 22% in 6th grade to 47% at the end of 8th grade. Still, fewer than half of the 78 students produced Level 2 or 3 responses on each written assessment. The overall performance of the longitudinal sample on this assessment item was better than on Item 1; 53% of students who completed all four assessments provided an adequate proof for Item 2 on at least one of the four assessments compared to 42% for Item 1. There is a noticeable drop in the Level 0 responses across the time points and an increase in completed proofs. Compared with Item 1, there are fewer Level 2 responses and an even higher count of Level 1 responses. Figure 2 illustrates the frequency of responses to Item 2 among the longitudinal sample across all four assessments.

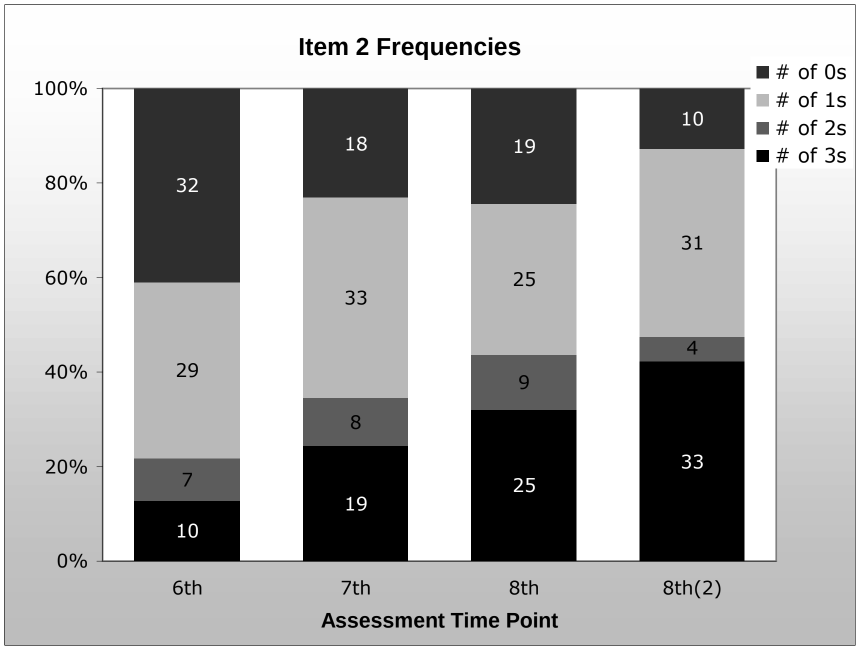

Figure 7

Assessment Item 3

One assumption underlying the design of Items 1 and 2 is that students possessed sufficient mathematical maturity to produce general arguments on their own for the given contexts. Item 3 attempted to further test students' beliefs about the value of empirical evidence, that is, whether students believe that examples suffice as proof or they are aware of the limitations of an examples-based argument. Students were presented two arguments, attributed to fictitious students, justifying the same statement. One argument provided only three examples as evidence of the truth of the statement, while the other argument used a deductive chain of reasoning with mathematical facts (i.e., a proof). Students were asked to select the argument that proves the mathematical statement.

Results of this assessment item indicate that, until the final assessment in the spring of 8th grade, many of the 78 students believed that the examples-based argument was preferable over the general justification. Only 18 (23%) students never chose the general justification over the examples-based argument on any of the four assessments. In the final assessment, nearly half of the students (38 or 49%) chose the general argument over the examples-based argument (29 or 27%). Other responses given include "Both," where the student believed both arguments suffice as proof, "Neither," where neither argument is believed to suffice as proof, and "No Response" or "Non-Codable" (NR/NC). Figure 3 displays the type of responses given by students for each assessment time point.

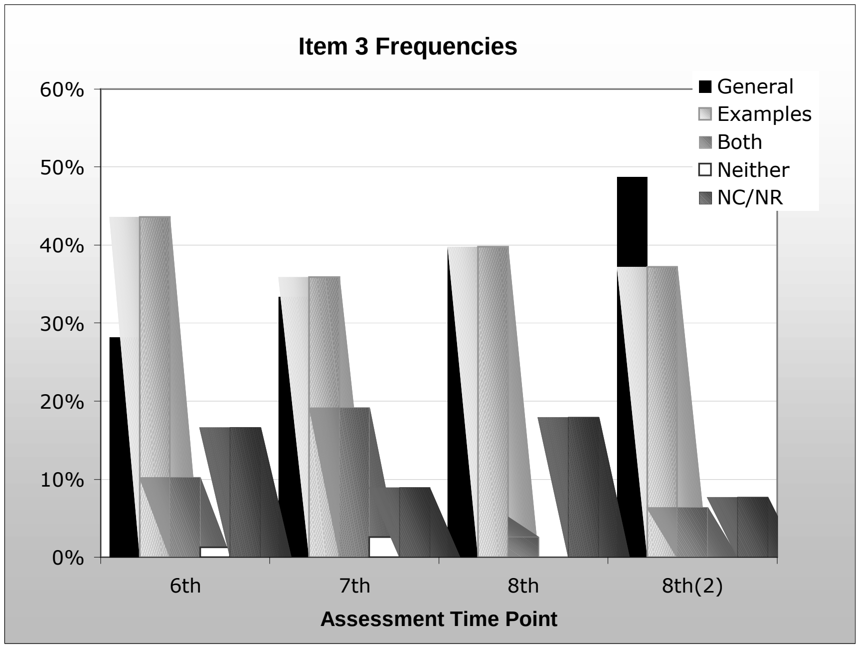

Figure 8

The figure clearly shows that, from Time Point 1 to Time Point 4, the percentage of students choosing the general argument (G) increases, while the pattern of among students choosing the examples-based argument (E) is less clear. Considering all analyses for this item, students appeared to have made gains in their understanding about what type of argument serves to justify a general mathematical statement from the 6th to 8th grades.

Discussion

In closing, our results suggest that many middle school students lack an understanding of generality, particularly when it applies to producing a general justification of a mathematical statement. The types of statements that students were asked to provide justification for, such as Item 2, are closely related to activities in *Connected Mathematics* units, particularly the *Prime Time* unit in 6th grade. However, a large number of students used examples-based justifications on these items, and that number did not decrease much, if any, throughout the duration of the study. It may be possible that the structuring of activities in *Connected Mathematics* promotes empirical investigation but does not offer as much opportunity for students to generate and evaluate abstract conjectures from concrete experiences. Further, an informal analysis of the *Connected Mathematics* curriculum reveals that it is the responsibility of teachers to exploit opportunities to engage students in proof-related activities; they are not a ubiquitous part of the materials.

On the other hand, some results, indicate that some students do possess an understanding of generality, particularly when they are asked to evaluate whether or not an argument is a proof. What is interesting about this finding is that the Connected Mathematics curriculum rarely asks students to determine whether an argument suffices as proof, but students show gains in performance from 6th to 8th grade on these types of tasks. We speculate that as students encounter more mathematics, they may learn what types of arguments are preferred by their teachers without necessarily having explicit instruction on what makes for a valid proof. Hence, while students may learn to recognize that examples alone do not suffice as proof, further *investigation would need to determine whether they could discern the difference between two general arguments where one of the arguments fell short of being a proof.*

Endnotes

[1] The research reported here is supported by the National Science Foundation under grant REC-0092746. The opinions expressed herein are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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TRACING MIDDLE-SCHOOL STUDENTS' ARGUMENTS

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The purpose of this paper is to deepen our understanding of how group discussion of middle-school aged urban students can facilitate the learning of mathematics. We illustrate how student challenges and counterarguments can lead them toward investigating the deep, underlying mathematical structure involved in the situation that they are investigating and the arguments that they are making. In particular, we investigated how students made judgments about fairness of dice by running computer simulations of die tosses. Our analysis revealed that although students initially offered unsophisticated justifications for their judgments on which dice were fair, challenges to students' justifications led to a lively debate on important mathematical principles such as the importance of sample size.

Introduction

Participating in discussions in which mathematical ideas are debated provide students with powerful opportunities for learning mathematics (Balacheff, 1991; Cobb et al., 2001). McCrone argues that discussions "allow students to test ideas, to hear and incorporate the ideas of others, to consolidate their thinking by putting their ideas into words, and hence, to build a deeper understanding of key concepts" (p. 111). For these reasons, influential organizations such as the NCTM (2000) and many researchers in mathematics education (e.g., Balacheff, 1991; Heibert & Wearne, 1993; Lampert, Rittenhouse, & Crumbaugh, 1996) recommend that discussion play a prominent role in reform-oriented mathematics classrooms. Analyzing students' discussions is also considered central in analyzing how students learn mathematics (Powell & Maher, 2002). Ongoing research is building an empirical base for the role of discussion in mathematical learning (e.g., Cobb et al., 1997; Manouchehri & Enderson, 1999; Stephan & Rasmussen, 2002; McCrone, 2005). The purpose of this paper is to contribute to this literature by presenting and analyzing a discussion in which African American and Latino middle school children from a poor, urban environment make, justify, and challenge statistical claims.

Our goal in this paper is to describe and illustrate one way that discussion can foster mathematical learning. In the mathematics education literature, several accounts for how discourse can contribute to mathematical learning have been proposed. Discussion can objectify students' previous experiences, thereby making these experiences objects that can be analyzed (Cobb et al., 1997), encourage students to take a more reflective stance on their mathematical reasoning (Manhouchehri & Enderson, 1999), require students to consolidate their thinking by verbalizing their thoughts (McCrone, 2005), and help students learn to communicate mathematically and participate in a wider range of mathematical argumentation (Lampert & Cobb, 2003). We propose that group discussion can also facilitate learning by inviting students to be explicit both about the ways in which they make new claims from previously established facts and about the standards they are using in deciding whether an argument is acceptable. Further, challenges from classmates can lead students to debate whether a particular method of argumentation is appropriate, provide students with the opportunity either to justify their methods when their reasoning is sound or to revise or abandon these methods when their reasoning is flawed. In the discussion that we will analyze, students debated what statistical conclusions legitimately could be drawn from a set of existing data. In the discussion, students considered central statistical issues such as the importance of sample size in drawing warranted conclusions.

Theoretical Perspective

Krummheuer (1995) and others (e.g., Stephan & Rasmussen, 2002; Rasmussen & Stephan, in press) have argued that Toulmin's (1969) model of argumentation can be a useful analytical tool for understanding the progression of students' arguments during collective discourse. In Toulmin's model, an argumentation consists of three essential parts, called the core of the argument: the claim, the data, and the warrant. When an individual presents an argument to his or her community, he or she is trying to convince the audience of a particular *claim*. To support the claim that is being made, the individual typically presents evidence, or *data*. The audience may ask the individual presenting the argument to explain why one should deduce the claim being made from the data being presented. In Toulmin's scheme, such an explanation is referred to as a *warrant*. In mathematical discussions, students often will not state their warrant when they are presenting their argument to their classmates. Frequently, it is a challenge by a teacher or classmate that prompts the student to explicitly state the warrant that he or she had, until that point, only used implicitly (Rasmussen & Stephan, in press). It is possible for the audience to accept the data put forth by the presenter but reject the explanation for why the data support the conclusion. If this occurs, the presenter is required to provide *backing* for why the warrant is acceptable and the core of the argument is valid (cf., Stephan & Rasmussen, 2002).

A central argument advanced in this paper is that classroom discussion may foster mathematical learning by requiring students to be explicit about the warrants that they are using and requesting that they provide backing for why these warrants are legitimate, thereby establishing that their modes of reasoning are valid. As Rasmussen and Stephan (in press) note, "eliciting backings is crucial for supporting the evolution of increasingly sophisticated mathematical ideas". Without such elicitation, students may come to believe that all means of deducing new information are equally acceptable. It is also possible that the class may present an argument that illustrates why the warrant in question is not valid and should not be used to present a purported convincing argument. The arguments from our study differ from those presented in Rasmussen and Stephan in that our students were not trying to obtain a mathematical solution to a problem but were asked to examine whether various six-sided dice were fair based on data obtained from computer simulations. The claims students made were judgments about the fairness of each of the dice used in our study. The data (in Toulmin's sense) usually, but not always, consisted of data (in a statistical sense) obtained from running simulations in which the die under consideration was "rolled" multiple times. Similar to the classrooms described in Rasmussen and Stephan, what was most interesting to us as researchers was not whether a student believed a particular die was fair (the student's claim) or even the data the student used to justify this conclusion (the data) but what principles the student was using to draw conclusions from examining data (the warrant) and the ensuing debates about whether these principles were legitimate (the debate about warrants and the construction of backings).

Methods

Setting

This study is an adjunct of larger, ongoing analyses that emerge from a multi-prong, three-year research endeavor, “Informal Mathematics Learning Project” (IML), conducted in an after-school program in a partnership between the Robert B. Davis Institute of Rutgers University and the Plainfield School District in New Jersey, an economically depressed, urban area, whose school population is 98 percent African American and Latino students (1). Two primary goals of the IML project involve investigating (a) how middle-school students (11 to 13 years old) develop mathematical ideas and reasoning over time in an informal, after-school environment and exploring the relationship between agency and students’ learning as well as (b) how teachers facilitate IML sessions and attend to students’ ideas and reasoning. For two and a half academic years, including the intervening summers, we facilitated 30 sessions, 60- to 75 minutes each, with a cohort of approximately 25 students. During these sessions, students were asked to work on open-ended, well-defined tasks involving topics such as fractions, combinatorics, and probability. Throughout the study, collaboration, justification, and sense making were encouraged, and both researchers and students attended to students’ ideas and took them seriously.

This report focuses on a culminating probability and data analysis task, “Schoolopoly,” That has been used in prior studies (Stohl & Tarr, 2002; Tarr, Lee, & Rider, 2006). In this task, six hypothetical dice companies produced die that may or may not be fair and students are challenged to decide which company should supply the dice for a Schoolopoly game. Each pair of students was assigned two or three companies, and was asked to judge whether each company produced fair dice by simulating rolling their dice using *Probability Explorer* software. Each die company was explorer by at least two groups. After their investigation, students produced a poster of their findings, including the data and graphical displays from their simulations and justifications for why they believed their judgment was correct. Students then explored the posters that each group constructed. Our analysis in this paper focuses on a 30-minute discussion that followed students’ analysis of posters as students debated which company to choose for the Schoolopoly game.

Analysis

All student activities were videotaped. The data were analyzed in a manner consistent with the first stage of the research methodology prescribed by Stephan and Rasmussen (in press). Members of the research team to get a strong sense of the data viewed the videotape repeatedly. The discussion was then transcribed. Next, a description (using the methods of Powell, Fransisco, and Maher, 2003) of the videotape was constructed. The description of the videotape is a relatively objective description of what transpired in which acts of interpretation are explicitly avoided. The data was parsed into specific student argumentations. Each argument that a student presented was coded according to Toulmin’s scheme as prescribed by Stephan and Rasmussen (in press); each argument was coded in terms of the claim being made, the data to support the claim, and, when given, the warrant for how the data implied the claim. In cases where no warrant was provided, we sometimes would make a note in our codings with our interpretation of the warrant that the student seemed to be implicitly using. Each coding and all interpretations were discussed within our research team until all disagreements were resolved. We also coded all challenges to a students’ argument by what part of the argument—the claim, data, or warrant—was being challenged. The result of our analysis was a coded chronological account of the arguments and challenges that the students raised during the 30-minute discussion.

Results

A description of the discussion among students about which die was fair is organized by presenting an analysis of three episodes in which a student presented an argument that was challenged by his or her peers.

Episode 1.

The first episode occurs at the beginning of the discussion. The researcher begins by asking Chris to explain why he would choose to buy dice from Delta's Dice Company.

Chris: Because, if you look at both of them [posters], they both like, like really explain the same thing. Like to me, I thought the first poster and the second poster were like about the same thing. They really explained it […]

R1: OK. Out of how many trials? What were they doing, do you remember?

Chris: Uh…. I think it was 600. But I don't know.

R1: OK. So it seems to me that one of the reasons why you're picking Delta's Dice is because the two groups agree.

Chris: Uh-huh. Because, like, in other ones [posters of other dice companies], like, one didn't agree and one did agree, or sometimes they didn't really explain enough.

We coded this excerpt in the following way:

Claim: Delta's Dice is fair

Data: Both posters agreed that Delta's Dice was fair and presented similar arguments. Posters for other dice did not agree, or did not give thorough explanations.

Warrant: None provided. We inferred Chris' warrant to be that if both posters for a die agreed, it was reasonable to conclude that die was fair.

Chris seems to be accepting that Delta's Dice was fair based on the authority of the posters that he examined, but not on the data or arguments contained in the posters. When the researcher asked Chris for details about the poster ("Out of how many trials? What were they doing?"), Chris could not recall details of the poster with confidence. He also agreed the agreement of the posters was one of the reasons he chose Delta's Dice. After Chris gave his argument, Chanel challenged it:

Chanel: I just have a real quick question. Why does on the one scribbly and stuff, why does it say that one [outcome] is lower, one might be lower and the rest are higher, and why, how is that fair? [Referring to statement on poster "the dice are close to each other except for one might be low like number 5"] Yeah, I don't get it […] that one, it might be lower, and the rest, the rest is just higher. So, how is, I don't get it, how is that fair?

In this excerpt, Chanel is questioning whether the data and claim made in one of the posters for Delta's Dice really is evidence that Delta's Dice is fair. Several other students raised similar questions. Chanel's challenge is arguably more sophisticated than Chris' argument since she examines the data and argument presented in the poster and not just the conclusion expressed in the poster. However, it is also important to note that Chanel's challenge was to Chris' conclusion, but not to his reasoning. She did not question whether it was appropriate to conclude a die was fair because both posters claimed it was; rather she challenged Chris' claim because she reached a different conclusion from her examination of data and claims on one of the posters of Delta's Dice. The next two students after Chris presented similar arguments that Delta's Dice was fair, primarily relying on the fact that the two posters evaluating Delta's Dice both concluded that it was fair.

Episode 2.

Tiffany deviated from the previous students by arguing that Calibrated Cubes would be the die most likely to be fair:

R1:	OK. Tiffany, what do you think?
Tiffany:	Um, I picked Calibrated Cubes.
R1:	OK. Can you tell us why?
Tiffany:	Because, um, I think it's fair because all the numbers were even, 'cause when I looked at the charts, all the numbers had 11, I think.

Here, Tiffany is referring to a table on one of the posters that showed that on one of the samples of 80 trials, 1, 2, and 3 each occurred exactly 11 times. The table on the poster was arranged in such a way that the number of times a 4, 5, and 6 occurred were not shown. We coded this argumentation as follows:

Claim: Calibrated Cubes is a fair die.

Data: The table for Calibrated Cubes shows that 1, 2, and 3 each occurred exactly 11 times.

Warrant: None given. We inferred the warrant to be that if three sides of the die were rolled the exact same number of times in a particular simulation, the die is likely to be fair.

Note that Tiffany differed from the previous students in that the data with which she supported her claim did not concern the judgment of the posters that she inspected, but rather their contents. Chanel immediately challenged Tiffany's argument:

Chanel:	I think that, um, like on Calibrated Cubes it just showed three 11's. It didn't show all of the cubes. 'Cause there were three more cubes, and those could have been 12, 13, or 14, or any other number. And it didn't show all the, um, numbers, it showed the three 11's. How do we know it wasn't like 34 or something on the other ones [outcomes 4, 5, and 6]?
Student:	(off camera) Or 117.
Chanel:	You agree with it!
Student:	(off camera) I know but still.
Tiffany:	But I have a question. Whoever that is, what was the other numbers? You don't have to lie.

In this excerpt, Chanel is challenging Tiffany's implicit warrant. It is possible that in the table that Tiffany looked at, the missing numbers could have been large, such as 34. Chanel appears to be implicitly arguing that if a 1 appeared 11 times and a 4 appeared 34 times, a plausible possibility given the data Tiffany alluded to, then it would be unreasonable to call a die fair based on this evidence. Another student, apparently one who initially agreed with Tiffany that Calibrated Cubes was fair, chimed in that one of the missing numbers could have occurred 117 times, an impossibility since the total number of trials was 80. It is not clear whether Chanel's challenge caused this student to change his mind that Calibrated Cubes was fair, but this student clearly was attending to Chanel's challenge and building upon her reasoning. Tiffany's request to know the missing values of the table suggests that she too is attending to and appreciates the merits of Chanel's challenge. Finally, note that Chanel's challenge here differs from the one that she and others posed to Chris. Chanel did not simply challenge Tiffany's claim because she arrived at a different judgment; here she is challenging the validity of Tiffany's reasoning.

Episode 3.

Jerel challenges Tiffany's argument for a different reason than Chanel.

Jerel: Well look. They only ran it 80 times. You'll never know if another number is gonna come up and pass it. Even though it was even, they ran it a small amount of times. You need to run it a lot of times. Because

Terrill: Why, why? I don't understand why.

Jerel: I didn't say you had to, I said you *need* to […] Because, you, all right, just like when we were doing Delta's Dice, we had ran it, um, I think a hundred times, and one number won by a lot. But when we ran it like one thousand times and all that, other numbers won….Because, like, um, other numbers won, but they were close to each other still, and the reason they got that all is because they had a little bit amount of numbers that they ran it, but when you like, I guarantee you if you ran it like 500 times, it would have been different. You ought to say it was unfair.

In this exchange, Jerel is arguing against a different aspect of Tiffany's warrant. Tiffany is drawing a conclusion based on the results of rolling the die 80 times. Jerel is questioning whether one can legitimately draw such conclusions from such a limited data set. In fact, he puts forth a counterclaim that one needs to run a large number of trials in order to reach a reliable conclusion, citing his experience in seeing discrepancies in his data when he ran simulations with 100 and 1000 trials. Jerel's counterclaim goes beyond the data Tiffany referred to and the argument that Tiffany made; it focuses on a central statistical issue—the importance of sample size. Jerel's counterclaim became the subject of intense debate among the students, with students offering arguments and counterarguments for why they did or did not need to examine data with a large sample size when judging whether a die was fair.

Summary

The first three students who offered their decision on which die to buy chose Delta's Dice, primarily because the two groups that inspected Delta's Dice both found Delta's Dice to be fair and no other dice company shared a similar level of agreement. The students making these arguments did not discuss the content of the posters. Only one of the three arguments was challenged. The challenges posed were not based on the reasoning that the student used, but on the conclusion that he reached. Tiffany was the first student to present an argument based on the data presented in the posters that she inspected. Challenges to Tiffany's arguments were based on the warrants that she appeared to be using to draw her conclusions. A central challenge to

Tiffany's data occurred when Jerel questioned whether it was appropriate to draw conclusions from a simulation that used a relatively small sample size. The issue of sample size subsequently led to a lengthy and lively debate. Subsequent challenges to students' arguments were based on the warrants that the students were using, or appeared to be using implicitly. In particular, arguments based on the conclusions of the posters but not the data presented in the posters were challenged by other students.

Endnote

(1) This research is a component of the National Science Foundation funded project, Research on Informal Mathematical Learning (REC-0309062). Any opinions, findings, conclusions and recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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PROMOTING TEACHER LEARNING OF MATHEMATICS: THE USE OF "TEACHING-RELATED MATHEMATICS TASKS" IN TEACHER EDUCATION (1)

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Research suggests that teachers need to have mathematics content knowledge that allows them to effectively deal with the particular mathematical issues that arise in their everyday practice. This implies the importance of providing teachers with learning opportunities that prepare them to both recognize situations in their practice where these mathematical issues arise and be able to apply their mathematical knowledge to successfully manage these situations. Yet, little research has focused on how such learning opportunities can be effectively promoted in teacher education. In this article we take a step toward addressing this limitation by discussing and exemplifying a special kind of tasks for use in teacher education which we call "teaching-related mathematics tasks." These are mathematics tasks that are connected to teaching and can foster the development of teachers' mathematics content knowledge that is important for teaching.

In recent years, there has been increased research attention to the mathematics content knowledge that is important for teaching (e.g., Ball et al., 2001; Ball & Bass, 2000, 2003; Davis & Simmt, 2006; Ma, 1999; Shulman, 1986). A major idea advanced in this body of research, especially in the work of Ball and Bass, is that teachers need to have mathematics content knowledge that allows them to effectively deal with the mathematical issues that arise in their everyday practice, which are generally different from the mathematical issues that arise in the everyday practice of other professionals who use mathematics. For example, the work of an engineer does not (normally) necessitate that the engineer knows different methods for dividing fractions, or how these methods correspond to one another. A teacher, however, needs to be able to reason accurately and quickly about different methods – both valid and invalid – for dividing fractions in order to be able to function effectively in teaching situations where this mathematical issue arises. In the context of a classroom discussion of the standard "invert and multiply" method, a student may ask the teacher whether it is correct to use a different method such as "dividing numerators and denominators." This situation raises for the teacher some critical questions: Is dividing numerators and denominators a valid method for dividing two fractions? If so, how does this method correspond to the standard invert and multiply method?

The idea that teachers need to have mathematics content knowledge that allows them to effectively deal with the mathematical issues that arise in their everyday practice implies the importance that teacher education programs design opportunities for teachers to learn mathematics that are tailored for the particular needs of teaching. Specifically, it implies the importance that these programs provide teachers with learning opportunities that prepare them to both recognize situations in their practice where different mathematical issues arise (like the situation described in the previous paragraph) and be able to apply their mathematical knowledge to successfully manage these situations. Despite the importance of these learning opportunities for teachers, little research has focused on how such opportunities can be effectively promoted in teacher education (both teacher preparation and professional development programs).

In this article, which is a continuation of Stylianides and Stylianides (2006), we take a step toward addressing this limitation by discussing a special kind of mathematics tasks – which we call "teaching-related mathematics tasks" – that can foster the development of teachers' mathematics content knowledge that is important for teaching. Our goal is to further elaborate on the nature and illustrate the utility of this kind of tasks. To promote our goal, we connect the notion of teaching-related mathematics tasks with existing literature and we offer a critical reflective account of our own personal experiences in designing and implementing one such task in a mathematics course for preservice elementary teachers (grades K-6). To investigate more systematically the utility of the tasks in teacher education we are in the process of conducting a design experiment, the findings of which will appear in a future report.

The Notion of "Teaching-related Mathematics Tasks"

Consider the following two versions of a mathematics task that aims to promote teacher learning of proof in the domain of multiples of a number. The first version is a standard mathematics task typically used in mathematics courses for preservice elementary teachers, whereas the second version is a teaching-related mathematics task that we use in our mathematics course for preservice elementary teachers.

Version 1 (standard mathematics task):

Develop three proofs for the claim: "A multiple of 3 plus a multiple of 3 equals a multiple of 3." One proof should use everyday language, another proof should use pictures, and the third proof should use algebra.

Version 2 (teaching-related mathematics task):

As the mathematics specialist in your school, you teach mathematics in all three fifth-grade classes: Class A, Class B, and Class C. The past week you have been working with all three classes on the notion of multiples of a number. The three classes developed appropriate definitions of multiples of a number, but in each class these definitions were represented in a different way. In Class A they were represented using everyday language, in Class B they were represented pictorially, and in Class C they were represented algebraically.

In the next lesson, you plan to engage your students in the three classes in proving the claim: "A multiple of 3 plus a multiple of 3 equals a multiple of 3."

In preparation for this lesson, you want to take into account the fact that each class shares definitions that are represented in a different way. So, for Class A you want to prepare a proof that uses everyday language, for Class B you want to prepare a pictorial proof, and for Class C you want to prepare an algebraic proof. Write the three proofs.

Although both tasks aim to promote teacher learning of proof in the domain of multiples of a number, they do so in substantially different ways. Contrary to Version 1 of the task, Version 2 facilitates connections between the intended learning and the work of teaching by situating the preservice teachers' mathematical activity in an authentic teaching situation where this learning is crucial. In particular, Version 2 of the task helps preservice teachers make connections between learning equivalent proofs that utilize different representations and teaching situations where only certain representational tools are available in the shared knowledge of a particular classroom community. By so doing, Version 2 of the task helps preservice teachers appreciate what this task intends to teach them and, thus, makes it more likely than Version 1 of the task that preservice teachers will learn the mathematical ideas embedded in the task (Harel, 1998). According to Harel, "[s]tudents are most likely to learn when they see a need for what we intend to teach them, where by 'need' is meant intellectual need, as opposed to social or economic need" (p. 501). In the case of preservice teachers, "need" can be defined as their interest in developing mathematics content knowledge that is useful for their work and in becoming able to apply this knowledge to successfully manage teaching situations where this knowledge is called for. The consideration of preservice teachers' need for learning is consistent with findings of research on the motivational aspects of cognition which suggest that a prerequisite for successful problem solving is that problem solving be situated in personally meaningful contexts (see, e.g., Mayer, 1998; Renninger et al., 1992; Weiner, 1986). According to the findings of this body of research, students learn better – that is, they think harder and process the material more deeply and with more likelihood of transfer – when they have an interest in the material.

Given our analysis of versions 1 and 2 of the task above, we argue that mathematics courses for teachers (both preservice and inservice) need to place emphasis on the use of tasks like Version 2 that we call teaching-related mathematics tasks. These are mathematics tasks that are connected to teaching, and have a dual purpose: (1) to foster learning of mathematics that is important for teaching; and (2) to help teachers see how this mathematics relates to teaching, thereby increasing the possibility that they will learn it and use it in their work. By "task" we mean a sequence of related activities (e.g., engagement with a mathematical problem accompanied by reflection on the work to solve the problem) that focus on a particular idea and aim to promote a particular goal.

Teachers' engagement with teaching-related mathematics tasks can be thought of as a process of mathematizing teaching. Our use of the term "mathematizing" follows that of Freudenthal (1973, 1991). Freudenthal, used mathematizing to describe a notion of mathematics as an activity – as schematizing, structuring, and modeling the world mathematically, rather than as a discipline of structures to be transmitted, discovered, or even constructed. Also, we use mathematizing to include both horizontal and vertical mathematization, i.e., the process of converting a contextual (teaching) problem to a mathematical problem that can be solved mathematically and the process of taking the mathematical content to a higher (meta) level, respectively (Treffers, 1987). We elaborate on these ideas below.

A teaching-related mathematics task engages teachers in studying (e.g., interpreting, analyzing) a teaching situation with a mathematical lens, and, in some cases, the task engages them additionally in studying their own mathematical activity in the task at a meta-level (e.g., reflecting on their own work in the task). Version 2 of the task we presented earlier provides an example of the former: it engages preservice teachers in studying mathematically the situation where a teacher needs to develop three proofs for a mathematical claim given the constraints in the representational tools available to different student populations. If the task additionally asked preservice teachers to reflect on their own mathematical activity in the task, this would facilitate studying the mathematical activity at a meta-level. Such a reflection could help preservice teachers identify their mathematical activity in this particular task as being part of the more general mathematical activity of producing equivalent ways to represent different mathematical ideas (e.g., arguments, concepts) within the constraints of given mathematical systems. Identification of this general mathematical activity would promote teachers' understanding of where and how in teaching they would need to engage in a similar mathematical activity, thereby increasing the possibility of them transferring the mathematics content knowledge to be acquired from their engagement in this teaching-related mathematics task to new teaching situations.

A Mathematics Course for Preservice Elementary Teachers

Over the past three years, we have developed a set of teaching-related mathematics tasks on different content areas (e.g., algebra, geometry, number theory) that we have piloted – with promising results – in five enactments of a mathematics course for preservice elementary teachers. The course places emphasis on the use of teaching-related mathematics tasks and aims to advance preservice teachers' content knowledge that is important for teaching elementary school mathematics. A major goal of the course is to promote a notion of mathematics as a sense-making activity – that is, activity characterized by meaningful learning – and to create a community of mathematical discourse where ideas are validated based on mathematical argument. As a result the notion of proof holds a prominent place in the course. To facilitate connections between the mathematics learnt in the course and the work of teaching, the course makes extensive use of records of classroom practice (e.g., video or written reports of lessons, student artifacts). These records are derived from various sources, such as books (e.g., Carpenter et al., 2003), research reports (e.g., Zack, 1997), and a database of the Mathematics Teaching and Learning to Teach (MTLT) project at the University of Michigan that documents an entire year of the mathematics teaching of Deborah Ball in a United States public school third-grade class.

An Example of A Teaching-related Mathematics Task From the Course

In this section we present a teaching-related mathematics task we designed and implemented in our mathematics course for preservice elementary teachers. The presentation of the task is based on data (video records of lessons and student artifacts) we collected during the last enactment of the course in 2006; the instructor was the first author.

Description

This teaching-related mathematics task comes from the content area of number theory. In the class prior to the implementation of this task, the preservice teachers analyzed textbook definitions of even and odd numbers, and developed equivalent definitions of these concepts that are accurate mathematically and appropriate for use in the elementary grades. With this teaching-related mathematics task, we aimed to help preservice teachers understand the utility of definitions in mathematical reasoning and, in particular, in proving true claims over infinite sets. The task had three parts:

- Individual work to prove the claim: An odd number plus an odd number equals an even number.
- Discussion of the mathematical issue raised in a videoclip from third grade where students try to prove the same claim.
- Individual and small group work to revisit their work in part A.

Part B is particularly important, because it uses a teaching situation to help preservice teachers realize the limitations of empirical arguments (that predominate in their solutions of part A), understand the importance of definitions in the development of proofs, and motivate them to revisit their work in part A to produce a proof. Below we consider each part separately.

Part A.

The course was taken by 18 preservice teachers. Their written responses to part A of the task are classified as follows: two responses did not offer any real argument for the claim to be proved (e.g., they were faulty or irrelevant responses); nine were empirical arguments; two were unsuccessful attempts for a general argument; and five were general arguments (see Table 1 for illustrative examples). By "general arguments" we mean arguments that address appropriately the general case and either qualify as proofs or require minor refinements (e.g., better justification of a step in the argument) before they qualify as such. By "proofs" we mean valid arguments from accepted truths (e.g., definitions, axioms) for or against mathematical claims. In sum, at the beginning of preservice teachers' engagement with the task, only seven out of 18 made an effort to produce arguments that address the general case.

Response	Commentary
An odd number multiplied by 2 always gives an odd number.	This is a faulty response that does not offer any real argument for the claim to be proved.
3+7=10, 1+5=6, 7+9=16, 3+37=40, -47+3=-44, -3+1=-2, -3+(-1)=-4, -37+(-51)=-88, 573+697=1270. Any odd number added to another odd number would equal an even number. Any way that you do it, it comes out to an even number.	This is an empirical argument because the claim follows as a generalization from the confirming evidence offered by the examination of a few particular cases (a proper subset of all the possible cases).
Any odd number on a number line that is moved an odd number of spaces to the right (i.e., an odd number is added to it), you always will land on an even number.	This is an unsuccessful attempt for a general argument because, although it deals with the general case, it does not produce a valid argument.
An even number can be divided equally by two. An odd number can be divided by two with one left over. If you add together two odd numbers, you are also adding the two leftovers, which will be able to be grouped by two without leftovers.	This a general argument that qualifies as a proof because it makes adequate use of definitions of even and odd numbers to deduce logically the truth of the claim "odd+odd=even."

Table 1. Illustrative examples of different types of responses to part A of the task

Part B.

After the preservice teachers completed their individual work on part A, and without discussing their solutions in the whole group, the instructor engaged them in watching and discussing a videoclip from the MTLT database. The videoclip shows the third graders in Deborah Ball's class investigating the same mathematical claim as in part A of the task, namely, that the sum of any two odd numbers is even. Ofala, like many other students in the third-grade class, asserts that the claim is true because she verified it in a few particular cases (e.g., 1+5=6). Jeannie, however, begins to worry about what it really means to prove a claim that involves an infinite number of cases. She says: "Numbers go on and on forever and that means odd numbers and even numbers, um, go on for ever and, um, so you couldn't prove that all of them work." (See Ball and Bass [2003] for a more detailed description of the classroom episode in the videoclip.)

The guiding question for the preservice teachers in watching the videoclip was the following: What was the mathematical issue raised in the videoclip and how could this issue be addressed? After some discussion in small groups and then in the whole group, the preservice teachers concluded that the main issue in the videoclip was that checking a few particular cases does not suffice to prove true mathematical claims like "odd+odd=even" that involve an infinite number of cases. Regarding the second part of the guiding question, one preservice teacher suggested that "there's a consistency in how odd and even numbers behave and this consistency guarantees that what happens in particular cases happens in all cases." The instructor highlighted this idea of consistency and asked the preservice teachers to think more about it. What exactly about even and odd numbers guarantees this consistency? Another preservice teacher then pointed out that the consistency resides in the definitions of even and odd numbers.

Part C.

After the discussion of the videoclip, the instructor introduced part C of the task: “Now that we’ve watched the clip and understood what the issue there was, I’d like you to think whether the proofs you gave [in part A] for the claim ‘odd+odd=even’ address this issue. If they don’t, how could you revise your proofs?” After the small group work, the instructor asked the different groups to present their proofs in the whole class. Out of five small groups, two tried unsuccessfully to produce a general argument and three produced general arguments. In other words, all groups made an explicit effort to avoid empirical arguments and to produce arguments that address the general case.

Discussion

A well-documented finding in the literature is that many students and teachers of mathematics demonstrate a reliance on the use of examples to verify the truth of general mathematical claims (e.g., Knuth et al., 2002; Martin & Harel, 1989; Simon & Blume, 1996). This finding is illustrated both in the work of our preservice teachers in part A of the task and in the episode used in part B of the task that showed the work of the third graders in Deborah Ball’s class. The teaching-related mathematics task helped us to engage our preservice teachers in thinking about the important mathematical issue of how one can prove true mathematical claims that involve an infinite number of cases. The innovative aspect of the task is that, instead of relying on the instructor and his/her authority to convince preservice teachers about the limitations of empirical arguments to prove general mathematical claims, it used a classroom episode from third grade that was raising the exact same issue. The set up of the task helped preservice teachers reconsider in productive ways their original approaches to proving the claim in part A of the task, resulting in significantly improved arguments in part C. Also, the preservice teachers appeared to be motivated to revisit their original approaches and to appreciate the value of what they were learning. The latter is encouraging, especially in the context of a task focusing on the notion of proof, because preservice teachers tend to consider proof as an advanced topic and, thus, they are often resistant to engage in activities that aim to develop their generally weak content knowledge in this important mathematical domain. Finally, we should note that part B of the task engaged preservice teachers in a process of mathematizing teaching, as it engaged them in interpreting and analyzing a teaching situation with a mathematical lens.

Conclusion

This article contributes to teacher educators’ understanding of how theoretical ideas on the mathematics content knowledge that is important for teaching can be used to design useful tasks that offer preservice teachers rich opportunities to learn mathematics in connection to the work of teaching. The reflective account of our own personal experiences in designing and implementing teaching-related mathematics tasks illuminates aspects of the role and nature of this kind of tasks, and suggests their promise in teacher education. Further research is needed to develop a comprehensive set of teaching-related mathematics tasks and investigate systematically their effectiveness by conducting experimental or quasi-experimental studies that will examine their impact on teachers’ learning of mathematics and on their teaching practice.

Endnote

1. The two authors had an equal contribution in writing this article.

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PRINCIPLES OF CONCEPTUALIZATION FOR IMAGE-BASED REASONING IN GEOMETRY

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Image-based reasoning, in which the image is an essential component of the reasoning process rather than merely a cognitive aid, is the basis of high-school geometry. The principles of conceptualization are a proposed framework for understanding geometrical reasoning as a synergy between image and concept. It points to a new "logic" for image-based reasoning.

Rationale

The term *image-based reasoning* refers to informal deductive geometry, in which image data must be used in the argument. It is a feature of high-school geometry. A better grasp of the cognitive processes of image-based reasoning would allow educators to comprehend student under-performance in geometry and lead to improvements in pedagogy.

Review of Literature

A *schematic property* is a geometric property that can be inferred validly from a geometrical diagram within the context of a geometrical argument; a deductive geometry without axioms is required of students in high school in many educational jurisdictions; schematic properties are an essential component of this type of informal deductive geometry (Handscomb, 2005). It is interesting to note also that the geometry of Euclid required the use of schematic properties (Netz, 1999).

A *conceptualization* of an image is a set of properties that are true of the image, for example schematic properties in informal deductive geometry. Aristotle's *qua* operator (Lear, 1982) and Godfrey's (1910) "geometrical 'eye'" (p. 197) both imply this notion.

According to Fischbein (1993), image and concept together form a hybrid third type of entity: "The objects of investigation and manipulation in geometrical reasoning are then mental entities, called by us *figural concepts*, which reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual properties—like ideality, abstractness, generality, perfection" (p. 143, author's italics).

Theoretical Model

A proof in high school geometry starts with a figural concept and ends with a figural concept—the given information and the conclusion, a statement that has been deduced from the given information. In between are a number of intermediate figural concepts. Laws governing the flow of these figural concepts would comprise a "logic" for image-based reasoning, a set of cognitive principles for non-formal deductive geometry.

It is necessary to modify and extend Fischbein's figural concept notion in two respects. Firstly, Fischbein's concepts exist in a formal framework, and this requirement should be relaxed to allow concepts from outside a formal, axiomatic framework. Secondly, the figural concept is a static construct, whereas I would like to understand it in terms of a dynamic interplay between image and concept, in which the geometer continually conceptualizes new aspects of the image.

Given these modifications to the figural concept idea, I have developed five principles of conceptualization. These notions were derived through an introspective analysis of the reasoning process that needs to happen for successful, informal geometric proof.

Principle 1

If C and D are two conceptualizations of an image I, then the union of the properties of C and D is also a conceptualization of I.

Principle 2

If C is a conceptualization of image I, and D is a subset of C, then D is also a conceptualization of I.

Principle 3

If C is a conceptualization of image I, and the properties of D can be deduced from the properties of C, then D is also a conceptualization of I.

Principle 4

If image I is contained in image J, and C is a conceptualization of I, then C is also a conceptualization of J.

Principle 5

If image I is contained in image J and C is a conceptualization of J, then there is a subset D of C such that D is a conceptualization of I.

A diagrammatic representation of these ideas is given in Figure 1. Principles 1 and 2 can be seen to be special cases of Principle 3. They are discussed separately for additional clarity. Note that Principle 4 is certainly not true in a formal mathematical sense. It should be emphasized that these are not mathematical propositions, but cognitive principles in the same spirit as the laws of Gestalt (Kosslyn, 1983).

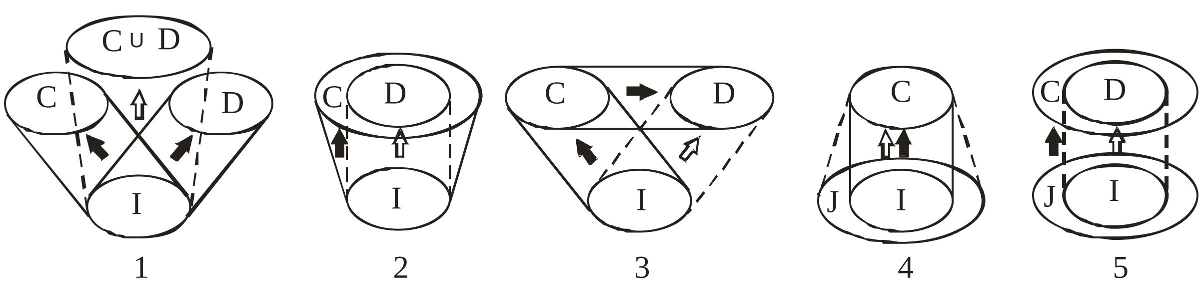

Figure 1. Principles of conceptualization

Discussion

A closer analysis of the processes involved in the principles of conceptualization leads to four geometrical reasoning skills. I have described these geometrical reasoning skills in Handscomb (2005) and compared them to the van Hiele levels discussed in van Hiele (1986). There is a close correspondence between the two frameworks, although the principles of conceptualization and the geometrical skills are considered here more purely as cognitive processes rather than as a developmental model of geometrical reasoning.

My next step is to attempt a corroboration and/or refutation of these principles and skills by means of empirical research. I will be using electroencephalographic (EEG) data gathered in the Engrammetron Laboratory at Simon Fraser University to seek reliable and valid brain-based correlates of these postulated cognitive processes.

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WHAT IS PREDICTION AND WHAT CAN PREDICTION DO TO PROMOTE REASONING?

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A recent study shows that prediction is the most prevalent grade level expectation on reasoning in state mathematics standards. In this presentation, we articulate characteristics of prediction and how prediction can be utilized to promote student reasoning using examples from state mathematics standards as well as 8th-grade classroom data.

The analysis of the grade level expectations (GLEs) requiring reasoning in state mathematics standards revealed that GLEs pertaining to prediction were the most prevalent reasoning expectations across grades as well as content strands in the state (Kim & Kasmer, 2006). This suggested that prediction be an important component of reasoning that could be easily pursued at all grade levels and in all mathematics content strands. It also implicated the sound rationale to investigate the potential of prediction in the mathematics classroom that encouraged mathematical reasoning. Prediction, however, has rarely been investigated in comparison to other aspects of mathematical reasoning, such as justification. It is speculated that part of the reason may be that prediction is considered in relation to conjecture or hypothesis and typically not seen as a unique feature of reasoning. For example, the *Trends in Mathematics and Science Study* (TIMSS) includes hypothesize, conjecture, and predict in one category in its assessment framework on reasoning (Mullis, Martin, Smith, Garden, Gregory, Gonzalez, Chrostowski, & O'Connor, 2001). Some researchers used prediction and conjecture interchangeably (e.g., Blanton and Kaput, 2005). On the contrary, a body of research has investigated prediction in the area of reading (e.g., Block, Rodgers, & Johnson, 2004). In such research, students were asked to make a prediction as they proceeded in a reading activity using questions akin to 'given the situation in the story, what will possibly happen next?' Research found that asking students to make such a prediction helped increase students' comprehension of reading.

As such, we began to look at how prediction could help student reasoning by observing classrooms where prediction was utilized in each lesson. For this paper, we search for aspects of prediction expectations in state standards in order to identify what prediction is. We also further elaborate characteristics of prediction as a tool to promote reasoning using classroom data.

Prediction in State Mathematics Standards

State standards present various prediction GLEs as seen in the examples below whereas conjecture GLEs are much less common and primarily target general properties. These examples delineate a range of possible predictions from simple to more sophisticated (P1 can result in a premature guess as well as a prediction with sound reasoning), from forming a generalization (P5) to using a generalization (P2), and from dealing with one particular event (P3, P4, P6) to possibly leading to general structures (P4, P5, P6).

P1. Predict the effect on the graph of a linear equation when the slope changes
P2. Make reasonable predictions using generalizations about patterns
P3. Predict what comes next in an established pattern and justify thinking
P4. Predict and evaluate how adding data to a set of data affects measures of center

P5. Make and test predictions about measurements, using different units to measure the same length or volume

P6. Predict and justify the results of subdividing, combining and transforming shapes

As seen above, prediction can be made by random guessing or based on plausible reasoning about mathematical relationships. In other cases, prediction can occur based on an established generalization of patterns observed. Prediction in this case can be a further application of generalization using known facts. Prediction can be used interchangeably with conjecture as seen in P5, and yet the two can have different focuses. A conjecture is to be tested to form a generalization (see Reid, 2002), but a prediction may or may not lead to a generalization and can be a simple application of a generalization. It can also be said that a prediction attempts to describe the outcome of a specific future event as well as a foundation of generality while a conjecture attempts to describe a general structure of events. In this sense, conjecture can be understood as a specific type of prediction.

A Classroom Example

The classroom episode described below demonstrates what prediction looks like in a classroom. Prior to the episode, students used dot grid paper to find areas of various geometric figures. In this episode, the teacher asked students to predict the length of a diagonal line segment connecting dots on a dot grid four across and one up, and justify their predictions. After students individually thought about the question and recorded their thinking, the teacher elicited responses about their predictions in the whole group. A summary of students' predictions and reasoning discussed is presented below.

Student	Prediction	Reasoning
S1	5	Count dots "4 over and 1 up" or "1 down and 4 to the left
S2	4	Multiply 4 (4 unit lengths across) by 1 (one unit length down)
S3	4	Count spaces in between dots (the number of unit lengths either on the top or the bottom) only across
S4	2	No particular reason
S5	4 1/3	"Since it's a diagonal, it's actually longer [than the line across, which is 4]." "It doesn't look like it's that long [5] if you spun it around."
S6	4.5	Same as S5

In this example, the teacher used prediction to motivate and engage students, which would eventually lead to properties of a Pythagorean triangle. Predictions students made in this lesson ranged from random guesses, such as 2, to more sophisticated ones noticing the diagonal being longer than each of the legs and shorter than their sum. Students provided justifications for their predictions using various support. Making predictions using this special case, students began to consider the relationship between the lengths of the hypotenuse and the legs of a triangle. These students' predictions are indicative of the range of prediction GLEs found in the state standards.

Conclusion

Prediction has the potential to promote reasoning. Even though some consider prediction part of conjecture, it has its own distinct characteristics that help students develop reasoning. Prediction can facilitate students' engagement in problems dealing with particular cases and yet it leads to general structures and properties in later explorations. When students offer their predictions with supported reasoning, various perspectives can come to the forefront of the discussion. Therefore, making a prediction affords students an opportunity to think about and organize what they know and what they notice and make possible connections between those and concepts in later explorations.

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STUDENTS' USE OF REPRESENTATIONS IN THEIR DEVELOPMENT OF COMBINATORIAL REASONING AND JUSTIFICATION

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This study documents the mathematical development of a group of eleventh-grade students who built representations to solve challenging combinatorics tasks, then refined and linked those representations to develop an understanding of the relationship among the tasks, combinatorial idea of (m choose n), and Pascal's Triangle. Students' use of their representations was critical for their development of combinatorial reasoning and justification.

The National Council of Teachers of Mathematics [NCTM] (2000) recommends that combinatorics be an integral part of the mathematics curriculum. Research indicates, however, that "Combinatorics is a field that most pupils find very difficult" (Batanero, Navarro-Pelayo, and Godino, 1997, p. 182), and Schielach (1991) asserts that students are often given many formulas, with little justification, and attempt to memorize these formulas with the result being "disastrous" (p. 137). This study documents the mathematical development of a group of eleventh-grade students who built representations to solve challenging combinatorics tasks and then refined and linked their representations in order to develop an understanding of the relationship among the tasks, the combinatorial idea of (*m* choose *n*), and Pascal's Triangle. The perspective underlying this research is based on the view that when students are presented with challenging problem-solving tasks in an appropriately supportive environment, and have opportunities to build, refine, and link representations for their ideas, they can build elaborate justifications (Tarlow, 2004; Warner & Schorr, 2004; Davis & Maher, 1990). The NCTM (2000) states, "It is important that students have opportunities not only to learn conventional forms of representations, but also to construct, refine, and use their own representations as tools to support learning and doing mathematics" (p. 68). In this manner, students develop understanding by building upon their experience, rather than by being told by the teacher.

Methods

In an ongoing seventeen-year longitudinal study involving the development of students' mathematical ideas[1], students were engaged in combinatorics investigations in grades two through eight, during which they worked together to find solutions to problems and build justifications for their ideas. During this component of the longitudinal study, eleventh grade after-school sessions, several identical and similar problems were posed to nine students, five of whom were a subset of the original group. The tasks that are the basis of this study are the Tower and Pizza Problems. In the Tower Problem, students are asked how many towers *n* unifix cubes tall could be built, given two colors to choose from. In the Pizza Problem, students are asked how many pizza choices a customer has, given *n* toppings to choose from.

Videotapes of each session, students' written work, field notes, and videotape transcripts provide the data for this research. A qualitative methodology for data analysis was employed. Students' representations, strategies, justifications, connections, and interactions, as well as the role of the teacher/researcher were coded, and the codes were used to identify and trace the students' development of representations, reasoning, and justifications.

Results and Conclusions

When presented with the Pizza Problem, one student, Stephanie, suggested that they employ a strategy that they had used with the Shirts and Pants Problem or the Tower Problem [referring to problems that they had explored in the study several years earlier], and Shelly recalled that they had used a tree diagram. The students used tree diagrams, which they modified to avoid duplicating pizzas, and then lists of letter codes to represent pizza choices. They organized their pizzas by cases according to the number of toppings and used a proof by cases to justify their solution. They found that for pizzas with four toppings available, there were 1 4 6 4 1 topping combinations, for a total of sixteen pizzas, and recognized these numbers as a row in Pascal's Triangle. They extended their representation and used the Triangle to determine the number of possible pizzas with five available toppings—the next row 1 5 10 10 5 1—for a total of thirty-two pizzas. The students then connected the numbers on Pascal's Triangle to the corresponding topping combinations and used pizzas to explain the addition rule for generating rows on the Triangle. Robert generalized the solution for the total number of pizzas as 2^n, for n available toppings, based on a doubling pattern that he observed when he summed the numbers in the rows of Pascal's Triangle. When Stephanie explained how pizzas could be moved to two different places on the Triangle—in one they remain the same and in the other they get an added topping, Amy-Lynn connected this "two" with Robert's 2^n to provide a justification for his generalization. Furthermore, the students connected the numbers on Pascal's Triangle to towers and used towers to explain the addition rule on the Triangle. They also explained the isomorphism between the Tower and Pizza Problems.

As the students explored the tasks, they retrieved, built, refined, and linked representations. They developed a progression of representations that became increasingly abstract and symbolic, and they moved back and forth between their representations as they developed their ideas. In this manner, they developed important ideas in combinatorics and justification for their ideas. The students' use of their representations was critical for their development of an understanding of combinatorics, a topic that has been noted to cause difficulty for students. This suggests that ideas of combinatorial notation should not be imposed on students, but rather students should be given time to develop these ideas in environments that encourage them to recall, produce, refine, and connect representations. By examining the nature and role of the students' changing representations, educators may better understand students' use of representations to develop and justify their combinatorial ideas.

Endnotes

1. This work was supported in part by National Science Foundation [NSF] grants MDR9053597 (directed by R. B. Davis and C. A. Maher) and REC-9814846 (directed by C. A. Maher) and by grant 93-992022-8001 from the NJ Department of Higher Education [NJDHE]. Any opinions, findings, conclusions or recommendations expressed in this work are those of the author and do not necessarily reflect the views of the NSF or the NJDHE.

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INTERPRETATION OF THE CABRI DRAGGING IN A LEARNING EXPERIENCE

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This paper reports the results of an exploratory study in the manner in which students of senior secondary school perceive dragging after multiple activities on the construction of geometrical figures with ruler and compass as well as with the software - Cabri–Géomètre

Many works have established the necessity of studying the student's perception on relying on geometrical elements for drawing and the sequence of drawing on a dynamic geometrical environment. (Hazzan & Goldenberg, 1997; Goldenberg & Cuoco, 1998; Mariotti, 2001, Talmon & Yerushalmy, 2004).

The ability to perceive the relationships that exist among the properties of given geometrical figure is a necessary condition to understanding a Mathematical proof. Without this it is impossible to know with which theory to explain the problem at hand. In the dynamic geometrical environment, the ability to drag an object is related to the notion above (Mariotti, 2001). The question then is: Can various activities using Cabri enable the participants develop enough skills to recognize the relationships that exist among different properties of geometrical objects?

Participants

The participants are 42 students. They are all in the third semester in the high school. Their ages range between 15 and 16 years. The participants are already familiar with the use of ruler and compass for drawing and were given instruction on the use of Cabri. One of the authors of this paper works as a teacher of Mathematics III – geometrical drawing, which allowed him to observe the students over a period of 30 hours. This work focuses on observation carried out on 19 students randomly selected from the participants.

Procedures

The construction of typical geometrical figures was carried out using ruler & compass and with Cabri, following some given steps, other activities include exploration of typical figures, discovery and conjectures. 20 hours were dedicated to the former and 10 hours to the latter.

Answer sheets were given to students for both drawings. Questionnaires were administered on each student after they had carry out to conjectures activities.

Instruments

Questionnaire of 3 questions:

- Why could some elements be dragged and some could not?
- Describe what happens when an element is dragged?
- What happens to the properties of some geometrical figures when they are dragged?

The answers to the above questions are complex to a certain degree, but the following answers, among others, which demonstrate to a certain degree the understanding of dragging a figure, could be expected from the participants.

- There are independent and dependent figures or objects. The latter can be constructed from the former.

- Certain objects remain constant, some change in some aspects. In some, certain properties are conserved; depending on how the were constructed.
- The properties that were established during the construction using the commands of the menu are conserved. In another case, some properties appeared on the some of which were conserved why others were not.

Results and Discussion.

Question 1. Even though the participants had dragged many objects during various activities, almost half of those examined did not attempt to respond to the question which asks about the difference between the objects that can be dragged and those that can not. Among those who answered this question, 4 gave an answer that was least expected of them; the "dependent object" -that were first constructed can not be moved while those that were constructed from the dependent objects –dependent figures, are movable. This idea runs contrary to the working of Cabri and its source could be traced to a static knowledge of geometrical drawing (Talmon & Yerushalmy, 2004).

Question 2. The answers indicate that participants could not immediately observe many of the properties of the figures drawn –the figures that change in whole, those that partially change, with some of its elements moveable while other are static. The participants lack the ability to spontaneously interpret or read meaning to some phenomenon of their activities. The more advanced student only pointed out that certain elements in the figures are movable, with no mention on those figures that could be movable in whole. The stragglers among the participants could only observe the movement of the object without being able to detect which elements or parts of the figures, remain constant.

Question 3. Most of the participants could not clearly distinguish the concept of geometrical properties from the perceived characteristics of the drawings. They identified the properties of the figures through its constituent elements. They were unable to understand the property of a geometrical figure as being an integral characteristic of the figure and not the constituent elements.

Conclusions

Based on the drawing activities, the participant, were able to observe some relationships that exist between the objects and the sequence of their construction, without a clear spontaneous idea or knowledge on the figures and their properties. This on one hand is as a result of mode of thinking about the objects, which sees them as static. On the other hand, the participants were able to identify the features in the drawing through the geometrical properties. However, the act of drawing with Cabri seems promising enough for students, to gradually perceive the properties that involve in the constructions. This facilitates an orientation towards identifying some differences in the interpretation of dragging by students which could help in understanding of dependent geometrical relationships that were dragging on the dynamic geometrical environment.

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STUDENTS' STRATEGIES FOR CONSTRUCTING MATHEMATICAL PROOFS IN A PROBLEM-BASED UNDERGRADUATE COURSE

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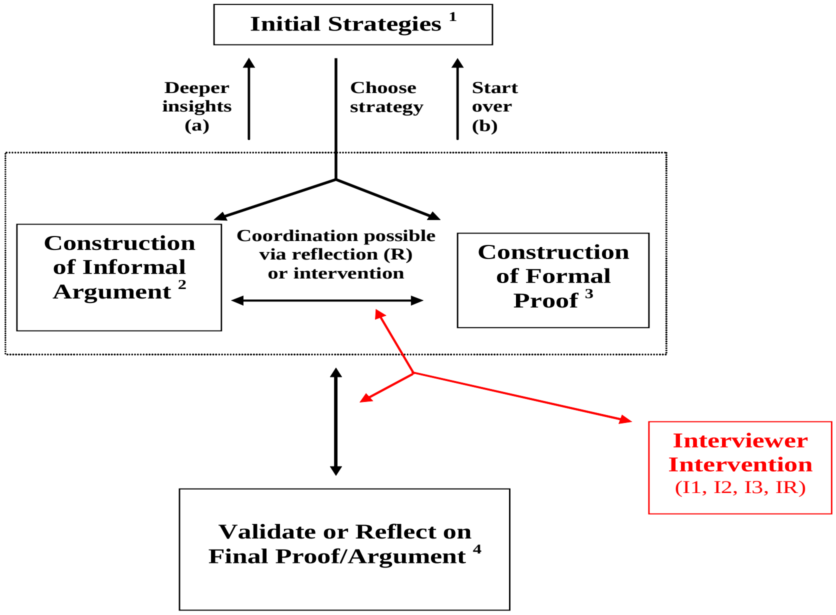
This paper reports the results of an exploratory study of students' proof strategies in the context of a problem-based undergraduate number theory course. The students' strategies for constructing proofs varied depending on context, but our analysis demonstrated that they were primarily engaged in making sense of the mathematics, rather than attempting to reproduce particular proof types or strategies.

Mathematics educators and education researchers have reported students' difficulties with mathematical proof and point out the conflict between the nature of this essential mathematical activity and current approaches to teaching it (Hanna, 1991; Harel & Sowder, 1998; Moore, 1994; Raman, 2003; Selden & Selden, 1995; Usiskin, 1980; Weber, 2001). This recent interest has led to an increased effort to teach proof in innovative ways. One instructional approach that emphasizes student-centered learning processes is "inquiry-based" or "problem-based" teaching, in which the central activity of the course is to engage students in mathematical inquiry or problem solving, rather than to present them formal mathematics in the form of a lecture.

Our study focused on students' proving processes and strategies for constructing proof in inquiry-based undergraduate number theory course at a large state university. In this particular course, students worked outside of class to solve problems and prove theorems. The students then presented their solutions during class meetings and the instructor led whole-group discussions of their work. The role of the instructor in the course was that of facilitator and advisor; he did not lecture or present himself as the arbiter of mathematical "truth". The students in the course were expected to determine the correctness of the presented solutions through their discussion. The course served as a "transition to proof" course at the university, so for most of the students, this was their first course in which formal mathematical proof was the primary focus. The instructor did not tell them what constituted a mathematical proof; rather, he expected the students to construct an understanding of proof by participating in the course.

In order to closely examine how students in this non-traditional context learned to construct mathematical proofs, we employed an exploratory case study design guided by two research questions: What processes do undergraduate students employ when proving mathematical statements? What strategies do students use to construct mathematical proofs in an inquiry-based undergraduate course? Six students were selected from those enrolled in the above-described course. During a series of four task-based interviews conducted over the span of the semester of study, the students were asked to construct proofs of various number theory statements.

The interview transcripts were analyzed using open coding techniques, and a framework of interconnected "paths" for proof emerged. A visual representation of these paths is shown in Figure 1. Students' proving processes consisted of four phases: use of initial strategies, construction of informal arguments, construction of a formal proof, and validating or reflecting on the final proof/argument. Our results demonstrated that students' proving processes were not necessarily hierarchical in nature, but shifted fluidly and frequently between these phases. The students were primarily engaged in making sense of mathematics and often used concrete examples to understand the statement, clarify the strategy or gain further insight into the problem.



Reflection was an important component in students' progress in constructing proofs and appeared to be the mechanism by which students shifted from one phase of the proving process to another. We found that the students did not tend to reflect on the final proof or argument spontaneously, so the influence of our intervention was included in the analysis.

Overall, our results also showed that individual students' strategies for constructing proofs varied greatly, in contrast to the more static tendencies for proof frequently seen discussed in the literature. The participants in our study did not seem to prefer the same sequence of phases for each proof attempted, nor did they appear to use the same strategies each time. We hypothesize that the problem-based structure of the courses facilitated the development of their relatively flexible and sophisticated strategies for proof.

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IMPORTANT MOMENTS IN CALCULUS: QUALITATIVE RESEARCH IN AN EXPERIMENTAL MATHEMATICS COURSE

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Historically, calculus has triggered and supported revolutionary shifts in how we see, think about, and work in the world, but can similar impacts occur on the students in a course where students were invited, through real-world problem tasks that raised central conceptual issues, to invent major ideas of calculus? Research in cognition and brain function has recently revealed the fundamental role played by the personal experience of perceptually guided action in the forming of many central mathematical concepts.

The research here centers on the following questions:

- *How might careful, explicit analyses of learners' problem solving offer opportunities to revisit and perhaps rethink important mathematical content*?
- *How might such studies offer insight into how more learners might come to understand and care about important mathematics concepts?*

In this poster I will describe some preliminary results from the analytic phase of a research study based on data from experimental teaching in a special section of Honors 250, *Mathematical Modeling*, taught in Fall 2005 by professor Robert Speiser. The analyses focus on learners' actions and reflections, considered in detail.

Honors 250 is a one-semester general education course where students outside the sciences can look closely at how mathematical ideas help clarify our understanding of the world. Problem-based instruction centered on key issues in the mathematical study of motion and change. The project's research focuses closely on work and thinking of the students, as they sought both personally and collectively to build key ideas, representations and compelling lines of reasoning.

The fourteen student subjects, all majoring in dance or dance related subjects, constituted one entire section of Honors 250. My work emphasizes a focus group working within the full classroom context. The class met three hours at a time, once a week, for fifteen weeks. I videotaped all class sessions and took field notes on each. Student journals were also collected. As students in the arts, these students were seemed to value creativity, technique, and analytic rigor. The videotaped record of the work and thinking of these fourteen students, supplemented by their drawings and inscriptions, constitute the data I will analyze.

Preliminary findings suggest strongly encouraging responses to both research questions mentioned above.

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CONFLICT AND MATHEMATICAL ACHIEVEMENT IN A GRADE 12 CLASSROOM

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Complexity theory focuses on complex learning systems which arise through the actions and interactions of individual agents and which engage in ongoing adaptation to a changing environment (Davis & Simmt, 2003). One of the necessary conditions for the emergence of a complex learning system is internal diversity. Activity theory likewise deals with evolving dynamic systems situated in an environment (Nelson, 2002) but stresses the role of internal incoherence and conflict in bringing about change. An important question in comparing the two theories is hence "in what ways do internal diversity and conflict differ?".

In this study a group of eleven grade 12 students worked on a problem using Geometer's Sketchpad. When students shared their findings, one student, who had created an incorrect formula, was challenged, and another student, with a useful data-collection idea, got little feedback. The first student went on to independently discover the correct formula and prove it with mathematical induction; the second student did not develop her work further.

The type of interactions in which both students engaged affected their mathematical behaviour. The challenge put to the first student was made tentatively, on the basis of differing results, and a blatant arithmetic mistake was ignored. The second student failed to engage her neighbours with her ideas. Both students experienced conflict, in the sense of a "disagreement between people with different ideas or beliefs" (Oxford Paperback Dictionary, 1991), but the first student experienced interest and respect while the second experienced hostility and indifference.

This finding suggests that the ways in which diversity or difference are both communicated and responded to have an important effect in subsequent mathematical achievement. It also points to the inadequacy of the general notion of conflict as a factor in change: further comparison between complexity and activity theory will require further elaboration of this notion.

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INFORMAL MATHEMATICS LEARNING: A CONTEXT FOR STUDYING KNOWLEDGE GROWTH IN STUDENTS AND TEACHERS

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The NSF-funded project "Research on Informal Mathematics Learning (IML)" is a three-year, multi-pronged study of mathematics learning in the context of an after-school enrichment program at an urban middle school in Central NJ. During the first year, a team of Rutgers researchers facilitated three cycles of six after-school sessions with the first cohort of 6th grade students. As part of the study's design, a group of 6-9 of district teachers became "teacher-interns" to the research, observing after-school sessions being facilitated by the research team and joining them in discussion about what happened at each session immediately following it. The teacher-interns then studied the video recordings of those sessions in preparation for the second year when they began to facilitate IML sessions in replication of the first year's activities with a new cohort of 6th grade students. The first cycle of activities for both cohorts of students centered on problem-solving tasks using Cuisenaire rods to study students' reasoning. For the students, this led to exploring proportional relationships, including fractional parts to wholes.

The data in this presentation come from the sixth day of the first cycle facilitated by the teacher-interns, after the students had explored several tasks in which they reasoned proportionally using Cuisenaire rods to justify their solutions. One of the teacher-interns co-facilitating the session introduced a new question building on relationships identified in previous tasks. Referring back to some models made with Cuisenaire rods via the overhead projector, he asked the students "Are all one-thirds the same?" He suggested they discuss the question at their tables and be prepared to offer justifications for their arguments. Students had about 15 minutes to work at their tables. The teacher-interns circled the room to observe whether students understood the task and to monitor their progress.

Video data collected from the three cameras that recorded the IML activities were analyzed using the methodology described by Powell, Francisco & Maher (2003). The events were then situated within the context of the full cycle through working with fellow researchers in a graduate-level mathematics education course who had carefully analyzed video data from the first five sessions and post-session discussions of this particular cycle.

All sixteen students worked on the problem at their tables (four students at each table), and two pairs of students made presentations at the overhead projector to share their reasoning. The first pair of students concluded that the answer is not "yes *or* no" but rather "yes *and* no", and the second pair agreed with them. The poster will illustrate representations shared by the students and how, through a particular mode of questioning that the teacher-interns learned how to emulate, students articulated that all one-thirds are the same in that for each instance it takes three to make one whole (unit), but that they are different in that they are one-thirds of different wholes (units).

Endnotes

1. This research was funded by National Science Foundation grant REC-0309062.

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SOLUTION STRATEGIES LINKED TO PROPORTIONALITY PROBLEMS: CASE STUDY

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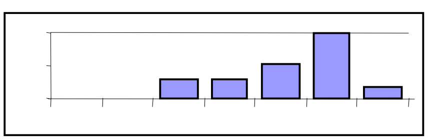
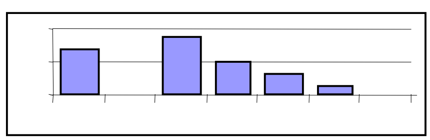
The following research is aimed to explore potential relations between the mathematic experience that elementary education students (10-11 years old) have accumulated in two contexts -within school and outside the school- and the complexity level of their strategies to solve proportionality arithmetic problems. The study was carried out in two schools: while "I" has a high institutional cohesion level, "Z" has an incipient organization. The students in "Z" have wide "out-of-school-mathematics" experience (due to their activities in informal economy) and students in "I" excel due to their performance achieved in official evaluations on mathematics. 5 students of each school took part (those with the highest performance) and were asked to solve 5 proportionality problems (missing value with some structural difficulties), by means of two semi-structured interviews (recorded on video and transcribed), and one written exam. Results of research on proportional thinking and Piagetian Genetic Epistemology, were used as the interpretative framework, in order to define, analyze and portray the complexity level of the strategies provided by the children. Thus, special attention was paid to the kind of objects involved in such strategies, the operations carried out by the students over such objects, as well as the degree of conceptual structuring. The following three levels and sublevels were identified:

Level 1. Absence of Proportional Thinking. 1.A. They focus on one single term isolated from a single Measure Universe (UM) to which they compulsively apply operations. 1.B. They take a single object from each UM, to which they symmetrically apply qualitative actions, with the purpose of compensating Level 2. Pre-proportional thinking. 2.A. They built ratio tables by means of counting processes, thoroughly considering all possible objects, which they represent in an iconic way. 2.B. They built ratio tables by means of recursive processes based upon multiples. The UM has a very incipient structure (multiplicative). Iteration and its representation point towards reasoning (still very attached to the object). Level 3. Proportional thinking. 3.A. They apply the scale factor to objects that belong to UM taken as totals with a multiplicative structure. There is coordination of arithmetic operations. 3.B. They apply proportionality factor to the UM considered as sets with a clear multiplicative structure. It is the immediate precedent to the direct proportional variation idea as linear function and the operativity over the complete totals (linked to reflexive abstraction processes). The poster will show typical answers from every sub-level.

During the research, there were group regularities detected regarding the chosen strategies: in "I", a clear tendency to use the more general and efficient strategies, located in the last two levels of cognitive complexity, while in "Z" the use of strategies that stick more to concrete (see graphics).





Our study suggests that a school like “I” provides the student the cognitive tools that less structured schools seem not to provide (like “Z”) or the informal mathematics experiences that such student has. This research provides evidence about the influence of context in the constitution of their cognitive schemes, and, as a result, in their development.

USING A LITERAL LENS TO INVESTIGATE MIDDLE SCHOOL STUDENTS' UNDERSTANDING OF A PROBABILITY EXPERIMENT

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This poster presentation highlights a selection of posters constructed by urban middle school students to justify their solutions to a set of probability investigations using data from computer simulations, and the students' written comparisons and critiques of each others' justifications. In this study, ten students, working in pairs, analyzed the fairness of dice. The students simulated the tossing of each die, supposedly produced by different companies, on computers using Probability Explorer (Probability Explorer (c) Hollylynne Stohl Lee, 1999-2005), a software tool that provides graphic and table representations of the data as the die is "tossed" for any given sample size requested by the students. Each company's die had been weighted by the computer program with the actual weights hidden from the students.

The authors studied the inscriptions, symbolic - graphic - linguistic that the students chose to include on their posters and the conclusions they made about the data, as well as the evidence they used in comparing the various solutions and agreeing or disagreeing with other students' conclusions. Of particular interest, was the evidence documenting the students' reasoning about the importance of sample size and their understanding of fairness. Subsequent interviews with these students gave further insight into their reasons for selecting a graph or table to constitute evidence. Some students indicated that there might be different definitions of fairness, and that a claim is valid as long as evidence supports it.

The framework for this study is guided by research on the development of representations (Davis & Maher, 1997). Students today are expected to be able to read, analyze, synthesize, write and discuss solutions to mathematical problems (NCTM, 2000). Their ability to solve mathematical problems depends also on their ability to connect with the natural language of the classroom. This research analyzes the students' posters and critiques through a lens that focuses on graphic, symbolic and written inscriptions as evidence of the students' mathematical thinking carried out in their urban context.

This research is part of a study on representations and reasoning from data that is one component of the "Informal Mathematical Learning" Project, a three year research project focusing on the mathematical thinking of urban, middle school students as they engage voluntarily in open-ended investigations in after school sessions and summer institutes, supported by the National Science Foundation (ROLE Grant REC0309062, directed by Carolyn Maher, Arthur Powell and Keith Weber).

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RESEARCH METHODS

DEFINING MATHEMATICS EDUCATIONAL NEUROSCIENCE

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Educational neuroscience is a potentially foundational new area of scientifically grounded, evidence-based research that promises to help integrate and add dimensionality to traditional forms of educational research. In particular, educational neuroscience seeks to combine theoretical and methodological orientations of educational research, such as those of developmental and cognitive psychology, with methods of cognitive neuroscience and psychophysiology. Using traditional methods of educational research informed by research in cognitive psychology and cognitive neuroscience, mathematics educational neuroscience aims to integrate and extend what is known about our biological and cultivated abilities to think mathematically. This paper explores some of the background, paradigms, and possibilities for this emerging new area of research in mathematics education.

Introduction

There has been much research in mathematics education that has addressed a wide variety of affective, cognitive, and social issues (e.g., Grouws, 1992), and there have been a wide variety of phenomenological, behavioural, cognitive, and social interactionist approaches taken to understanding these issues (Sierpinska & Kilpatrick, 1998). To date, however, researchers and practitioners in mathematics education remain largely unaware of and uninformed by other growing bodies of research into the more biologically grounded nature and processes of mathematical cognition and learning, especially in the areas of cognitive psychology (e.g., Campbell, 2004), cognitive neuroscience (e.g., Dehaene, 1997), and in the fast emerging area of neurogenetics (e.g., Gordon & Hen, 2004). Basically, related phenomena pertaining to mathematics education are being studied from a variety of humanistic, social, and scientific perspectives. These disparate areas need to be focused, integrated, and extended to more effectively inform and expand upon current research and practice in mathematics education. This research report introduces mathematics educational neuroscience as a new field of inquiry with that aim in view.

Educational neuroscience is a fast emerging and potentially foundational new area of educational research. There is a general consensus on two basic points. First, educational neuroscience is characterized by soundly reasoned and evidence-based research into ways in which the neurosciences can inform educational practice, and vice versa. Secondly, educational research in cognitive psychology informed by, and informing, cognitive neuroscience constitutes the core of educational neuroscience (*cf.*, Bruer, 1997).

Background

Educational neuroscience is viewed here as a new area of educational research, not so much in terms of building a bridge between the neurosciences and education, but rather, as filling a gap between these vast areas. In filling this gap, foundations for such a bridge can be put in place. Educational neuroscience prioritizes learners. It is informed by, but is not geared toward identifying neural mechanisms underlying or accounting for cognitive behavior. Such is the task of cognitive neuroscience, not educational neuroscience.

Accordingly, differences between educational and cognitive neuroscience can be exemplified by the latter's quandaries regarding the function of consciousness and how it arises from, and even how it can *possibly* arise from the activity of neural mechanisms. Educational neuroscience, on the other hand, takes the reality and utility of consciousness as given; something to work with, not something to explain, or to explain away.

Perhaps the major area of potential overlap and common interest between cognitive neuroscience and educational neuroscience are cognitive and educational psychology, respectively. Both have been engaged, from various philosophical perspectives, in developing models of cognition and learning. Both are concerned with identifying and establishing reliable correlations between these models and brain behavior.

Over the past couple of decades, collaborations between cognitive psychologists and neuroscientists have been forging ahead (Byrnes, 2001) resulting in the vibrant and rapidly expanding new field of cognitive neuroscience. Cognitive neuroscience offers new alternatives and greater dimensionality in moving toward more scientific evidence-based educational research. Driven by new brain imaging methods, and informed by decades of lesion studies, cognitive neuroscience is making great strides in correlating cognitive function with brain and brain behaviour. Mathematics education can benefit greatly from these developments. This suggests the possibility of a truly *educational* neuroscience, as a bona fide new area of educational research that is both informed by methods, theories and results of cognitive neuroscience. Of specific interest here is the possibility of a bona fide *mathematics educational neuroscience* as a new area of research in mathematics education.

There have been previous initiatives and efforts to incorporate cognitive science and cognitive technologies into research in mathematics education (e.g., Davis, 1984; Schoenfeld, 1987. Pea, 1987). Until very recently (e.g., Campbell, forthcoming), however, there has been very little to be found in the literature exploring or drawing out possible implications of neuroscience or cognitive neuroscience for research in mathematics education. Indeed, the term "neuroscience" is not to be found in the indexes of the following publications: Grouws, 1992; Sierpinska & Kilpatrick1998. Perhaps more surprisingly, despite much hoopla over the '90's being designated as "the decade of the brain," and a naïve though quite popular "brain-based education" movement, which is in dire need of critique, there is no mention of that term (viz., "neuroscience") whatsoever in Guitierrez & Boero, 2006.

Embodied Cognition

Setting aside foundational dualisms that have traditionally served to undermine a unified studies of subjective human experience and objectively observable behaviour (Campbell & Dawson, 1995), educational neuroscience adopts a view that: 1) embeds mind in body (with a special emphasis on brain); 2) situates embodied minds within human cultures; and 3) recognizes the biological emergence of humanity within and our dependence on the natural world (Merleau-Ponty, 1962, 1968; Varela, Thompson, & Rosch, 1991; Campbell & Dawson, 1995; Núñez, Edwards, Matos, 1999). To help illustrate the unifying power of this view, consider Eugene Wigner's famous reflections on the "unreasonable effectiveness of mathematics in the natural sciences" (1960). If mind is fundamentally (i.e., ontologically) distinct from the material world, it remains a mystery as to why mathematics can be applied to the world so effectively. If mind is embedded within the material world, as the embodied view suggests, mystery dissolves into expectation (Campbell, 2001). Moreover, in considering embodied mind as an ontological primitive, there is no need to treat consciousness as an inexplicable and apparently useless epiphenomenon of erstwhile mechanical neural processes (cf., Jackendoff, 1987). We can then expect the subjective experience of mind to share and participate in the kinds of objectively observable structures and processes that are evident in the natural world (Campbell, 2003a, 2002a, 2001).

In accord with this naturalistic embodied, situated, and emergent view, when meaning is constructed, transformations take place in the minds that are manifested through bodies (especially through changes in brain behaviour). It remains possible, of course, that such embodied, i.e., objectively observable and measurable, manifestations of mind, remain but shadows of subjectivity, analogous in a sense to the way in which the exterior of an extensible object is but a external manifestation of that object’s interior. Scratch away at the surface of an extensible object as much as one might, some aspects of the interior will always remain hidden. The bottom line here is that brain and brain behaviour are made progressively more manifest to investigation through close observation and study of embodied action and social interaction in both clinical, classroom, and ecological contexts. With the advances in brain imaging, the shadows of mind are becoming much sharper.

Embodied cognition provides mathematics educational neuroscience with a common perspective from which the lived subjective *experience* of mind is hypothesized to be manifest in objectively observable aspects of embodied *actions* and *behaviour*. Such framework may enable educational neuroscience to become a bona fide transdisciplinary inquiry (Gibbons, Limoges, Nowotny, Schwartzman, Scott, & Trow, 1994), in that it has the potential to integrate and to extend well beyond traditional ontologically disjoint frameworks, be they of mind (i.e., phenomenology), brain (i.e., neuroscience), function (i.e., functionalism), or behaviour (i.e., behaviourism). Moreover, there is no need to attempt to reduce mind to brain (physicalism), or brain to mind (idealism).

New Questions and New Tools

Embodied cognition introduces new research questions pertaining to investigations into teaching and learning. Consider, for example, what kinds of detectable, measurable, and recordable psychophysiological changes are occurring in learners' minds and bodies during mathematical concept formation — that is, when various mental happenings coalesce into pseudo or bona fide understandings of some aspect of mathematics. For instance, what observable embodied changes in brain activity detectible using electroencephalography (EEG) occurred in a student working with Geometer's Sketchpad™ as she came into a realisation that all right triangles inscribed in a circle must pass through the centre of that circle (Campbell 2003b)? And what of the student at ease with graphs, yet who cringed at the sight of mathematical symbols on the screen (ibid.)? How might eye-tracking technology, electrocardiography (EKG), and galvanic skin response (GSR) have helped to quantitatively capture embodied responses what is but qualitatively described in the foregoing as a “cringe.” Capturing embodied behaviours at moments such as these may provide rich and important insights into subjective experience and afford exciting new venues for mathematics educational neuroscience research.

What is gained from using methods of psychophysiology and cognitive neuroscience, such as EEG, EKG, eye-tracking, and GSR, are new means for operationalising the psychological and sociological models educational researchers have traditionally developed for interpreting the mental states and social interactions of teachers and learners in the course of teaching and learning mathematics. This statement holds for qualitative educational researchers and quantitative educational psychometricians alike. It bears emphasis that educational neuroscience can augment traditional qualitative and quantitative studies in cognitive modelling in general, and more specifically, in research in mathematics education. McVee, Dunsmore, & Gavelek (2005) have recently argued quite compellingly that schema theory, the mainstay of cognitive modelling, remains of fundamental relevance to contemporary orientations towards social and cultural theories of learning. Holding fast to a *humanistic* orientation, educational neuroscience concerns both psychological, sociological, and naturalistic dimensions of learning, only now, using methods of cognitive neuroscience, all the while guided by, and yet also serving to test and refine, more traditional educational models, questions, problems, and studies.

Research in mathematics education to date has taken a wide variety of phenomenological, behavioural, cognitive, and social interactionist approaches to mathematical cognition and learning, and not surprisingly so. Mathematics education lies at the crossroads of many well-established areas, including mathematics, psychology, epistemology, cognitive science, semiotics, and sociology (Sierpinska & Kilpatrick, 1998). A central focus of this author's recent foray into mathematics educational neuroscience concerns preservice teachers' understandings of mathematics (e.g., Campbell & Zazkis, 2002; Zazkis & Campbell, 2006). Most of this cultivated understanding goes well beyond what is known about the biological and psychophysiological groundings of mathematical cognition studied by cognitive neuroscientists (e.g., Dehaene, 1997) and cognitive psychologists (e.g., Campbell, 2004). It seems important that these culturally inculcated understandings should be consistent in connecting with and building upon their biological and psychophysiological underpinnings (e.g., Dehaene, Piazza, Pinel, & Cohen, 2004). What is more likely the case, and this is the central and guiding hypothesis for this initiative, is that there are a range of "disconnects" between our inherited biological predispositions for mathematics and the culturally derived mathematics comprising the K-12 mathematics curriculum.

Cultivating mathematical ability is the main task and mandate of mathematics education. Most of this cultivated understanding goes well beyond what is known about the biological and psychophysiological groundings of mathematical cognition studied by cognitive neuroscientists, neurogeneticists, and cognitive psychologists. It seems important, however, that these culturally inculcated understandings connect and are consistent with their biological and psychophysiological underpinnings. A guiding hypothesis in defining mathematics educational neuroscience is that there are a range of "disconnects" between our inherited biological predispositions for mathematics and the culturally derived mathematics comprising the K-12 mathematics curriculum. Pursuing the implications of this foundational hypothesis of disconnects in understanding between biologically and culturally developed aspects of mathematical cognition has inspired, motivated, and guided this embodied approach to defining mathematics educational neuroscience.

As a case in point, there is an emerging consensus in cognitive neuroscience that the human brain naturally supports two key mathematical processes: a discrete incrementing process, which generates countable quantities, and a continuous accumulation process, which generates continuous quantities. Gallistel & Gelman (2000) have noted an emerging synthesis between these two processes, and the tensions between them, have been "central to the historical development of mathematical thought" and "rooted in the non-verbal foundations of numerical thinking" in both non-verbal animals and humans. These processes also appear to be implicated in Lakoff and Nüñez's (2000) four fundamental "grounding metaphors" of object construction and collection (viz. discrete) and measuring and motion (viz. continuous). In research in mathematics education, it is well documented that many children and adults have notorious difficulties in moving from whole number arithmetic (working with quantities) to rational number arithmetic (working with magnitudes). It is common practice in mathematics education, in accord with a relatively quite recent development in the history of mathematics, to view whole numbers as a “subset” of rational numbers. This may constitute a classic disconnect between our natural biological predispositions and the mathematics curriculum and instruction, and potentially explains why this shift is so problematic for learners from early childhood into adolescence and beyond. Identifying and reconciling disconnects such as these can be taken as central issues and concerns in defining mathematics educational neuroscience.

Methodologies and Applications

Electroencephalography

Electroencephalography (EEG) is a clear method of choice for educational researchers who would like to augment their research with quantifiable observations of brain and brain behavior. One reason for this is that EEG is amongst the least expensive of brain imaging technologies. Another is that EEG is quite adept for capturing the dynamics of thought in action. It offers temporal resolutions at the speed of thought and places fewer spatial constraints on learners than other methods. Furthermore, as evidence of increasing confidence in both the reliability and robustness of the method, many “turnkey” acquisition and analysis systems are now readily available, placing fewer technical burdens on researchers using these systems.

Cognitive neuroscientists have developed a viable approach to studying complex cognitive phenomena through electromagnetic oscillation of neural assemblies (e.g., Klimesch, 1999). The key to this electroencephalographic approach is the notion of event related desynchronization/synchronization (ERD/S) (Pfurtscheller & Aranibar, 1977). In the course of thinking, the working brain produces a fluctuating electromagnetic field that is not random, but rather appears to correlate well within distinct frequency ranges with higher cognitive function in repeatable and predictable ways. With the elements of such a "neural code" falling into place, well-developed and reliable tools and methods for data acquisition and analysis of EEG correlates of cognitive and affective function are available to mathematics educational neuroscientists to study these phenomena.

Eye-tracking and eye-gaze

Non-intrusive, methods have been developed for remotely measuring eye movements in human-computer interactions. These remote-based methods have become very reliable, quite robust, and easy to set up. Most instructional software today can be offered in web-based environments. Remote-based eye-tracking, therefore, offers reliable means for evaluating the design and usage of computer enhanced mathematics learning environments (CEMLEs). Specific measures that can be analysed and correlated with EEG include the real-time tracking and video recording of users’ hand (mouse) and eye movement, including dwell time of eye-gaze on various aspects of the web site, both in serial time and cumulative time modalities. Results from such analyses can serve to inform understandings of CEMLEs and contribute to improving and expediting their design and development.

Psychophysiology

It is well established that affective and cognitive factors are deeply interrelated (e.g. Bell & Fox, 2003). Therefore, it seems essential to contrast and conjoin data, say, acquired in moments of insight during problem solving activities using CEMLEs with psychophysiological data functionally correlated to various manifestations of math anxiety. One preservice teacher encountering algebraic symbols in a CEMLE designed to investigate learners' reactions to symbolic and visual representations of division and divisibility reported "... when I look at all of this... all I want to do is throw the screen away," while another "loved it!" (Campbell, 2003b). What functional correlates of brain behaviour are implicated in moments of enlightenment and aversion in the learning of mathematics? Learners interacting with CEMLEs provide rich opportunities for capturing such moments while monitoring their psychophysiological states.

Psychometrics

Research in both cognitive neuroscience and mathematics education reveals anxiety to be a multifaceted construct. Mathematics educational neuroscience research in this area can benefit from being informed by results from both fields. Factors implicated with anxiety include decreased working memory capacity and a general shift away from symbolic toward figurative cognitive functions. Research into math anxiety in mathematics education, on the other hand, has identified anxieties associated with calculating in everyday contexts, performance anxieties associated with being observed in the act of doing mathematics, and various other kinds of anxieties associated with taking tests, using computers and problem-solving. Methods of mathematics educational neuroscience should provide invaluable new resources for detailed and comprehensive research into functional correlates of this complex phenomenon and its inhibitory effects on concept formation. It is desirable, if not imperative, to guide these studies with well validated psychometrical instruments (e.g., Hopko, 2003).

Concluding Remarks

Educational neuroscience is an emerging new area of educational research that promises to integrate educational psychology and cognitive neuroscience in ways that are likely to carry significant new possibilities for research in mathematics education. One set of possibilities indicated here is to apply these frameworks and methods to the study of teachers’ understandings of and aversions toward mathematics, particularly in mathematical problem solving contexts using computer enhanced mathematics learning environments.

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CHILDREN AND ADULTS TALKING AND DOING MATHEMATICS: A STUDY OF AN AFTER-SCHOOL MATH CLUB

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This paper outlines examples of children learning mathematics in a bilingual and culturally inclusive environment utilizing the lens of student identity as a way to understand the intersection of multiple factors affecting their learning. The research is situated in an after-school setting located in a majority Latino/a school. Several studies highlight the importance of community knowledge, situating it on the same level as academic knowledge. Our findings indicate that the inclusion of community knowledge and use of home language are elements that mediate mathematics learning, particularly in the context of social justice mathematics. The study has implications for teachers who wish to create more inclusive and potentially transformative learning environments for all students.

Objectives

Schools in the United States are increasingly becoming locations where children from different cultures of origin and backgrounds come together. These multicultural contexts raise important challenges to professionals who work in educational settings, particularly in light of inequitable educational outcomes for marginalized populations. In this paper we address these challenges within the context of mathematics teaching and learning, in the hopes of uncovering how apparent "challenges" can be transformed into assets by creating learning environments in which they are valued and drawn upon. In particular we focus on the relationship between mathematics education, language, and culture through the lens of identity. With our research we seek to provide insight into what a linguistically and culturally inclusive learning environment might look like as well as to describe the impact of embedding mathematics learning within social justice mathematics.

Theoretical Framework

This research takes place in an after-school "math club" for fourth and fifth graders in a predominantly Latino/a (primarily of Mexican origin), border community. Our approach to the activities in the Math Club is grounded in our belief in education as a tool to transform exclusionary situations and to fight for social justice (Freire, 1998). To this end, many of the activities in this mathematics club have a social justice component. Our main goal is to analyze the teaching and learning of mathematics in this after-school setting, in order to identify strategies that provide learning opportunities to all students. In this particular setting, we analyze the relationships that exist between mathematics, language, culture, and social justice.

We focus our analysis on student identity to attempt to understand the impact of an inclusive and transformative learning environment. Several authors emphasize the importance of considering identity in the form of narratives about persons in analyzing sociological and educational processes related to learning and teaching. A student's evolving identity influences how they learn and how they make sense of mathematical ideas. Sfard and Prusak (2005) view "identity-making as a communicational practice" (p. 16), and therefore interaction would be a

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natural place to understand the development of a student's identity in relation to learning. Identity is a concept that allows us to focus on lived realities in context, such as the process of mathematics learning of Latino children in schools, as well as the impact of a unique learning environment such as the Math Club.

Recent research shows that an individual's home, related to history and the social context, is the source of funds of knowledge (Moll & Gonzalez, 2004) that students could potentially bring to learning situations if the learning environment encouraged it. Researchers in critical pedagogy (Gutstein, 2005; Flecha, 2000) claim that these funds of knowledge should be given the same value as traditional "academic" knowledge. Researchers have explored the idea of funds of knowledge specifically in relation to teaching and learning mathematics (Civil & Andrade, 2002), highlighting the fact that individuals have multiple forms of understanding mathematics that are not only in formal ways. There are several experiences around the world that demonstrate a learning environment based on critical pedagogy and the incorporation of students' funds of knowledge, such as Learning Communities (Elboj, Puigdellívol, Soler, & Valls, 2002) and dialogic learning (Flecha, 2000) in Spain, and Accelerated Schools (Levin, 1998) in the United States. In addition, prior research demonstrates the impact of including mathematics funds of knowledge in classroom environments (Civil, 2002; González, Andrade, Civil, & Moll, 2001). These experiences and research demonstrate the need to understand the intersection of mathematics, language, culture, and identity and inform our research.

Methods

Our methodology is based on the communicative paradigm (Flecha & Gómez, 2004; Gómez, 2001) and includes participant observations of students interacting with others as they learn mathematics. According to this methodological paradigm, research is a tool to transform the situations that we are researching. In particular, methodology is seen as a systematic way to understand the inter-subjective relationships that constitute our world in order to find elements with which to transform exclusionary situations. Through our research we seek for ways in which pedagogy can be transformative for students who are often marginalized in our educational system.

Over the course of the past school year, we conducted after-school sessions twice a week during which students engaged in mathematics activities. All sessions were videotaped in order to conduct in-depth reviews of the interactions amongst participants. A graduate student, two post-doctoral students and four undergraduate facilitators were participant researchers who assisted in facilitation and took detailed field notes. Most sessions involved several small groups of students interacting, and so it was imperative that each researcher took field notes to be able to capture all conversations and interactions that might not have been the focus of the videotape.

Evidence

In reviewing our data, we focused on narratives about students in order to understand the multiple elements of language, mathematics learning and culture interacting in the context of social justice mathematics activities. This analysis allowed us to focus on student identity as a way to analyze how these elements played out in interaction in the immediate context while considering social and historical factors that affect student learning. Our examples illustrate the role that funds of knowledge, social justice mathematics, language, culture, and dialogue played in the Math Club learning environment.

Funds of knowledge

The learning environment of the Math Club was characterized by group work, embedding of activities in real life experiences, as well as facilitators being seen as resources rather than experts. This environment encouraged students to draw on their funds of knowledge easily as they worked to solve mathematics problems related to gardening, such as measuring and evenly dividing up rows, calculating seed depth, and charting plant growth. As students engaged in planning out how to divide up the plots for a garden, they spoke fluidly in Spanish about their prior gardening experiences. One student spoke about how she had learned to add sand to her Nana's garden to save water and ran to the adjacent sandbox to add some to our garden. Another student spoke about knowing how to plant seeds because of work he had done on his uncle's *finca* (farm). Being able to speak in Spanish and positioning all participants as experts enabled these two children who were often distracted in classroom activities or when it comes to mathematics tasks, to engage in the activity and subsequently the mathematics problems.

Identity, language and mathematical understanding in the context of social justice mathematics

During the past school year in which we conducted the Math Club, recent immigration protests began to take place across the United States, in which immigrants voiced their objection to legislation introduced that would criminalize illegal immigrants, among other concerns. During this national mobilization, several participants in the Math Club showed their knowledge of and preoccupation with the issue, prompting us to include an immigration project as a social justice mathematics activity. The direction of the project was open to student input, and we began by questioning the students about the theme of immigration from a mathematics perspective.

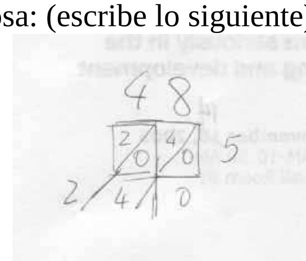
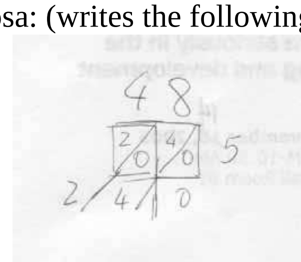
The children became very engaged with this project, and because many of them are immigrants or children of immigrants, they drew on resources from their own lived experiences. Two sisters in the program, whose family includes immigrants from Mexico, decided to carry out a survey with family, friends, and Math Club participants to capture their community's opinions about the legislation and protests. They constructed a table with the data they gathered in collaboration with other participants. Several other students decided to calculate the time it takes for a person to travel on foot from the nearest Mexican border city, Nogales, to their own U.S. border city, Tucson. These students used a map, interpreting the scale in order to calculate the distance between cities. They then calculated the total time it would take to walk that distance, using a standard rate for walking in kilometers and miles per hour. The attached picture illustrates one student's calculations and results. In this example we can see how she uses English (and the U.S. system of measurement) as well as Spanish (in terms of vocabulary and in the use of the metric system). Finally, one student sought to represent the distribution of immigrants by gender, in which he discovered that there are more male immigrants than female. In discussing these results with his

classmates, the students explained that in their experience, seeking jobs is a motive of immigration and that often the male head of household is more likely to be the breadwinner than the female.

Mathematical strategies and dialogue

Dialogue became a natural way to share and learn different strategies to solve mathematics problems, validating students' multiple approaches to formal mathematics. The following dialogue (Table 1), which primarily took place in Spanish, is between a monolingual Spanish-speaking student, a bilingual student, and a bilingual facilitator:

Table 1: Methods of Multiplication

Jenny: (va a la pizarra y apunta 48 x 5 –en vertical-) ¡Dos cientos cuarenta! Facilitador: Muy bien. ¿Sabes otra manera más rápida de hacerlo? Jenny: (mira dudando) Facilitador: Imagínate que tienes billetes de 10. Entonces, 48 son 480, ¿no? Pues si tienes billetes de $5, entonces es la mitad... y la mitad de 0 es 0, no? La de 8 es cuatro, y la de 4 es 2, Con lo cual, qué tienes? (escribe 240). ¿Es el mismo resultado, no? Jenny: (va a buscar a Rosa y le hace la misma pregunta) *Do you know another way to do that?* Rosa: (escribe lo siguiente):  Jenny: (le explica el método que acaba de aprender). Rosa: (se sorprende con el método "nuevo"). (Researcher fieldnotes, 14/11/2005)	(Context) Rosa and Jenny are playing with copies of dollar bills. The facilitator has 48 $5 bills. He asks Jenny how much money he has. Jenny: (goes to board and writes 48 x 5 –vertically-) Two hundred forty! Facilitator: Very good. Do you know faster way to do it? Jenny: (looks doubtful) Facilitator: Imagine that you have $10 bills. So 48 would be 480, right? So if you have $5 bills, then it is half... and half of 0 is 0, right? Half of 8 is 4, and of 4 is 2. And so, what do you have? (writes 240). It is the same answer, right? Jenny: (goes to look for Rosa and asks her the same question) Do you know another way to do that? Rosa: (writes the following):  Jenny: (explains the method she has just learned). Rosa: (is amazed by the "new" method). (Researcher fieldnotes, 11/14/2005)

This example shows a dialogue between two children and a facilitator about a multiplication problem. The first student uses a formal method most often used in Mexican schools to solve the problem (presumably because she is a recent immigrant). The second student uses the lattice method (the method that teachers use in her school) to solve the same problem. The facilitator proposes a third way to do the same problem that demands applying logical reasoning based on knowledge of using ideas such as "double" or "half". The importance of this dialogue is that the students were able to communicate in the language they are comfortable in, and that dialogue took place in which three different methods of solving the same problem emerged, each with its own cultural origin and seen as equally valid. Deeper understanding and an inclusive learning environment are facilitated as a result of using cultural resources.

Mathematics and language

Over the course of the school year, a shift took place in terms of language use. In the beginning, students often spoke in English, the language of instruction in their school, when communicating about mathematics or as a group. Language use became much more fluid and occurred naturally in context depending on participants, purpose and relationships. On one occasion in the Math Club, a boy who is often distracted and unengaged during his classroom mathematics discussions and activities, tried to explain a concept in English. Another student turned to him and said, "Speak Spanish." He then started to explain his mathematical thinking clearly and with ease. Also, students were particularly vigilant about translating mathematics books they read at the beginning of the Math Club when Spanish monolingual students were present, a situation that does not happen in their classrooms. Students often grouped together based on language use so that the Spanish monolingual students were able to communicate in Spanish to group members. These situations evolved in the Math Club environment where code switching and language choice happened naturally and fluidly in all aspects of communication, whether it was social or academic discussion.

Results and Conclusions

Our findings demonstrate the importance of language as a cultural mediator in learning (Vygotsky, 1978; Schliemann, 2002) as well as the need to incorporate students' funds of knowledge in mathematics learning, particularly for students whose language and culture are marginalized in typical classrooms. When the after-school Math Club first started, these primarily bilingual children used only English to participate in the activities. Children asked questions, provided answers and explained their thoughts in English, which is the language used in classrooms for teaching, as dictated by State law. Through observations, we have seen how Latino/a students, many of them recent immigrants and Spanish dominant, began to group with other students who spoke Spanish in order to communicate during mathematics activities. This indicates that language is not only a cultural mediator in terms of an "instrument" that allows us to gain/transmit understanding (in this case in the area of mathematics); language is also a mediator in the range of inter-subjective relationships. Knowledge (or lack of) a language shapes group dynamics in terms of how students interact with each other and in terms of participation and student engagement with activities. During group activities, several Spanish-speaking students who would often appear distracted, quiet or disruptive in their English-dominant classrooms would engage in the activities.

The diverse cultural and linguistic backgrounds of the students also explain the use of different strategies to approach the same operation, as shown in the third example. This demonstrates that prior experience, educational history, and problem-solving methods used by students have a clear impact on the strategies that students use (from the cognitive point of view of mathematical understanding). Here, ideas such as funds of knowledge (Moll & Gonzalez, 2001) or identity development in communication with others (Sfard & Prusak, 2005) play an important role in terms of their contribution in demonstrating and uncovering the social aspect of learning. We argue that this is essential when planning curriculum and in developing a classroom environment and pedagogical strategies for use in diverse classrooms.

In relation to our second area of interest, the impact of embedding mathematics learning within social justice activities, we have found strong evidence of increased student engagement when using mathematics as a tool to read and write the world (in the Freirean sense) from a critical perspective (Gutstein, 2005; Skovsmose & Valero, 2002; Frankenstein & Powell, 1994). The "immigrant" experience that is a part of the evolving identities of most participants in the Math Club played an important role in involving these students in the activities related to immigration. This kind of activity shows evidence of the importance of embedding mathematics in their lived social context in order to engage students and develop their sense of the importance and power of mathematics. Identity appears as a key element in understanding group dynamics and individuals within a learning environment, and allows us to focus on the development of a learning environment that is inclusive of and empowering to all students.

All of these findings suggest that introducing strategies such as incorporating student culture, language and other funds of knowledge into curriculum and classroom teaching is one way to counter dominant ideologies that create barriers to learning and to overcome inequalities in educational opportunities. Our current research focuses on gathering more evidence to prove (or disprove) these findings and on continuing to uncover methods that will prove useful for teachers in developing their practice in diverse classrooms.

Relationship of Paper to Goals of PME-NA

This research project is the result of the collaboration between a mathematician, mathematics education researcher and educational researcher with a background in educational psychology, all with experience in mathematics teaching. This demonstrates the power of interdisciplinary research in order to understand the complex factors influencing the mathematics educational experiences of diverse students. In order to understand how to reverse the achievement gap for students who are marginalized from the educational system in the United States, we must understand the complex factors that affect and shape student identity, and how this can be affected through culturally and linguistically inclusive mathematics learning environments. Through an examination of the changing identities of students in the context of social justice mathematics learning, we highlight the need to include students' funds of knowledge (including home language and culture) in diverse mathematics learning environments.

Acknowledgements

We would like to thank CEMELA (Center for Mathematics Education of Latinos/as) for allowing us to conduct this study as a part of their overall research agenda. CEMELA is funded by the National Science Foundation under grant ESI-0424983. We would also like to thank the Fulbright Commission for the support given to one of the researchers and authors of this study, through the Fulbright Visitor Program. The views expressed here are those of the authors and do not necessarily reflect the views of the funding agencies.

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ARTICULATING THE RELATION BETWEEN TEACHERS' LEARNING IN PROFESSIONAL DEVELOPMENT AND THEIR PRACTICE IN THE CLASSROOM: IMPLICATIONS FOR DESIGN RESEARCH

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In this paper, our goal is to address a conceptual challenge that arises as researchers conduct design experiments to support and understand teachers' learning. This challenge centers on articulating the relations between teachers' learning in professional development and their practice in the classroom. In our view, designs for supporting teachers' learning necessarily involve suppositions and assumptions about such relations. These suppositions and assumptions shape not merely the goals for teachers' learning but the actual process of their learning and the means of supporting and organizing it. By drawing on our own design research experience we propose a bi-directional conceptualization that, in our view, profoundly influences all three phases in a design experiment.

The design research methodology has become increasingly prominent in mathematics education and related fields in recent years. A program of design research that is aimed at supporting teacher learning involves *engineering* the process of supporting teacher change through iterative cycles of design and research (Brown, 1992). In this process, conjectures about the trajectory of the teachers' learning and the means of supporting it are continually tested and revised in the course of the experiment. In this highly interventionist activity, decisions about how to proceed are informed by ongoing analyses of the participating teachers' activity. As noted by Wilson and Berne (1999), design research is an appropriate methodology to investigate teacher professional development as little is known about systematically designing professional development to support teacher learning and as a result, teacher educators are "researching a phenomenon while they are trying to build it" (p. 197).

Research of this type involves a "bifocal" attention encompassing both "designing meaningful professional development and conducting rigorous research" at the same time (Wilson & Berne, 1999, p.197). This interdependence between design and research is reflected in all three phases of a design experiment: preparing for the experiment, experimenting to support learning, and conducting retrospective analyses of the data generated in the course of the experiment (cf. Cobb, Confrey, diSessa, Lehrer & Schauble, 2003).

In this paper, our goal is to address a conceptual challenge that arises as researchers conduct design experiments to support and understand teachers' learning. This challenge is inherent to teacher development experiment as the primary intent is to engage teachers in activities in professional development sessions with the goal of supporting the reorganization of their activity in another setting, the classroom. Thus, the coordination of teachers' learning across two differing settings is an distinctive characteristic of teacher development experiments that distinguishes them from classroom based design experiments aimed at supporting students' learning of mathematics. The conceptual challenge that we address in this paper therefore centers on articulating the relations between teachers' learning in professional development and their practice in the classroom.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

In the following sections, we first clarify the implications of a conceptualization as such for all three phases in a design research. We then draw on our own design research experience with a group of middle-school mathematics teachers to illustrate how our own conceptualization of this relation evolved as a result of our ongoing effort to support the teachers' learning. We conclude by discussing the specific implications that the resulting conceptualization carries for more effective design.

Implications for the Three Phases in a Design Research

A research team's conceptualization of the relations between teachers' activities in these two settings profoundly shapes all three phases of an experiment even in cases where the nature of these relations is implicitly assumed rather than explicitly articulately. The design conjectures formulated in the preparation phase of a design research necessarily involves assumptions about the specific ways in which teachers' learning in professional development sessions might influence their classroom practices and vise versa. It is possible to infer how these relations are conceptualized in various designs for supporting teachers' learning even when underlying assumptions are not made explicit. In most cases, the relations are conceptualized in uni-directional terms (Borko, 2004; Clarke & Hollingsworth, 2002). It is assumed that teachers will develop insights into their instructional practices and their students' learning in professional development sessions and then apply them in their classrooms. Designs for supporting teachers' learning that reflect such a conceptualization typically focus on equipping teachers with forms of expertise that researchers believe are important their development of effective instructional practices.

A number of researchers have challenged this uni-directional conceptualization by arguing that teacher professional development should be situated in the context of teaching. For example, Ball and Cohen (Ball & Cohen, 1999) call for teacher development activities to be centered on the use of artifacts and practices that are directly relevant to teachers' daily practices. This proposal is underpinned by the claim that teachers' classroom practices constitute a valuable resource on which researchers can draw as they formulate design conjectures. In this conceptualization, what counts as an effective design for supporting teachers' learning depends on how closely it is tied to teachers' classroom experiences, needs, and practices (Ball & Cohen, 1999; Franke, Carpenter, Fennema, Ansell, & Behrend, 1998; Nelson, 1997; Borko, 2000).

The second phase of an experiment, experimenting to support learning, involves testing and revising design conjectures about both the learning of a group of teachers and the specific means of supporting that learning. In this phase, the research team's assumptions about the relationship between teachers' learning in the professional development sessions and their classroom practices circumscribe the ongoing design decisions to a considerable extent. For example, when this relationship is conceptualized in uni-directional terms, the revisions made to design conjectures are likely focus on 1) additional skills or insights that researchers think are crucial for effective instructional practice, 2) new tools or technologies that can be used to support teachers' development of these skills or insights, and 3) the specific activities in which teachers should engage in professional development sessions in order to develop these skills or insights. In such cases, the iterative design cycles focus primarily on what can be accomplished in the professional development setting, and the teachers' classrooms are viewed as settings in which the consequences of their learning in the professional sessions can be assessed.

A research team might scrutinize its assumptions about the relations between teachers' learning in professional development sessions and their classroom practices as it tests and revises design conjectures in the second phase of an experiment. However, it is unlikely that this will occur unless the research team has explicated its assumptions about these relations and is aware of how they shape design conjectures. In cases where the assumed relations have not been articulated or are not considered central to the design process, it is doubtful that they will be implicated in the success or failure of the design conjectures.

The last phase of a design experiment involves conducting retrospective analyses. One of the goals of this phase should, in our view, be to contribute to the development of a domain-specific teacher development theory (cf. Cobb & Greveimeijer, in press; Cobb et al, 2004). Assumptions about the relations between teachers' participation in professional development sessions and their classroom practices will be inherent in the retrospective account of the teachers' learning and thus in the resulting teacher development theory.

Our Evolving Conceptualization

Having clarified the importance of explicating assumptions about relations between teachers' activity in professional development session and the classroom, we now illustrate how our conceptualization of these relations evolved in the course of a five-year collaboration with a group of middle-school teachers. The school district in which the collaborating teachers worked is a large urban district located in a state with a high-stakes accountability program. Our long-term goal in working with the teachers was to support their development of instructional practices that place students' reasoning at the center of their instructional decision making. To this end, we engaged the teachers in activities from a statistical data analysis instructional sequence that was designed, tested, and revised during prior NSF funded classroom design experiments conducted with middle grades students (Cobb, 1999; McClain & Cobb, 2001). During the five years of our collaboration with the teachers, we conducted six one-day work sessions each school year and three-day sessions each summer.

About 18 months into the collaboration the group evolved into a genuine professional teaching community that satisfies Wenger's (1998) criteria for a community of practice indicated by joint enterprise, mutual engagement, and a shared repertoire. As we have documented elsewhere (Cobb, McClain, Lamberg & Dean, 2003; Dean, 2005), the activities during the first two years supported the deprivatization of the teachers' instructional practices and the evolution of the teacher group into a community. Against this background, we engaged the teachers in activities in which they analyzed their students' work.

At the outset of our collaboration with the teachers, our conceptualization of the relation between their activity in the two settings was consistent with Ball and Cohen's (1999) view that professional development should involve the use of artifacts or practices that originate in the teachers' classrooms. More specifically, this design decision was based on three rationales. We conjectured that because students' work is an indispensable aspect of teachers' instructional practices, making it a focus of activity would enhance the pragmatic value of the professional development sessions in relation to the teachers' classroom practices. In addition, we conjectured that the teachers would openly critique and challenge each other's interpretations of student work because teaching was now deprivatized. Finally, we conjectured that open discussions of this type would give rise to opportunities for the teachers to gain insight into the diversity of their students' reasoning that would be useful when they attempted to build on their students' solutions while conducting whole class discussions. These interrelated rationales reflect our conscious effort to build on the teachers' classroom practices and indicate our conceptualization of the relations between the teachers' activity in professional development sessions and their classrooms at that time. The specific questions that we posed in order to orient the teachers' analysis of their students' work were as follows:

- What are the different solutions that you can identify from your students' work?
- How would you categorize students' solutions according to their level of sophistication?
- How would you, as a teacher, build on these different solutions? Which solutions would you choose to focus on in class and why?

Our design conjectures proved to be unviable despite our detailed preparations. The teachers seemed to find the activity engaging and discussed their interpretations of the student work openly. Furthermore, most were able to discriminate between students' solutions in terms of levels of sophistication. However, it became apparent that they did not view this activity as relevant to their classroom instruction. The teachers' primary orientation was evaluative in that they assessed whether the instructional activity had been successful or not. Students' work, for these teachers, was an assessment tool rather than a resource for instructional planning. The orientation that teachers took towards students' work was particularly evident when our question of "how are you going to build on students' different solutions" received puzzled looks and almost no response from the teachers. The conversation within the work group came to a halt at this point.

The teachers' orientation towards students' work indicated that there was something about the teachers' classroom practices that we had yet understood. This realization in turn led us to reexamine our assumptions about the relations between teachers' activity in the professional development sessions and their classroom practices. We generated data to that might enable us to address these issues by conducting an unscheduled series of modified teaching sets (Simon & Tzur, 1999) with all the participating teachers. These modified teaching sets involve observing one or more lessons and then conducting an interview in which questions are grounded in specific activities and events that occurred during the observed lessons. A central principle that guided our analysis of these and other teaching sets was that the teachers' instructional practices were reasonable and coherent within their landscape of teaching and learning.

The analysis of the modified teaching sets revealed that the process of students' learning and what supported their learning was, for the teachers, a black box. We conjectured that their repeated observation that students' engagement in the same classroom activity typically resulted in different learning outcomes for different students only served to mystify the process of students' learning. The teachers indicated that they had a limited sense of control in how they could influence their students' learning and identified two ways in which they believed they could support student learning. The first was to ensure that students had sufficient opportunities to engage in instructional activities as intended. The common strategies that the teaches employed included using different forms of presentation (e.g. different visual supports or manipulatives), breaking mathematics problems down into smaller steps, and providing students with sufficient time and enough problems of a similar type. The second way that the teachers believed they could influence student learning was to make sure that the students attended to the learning opportunities that would arise if they engaging in tasks as intended. All the teachers valued students' engagement highly and, for many, staying on task was synonymous with learning. The teachers typically accounted for students' failure to learn in terms of their lack of focused attention or, sometimes, their unwillingness to concentrate on the mathematical intent of tasks. As a result, the overriding challenge that the teachers attempted to address in their instruction was that of ensuring that students were on task.

This analysis of the teaching sets suggested that the teachers' classroom practices might have been influenced by the institutional settings in which they worked more deeply than we had initially assumed. Their orientation to teaching was largely shaped by the fact that the school leaders assessed the quality of their instruction in terms of content coverage and the extent to which students were on task. The teaching sets also revealed that students' reasoning was largely invisible to the teachers as they engaged in classroom instruction. This finding explained why the analysis student work was irrelevant to the teachers' classroom practices. From the teachers' perspective, the diversity in their students' solutions served to confirm that learning was an elusive phenomenon. For most of the teachers, students' work was a product of learning rather than a record of students' reasoning and indicated whether the instructional activity was successful or not. In other words, the teachers viewed students' work as a tool for retrospective assessment rather than as a resource for prospective planning.

Our analysis of the teaching sets resulted in two important insights. First, it enabled us to understand why the teachers took an evaluative orientation towards the use of students' work in the sessions. Second, we came to realize that the ways in which we assumed student work would be used in the sessions did not fit with how the teachers used student work in their classrooms. In Wenger's (1998) terms, student work was a reification of students' reasoning within the context of our practices as researchers and teacher educators. In contrast, student work was a reification of the outcome of instruction for the teachers within the context of their classroom practices. These insights led us to explicate and question our assumption that teachers' learning in professional development sessions and in their classrooms could be related by focusing professional development activities on artifacts that originated in their classrooms.

We found Beach's (1999) notion of *consequential transitions* particularly useful as we attempted to rethink the relations between teachers' activity in professional development sessions and their classroom practices. In Beach's terms, transitions between settings occur when teachers shift from engaging in classroom teaching to participating in professional development activities, and vice versa. For Beach, these transitions are consequential if and only if teachers' participation in professional development sessions is oriented towards reworking their classroom practices, and if their classroom teaching constitutes the context in which they make sense of their engagement in professional development activities. This perspective gives rise to two implications for professional development. The first implication is that professional development activities should be designed so that teachers can relate their participation in sessions to their classroom practices. In the case of the teachers with whom we worked, our design conjectures implicitly assumed that the teachers used student work as a reification of student reasoning in their classrooms. As we have illustrated, this assumption was unviable.

The second related implication of Beach's perspective on people's activity in different settings is that teachers' activity in professional development sessions should be interpreted against the background of their classroom practices. This implication clarifies that when the same artifact is used in activities in different settings (e.g., students' work is used both in professional development activities and the classroom), its constitution in one setting needs to be understood in relation to how it is used in the other setting. In the case of student work, the questions that might be addressed when conducting an analysis of this type include: How do the participating teachers typically use students' work in their classroom practices? What pedagogical value do they attribute to students' work in the context of those practices? Are there significant differences between the teachers' use of student work in their classrooms and the ways in which the researchers envision it being used in professional development sessions? Answers to these questions clarify whether the planned use of an artifact such as student work constitutes a viable means of supporting the teachers' learning across the settings of the professional development session and their classrooms.

In summary, when we began the teacher development experiment, we assumed that the two-way movement of artifacts between the professional development sessions and the teachers' classrooms would support their learning across the two settings. In attempting to understand why design conjectures based on this assumption were unviable, we came to conceptualize the relations between the teachers' activity in the two settings as involving a bi-directional interplay. This conceptualization focuses not on the movement of artifacts per se, but on the relations between teachers' *use* of artifacts in professional development sessions and the classroom.

Implications of the Bi-directional Conceptualization

This bi-directional conceptualization is consequential for all three phases of a teacher development experiment. In preparing for an experiment, it indicates the importance of developing relatively detailed accounts of the collaborating teachers' instructional practices and thus of the ways in which they use key artifacts. In our view, two aspects of teachers' classroom practices are particularly worthy of attention. The first concerns the extent to which students' reasoning is visible in teachers' classroom practices whereas the second involves identifying issues that are pragmatically relevant to the teachers in the context of their instructional practices and that can be leveraged to achieve the professional development agenda of supporting their learning across the two settings.

The bi-directional conceptualization implies that during the second phase of experimenting to support learning, the ongoing process of testing and revising the design conjectures should be informed by analyses of the collaborating teachers' developing classroom practices as well as by analyses of their activity in the professional development sessions. Recall again that our design conjectures for analyzing student work proved to be unviable. We would not have understood why the teachers took an evaluative stance towards student work and thus did not view the activity as relevant to their classroom practices had we not conducted an additional round of data collection in order to analyze those practices.

In the final phase of conducting retrospective analyses, the bi-directional conceptualization shapes the explanation of the teacher groups' learning and also results in credible accounts for why particular design decisions did not work as expected. For example, to account for why students' work did not support the learning of the teachers with whom we collaborated, we focused on the lack of alignment between how we envisioned student work might be used in the professional development sessions and how it was constituted in teachers' classroom practices. This type of explanation is potentially generalizable to other cases in which there is a similar lack of alignments between the use of artifacts in professional development sessions and the classroom. In this regard, the bi-directional conceptualization structures the aspects of a design that are viewed as necessary and as contingent in supporting a group of teachers' learning, and thus what is potentially generalizable and replicable.

Conclusion

Designs for supporting teachers' learning necessarily involve suppositions and assumptions about the relations between teachers' activity in the setting of professional development and the classroom. These suppositions and assumptions shape not merely the goals for teachers' learning but the actual process of their learning and the means of supporting and organizing it. In our view, it is therefore crucial for researchers to scrutinize their assumptions and to be explicit about how they conceptualize the relations between teachers' activities in the professional development sessions and their classroom instructional practices. In this paper, the bi-directional conceptualization that we have proposed to guide teacher development experiments reflects the view of these relations that we developed while collaborating with a group of teachers to support their learning.

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MATHEMATICAL REPRESENTATIONS AS CONCEPTUAL COMPOSITES: IMPLICATIONS FOR DESIGN

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Positing that mathematical representations are covert conceptual composites, i.e., they implicitly enfold coordination of two or more ideas, I propose a design framework for fostering deep conceptual understanding of standard mathematical representations. Working with bridging tools, students engage in situated problem-solving activities to recruit and insightfully recompose familiar representations into the standard representation. I demonstrate this framework through designs created for studies in three mathematical domains.

This design-theory paper presents a framework that spells out intuitive aspects of the craft of design for mathematics education so as to formulate these aspects, giving designers tools for progressing from domain analysis and diagnosis of learning problems toward design, implementation, and data analysis. The proposed framework focuses on mathematical representations and attempts to provide specificity, a “design template,” for implementing radical-constructivist philosophy of didactics in terms of actual objects, activities, and facilitation guidelines for mathematics learning environments. The paper emanates from reflection on a decade of design-based research my collaborators and I have conducted on students’ mathematical learning, in three separate projects with designs for the content domains of fractions, ratio and proportion, and probability, respectively, with K-16 participants (Abrahamson, 2000; Abrahamson & Wilensky, in press; Fuson & Abrahamson, 2005).

The foundations of the proposed design framework are in the philosophies of constructivism and phenomenology (Freudenthal, 1986; Heidegger, 1962; Piaget & Inhelder, 1952). Also, I regard effective learning as acts of creativity, so I build on creativity studies (Steiner, 2001), which describe insight as the act of imaginatively combining ideas. Mathematical representations, I posit, are conceptual composites, i.e., they enfold a historical coordination of two or more ideas. For example, part-to-whole diagrams representing the idea of a fraction (see Figure 1) integrate the multiplicative relation between a part and a whole, e.g., 2-to-3, and the logical relation of inclusion, i.e., the part is integral to the whole. The composite nature of mathematical representations is often covert—one can use these representations without appreciating which ideas they enfold and how these ideas are coordinated. Consequently, learners who, at best, develop procedural fluency with these representations, may not experience a sense of understanding, because they lack opportunities to bridge the embedded ideas, even if these embedded ideas are each familiar and robust.

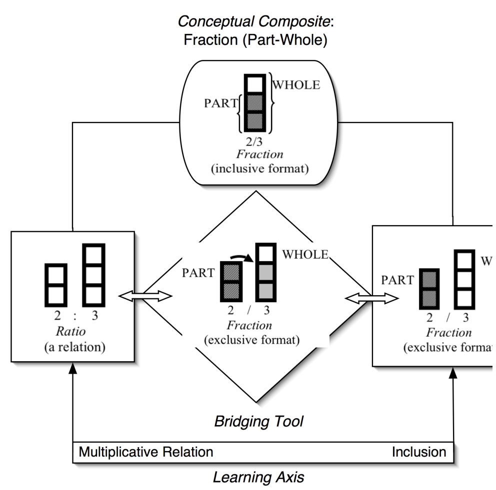

Figure 1. A part-to-whole fraction as a conceptual composite.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

In the proposed framework, a designer creates a cluster of mathematical representations that decompose and "satellite" the target representation, highlighting its covert conceptual components. The teacher leads classroom discussion of situated problems to illuminate how the satellite representations are embedded in the target representation. Working with bridging tools (Abrahamson, 2004)—"ambiguous" representations interpretable as either of the complementary composite components—students recompose the components insightfully, as a reconciliation of the tension caused by the ambiguity, into the composite captured in the standard representation. Figures 1 and 2 demonstrate, for three designs, the standard representation interpreted as a conceptual composite and the bridging tools that help students build on their previous and emergent understandings and support students in seeing and coordinating these understandings.

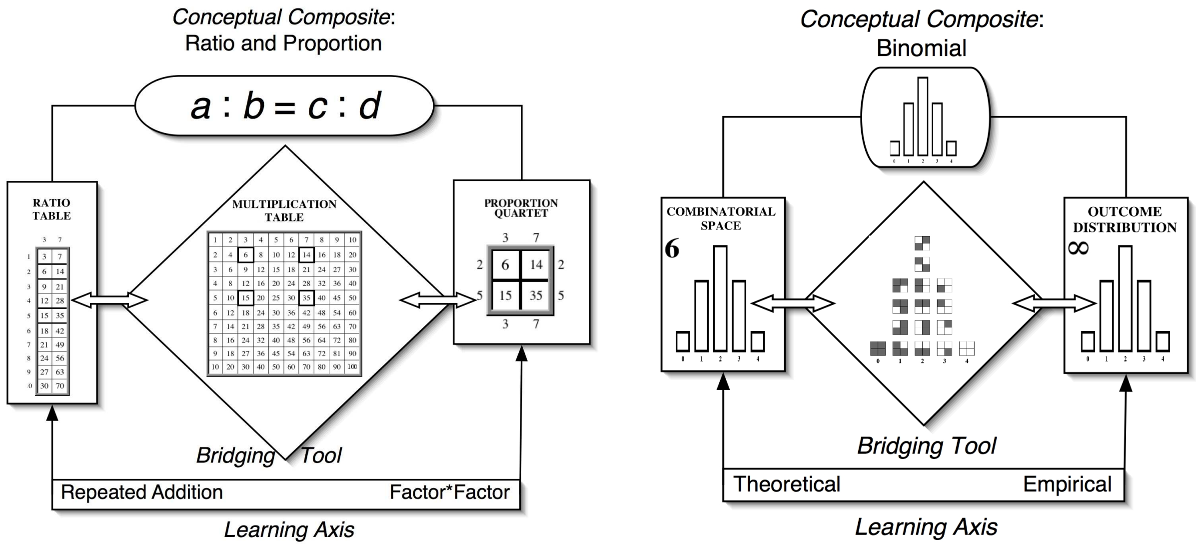

Figure 2. De-/re-compositions of ratio-and-proportion and probability representations.

The emergent framework may contribute to the work of researchers and practitioners: (a) For design-based researchers, the framework may guide both analyses and design in further mathematical domains; and (b) the framework may inform guidelines for professional development and, specifically, it may sensitize teachers to the possible opacity of some taken-as-shared mathematical constructs that are in fact historical composites—teachers informed by this framework may have new facilitation tools for seeing through the "smokescreen" of procedural fluency and helping students rebuild conceptual knowledge on their own robust understanding.

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SOCIO-CULTURAL ISSUES

DESIGN RATIONALE: ROLE OF CURRICULA IN PROVIDING OPPORTUNITIES FOR TEACHERS TO DEVELOP COMPLEX PRACTICES

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This study analyzes the potential of two similar tasks to generate dialogic classroom interactions. Although both tasks were similar in context and outcome, one affords teachers' actions to elicit and build from diverse student explanations. This would require greater teacher expertise – both mathematically and pedagogically – and an articulation of conditions when more potentially dialogical tasks should be implemented.

Introduction

An enduring challenge in mathematics education reform is to help teachers develop competency at new and complex practices (cf. Smith, 1996). The mathematics reform represented by the NCTM Standards implicates teaching practices that involve highly developed knowledge packages and skills that are demanding and complex. Large-scale studies have indicated that few teachers develop such skills (cf. Stigler & Hiebert, 1997; Jacobs et al., 2006); consequently, attention must be paid to conditions which foster the development of new and complex skills.

Curricula can play a role in teacher learning of complex practices, especially curricula which speak to, rather than through, teachers (Remillard, 2005). A number of curricula, particularly NSF-supported 'reform' curricula, include characteristics of educative curricula (Ball and Cohen; 1996; Davis & Krajcik, 2005) such as descriptions and analyses of a variety of student responses to a particular problem, mappings of student learning in a particular content strand over time, and consideration of various representations of and connections between mathematical concepts. Such elaborations of the 'design rationale' "help teachers to see connections between suggested activities in the curriculum and their own understanding of mathematics and what they believe is important for students to come know and understand in mathematics, thereby moving them away from teaching a list of unconnected, isolated topics and toward teaching mathematical concepts and ideas" (Stein & Kim, 2006, p. 17).

Curriculum designers face a tension regarding the design rationale. Curricula designed to promote learning primarily through the interaction between the written task and students (procedure-centric curricula) require less expertise on the part of the teacher than curricula which promote learning through the interaction between teacher and student (resource-centric curricula) (Stein & Kim, 2006). The tension faced by curriculum designers can be stated in terms of two competing forces: on the one hand, there is the vision of mathematics instruction which highlights conditions for learning with understanding; on the other hand, there is the reality of teachers' familiarity and skill with complex practices noted above.

This article articulates a framework which illustrates the interaction between features of curriculum design, teacher learning, and opportunities for students to learn with understanding. The underlying perspective is that there is a participatory relationship between teachers and curricula (Remillard, 2005), no matter the intended design. That is, teachers' engagement with curricula varies according to available resources, individual traits such as beliefs and knowledge, and the context in which the teacher works.

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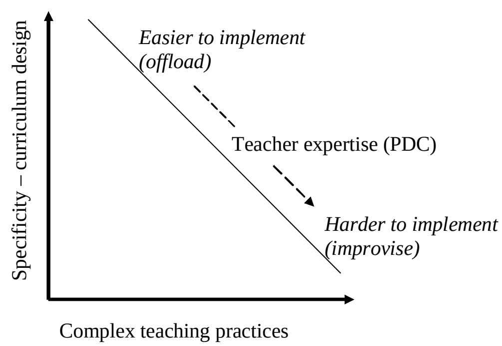
The goals of this article are to elaborate the role of design rationale in providing opportunities for teachers to learn complex practices and to highlight the importance of analyzing curricula in terms of the nature and elaboration of their design rationale. I use examples of two very similar tasks to illustrate characteristics of tasks in relation to design rationale.

Framework

The underlying notion for this article is that curricula can afford or constrain complex teaching practices and that complex teaching practices are associated with creating opportunities to learn with understanding, which Carpenter et al. (2004) define as engaging in four types of activities: (a) constructing relationships, (b) extending and applying mathematical knowledge, (c) justifying and explaining generalizations and procedures, and (d) developing a sense of identity related to taking responsibility for making sense of mathematical knowledge. I emphasize the interactive nature of learning with understanding, both in the sense of students interacting with mathematical ideas but also with each other.

I make two assumptions: (1) curricula which strongly prescribe teacher and student actions require less complex teacher practices (those associated with recitational forms of instruction) and are less likely to engender learning with understanding; and (2) curricula which focus more on the mathematical ideas to be developed rather than highly specified task structures and sequences require more complex teaching practices and are more likely to engender learning with understanding. The figure below illustrates this relationship.

Figure 1: Diagram of framework



Borrowing from Brown and Edelson (2003), I would say the latter curriculum requires a greater degree of pedagogical design capacity (PDC), which refers to a teacher's ability to use resources, including curricula, to design and enact tasks that engage students in disciplinary thinking. Brown and Edelson describe a continuum of three teacher actions which exemplify a teacher's PDC. The first practice, offloading, suggests that teachers follow the task design and sequence specified in the curriculum without adaptation. Although this adherence to task design and sequencing affords types of student activity which the teacher may be unable to manage on her own, it also constrains efforts to build from local resources, including how students are thinking about a topic. As teachers become more experienced with the curricular content and how students think about the content in regards to the particular task design and sequencing, they begin to adapt the curriculum to make it a better fit. Through an extensive process of adapting the curriculum, teachers begin to improvise task design, focusing on more organic and emergent processes in relation to the big ideas.

The framework suggests a fairly straightforward (and possibly naïve) correspondence between offloading and less complex teaching practices and an even more problematic correspondence between complex teaching practices and improvising. I make strong assumptions in the model that a teacher's adaptation process will result in the development of, for example, the ability to orchestrate student discourse around the big mathematical ideas implied in tasks. This assumes that a teacher has the appropriate disposition, epistemological beliefs, and mathematics knowledge to build from and guide the adaptation process.

Given these assumptions, however, the design rationale and task design of curricula can serve to inhibit or afford growth in a teacher's PDC. Brown (in press, cited in Kim & Stein, 2006) discusses the notion of procedure-centric and resource-centric curricula in relation to how curricula speak to teachers. A procedure-centric curriculum "focuses on the actions involved in carrying out the lesson" while a resource-centric "approach to the design of teacher materials … emphasizes the key building blocks of a lesson and tries to make visible the pedagogical affordances of such building blocks" (Stein & Kim, p. 29).

Below, I analyze the design of two highly related tasks, one from the Connected Mathematics Project (CMP) (Lappan et al., 1998) curriculum, and the other from the book Connected Mathematical Ideas (CMI) (Boaler & Humphreys, 2005). I use the analysis to point out how the structure of the task design and sequence requires different levels of teacher expertise and affords certain aspects of learning with understanding. The analysis is intended to suggest that teachers ultimately need to adapt CMP tasks to afford greater interactivity. The process of adaptation is reflexive: adapting tasks will foster growth in understanding of mathematics and of student thinking, while the adapted tasks will require a greater understanding of student thinking and a strongly connected understanding of the underlying mathematical ideas.

A tale of two task designs

The two tasks have highly similar goals, but their designs differ in terms of the opportunities they provide for generating multiple appropriate solutions. Consequently, the opportunities for interactivity differ across the two tasks. The purpose of this analysis is: (1) to highlight the impact of task design on opportunities to generate interactive student engagement; (2) to suggest that published curricula are intended to scaffold the teaching as well as learning of mathematical concepts; and (3) therefore, to consider how experienced teachers can adapt tasks to enhance opportunities to learn with understanding. The task is labeled the 'Border Problem' (Boaler & Humphreys, 2005). In the task, students are asked to determine the number of tiles that it would take to make a border around a square. Below, I summarize the two task designs.

CMP task design.

The CMP version of the border problem introduces the task by emphasizing the notion of equivalency of two symbolic expressions. The text includes a familiar context, the area formula of a rectangle, so that students can easily verify the equivalency of the two expressions. This textual cue points to the goal of the activity as determining symbolic equivalency. The text then presents the scenario of finding the number of square tiles (1 foot x 1 foot) necessary to create a border around a 5-foot by 5-foot swimming pool. The example provides the appropriate quantity, 24, to help students understand the nature of the problem (can contact curriculum designers about this).

Problem 2.1 follows the introduction and states that students should calculate the tiles in the border of square swimming pools that have side length 1, 2, 3, 4, 6, and 10 feet. Students are asked to create a table and then write an equation from the table that relates the number of tiles to the side length of the pool. As a matter of background, students have generated equations from tables in the Variables and Patterns unit of the CMP curriculum (need to make sure that this typically precedes Say it With Symbols).[Should get a scanned version for Problem 2.1]

In the last part of Problem 2.1, students are asked to generate another equation from the same table. The follow-up to problem 2.1 is designed to get students to write quadratic expressions relating the areas of the pool and the border and to compare these expressions with linear expressions. Problem 2.2 begins with a mini-introduction, similar to the introduction to the entire investigation involving the area formula for a rectangle (Might need a scan of the area formula). The mini-introduction relates one expression for the length of the border to its geometric representation. The sub-parts of problem 2.2 provide three more expressions and ask that students generate geometric representations related to each of the symbolic expressions. The follow up to Problem 2.2 then asks students for ways to show the various expressions are equivalent.

Analysis

Although the sequence of tasks affords opportunities for students to consider multiple ways of expressing the length of the border, each subpart of a task (or subtask), is highly constrained in terms of the variation of appropriate (within the context of the instructional goals of the unit) student responses. This acts to scaffold the enactment in terms of the teacher's choice of task sequence and focus of each task, but it also constrains the generative characteristics of the task. For example, in Problem 2.2, part b, students are asked to relate the symbolic expression to the geometric figure. There is really only one appropriate way to relate the two.

The investigation also sequences the order with which representations are introduced. In Problem 2.1, the students are to make a table of values and then generate an equation from the table. In problem 2.2, the students then take symbolic equations and relate them to geometric diagrams. In each case, the task design stipulates the representations with which students work. The unit design also constrains generativity by focusing on representations that provide fewer appropriate choices. For example, Problem 2.1 has students generating equations from a table, which provides fewer choices than if they were generating equations from the geometric diagram.

Boaler & Humphreys task design.

In the Boaler and Humphreys version of the Border Task, students were first tasked to determine the number of squares in a border surrounding a 10x10 foot pool. The students were to do this without counting and without using pencil and paper. These restrictions were intended to focus students on finding a strategy connecting the sides of the pool to the length of the border. The students then shared their strategies with the class, which Humphreys listed on the board, thus creating a shared public record of the set of strategies. Students who shared their solutions were required to explain their strategy by explicitly relating it back to the diagram. Humphrey's intent of this initial task was to get students to get students to think of the functional relationship between the length of the border and the length of the pool.

The next task was for students to use the strategy they had developed on the 10x10 square on the 6x6 square. The intent of this was to help students understand what was generalizable about their strategies; i.e. what changed and what remained the same. The following lesson began with students having to verbalize their strategies, keeping in mind what had changed and what had stayed the same between the two different grids. The verbalizing was designed to help students generate formulas by explicitly expressing the components of their strategy. The next step was to have students generate symbolic expressions for their strategies.

Analysis

Students were provided the opportunity to generate and then compare the 6 ways of finding the sides of the square. A considerable amount of discussion was generated around comparing the numeric expressions structurally to determine equivalency. Similarly, students selected their own strategy, numerically and verbally represented, and create a formula from them. Even within strategies, there was some useful variation for different formulas, mostly centering on how many letters to use. Given a uniform decision to use one variable in a formula, then there would be 6 different formulas to represent the different strategies. Going from geometric to numeric provided maximum variation (6 different ways), which Cathy later used to illustrate the structural equivalency. As students develop symbolic expressions for these, they will already have established their equivalence and can then look to see what rules should apply in the manipulations to make them equivalent.

Comparison of the two tasks

Over the sequence of tasks, the students connect tabular, symbolic, and geometric representations for the length of the border. By doing so, the tasks support the development of the notion of equivalence for linear and quadratic expressions. This helps to ultimately establish rules for simplifying algebraic expressions through combining like terms and using the distributive property. The multiple representations act to provide anchors and rich connections for establishing equivalence.

Although the outcome of the two task designs were quite similar – the students are to connect patterns found in the border problem to equivalent algebraic expressions which highlight the distributive property – the opportunities for student interaction are afforded in Humphrey's design and constrained in the CMP design. In the case of CMP, the sequenced nature of the subtasks affords a trajectory from the border problem to the development of the distributive property. Although the task as a whole supports students' consideration of multiple representations and multiple ways to generate the border, each subtask provides little room for students to generate multiple solutions or explanations.

Humphrey's task design explicitly elicits and builds from multiple interpretations or strategies. In fact, it is the diversity of explanations which drives the development of content. Humphreys displays the students' strategies numerically in a way that elicits comparisons and then facilitates the development of multiple algebraic expressions. From their prior work, students know that the expressions must be equivalent and must therefore come to understand the mechanism that allows $4n - 4$ to be equivalent to $4(n - 1)$. The CMP task also requires that the students consider equivalent algebraic expressions, but these expressions are given and not explicitly generated by the students.

Discussion

The task in Boaler and Humphreys demands more expert teaching practices and affords the kinds of interactions associated with learning for understanding. The CMP task was more structured and, if the teacher fully offloaded the task design onto the written text, would require less complex teaching practices and more constricted forms of student engagement and interactivity. A goal of this analysis was to locate tasks at either end of the oblique line given in figure 1. The next part of this discussion is to conjecture how teachers would adapt the CMP tasks to afford the kinds of generative possibilities inherent in the Boaler and Humphreys' task. That is, how could a teacher understand the characteristics of the CMP task design in a way that afforded adaptations toward a task that required more complex teaching practices and that afforded greater opportunities for students to learn with understanding?

The teacher's edition of the Say It With Symbols unit of CMP, in which the border task is located, provides a general overview of the big mathematical ideas, the representations that will be emphasized, and a trajectory of learning within the content strand. In that sense, it provides a fairly comprehensive description of the design rationale of the unit. The teacher's edition provides a very brief set of cited resources to which a teacher could seek greater elaboration of some of the tasks, but it does not provide citations for research about student learning in the content area. Teachers read and utilize curricular material in variable ways (Sherin & Drake, 2005), so even the inclusion of features of design rationale is insufficient to guide the process of adapting tasks.

Cathy Humphreys, the teacher who designed and implemented the Border Problem, describes in some detail the research as well as the mathematical goals that guided her design and enactment of the Border Problem. It is clear from the description of Cathy's thinking that the process of adapting the Border Problem has been a lengthy, iterative, and reflective process for her.

The written tasks and accompanying resources provide a starting point for teachers to consider the enactment of a task. The process of adapting tasks to accomplish more demanding and productive learning requires a continual effort on the part of the teacher. This effort is much like a design experiment, in that teachers make conjectures based on prior experience and reading of research, test out new adaptations, note how students react to them, reflect on the task design, and revise the task for the next iteration. The goal of these design experiments is to create opportunities for students to learn with understanding.

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INSTRUCTIONAL-DESIGN SOUTH OF THE PISA BORDER

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In this paper I analyze 37 clinical interviews of 13-year-old students from the same classroom. The interviews were conducted for the purpose of documenting students' quantitative understanding about fractions. The interviewed students attended a school that belongs to a branch of the Mexican educational system ("Secundaria Técnica") ranked as low-performing according to results from PISA 2003. The analysis serves to identify some important challenges in designing meaningful mathematical instruction for students attending this kind of educational institutions. It suggests that strategies should be developed to help students build from early types of proportional notions, given that many might not have developed satisfactory understandings about fractions from prior instructional and/or out-of-school experiences. The analysis also suggests that some students might also need help in developing relatively basic number-sense.

Since the year 2000, the Organization for Cooperation and Economic Development (OCED) has implemented the Program for International Student Assessment (PISA) for the purpose of assessing the quality of the educational systems of its 30 member countries, as well as those of some partner countries. The assessment is conducted every three years. It is based on representative samples of the fifteen-year-old population enrolled in school at the time. It assesses students' performance in reading, science, and mathematics, specializing in one of these areas in each implementation. In the 2003 assessment the specialized area was mathematics.

The PISA 2003 was meant to assess students' mathematical skills for meeting the challenges of today's knowledge societies (Learning for tomorrow's world: First results from PISA 2003, 2004). According to their performance in the assessment, students were placed at one of seven performance levels (from 0 to 6). The main borderline was set between levels 1 and 2. Following the PISA rationale, students at Levels 0 and 1 (i.e. under the main borderline) failed to demonstrate consistently that they had "baseline mathematical skills, such as the capacity to use direct inference to recognize the mathematical elements of a situation, use a single representation to help explore and understand a situation, use basic algorithms, formulae and procedures, and the capacity to make literal interpretations and apply direct reasoning" (p. 91). These students were considered unlikely to develop the mathematical skills necessary for insertion in tomorrow's workforce and for participation in democratic societies as active and informed citizens.

The PISA 2003 serves to identify the challenges that participating countries face in terms of providing all of their students with the mathematics education necessary for becoming active and productive citizens. The results show that in every country there is much work to do. The average score among the OCDE member countries showed 21.4% of students at Levels 0 and 1 (In Canada 10.1% and in the USA 25.7%). However, the proportion was significantly larger in some of the developing countries that participated. In particular, Brazil, Indonesia, Mexico, Thailand, Tunisia, and Uruguay had more than 50% of their students placed south of the PISA border.

In this paper I analyze 37 clinical interviews of students from a primero de secundaria classroom (13 year-olds) in the Mexican state of Chiapas. These students attended a Secundaria Técnica, a middle school that belongs to one of four branches within the Mexican public school system offering middle school education. Schools in this branch offer, in addition to the core national curriculum, 4 hours per week of technical education (e.g., electricity, computing, carpentry, etc.). It is worth clarifying that in Mexico there is no tracking in the educational system (at least not officially). As a consequence, it would not be adequate to consider the Secundaria Técnica branch as part of a non-college track. The other three branches of the system are (a) “Secundaría General,” a branch that offers the same curriculum as Secundaria Técnica, except for the 4 hours per week of technical education; “Telesecundaria,” a branch that services rural populations, where 1 instructor teaches all the subjects with the support of television broadcasts; and “Secundaría para Trabajadores,” a branch that offers classes in the evenings for teenagers and adults that work.

The Secundaria Técnica schools service 28% of the national middle-school student body. In PISA 2003, 82.7% of the students attending Secundaria Técnica were placed at Levels 0 or 1 (these findings are statistically representative, Vidal & Díaz, 2004). The purpose of this analysis is to specify some of the main instructional-design challenges involved in supporting significant mathematical learning in classrooms that belong to low performing middle-school educational systems such as the Mexican Secundaria Técnica.

Methodology and Data Collection

The 37 interviews were part of the planning phase of a classroom design experiment (Cobb, 2000b). They were conducted with the intent of identifying a viable starting point for a conjectured learning trajectory (Gravemeijer, 2004 Simon, 1995) aimed at supporting students’ understanding of basic proportional concepts used for analyzing data (fractions, percents, and ratios). The overarching instructional goal of the classroom design experiment was to help the students develop relatively sophisticated understandings of proportionality, thereby enabling them to make sound quantitative sense of statements such as the following: “23% of the adult population in Chiapas is literate, 2/3 of whom are women.” “In Chiapas there are 1100.5 people per doctor, whereas in Mexico City there are 700.2.”

The interviews were individual and were video-recorded. I conducted all of the interviews in April 2005, following the general guidelines of clinical interviews as they have been used in mathematics education (Cobb, 1986). Each interview lasted about 35 minutes. During the interviews, students were presented with 4 problems, 2 of which I will discuss in this paper.

The first problem was based on a narrative about a bus traveling from San Cristóbal de las Casas (the students’ hometown) to Comitán, a city that is 90 kilometers away. Students were shown a diagram representing the road that connects the two cities (see Figure 1) and asked, first, to identify and make a mark at the point where the bus would be upon having traveled 1/2 of the total distance. They were also asked to determine that distance in kilometers (i.e. 45 km).

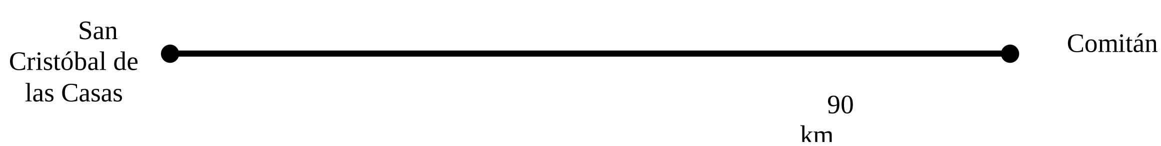

Figure 1. The road between San Cristóbal de las Casas and Comitán

Students that were able to identify half the distance were then asked to do the same with 1/3 the distance, and likewise to determine its correspondence in kilometers (i.e. 30 km). Those that were readily able to do so, or that required little support, were then asked to identify 3/5 of the distance, once again both in the diagram and in kilometers (i.e. 54 km). If a student was not able to do so, the interviewer explored if he could at least identify 1/5 of the distance in the diagram and in kilometers (i.e. 18 km). The interviewer (me) always tried to make certain that students understood the situation by asking questions and making as many explanations as necessary. I also made probing questions to determine how sure students were about their answers and to explore how they were reasoning.

In the second problem, students were shown a drawing of two bags of cement leaning against a wall (see Figure 2). They were first asked to determine which of the two bags had more cement, given that one was filled up to 4/9 and the other to 3/4. Students that gave adequate responses were then told the sum of adding the contents of the 2 bags (i.e. 43/36) and asked if they thought it would be enough to fill one bag, and to explain their answer.

The students who said that they did not know which of the two bags would have more cement, that gave an incorrect answer, or that seemed uncertain about their answer, were asked to mark the level of the cement in the bag that was filled up to 3/4. Those that still gave an incorrect answer or seemed uncertain about their answer were asked to identify the 1/2 and then the 1/4 levels. Some of them were also asked to clarify if they thought that 2/4 would be more, the same, or less than 1/2.

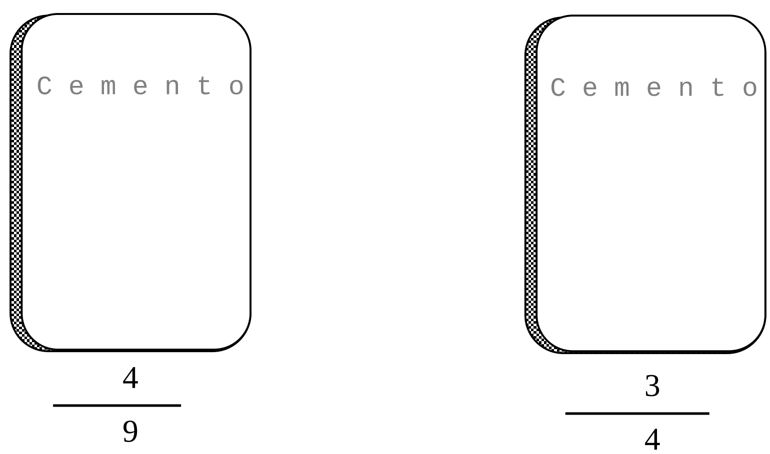

Figure 2. Two bags of cement leaning against a wall

The video-recorded interviews were coded, identifying whether a student could adequately respond to the question or not, given the social context of the interview (Cobb, 2000a). Students that could readily respond to a question and justify their answers were considered to understand the notion involved (e.g. 1/2). Students that with relatively little help could produce an adequate response and justify it were also considered to understand the notion involved. For instance, in the road problem, some students seemed uncertain about where the bus would be upon having traveled one third of the distance (un tercio) but could identify the spot and its equivalence in kilometers when asked where the bus would be upon having traveled one of three parts of the trajectory (una tercera parte del trayecto). These students were considered to understand the notion of 1/3. Students that required more help to produce an adequate answer or that did not produce an adequate answer were considered to not understand the notion involved, given the problem that was presented and the social context of the interview.

In addition to the coding, the mathematical reasoning of the three students whose performance was the poorest during the interviews was analyzed in detail. As I further explain below, the analysis of these interviews suggests that these students not only struggled when dealing with simple proportional tasks (e.g. identify the 1/4 point of a bag of cement), but also seem to have relatively poor number sense.

Results

With regard to the road problem (see Figure 1), all of the students were able to identify the point in the diagram were the bus would be upon having traveled 1/2 of the distance, to establish its correspondence in kilometers (45 km). However, three students struggled with calculating the correspondence of 1/2 of 90 km. 16 of the 37 students (43%) were able to adequately identify the spot where the bus would be upon having traveled 1/3 of the distance (i.e. some place before 1/2 but after 1/4) and to establish its correspondence in kilometers (30 km). Of these 16 students, 5 could adequately identify the spot where the bus would be upon having traveled 3/5 of the distance (e.g. where the bus would be upon having traveled 3 of 5 equal segments), and 2 were readily capable of identifying the correspondence in kilometers (54 km). Of the 11 students that did not adequately identify the 3/5 spot, 6 could do so with the 1/5 spot and establish its correspondence in kilometers (30 km).

With regard to the cement-bags problem (see figure 2), 6 of the 37 students (16%) identified the bag that was filled up to 3/4 as having more cement than the one filled up to 4/9, and soundly explained their answer. Of the remaining 25 students, all could identify the point where the cement would reach if the bag were to be filled up to 1/2; 20 could identify the 1/4 point, and 18 of them the 3/4 point. 18 of the 25 students who did not identified the bag with 3/4 as having more cement that the one with 4/9 were asked about where the cement would be if a bag were to be filled up to 2/4, and to clarify if it would have more, the same, or less cement than if it were to be filled up to 1/2; 10 of them gave an adequate response.

Table 1 shows the number of students that seemed to be capable of reasoning satisfactorily about the different notions involved in the problems, given the nature of the problems and the social context of the interviews.

Notion	1/2	1/4	3/4	1/3	1/5	3/5	4/9	Improper Fraction
Absolute Frequency	37	32	30	16	11	5	6	6
Relative Frequency	100%	86%	81%	43%	30%	14%	16%	16%

Table 1. Students who were capable of reasoning about different notions

The cases of three students (all of them male) were of special interest since, in addition to struggling with the relatively simple problems of the interview (e.g. identifying the 1/4 point in a cement bag), they also seemed to experience difficulties when addressing relatively simple arithmetic tasks. The ensuing exchange is representative of these students' difficulties.

Researcher: (After the student identified half the distance in the diagram representing the road; see Figure 1) And how many kilometers would that be?

Student: Sorry?

Researcher: If from here to here (pointing at the start and end of line in the diagram) there are 90 kilometers, when the bus is half of the way (pointing at the mark made by the student), how many kilometers would that be?

Student: (Silent for a few seconds) 25?

Researcher: 25?
Student: Yes?
Researcher: If you travel 25 kilometers twice would you have traveled 90 kilometers?
Student: (Silent)
Researcher: Does 25 plus 25 equal 90?
Student: Yes (seeming doubtful).

(Next, the student and the researcher spend a few minutes with a calculator, seeking for a number that added twice would become 90 [i.e., 45 + 45 = 90]).

It is not possible to establish to what point the social context of the interview (as experienced by the child) may have a been a determining factor in this case and in the cases of other students' who exhibited rather poor mathematical reasoning, nor how much better some of them could have performed in other social contexts. However, it is worth noticing the relative simplicity of the tasks with which three of the students struggled (e.g. determining what would be half of 90). It is reasonable to conjecture that the number-sense developed by some of the students—either in their prior instructional experiences or in their out-of-school experiences—might not be enough to allow them to readily benefit from participating in instructional activities that involve dealing with relatively simple proportional tasks.

Discussion

The analysis of the interviews is useful for understanding what might be happening in terms of mathematical learning in educational systems like Secundaria Técnica, and for formulating instructional-design conjectures about how to seek improvement. It is worth noticing that the relatively low-performance of the interviewed students is, in general, consistent with the results of students attending the Secundaria Técnica branch on the PISA 2003 exam (Vidal & Díaz, 2004). It is then reasonable to expect that many classrooms of this grade level—in this kind of educational system—would be composed of students that would perform similarly to what was documented in the interviews. In this sense, the analysis of the interviews is useful both for understanding why students in this kind of educational institution perform poorly on tests, and for informing instructional design that can better support students' mathematical learning.

The Mexican educational system operates under a national curriculum (Secretaría de Educación Pública, 2006). Much of the content-knowledge addressed in the curriculum for primero de secundaria (the grade level of the interviewed students) requires an understanding of proportional notions that only very few of the interviewees seemed to have developed at the time of the study. For instance, the national curriculum requires that, at the beginning of primero de secundaria, teachers address tasks that involve finding correspondences between fractions and decimals in the number line. The analysis suggests that very few of the interviewed students (no more than 6) would have had an understanding of fractions comprehensive enough to allow them to readily and meaningfully engage in instruction involving such tasks.

The results of the interviews make it reasonable to conjecture that a large portion of students attending the Mexican educational system might not have developed the mathematical understandings that the national curriculum expects them to have mastered upon arriving at primero de secundaria. If this is true, it is possible that many students in educational systems like Secundaria Técnica are being asked to deal with mathematical tasks about which they have not yet developed the necessary understandings to benefit from. Such a situation would imply that many students might not be having many opportunities to learn as a result of dealing with the kind of tasks that the curriculum prescribes.

The analysis suggests that the majority of the interviewed students could have benefited from participating in instruction that addressed basic fractional concepts. The analysis also suggests that, with proper instructional support, many of them could have rapidly advanced their understanding of proportionality.

With regard to instructional design, the analysis suggests that it would be worthwhile to develop resources that allow primero de secundaria teachers to detect how their students understand basic fractional notions, so that they could take this information into consideration when making instructional decisions. In addition, it also seems worthwhile to develop instructional resources that would help teachers in these contexts and at this grade level support their students' understandings of basic proportional notions.

It is also important to mention that three students were detected in the interviews that not only might have needed help in developing basic understandings about fractions but also might have required support in developing basic number sense. The results of PISA 2003 are not helpful for formulating conjectures about how widespread the situation of these students might be within an educational system like Secundaria Técnica. However, it seems important not to lose sight of the fact that some students in these systems might not be readily capable of benefiting from engaging in instruction that addresses rather basic proportional notions, and that provisions should be made to help them develop basic number sense.

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INEQUITY IN MATHEMATICS EDUCATION: MOVING BEYOND INDIVIDUAL-LEVEL EXPLANATIONS OF DIFFERENTIAL MATHEMATICS ACHIEVEMENT TO ACCOUNT FOR RACE AND POWER

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This paper discusses the entailments of mathematics education research that focuses on cognition and learning, culture and learning, and issues of race and power in mathematics education for understanding and addressing the achievement gap. Research on the achievement gap that provides individual-level accounts of cognitive or social differences between students from dominant and non-dominant backgrounds often misses group-level processes shaping opportunities to learn. As part of a review for the Second Handbook of Teaching and Learning in Mathematics Education, this paper conjoins theories of race and power with research in mathematics education to propose recommendations for research on inequity and injustice in mathematics education.

The 2005 NAEP reports that the scores of African American and Latino students in mathematics continue to be substantially lower than their White and Asian counterparts. This report simply echoes what teachers, students, and parents of students experience in increasingly diverse schools and mathematics classrooms--that students of color are disproportionately placed in lower tracks (Oakes, 1985) and often feel marginalized (Martin, in press; Cobb & Hodge, 2002). Bob Moses, an esteemed civil rights leader and mathematician, argues that students who face these unjust circumstances need to “demand to understand” mathematics in order to pass college entrance exams and thus have access to a broader array of social and capital resources (Moses & Cobb, 2001). Similarly, a number of mathematics researchers from a social justice perspective are calling for teachers to focus not only on teaching students mathematical content, but also on how mathematics can be used as a tool for social empowerment (Gutstein, 2005; Gutierrez, 2002). These researchers, educators, and professionals are acutely aware that social, educational, and thus professional differences in opportunities to become math doers among racial groups in America are deeply entrenched in a broader social and political system that serves to perpetuate inequity and social injustice in society (Kozol, 2005). Our aim in this paper is to review research in mathematics education that can speak to the achievement gap in mathematics education by conceptualizing these differences through group frameworks such as culture and race.

Individual vs. group level explanations

Despite the portrayal of the “achievement gap” as a group phenomenon, much of the research conducted on equity in mathematics has tended to focus on individual students, teachers, or schools. There are several limitations to this approach. First, by comparing the characteristics of individuals and/or discrete programs to what is considered normative or “adaptive”, researchers fail to note the fact that these characteristics inherently mirror those of

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

the dominant "White" culture (Tate, 1994). Thus, instead of conducting analyses of learning and development across various communities and contexts, these accounts of mathematics learning and teaching tend to focus on what's missing or defective in non-normative (i.e. non-White) contexts, "blaming the victim" for their own problematic condition,

A second characteristic of this research is the tendency to focus heavily on the relation of cognition to achievement without attending to broader social and cultural processes. Of course, this critique often plays out in heated debates over particular theoretical perspectives (Sfard, 1998). For the purposes of this paper, we contend that while each perspective is certainly valuable, research on individual psychological processes, motivational orientations, or knowledge structures does not address the issue of differences in and among groups and more importantly, the relations between them. In order to understand group differences in access to educational resources (material as well as ideational), the field of mathematics education research must begin to examine the broader sociopolitical power structures through which some groups experience discrimination and marginalization in our mathematics classrooms. An unintended consequence of the limitations we point out in the current research is that explanations for school failure and success tend to be situated within individuals instead of acknowledging how groups of students, teacher and schools are positioned relative to each other and various resources, opportunities and constraints. In addition, framing differences in achievement outcomes across groups as individual differences tends to support deficit models for students of color and the poor. This paper deconstructs current explanations for differential achievement in mathematics with respect to particular lines of research and the assumptions of this research about individuals and groups in relation to broader social and cultural structures.

Race, mathematics, and inequity

Another aspect of differential achievement in mathematics among groups of students is that it is generally framed in terms of race or ethnicity. Thus, although race is fundamentally a social construction that is used to classify groups of people, it has tremendous power in shaping explanations for why some students are better or worse at mathematics. However, researchers in mathematics have only begun to study what Martin (in press) calls "racialized experiences" of groups of students (and teachers) in mathematics education. Perhaps one reason why researchers have shied away from group-level analyses that focus on race, ethnicity, or language is the danger of essentializing, or making attributions to individuals based on their group affiliation without attending to how individuals take up, adapt, or reject them. Gutierrez and Rogoff (2003) warn against stereotyping individual and groups of students in terms of, for example, learning styles or motivational orientations. However, as Martin's (in press) research illustrates, individuals continually negotiate racist structures and ideologies that influence both their access to high quality mathematics as well as their perspectives of themselves as mathematics learners and doers.

Another reason why less is known about the relations between groups is that this type of research has been viewed as under the purview of research traditions outside of education, including sociology and anthropology. Although an increasing number of mathematics education researchers draw from these traditions to inform their research (Cobb & Nasir, 2002), the field as a whole has not yet determined how best to manage cross-disciplinary research such that it receives the same attention as conventional educational research. This is most evident in methodology. Research that draws on multiple disciplines to analyze educational contexts requires new ways of understanding educational phenomena and these ways of doing research are often not as well understood by researchers in mathematics education. In an effort to further the field's understanding of the importance and complexities of group-level analyses, this paper examines research that focuses on issues of race and power in mathematics education that makes allowances for both individual agency and social structure.

Overview of the paper

In this paper, we review research in mathematics education that has attended to culture and examine what it contributes to an understanding learning inequities. Second, we consider what a perspective on race and power affords the field of mathematics education in understanding how to address learning inequities among groups of students. Finally, we offer recommendations for conducting research that focuses squarely on issues of race and power in mathematics education. Our goal here is not to propose a new set of theories, but rather to offer different lenses to the field of mathematics education research where power and race are central.

Methods

This paper is part of a broader review of research on equity in mathematics education compiled over the last ten years in preparation for a chapter in the *Second Handbook of Teaching and Learning in Mathematics Education*. An extensive literature review was conducted of both theoretical and empirical work that analyzed the affordances and constraints of this research for understanding group-level differences in mathematical achievement.

Cultural Activity, Community, and Opportunity to Learn

The relation between culture and individual differences in cognition, learning, and development has been a subject of study for many years. In the past ten years, research on culture, formal and informal mathematical practices, and identity have gained prominence and offered alternative perspectives on the role of culture in learning, in terms of both what counts as learning and who has access to it. What we summarize as *theories of cultural activity* has afforded an understanding of knowing and learning as a function of what an individual accomplishes over time and across the various communities and practices in which he or she participates Theories of cultural activity include situated cognition, activity theory, cultural historical activity theory, and sociocultural theory. These theories point to the fact that mathematics classrooms are necessarily cultural and social spaces that can perpetuate social inequities by privileging certain forms of discourse and ways of reasoning or reorganize them by positioning multiple forms of learning and knowing as "having clout" (Cobb & Hodge, 2002; Gutstein, 2005).

Understanding the cultural entailments of mathematics learning requires complicated analyses of how people live and learn culturally both within and outside of the mathematics classroom, individually and as part of groups. Group membership does not require that individuals directly take up the roles and relationships within the communities in which they participate. Rather, research has illuminated the diverse ways that students from similar and different backgrounds create, contest, and reconfigure learning within and outside of mathematics classrooms.

One major contribution of this research is the conceptualization of mathematics classroom as communities, where various curricular and participation structures afford and constrain students development as learners and doers of mathematics (see, e.g., Boaler & Greeno, 2000; Cobb & Hodge, 2002; Martin, 2000). This perspective highlights the importance of arrangements for competence, ownership, and authority in the classroom to the social and cultural practices and identities of classroom participants. In particular, alignment between the practices and identities of home and school has implications for whether students negotiate ways of participating that serve their individual goals, as well as the goals of the classroom community (Cobb & Hodge, 2002; Hand, 2003).

A second and growing contribution of this research is an expanded conception of competent classroom participation as supported by the wide variety of mathematical practices and identities that students bring to the classroom from their home and local communities (see, e.g., Nasir, 2002; Taylor, 2004). In mathematics classrooms where teachers rely on traditional scripts of, and formats for, classroom instruction, students' practices and ways of reasoning couched in everyday discourse can be inadvertently marginalized. Broadening mathematical activity to recognize and value the multiple ways that students participate in mathematics can draw in students who may normally be sidelined.

Finally, this research offers promising models of classroom learning environments that begin to address issues of race and power in the mathematics classroom by focusing squarely on issues of cultural relevancy and social justice (see, e.g., Gutstein, 2005 Ladson-Billings, 1994; Moses & Cobb, 2001).

Race, Power, and Opportunity to Learn

Orfield, Frankenberg, and Lee (2003) stated that the level of segregation of schools is worse now than in 1968. Students of color and Whites are increasingly not in the same schools. Moreover, only 15% of the intensely segregated White schools have populations in which more than half are poor enough to receive free and reduced lunches. For Black and Latino students the percentage is 86%. Schools in communities predominantly consisting of Blacks and Latinos are poorer, and they generally have fewer AP courses, fewer credentialed teachers, more out-of-field teachers, and buildings in worse conditions. Kozol (2005) and others (Frankenberg, Lee & Orfield, 2003; Hunter & Donahoo, 2003) have ascribed this situation to a new form of apartheid in the U.S. school system, where low-income public schools have become hypersegregated with populations up to 99% students of color.

Along with the material conditions in "apartheid" schools, Kozol (2005) notes that in urban schools there are another set of conditions around how we talk about students and the ways they are expected to participate. Kozol points out that he has heard hypnotic slogans like "I'm smart! I know that I'm smart," repeated everyday, "but rarely in suburban schools where potential is assumed" (p. 36). These non-material conditions shape the opportunities of students of color – often blaming them for their own failure. At the same time that these students are blamed for their failure, the system of mathematics education continually fails them.

The literature on access and opportunity to learn mathematics documents how experiences differ along racial lines. Overall, segregated minority schools offer less access to upper-level math and science courses, many not offering courses beyond Algebra II. Oakes, Muir, and Joseph (2000) wrote that,

> A student can only take a high level class in science and mathematics if his or her school offers such classes or if his or her school opens up access to these courses to all students. In other words, how far a student can go down either the mathematics or science pipeline depends on his or her access to particular courses. (p. 12)

On the basis of a student's race, he or she can expect to experience mathematics education differently (Hunter & Donahoo, 2003). Furthermore, as described in earlier sections, White culture also often determines what is "normal" and also constructs the dialogue or ideology for understanding the "other." This dialogue is constructed and reinforced in mathematics education, for example, when achievement scores are reported in terms of race, and lower test scores are ascribed to race (ignoring the fact that "White" is also a racial category). Educators fail to ask how the racial and cultural entailments of whiteness provide opportunities for large groups of White students to be consistently ahead of their Black and Latino/a counterparts. Instead, the success or failure of a White student often gets framed as an individual act, acclaiming or pathologizing the individual rather than the race.

Ideologies are embedded within language and ways of talking that perpetuate stereotypes of the "other." These broad Discourses, as Gee (1990) and colleagues call them, structure the ways of talking about children of color, communities of color, and structure our individual actions. Gee (1990) calls this dialogue Discourse with a big *D* because it contains ideologies, beliefs, practices, and ways of being that further the power of the dominant culture. There is more going on in individual success or failure, or individual interaction, than what is actually seen in front of us. Individual interaction sits inside of a historical reality; it sits within history, within a context, and within a relation of power. The stories embedded in these Discourses limit the ways of talking and thinking about people of color and can limit how one thinks about their intelligence and abilities, quality of family life, and cultural resources (Warren, 2005).

The increasing segregation, decreasing access, and pervading Discourses place race and educational structures as central in educational opportunity. These areas are important for the field to explore in relation to mathematics education in particular.

Theories of race and power provide a number of methodological techniques, which when applied to educational contexts, can highlight practices and policies that serve to perpetuate inequitable situations. For example, examination of counternarratives identifying structural causes of school failure or successful students from non-dominant backgrounds challenge prevalent Discourses that shape research studies and agendas. Case studies that examine educational policies and practices from multiple levels (from school districts to classrooms) can illustrate how Discourses and schooling practices are taken for granted, normal, and neutral because they are part of the schooling institution (Spencer, 2006). Understanding and unpacking the social systems, policies, and narratives that structure classroom learning mathematics educators can better implement new reforms, navigate the political system, and develop policies that work for, rather than against, African Americans, Latinos, and the poor. This requires new understandings about how power employed through policy, states, districts, Discourses and Whiteness influences the learning that goes on in classrooms. It also means listening and taking up the concerns of those not empowered by the current system in researching mathematics education.

Researching Culture, Race, and Power in Mathematics Education

The examination of race, culture, and power with respect to student achievement and learning in mathematics raises different questions for our current system of mathematics educators. Research questions are needed that can help guide studies of mathematics education in both untangling and challenging processes that perpetuate current inequity and injustice in mathematics education and in society writ large (see, e.g., Gutierrez, 2002; Gutstein, 2006; Martin, in press). These questions stem from the perspectives that these researchers hold about the relations of race, culture, and power in mathematics, the nature of mathematics teaching and learning, as well as the role of mathematics education in society. These researchers share a concern with analyzing what counts as mathematics learning, in whose eyes, and how these culturally bound distinctions afford and constrain opportunities for students of color to have access to mathematical trajectories in school and beyond.

Different kinds of questions often require different methodologies. Asking questions about systematic inequities leads to methodologies that allow the researcher to look at multiple levels simultaneously. This means that mathematics education research should take a multifaceted approach, aimed at multiple levels from the classroom to broader social structures, within a variety of contexts both in and out of school, and at a broad span of relationships including researcher to study participants, teachers to schools, schools to districts, and districts to national policy. It is important, then, that researchers understand that policies *do* play out as well as *the ways in which* they play out at the classroom, school, district, and state levels.

Both qualitative and quantitative research has strengths and limitations. Work using narratives, ethnographies, and historical analyses allow research to speak to multiple levels of practice in order to see nuanced details. Although these methods are sometimes discredited, not counted as research, or not given the same respect as other forms, their multilevel nature situates them as particularly powerful in understanding the details of relationships and structures. Similarly, quantitative methods such as multilevel modeling can uncover systematic issues of inequity. Although multilevel modeling such as Hierarchical Linear Modeling (HLM) is already respected as a form of research, when using such techniques we must be just as thoughtful and careful that we are actually measuring what we intend. The measures must be sensitive enough to allow for the subtle ways that culture, race, and power can influence teaching and learning in mathematics classrooms. When this is the case, multilevel modeling allows researchers to understand complex causal relationships that can uncover power dynamics within social structures that shape the experiences of groups.

In addition to different kinds of questions and methodologies, this work will push the field to develop new ways of understanding results. This is not an argument for a particular framework; rather, multiple lenses and theoretical perspectives will be needed to understand inequity in mathematics education in relation to social structures and cultural and racial histories. New frameworks for understanding the interactions between culture, race, and power would shape how we discuss and understand the work of mathematics education.

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HOW PARENTS ARE FRAMED IN REFORM-ORIENTED ELEMENTARY MATHEMATICS CURRICULAR MATERIALS

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We present findings from textual and readability analyses of how two reform-oriented elementary mathematics curricula frame and communicate mathematical ideas to parents. The materials provide parental roles and reflect assumptions that parents have particular sets of knowledge and skills that they bring to their work with their children. We question the accessibility of these materials to all parents, particularly in light of the current educational policies which target parents in low-income areas.

Introduction

Over the past 15 years, with the adoption of the National Council of Teachers of Mathematics (NCTM) *Standards* (1989; 2000), the development of the National Science Foundation supported reform-oriented mathematics curricular projects reflecting the vision of the *Standards* documents, and the adoption of reform-oriented *Standards*-based curricula by about half of the school districts in the ten most populous U.S. cities, there has been a simultaneous increase in the attention given to the role of parental involvement in education. Federal, state, and local governments have mandated that districts make efforts to involve parents. An assumption guiding such calls for parent involvement is that students who have more "involved" parents will achieve at higher rates than other students with similar backgrounds. Consequently, parents, particularly in low-income communities, are being called on to support their children more directly as they encounter school mathematics that looks quite different from the mathematics they learned in school (Jackson & Remillard, 2005). In this paper, we focus on the textual materials that the designers of *Standards*-based elementary mathematics curricula created to "involve" parents.

All *Standards*-based elementary mathematics curricula include material designed to support parents' understandings of how best to support their children's learning of mathematics and completion of homework. However, there has been no research as to the roles these curricular materials actually afford for parents with regard to their children's homework. Nor has there been research about the assumptions these materials make as to the background, skills, and knowledge that parents need to assume valuable roles in the mathematics education of their children. We assert that this sort of research is critical to undertake, particularly in the context of reform and in the context of increased calls for parental involvement that target low-income families.

Based on a textual and readability analysis of two elementary *Standards*-based mathematics curricula, this study is designed to answer the following questions: What sorts of roles do these materials afford and constrain for parents? What do these materials assume parents need to know and do to support their children's mathematical learning? What are the readability levels of these materials and how might this relate to the assumptions made about parents' roles in supporting the mathematical learning of their children?

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Theoretical Framework

To guide this study, we draw from Hoover-Dempsey and Sandler's (1995) model of parent involvement in education in general and in homework in particular (Hoover-Dempsey, Bassler, & Burow, 1995). They provide a theoretical model of *why* and *how* parents are involved in their children's education and how involvement influences children's school outcomes. We use this account because it views parent involvement as a dynamic *process* that occurs over time involving parents, school, child, and societal contributions. They posit that there are three mechanisms, *modeling*, *reinforcement*, and *direct instruction*, which may have a positive effect on children's educational success. Though none are necessary or sufficient, these mechanisms can create opportunities for children's success.

We also draw upon the work of Cai (2003; Cai, Moyer, & Wang, 1999), who in contrast to Hoover-Dempsey and colleagues, considered parental roles *specific* to involvement in mathematics homework. Cai and his colleagues identified five roles parents may play in their children's mathematics learning (Cai, 2003, p. 89): motivator, resource provider, monitor, mathematics content advisor, and mathematics learning counselor. These roles overlap with the practices discussed by Hoover-Dempsey and her colleagues but are somewhat more specific to mathematics.

We also sought to locate parent involvement in mathematics homework *in the context of mathematics education reform*. Prior research suggests that parents may feel less able to act in roles that require mathematics content and pedagogical content knowledge particular to reform-oriented approaches than they might with traditional approaches to the teaching of mathematics (Civil, 2001; Jackson & Remillard, 2005; Peressini, 1996, 1998; Remillard & Jackson, in press). In particular, we heed Peressini's (1998) summary review of documents about prominent parent involvement policies in mathematics education: "It is clear throughout these documents parents have not been recognized as significant contributors to the mathematics education of their children" (p. 569).

Method

We completed a textual analysis and a readability analysis of the *Everyday Mathematics* (EM) (University of Chicago School Mathematics Project, 2001) and *Investigations* (INV) (TERC, 1998) curricular materials designed for parents, grades 1-3. We chose these two curricula to analyze because they are the most popular *Standards*-based curricula adopted across school districts in the United States. We focused on grades 1-3 because part of our analyses included cross-curricular comparisons, and EM does not include parent materials beyond grade 3. By parent materials, we mean the written directions that are given to parents as part of the homework assignments that are sent home with the children.

Description of Parent Curricular Materials

EM Documents

Common to both curricula is the use of a letter to family members which begins each unit. Family Letters in EM are usually multi-page documents. The term "letter" for these documents may be misleading. They do not begin with a salutation such as "Dear family member." Rather, Family Letters tend to present an overview of the unit. Embedded within the EM "Family Letter" are two other documents. The first is a list of vocabulary words that represents important terms used in the unit. Each of these terms is defined and in some cases is accompanied by an illustration to help further clarify the definition. The second embedded document is called "Do-anytime Activities." These activities consist of suggested tasks that parents can do with their children to support their mathematics learning. The final section of the EM "Family Letter" is called "As you help your child with homework." In this section, parents are given the answers to the homework problems that are given to the child in that unit. They are instructed as follows: "As your child brings home assignments, you may want to go over the instructions together, clarifying them as necessary. The answers listed below will guide you through this unit's Home Links." At the conclusion of this "Family Letter," the activities are presented and designated with a number such as Home Link 2.1. Typically, each of these Home Links will begin with a brief "Family Note," located at the top of the Home Link homework task. The letters and notes are integrated into the curriculum's *Home Links* workbook that contains all homework assignments (as prescribed in the EM teacher's guide for each lesson). It is assumed that this goes home with the child on a daily basis.

INV Documents

INV's parent materials are available for purchase *separately* from the classroom curricular materials as a series of booklets that if a school chooses to purchase, the teacher sends home with the child. INV's booklets, *Investigations at Home*, correspond with the INV classroom units. Each booklet begins with a "Family Letter." In contrast to the EM Family Letter, the INV letter has the format of a letter. The letter includes information about what happens in the unit. Following this there is information about the importance of playing games with the child.

Each INV unit follows the same format. It is divided into several separate investigations (typically four or five). Each investigation includes several sections. First, there is a section titled "What Happens" in which parents are given descriptions of what students will be doing in class during each session. The next section is called "Mathematical Emphasis" that highlights the mathematical content embedded in the investigation. The next part of each unit is called "Homework Notes." These notes are presented for each session and include a section titled "Math Content," a section titled "Materials", and then the description of the session. Finally, at the end of each unit there is a section called "Related Activities." This section is similar to EM's "Do-Anytime Activities."

Readability Analysis

All first through third grade units of EM *Home Links* and INV *Investigations at Home* were scanned and converted to Microsoft Word™ text documents using OmniPage Pro™ optical character recognition (OCR) software. We chose those sections that consisted mainly of text to use for conducting readability analyses. In EM, we scored (1) Family Letter (not including the vocabulary definitions or the answers to the math problems), (2) Do-Anytime Activities, and (3) Family Notes. In INV, we scored (1) Family Letters, (2) What Happens, (3) Mathematical Emphasis, (4) Homework Notes, and (5) Related Activities. Each of these sections was highlighted in Microsoft Word, and, using the built in Word Readability Statistics program (part of the spelling and grammar checking function), we calculated Flesch-Kincaid Grade Level and the Flesch Reading Ease statistics. These data were then compiled as an SPSS data file and analyzed using analysis of variance procedures which used curriculum, grade, and type of text (letters versus other types of texts) as independent variables and Flesch Grade Reading Level and Reading Ease as dependent variables.

Textual Analysis

Textual analysis involved inductive coding of the documents; this was an iterative process and involved several cycles of refining our codes. We first downloaded all documents, grades 1-

3, for both EM and INV into QSR NVIVO, a qualitative data analysis software package. Using QSR NVIVO, we coded for the sorts of roles that each of the documents afforded parents. We drew on the work of Hoover-Dempsey et al. (2001) and Cai (2003) to identify the types of parental roles/practices afforded by the curricular documents. However, we did not limit our coding to categories provided by Hoover-Dempsey et al. and Cai. Based on our reading and re-reading of the documents, we identified roles and their respective practices unique to the documents. Once we had established what we deemed to be a fairly rigorous coding schema, we had a second coder code sample documents representing each grade and each curricula. Interrater reliability ranged from 0.72 and 1.00 with an average agreement (Cohen's Kappa statistic) of 0.91. For each unit of each curriculum, the frequency of each coding category was entered into an SPSS data file. Analyses of variance using curriculum and grade as independent variables and the frequencies of various roles were computed.

Results

Readability Analysis

Table 1 presents reading ease data and Table 2 presents reading level data. In Table 1, higher numbers mean greater ease of reading. In Table 2, numbers refer to grade levels.

Table 1. Reading ease

Grade	*Everyday Math* Mean and (SD)	N	*Investigations* Mean and (SD)	N	Over Both Curricula Mean and Standard Error
ALL DOCUMENTS COMBINED					
1	64.8 (14.2)	123	54.9 (18.7)	77	60.9 (1.1)
2	59.1 (11.5)	126	48.3 (21.1)	103	54.4 (1.1)
3	57.0 (14.7)	137	50.9 (14.2)	102	55.0 (1.1
Over All 3 Grades	60.2 (13.9)	386	51.0 (18.3)	282	
FAMILY LETTER AND ALL OTHER DOCUMENTS TREATED SEPARATELY					
1 Letter	53.7 (11.7)	13	67.2 (6.1)	6	60.4 (2.6)
1 Other	66.1 (13.9)	110	53.9 (19.1)	71	60.0 (1.2)
2 Letter	45.1 (9.5)	12	61.4 (7.1)	8	53.2 (2.4)
2 Other	60.5 (10.7)	114	47.1 (21.5)	95	53.8 (1.1)
3 Letter	42.1 (13.9)	12	63.6 (7.4)	8	52.8 (2.4)
3 Other	58.4 (14.0)	125	49.8 (14.1)	94	54.1 (1.1)
Over All 3 Grades: Letter	47.1 (12.6)	37	63.8 (7.0)	22	
Over All 3 Grades: Other	61.5 (13.3)	349	49.9 (18.6)	260	

Based on modified population marginal mean; Significance level of comparisons: Grade 1 vs. grade 2: <.001; Grade 1 vs. grade 3 <.001; Grade 2 vs. grade 3:n.s.; Over All 3 Grades: INV vs. EM: <.001

Table 2. Reading grade level

Grade	*Everyday Math* Mean and (SD)	N	*Investigations* Mean and (SD)	N	Over Both Curricula Mean and Standard Error
ALL DOCUMENTS COMBINED					
1	7.9 (2.0)	123	9.0 (1.8)	77	8.3 (0.13)
2	8.8 (1.8)	126	9.7 (1.7)	103	9.2 (0.12)
3	8.9(1.9)	137	10.3 (1.8)	102	9.5 (0.12)
Over All 3 Grades	8.6 (1.9)	386	9.7 (1.8)	282	
FAMILY LETTER AND ALL OTHER DOCUMENTS TREATED SEPARATELY					
1 Letter	9.5 (1.6)	13	8.1 (1.4)	6	8.8 (0.43)
1 Other	7.7 (2.0)	110	9.1 (1.8)	71	8.4 (0.13)
2 Letter	10.8 (1.0)	12	8.7 (1.2)	8	9.8 (0.40)
2 Other	8.6 (1.7)	114	9.8 (1.7)	95	9.2 (0.12)
3 Letter	10.9 (1.4)	12	8.2 (1.4)	8	9.6 (0.40)
3 Other	8.7 (1.9)	125	10.4 (1.7)	94	9.6 (0.12)
Over All 3 Grades: Letter	10.4 (1.5)	37	8.3 (1.3)	22	
Over All 3 Grades: Other	8.4 (1.9)	349	9.8 (1.8)	260	

Based on modified population marginal mean significance level of comparisons: Grade 1 vs. grade 2: <.001; Grade 1 vs. grade 3 <.001; Grade 2 vs. grade 3:n.s.; Over All 3 Grades: INV vs. EM: <.001

Overall, EM documents are easier to read than those from INV. However, EM Family Letters are significantly more difficult to read than INV Family Letters. In contrast, other INV documents are significantly more difficult to read than the other EM documents. For both curricula, parental reading material is easier in first grade than it is in second or third grade.

Overall, 18% of all documents were at the 11th grade reading level or higher. 13% of EM documents exceeded the 11th grade reading level, and 25% of all INV documents exceeded the 11th grade reading level. In fact, 15% of the INV documents were written at grade 12 reading levels.

It should be noted that while we reviewed the materials available in English, EM is available in Spanish, and INV is available in Spanish, Vietnamese, Cantonese, Hmong, and Cambodian.

Textual Analysis

All passages of text from the various units were coded for parental roles. We collapsed initial subcategories to form five major roles. The roles and their definitions were:

1. Instructor: Parent is expected to engage in a behavior designed to teach the child and/or review with the child a concept or skill (mathematical and/or non-mathematical), and/or evaluate the child's understandings.
2. Monitor: Parent expected to oversee an aspect of the child's activity, but not provide instructional support.
3. Game Player: Parent asked to play math game with child prescribes by curriculum.
4. Engaged Audience Member: For example parent listens to something the child says or looks at something the child shows to the parent.
5. Material Resource Provider: Parent is expected to provide materials for child to take to classroom or is asked to gather materials for the child to use in the home.

Table 3 presents data about the frequency of various roles for each curriculum.

Table 3. Comparison of roles

ROLE	*Everyday Math* Mean and (SD)	N	*Investigations* Mean and (SD)	N	Over Both Curricula Mean and (Standard Error)
FIRST GRADE					
Instructor	**7.6** (3.6)	10	**13.3** (6.0)	6	**9.8** (5.3)
Monitor	**10.3** (2.9)	10	**3.5** (2.7)	6	**7.8** (4.4)
Game Player	**0.6** (1.6)	10	**7.7** (7.2)	6	**3.3** (5.6)
Audience	**4.3** (2.8)	10	**3.8** (3.2)	6	**4.1** (2.8)
Material Provider	**2.9** (2.4)	10	**3.5** (3.0)	6	**3.1** (2.6)
SECOND GRADE					
Instructor	**6.7** 3.5)	12	**6.4** (2.8)	8	**6.6** (3.2)
Monitor	**9.6** (3.5)	12	**2.0** (1.4)	8	**6.6** (4.7)
Game Player	**0.0** (0.0)	12	**2.8** (2.8)	8	**1.1** (2.2)
Audience	**3.8** (3.1)	12	**2.3** (1.8)	8	**3.2** (2.7)
Material Provider	**2.0** (1.7)	12	**2.5** (1.4)	8	**2.2** (1.5)
THIRD GRADE					
Instructor	**6.7** (3.3)	11	**4.4** (2.4)	10	**5.6** (3.1)
Monitor	**12.3** (2.6)	11	**2.4** (2.4)	10	**7.6** (5.6)
Game Player	**0.6** (0.8)	11	**1.3** (1.6)	10	**0.9** (1.3)
Audience	**3.6** (2.0)	11	**3.1** (1.7)	10	**3.3** (1.8)
Material Provider	**1.6** (0.8)	11	**1.8** (1.8)	10	**1.7** (1.4)
ALL GRADES COMBINED					
Instructor	**7.0** (3.4)	33	**7.3** (5.1)	24	**7.1** (4.2)
Monitor	**10.7** (3.2)	33	**2.5** (2.2)	24	**7.3** (4.9)
Game Player	**0.4** (1.0)	33	**3.4** (4.6)	24	**1.6** (3.4)
Audience	**3.9** (2.6)	33	**3.0** (2.1)	24	**3.5** (2.4)
Material Provider	**2.1** (1.8)	33	**2.5** (2.1)	24	**2.3** (1.9)

Significance of comparisons: Main Effect for Roles: <.001; Main effect for curriculum: n.s.; Main effect for grade: <.002; role X curriculum interaction: <.001; role X grade interaction: <.001; role X curriculum X grade interaction: <.009; Pairwise comparison of roles using Duncan's New Multiple Range Test: instructor vs. monitor:n.s.; instructor vs. gameplayer: <.001; instructor vs. audience: <.001; instructor vs. material provider: <.001; monitor vs. gameplayer: <.001; monitor vs: audience: <.001; monitor vs. material provider: <.001; gameplayer vs. audience: <.005; gameplayer vs. material provider: n.s.; audience vs. material provider: <.029

For parents whose children are using INV, the role of instructor is the most frequent role available. However, both curricula provide comparable numbers of opportunities for parents to assume an instructional role. In general, when parents were asked to be instructors, the materials assumed that parents had a relatively solid mathematics education background; in most cases it was assumed that a parent should understand a particular concept with little explanation. Further, most of the materials assumed that the parents were familiar with reform-oriented ways of teaching mathematics.

EM offers a comparable number of opportunities for parents to assume an instructional role in each of the three grades. In contrast, INV offers extensive opportunities for parental instruction in the first grade, but by second grade, it offers less than half as many opportunities, and that number declines further by third grade. Likewise, INV offers parents frequent opportunities to serve as a game playing partner to their child in the first grade. However, those opportunities declined by more than half in the second grade and continued to decline further in the third grade.

Monitoring one's child is the most frequently available role to a parent whose child uses EM. The level of monitoring activities available remains relatively constant over each of the first three grades of EM. In contrast, INV is consistent in that it offers relatively few opportunities for parents to monitor the work of their child over each grade.

Discussion: Issues of Access, Instruction, and Monitoring

Both curricula provide parents roles of Instructors. However, INV offers parents virtually no information as to *how to assume the role of an instructor* thereby effectively excluding most parents from being able to take on this role. With no other roles to play, INV affords parents little opportunity to be involved in homework activities. INV Family Letters, much easier to read than Mathematical Ideas Homework Notes, *tell* the parents what the teacher is doing with their children in school and offer very few ways in which parents can be involved in the homework activities. We argue that this relationship between the readability and what parents are asked to do reflect Peressini’s (1998) critique that parents are *told* what is best for their children and are given few opportunities to learn about the reform-oriented approaches and or to question them. Parents are explicitly told *not* to share the algorithmic methods of calculation that they learned in school. The sense one gets from reading the materials *other than* the letters in INV is that they are offered as evidence that teachers have a well-planned curriculum and that the children are in the hands of competent professionals.

While (at least in the first grade) EM does not offer as many opportunities for parents to take on roles as Instructors as does INV, it does dedicate more space to attempts to communicate some of the reform-oriented approaches, as in providing vocabulary terms with definitions and Family Notes explaining EM curricular conventions, recognizing that many parents have not been schooled in reform-oriented approaches to the learning of mathematics.

The most frequently offered role by EM is monitoring, often through the provision of homework answers to the parent, indicating that while EM recognizes that a parent may not, herself, know how to solve the problem, she has an interest in being able to monitor her child’s work and ascertain whether the child is getting the correct answer. While both reform curricula place less emphasis on “getting correct answers” than on engaging in mathematical thinking, EM provides a role that allows parents to take on a role that does not require particular mathematics content knowledge. By providing mechanisms (such as the correct answers for parents to consult), EM offers a way for parents to feel efficacious and to engage in behaviors consistent with their views of what a concerned parent ought to do. INV provides few opportunities for parents to engage in monitoring and thus affords fewer opportunities than EM for parents to experience a sense of efficacy in connection with possible involvement in their child’s mathematics homework.

However, while EM provides the role of Monitor for parents, the readability analysis shows that EM’s explanations of mathematical content and reform-oriented approaches (conveyed in the Family Letters) are generally at the mid-tenth grade reading level. EM’s Family Letters are much more difficult to read than their general instructions for parents in the form of the Family Notes (the instructions given to parents for each homework assignment), which tend to suggest the parental role of Monitor. In other words, parents can more easily access information on how to act as Monitors than as Instructors. This again confirms Peressini’s (1998) critique that parents are effectively being constricted in the sorts of roles that the mathematics education community formally invites.

In addition, for both curricula, as the grade levels increase, so does the readability level of difficulty. We assume that this happens because the mathematics content increases in its sophistication. However, this finding raises critical questions about the ability of parents to support their children with their homework in ways consistent with that of INV and EM, particularly as the children grow older and encounter more difficult material.

Implications and Future Research

This analysis raised several questions about the sort of mathematical, pedagogical, and pedagogical content knowledge parents are assumed to have and how parents are positioned via the reforms in materials that intend to provide parents a way to support their children. We assert that these questions are critical to explore in contexts where parental involvement is increasingly being mandated.

This textual analysis is phase one of a larger project designed to understand how these curricular materials position parents, how parents interpret such materials, how parents work with their children on mathematics homework, and how teachers make use of these materials. While the findings reported on in this paper are only a part of the larger picture, they suggest strongly that parents are not invited to take on roles that will maximize modeling, reinforcement, and instruction—mechanisms that positively influence student outcomes (Hoover-Dempsey & Sandler, 1995). Further, the findings suggest that these materials are not sufficient for parents to learn about how best to support their children in the context of reform-oriented approaches to the teaching and learning of mathematics.

Endnotes

1 MetroMath, a Center for Learning and Teaching funded by the National Science Foundation, supported this research under Grant # ESI0333753. Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors, and do not necessarily reflect the views of the National Science Foundation. A consortium of Rutgers University, the University of Pennsylvania, and the City University of New York, MetroMath aims to improve the learning of mathematics of inner-city children. More information about MetroMath is available at http://www.metromath.org.

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TEACHER COMMUNICATION BEHAVIOR IN THE MATHEMATICS CLASSROOM

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This paper describes the results of a questionnaire, the Teacher Communication Behavior Questionnaire (TCBQ), distributed to 178 students. The TCBQ assesses students' perceptions of five teacher communication behaviors. In this exploratory study, we show that the TCBQ, which was developed for science classrooms, can provide useful information about teacher communication behavior in mathematics classrooms as well.

Introduction and Current Literature

To promote effective discourse in the mathematics classroom, teachers should encourage participation by *all* students and provide feedback about the mathematics as well as the quality of communication (Adler, 1999). As Ambrose, Levi & Fennema (1997) argue, when teachers change their practices to encourage participation by all students, some equity issues will lessen while other unanticipated issues may arise (e.g., participation, peer affiliation, listening). Teachers must tend directly to issues related to gender inequities, for example, by reflecting on their decisions about orchestrating classroom discourse and balancing pragmatic and idealistic stances (Ambrose *et al.*, 1997). Affective aspects of underlying interpersonal relationships in classrooms are no less important for children's learning than cognitive aspects (Cazden, 2001).

In this paper, we report the results of the implementation of a questionnaire on students' perceptions of their teachers' communication behaviors that has been used in science classrooms. We argue that more research needs to be done to examine the students' and teachers' perceptions of the communication environment in the classroom. If we want teachers to facilitate effective classroom communication, we must also understand their students' perceptions.

Research Methodology

This study takes place within a larger, five-year NSF-funded project[1] in which eight middle grades[2] mathematics teachers from seven different schools are involved. Data for this study were collected during October, 2005.

This exploratory study is based on the administration of the Teacher Communication Behavior Questionnaire (TCBQ) (She & Fisher, 2000), which assesses student perceptions of the following five categories of teachers' communication behaviors: Challenging, Encouragement and Praise, Non-Verbal Support, Understanding and Friendly, and Controlling. Each category included eight statements, which were scored on a Likert scale (where the values 1-5 correspond to Almost Never, Seldom, Sometimes, Often, and Almost Always, respectively). Examples of the statements include: *This teacher encourages me to discuss the answers to questions and This teacher nods his/her head to show his/her understanding of my opinion.*

The data were analyzed using the mean scores from the five categories provided on the questionnaire. We report the findings related to the entire group of students (n = 178), the entire group of teachers (n = 8), the eight classes separately as entire classes and by gender, and the relationship between gender of student (n = 102 males, n = 75 females, n = 1 missing) and teacher (n = 3 males, n = 5 females). The teachers were included in the data file and variables that would calculate a mean score for each of the five categories were created. Using the five

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communication behaviors as the dependent variables and the teacher as the independent variable, student-reported means were calculated for every behavior category for their respective teachers.

Results

The results of a regression analysis indicated a high positive correlation (r =.720) between the Non-Verbal Support and Encouragement and Praise variables. In addition, moderate positive correlations were found between the following variables: Challenging/Encouragement and Praise (r = .681); Challenging/Non-Verbal Support (r = .602); Understanding and Friendly/Encouragement and Praise (r = .618); and Understanding and Friendly/Non-Verbal Support (r = .650).

The gender variable also produced interesting results. Although the *n*-size of any one class is too small to draw conclusions of statistical significance, bar graphs were used to provide suggestive evidence of differences between the perceptions of male and female students within teachers' classrooms. Most often, differences were noticed in the Non-Verbal Support variable.

Summary and Implications

This study shows that the TCBQ can provide useful information about students' perceptions of teacher communication behavior in the mathematics classroom. Because teachers are "usually too busy teaching to have the leisure to observe the details of the ebb and flow of students' engagement" (Lemke, 1990, p. 135), the TCBQ results can be used to uncover the students' perceptions of their teachers' communication behaviors.

The perspectives and voices of students themselves need to be considered more seriously in research on teaching and learning (Erickson & Shultz, 1992). Mutual trust between students and teachers takes on many forms and is dependent on both individual and cultural histories and preferences (Cazden, 2001). Deeper understanding of teacher communication behavior from both teachers' and students' perspectives, can assist in promoting effective classroom communication.

Endnotes

(1) This paper is based upon work supported by NSF (Grant # 0347906, Herbel-Eisenmann, PI). Any opinions, findings, and conclusions or recommendations expressed in this paper are those of the authors and do not necessarily reflect the views of NSF.

(2) In this study, "middle grades" refers to grades in which students' ages range from 11 to 16.

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CHILDREN'S OTHER TEACHERS — THEIR PARENTS

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This study focused on the interactions between urban parents with limited mathematics education and their 3rd and 4th grade children as they worked together on mathematics homework. A variety of parent behaviors, instructional strategies and support mechanisms emerged during the parent/child homework sessions, even though parents were not always able to complete the assignments themselves.

This study seeks to explore what happens when parents, who did not complete high school, work with their children on math homework. This population is of great interest because, in some urban areas, up to 40% of the adults have not graduated from high school and their children are at higher risk of poor school performance (Grissmer, Kirby, Berends, & Williamson, 1994). We hypothesize that parents with limited mathematics education will display behaviors and patterns of interaction that differ from those described in studies of relatively well educated middle class parents (Shumow, 1998, 2003).

Conceptual Framework

This research is informed by Bronfenbrenner's ecological framework (1979) that envisions children simultaneously developing as learners in multiple settings or microcosystems. The settings are both interconnected but also independently influenced by their own external environments. Through interactions around homework, we can see the extent that the activities, processes and behaviors developed in the classroom and family microcosystems are or are not aligned.

In studies in which parents' and teachers' interactions with children were compared (simple model building tasks, Wertsch, Minick, & Arns, 1984; scientific reasoning, Gleason & Schauble, 2000; arithmetic word problems, Lehrer & Shumow, 1997), parents were found to be more directive and tended to take responsibility for conceptualizing the tasks. This study questions whether the nature of the parent-teacher alignment differs when parents are less confident of their own mathematics content knowledge.

Methodology

Eight parents/caregivers of 3rd or 4th grade children were recruited from two adult basic education classes in large east coast cities. All adult students with appropriately aged children were invited to participate in the study; one mother declined to participate citing time constraints. The 7 mothers and one grandmother ranged in age from 34 to 61 and included 5 African-Americans, 2 Latina immigrants and one Caucasian. Their children all attend city schools that use the reform Everyday Mathematics curriculum.

Each parental figure worked with her child on a sequence of tasks drawn from homework materials provided with the classroom curriculum materials. Parent/child sessions took place at a time and location that was convenient to the parent. Five parents chose to meet at their homes and three chose to meet with their children at the adult education site. Sessions lasted about one hour and were videotaped.

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A coding scheme was developed by the authors through an iterative process, initially starting from Lehrer & Shumow's (1997) codes for level of control and function of parental assistance and then adding and modifying the codes to capture the nature of the parent/child interactions captured on the videotapes.

Findings

Many unproductive interactions resulted from parents' limited understanding of the mathematics. That said, there were also productive situations and interactions that were common across multiple parents. There were three categories of interactions that emerged when parents were unsure of the mathematics needed for the task: parallel work, positioning the child as teacher, and collaborative work with justifying conjectures.

One issue that emerged was children's use of alternative algorithms learned in school, that parents had been unable to understand. Many of the parents and children have negotiated a practice of parallel work, during which parents use a traditional algorithm to check the children's answer that was derived with an alternative algorithm.

Some parents use homework time as an opportunity to learn from the child about what he or she is learning in school. This also serves as a learning opportunity for the parent when she is unsure of the content. The parent puts the child in the role of the "explainer," answering the parent's questions and explaining how he or she is approaching and solving the problems.

Sometimes, when both parents and children are unsure of how to solve a problem, they work together to come to an agreed upon solution, with one or both putting forth suggestions and justifying their methodology and conclusion. When one mother was unsure of a strategy to solve the problem and did not immediately understand what her son was doing, she questioned his conclusion so he explained his reasoning to her, convincing her of its logic.

Discussion

The parent/child interactions described here are reflective of the types of interactions that are promoted for the classroom by the NCTM Standards. These include making and investigating mathematical conjectures, communicating mathematical thinking, flexibly shifting among various representations, cooperative learning and sharing ideas through discussion (NCTM, 2000). In an environment in which both parent and child are struggling to understand a mathematical idea, it seems that the parent can take on the role of a peer as envisioned and enacted in reform classrooms, rather than the role of a teacher.

Acknowledgement

This research was supported by the National Science Foundation under Grant No. ESI-0333753.

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IDENTITY AND MATHEMATICAL SUCCESS AMONG FIRST GENERATION IMMIGRANT LATINAS

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This study explores the life stories of four young immigrant women from El Salvador, Guatemala, Mexico, and Nicaragua who began high school as English Language Learners and successfully completed four years of college preparatory mathematics. We learn that these students' mathematical successes had much to do with the identities they constructed as women and the ways in which their mathematics classrooms supported these identities through socio-mathematical norms. These stories provide powerful counter narratives to a mainstream story which often suggests that students who immigrate to the United States and are English Language Learners cannot learn rigorous mathematics or that their parents and broader cultures do not support them in doing so.

Purpose and Background

Attention to the ways in which students participate in mathematical learning communities provides information about not only what students learn but also the forms of knowledge and the identities they create in relation to mathematics (Boaler & Greeno, 2000; Boaler, 2002; Martin, 2000; Nasir, 2002). This is important work as attention to the identities students create within mathematics classrooms has the potential to illuminate how students make sense of their mathematical experiences and then make choices about how to act in relation to them. Additionally, understanding learning as a process that encompasses the construction of new ways of being provides for a unique balance between personal agency and influence from the broader communities in which students participate. This perspective prevents us from completely attributing students' achievement or failure to culture and simultaneously recognizes the role of the individual in academic pursuits.

The gap in mathematics achievement and enrollment in higher-level mathematics courses between young women and men, between immigrant students and U.S. citizens and between Latinos, African Americans and White students continues today (NCES, 2005). While much work has been done to narrow this gap, too often efforts to do so rely on the homogenization of students. For example, it is frequently assumed that all girls learn better in a collaborative environment or that all Latina/o students require a certain kind of academic support. This study complexifies constructions of gender, ethnicity and the immigrant experience by including young women who immigrated to the United States from four different countries and who currently reside in a large urban area in Northern California.

Method

This project is an analysis of in-depth interviews and focus group discussions with four Latinas who were enrolled in and successfully completed four years of college preparatory mathematics courses. Each of the women immigrated to the United States from Mexico or from countries in Central America and began their high school mathematics careers as English Language Learners. This study contributes to a four-year longitudinal research project conducted

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by the Stanford Mathematics and Learning Study (2000-2004), that monitored approximately 700 students through four years of mathematics at three different public high schools in Northern California. When compared to students from the two other schools, Boaler and Staples (in press) claim that "students [at Railside] achieved more, enjoyed mathematics more and stayed with mathematics to higher levels" (p.1). It is also important to note that the mathematics program at Railside School is detracked and reform-oriented and has been shown to promote a view of mathematics focused on connections and meaning, relational equity, and high academic achievement amongst its ethnically diverse and working-class student population (Horn, 2006).

I borrow from McAdams' (1985) construction of identity as life-story and use a narrative approach to gather the life-stories of these particular young women; stories that explore the role of school and mathematics in their families and communities both in their home countries and in the United States. Narrative methods and the analysis of life stories provided access to the women's personal identities and the cultural and social worlds in which they are created (McAdams, 1993). This approach afforded insight into the personal constructions of self, the women's relations to mathematics in general and to school mathematics in particular rather than relying on second or third-person accounts.

Results

Results from this project provide powerful stories from young immigrant Latinas. These stories run counter to a mainstream narrative in the United States, which often suggests that students who immigrate to the United States and are English Language Learners cannot learn rigorous mathematics or that their parents and broader cultures do not support them in doing so. From the young participants in this project, we learn that their mathematical successes have much to do with the identities they have constructed as women and the ways in which their high school mathematics classrooms supported these identities through the socio-mathematical norms created in their math classrooms. Given the rapidity with which immigrants are joining schools across the United States and the unacceptable rate at which these same students are rejecting mathematics or dropping out of school altogether, these findings are crucial for mathematics teachers and anyone working to improve the quality of education for immigrant students.

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FOSTERING MAWIKINUTIMATIMK IN RESEARCH AND CLASSROOM PRACTICE

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While engaging aboriginal elders from eastern canada in ethnomathical conversations, we discovered challenges trying to develop a sense of mawikinutimatimk – learning together – which was our guiding principle. We reflect upon these initial challenges and how they correspond with similar issues in creating a community of learners in mathematics classrooms, especially as it pertains to aboriginal students.

The National Council of Teachers of Mathematics (NCTM) asserts that "the need to understand and be able to use mathematics in everyday life and in the workplace has never been greater and will continue to increase" (2000, p. 4). Today's changing world demands that students have the opportunity to "learn significant mathematics with depth and understanding" (NCTM, 2000, p.5). Mathematics learning is especially important for children in Canada's Aboriginal communities. Aboriginal leaders look to the younger generations to acquire the knowledge and skills to address community challenges such as developing sustainable economies as they move towards greater self-determination. However, currently, too few Aboriginal students are choosing to pursue studies in essential skill areas such as mathematics and science. Aboriginal people in North America have the lowest participation rates of all cultural groups in advanced levels of mathematics (Trumbull, Nelson-Barber, & Mitchell, 2002).

Methodology

Our research initiative emerged both from our desire to address the Mi'kmaw people's call for improvements to mathematics education within their schools and from the increased national concern about the disengagement of Aboriginal students from mathematics and science. The initial goal of our research was to bring together community elders, adults and youth in dialogue about the role of mathematical processes within Aboriginal culture. The groups discuss both historical and current cultural practices that involve an informal knowledge of mathematics. Participants also consider ways to engage more Aboriginal youth in the study of mathematics. The key questions are: *What mathematics is already present in Aboriginal culture?* and *How can this Indigenous knowledge be incorporated into the learning and teaching of mathematics to meet the needs of Aboriginal students better?*

The Mi'kmaw Ethics Committee guidelines declare that Mi'kmaq people must be treated as equals when participating in research, and that knowledge must be collectively discovered. Thus research needs to be done collaboratively with participants, respecting the knowledge, values and traditions of the communities. In a casual dialogue, Lisa asked an elder for a word that describes this kind of interaction – the act of people coming together to talk about an issue or solve a problem. The word he suggested, *mawikinutimatimk,* literally means "coming together to learn together". This spirit informs our methodology: all members of the group have something to share and something to learn. Our research comprises conversations, which reflect the long-standing Mi'kmaq tradition of coming together to share stories and ideas, and to deal with concerns (c.f. Joe & Choyce, 1997).

Findings

Early in our initial discussions with elders, we discovered the challenge of fostering *mawikinutimatimk*. The elders frequently asked if what they were saying was what we wanted. "Is this what you want? Are we telling you stuff you wanted to know?" Such questions demonstrate an unequal conversation, in which one group's questions and understanding are felt to be privileged over another's. The elders came with an understanding of what mathematics is and were unclear how that related to their everyday lives. One member of the group later said he was shocked to see everyday activities as "math" because to him they were "just problem solving in a way." Ironically, our attempts to value Indigenous mathematics knowledge seemed to be challenging these Indigenous people's sense of what mathematics is.

As we faced the challenges of *mawikinutimatimk* in research, we realized that similar challenges exist in mathematics classrooms. We wondered what prevents mathematics classes from becoming communities of learners – communities marked by collective discovery and in which each participant's contributions are valued and everyone, including the teacher, can learn something new. How many students harbour the same questions for their teachers that these elders asked of us: "What do you want? What do you really want? What are you going to do with the information I give to you?" Such questions suggest a sense of inequality amongst participants (even if the teacher truly wants a sense of equality), a barrier to a classroom culture that would reflect both *mawikinutimatimk* and the NCTM's communication principle: "Conversations in which mathematical ideas are explored from multiple perspectives help the participants sharpen their thinking and make connections" (p.60). All students must feel that their ideas and beliefs are valued within such a classroom culture. This may be especially important amongst Aboriginal children because when they do not see their cultural values reflected in the mathematics they are learning, they are not inclined to think that their ideas belong (Aikenhead, 2002; Trumbull, Nelson-Barber, & Mitchell, 2002).

Unfortunately, attempts to include cultural contexts (e.g. exploring patterns in baskets, rugs, and beadwork) often position "Western" mathematics as acting on indigenous cultural artefacts. Such trivializations insufficiently address the colonization embedded in the life experiences of Aboriginal students. A substantive, decolonized approach needs to address Indigenous *world views* in addition to artefacts: "Aboriginal children are advantaged by their own cultural identity and language, not disadvantaged in some deficit sense"(Aikenhead, 2002, p.3).

It is important for mathematics educators at all levels to be conscious of the effects of colonization on Aboriginal learners and to seek ways to include indigenous knowledge in a substantive way. Our conversations with elders have shown that mathematical knowledge was and still is embedded in the daily activities of Aboriginal people. This knowledge needs to be valued if we are to have *mawikinutimatimk* in mathematics classrooms.

In our presentation, we will describe the elders' responses to our reflections on our conversations.

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WHO BENEFITS?: EXPLORING ISSUES OF EQUITY AND A STANDARDS-BASED ELEMENTARY SCHOOL CURRICULUM

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We compare the performance of students of different ethnic and socioeconomic groups using the revised Investigations in Number, Data, and Space curriculum to the performance of students using other, non-Standard-based curricula. This study also examines gender differences in achievement and strategy use. Moreover, we explore the following question: How does the revised Investigations curriculum influence gender-, ethnic-, and socioeconomic-related achievement gaps?

The National Council of Teachers of Mathematics (NCTM) strives to empower all students by providing them with an equitable, high-quality mathematics education. Because NCTM placed equity first among their core principles in the *Principles and Standards for School Mathematics* (NCTM, 2000), implementing Standards-based curricula carries an implied promise to produce the positive, meaningful changes necessary to eliminate mathematics achievement inequities often linked with larger societal inequities. However, the field of mathematics education lacks research documenting the benefits and limitations of using Standards-based curricula with all students and needs a clearer, more complete description of equitable teaching practices.

In this short oral session, we describe research comparing the performance of students of different ethnic and socioeconomic groups using the revised *Investigations in Number, Data, and Space* curriculum (*Inv*.) to the performance of students using other, non-Standards-based curricula. We also examine gender differences in achievement and strategy use. Moreover, we explore the following question: How does the revised *Inv.* curriculum influence gender-, ethnic-, and socioeconomic-related achievement gaps?

The study is part of a large-scale, longitudinal evaluation of the revised *Inv*. curriculum. The evaluation compares the mathematical learning and achievement of students in classrooms using the revised *Inv.* curriculum with that of students in classrooms not using *Inv.* While the larger evaluation study follows two student cohorts over three years, beginning in the first and third grades, this report focuses on the performance of students in a subset of the third-grade cohort as they progressed through three years of school. The approximately 380 participants attended eight elementary schools (four revised *Inv*. and four non-*Inv*. schools) located in an ethnically, socioeconomically diverse large Midwest urban district.

To gather achievement (performance) data, we used four primary instrument types: (a) the Iowa Test of Basic Skills (ITBS), (b) content-focused assessments, (c) state-wide assessments, and (d) a task-based interview. We administered the ITBS mathematics and reading subtests in the fall of the 2002-2003 school year and used the results to account for initial achievement differences between the *Inv*. and the non-*Inv*. students. The students also took the initial content-focused assessment in the fall of 2003 and spring of 2004, at the beginning and end of their third-grade year, and the second and third content-focused assessments at the end of their fourth- and fifth-grade years. These assessments capture students' growth in number and operations and algebraic reasoning. They emphasize problem-solving contexts and authentic tasks, but include

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some symbolic computation items. Additionally, in the late fall of each school year, the students completed the state-wide assessments, and during the middle of their fifth-grade year, we conducted the task-based interviews with a subset of approximately 50 *Inv.* students. Through the interviews, we investigated gender differences in strategy use found in previous research. In addition, the teachers completed curriculum logs and a pedagogical survey to record and describe the implemented curriculum. We analyzed each data source using standard procedures of coding and statistical analysis and interpreted the multiple data sources in relation to one another.

In this short oral session, we add to the discussion about the potential of Standards-based curricula and teaching to promote equitable outcomes in our schools. Along with previous research, the study's results suggest that implementing Standards-based curricula and pedagogy without paying particular attention to poor and minority students' strengths and needs does not automatically narrow the achievement gap. Furthermore, they suggest using the *Inv.* curriculum has the potential to increase the achievement gap among diverse ethnic and socioeconomic groups. However, the findings also suggest that using the *Inv.* curriculum does not promote gender differences, and thus, contrast the results found by Fennema et al. (1998) in their study of early elementary students whose teachers used Standards-based teaching methods. In their study, Fennema et al. found boys' and girls' achieved at the same level, but girls were more likely to use modeling, counting, and standard algorithms, and boys were more likely to use abstract strategies.

Although reasons for the inequities in schools extend well beyond the nature of curriculum and instruction, this study focuses on curriculum—one of the elements of the student-teacher relationship teachers and schools can effectively control (Knapp et al., 1995). The results highlight the need for more research focusing on Standards-based curricula, teaching practices, and equity to further the discussion about equitable teaching practices (The National Research Council, 2004). Moreover, the findings reaffirm Lubienski's (2000) assertion equity must be kept at the forefront of discussions regarding curriculum and pedagogy; a curriculum or pedagogical practice promising for many students may not be promising for all.

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COMPARING THE PERCEPTIONS OF TEACHERS IN HIGH- AND LOW-SES CONTEXTS TOWARDS THE ROLE OF THE GRAPHING CALCULATOR IN MATHEMATICS INSTRUCTION

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This study investigates teachers' perceptions of the role of graphing calculators in the mathematics instruction of students from different SES schools. Findings showed that the nature of graphing calculator use was strongly influenced by the various contexts and that the low-SES school's respondents appeared not to involve their students in lessons that capitalized on the powerful characteristics of graphing calculators.

Introduction and Rationale

Race and socioeconomic status (SES) are equity factors, which have long been associated with the disparities and achievement gap, amongst students in mathematics. Although there has been a multiplicity of meanings of the term equity in relation to mathematics learning, the general consensus has been the acknowledgement of the existence of this achievement gap. Moreover, technology has been one of the tools recommended for achieving equity in the mathematics classroom. Compared to other forms of technology, however, the physical access to graphing calculators is high in general and issues pertaining to equity when using graphing calculators arise more from the experiential access; that is, the nature of graphing calculator use. This study investigates teachers' perceptions of the role of graphing calculators in the mathematics instruction of students from different SES schools.

Perspectives and Frameworks

A sociocultural perspective enabled me to examine teachers' perceptions of graphing calculator use as a mediating tool to facilitate the mathematical learning of low-SES and high-SES students situated within different sociocultural classroom and school contexts. According to sociocultural theory, learning is socially and culturally situated in contexts of everyday activities (Vygotsky, 1978; Wertsch, 1991) and is the result of a dynamic interaction between individuals, other people, and cultural artifacts or tools, all of which contribute to the social formation of the individual mind and lead to the realization of socially valued goals. These activities include the everyday cultural experiences that are subject to social conditions, such as SES.

The goal of this study was to learn about, and draw to the attention of mathematics educators, some of the potential red flags that stand in the pathway of ensuring not only the availability but also the appropriate use of graphing calculators that can in turn promote equitable mathematics education. The research questions that guided the study were:

1. What are the perceptions of teachers regarding the role of graphing calculators in the mathematics instruction of students with different SES? What are their perceptions of the local constraints pertaining to the use of graphing calculators?
2. What are the teachers' perceptions of the factors that influence their decisions regarding the use of graphing calculators in different SES contexts?

In addressing these questions, I compared the perception of the respondents, at both high-and

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low-SES schools, of the role or use of graphing calculators in mathematics instruction. In addition, I investigated how the situational context appeared to have enhanced or constrained the use of graphing calculators at both the high-and low-SES schools. Pertinent to this discussion and of prime importance is how the situational context appeared to have influenced the way the graphing calculator was used**.**

Data Collection

In this study, I used both quantitative and qualitative methodologies to investigate the research questions. The quantitative part was comprised of a Likert scale survey instrument, while the qualitative component was comprised of classroom observations and semi-structured interviews.

Results and Conclusion

The results of this study indicate that the participants' perception of the nature the role of graphing calculator is dependent on the context within which it is used and that the low-SES school's respondents appeared not to involve their students in lessons that capitalized on the powerful characteristics of graphing calculators. In my analysis, I conceptualized a four-component framework, which helped to tease out the role of the situational context (see figure 1).

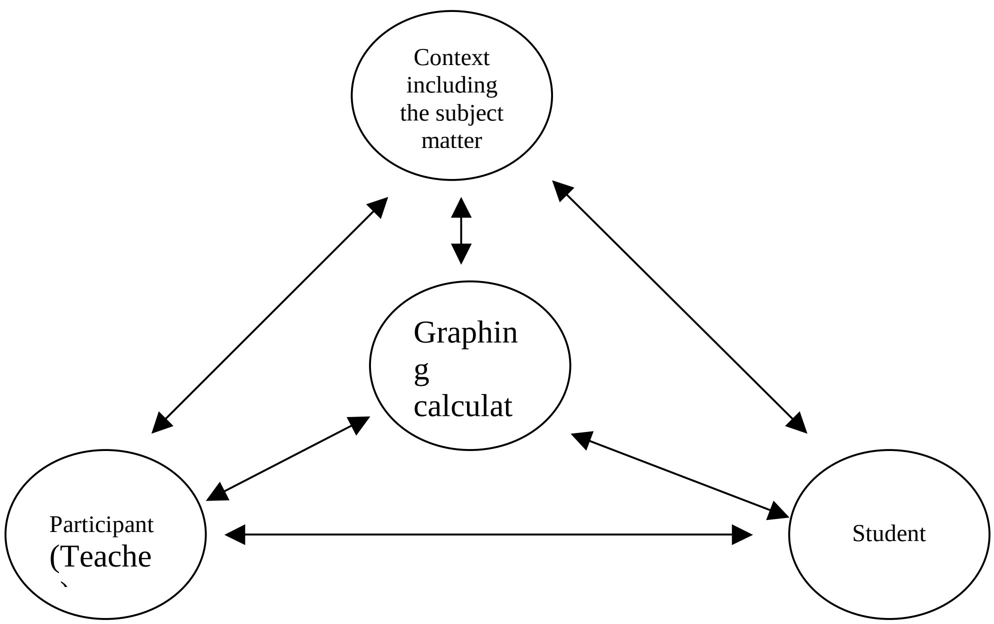

Figure 1. A Four-component Model of Mathematical Learning using a Graphing Calculator

Moreover, I assumed that the components of this framework are continuously in interaction with one another which implies that a change or perturbation in one of the components perturbates all the other components. The continuous interactions of the components of this framework suggest that equity issues in connection to the nature of graphing calculator should be an ongoing process that is continuously locating for strategies that will afford all students appropriate access and use of graphing calculators. This is important for the use of the graphing calculator, as a tool of educational reform, to achieve the NCTM's (2000) equity goal, rather than to end up exacerbating the already existing inequities between students of high-SES and low-SES schools.

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CHALLENGES OF INSTRUCTING SECONDARY ENGLISH LANGUAGE LEARNER STUDENTS IN MATHEMATICS: A SURVEY OF TEXAS TEACHERS

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The results of this paper are part of a multiyear project focusing on developing instructional resources designed to increase the effectiveness of mathematics instruction for students who are English Language Learners. The project is a partnership between the Texas State University Mathematics for English Language Learners Initiative, Texas State University System universities, and the Texas Education Agency.

Introduction

English Language Learner (ELL) students, students whose native language is not English and who are in the process of developing English speaking and writing skills, experience significant challenges in the typical United States mathematics classroom. As a group, secondary ELL students consistently score among the lowest of any student group on the Texas Assessment of Knowledge and Skills (Texas Education Agency, 2000). Growing evidence suggests that low performance on standardized assessments by ELL students has little to do with innate mathematical ability and much to do with cultural differences in the ways mathematics concepts are taught in other countries and with linguistic (vocabulary) barriers commonly found among non-native English speakers (Richardson & Wilkinson, 2005).

Such issues are often coupled with the problem of limited professional development available to teachers of such students. The rationale for this study is to improve understandings about how to teach mathematics to ELL students. More specifically, the goal of this study is to support mathematics instruction for ELL students in Texas, specifically students at the secondary level, through citing secondary mathematics teachers' perceptions of needed professional development in this area. Such professional development training needs may include, but are not limited to, in-service training, supportive instructional resources and tools, and implementation of specific professional development programs.

Data Collection

Researchers ascertained secondary mathematics Texas teachers' perceptions of needed professional development to better support mathematics instruction for their ELL students through the collection and analysis of data from two key sources: survey data and focus group findings. The targeted audience included primarily high school mathematics teachers who were currently teaching or had taught a significant number of ELL students. However, researchers also interviewed other educators, including secondary ELL/ESL teachers who taught no specific content discipline, middle school and high school mathematics curriculum coordinators, middle school mathematics teachers, elementary ELL/ESL teachers, and school administrators. As a first step in identifying secondary mathematics teachers' perceived professional development needs to better support mathematics instruction for their ELL students, survey forms were mailed to 130 randomly selected high schools throughout the state of Texas, including each of the 20 Texas Education Service Areas. In each packet, the distribution was addressed directly to a high school principal with a cover letter requesting that the principal distribute the survey to three mathematics teachers, with emphasis on teachers with high populations of ELL students. Postage-free envelopes were provided for return of the surveys.

Approximately 25% of the surveys distributed were returned and analyzed. Select high school mathematics teachers, some of whom completed the survey, were invited to participate in small focus group interviews. Researchers hosted focus groups throughout the state, including focus groups in central Texas, south Texas, southeast Texas, and west Texas. The number of participants in each focus group varied, ranging from five to fourteen participants. Participants for each focus group were selected based on their documented interest expressed in the surveys distributed prior to formation of the focus groups, recommendations made by school principals and vice-principals, and recruitment from researchers based on their prior knowledge of schools with high ELL student populations. Because research indicates that focus group participants are generally more open and less guarded with people they do not know (Morgan, 1988), care was taken to ensure that teachers who knew one another were not recruited for the same sessions. The composition of each focus group consisted of teachers from different schools and/or school districts.

Results

Analysis of data revealed that teachers often engage in general forms of professional development that they couple with other strategies to assist their ELL students in the learning of mathematics, but all data indicate that participants had little to no professional development that specifically addressed improving <u>mathematics</u> instruction for ELL students. Findings also revealed that mathematics teachers were in need of training that separated learning mathematics from learning English and many school districts were in need of certified secondary bilingual mathematics teachers.

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CREATING OPPORTUNITIES FOR ALL:UNPACKING EQUITABLE PRACTICES IN MATHEMATICS CLASSROOMS

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Research on equity in mathematics and other school subjects has tended to focus on what is meaningful for particular racial and ethnic groups. With increasingly diverse classrooms and concerns about the danger of essentializing, there is a need for the field of mathematics education to identify classroom practices that provide equal access to opportunities to engage in mathematical inquiry and develop positive dispositions towards mathematics for a range of students. This paper reports three studies that examined the structure of activity in mathematics classrooms that were successful at promoting broad-based participation among their students. The analyses focus on four aspects of activity systems including instructional tasks, classroom discourse, and the roles of the teacher and students. Additionally, the analyses illustrate how interactions among these elements created access to opportunities for all students to be successful.

Introduction

The purpose of this paper is to problematize a current conceptualization of equitable teaching that focuses primarily on what is *meaningful* for particular racial and ethnic groups, and encourage a focus instead on how classrooms create opportunities for all students to negotiate competent participation and thus positive dispositions towards mathematics learning. The motivation for this paper stems from several pressing concerns. First, we are wary of approaches that simplify a complicated and rich process of locally constructing cultural practices in a particular setting by reducing them either to a list of topics to include in a curriculum or practices that may be easily stereotyped. Second, we believe that accommodating every student's home experiences is often untenable in America's increasingly heterogeneous schools, wherein one classroom may include students from many different cultural backgrounds. For these reasons, we believe that researchers who are concerned with issues of equity need to expand their focus to explore how particular classroom practices create opportunities for all students to be successful (Delpit, 1995; Gutierrez, 1999; Moschkovich, 2002). Doing so requires thinking about success as more than achievement scores to include a consideration of both the ideas that students learn and how they come to appreciate and see value in mathematics.

Our examination of opportunities to learn in the local space of the mathematics classroom shapes our conceptualization of equitable teaching in terms of students' *access* to classroom mathematical practices. We consider the processes by which students gain or lose access to mathematical practices in the design of classroom tasks, in classroom discussion, and in the broader classroom practices and norms that get organized over time Cobb & Nasir, 2002). This perspective of equity stands in contrast to one in which broader socio-political structures and processes around race, culture, and power outside of the classroom are considered with respect to classroom life (Martin, 2000, 2003; Tate, 1994)

In the proposed paper, we will present three studies that investigate aspects of the classroom social context in supporting students' access to ideas and forms of reasoning as well as a sense of

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affiliation with mathematics. These three studies are united in their approach to the study of equitable teaching both in their perspective of the classroom as a system of activity comprised of different elements (Greeno,& MMAP, 1998;), as well as the use of research methods that capture the functioning of these elements both in moment-to-moment classroom interaction and over time.

Considering Access in the Classroom Activity System

Creating access to opportunities to engage deeply with mathematical content is a theme that cuts across these studies. Focusing on the distribution of access in this way highlights the role of the organization of activity systems (Cohen & Lotan, 1995; Engle & Conant, 2002), and the nature of students' experiences in mathematics in order to account for students' successes and difficulties (Boaler, 1998; Martin, 2000). This perspective draws attention to the complex interactions between elements of a classroom system in constructing and reifying mathematical practices, and thus what counts as competent participation (XXXX). From the perspective of access, aspects of the classroom system can be seen to support or delimit students' experiences to be successful in learning mathematics. Considering the construction of equitable access to meaningful engagement ensures that practices of engaging with content become the focus of analysis, rather than assumptions of what may be meaningful, and thus "motivating," for particular groups of students (XXXX).

In the proposed paper, we will draw on data from three studies to delineate aspects of the classroom activity system that contributed to of the creation of opportunities for students to engage competently with mathematical content. In doing so, we focus our discussion on four themes: instructional activities, classroom discourse, the role of the teacher, and the role of the student. These aspects are closely related and can be seen to contribute to how mathematics becomes realized in particular classrooms (Bowers & Nickerson, 2001; Cobb, Yackel, Wood & McNeal, 1992; Lampert, 2001). The sorts of mathematical conversations that take place in a classroom can be viewed as situated within a space of possibilities that are afforded by instructional activities. In turn, classroom discourse can be seen as developing in interaction in which the teacher's role, students' roles, and their negotiation of what counts as mathematical competence are critical.

Data Sources and Method of Analysis

All three studies took place in the United States in urban middle and high school classrooms. The duration of the studies ranged from one semester to one year. All analyses drew on field notes, videotaped recordings of class sessions, and student interviews. The studies share common aspects of systematically analyzing data through multiple stages of coding (Glaser & Strauss, 1967). These common and overlapping phases include first working through field notes in order to identify critical incidents and potential shifts in patterns of classroom discourse, the role of the teacher, and the role of the students. A second phase involves working through videotaped recordings session by session in order to isolate critical incidents and to make conjectures on patterns of participation. A third phase involved examining the analyses of sessions at a meta-level in order to test conjectures about patterns and to isolate sessions that illustrate significant shifts in discourse, roles, and the opportunities that were afforded students to engage with mathematics in substantial ways. Particular attention was paid in each of the studies to situations that seemed to contradict ongoing conjectures and to offer explanations for these situations and how they informed the ongoing analyses.

Results

A number of significant findings emerged from each study, and the purpose of the proposed paper is to draw attention to connections as well as significant differences across studies in order to inform more equitable classroom practices. We discuss key findings in terms of the four themes identified above. Our analyses revealed that instructional activities were critical in supporting substantial conversations about mathematics in that they opened up a space of possible mathematical topics that could become the focus of discussions. The structure of activities served both to support students' initial engagement with the task and their later use of mathematics for solving problems.

Classroom discourse and in particular what counted as a mathematical explanation afforded students access to multiple task interpretations from which they could become interested in informing the problem situation and in teasing out important math ideas as they compared analyses. As the analyses show, classroom discourse also presented students with opportunities to make their thinking and mathematics public, thereby creating opportunities for revision and contributing to a shared purpose and a sense of community in supporting everyone's learning.

Both the roles of the teacher and students proved to be critical as well in opening up spaces of opportunities. For instance, all three teachers adeptly included students' comments as part of discussions while advancing the mathematical agenda. At the same time, all three teachers drew on different strategies and practices to navigate this tension. The roles that became constituted for students differed across studies, but one key commonality was that students were active contributors to the ideas that came to matter in each class, and in this process students were able to experience voice while engaging deeply with mathematics.

Significance

The proposed paper is positioned to make significant theoretical and pragmatic contributions. Theoretically, the paper will contribute to an understanding of how the dynamics of equity and processes involving access and opportunities are created, developed and sustained in classrooms. Pragmatically, the proposed paper will contribute to an understanding of more equitable instructional practices and resources that support students' access to mathematical ideas and to a view of mathematics as a worthwhile and important activity.

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INFINITESIMAL PROCEDURES IN MODERN AND MEDIEVAL MATHEMATICS TEXTBOOKS

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The paper presents the results of an investigation of the infinitesimal procedures for the calculation of the area of a circle depicted in 19 modern Canadian, Taiwanese, and Russian secondary schools textbooks and in their medieval counterparts. The authors conjecture that the procedures found in the textbooks are related to mathematical methods found in medieval mathematical writings, in particular, in those of Gerard of Brussels and of J. Kepler.

Recent investigations of the processes taking place in the mathematics classroom have focused on various aspects of the actual interactions within the groups composed by the teachers and learners thus often unintentionally downplaying the role of the textbooks used by the teachers. However, the mathematics textbooks arguably provide the frame of reference for the course design even in the cases when the approaches they suggest are not referred to by the teachers or even are opposed by them. In our study, we explored the hypothesis according to which a historical investigation of the concepts, strategies, and tools found in mathematics textbooks is relevant to the investigation of the conceptual framework shaping the modern classroom activities and to the description of the representations determining the most general trends of the mathematics education.

Our investigation focused on the infinitesimal procedures for the calculation of the area of a circle depicted in modern Canadian, Taiwanese, and Soviet/Russian textbooks for secondary schools along with their potential didactical transposition for classroom activities. The approach is similar to that applied in (Freiman and Volkov 2004) to the study of the concept of fraction employed in textbooks representing various didactical traditions. We analyzed 12 textbooks published in Quebec province in 1969-1994, one textbook from Ontario (1997), five textbooks from USSR/Russia (1979-2001) and one Taiwanese textbook (2003). It was found that the majority of the procedures have been based on "intuitively clear" geometrical constructions intended to justify the formula of the area without using sophisticated formal tools such as the classical definition of limit involving the "ε–δ language." These constructions were usually presented with one or several diagrams accompanied by short explanations appealing mostly to the intuition of the student. Among the especially popular diagrams there are (1) the circle covered with a large number of squares of small size (the surface area of the circle is thus being introduced as the number of the squares contained inside the circle when the squares are small enough); (2) the inscribed regular polygon (with the length of one side small enough to make the area of the polygon close to the area of the circle); (3) the circle subdivided into a number of identical sectors which can be rearranged in order to produce a curvilinear figure that can be approximated with a parallelogram if the number of the sectors is large enough; (4) the circle subdivided into a number of concentric strips that can be straighten up to produce a rectilinear figure (usually a triangle). Three latter constructions lead to the formula of the area of a circle, while the former one can only be used to introduce the very concept of the area of a circle.

Even though these "intuitively clear" constructions were seemingly designed for the modern schoolchildren with purely didactical purpose, a number of them are actually much older and can

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be found in medieval Western and East-Asian mathematical treatises. For instance, the configuration including the squares covering the surface of a circle is to be found in a number of Chinese treatises, for example, in the *Hushi suanshu* 弧矢算術 (The procedure of calculation of arcs and sagittas) by Gu Yingxiang 顧應祥 (1483-1565), as well as in Korean treatises, for example, in the *Sanhak yipmun* 算學入門 (Introduction to the science of calculation) by Hwang Yun-Syo 黃胤錫 (1729-1791), even though any evidence of the direct connections between these manuals and their modern counterparts can hardly be provided. However, the method of subdivision of a circle into sectors that became popular in Western textbooks no later than the early 20th century (Hall and Stevens 1910, p. 203) can be found in much earlier mathematical texts, in particular, in the works of J. Kepler (1571-1630), while the subdivision of the circle into a large number of concentric strips was suggested by Gerard of Brussels (1187-1260) in his *Liber de motu*, although in a rather different context (Clagett 1956).

Historically, the first mathematically strict demonstration of the formula for the area of a circle was offered by Archimedes (287-212 BC) who carefully avoided any operations involving infinitely small and infinitely large magnitudes. However, our brief inspection of the history of European approaches to the topic suggested that the highly technical Archimedes' method was not always enthusiastically embraced by the authors of medieval mathematics textbooks. Their didactical approach looked very close to that adopted in the school textbooks written in the spirit of the "intuitive geometry" of the late 19th – early 20th century widely employing visual representations similar to those of Gerard of Brussels and J.Kepler. One can conjecture that these representations, sometimes with slight modifications, eventually found their way to the recent textbooks whose authors most likely were not aware of the complex history of the methods they offered to the schoolchildren. Moreover, despite the apparent intention of the authors of the modern textbooks to avoid mathematically strict yet highly technical demonstrations and to offer seemingly simple visual explanations instead, the diagrams they provide may lead to various misconceptions concerning the concepts of infinity and of the areas of curvilinear figures.

One can suggest that the similarities between the approaches found in today's school mathematics textbooks and in their medieval counterparts do not represent a mere coincidence but are partly due to particularities of the history of the formation of the modern school textbooks. In the case under investigation, the visualizations of the concept of infinitesimals positioned as didactic approaches in modern textbooks are arguably related to medieval mathematical methods.

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COMMON SENSE, NECESSITY, AND INTENTION IN ETHNOMATHEMATICS

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In ethnomathematical conversations with Aboriginal elders in Eastern Canada, we examine conflicts in values and intentions between the cultural mathematical practices in Aboriginal communities (both traditional and modern) and Western-oriented schools. Elders' accounts of their mathematical practices highlight common sense, which cannot be applied in a school setting abstracted from community issues and needs.

"You just take a [piece of birch] bark and hold it over the circle. Fold it in half and fold it in half again to get the centre." Mi'kmaw elder, Diane Toney, was well-known for the quality of the boxes she made out of porcupine quills. For her, folding a round piece of bark to find the centre of the circle was common sense; it was not mathematics.

As part of a large-scale project investigating mathematics and science learning in informal contexts in Atlantic Canada, we have been interviewing Aboriginal elders to identify some of their everyday practices (both traditional and current) that could be deemed mathematical. This typical approach to ethnomathematics research (c.f. Powell & Frankenstein, 1997) relies on Bishop's (1988) definition of mathematical activity (practices that involves counting, measuring, locating, playing, designing or explaining) and on the assumption that any mathematics is an artefact of a particular culture.

In these ongoing conversations, it is our intent to get beyond the identification of mathematical practices to consider differences in values and intentions as well as the changes Aboriginal children experience when they are encultured in their classroom mathematics. Thus, our focus question is: *What is lost and what is gained in a move from the community's cultural practice to a "Western" mathematical practice?* Our aim is not to deem such moves inappropriate, but rather to raise awareness to their socio-cultural effects. We believe that awareness of both potential losses and possible new opportunities could mitigate the losses to some extent. Also, awareness may encourage young Aboriginal people to engage more with Western mathematics and science because they will have had the opportunity to explore the issues behind the nagging feelings of inappropriacy that accompany such cultural transitions.

Conflicts Between Aboriginal and School Mathematics

In the interviews, the elders have been quick to identify cultural mathematical practices after we suggest the unacceptability of our larger society's tacit definition of mathematics – things done in mathematics classrooms – and outline Bishop's (1988) alternative definition. For example, when Diane Toney (who died May 15, 2006) made quill boxes (which are circular) she knew that "To make a ring, you need to go across the centre of your birch bark [the diameter] three times and allow about the width of your thumb [i.e. π] to make a perfect round." She could also find a circle's centre as described above.

When we asked about conflicts between cultural mathematical practices and school mathematics, elders responded saying that children take things for granted too much: They only flick a switch to get light. Because different groups of elders responded in the same way, we see that taken-for-granted aspects of our modern world must be at the heart of the perceived conflict

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between traditional practices (some of which remain current practices) and the mathematics the elders see children take in school.

In our interpretation of these interviews, we noticed the frequent reference to "common sense" when the elders described how they know what to do in the situations they described as mathematical. How much wood would they haul home for fuel? "Enough." How did they know how much to bring? "Common sense." By contrast, we might consider a school mathematics word problem that asks, "Bob's wood pile for a week of fuel is about 2 x 5 x 3 feet. What would the dimensions of the pile be for two weeks of fuel?" It is easy to imagine a child in school answering 4 x 10 x 6, doubling each dimension. But it is hard to imagine someone who is cutting, hauling and burning the wood making the same error.

The person who needs wood for fuel draws on common sense, which includes a sense of the situation, a sense of the family's needs and a sense of the work it takes to meet these needs. In such situations, the answer to our mathematical questions can be "enough". How many potatoes would you cook? "Enough. That way you didn't waste any." These kinds of answer may seem unmathematical because we may wonder how much enough is. But a typical mathematical word problem answer, like "9 potatoes are needed for a family of 6," ignores the reality of variance in potato size. For the answer to the potato question, a gesture showing an imagined volume (roughly spherical) accompanies the elder's "enough." Likewise, for the wood-fetching question, the elder marks a height off with his hand as he says "enough." The natural gesture, which is part of his answer, does not tell us how much wood was needed, but it does show us that he knew how much enough was.

"Enough" implies a sense of what is needed. For this kind of sense, the question needs to be situated in a problem – a real problem. Children who have everything they need at the their fingertips cannot have a sense of necessity. To ground classroom mathematics in such necessity, we, like D'Ambrosio (1998), suggest that class activity begin with an issue faced by the children's community. With mathematical activity that begins in local issues, students can begin to use their mathematics to exercise their intentions within these issues. This kind of personal (and communal) agency is different from agency that arises in classroom contexts in which the mathematical starting points relate to other people's concerns.

When Students cannot use Common Sense

When their mathematics is not grounded in their experience, students cannot apply common sense. They need something else. Perhaps this something else is what some educators call spatial sense and number sense. It seems to be expected that children learn to understand space and number before addressing their community's issues. The Aboriginal elders who we have been interviewing seem to be saying that this is backwards. Mathematics should *begin* with common sense. Brown (1996) asserts that the emphasis should not be "on students re-creating the teacher's intention but instead […] on students' production of meaning in respect to their given task" (p. 64). We suggest that students' production of meaning should rather relate to their tasks as humans, addressing community needs.

In our presentation, we will describe the elders' responses to our interpretations.

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MATHEMATICS TEACHING ASSISTANTS: DETERMINING THEIR INVOLVEMENT IN COLLEGE TEACHING

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For many years, Graduate Teaching Assistants (GTAs) have been employed to perform teaching related responsibilities. Studies from the early 1990's found that about 2/3 of all GTAs serve as sole instructor for their course (Buerkel-Rothfuss & Gray, 1991), teaching an estimated 33% of all undergraduate classes (Butler, Laumer, & Moore, 1993).

Because of the potential impact that Mathematics Teaching Assistants (MTAs) have on college mathematics instruction, researchers are trying to determine how best to prepare MTAs. The problem is that we do not know how representative these studies are of our MTA population, specifically regarding their demographics and responsibilities.

To address this problem, we conducted a nationwide study of mathematics departments in order to determine: a) the extent and nature of MTA involvement in college instruction during the fall 2005 quarter/semester, and b) what is currently being done to prepare them for these responsibilities. This poster presents those findings related to the following questions: 1) How does the MTA population compare to the GTA population in general? 2) What type of responsibilities do MTAs typically have?

Data analysis revealed major differences between MTAs and the GTA population at large, as well as great diversity among MTAs' responsibilities. Comparing MTAs and GTAs, we found that MTAs taught less frequently and had less autonomy than their counterparts in other disciplines. We further noted differences between institution type and the nature of MTA involvement in instruction. Analysis also revealed that departments differ by several non-correlating variables; these variables, which include MTA population size, cultural diversity, and workload/autonomy, have the potential to create seriously different contexts and thus impact the social networking opportunities and thus the teaching development of MTAs in these departments.

Results provide a variety of implications for researchers interested in understanding the experience and preparation of MTAs. By showing that cross-disciplinary research is not representative of the MTA population, this study supports the need for discipline-specific inquiry. Existence of the above variables suggests caution regarding the generalization of small scale, qualitative research; this implies that such studies provide this information to aid in interpretation of study results. Further details will be provided in the poster presentation.

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UNDERGRADUATES' USE OF MATHEMATICS TEXTBOOKS

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While most mathematics textbooks are written to help the reader develop an understanding of the mathematical content, our hypothesis is that many undergraduate students do not use textbooks in ways that help them gain this understanding. Many college mathematics teachers have anecdotal accounts of the ways students use textbooks but this has not been studied in detail. Our goal is to describe the ways undergraduate math students use their textbooks.

In our pilot study, we administered a survey to students in 1st and 2nd year mathematics courses at three large universities and a liberal arts college. Participating students were enrolled in a math class, including college algebra, calculus, statistics, and math content courses for elementary teachers.

The survey questions asked students to identify the parts of the textbook they use, the times they use them, and their goals in using them. For example, we found that 90% of students look at the examples in the book, and students in math courses for pre-service elementary teachers were more likely to rephrase or summarize these examples while doing homework than students in other classes. We found that students thought the most important qualities of a textbook were including many examples and highlighting important equations. The survey also asked how the textbook is integrated into the course. For example, we found that within a class students' perceptions of how they are asked to use their textbook vary considerably However, when students believe they are asked to read their textbook daily or weekly, they are more likely to use the book to prepare for class than if they are asked less frequently.

Researchers have previously investigated mathematics curricula at the K-12 level. There has been considerable research on the teacher-curriculum relationship (e.g. Remillard, 2005). Studies have also investigated the impact of curricula, in particular standards-based or reform curricula, on student learning (e.g. Riordan and Noyce 2001). Some studies have attempted to characterize various aspects of textbook content, such as types of definitions or control structures (e.g. Mesa, 2004).

We believe our study adds to previous research by addressing several characteristics of undergraduate education that differ from K-12. Because of the lack of extensive curricular materials and college instructors' considerable freedom in structuring courses, the textbook is effectively the unifying curricular element in undergraduate mathematics classes. Also, the student's relationship with the textbook exists primarily outside of class, and the instructor is involved only indirectly in this relationship. In addition, our study informs textbook analyses by describing how students actually use different aspects of the textbook.

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HOW DO GIRLS EXPERIENCE SMALL GROUPS IN THE MIDDLE SCHOOL MATHEMATICS CLASSROOM?

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Much of the research indicates that teachers who adopt classroom methods of cooperation versus competition "establish learning environments that change the traditional dynamics of classrooms" (Streitmatter, 1994, p. 98). For many girls and women, an atmosphere that enables students to enter into mathematics through "connected knowing" promotes successful learning (Wickett, 1997). In this poster, I present the results of a qualitative study that examined the experiences of two female middle school students working in small groups in a reform-oriented mathematics classroom.

For this study, I observed two different eighth-grade mathematics classes taught by the same teacher. Study participants completed a brief open-ended questionnaire about their attitudes towards mathematics class and their feelings about working in small groups. After interviewing the students, I performed member checks by having the participants read my interpretations of the interviews. I then conducted follow-up interviews.

Although I was able to identify numerous themes, the two prevalent themes that I highlight in the poster are the themes of "helping" versus "giving answers" and what I call, "getting it done." Both of the girls in the study, who I refer to as Amy and Janelle, spoke about *helping* students in a way that they were not just *giving* answers. They both emphasized that it is important to *understand* the mathematics rather than just obtain the answer. Amy and Janelle frequently spoke of "getting it done." By "it," the girls were referring to the tasks that their teacher assigned for them to do in groups.

Overall, in this study, the students reported that their experiences of working in small groups in mathematics class were positive and productive. The two students seemed to have a strong sense about the type of helping behaviors that are beneficial to themselves and to their classmates. Teacher-led conversations about giving help versus giving answers might be fruitful in fostering more productive work groups. Last, the results indicate that teachers may want to think more carefully about the importance of group and individual accountability.

Acknowledgement

This material is based upon work supported by the NSF under Grant No. 0347906 (Beth Herbel-Eisenmann, PI). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the NSF. I would also like to thank Beth Herbel-Eisenmann for her feedback on this research.

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EXPLORING THE CONNECTION BETWEEN MATHEMATICS ATTITUDES OF AFRICAN AMERICAN STUDENTS AND CULTURALLY RELEVANT PEDAGOGY

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This study examined how pre-college experiences with equity in the mathematics classroom connect to mathematics attitudes in undergraduate African American students. The participants, ages 18 to 19, were comprised of varied backgrounds in terms of majors, type and demographics of K-12 schools attended, mathematics performance, teachers, and instructional experiences. Semi-structured interviews were the primary data source for this qualitative study.

Results suggested that pre-college perceptions of equity are integrally connected to the mathematics attitudes of African American undergraduate students in that the kind of mathematics experiences they had either encouraged or diminished positive attitudes toward mathematics and toward themselves as learners of mathematics. The constructs that affected attitudes were: teacher characteristics (the type of math teachers the student encountered), student resiliency (personality and other individual traits), and student ideology (how the student viewed himself and the world). This paper focuses specifically on the findings related to the dimension of teacher characteristics. Characteristics are considered within the context of Culturally Relevant Pedagogy (Ladson-Billings, 1995). The propositions: conceptions of self and others, social relations, and conceptions of knowledge were evidenced in the findings. Positive teacher-student relationships, high expectations, supportive learning communities, and dynamic teaching and scaffolding were salient in participants' perceptions of what constituted a "good" mathematics teacher.

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SOCIAL CONSTRUCTION OF THE TRIGONOMETRICAL FUNCTION

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The phenomenon associated to the trigonometrical function has been contemplated in investigations of the cognitive perspective (De Kee, et. al, 1996) and didactic perspective (Maldonado, 2005), showing the narrow relationship of their results with the organization of the study programs, the exhibition of the text books and the teaching mathematical discourse.

We propose a model on the social construction of knowledge associated to the trigonometrical function based on the germinal ideas, the context and the problems that give meaning and functionality to the related concepts. The elements of social construction we consider in our model are: the activities, the reference practices and the social practices bounded to the trigonometrical function (Montiel, 2005).

	Social practice		
	Anticipation	**Prediction**	**Formalize**
Reference Practice	Mathematizing Astronomy activity	Mathematizing Physic activity	Mathematizing the heat transference
Natural Context	Static – Proportional	Dynamic – Periodic	Stable – Analytic
Associate Mathematic tool	Trigonometric Ratio	Trigonometric Function	Trigonometric Serie
Related variable	$sen\,\theta$ θ is an arc $sen\,\theta$ is a (chord)	$sen\,x$ x is time (radian) $sen\,x$ is distance	$sen\ t$ t is time $sen\,t$ is temperature

Basic Principles for the social construction of the trigonometrical function

In the poster we are going to show the practical and scientific activities related to these reference practices, that give sense and meaning to the trigonometrical objects. At the moment we are designing didactic sequences, based on the model, in order to generate a "fiction genesis" on students, in the way of theory of didactical situations (Brousseau, 1997).

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FACTORS THAT MOTIVATE THE CAREER GOALS OF WOMEN MATHEMATICS STUDENTS

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Research studies show that while women and men have similar achievement levels in mathematics (Boaler, 2002; Fox & Soller, 2001), women do not choose mathematics-related careers at the same rate as men (Gordon & Keyfitz, 2004; Sharpe & Sonnert, 1999). Although substantial research has been conducted to investigate women's attitudes towards mathematics and gender bias in mathematics education, most of this work has been at the primary and secondary school levels and with the purpose to explore why women do not choose to study mathematics. A considerable number of women, however, do choose to study mathematics at the undergraduate level (NCES, 2003). Therefore, in addition to investigating why some women do not choose to pursue mathematics-based careers, it is also important to consider why certain women do choose mathematics. Such knowledge may assist in designing appropriate intervention programs to help support other young women in pursuing careers in mathematics. As of the present, few studies have been conducted in this area.

This research study is intended to help fill this gap in the literature by investigating what has influenced undergraduate women mathematics majors to pursue mathematics at the undergraduate level and whether or not they intend to pursue a mathematics-based career. Through a series of in-depth, phenomenologically based interviews with undergraduate mathematics majors at two different universities, we investigated the following research questions:

- What has influenced these women to study mathematics at the undergraduate level?
- At what stage of their life did they make this decision?
- What are these women's future career goals?
- What experiences (if any) have these women encountered during their undergraduate education that has influenced their choices for future careers?

In this poster we will discuss our preliminary results with regards to these research questions.

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MAKING MATHEMATICS RELEVANT IN AN URBAN CONTEXT

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Making Mathematics Relevant in Brooklyn

This research is situated in New York City, whose school system is the largest in the U.S.A. Within a climate of high stakes testing, mathematics teacher shortages, and an unacceptably low graduation rate, the call for "mathematics for all" is especially serious. In addition, New York City is an exemplary model of urban-ness in terms of its linguistic, cultural, ethnic, and racial diversity alongside other urban factors like, say, high population density or paucity of resources. How should we be preparing and training mathematics teachers to teach in urban contexts? In addition to the requisite subject matter and pedagogical content knowledge (Shulman, 1986), a third dimension, that of equity pedagogy (Banks and Banks, 1995), is crucial. We opt to utilize Ladson-Billing's (1995) specific construct of culturally relevant pedagogy (CRP) as our foundation, with its three attributes: it i) emphasizes students' academic success, ii) encourages the development of cultural competence, and iii) facilitates development of critical consciousness.

There is a powerful set of examples in the literature of teaching culturally relevant mathematics (e.g., Gutstein, 2006; Skovsmose, 1994; Vithal, 2003). These studies provide important existence proofs, typically taking place in individual classrooms in which the researcher acted as a teacher or as a co-teacher. In most cases, these projects employ a research model whereby the investigation and determination of students' out-of-school activities and interests are conducted by university researchers. However, the local and temporal bounds of cultural and social relevance prompt a critical question as to how teachers might learn to identify what is relevant or meaningful with respect to their own students. The literature does not provide us with answers to this question and it is this opening that forms the basis of our work.

This particular project is organized around the following research questions. The first is, how can urban mathematics teachers learn to teach mathematics with a culturally relevant approach? The second research question is, what are the complexities inherent in teacher learning about CRP when students come from a variety of cultural and/or linguistic backgrounds, all of which may differ from the teacher's background? Finally, the third research question is, how does a structure of a professional community of learners, organized around the goal of learning to teach mathematics with a CR approach in cities, contribute to teacher learning? The poster will describe the collaborative project, trace several teacher learning trajectories, and identify a set of emergent questions and tensions.

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TEACHER BELIEFS

PICTURES AS A MEANS FOR INVESTIGATING MATHEMATICAL BELIEFS

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This paper presents an innovative design for investigating mathematical beliefs. Students of grades 5, 9, and 11 were asked to express their views on mathematics on a sheet of paper. Further data was collected and qualitative methods were employed to identify the beliefs encoded in these works. The data was analyzed according to established categories describing mathematical beliefs. Typical features of each category were found in the pictures. Concrete examples that support these features are provided as evidences for the represented mathematical beliefs. Interestingly, older students, from grade 9 in this study, often include affective aspects in their pictures and texts on mathematics.

Introduction

The importance of beliefs in mathematics learning is nowadays widely acknowledged (Leder, Pehkonen & Törner, 2002). Traditionally, mathematical beliefs are investigated with the aid of questionnaires or interviews. This approach is well established. However, especially for younger students, who are not yet used to this technique and who might have difficulty in reading a long questionnaire attentively, alternatives could be helpful. Bulmer and Rolka (2005) introduced pictures as a means to understand university students' views on statistics. In our study, we used a combination of pictures as well as written and oral statements for investigating student beliefs.

Mathematical Beliefs

Dionne (1984) suggests that mathematical beliefs are composed of three basic components called the traditional perspective, the formalist perspective and the constructivist perspective. Similarly, Ernest (1989; 1991) describes three views on mathematics called instrumentalist, Platonist, and problem-solving which correspond more or less with the notions of Dionne.

In this work, we employ the notions of Ernest (1989; 1991) and use this section to briefly recall what is understood by them. In the instrumentalist view, mathematics is seen as a useful but unrelated collection of facts, rules, formulae, skills, and procedures. In the Platonist view, mathematics is characterized as a static but unified body of knowledge where interconnecting structures and truths play an important role. In the problem-solving view, mathematics is considered as a dynamic and continually expanding field in which creative and constructive processes are of central relevance.

Methodology

In this study, we extended the approach of Bulmer and Rolka (2005) using pictures as a means for investigating student beliefs. Additionally to the pictures, we asked the students to give an explanation of their work. The first task, scheduled for one week, was close to that in the study mentioned above:

> *Imagine you are an artist or a writer and you are asked to show on this sheet of paper what mathematics is for you.*

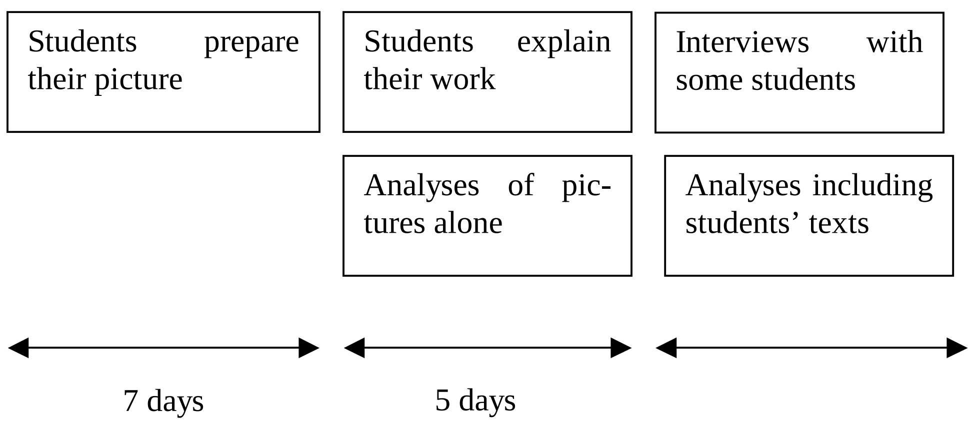
Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

After the submission of their work, a second task was given over a period of five days:

Explain your work by answering the following questions:

- *In which way is mathematics included in your work?*
- *Why did you choose this style for your presentation?*
- *Is there anything you would have liked to show but which you were not able to express?*

During these five days, the authors independently tried to classify the pictures according to the three mentioned views: instrumentalist, Platonist, and problem-solving. This classification was repeated later, when both pictures and texts were available to the authors. In certain cases, a third step was carried out by the first author, who interviewed the students individually. This was considered necessary when the answers based on picture and text remained unclear. The following illustration shows how the project proceeded:



The tasks were given to 84 students of grades 5, 9, and 11 from two schools in Germany. Among these, 61 students submitted pictures. The picturing task was integrated in the coursework of the classes. The students involved were used to work on projects over a longer period of time. In Halverscheid & Rolka (2006), the method was illustrated for students of grade 5. In this paper, we want to investigate differences and common features which appeared in the comparison of the grades 5, and 9.

Results and Discussion

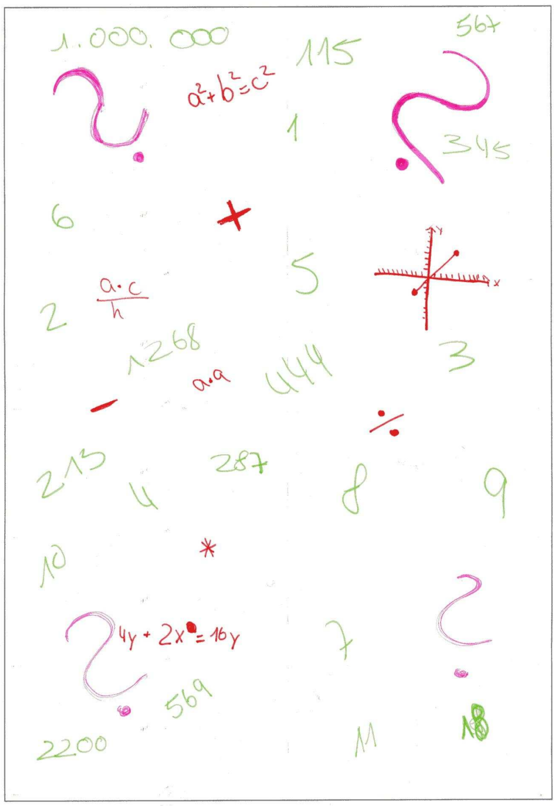
The students take different views on "what mathematics is for" them. Some consider and describe mathematics as a science without direct links to their own learning experiences. For others, the learning experiences in mathematics are given much more attention than the scientific viewpoint.

In the following, we first give three examples that serve as illustrations for the classification according to the three categories. We then focus on some affective aspects that can be found in the works.

Illustrations of the three categories

Pictures consisting of several non-connected sequences, such as symbols, objects, and situations seem to parallel an instrumentalist view on mathematics. The appearance of important people in the history of science is often an indication for a Platonist view, which can also mix with an instrumentalist view or a problem-solving view. Pictures telling a story or delivering objects for mathematical activities tend to correspond to a problem-solving view.

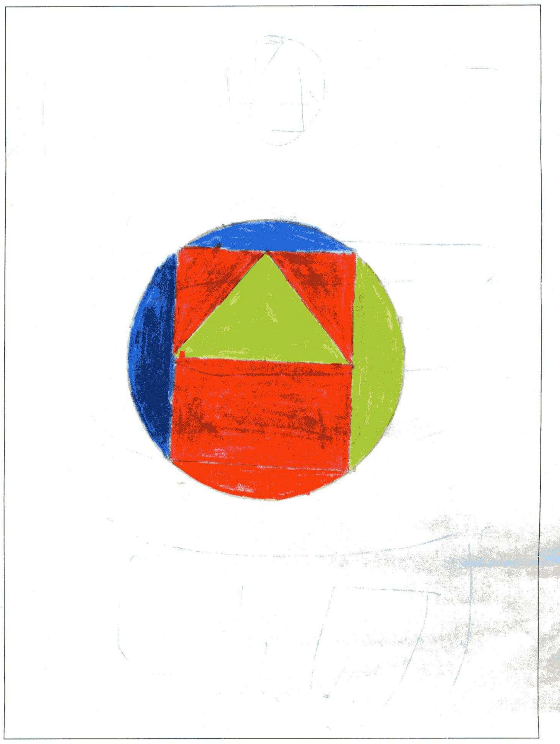
The following picture has been classified as belonging to the catogory of the instrumentalist view:



Student (grade 9, age 15, female): “For me, mathematics means many formulae and that it is illogical for me. I have decided to design my picture like this, because I think that mathematics cannot be explained and that there are just numbers and formulae…
I would have liked to show more exactly how illogical mathematics is for me. Is it possible to explain mathematics at all? Not for me.”

In the picture a lot of different symbols, numbers, and some question marks are depicted. The text stresses the point of view that mathematics is not logical and that it cannot be explained. This supports the impression of a disconnected gathering of objects, which are not linked by a common thought. The text describes mathematics as a static, “illogical” field. Note already here that this student of grade 9 broaches the issue of her relationship to mathematics.

The Platonist view is sometimes related to historic figures. Albert Einstein - particularly present in the media in 2005 - appears several times as a mathematical protagonist. Here is an example of a student who stresses the context of the geometrical objects he painted. It is hard to make sense of the student’s picture alone; it is the interplay with his text which makes it possible to understand it better.



Student (grade 5, age 11, male):
"Mathematics is shown in my picture by geometrical forms. The circle is a figure that has no edges and corners. With π one can calculate the perimeter, the area, the diameter and the radius of a circle.

The triangle is a figure whose angles add up to 180 degrees. The rectangle is a figure with four right angles. It has two longer and two shorter sides.

I have picked this presentation because I have liked geometry at primary school very much. […]

Originally, I wanted to draw the pictire ´Proportions of the Human Figure` by Leonardo da Vinci."

To illustrate a typical feature of the problem-solving view, we include here the following work of grade 5. This picture too illustrates how essentially the analysis may depend on the use of a student's text.

The boy, 11, writes: "I wanted to sketch different fractions, such as 7/9 in the case of the set square or 9/10 for the pair of scissors. You can imagine any object which is not sketched entirely. In this way it is easier to memorize fractions. At least this is the way I do it."

The picture shows tools used in mathematics: an exercise book, pencils, a ruler, a set square, scissors. Parts of certain objects are missing like in the cases of the exercise book, the set square and the pair of scissors. For the other objects, it is not that clear whether they are sketched entirely. The facts that there is a number of objects and that a certain proportion of them is sketched come to mind as relations to mathematics. But the exact nature of the mathematical contents remains vague as far as the picture is considered alone.

The text stresses the parts of the objects which are sketched. Finally, it does not matter to the student whether tools or other things are used. It is the creative and thus dynamic mathematical process itself which is of interest to the student: he takes any object and associates a rational number with it. For this reason, his work and explanations contain the main features of the problem-solving category established by Ernest (1989; 1991).

In grade 5, the pictures jointly with the texts could be classified along the above-mentioned categories independently by the authors. In 26 pictures, we observed 14 pictures and texts with an instrumentalist view, 5 pictures and texts belonging to the Platonist view and 5 pictures and texts showing a problem solving view; 2 could not be classified. In some cases it was necessary to interview the students before classifying their works in the category of the Platonist view. As explained in more detail in Halverscheid & Rolka (2006), the Platonist view is related to both other views; the works often contain elements of the instrumentalist and, resp. or, problem solving view. Without the texts, a classification appears often impossible.

Affective statements in the students' texts

The tasks leave a lot of freedom to the students to express their views on mathematics. The first example given above for the instrumentalist view represents also another aspect we observed in the data. In particular, the affective side of mathematics – often present in the works - has many facets: Admiration for the beauty of mathematics, pride of understanding certain mathematical ideas, exam nerves, frustration about difficulties in maths, just to name a few.

The young students in grade 5 answer the question as to what mathematics is for them differently than the older students of grades 9 and 11. It is the science and the cultural aspects of mathematics for everyday life which are in the focus of the fifth-graders. Some of them refer to their learning experiences in mathematics, but with one exception all use these to describe their views on mathematics as a science or as a useful tool.

It is only one student girl of grade 5 who considers the learning of mathematics as the major topic: She sketches a girl lying in bed at night with a text book on mathematics on her pillow, and she explains that the girl still has to prepare for a maths test.

In grade 9, the focus of the pictures shifts towards a reflection on the learning of mathematics. In this study, about half of the students in grade 9 picked the learning of mathematics out as a central theme. Some very explicitly mention their frustration about mathematics at school and certain learning experiences.

Difficulties with mathematical exercises are mentioned several times and are considered as something which can hardly be learnt. Frustrating experiences can be seen both in pictures and texts.

One student, for instance, shows a machine which delivers in the students' case always the wrong answer. In another picture that shows the frustration about problem solving in mathematics education the student describes problem solving tasks as a sort of vicious circle:

After every test, the student says, she smiles and thinks she has done a good job. When the results are published, she is always disappointed. Others use the texts to express their happiness in case of success or their disappointment in case of failure.

The topics raised by the students can also be found when looking at affective statements which the students make to explain their pictures. These affective statements appear in the following ways:

- Attitude towards mathematics
- Emotions while learning mathematics
- Self-confidence in mathematical abilities
- Aesthetic comments on the picture.

Certain components, for example the aesthetic comments, are expressed somewhat indirectly in the choice of words, or concerning the self-confidence in their mathematical abilities. Roughly speaking, the young students express more positive feelings on mathematics and a certain easiness concerning the challenges of mathematics.

We want to pick out two aspects which might illustrate this when the fifth-graders and the nineth-graders are compared.

The fifth-graders mention rather rarely emotions while learning mathematics, whereas the nineth-graders have chosen it in several cases as the main feature. We mentioned above already that only one girl among the fifth-graders has chosen this as the main topic. Self-confidence in mathematics does not seem to be a feeling which is expressed by the nineth-graders investigated in this study. It is not clear whether this is an indication of a lack of self confidence in mathematical abilities or or because they consider it inappropriate to mention it.

The idea of mathematics as something useful seems to change over the years, too. The young students regard mathematics as a useful tool for everyday life and for future jobs. In grade 9 or11, this idea can still be found. In all ages it is one of the key indicators of an instrumentalist view on mathematics. However, the older students stress more the use of mathematics at school. It seems that mathematics is less present in their everyday life.

Conclusion

Combining pictures and texts improves the empirical basis for decoding views on mathematics considerably. Furthermore, it makes better use of different students' abilities which are often neglected in everyday mathematical activities at school. Since students are normally not used to pictures and texts in mathematics, the approach presented here is appealing to those who are talented in painting and writing. Already fifth-graders are able to express their views on mathematics without reading and filling in questionnaires which implies certain problems regarding their age.

Students' views on mathematics have many facets and allow both a classification of common features and a discussion of every individual attitude towards mathematics. The simultaneous use of students' pictures and their own interpretation of their works and, if necessary, of short interviews to clarify certain aspects has proved to give insights into the views on mathematics of every individual student.

An analysis in several steps, starting with the pictures alone and passing to the pictures and the students' texts afterwards, gives a basis of data which allows to get often a distinctive idea of the student's views on mathematics. For the students and the researcher, the picture is the starting point for a discussion on mathematics and the learning of mathematics which, in the cases considered here, was continued in the classrooms. In this sense, the approach seems more rewarding for students than answering a questionnaire which is more difficult to integrate in classroom activities.

One result concerning the differences between grade 5 and 9 is that younger students seem to express more positive attitudes towards mathematics than older students. In grade 9, the view on mathematics is dominated by the view on mathematics as a school subject or by their learning experiences. This is often accompanied by negative emotions like frustration or disappointment - a tendency which seems to continue in grade 11. This result that attitudes towards mathematics develop in negative direction when students progress through the grades has been observed already in other studies (Kislenko, to appear; Nurmi et al., 2004).

The findings suggest that it is worthwhile to use this approach for investigating mathematical beliefs. If this approach is examined in more detail and with a variation of tasks in the future, it might have a potential to serve as a means of getting to know students better "beyond the purely cognitive" (Schoenfeld, 1983).

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NOVICE MATHEMATICS TEACHERS, STRESS, AND TEACHING IN THE SPIRIT OF THE NCTM STANDARDS

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This study explores challenges novice secondary mathematics teachers experienced while attempting NCTM (1991) Standards-based teaching. The ongoing analysis has uncovered links between stresses novices faced and attempts at NCTM Standards-based teaching, including classroom discussions, mathematical explorations, and alternative assessments. The emerging results imply that goals that reform-oriented teacher preparation programs encourage novices to adopt may play into stresses they experience.

In this paper, I introduce my dissertation research on how six novices' attempts to teach in the spirit of the NCTM (1991; 2000) Standards related to the teaching stresses that they experienced. The data analysis is ongoing; however, since methodology and results could prove useful to teacher educators, I discuss both here. The methodology that I employ has resulted in 21 categories of novice teacher stress to date, some of which are unique to novices attempting teaching consonant with the NCTM Standards. I also describe a closer analysis of 4 carefully selected novice teacher stress categories that strongly and directly relate to specific standards in the NCTM (1991) *Professional standards for teaching mathematics*.

Objectives

Teaching is among the most stressful of human service occupations (Travers & Cooper, 1996). Novices in the U.S. must carry the same workload as experienced colleagues. Borko & Putnam (1996) suggest novices may be too stressed to learn efficiently from their teaching experiences. Efforts to enact some of the teaching practices described in the NCTM (1991) *Professional standards for teaching mathematics*, or to attempt, to some degree, NCTM Standards-based teaching [SBT] may add to those challenges. Because novices are often most familiar with "teaching as telling," efforts to set "telling" aside to attempt SBT, may increase novices feelings of uncertainty (Smith, 1996). The literature is peppered with examples of novices struggling to enact SBT (e.g. VanZoest & Bohl, 2000). Lack of SBT-attempting colleagues in novices' first teaching contexts may also prevent them from attempting SBT (Wilcox, et al., 1991). Existing research provides little insight into how general stresses of beginning teaching interact with attempts at SBT, which appears crucial knowledge for teacher educators advocating such practices.

Through the lenses of teacher stress, novice teacher learning and development, and the NCTM (1991) Standards, I explore which stresses novice teachers attempting SBT faced, which of those stresses are related to their attempts at SBT, and whether the novices had resources to meet those stresses. As stated above, this paper focuses on the first two issues.

Theoretical Framework

While researchers have typically studied teacher stress via surveys, I situate this study in novices' classrooms, as some survey researchers have advocated (e.g. Manthei, et al., 1996).

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I define novices as teachers with less than 3 full years of classroom teaching experience. I characterize SBT using the six categories of high quality teaching identified in the NCTM (1991) *Professional standards for teaching mathematics*, namely Standard 1: Worthwhile mathematical tasks, Standard 2: Teacher's role in discourse, Standard 3: Students' role in discourse, Standard 4: Tools for enhancing discourse, Standard 5: Learning environment, and Standard 6: Analysis of teaching and learning.

Based on Kyriacou and Sutcliffe's (1978) definition, I define teacher stress as a teacher's perception that an aspect of the job is demanding or induces negative affect, often mediated by coping mechanisms acting to reduce the perceived threat. This definition aligns with my research goals and coincides with my ability to measure the perceived stresses, as well as acknowledging that I also plan to identify where teachers report using coping mechanisms or resources to mediate those stresses.

I define coping mechanisms or resources as means of either alleviating the immediate effects of stress (such as meditation) or resolving the situation or event that caused the stress (such as a colleague who shares advice about dealing with a particular student's behavior).

Research Question*s*

Framed by the aforementioned definitions, this study addresses the following questions:

- What stresses do novice secondary mathematics teachers experience as they attempt to enact elements of the NCTM (1991) *Professional standards for teaching mathematics*?
- During these attempts, which stresses are predicted by the existing literature and which are new? Which are related to their attempts at Standards-based teaching and how?

I answer the first question by describing the list of categories that has emerged from my data analysis to date. To answer the second question, I discuss the results of my data analysis for the four coding categories most strongly and directly related to the NCTM (1991) Standards.

Methods

I asked the fifth-year content instructor at a highly ranked, progressive Midwestern University [hereafter, 'MU'] for a list of recent graduates who were likely attempting SBT, based on observations of their MU classroom participation, student teaching, and mentor teacher's orientation towards SBT. I sat in on 11 novice mathematics teachers' classrooms who lived close enough to MU to make participation practical and who agreed to participate. Of those 11, I observed 6 secondary teachers attempting SBT, all of whom participated. 5 of the 6 were female; 1 was male. Their teaching experience ranged from 1.25 to 2.75 years.

I conducted a preliminary audiotaped interview, a minimum of three videotaped observations in a class where the teacher felt comfortable and the students were described as constructively talkative about mathematics [hereafter, 'focus' class], a single videotaped observation of a class where the teacher's challenges were least like those in the focus class [hereafter, 'contrast' class], and a final audiotaped interview. I asked the teacher to keep a teaching log during my observations in the focus class, noting surprising, unexpected, and challenging events. After the observations were complete, I reviewed the videotapes to create a list of events where the teacher showed observable signs of stress; then using the teacher's and my perceptions of those events, I prepared a videotape to review at the final interview of three salient, representative, and potentially stressful events by melding the teacher's and my own observations, giving priority to those that I judged to be related to attempts at SBT.

During the two interviews, I asked teachers about their teaching challenges. During the first interview, I asked general questions about those challenges. During the final interview, I asked the teacher to discuss their thoughts on the three specific classroom events that we viewed together from the videotapes, in particular what was going through their minds and whether they perceived those events as challenging. Then we discussed specific categories from the literature that novice teachers generally find challenging. Finally, we talked about how representative the challenges we had discussed were of those encountered regularly.

Using this method, I learned about the bigger picture of the teachers' stresses in the first interview. In the second interview, I attempted to get the teachers to create a picture of the specific challenges they perceived in their current teaching context. During this process, I subtly probed, when necessary, whether those challenges were related to their attempts at SBT. Finally, I broadened the discussion to discover how common and salient those specific challenges were in the context of all their teaching challenges.

Data and Results

As I began the analysis, I operationalized my definition of teacher stress as a teacher statement that something is challenging or is accompanied by negative affect. In my analysis, words that indicated that a situation or event was challenging for the teacher included: hard, difficult, stress, challenge, problem, etc. Some of the words that indicated that a situation or event was accompanied by negative affect included: hate, frustrate, disappoint, bother, peeve, irritate, annoy, etc. Also, I considered a negative description of an event to indicate negative teacher affect, including phrased such as "a bad [situation]" or "my class was out of control," rather than neutral descriptions such as "they are not very organized students" or "they're not really self motivated," which did not necessarily convey negative teacher affect.

I analyzed the interview data beginning with the coding categories found in the literature (Kyriacou, 1989; Lewis, 2004). I used an open coding (Glaser & Strauss, 1967) strategy, moving in recursive fashion between the data and the coding categories to refine them to fit the data. To date, 21 categories of teacher stress have emerged from the data, as follows:

- *Directing student-led explorations of mathematics:* Stresses deriving from engaging students in activities where they explored, alone or in pairs or small groups, a problem or situation.
- *Implementing NCTM Standards-based assessment strategies:* Stresses deriving from implementing assessment strategies consonant with the NCTM Standards.
- *Managing classroom discussions:* Stresses deriving from facilitating classroom discussions by actively involving students in contributing expected ideas.
- *Responding to unexpected student ideas:* Stresses deriving from teachers' attempts to engage with and pursue unexpected student ideas during classroom discussions.
- *Working with technology:* Stresses deriving from teachers' attempts to implement technology, including the time that it took to teach technology use or the teachers' lack of knowledge about technology.
- *Creating, aligning, modifying, or implementing curricula:* Stresses deriving from organizing, creating, or implementing the school's chosen curriculum, or from aligning it with the NCTM (2000) Standards or the state's curriculum standards.

- *Relationships and communications with colleagues:* Stresses deriving from teachers' interactions with their peers, often related to differences in beliefs about classroom management or pedagogical issues.
- *Lack of student interest/motivation:* Stresses deriving from attempts to engage students and/or students' resistance to engaging in classroom activities.
- *Dealing with class length or schedule:* Stresses deriving from the length of class periods or the demands of a teacher's particular schedule, including number of preps, types of classes, scheduling of classes (block/semester/trimester), etc.
- *Teaching students of varying ability levels:* Stresses deriving from attempts to teach students with a variety of perceived ability levels within a single class.
- *Learning and teaching unfamiliar content:* Stresses deriving from teachers' attempts to teach content that they had never seen (e.g. graph theory), had not seen recently (e.g. geometry), or had had limited opportunities to learn (e.g. calculus).
- *Preparing for, administering, or evaluating standardized tests:* Stresses deriving from preparing students for locally or federally mandated tests, including modifying curriculum, administering tests, evaluating tests, or utilizing feedback.
- *Relationships and communications with parents:* Stresses deriving from teacher-parent interactions, often relating to student misbehavior, assignments, or grades.
- *Responding to problematic student behaviors:* Stresses deriving from student behaviors that teachers perceived as inappropriate for the classroom.
- *Finding resources:* Stresses deriving from teachers' attempts to find physical, curricular, or human resources to meet their pedagogical goals.
- *After hours work / Long hours:* Stresses deriving from teachers' attempts to manage extracurricular work hours so that they could complete preparations and grading for their classes, while also fostering a personal life outside of school.
- *Planning lessons:* Stresses deriving from teachers' attempts to prepare lessons. Sometimes this involved creating or adapting existing lessons to their curricular objectives [with a pedagogy not clearly described as NCTM Standards-based].
- *Challenges associated with professional development:* Stresses deriving from teachers' feelings about their professional development experiences, including its adequacy to meet their needs and how efficiently it used their time.
- *Student diversity issues:* Stresses deriving from teachers' attempts to relate to and involve all students, including students' whose racial and/or ethnic background was different from the teacher's and/or some of their peers.
- *Controlling emotions:* Stresses deriving from trying to control one's emotions in difficult circumstances in order to teach effectively.
- *Teacher boredom:* Stresses deriving from trying to deal with one's own boredom when the content, lesson format, or students were not particularly engaging.

Stresses contained in almost any of these 21 categories could connect in some way to SBT, but as I will soon explain, the connection was clear and direct for the 4 categories listed in Table 1. I also looked for evidence in the data to link each of those 4 categories to the most relevant of the six NCTM (1991) Standards related to each in the subsequent discussion.

Category	**Definition**	**Example**
Directing student-led explorations of mathematics	Stresses deriving from engaging students in activities where they explored, alone or in pairs or small groups, a problem or situation.	[Ms. Grant felt frustrated by students' lack of reflection on the results of an in-class, small-group exploration of how changing the coefficients in the equation alters the graph of $y=ax^2+bx+c$.] Ms. Grant: So making that table and having the different a, b, and c values…, I thought that they'd start to recognize patterns faster, but …I didn't anticipate that when they entered it into their calculators… that they would enter x^2+x, x^2+2x, x^2+3x, all like Y1, Y2, Y3, Y4, Y5**.** So then when they get the picture, they're all just there. So it's hard for them to say which one's which. And so they couldn't see a progression… I thought they would graph one, then they'd change the coefficient and graph it again. Then they'd start to see, "Well every time I'm just moving down, or every time, I'm just moving up." [Post, 2.15-3.10]
Implementing NCTM Standards-based assessments	Stresses deriving from teachers' attempts to implement assessment strategies consonant with the NCTM Standards.	Mr. Jones: We have a performance assessment, which is where students get into groups of three or four and they work on a set of eight problems… Each group is going to have to present [a] problem for their final exam. And they have to teach it to the class… So this is really stressful for me, because I have to get all the problems copied. I have to make sure that I'm getting the class engaged enough to work on it. And when the group presents, I have to ask really thoughtful questions to make sure that I know what the students know [Pre, 7.30-7.42].
Managing classroom discussions	Stresses deriving from facilitating classroom discussions by actively involving students in contributing expected ideas.	Mr. Jones: Mmm, classroom discussions… would be probably a big one. We talked about that as far as who should be talking and how much work I should be doing as far as the classroom discussion. And I really feel like I should be the facilitator in a discussion, but sometimes it just doesn't happen. And I need to work on some different ways of having discussions. And I think that's probably a big challenge [Post, 21.35-21.39].
Responding to unexpected student ideas	Stresses deriving from teachers' attempts to engage with unexpected student ideas during classroom discussions.	Ms. Boone: When she said that, like I looked, because I was like, 'Okay, I'd better figure out right now if this is true or not.' So I sat there and I looked. And I was thinking through a couple of examples really fast in my head. And I was like, 'Okay, that would be true'… I: Was there anything challenging about that at all? Ms. Boone: Well, trying to figure it out within 30 seconds to see if it's right or not! [Post, 5.28-5.42]

Table 1. Stress coding categories and examples

Directing student-led explorations of mathematics directly relates to the NCTM (1991) Standards. When teachers created opportunities for students to explore problem situations, they encountered challenges as they worked to write questions that students could understand (see Standard 1: Worthwhile mathematical tasks) and trained students to actively explore problem situations (see Standard 2: Teacher's role in discourse). The example in Table 1 illustrates how Ms. Grant struggled to get students to think during and thoughtfully reflect on the results of a task exploring how changing the coefficients in the equation $y=ax^2+bx+c$ affects the graph, in anticipation of a whole-class discussion.

Implementing NCTM Standards-based assessments implicitly relates to the Standards. When teachers employed assessments that involved written or verbal communication of mathematical ideas (see Standard 1: Worthwhile mathematical tasks), they experienced difficulties with asking questions requiring mathematical justification (see Standard 5: Learning environment), dealing with fairness issues in group work situations, etc. The example in Table 1 illustrates how Mr. Jones felt stressed about needing to ask questions that would gauge the depth of his students' understanding during each group's problem presentation during their final exam, in addition to other concerns.

Managing classroom discussions related primarily to attempts at SBT, because most of the aspects of discussion that stressed teachers dealt with active student participation. When teachers attempted to facilitate classroom discussions (see Standard 2: Teacher's role in discourse) that involved building mathematical understanding as a learning community (see Standard 5: Learning environment) through active, meaningful student contributions to the discussion (see Standard 3: Students' role in discourse), the teachers struggled with how to facilitate such participation. The example in Table 1 shows that Mr. Jones felt like he spent more time talking than he wanted, rather than facilitating the discussion. So Mr. Jones' stress appeared to be as much about his teaching goals as it was about actual classroom events.

Responding to unexpected student ideas relates to the NCTM (1991) Standards, because when teachers encountered unexpected student ideas, the flexibility of their mathematical knowledge was tested if they engaged with those ideas. Only teachers that wanted to use students' ideas meaningfully in instruction (see Standard 2: Teacher's role in discourse) felt stressed when students expressed them unexpectedly (see Standard 3: Students' role in discourse), often because they did not know how to use them to continue building the mathematical discussion. The example in Table 1 illustrates how Ms. Boone scrambled to decide if a student's conjecture was true. The topic had not arisen in any other section of the class, even though this was her third year teaching it.

In the sense that all of these categories of novice teacher stress are related to the NCTM (1991) Standards, all are new to the literature; moreover, only *Managing classroom discussions* overlaps meaningfully with teacher stress categories existing in the literature as a specific form of classroom management.

Names / **Codes**	Ms. Boone	Ms. Grant	Mr. Jones	Ms. Riley	Ms. Price	Ms. Wells	**Total**
Directing student-led explorations of mathematics	0	3	1	0	0	1	**5**
Implementing NCTM Standards-based assessments	0	0	2	0	0	0	**2**
Managing classroom discussions	1	3	4	4	5	0	**17**
Responding to unexpected student ideas	1	0	0	0	1	0	**2**
TOTALS	**2**	**6**	**7**	**4**	**6**	**1**	**26**

Table 2. Stress frequencies by category and participant

Among the four stress categories strongly related to SBT, nearly all teachers reported the greatest number of stresses related to *Managing classroom discussions* (see Table 2). While some of those stresses had more to do with generic concerns like wait time and transitioning from extracurricular to mathematical topics, most novices' struggles related to creating a classroom where students shared their own mathematical ideas and conjectures. They also reported several stresses coded as *Directing student-led explorations of mathematics* (see Table 2), generally related to students not completely understanding or not actively exploring a given task. *Responding to unexpected student ideas* and *Implementing the NCTM Standards-based assessments* (see Table 2) also occurred, although much less frequently than the other categories.

Conclusions

The list of 21 stress categories shows the variety of stresses experienced by these novice teachers who were attempting SBT. I have also demonstrated that novices face many challenges while managing classroom discussions in which they expect students to participate and contribute meaningfully, when they encounter unexpected student ideas or conjectures during those discussions, while directing student-led explorations of mathematics, and while attempting to implement Standards-based assessments. These categories are all new to the teacher stress literature. These categories also relate to much of the terrain that at least four of the six NCTM (1991) Standards encompass, namely Standard 1: Worthwhile mathematical tasks, Standard 2: Teachers' role in discourse, Standard 3: Students' role in discourse, and Standard 5: Learning environment.

As teacher educators grow to understand the challenges that developing teachers face and how those challenges relate to their attempts to engage in a powerful pedagogy like the one outlined by the NCTM (1991; 2000) Standards, we can to prepare them for such challenges and help schools to support novices in those challenges. Arguably, Ms. Grant could have benefited from support in dealing with her frustration when students were supposed to find patterns in the graphs as they changed the coefficients in the equation $y=ax^2+bx+c$, but they either did not understand the task or failed to reflect deeply on the results of their explorations. While many challenges novices face are not related to their attempts at SBT, many clearly are, as in Ms. Grant's case. An understanding of the challenges novices face as they attempt to enact SBT can help teacher educators, school administrators, and their teaching colleagues offer effective support to maximize the learning accompanying stresses that novices' teaching challenges and goals related to the NCTM Standards impose.

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ELEMENTARY PRESERVICE TEACHERS' CHANGING PEDAGOGICAL AND EFFICACY BELIEFS DURING A DEVELOPMENTAL TEACHER PREPARATION PROGRAM

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This study examined changes in the mathematics pedagogical and teaching efficacy beliefs of elementary preservice teachers during a developmental teacher preparation program that included a two course mathematics methods sequence. Preservice teachers' beliefs became more cognitively-aligned during the first methods course but had mean decreases during the second methods course and student teaching. Preservice teachers also had significant shifts in their personal efficacy for teaching mathematics, with these changes largely occurring in the second methods course. The results of this study suggest that as the preservice teachers continued to study, experiment with, and reflect on ways to implement standards-based pedagogy, they became more confident in their abilities to teach mathematics effectively.

Theoretical Perspectives

The relationship between beliefs and teaching is well-established. Beliefs influence teacher behavior and decision-making (Thompson, 1992; Wilson & Cooney, 2002). It is also known that beliefs develop over time (Richardson, 1996), that they are well-established by the time a student enters college (Pajares, 1992), and that they develop during what Lortie (1975) terms the *apprenticeship of observation* which occurs over years as a student. Teacher preparation programs have a limited amount of time to impact change in preservice teacher beliefs—usually two years or less. It is imperative that preservice teacher development and program effectiveness be assessed, at least in part, by development of beliefs that are consistent with a program's philosophy of learning and teaching.

The National Council of Teachers of Mathematics (NCTM) has proposed a vision of teaching mathematics based on a constructivist theory of learning which provides the basis for many teacher education programs in mathematics (NCTM, 2000). Introducing preservice teachers to the research-based professional development materials from the Cognitively Guided Instruction (CGI) Project (Carpenter, Fennema, Franke, Levi, & Empson, 1999) is one way for teacher preparation programs to develop mathematics pedagogical beliefs that are consistent with NCTM reforms. CGI is an approach to teaching and learning mathematics which focuses on teachers using knowledge of children's mathematical thinking to make instructional decisions (Carpenter & Fennema, 1991). CGI includes four tenets: (a) children can construct their own mathematics knowledge, (b) mathematics instruction should be organized to facilitate children's construction of knowledge, (c) children's development of mathematical ideas should provide the basis for sequencing topics for instruction, and (d) mathematical skills should be taught in relation to understanding and problem solving (Peterson, Fennema, Carpenter, & Loef, 1989). In CGI, the essence of teaching mathematics becomes posing word problems, facilitating students' problem solving, and orchestrating discussions of students' thinking and solution strategies.

Documentation of CGI's effectiveness in changing inservice teachers' beliefs about the teaching and learning of mathematics is abundant (Fennema, et al., 1996; Fennema, Franke,

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Carpenters, & Carey, 1993; Peterson, Fennema, Carpenter, & Loef, 1989). Although fewer studies have been conducted with preservice teachers, these have also indicated significant changes in preservice teachers' beliefs about mathematics instruction (Vacc & Bright, 1999). A challenge for reform in mathematics pedagogy is the tension between teachers' development of constructivist-based beliefs and their efficaciousness toward such (Smith, 1996). Most teachers' past experiences with mathematics are based upon traditional, behaviorist methods of mathematics instruction that rely on transmission by the teacher and absorption by the student (Battista, 1994). Such experiences contribute to the tension between teachers' sense of efficaciousness toward teaching mathematics and embracing constructivist pedagogical beliefs (Smith, 1996). Teachers' possessing a strong sense of efficaciousness is of critical importance as teacher efficacy has been linked with classroom instructional strategies, willingness to embrace educational reform, commitment to teaching, and student achievement.

Using Bandura's theoretical framework of self-efficacy, teacher efficacy is considered by many researchers to be a two-dimensional construct (Enochs, Smith, & Huinker, 2000). The first factor, *personal teaching efficacy*, represents a teacher's belief in his or her skills and abilities to be an effective teacher. The second factor, *teaching outcome expectancy*, is a teacher's belief that effective teaching can bring about student learning regardless of external factors such as home environment, family background, and parental influences.

Bandura's theory suggests that efficacy beliefs may be most malleable early in learning, thus the first few years of teacher development could be critical to the long-term development of teaching efficacy (Hoy, 2004). Once teaching efficacy beliefs are established, they are highly resistant to change. Studies suggest that coursework and the student teaching experience have differential impacts upon the personal teaching efficacy beliefs and teaching outcome expectancy beliefs of preservice teachers. Personal teaching efficacy increases during coursework and continues to increase during the student teaching experience (Hoy & Woolfolk, 1990; Plourde, 2002). However, teaching outcome expectancy beliefs increase during college coursework but decline during student teaching. This decline has been attributed to the unrealistic optimism preservice teachers have prior to student teaching about teachers' abilities to overcome negative influences (Hoy & Woolfolk, 1990). Although there have been numerous studies on generalized teaching efficacy, there has been less research specifically into the mathematics teaching efficacy of elementary preservice teachers. Most of the previous studies that have examined the effects of mathematics methods courses have indicated significant increases in mathematics teaching efficacy upon completion of the course (Huinker & Madison, 1997; Utley, Moseley, & Bryant, 2005).

Research Objectives

1. To examine changes in the mathematics pedagogical and teaching efficacy beliefs of elementary preservice teachers during a developmental teacher preparation program
2. To investigate the relationship between the mathematics pedagogical and teaching efficacy beliefs of elementary preservice teachers during a developmental teacher preparation program

Methodology

This study involved 24 elementary preservice teachers (23 females and 1 male) at a large urban university in the southeastern United States. The participants were enrolled in a two-year undergraduate teacher education program during their junior and senior year. The group was admitted as a cohort and completed all education courses together. The program consists of four semesters of coursework with three semesters of two-day-a-week field placements followed by a semester of student teaching. The field placements are considered a developmental model since preservice teachers start their placements in prekindergarten and finish in fifth grade prior to student teaching. The development model is outlined in Figure 1, which shows the sequence and length of placements as well as when the two mathematics methods courses were completed. Other mathematics requirements in the program included three mathematics content courses for teachers taught through the mathematics department (number and operations, geometry, and statistics).

	Semester 1	Semester 2*	Semester 3*	Semester 4*
Mathematics methods courses	None	Focus on PreK-2 mathematics	Focus on 3-5 mathematics	None
Field Placement	PreK – 5 weeks K – 9 weeks	lst – 7 weeks 2nd or 3rd – 7 weeks	4th – 7 weeks 5th – 7 weeks	Student teaching
Administration of MTEBI & MBI (Four times)	None	Week one – Initial Week fourteen – Post one	Week fourteen – Post two	Week fourteen – Final
Interviews	None	None	Week fourteen	None

*Asterisks denote semesters included in this study.

Figure 1. Sequence of teacher preparation program and data collection.

During this study, the second author served as instructor of the mathematics methods courses throughout the two-semester sequence. Thus, the philosophical focus and the emphasis on teaching for understanding were consistent across the sequence. Important goals of the courses included (a) understanding children's thinking about important mathematical concepts, (b) creating interest in changing curriculum and pedagogy, (c) understanding available alternatives to traditional instructional practices, (d) and inviting experimentation and reflection on the benefits to children of standards-based practices (see Smith, Smith, & Williams, 2005).

During the first methods course, the students were assigned to read most of the general and P-2 sections of the NCTM Principles and Standards and *Children's Mathematics: Cognitively Guided Instruction* (Carpenter et al., 1999). Classroom discussions and learning activities focused on social-constructivist pedagogy, supported by viewing videotapes of classrooms and clinical interviews with children. Students were also introduced to Standards-based curriculum materials, such as *Investigations in Number, Data, and Space* (TERC, 1998). Course assignments included three clinical-style interviews of children's understandings of number and operations and two reports of students' field experiences using this type of pedagogy in teaching mathematical concepts, including analysis of these lessons for coherence with the NCTM *Principles and Standards*.

During the second methods course, the students were assigned to read *Thinking Mathematically: Integrating Arithmetic & Algebra in Elementary School* (Carpenter, Franke, & Levi, 2003) and selections of cases from two units of the *Developing Mathematical Ideas* materials (Schifter, Bastable, & Russell, 1999a, 1999b). Classroom discussions and learning activities continued to focus on social-constructivist pedagogy, supported by viewing videotapes of classrooms and clinical interviews with children. Course assignments included two clinical interviews of children's early algebraic thinking and two more analyses of the field experience lessons for coherence with the NCTM *Principles and Standards*.

Data Collection

Two instruments, the Mathematics Teaching Efficacy Beliefs Instrument (MTEBI) and the Mathematics Beliefs Instrument (MBI), were administered to the participants four times during the teacher preparation program (see Figure 1). The MTEBI consists of 21 items, 13 on the Personal Mathematics Teaching Efficacy (PMTE) subscale and 8 on the Mathematics Teaching Outcome Expectancy (MTOE) subscale (Enochs, Smith, & Huinker, 2000). The two subscales are consistent with the two-dimensional aspect of teacher efficacy. The PMTE subscale addresses the preservice teachers' beliefs in their individual capabilities to be effective mathematics teachers. The MTOE subscale addresses the preservice teachers' beliefs that effective teaching of mathematics can bring about student learning regardless of external factors. The instrument uses a Likert scale with five response categories (strongly agree, agree, uncertain, disagree, and strongly disagree). Thus, possible scores on the PMTE subscale range from 13 to 65; MTOE subscale scores range from 8 to 40. Reliability analysis produced an alpha coefficient of .88 for the PMTE subscale and an alpha coefficient of .75 for the MTOE subscale. Confirmatory factor analysis indicated that the two subscales are independent, adding to the construct validity of the MTEBI (Alkhateeb & Abed, 2003; Enochs, Smith, & Huinker, 2000).

The MBI is a 48-item Likert scale instrument designed to assess preservice teachers' beliefs about the teaching and learning of mathematics (Peterson, Fennema, Carpenter, & Loef, 1989). The four subscales include (a) role of the learner, (b) relationship between skills and understanding, (c) sequencing of topics, and (d) role of the teacher. Possible total scores range from 48 to 240. Peterson, et al. (1989) reported that internal consistency estimates for the total scores were .93 and for the subscales ranges from .57 to .86.

After the two methods courses (see Figure 1), six participants were selected and interviewed as representatives of the following two groups of participants: Those with the greatest positive change in personal teaching efficacy scores and those with no change or a decrease in personal teaching efficacy scores. The interview questions were developed to elicit beliefs of the teachers around the same domains as the survey instrument (efficacy for teaching mathematics and beliefs consistent with the reform pedagogy) and to illuminate the survey responses for the two groups of participants these interviewees represented.

Members of the research team initially analyzed the interview data individually looking for statements of beliefs about teaching and learning mathematics, beliefs about skills and abilities to teach mathematics effectively, and beliefs about the usefulness and appropriateness of the reform pedagogy that was introduced to them. After individual analyses were complete, the team engaged in recursive dialogue to verify their findings against each other and the data.

Results

Mean scores and standard deviations across the administrations are provided in Table 1. Table 2 indicates the statistical significance of the differences between these means using Wilks' Lambda and its associated F-statistic. As indicated in Table 2, the preservice teachers had significant increases in overall MBI scores. The preservice teachers' beliefs became more cognitively-aligned throughout the teacher preparation program with these changes largely occurring in the first methods course. The scores had mean decreases during the second methods course and significant decreases during student teaching. Data from the PMTE revealed the preservice teachers had significant increases in their beliefs in their skills and abilities to teach mathematics effectively throughout the teacher preparation program. These changes largely occurred during the second methods course with the mean scores remaining constant during student teaching.

Table 1. Means and Standard Deviations for Mathematics Teaching Efficacy and Pedagogical Beliefs Scores

	Means				Standard deviations			
Scale	Initial	Post one	Post two	Final	Initial	Post one	Post two	Final
PMTE	48.46	49.96	54.38	54.38	7.30	7.30	5.58	8.26
MTOE	26.75	28.33	29.75	29.46	4.33	3.70	3.25	4.42
MBI	154.22	185.09	183.50	175.05	17.82	21.52	19.37	22.70

Table 2. F-Values (p-values) for Mathematics Teaching Efficacy and Pedagogical Beliefs Scores*

Scale	Overall	Initial to post one	Post one to post two	Post two to final
PMTE	6.09 (.004)	.93 (.346)	12.58 (.002)	.00 (1.00)
MTOE	5.18 (.008)	4.10 (.055)	2.89 (.103)	.140 (.712)
MBI	25.40 (.000)	79.53 (.000)	.22 (.646)	7.85 (.01)

*For the overall comparisons, df = 3, 21; for all other comparisons, df = 1, 23

The results of a Pearson product moment correlation analysis across the administrations are provided in Table 3. At the beginning and end of the first course, there are no significant relationships between personal mathematics teaching efficacy, mathematics teaching outcome expectancy, and pedagogical beliefs. However, at the end of both courses and student teaching the PMTE and MBI data reveals that the preservice teachers who had stronger beliefs in their skills and abilities to teach mathematics effectively generally had more cognitively-oriented beliefs toward the teaching and learning of mathematics. In addition, the preservice teachers with more cognitively-oriented beliefs had stronger beliefs that effective mathematics instruction can bring about student learning regardless of external influences at the end of the second course but this relationship is not evident at end of student teaching.

Table 3. Pearson Product Moment Correlations Comparing Mathematics Teaching Efficacy and Pedagogical Beliefs Scores*

MTEBI	MBI (initial)	MBI (post one)	MBI (post two)	MBI (final)
PMTE subscale	.289	.377	.629**	.685**
MTOE subscale	-.032	.033	.473**	.389

*n = 24 **Correlation is significant at the 0.05 level (2-tailed).

The interview data revealed that all six students had a solid understanding of the alternative, standards-based pedagogy emphasized in the courses. Five out of six interviewees maintained or increased their overall personal teaching efficacy scores. The one whose efficacy decreased by

13 points began with an initial personal teaching efficacy score of 62 (out of a maximum of 65) and declined to a more typical level of efficacy.

All six students interviewed indicated an initially skeptical attitude toward the standards-based instructional model, largely because they had not personally experienced this model and had not seen it used in schools during their field experiences. By the end of the two methods courses, those interviewed expressed some variations in personal preference for and beliefs about the usefulness of constructivist pedagogy, reflecting a need for continued experimentation with and reflection on the results of teaching for conceptual understanding. Many of the interviewees' responses connected their confidence in teaching mathematics with their confidence in understanding mathematics.

Three profiles of change emerged among the interviewees. Profile 1 (Participants #33, #2, and #1): Weak to uncertain confidence in understanding of mathematics for teaching and uncertain overall personal teaching efficacy beliefs; substantial improvement to solid confidence in understanding of mathematics for teaching and large increase to solid overall personal teaching efficacy beliefs.

Profile 2 (Participants #17 and #6): Solid initial confidence in understanding of mathematics for teaching and solid overall initial personal teaching efficacy beliefs; maintained solid confidence in understanding and solid overall personal teaching efficacy beliefs.

Profile 3 (Participant #15): Unrealistically strong initial confidence in understanding of mathematics for teaching and overall personal teaching efficacy beliefs; confidence in understanding and overall personal teaching efficacy beliefs moderated significantly toward a more realistic and slightly below average level, indicating a better understanding of the challenges of teaching mathematics for understanding.

Conclusions

The 24 preservice teachers in this study demonstrated significant increases in their personal efficacy for teaching mathematics and significant shifts in their pedagogical beliefs. The experiences in the courses emphasized views of mathematics learning and teaching that were unfamiliar and challenging. The shift in preservice teachers' pedagogical beliefs toward constructivist methods largely occurred in the first course with small decreases in these scores during the second methods course and student teaching. For most of the preservice teachers, the first of the two semesters presented a paradigm shift in what it means to know and do mathematics and seems to have resulted in some "unfreezing" of beliefs. The second of the two semesters produced significant shifts in personal efficacy beliefs about teaching mathematics. The student teaching experience had no effect on the beliefs measured in this study. These results suggest that as preservice teachers continued to study, experiment with, and reflect on ways to implement standards-based pedagogy during the second mathematics methods course, they became more confident in their abilities to teach mathematics effectively using constructivist methods they had learned during the first methods course.

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DEVELOPING PRESERVICE TEACHERS' BELIEFS ABOUT MATHEMATICS USING A CHILDREN'S THINKING APPROACH IN CONTENT AREA COURSES

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This paper measures changes in preservice elementary teachers' beliefs in math content courses when utilizing a supplement designed to help them understand how children learn mathematics. Results show that prospective elementary teachers can change their beliefs about mathematics and its teaching by focusing on how children learn and think about mathematics.

Recent research in teacher education has focused on the importance of understanding and changing preservice teachers' beliefs about mathematics (Stuart & Thurlow, 2000). It has long been known that those entering a teacher education program come with a variety of beliefs and conceptions about content knowledge and teaching based on their past experiences (Lortie, 1975). Recent studies have demonstrated that these beliefs about mathematics may be more resistant to change than beliefs about teaching in general (Raymond, 1997). Therefore, it is essential that beliefs about mathematics be addressed as soon as possible in teacher education. Since most methods courses focus primarily on teaching, it may be important to address these issues in content courses (Thompson, 1992).

This paper will focus on changes in preservice elementary teachers' beliefs in math content courses. These prospective teachers were enrolled in mathematics content courses using the Connecting Mathematics for Elementary Teachers (CMET) supplement designed to help them understand how children learn mathematics. Helping preservice teachers understand how children learn and think about mathematics may be one means of influencing their beliefs about mathematics, teaching mathematics, and how children learn mathematics (Vacc & Bright, 1999).

The CMET curriculum development project attempts to help preservice elementary teachers connect the mathematics they are learning in content courses with how children learn and think about mathematics and in so doing ties research on children's learning of mathematics to practice. For this reason, a supplement was developed that parallels the typical mathematics content course topics. The CMET materials primarily consist of descriptions, written for prospective elementary teachers, about how children think, misunderstand, and come to understand mathematics.

Methods

Evaluation studies are underway to test the effects of the use of the CMET supplement on prospective teacher preparation. In particular, we are examining changes in prospective teachers' beliefs about mathematics and its teaching, their efficacy for mathematics subject matter and its teaching, and their knowledge of the mathematics necessary for teaching. This paper reports results pertaining to changes in prospective teachers' beliefs about mathematics and its teaching.

During the fall 2005 semester, the CMET supplement was used by two of its authors in typical Mathematics for Elementary Teachers courses at a small Midwestern university. Students enrolled in the two sections of course one and one section of course two agreed to participate in this study. The students enrolled in course two had been recipients of course one at the same university.

To examine effects of the courses on participant beliefs, a fifteen item Likert-style questionnaire that asked participants the extent to which they agreed or disagreed with belief statements was administered. This questionnaire was administered pre-course one, post-course one and post-course two, the purpose of which was to test the effects of the courses on participants' beliefs as well as to see if the effects of the change (should any occur) were cumulative, i.e., did beliefs continue to change after participating in course? Some items were positively worded, such as: *Children's own methods of solving mathematical problems are useful in their learning*; some were negatively worded, such as: *Children should master the basic facts in mathematics before doing problem solving*. Negatively worded questions were reverse scored.

Principal component factor analyses were run before and after the courses to examine the structure of participants' beliefs. Cronbach's alpha were calculated on scales and resulting subscales to test for reliability. Two variables corresponding to the emergent factors math as *procedures* and math as *creative activity* were created and used to test for pre- and post-course differences. A Multiple Analysis of Variance (MANOVA), using SPSS was used to obtain these results.

Results

A two factor solution to the principal component analysis indicated that before taking the Math for Elementary Teachers course, participants had somewhat conflicting beliefs about mathematics and its teaching. Participants on average tended to agree or strongly agree with statements such as: *Children learn mathematics best through extensive drill and practice* (*procedures*); while agreeing with statements such as: *Problem solving is an important aspect of mathematics* (*creative activity*). Table 1 summarizes these results.

	Factor Loadings	
Item	procedures	creative activity
Mathematics is primarily a step-by-step mechanical process.	.772	
Mathematical skills should be taught before concepts.	.718	
Children best learn mathematics through extensive drill and practice.	.620	
Mathematics is mainly about learning rules and formulas.	.535	
An elementary teacher should immediately explain the correct procedure when a child makes a mistake.	.522	
A good textbook is more important for helping students learn mathematics than using manipulatives.	.377	
Children should be able to figure out for themselves whether an answer is mathematically reasonable.	.388	.353
Children should master the basic facts before doing problem solving.	.326	.410
Frequently when doing mathematics one is discovering patterns and making generalizations.	.306	.303
Children's own methods of solving mathematical problems are useful in their learning mathematics.		.764
Problem solving is an important aspect of mathematics		.721
In mathematics there is one correct answer.		.507
In mathematics there is always one best way to solve a problem.		.464
For elementary school children it is not important to understand why a mathematical procedure works.		.452
Children are often creative when solving problems.		.339

Table 1. Factor Loadings for beliefs about math survey items.

Post-course, all items loaded on a single factor indicating that this conflict had been somewhat resolved as a result of taking the course. Table 2 displays the results the mean scores for procedures and creative activities from pre to post courses.

Time	Pre course 1	Post course 1	Post course 2
Dependent Variable			
Math as procedures	2.8	3.3	3.6
Math as creative activity	3.9	4.1	4.2

Table 2. Means for procedures and creative activity.

Results of the MANOVA that tested for differences in variables between response time indicated an overall significant result ($F(4,292) = 10.3$, $p<.001$). Subsequent univariate tests indicated that both *procedures* and *creative activity* were contributing to this difference ($F(2, 146)=23.8$, $p<.001$; $F(2, 146)=7.9$, $p<.001$, respectively). Significant differences in these beliefs were found between pre course 1 and post course one ($p =.01$) and between post course one and post course two ($p=.03$) indicating that not only did participants beliefs change (making them more consistent with the PSSM) but they also become stronger as a result of a second course using the same pedagogy.

Discussion and Conclusion

Prospective elementary teachers can change their beliefs about mathematics and its teaching by focusing on how children learn and think about mathematics in content courses for elementary teachers. Furthermore, these beliefs tend to get stronger as a result of participation in a second such course. These results are important to consider based on the recent research regarding the difficulty in changing mathematics beliefs. Perhaps if more focus was placed on a children's thinking approach earlier in a preservice program (i.e., content courses), it would provide a better opportunity to affect these beliefs. More research needs to be done, but perhaps more significantly, this approach may also improve students' future teaching of mathematics to children.

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DEVELOPING FUTURE MATHEMATICS TEACHERS: CREATING "SPACES OF DIFFERENCE"

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The results of case study research, focused on one pre-serve teacher, suggests the importance of pre-service teachers working in classrooms wherein the mentor teacher and his/her students participate in ongoing and evolving mathematical conversations within an emergent curriculum. This experience had a profound impact on her beliefs about mathematics teaching and learning and on her emerging teaching practices

Creating spaces for teacher candidates to be challenged and to grow, to be transformed, is an important goal for most teacher preparation programs. Often these experiences occur in methods classes, field experiences, and during the student teaching experience. However, despite good planning and thoughtful placements for student teaching experiences, teacher candidates often find themselves with mentor teachers who teach using pedantic, traditional "stand and deliver" methods. In such cases, teacher candidates are inducted into teaching only having experienced limited pedagogical practices, thus growth and transformation are not likely to take place. Deleuze discusses the notion of one being "place[ed]… in a transformational matrix, a space of potential difference through which passes, from time to time, a spike of lightning that is the active realization of the transformative power of life" (Roy, 2003). How can we create "spaces of difference" for teacher candidates – ones that have transformative potential? What opportunities and experiences can challenge the beliefs and perhaps transform pedagogical practices of beginning teachers?

Research on teacher beliefs indicates that teachers' specific content and pedagogical knowledge is filtered through their beliefs about the nature of learning and about the nature of the specific content area (Swafford, 1995). It is widely understood that teacher held beliefs have a strong impact on the choice of actions and behaviors related to instructional approaches in mathematics (Enochs, Smith & Huinker, 2000; Foss & Leinasser, 1997; Raymond, 1997). Thompson (1984) purports that there exists a consistent relationship between teachers' instructional practices and their beliefs about the nature of mathematics while Cobb, Wood and Yackel (1990) contend that there exists a relationship between teachers' beliefs about teaching and their beliefs about learning.

The study sought to undertake the complex task of examining the beliefs of one teacher candidate during her student teaching experience and to better understand the impact on her beliefs about teaching and learning as she worked in constructivist mathematics classroom wherein the curriculum emerged and evolved as a part of an ongoing conversation between and among the students and the teacher.

Methodology

Using case study methodologies, narrative inquiry was employed in a desire to capture the wholeness of the teacher candidate's experiences in her own telling "filled with narrative fragments, enacted storied moments of time and space, and reflected upon" (Clandinin & Connelly, 1999, p. 17). Additionally, an interpretational approach combined with a method of

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constant comparison of data was used to both guide data collection and analysis of data throughout the study (Gall, Borg & Gall, 1996).

Data Sources

Data for this study were collected from a variety of sources. Observations were made on a weekly to bi-weekly basis over a four month period. Extensive interviews were conducted following each observation with both the teacher candidate and the mentor teacher following each classroom observation. Both interviews and observations were audio taped and later transcribed. Additionally, a daily journal was kept by the teacher candidate participant and was submitted weekly via e-mail.

The Classroom and Mentor Teacher

The school where Julie (participant pseudonym) student taught was a middle to upper suburban public school adjacent to one of the state's larger cities. The classes were 7th grade mathematics, pre-algebra and Algebra I. Her mentor teacher, Wesley, is a veteran teacher of 28 year. Wesley's pedagogic practices began to transform after his attending a workshop on Problem Centered Learning about 14 years ago. His classroom today is a problem centered learning environment wherein he describes himself as participating in an ongoing, evolving and emerging conversation in which mathematical meaning is made.

The Case of Julie

Julie was an elementary education major at a state land grant university in the Mid-western United States. In addition to her elementary education certification, she had passed the state certification exam to teach middle school mathematics through geometry. Her background in mathematics was typical of most students graduating from public schools. She believed she was "strong" in mathematics, stating that she "made good grades in high school and college math courses."

Perturbation and Change in Beliefs for Julie

Perturbation in this experience occurred for Julie on many occasions. She expressed concerns about what the student's understand (or do not understand), issues related to middle school students, working with 140+ students during one day, the tasks (mathematics content embedded), asking questions (most frequently mentioned in journal and interviews – "How do I ask better questions?") and finally listening to students – trying to understand what they are learning and what they are struggling with.

The impact of this experience and its transformative power, however, can best be understood in Julie's words:

- I really used to think mathematics was just about numbers and learning all the relationships and how they fit together which means there are a lot of rules to learn. After this experience I am thinking differently about mathematics more as patterns and relationships and not so much about the rules.
    - I am not sure that I can go out at first and teach the way I have had the opportunity to this semester but I know that I cannot teach the way I was taught (traditionally) because I know I would be doing an injustice to my students!
- I have had the most amazing conversations this semester with 7th graders and with Wesley. Conversations about enlarging area, common sense ways for determining discounts and sales

tax, and conversations about fractions, decimals, and percents. Absolutely amazing conversations – I never would have guessed something like this would have happened!

For Julie, as with most transformative experiences, a variety of issues and opportunities gave rise to her questions and quest for understanding. This student teaching placement created a "space of difference" wherein multiple "spikes of lightning" occurred for Julie such that she began to question her ideas about teaching and learning and what it means to understand mathematics.

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THE ROLE OF CURRICULUM MATERIALS IN NEW TEACHERS' PRACTICE

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This study investigated the effect of curriculum materials used in beginning teachers' classrooms on their ability to meet their stated goals of implementing the Standards based teaching practices utilized in their university coursework and internship experiences. Seven teachers were observed, interviewed, and videotaped during their second year of teaching. We conclude that Standards-based curriculum materials are necessary, but not sufficient, to support beginning teachers.

The design and dissemination of curriculum materials has been a major means of attempting to change classroom instruction, both historically and in recent years (e.g., Ball & Cohen, 1996). In the United States, the National Council of Teachers of Mathematics' *Standards* documents (e.g., 2000) provide a focus for current change efforts. For preservice teachers, using *Standards*-based curriculum materials in a teacher preparation program may provide a vision of what it looks like to teach in a reform manner. Manouchehri & Goodman (1998) found that beginning teachers whose teacher preparation programs had emphasized reform-based ideas about teaching were more confident about, and more committed to, using reform-based materials. Other researchers have found that the curriculum materials used in early teaching positions have an effect on new teachers' practice. Specifically, a reform-based textbook seems to be a critical tool for implementing reform-based practices in the classroom (Steele, 2001). The study reported here examined the effect of the curriculum materials used in beginning teachers' classrooms on their ability to meet their stated goals of implementing the *Standards*-based teaching practices utilized in their university coursework and internship experiences.

Data sources were interviews and classroom observations from seven teachers' 2nd year of teaching—after they had completed their first "survival" year of teaching and had begun to establish patterns of instructional practice. All seven teachers had completed a mathematics education program designed to prepare *Standards*-focused teachers who could serve as change agents in their future schools. Their program included a semester-long teaching internship under the mentorship of classroom teachers supportive of the *Standards*. Participants were observed for three consecutive teaching days and interviewed before and after each observation. An observer took field notes and videotaped. An independent evaluator used the LSC Observation Instrument (Horizon Research, 2000) to evaluate their teaching. The completed instruments and the observer's field notes were used to develop a picture of each teacher's classroom instructional practices. The interviews were audio-taped and transcribed, and then coded to identify dialogue related to instructional planning, classroom activity, student thinking and understanding, and the participants' use of, and beliefs about, the curriculum materials they were using.

The Curriculum Difference

Five of the participants in this study used curriculum materials in their classrooms that had been rated exemplary by the U.S. Department of Education 1999 Math and Science Education Expert Panel, with four of these five teachers observed to display elements of effective instruction. The other two least effective instructors used curriculum materials that were not

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rated exemplary. Based on these observations, it appeared to us that using exemplary curriculum materials might be critical to supporting the novice teachers in their efforts to teach in a manner consistent with the practices described in the *Standards*. This raised the question of why the instruction of one teacher, Elliot, was rated ineffective when he also used exemplary materials.

Although Elliot used an exemplary textbook series, his practice was quite different than that of the other teachers who used exemplary materials. Elliot's verbal enthusiasm toward his curriculum might lead one to assume that he would wholeheartedly embrace it. In the classes that were observed, however, Elliot used instead pages that he had copied from another textbook to expose his middle school students to the kind of material he believed would prepare them for their high school courses—material very different from the curriculum which had been adopted for use in his classroom. In contrast to the other teachers who made *adaptations* to the curriculum, Elliot *replaced* the curriculum.

When asked specifically about his curriculum, Elliot expressed his support for it; however, he made comments at other points in the interview that suggested otherwise. For example, Elliot expressed concern that he was preparing kids to fail by using too much cooperative learning when they were likely to be expected to work independently at the high school level. He added that his students got tired of explaining, having to go the extra mile. His top students, especially, were "just traditional math students…they need the drill and practice; that's how they want to learn." He felt that there was not enough of this type of learning in his curriculum and thought that the students' basic skills were going to be weak in the long run. Elliot summed up his beliefs in the following dialogue:

> I think that for an advanced math class, for about 75% of the kids, it's not right for them. Because the real traditional, hard core math students can learn faster, can learn more, by doing it the traditional way. And that's one of the weaknesses, I think, of [his curriculum].

Conclusion

Using exemplary curriculum materials seems necessary to support novice teachers in achieving reforms advocated by organizations such as NCTM. In the words of one of our participants who did not have access to such materials, "I felt like I was taught all these wonderful things and all these wonderful methods, but unless I have a curriculum to support it, it's hard. I mean, I try. I honestly do try." However, we argue that the case of Elliot illustrates that exemplary curriculum materials are not sufficient on their own to ensure effective reform-based instruction. Instead, the use of such curricular materials is mediated by teachers' beliefs about learning mathematics and the needs of their students (Wilson & Lloyd, 1995). This study illustrates the importance of providing teachers with curricular materials that support their stated goals and the mediating effects of participants' deep-rooted beliefs on their use of such materials.

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FROM THE TEXTBOOK TO THE ENACTED CURRICULUM

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The study reported here describes how teachers use district-adopted mathematics curriculum materials and other curricular resources. In particular, this study addressed the following questions: How do teachers utilize district adopted school mathematics textbooks? Why do teachers make the instructional decisions they do regarding the use of the district-adopted textbook? The conceptual framework for this study was based on Remillard's model of the teacher's role in curriculum development (Remillard, 1999). The model includes three arenas: a design arena, a construction arena, and a curriculum mapping arena. By looking at the three stages in which teachers interact with their textbooks, clearer descriptions of the factors that might determine how teachers use their textbooks can be explored.

Three middle school teachers were chosen among 53 teachers participating in the Middle School Mathematics Study (Tarr et al, 2006). Analyses of survey data, textbook diaries, interviews, and classroom observations were used to describe the use of mathematics textbooks by these teachers. Two of them used Mathematics in Context (MiC), a comprehensive middle school curriculum developed by researchers at the University of Winsconsin and the Freudenthal Institute (Netherlands) with funding from the National Science Foundation. MiC was developed to reflect NCTM's vision, emphasizing real-world contexts, student interaction, and multiple strategies. The third teacher used Saxon Math, a program based on incremental development, continual practice and review, and frequent cumulative assessment. Skills and concepts are taught through direct instruction.

Teachers' view of the curriculum and the match, or lack of it, between their own views about mathematics and mathematics teaching and the philosophy of the textbook —explicit or not— were the primary factors that determined how the textbook was used. However, the primary factor that determined what tasks were presented to students was the textbook itself. As a result, the enacted curriculum in each of these three teachers' classrooms was shaped as much by the textbook as by the teachers' beliefs about mathematics and mathematics teaching. Their broader goals for their students were reflected on their teaching but also on their interaction with the textbook as a tool for instruction. Teachers with similar views about mathematics, but teaching with different textbooks in different schools, enacted very different curricula due to the influence of textbooks that were developed with different assumptions about the roles of the teacher and students. At the same time, teachers within the same school and using similar textbooks also enacted very different curricula, as a consequence of decisions based on their views about mathematics and their stance towards the curriculum materials they were using.

This study illustrates the fact that textbook adoption, by itself, does not necessarily change teachers' practices. However, to the extent that textbooks influence topic determination, the impact of textbook choice on students' opportunities for learning mathematics is certainly relevant.

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WORK IN-PROGRESS: A THEORETICAL FRAME DESCRIBING THE LOGIC TEACHERS EMPLOY WHEN VERBALLY COMMUNICATING TO STUDENTS

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This poster shares the current draft of a conceptual framework representing the logic teachers' employ when verbally communicating to students. The research studies implemented to create and support this framework are also outlined. Attention is given to interpersonal communication and mathematics education literatures, the foundations for the framework and studies. Questions are included to promote peer discussion and feedback.

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PERCEPTIONS OF MATHEMATICAL ABILITY AND MATHEMATICAL INTELLIGENCE: AN EXPLORATION OF PSYCHOLOGICAL FACTORS THAT IMPEDE SUCCESS IN MATHEMATICS

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Understanding the causes of the achievement gap between blacks and whites is a truly complex multidimensional task. According to Ogbu (1992), African Americans have adopted an attitude of resistance against that which is remotely "White". Others believe that poverty, not race is responsible for the achievement gap, however; even when class is controlled for, whites still outperform blacks (Trent, 1997). Race and class however, are just the tip of the iceberg. There are issues that are deeper than race and class which contribute to the black/white achievement gap in mathematics. Steele (1999) has shown that African American students feel pressure when taking standardized tests because they feel as though if they fail, then their race is a failure. This view of their race as a failure stems from the negative images and stereotypes portrayed by the media, and commonly accepted by society, thus creating a need for social acceptance. Factors such as math anxiety, test anxiety, low expectations, resources, and cultural sensitivity also affect mathematical performance (Gay, 2000; Green 1990; Oakes 1990). This presentation relates to the goals of PME-NA by seeking to expand on the factors that contribute to African American underachievement in mathematics by exploring the mental images and perceptions of teachers and black and white students of different socioeconomic backgrounds.

This poster will describe how black and white students and teachers feel about the ability of blacks and whites to do well in mathematics, and the characteristics that are attributed to "smartness" in mathematics by these students and teachers. Tasks were aimed at perceptions and images held by the participants. These were black and white elementary and middle school students of low SES in low track and high track math classes, and their teachers; and black and white elementary and middle students of high SES in low track and high track math classes, and their teachers.

These tasks were administered across grade levels ranging from third through eighth to see how perceptions might change from one grade to the next. It was necessary to have someone else administer the task for teachers as my race and/or gender could interfere with their comfort in being open and honest. Individual clinical interviews, generative in nature, were conducted to get some sense of how the students' teachers made the students feel as math learners, and to understand how students defined "smart' in the math classroom.

The distribution of student responses to my tasks and responses to interview questions provides some insight into why African Americans under-perform in mathematics. Rather than analyzing the results through a lens of black and white disparity, results were viewed in terms of similarities and/or differences in perceptions among blacks and whites in lower tracks of mathematics and among blacks and whites in higher tracks of mathematics. The tasks administered during the study reveal the ways in which black and white students and their teachers assign high and low mathematical ability and intelligence to racial groups and/or other categories and why they assign them in the manner that they do. Also, the ways in which students perceived mathematical ability and intelligence was observed across grade levels. I hypothesize that the results of this study may have broader implications for tracking students.

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DEVELOPING PRESERVICE ELEMENTARY TEACHERS' EFFICACY THROUGH MATHEMATICS CONTENT COURSES FOCUSED ON CHILDREN'S THINKING

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This paper examines how the Connecting Mathematics for Elementary Teachers (CMET) project is connected to the self-efficacy of preservice teachers. The primary goal of the CMET project is to connect the mathematics that preservice elementary teachers are learning in their content courses with how children learn and think about mathematics. While several studies have focused on innovative curriculum projects and their effects on preservice teachers (Hill et. al., 2004, Lloyd, 1999) and recent research has shown that preservice teachers' efficacy can be positively influenced in methods courses and in-service teachers' efficacy through professional development (Ross & Bruce, 2005), we ask: "Can preservice elementary teachers' efficacy be positively influenced earlier, in mathematical content courses, through a focus on children's mathematical thinking?"

Findings

Statistically significant results for students at University I on all pre- and post-efficacy items indicate that participants felt more confident in all content and teaching areas after the course than before. Again, statistically significant results were found for students at University II between pre- and post-scores indicating that participants felt more confident in all content and teaching areas after the course than before.

At both universities, prospective teachers felt more confident about their own mathematics ability than their ability to teach elementary mathematics to children before taking the course. After the participating in a course that focused on the children's thinking this gap narrowed and for some items disappeared. In addition to analyzing the self-efficacy of individual participants, a t-test was run on the difference scores for Teaching Efficacy to test for differences between universities. No significant differences were found (t=1.8) indicating that having one of the supplement authors teach the course was not the cause for the increase in prospective teachers' teaching efficacy.

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PRESERVICE MATHEMATICS TEACHERS' PERCEPTIONS OF THE INTEGRATION OF DISCRETE MATHEMATICS INTO SECONDARY CURRICULUM

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Introduction

The integration of discrete mathematics [DM] into the secondary school curriculum (grades 7-12) is an important consideration because the mathematical area is dynamic and interesting, but also because it provides opportunities for teachers to develop innovative strategies and for students to experience non-standard mathematical approaches to solve real world problems. Rosenstein (1997) expressed that discrete mathematics is applicable, accessible, attractive and appropriate. Others have suggested that discrete mathematics is an effective approach for illustrating and emphasizing the five National Council of Teachers Mathematics [NCTM] (2000) process standards: problem-solving, communication, reasoning and proof, representation, and connection (Kenney & Bezuszka, 1993). Discrete mathematics also can be used as an investigatory approach in mathematics classrooms (Burghes, 1985).

The purpose of this study is to identify and describe preservice secondary mathematics teachers' perceptions about integrating discrete mathematics into the curriculum. A case study design is used to describe preservice teachers' perceptions. Data sources include a selected coursework, and an online survey. The results will include key issues and themes emerging from analysis of preservice teachers' comments about the role of discrete mathematics in the school curriculum.

Background

In its Curriculum and Evaluation Standard for School Mathematics, from 1989, NCTM recognized the significance of discrete mathematics topics in the secondary curriculum by including it as one of the Standards. NCTM stated that:

> As we move toward the twenty-first century, information and its communication have become at least as important as the production of material goods. Whereas the physical or material world is most often modeled by continuous mathematics…the nonmaterial world of information processing requires the use of discrete (discontinuous) mathematics… it is crucial that all students have experiences with the concepts and methods of discrete mathematics. (p.176)

Recently there have been attempts to integrate DM into textbooks and curriculum materials for middle school and high school. For example, the *Contemporary Mathematics in Context* textbook series (for grades 9-12) dedicate several units to DM topics such as graph theory, combinatory, permutations; Hart, DeBellis, Kenney and Rosenstein have a forthcoming NCTM book titled *Navigation through Discrete Mathematics in Pre-Kindergarten to Grade 12*. There are also many websites available that contain activities on DM (for example, http://mathforum.com). In addition, a textbook for teachers, *Discrete Mathematics for Teachers* (Wheeler & Brawner, 2005), was recently published.

Some of the arguments made about the importance of teaching and learning DM in the schools are: (a) it is accessible for all students at all levels (Kenney, 1996; Rosenstein, Franzbalu,

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& Roberts, 1997); (b) it encourages an investigatory in mathematics teaching (Burghes, 1985; DeBellis & Rosenstein, 2004; Heinze, Anderson, & Reiss, 2004); (c) it is a good approach for illustrating and emphasizing the five NCTM (2000) process standards (Kenney & Bezuszka, 1993); (f) it allows teachers to see mathematics in a new way and to rethink about traditional mathematical topics (DeBellis & Rosenstein, 2004; Kenney, 1996). These arguments might sound solid, but they are mostly based on speculation, although some of them have grounds in observation and teachers' experiences (DeBellis & Rosenstein, 2004; Friedler, 1996).

As a result of this need for information, this study identifies and describes preservice mathematics teachers' perceptions of teaching and learning DM in the secondary school mathematics curriculum.

Methodology

The study involves a case study design in order to obtain an in-depth qualitative description of preservice teachers' views of DM in the secondary curriculum. I intend to obtain an accurate snapshot of how a specific group of students/preservice teachers reacts to concepts of DM. My research question will be: How do preservice secondary teachers perceive and react to the integration of discrete mathematics in the secondary curriculum?

In my case study I examine students enrolled in the course *Mathematics for Secondary Teachers I* (MATH 4625), a mathematics course at Virginia Tech, Blacksburg VA in fall of 2005. In this course, students analyzed topics in discrete mathematics and algebra from a secondary teaching perspective. In addition, students developed classroom activities and methods that involve the NCTM (2000) process standards (problem solving, reasoning and proof, communication, connections, and representation).

Multiple sources of data, selected coursework and an online survey, will be used to determine preservice teachers' conceptions about DM in the secondary curriculum.

A collection of selected homework assignments from Math 4625 related to the integration of DM into the traditional secondary mathematics class (e.g. algebra, geometry) was be used to identify preservice teachers' experiences and opinions about this topic. Part of the selected homework assignments included a survey. The purpose of the survey was to provide broad background information related to teaching and discrete mathematics beliefs among the students. The survey was given at the beginning of the course (August 2005). A second online survey focused on beliefs about teaching discrete mathematics had the purpose of identifying any changes of students' views about teaching DM after they examined in the course.

Implications

Considering that there is a gap in the literature concerning teachers' views of the relevance of discrete mathematics in the secondary classroom, different groups could benefit from this study:

- Educators: the study will provide preservice teachers' views about the integrating DM in the classroom. It will help to identify obstacle and resources for integrating DM into the curriculum.
- Students at different levels: the study will offer a new way to perceive mathematics as a practical way for problem solving.
- Researchers: the study will inspire them to conduct investigations leading to new journal articles and conference presentations related to this topic. Future studies may build on the findings of this investigation.

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A VIEW OF HIGH SCHOOL MATHEMATICS CURRICULA THROUGH THE LENS OF SCHWAB'S COMMONPLACES

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According to Schwab (1973), "defensible educational thought must take account of four commonplaces of equal rank: the learner, the teacher, the milieu, and the subject matter. None of these can be omitted without omitting a vital factor in educational thought and practice." (p. 508) In the creation and enactment of curriculum, Schwab's commonplace framework stresses the importance of focusing on all four commonplaces and their interactions with one another within particular classroom settings and with the outer environmental structures.

My dissertation study, a narrative inquiry, describes a quest to help mathematics teachers become empowered to deal with instructional challenges in the face of systemic impediments within their low-performing, urban high school. Initially, I sought to improve teachers' instructional approaches to a new high school Geometry program which I had written to meet the needs of the high-stakes test demands of the No Child Left Behind Act (NCLB) and the student-centered, cognitively-based standards set forth by the National Council of Teachers of Mathematics (2000). My study began with an imagined end in mind: With appropriate instructional support, teachers can improve their classroom practices within a professional development environment that provides them opportunities to share experiences and become reflective practitioners. Several weeks of classroom observations, post-observation discussions and suggestions for improvement indicated to me very little observable change in teachers' practices. During this time, I observed several critical incidents among teachers, students and administration which pointed to a school-wide milieu in disarray. I realized that attempts to improve the teacher-subject matter connection would be futile until teachers came to terms with the disconnections within their school. I decided to focus on improving the teacher-learner-milieu connections within the classroom, hoping that in turn teachers would recognize the need to work toward improving their repertoires of instructional approaches to better serve their diverse student population.

This extended experience and subsequent analysis using Schwab's commonplaces have provided a framework and tool for subsequent needs analyses when invited to provide professional development in other schools. In addition, when planning and enacting professional development programs, including subject matter courses for in-service mathematics teachers, the commonplaces play an important overarching framework in discussions about classroom culture and lesson design.

In my poster presentation, I describe how Schwab's commonplace framework became an interpretive tool for my dissertation study and now serves as a diagnostic and instructional tool in my professional development work with teachers and in schools.

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TEXTBOOK USE AND CLASSROOM PRACTICES: A CLOSE LOOK AT ONE CLASSROOM

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In this poster, we report the findings of a study that has a goal of better understanding the relationship between a teacher's conceptions (e.g., beliefs about the textbook, knowledge of mathematics) and what she actually does with a textbook when she uses it in her classroom. Remillard (2005) argues that we need to understand how teachers interact with curriculum materials in deeper ways than we currently do. It is also important to gain a richer understanding of the relationships between a teacher's conceptions and recommendations made within curriculum materials on how to use the materials (Lloyd, 1999). Teachers interact with their curriculum materials in different ways (e.g., such as following the teacher guide or drawing ideas from the materials) and there are dynamics, such as beliefs about knowledge, that influence this interaction (Brown, 2004). We draw on ideas from sociolinguistics (e.g., systemic functional linguistics) and sociocultural theory (e.g., Brown, 2004) to highlight the ways in which the teacher refers to and uses her textbook, thus closely examining the role of this tool in her classroom practices.

As part of a larger project on middle grades mathematics classroom discourse, we observed one teacher during fall 2005 for two weeks in one of her 6th grade mathematics classes. The teacher has taught in a rural school for 20 years, of which she has been teaching the *Connected Mathematics Project* for the past five years. In addition to the observations, she also took part in three one-hour long interviews that focused on her textbook as well as how she uses it in her classroom. In the analysis, we identified instances in the classroom observations in which the textbook was either being referred to (e.g., "Let's go over the ACE questions" or "Have out problem 1.3 from Bits and Pieces") or explicitly being used (e.g., "Turn to page twelve" or "You should be on page 43. Read that first paragraph."). We describe the nature of these instances and account for the ways the teacher refers to and uses her textbook based on the information provided in her interviews.

The preliminary findings show that this teacher uses the textbook everyday and often refers to it explicitly. One of the factors that influence her use of the textbook is how comfortable she is with the content that she presents that day. When she feels comfortable with the mathematics, she tends not to rely on the textbook as much; if she is not confident about a mathematical topic, she will do exactly what the textbook recommends. Other things that influence how she uses the textbook are the feelings of parents and others in the community (e.g., Parents have complained that they don't know what their students should be doing in math, so she now gives very explicit instructions to the students on each problem). In this poster, we will give examples of the different ways she refers to and explicitly uses the textbook in her classroom. We connect these linguistic elements to the construction of the textbook as a tool in her classroom. We also report some of the reasons she provides for her textbook use and reference.

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TEACHER EDUCATION – INSERVICE / PROFESSIONAL DEVELOPMENT

DO THEORETICAL TOOLS HELP TEACHERS MANAGE CLASSROOM SITUATIONS? A CASE STUDY

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Connecting Theory and Practice in school mathematics is a crucial issue for investigation. This report focuses on the role that theoretical models, as emerged from the observation of students at work, can play on the part of the teacher. In other words, how teacher training can orient teachers in studying and using some theoretical findings coming from research.

Introduction and theoretical framework

For a long time, theory and practice (i.e. the theoretical approach to the discipline and its transposition in school practice) have been considered as opposite poles. In recent decades, the dialectic nature of the theory-practice relationship has become increasingly recognised and embodied in research studies, which have highly fostered the dialogue between the two poles (Bartolini&Bazzini, 2003). There is general consensus on the assumption that the theoretical education of perspective and in-service teachers must not separate theory and practice, but has to try to develop the teacher's theoretical abilities to look beyond the practical surface of the problems they will encounter later in their teaching practice. In the search for boundary conditions to mediate knowledge between the two poles, there is evidence that any conception which assigns "theory" the instructing role, dooms "practice" to fail (Steinbring, 1998). The necessity of developing the notion of cooperation comes as a consequence. Following this position, existing literature provides interesting contributions supporting the idea of blending mathematical content with pedagogical knowledge (Ponte et al.,1994). We add that findings of research in Mathematics Education are of basic importance in teachers' education. As we will discuss later, this issue is of special use in the proceedings of our experiment, both during teacher training and in teaching practice. In our study we have adopted the modality of co-learning partnership (Jaworski, 2003), i.e. a tight co-operation between academic researchers and school teachers, which is a long-time shared component of our research methodology (Bazzini, 1991). Concerning the teachers' knowledge of learning theory, following Jaworski's suggestions, we addressed the question of how the teacher's knowledge on research findings is surfaced and used in practice.

The research problem: position and proceedings

We focus our attention on some theoretical models, which emerged from the analysis of students' behaviours and are reported in literature. We are concerned with the role they play on the part of the teacher during the teaching process. We assume that such theoretical models, when deeply owned by the teacher, can influence the teacher's behaviour and produce valid interventions in the teaching phases (i.e. planning, acting and reflecting).

This report develops an analysis of the professional behaviour of a teacher, after a given instruction on theoretical issues, as provided by existing literature in Mathematics Education. The teacher under discussion graduated in Mathematics and has been teaching in junior secondary school for several years. She is smart, open minded and sensitive to new ideas and suggestions. She was selected to attend a two year training course on "Approaching Algebra in

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Junior Secondary School", organized by the Regional Institute for Research and Teacher Training (IRRE Lombardia) and held in Milan, 1998-2000. This course treated general issues on the didactics of algebra, theoretical reflections on the nature of algebraic thinking and practical suggestions for teaching. Special attention was given to the work by Sfard (1991) on the dual nature of mathematical concepts (i.e. procedural and structural aspects) and to the work by Arzarello, Bazzini and Chiappini (1994, 2000) dealing with the distinction between sense and denotation of algebraic expressions.

According to Sfard, true understanding relies in mastering the passage from operational aspects to structural ones and vice versa. The dialectic relationship between the two aspects pervades learning in all of its phases: procedures prepare the ground for the "reification" of mathematical objects and, in turn, mathematical objects and their relations allow the student to look at procedures from a more general perspective.

Arzarello, Bazzini and Chiappini, point out that algebraic thinking, as any other kind of thinking, lives in the interplay between mental activity and linguistic expressions. In a more general perspective, representations and symbols of mathematics establish semiotic systems which are of fundamental importance in doing algebra. The denotation of an expression is the object to which the expression refers, while the sense is the way in which the object is given to us. The dialectic between sense and denotation allows one to consider algebraic reasoning as a game of interpretation: a given formula can activate different senses and symbols' manipulation is promoted by the passage from one sense to another depending on to the goal of the problem.

Finally, from a methodological point of view, the training course emphasized the role of mathematical discussion as powerful instrument for knowledge construction in social interaction (Bartolini Bussi, 1996).

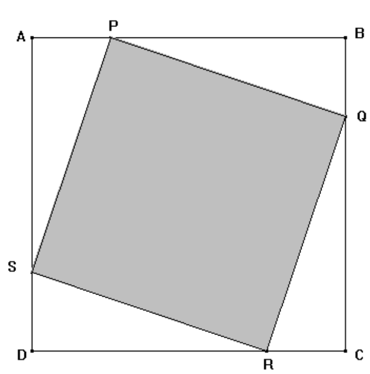
The teaching experiment: an overview

The teaching experiment we are going to review is framed in the above references. It is part of a wider research study which was carried out in the 8th grade in the period January-March.

During the three year course (grades 6th-8th), the teacher conceived an early approach to algebraic thinking: activities have been already planned in the first year of the course. Those activities were developed slowly and carefully, thus permitting students to grasp the meaning of what they were learning.

We underline that the students already knew the software Cabri Géomètre and were familiar with such practices as verbalizing, expressing opinions, formulating conjectures, doing critical comparison of products and procedures, sharing results and discoveries.

Thanks to this way of conceiving mathematical lessons, the teacher had created a group of "young mathematicians" willing to take part in the social construction of knowledge.



Furthermore, we observe that the students were already aware of the symbolic manipulation of symbolic expressions: this made the proposed activity particularly fruitful.

Our teaching experiment consisted of 7 lessons (totally 14 hours). During the lessons, the students were proposed activities, which took place in different environments (paper & pencil, Cabri Géomètre) and which were realized in different forms (individual work, group work, classroom discussion). The starting point was a geometric problem (see figure): the pupils were given the figure of a square inscribed into another square and were required to inscribe other squares and to reflect on possible configurations. The activity was guided by 5 schedules, which proposed:

- construction of geometric figures
- exploration of their characteristics
- determination of variables and constants
- determination of relations between the variables in question
- analysis of a phenomenon (how the area of the squares inscribed in the square changes)

The first series of activities (schedules 1, 2, 3 and relative class discussions) dealt mostly with geometric arguments and took place in the two environments: paper & pencil and Cabri Géomètre; (for a first analysis of the integration of the two work environments see Bazzini, Bertazzoli, Morselli 2003). This series of activities was conceived to make the students to get in touch with the problem, to think carefully on the configurations, to find out the variables and to make hypothesis on the relations between variables. During these activities the students singled out variables and made the first remarks on covariance (relationship between the length of AP and the area of PQRS) and on the existence of special cases (the so called limit-cases): minimal area and maximal area.

The subsequent activity (schedule 4) focused on the calculation of the area of PQRS. Students were asked to calculate the area in different situations (i.e. when AP is given different values). The students were required to calculate the area of squares built under given conditions. Then they were asked to reflect on the data in order to find the minimum value for the area.

The schedule no. 5 introduced the use of the sign x to name the length AP and to condense in one formula the calculation of the area of PQRS.

Case study: focus on the classroom discussion

For the aims of this report, we focus on two different but intertwined issues related to the introduction of Algebra in junior secondary school: first, the symbolic translation of a given geometric situation and, second, the mastery of symbolic expressions. Our attention focuses on activities 4 and 5, which mark the passage from practice (construction and analysis of geometrical configuration, calculation of the area) to the sign (introduction of the algebraic notation to represent and to study the phenomena of area variation and covariance).

In the schedule 4, students were asked to calculate the area of PQRS, for given (integer) values of AP. The schedule is aimed to guide the students in the passage from the geometric situation to its algebraic representation. A careful work on calculation procedures is aimed at finding the meaning of the subsequent algebraic formulation. Furthermore, the reflection on the schedule stimulates considerations on covariance and induces to consider the configuration in its global structure.

Here we report some extracts from the groups work (the calculation refers to the following values: AB=8 and AP=2):

Group 1 (Marta and Andrea):

We use Pythagoras $\sqrt{22+62}=\sqrt{40}$

In order to find the area [of PQRS] we should work out the square root, but it would be useless because, after, in order to get the area we should work out the square.

Group 2 (Pietro and Paolo):

We can do it in different ways.

We can do: $\sqrt{62+22}=\sqrt{40}=6.32$

Area=6.322=39.94

Otherwise: 82-6*4=64-24=40

We may observe that two different procedures were adopted by the students:

- difference of areas (square ABCD - 4 triangles)
- using the Pythagorean theorem to calculate PS and, then, calculating the area of PQRS directly

In the schedule 5, the teacher asked the students to name x the value of AP and to create an expression for the area.

In this way, students arrived at the expression through computation (procedural aspect, in Sfard's perspective) and successively could reflect on the expression as a mathematical object (structural aspect).

The whole process developed differently with different students and different expressions are given. This is due to the fact that different procedures of calculation emerged during the work on the schedule 4.

At this point the teacher profited from the situation and, following Arzarello, Bazzini & Chiappini's model, stressed the point that different expressions can be seen as carrier of different senses for the same denotation.

The discussion consists of two parts: at first the teacher asked the students to report the expressions used for calculating the area, and fostered a careful reflection on the algebraic formalization of the various elements of the area. Thereafter, the teacher promoted a reflection on the various expressions which were found and asked the students to verify – through symbolic manipulation – if different expressions (and different senses corresponding to the different methods of calculation) can represent the same area, i.e. the same denotation. The students succeeded in grasping the equivalence of the two expressions.

Here we report some excerpts from the discussion transcript.

12. Teacher: You know that the measure of the side is 8. If you indicate with x the length of the side AP, you can use the letter x to express the length of the side AS. What did you answer? Paolo?
13. Paolo: 8-x
14. Teacher: did you all answer like that?
15. Thomas: I did AB-BP=x
16. Teacher: Yes, but using the letter x, if AB is 8 you can express AS as 8-x. Afterwards… the area of the triangle APS.
17. Anna: (8-x)x/2
18. Teacher: very well. I would like that you read me your expressions for the area of the square PQRS. This is the part that interests me the most.
19. Francesca: 64-(8-x)x2
20. Teacher: are your expressions all alike? No? Then, I would like to hear them. Valentina and Maria?
21. Valentina: 64-[(8-x)x/2]4
22. Teacher: I ask: are these expressions equivalent? Paolo?
23. Paolo: finally, the result doesn't change.

24. Teacher: ok, but I would like also a formal equivalence, not only that finally the result doesn't change. Doing (8-x)x or doing x(8-x), does something change?
25. Andrea: dividing by 2 and multiplying by 4, is the same as multiplying by 2.
26. Teacher: ok, multiplying by 2 or dividing by 2 and multiplying by 4 is the same thing, I agree, all right. Another thing: the parenthesis and the x have their place inverted. Is it allowed?
27. Andrea Z: it is the same, it is the commutative property.

This first excerpt shows the choice of the teacher to exploit the fact that the students proposed different methods. She wants to exploit this richness of methods in order to make the students to work on the algebraic equivalence of the expressions. She encourages the work of algebraic transformation of an expression into another one. The geometric situation warrants the validity of the expressions and gives meaning to the transformation (from a geometrical standpoint, the two expressions are equivalent, then they must be equivalent also from an algebraic standpoint).

28. Teacher: ok, then these two expressions are equivalent. Are all your expressions absolutely equivalent? No? Marta, Andrea?
29. Andrea S.: √(8-x)2+x2
30. Teacher: I don't write the square root, we'll discuss why, later. So, listen, it is very different for the other expressions. Can you understand what are they doing? While the first two expressions were easily changed one onto the other, we understood that they are equivalent, this one is really different, it has a completely different style. Do you agree? How did they reason? Why did he told me a square root and I didn't write it?
31. Andrea Z.: 82 for me is 64
32. Teacher: of course. But they write x2+(8-x)2. What do they want to say? What does remind us this expression? Pay attention, in your worksheet x is AP and 8-x is…?
33. Andrea Z.: PB
34. Teacher: No, in your worksheet x is not PB, it is AS, which has the same length as PB….., AP is x and AS is 8-x. but those two, AP and AS, what are they?
35. Francesca: they used the theorem of Pythagoras.
36. Teacher: very well. Actually the theorem of Pythagoras would have the square root, but … you should use the square root in order to find what?
37. Andrea: the side.
38. Teacher: Yes, the side. Why did I take it away?
39. Anna: because in order to find the area we should do the square.
40. Teacher: perfect. Then, these two are more or less the same, this is different. Let's go on. Good idea, the fact that 82 is 64, but we'll come back to that later …. Pietro and Paolo?

In [30] we may have the impression that the teacher acts in a very traditional way, omitting the square root (thus, correcting the pupil's answer!). Actually, she wants to be sure that all the pupils understood the method that is linked to the expression, that is the meaning of the expression. Once she is sure of that, she takes into the discussion the matter of omitting the square root.

In [40], we see the micro-decision of the teacher: for the moment, she wants to keep the focus of the discussion on the meaning of the expressions. She plans to focus on algebraic equivalence afterwards. She asks to Pietro and Paolo, because she knows from their worksheet that they wrote a different expression. We note here the importance, for the teacher, of having at disposal the worksheets of the groups before the discussion, in order to have a clear idea of the situation of each group.

41. Paolo: we wrote like the first: 64-...
42. Teacher: You did not do 64 but 82, I've got the proof, but basically it is the same. Francesca and Simone?
43. Francesca: 64-4[(8-x)x/2]
44. Teacher: basically it is still like that by Valentina, we just have top ut things in a different way. So, we have these three expressions (64-(8-x)x2 ; 64-[(8-x)x/2]4 ; (8-x)2+x2 , our note): which is the most economic, between the first and the second? (VOICES: The first). Then, I do not offend anybody if I erase the second one, we understood that they are logically equivalent. Now I ask you: litteral calculation, we can write it in a better way. Because: we have 64-(8-x)x2, instead of x2 what can we write? (VOICES: 2x), well, I write it before the parenthesis. Who is going to write it in a cleaner way, without parenthesis?
45. Stefano: 64-16x+2x2.
46. Teacher: is it ok? Can somebody write this one differently? (she now refers to (8-x)2+x2)
47. Thomas: 64-x2.
48. Teacher: minus? The squares?
49. Thomas: 64+x2+x2.
50. Teacher: pay attention! I just said to the observer that you are good in algebra… here, another term is missing. When I square…
51. Stefano: Ah, we must write it twice!
52. Teacher: yes, it is true, we should think of it as… then, check what is missing. 64… are all the terms there?
53. Thomas: +8...
54. Teacher: Plus ?
55. Thomas: ah, the sign after… 64x, I guess.
56. Teacher: No, 64x no because 8*8=64, that is what Andrea Z. said before. Yes, Paolo?
57. Paolo: -8x-x2
58. Teacher: Not minus, it is x2. but one term is still missing.
59. Thomas: 8x.
60. Teacher: 8x is still missing, that's true. Are they all there now? Well. But are these two expressions alike? (VOICES: Yes). Why, Anna?

At this point, on the blackboard there are different signs (the expressions) and different senses (the methods). The teacher introduces here [44] the algebraic work (transformation).

61. Anna: because -8x-8x is –16x and x2+x2 is 2x2.
62. Teacher: Perfect, then I keep this one that is already fine. But what is this, then? It is the area, isn't it? (VOICES: yes). And how did we state to call the area? In algebraic language. (VOICES: A). NO, in our worksheet we did not call it A. (VOICES: y). y, very well. y= Write it on your sheet, it is very important.

Here the two expressions are clearly recognized as being equivalent, thanks also to their geometric interpretation (the area of the same square).

In [62], the teacher stresses that the algebraic expression stands for the area. She also corrects the students' answers as regards the "name" of the area (the area is y and not A). Why? Not just in order to set authoritatively the conventional use of letters, but because A had a different meaning on the worksheet (A was one of the vertexes of the square); she tries to prevent misunderstandings.

Discussion

The analysis of the lesson transcript gives evidence of the basic role of the teacher in planning, leading and coordinating the classroom discussion.

First of all, we discuss some macro-decisions of the teacher. We point out here the influence of theoretical tools (namely, the theory of Sfard) in planning the sequence of schedules (4 and 5) that guide students form the calculation to the reflection on the mathematical object.

Other theoretical tools (namely, Arzarello et alii.'s model) influenced the choice of the teacher to foster a reflection on the different expressions having the same denotation.

Furthermore, the theoretical tool concerning the potentialities of mathematical discussion helped the teacher to choose the suitable scenario for the reflection.

It is also interesting to compare what has been planned (before the classroom experiment, in co-operation with the university researchers) and what really happened in the classroom, when the teacher has to face "classroom life", including students' reactions, sometimes not foreseen. Within the overall perspective of long-term decisions, micro-decisions play a very important role (Malara, 2005), and reveal the inner knowledge of the teacher.

The analysis of our case study shows that the teachers' interventions are very helpful in grasping students' intuitions, fostering their potential and orienting classroom discussion towards profitable reflection.

There is clear evidence that the teacher acquired and elaborated the theoretical models studied during the course; this allows her to suitably use such models in the classroom activity.

The analysis of this case study provides useful suggestions to identify strategies which are in accordance with theoretical models (and probably inspired by them). However we do not exclude that similar situations can produce different strategies, in accordance or not with the previously shared theory. This is usually due to a variety of reasons, including tacit beliefs, original imprint, school constraints and so on.

The study of these variables is very interesting for us, but in need of further investigation.

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STUDENTS' THINKING ABOUT DOMAINS OF PIECEWISE FUNCTIONS

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This two-phase study investigated high school student difficulties with graphing and understanding piecewise functions, with a focus on how students thought about multiple domain statements for a single function. The report on the first phase details the essential aspects of student thinking and highlights underlying reasons that begin to account for the difficulties that students encounter. The report on the second phase outlines how modifications to classroom instruction, based on the findings from the first phase, impacted student thinking about piecewise functions and multiple domain statements.

This two-phase study investigated student difficulties with graphing and understanding piecewise functions. The focus was on how high school students thought about multiple domain statements for a single function, with the perspective that multiple domains may be central to the confusion that they encounter with piecewise functions. In phase 1 of this report, essential aspects of the students' thinking about piecewise functions are detailed, as are the emerging themes surrounding their difficulties. In response to the implications of phase 1, instruction of function was modified for an Intermediate Algebra class. In phase 2, the thinking of several of those students is outlined.

Many students struggle with graphing piecewise defined functions. This is a finding supported by the literature (Chazan & Yerushlamy, 2003; Markovits, 1986) and by my experiences as a secondary school mathematics teacher.

Learner's difficulties with piecewise functions often occur in spite of the students possessing necessary prerequisite skills, such as the ability to graph individual functions from equations and the ability to apply the vertical line test for functions. Graphing calculators also tend to be limited in their support for graphing and hence, reasoning about piecewise defined functions. Even when graphing calculators are used for displaying piecewise function graphs and tables, "…graphing representations are not transparent" and "…graphing representations are complex" (Chazan & Yerushlamy, 2003, p. 131).

Graphing and reasoning about piecewise defined functions is important for at least two reasons. First, analysis of discontinuous functions and one and two-sided limits found in Calculus cannot be fully appreciated without an appreciation for piecewise functions. Second, a strong understanding of piecewise functions is important because it leads to a deeper understanding of domain that more regular functions cannot offer (Oehrtman, Carlson & Thompson, in press).

My personal motivation for investigating student difficulties in this area is based on my work as a classroom teacher, watching students struggle. I have witnessed what could describe as a paralysis of action. Students see the piecewise function equations and begin to graph one of the pieces. But then, they seem to encounter a block and they soon stop, unable to manage the additional pieces. Multiple domain statements appear to be a main source of confusion.

For most functions that students encounter in high school, the domain statement is a singular expression. It states what the entire domain is or in some cases what the domain is not. Even

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with periodic functions like f(x)=tan(x) the domain can be represented in one statement by making reference to periodicity. As a result, multiple domain statements may seem foreign to learners, which may lead them to ignore multiple domain statements.

This study was created to address the following goals: (1) To gain a better understanding of student thinking on how domain, in general, is related to its range, (2) To determine how students view multiple domain statements for one function and (3) To gain insight into how students think about the domain of one piece, in the context of multiple domains.

Theoretical Background

Several theoretical considerations have influenced this study. The first is a radical constructivist perspective on learning, in which learners form schemes about the external world based on experiences, activities and reflections (von Glasersfeld, 1995). From this perspective, it is the researcher's job to build viable models of student thinking.

The second theoretical influence on this study is previous research on student understanding of function. Markovits (1986) found, for example, that translations of most functions were easier for students going from equation to graph than from graph to equation. However, for piecewise functions the translations were equally difficult. Markovits also found that students showed a lack of attention to the domain and range restrictions when asked to decide if a discrete function was in fact a function and whether the points of a discrete function should be connected. This lack of attention may compound in graphing piecewise functions. In another study by Bell and Janvier (1981), it was shown that students focus more on individual points than on functions or graphs in a more global way, such as by considering the domain.

The third theoretical consideration is that many students have more of an action conception of functions than a process conception, a classification system developed by Breidenbach et al (1992). According to Oehrtman, Carlson, & Thompson, "from an action view, input and output are not conceived except as a result of values considered one at a time, so the student cannot reason about a function acting on entire intervals" (p. 8). Alternately, with "a process view, students are freed from having to imagine each individual operation for an algebraically defined function" (p. 9). Students would most likely experience difficulties in understanding piecewise function domain statements if they thought about each input value separately.

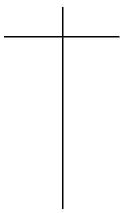
Methods

The subjects for phase 1 of this study were two high school Precalculus students in a mixed Precalculus/AP Calculus class from an inner city school with highly diverse ethnic representations. The students that participated would be rated as average, in mathematical ability, in relation to the rest of the class. Both students' algebra skills seemed to be founded on memorized procedures rather than on conceptual understanding.

Interviews were conducted one-on-one, with the interviewer presenting 4 tasks/activities and the student thinking aloud as she/he attempted each task. Interviews were audio taped and transcribed for subsequent analysis. Transcripts were coded for student difficulties and emergent themes were identified through cycles of searching for confirming and disconfirming evidence (Strauss & Corbin, 1990).

The first task was for students to look at a piecewise discontinuous graph, a piecewise continuous graph, a discrete graph and a table of values that strongly suggested a discontinuous piecewise function. The students were asked to decide which examples depicted functions. The second task was for students to look at a piecewise function equation and to explain their thinking around the domain statements. The third task was for students to examine three continuous smooth-curve functions that had been drawn on a whiteboard. They were instructed to erase some of each function so that what remained could be one function. They were also asked to explain and write down their ideas about the domain for each "piece." The fourth task involved three transparencies with a different function graphed on each one. The three transparencies were stacked and the graphs lined up and stapled together. The students were instructed to make two vertical cuts in the transparency stack and then to take one piece from each graph and create a new graph. They were also asked to share their ideas about the domain for each piece.

Activity 1 – How, if at all, do the following relate to functions?



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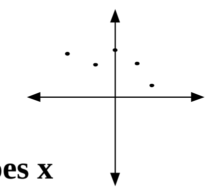

Activity 2 - Look at the following. What does x represent in each domain "zone?"

$$f(x) = \begin{cases} x^2 + x & x \le -2 \\ x - 2 & 0 < x \le 3 \\ x^3 & x > 3 \end{cases}$$

Activity 3 - Study these three functions (drawn on the same whiteboard in three different colors). Erase some of *each* function so that what remains is a function. Write down the domain for each piece that remains.

$f(x) = (x+4)^2$ $g(x) = -(x+2)+4$ $h(x) = \sqrt{x} - 4$

In light of the emergent themes from phase 1 and conjectures about what would improve student understanding, which will be discussed later in this paper, I designed an alternative approach to teaching functions to my Intermediate Algebra class, in the same school that the phase 1 data collection occurred in. I devoted the majority of class time to investigating piecewise function notation, piecewise function domains and ranges, boundaries between pieces of a piecewise function, piecewise graphs, piecewise tables and piecewise distance/time graphs. I also created activities that resembled the phase 1 tasks, where students created piecewise functions by cutting and pasting. I anticipated that this would aid in their developing a process view of piecewise functions. Only after exposing students to piecewise functions did I introduce the definition of a function, function notation, operations with functions, composite functions and the vertical line test.

In phase 2 of this study, I chose three Intermediate Algebra students to participate in similar interviews as in phase 1. Based on their unit tests on functions and on their classwork, I chose one student that showed proficiency with functions, one that showed basic skills and one that showed below-basic skills. I also chose students that I felt would be able to communicate their ideas well. The interviews in phase 2 occurred approximately three months after the unit on functions was taught and no review of piecewise functions was conducted prior to the interviews.

Discussion

In phase 1 of this study three themes emerged with respect to student difficulties with piecewise functions. The first theme was that students relied on recognition of functions more than they did on a definition of function. This was clear in several instances. During the first activity, L concluded that the discontinuous graph was not a function because "…it just looks like lines. I've never seen a function like this before." Similarly, L was looking for a pattern in the table. "It looks like one [a function] up to here." But then she pointed to the pattern change and concluded, "if it would be graphed out it would look crazy." L was looking for recognizable features rather than applying a definition or a criterion for what makes a function.

D also used recognition to determine which representations in the first activity were functions. For the continuous piecewise graph, he recognized the parabolic piece and concluded that this was also an example of a function. He also concluded that the discrete graph, "looks like a polynomial function," because it went "up and down."

In activity 3, D created a graph without discontinuities. However, his graph violated the vertical line test for functions. When asked about how he decided where to erase, D said, "I tried to make it look like a parabola…(pointing) part looks like a parabola." He also said, "I know it's a function because it crosses the x and y." These remarks are similar to his responses to activity 1, where he was looking for recognizable features to help him make choices for what to erase.

Surprisingly when both students were asked for a definition of function, L responded that, "…a function is something that you can put into a problem. If you put something in, a function will come out." She started to draw a picture to represent a machine as metaphor for a function and also spoke about a relation between x and y. D had a more formal definition. He stated that a function is "an equation with a domain and range in it…always a different output for a function." It seems he had some notion of mapping. Nevertheless, both students failed to enlist these ideas in their discussion of the graphs and table.

The second theme that emerged with respect to student difficulties with piecewise functions was that students felt overwhelmed when dealing with multiple domain statements. This sense of being overwhelmed manifested itself in the students as tentativeness. As a result, both students employed simplification strategies, which I took to be a sign of coping with the overload.

For example, although L showed that she had some understanding about multiple domain statements, when she was asked to write domain statements for her piecewise functions in Activities 3 and 4, she was very hesitant. After thinking for quite a while, she chose to write just one number in each domain. If a root was present, she chose that as the domain and if there was no root she chose a boundary point or the maximum value. She knew that x should represent more than one number, but still simplified the task to finding one number. It seemed that L was still engaged in considering functions as an action, dealing with one number at a time and that multiple domain statements were overwhelming.

At first D was also very tentative and unsure about multiple domain statements. He viewed the inequality statements in step 2 as a system of inequalities where there was one number that satisfied each statement. Rather than having to think about sets of numbers, D reduced the task to thinking about one value. However, when D realized that the domain zones did not overlap and thus no solution was possible, he changed his mind and said “No, three numbers, one in each…because it can’t be greater than three and less than three.

Interestingly, during Activities 3 and 4, D changed his mind. When asked to write the domains for the piecewise functions he had created, he now thought that x represented many numbers. He was also able to write the domains of each piece in both activities, although he was not confident enough to write them as inequality statements. For example, for one piece he wrote “-5 to -2,” which suggests that he may still not have been connecting the set of real numbers from -5 to -2 with x but may have been reducing the domain issue to one of determining the boundaries of a piece. Also, D did not realize that the overlap of domains meant that one of his pieces violated the vertical line test. But at least, it seemed like he was ready to shift his perspective whereas L did not seem as ready.

The third theme that emerged was that the students did not consider domain when they thought about or graphed functions. In Activity 1, neither student thought about domain in order to evaluate whether the tabular and graphical displays were functions. In Activity 2, neither student was able to connect the domain statements with the function equations in a meaningful way. To L, they were unrelated inequality statements. To D, they were a separate problem to be solved. In Activities 3 and 4, L clearly showed that she made no meaningful connection between the domain and the functions she created. Although she had talked about inputs and outputs for functions and did complete both activities, domain was reduced to single values. D showed some signs that he was beginning to think about domain in a new way, but more as an afterthought rather than as an important characteristic of a function. His creations were mainly driven by a desire to create a familiar function and a continuous function.

The three themes led me to make two conjectures about student experience with piecewise functions. First, I felt that students being exposed to special functions or what Schwartz and Hershkowitz call “prototypical” functions (1999), like linear and quadratic functions, first, makes them less inclined to consider piecewise functions as functions. Second, I felt that the way domain is normally taught, where regular graphs can be graphed without consulting the domain and often the domain is determined as a post-graphing exercise, leads students to view domain as superficial descriptor. I therefore altered the way I taught function, so that students would be exposed to piecewise functions sooner and more often, with the conjecture that piecewise functions will be seen as more normal and that domain be seen as a vital component of a function.

In the phase 2 interviews I reexamined the themes from phase 1. The first theme, that students rely more on recognition instead of a definition of function, was far less evident in the phase 2 interviews. The below basic student, Y, incorrectly rejected the three graphs in Activity 1 as not being functions “...’cause this one’s not like connected…and this one is not like a straight line…same with this one.” However, when asked when a graph is a function, he responded, “I would say a graph that…have numbers that keep going up…as the x increases the y increases.” This seems to show he had less of a reliance on the prototypical functions than the students in phase 1 had.

Interestingly, the basic level student, M, was so comfortable with piecewise functions that his criterion for choosing all three graphs as functions, was based on the fact that they were piecewise. “This one is a function because it has three equations.” “This one too…because it has…two starting points and two end points…A function should have one or two equations.” It seemed that for M, a graph was a function, by default, if it was piecewise. He was unsure about the table however. "I am not sure about this one now…because it looks like the dots are just going up, I'm not sure. I think no…because like I said a function should have two or three equations and…I think this one have only one equation." I can see the influence of the change in order of instruction in M's responses. While some of his thinking was flawed, there is a big difference between his criterion and the phase 1 students.

The proficient student, T, showed a fairly advanced understanding of functions. "I think all of them are (functions)…'cause they don't have repeat of the same numbers with different points (table) and this passes the vertical line test which no two points hit and the same with C and D…no two points on the same x axis when you draw a straight line down…if it has two points on x with a different y, yeah that's not a function." T's criterion relied on function characteristics rather than on recognition of familiar functions. In summary, all three students showed more understanding about function then the phase 1 students, although Y's understanding was only marginally so.

The second theme showed a similar trend. The two more advanced students showed less indication of being overwhelmed by multiple domain statements for functions that were presented to them and therefore relied less on simplification strategies. Y, the weaker student, began thinking that x in the domain statements was an unknown to find. "The x here you have to find out what is greater than three like four…x is like a number that you need to find." However, he later changed his mind. "Oh, four to infinity. I'm just saying it couldn't be a negative number or like two or less than three." His explanation for how the domain statements and the function equations are related was to say, "…it's the same thing. x could be this answer or this answer." I took it that Y meant that the domain statement and the equation are just repeating the same information.

M had no difficulty with there being three domain statements. "If you graph it, this is just how they are showing you where you should start and where you should stop…no, where the line should stop." Where he got confused was in the inequality statements. He was somewhat careless in reading the inequalities and made a conclusion that $x\leq-2$ was superfluous since $x<3$ overlapped $x\leq-2$. This would have been true except that $x>3$.

T also showed no signs of being overwhelmed by multiple domain statements. "I guess its trying to show you that any number you put in here…like negative five is lesser than negative two…and for the second one, whatever minus two is gonna be greater than zero but lesser than three or equal to, and x cubed, yeah, is always gonna be greater than three." He was the only student of the five who was able to talk about the equations and the domain statements simultaneously. M and T's responses indicated to me that they were on their way to developing a process view of function.

The third theme, that students don't consider the domain when graphing or thinking about a function, remained true for all three students. In Activity 3, Y, M and T each created a graph that violated the vertical line test, which shows that they were not attending to domain issues. As well, the students were unable to correctly identify the domains of each piece. Y's domain statements seemed to be an attempt to create similar looking domain statements with random numbers. He was unable to justify or give reasons for his statements $x<4$, $3>x>5$ and $x<5$ and the graphs did not reflect these values. M identified a boundary point for each piece rather than a set of values, similar to L from phase 1. For Activity 4, M again chose a boundary point for the left most and right most pieces and wrote $x<1>7$ for the middle piece (which should have been

$1<x<7$). M was probably trying to express both end points in a way he had seen in a previous activity without a full understanding.

T realized that his graph violated the vertical line test, even though the instructions said to create a function. I interpreted this to mean that T saw the vertical line test and the domain as things one thought about after a graph was created, rather than before or during. He also misidentified the domain, giving x and y coordinates of end points or local extrema instead of a set of values. M and T had a much better sense of domain when presented with a pre-made piecewise function than when thinking about a piecewise function of their own making.

Conclusion

The new approach to teaching function seems to have had some impact on student thinking. The students in phase 2 were more inclined to apply function rules to determine what was and wasn't a function then the phase 1 students. The phase 2 students also appeared to be less overwhelmed with multiple domain statements for piecewise functions that were presented to them and thus did not employ simplification strategies. However, when the phase 2 students were asked to create their own functions, the action view of function returned. They seemed unable to maintain a more holistic perspective of their function. This suggests the conjecture that it may be helpful for students to work in pairs. One student could create a function and the other could describe the function, thus helping the first student see their function from a process view.

While these conclusions are not completely positive, they are encouraging, especially in light of the fact that the phase 2 students were enrolled in a lower level class then the phase 1 students. It would be interesting to see the effect on student thinking of several years of a greater emphasis on piecewise functions and multiple domain statements.

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THE RELATIONSHIP BETWEEN TEACHERS AND TEXT: IN SEARCH OF GROUNDED THEORY

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The lack of current theories to guide analyses of the interplay between teaching and texts (cf., Ball and Cohen, 1999; Goldsmith and Schifter, 1997; Simon, 1997) became problematic in our ongoing professional development work with teachers. In order to address this problem, we began the development of an interpretive framework and associated analytic constructs along with a data collection method that defines the data corpus needed for such an analysis. As a result, our current work focuses on the refinement of the framework and associated constructs in the course of our ongoing analysis. Our goal is therefore to articulate by example our interpretive framework and constructs in the context of design research. In doing so we provide tools for clarifying the relationships between teachers and text.

Introduction

Both current and historical approaches to textbook implementation have been premised on the belief that teachers can be trained to implement instructional texts with fidelity and that this *fidelity to the curriculum* will lead to increased student achievement (Fullan & Pomfret, 1977; Snyder, Bolin, & Zumwalt, 1992). Snyder et al. state that a focus on fidelity entails "(1) measuring the degree to which a particular innovation is implemented as planned and (2) identifying the factors which facilitate or hinder implementation as planned" (p. 404). In this approach, support resources are designed to ensure the developers' intended enactment of the text. Teacher decision making is relegated to following scripted procedures outlined in teacher guides. In these settings, teachers can be de-professionalized and the text can become the primary means of students' learning. A fidelity approach to implementation gives agency to text resources and places strict adherence to the text as the goal of teaching. This approach stands in stark contrast to the goals of an approach to implementation that characterizes the text as a tool (cf. Meira, 1995, 1993; van Oers, 1996, 2000) and teachers as designers. In these latter settings, teaching is responsive to students' contributions and the interplay of text resources, mathematically significant discussions, and teacher intervention creates the setting for learning to occur.

When the emphasis of instruction is on building from students' current understandings, classroom interactions cannot be scripted. As a result, this type of complex engagement cannot be reduced to manuals, text resources or guides. This sentiment is captured by Carpenter and colleagues when they claim, "teaching is complex, and complex practices cannot, in principle, be simply codified and then handed over to others with the expectation that they will be enacted or replicated as intended" (Carpenter, Blanton, Cobb, Franke, Kaput, & McClain, 2004). This makes the notion of codifying teaching and handing it over in the form of teacher guides as an image of implementation untenable.

However, in this paper we argue that the degree to which administrators and designers (and teachers) hold a fidelity view influences the manner in which teachers are forced to grapple with implementation. In our ongoing work in schools, we have in fact documented the tensions

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inherent in conflicting views of implementation (cf. Cobb & McClain, 2004). We have found evidence of these tensions in analyses of (1) administrators' views and beliefs, (2) teachers' perceptions of district expectations, and (3) teachers' classroom instructional practices related to the use of text resources.

The lack of current theories to guide analyses of the interplay between teaching and texts (cf., Ball and Cohen, 1999; Goldsmith and Schifter, 1997; Simon, 1997) necessitated our development of an interpretive framework and associated analytic constructs along with a data collection method that defines the data corpus needed for analysis. As a result, our current work focuses on the refinement of the framework and associated constructs in the course of our ongoing analysis. Our goal in this paper is therefore to articulate by example our interpretive framework and constructs.

In order to understand how the framework and constructs emerged from our work, it is necessary to situate our development efforts in the context of our commitment to a design research perspective (cf. Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2004; McClain, 2004). We employ a design perspective in our work in schools so we naturally took a design perspective in the development of an interpretive framework. Our perspective on design research entails the development of a conjectured trajectory to guide initial activity. In the case of the development of an analytical framework, this involved conjectures about both the setting of teachers' work and their orientation to mathematics instruction. Our goal in the course of data collection was to document both of these aspects of practice. The next step involved developing constructs to use in the analysis. During the first round of analysis, we were able to operationalize the constructs. However, it was only in the course of iterative cycles of conjecture, data collection and analysis that we were able to refine both the framework and the constructs. As a result, we engaged in cycles of conjecture and revision. This was made possible by our work in multiple sites. A conjecture that resulted from analysis at one site was tested and refined in the course of subsequent analyses at another site as shown in Figure 1.

Our framework has therefore been developed and refined in the course of three iterations of conjecture, data collection and analysis. diSessa and Cobb (2003) make a strong argument for design-based theorizing in their characterization of a genre of theorizing that they claim is "strongly synergistic with design-based research" (p. 177). What we offer here is then a mezzo-cycle of design in building towards theory.

Although the constructs we have used in our analyses have proved helpful, we find that the lack of theory to guide our analysis places our work in the space between analysis and anecdote. Remillard (in press) shares this concern in her review of the literature on research on teachers' use of mathematics curricula.

> . . . a number of scholars over the last 25 years have studied how teachers use curriculum materials and the role that textbooks and curriculum materials have played in mathematics classrooms. However, findings from these studies have not been consolidated to produce reliable, theoretically grounded knowledge on teachers' interactions with curriculum materials or to guide the work of those involved in the design or implementation of curriculum.

She continues with a call for theoretical work in the field. "My primary assertion is that the current body of literature rests on under developed theoretical ground."

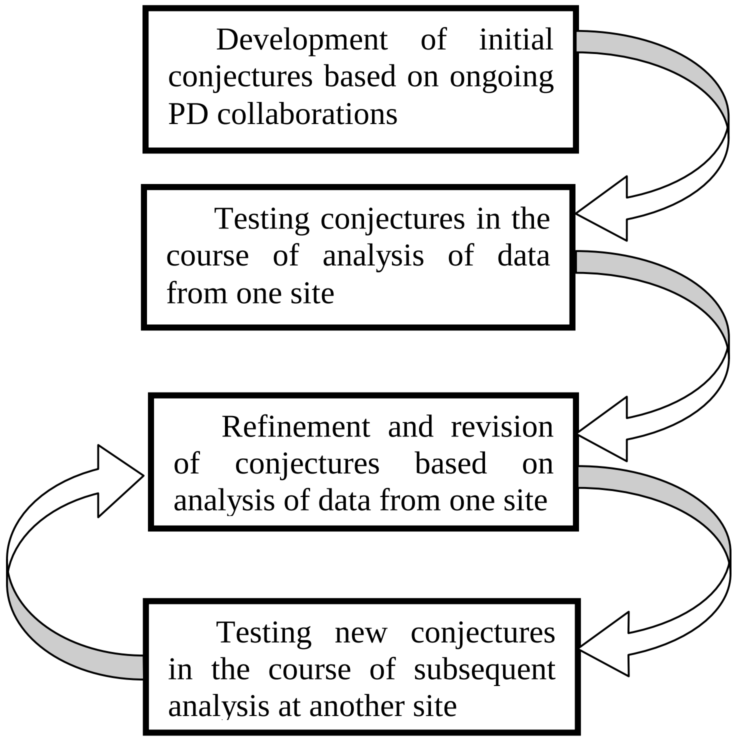

Figure 1. The design research cycle of the development of constructs.

We concur with Remillard and use our analyses and the development of constructs to propose an interpretive framework, or what diSessa and Cobb (2003) call an *orienting framework*. diSessa and Cobb make a distinction between orienting frameworks, frameworks for action and grand theory. They note that the development of grand theory emerges from a series of testing and revising frameworks and constructs in multiple settings. The strength of a framework or a construct lies in its ability to be predicable across settings. Therefore, by continuing this iterative approach to analysis and theory development, we propose that this orienting framework has potential as a theoretical tool for analyzing the interplay of mathematics teachers' practice and their instructional texts. We believe that all work should be in service of or in the development of theory. Hence our commitment to a design research approach to the development of a framework and constructs.

Theoretical Perspectives that Guided Analyses

We incorporated two theoretical perspectives into our analysis of the data in order to make sense of the complex dynamics involved in teaching and the institutional contexts through which it is enacted. First, we view teaching as a *social practice*. That is, we see the relationship between *social structures* (e.g. institutional settings – the classroom within the school and the school within the district), and local *events* (i.e. teachers' enactment of current instructional decisions within the context of the classroom) as mediated by the social practice of *teaching* (Fairclough, 2004). Second, we view teaching as a *distributed activity* and therefore situate teachers' instructional practices within the institutional settings of the school and school district.

We know from both first-hand experience and from a number of more formal investigations that teachers' instructional practices are profoundly influenced by the institutional constraints that they attempt to satisfy, the formal and informal sources of assistance on which they draw, and the materials and resources that they use in their classroom practice (Ball, 1993; Brown, Stein, & Forman, 1996; Cobb, McClain, Lamberg, & Dean, 2003; Feiman-Nemser & Remillard, 1996; Nelson, 1999; Senger, 1999; Stein & Brown, 1997). We therefore situate the analyses of the relationship between teacher and text in the broader analysis of the institutional context by drawing on the analytic approach proposed by Cobb, McClain, Lamberg and Dean (2003). In particular, we focused on the relationships between groups (or communities) within the district by analyzing the *boundary objects*, *boundary encounters*, and *brokers*. In each school district the analysis resulted in different institutional contexts. In two instances, there were no brokers. This made it difficult for the boundary objects to carry meaning across communities. In addition, the lack of boundary encounters in these settings was therefore limited. Our most successful setting was therefore the one in which brokers served a viable role in linking communication between teachers and administrators.

Data Corpus

Our data corpus is taken from three different school districts. One is from a southern east coast state, two is from a southeastern state, and three is from the west. Our data consists of modified teaching sets (cf. Simon & Tzur) of each teacher conducted twice a year. This set of data consists of a pre-observation interview with the teacher in which we have the teacher outline the goals for the lesson and her expectations for her students. We then conduct the observation, taking copious field notes. The observation is followed by a post-observation interview in which we ask the teacher to reflect on the lesson. In particular, we are interested in modifications made to the lesson (if any) in response to students' contributions. All activities are audio- or video-taped.

Proposal of Constructs

Although our primary work in each district is the ongoing professional development of communities of teachers, we have been unable to achieve our goals for teachers without understanding the role that text resources play in their instructional practice. We therefore placed the analysis of the institutional context in the background as we began to analyze the data in search of explanations for the relationships between the teachers and their text. In doing so, the first construct that emerged was that of the teachers' *instructional reality*. [Elsewhere we have clarified in detail this construct (see Zhao, Visnovska, & McClain, 2004), so we will limit our discussion here.] By instructional reality we mean the teachers' perceptions of their institutional context. This included the constraints and affordances that they perceive as both supporting and hindering their ability to develop their practice according to their personal philosophies. As an example, in many schools in the United States, teachers do not feel they are empowered to deviate in any way from the mandated curriculum due to high-stakes accountability testing – even when they perceive their current practices as not being in the best interest of their students.

- While the notion of instructional reality was instrumental in our understanding our first site, it did not provide enough analytic power to explain the similarities and differences across sites one and two. For this reason, the construct of *agency* was introduced in our analysis. By agency we mean the location of authority as perceived by the teachers. This is directly related to instructional reality, but further teases out the role of texts in teaching.

- At present, we have only introduced our third construct, *professional status.* Again, the need for an additional construct emerged as we began analysis of the third site. While there were many similarities in teachers' instructional reality and where they placed the mathematical authority, we were unable to explain why the teachers at site three were willing to modify the text, delete sections of the text, and even introduce additional text resources. Our current conjecture is that the difference can be explained in terms of professional status. By this we mean the manner in which the district values or does not value teachers' independent instructional decision making.
- At this point it is important to clarify the interrelated nature of the three constructs. The relationship between teachers and text cannot be separated from teachers' instructional reality, nor from their placement of agency that is tied up in their professional status. We therefore argue that while these constructs can be analyzed independently, it is the interrelated nature of these analyses that allow us to better understand the relationship between teachers and texts. Further, at times it may be difficult to determine if an analysis of certain data lies within just one construct. For that reason, we propose an overlapping view of the constructs as shown below.

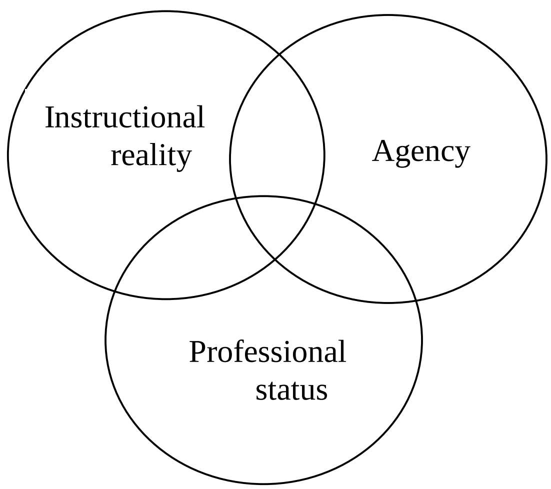

Figure 2. The interrelated nature of the three constructs.

Conclusion

We view the offer of our interpretive framework or *orienting framework* as a first step toward the development of a guiding theory. By taking a design approach to theory development, it then becomes possible for theory to "delineate classes of phenomena that are worthy of inquiry and specify how to look and what to see in order to understand them" which, in turn, "teach[es] us how to see" (diSessa & Cobb, 2003, p.79). The development of theories to guide analyses of the relationship between teachers and texts therefore requires the field to engage in serious critique and analysis of our own and other's work. The cyclic process of analysis and critique allows the field to consistently build from what is already known and therefore move forward. This process then creates the opportunity for theory to emerge from practice in a systematic, disciplined manner.

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HOW FIFTH GRADE TEACHERS USED INVESTIGATIONS IN NUMBER, DATA, AND SPACE: A STANDARDS-BASED CURRICULUM

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Achieving the vision of standards-based curriculum is problematic for elementary school practicing teachers in urban school districts. This study explored the salient instruction features of what goes on in classrooms that use a standards-based curriculum. This include lesson plan, activity sequence, questioning, classroom management, discipline, class environment, discussion, small group work, reflection, journaling, modeling, respect, humor, etc. This study found that teachers' content knowledge, dynamic and vibrant use of artifacts, and respectfulness were important components of inquiry-based instruction perceived to enhance conceptual understanding in mathematics teaching and learning. Questioning, responding, negotiation of meanings, and active listening were other significant components for successfully implementation of a standards-based curriculum. It also found lesson debriefing vital no matter how stretched of time a teacher was

Generally, in the USA, concerted efforts are under way to move students' thinking from an instrumental and procedural understanding of mathematics to a relational and conceptual understanding (e.g. Senk & Thompson, 2003; Stigler & Hiebert, 1997). Studies have shown that inquiry-based instructional strategies enhance students' conceptual understanding (Boaler, 1998; Flower, 1998; Reys, R. Reys, B., Lapan, & Holliday, 2003) and individual meaning-making. Inquiry-based instruction is better than traditional mathematics which is seen as dominated by teacher direction, student mastery of rules, and procedures learned by memorization.

An important yet little understood question is - how do teachers negotiate the demands of a new curriculum with their established practice and their own pedagogical beliefs? What are the competing factors that teachers have to negotiate to effectively implement standards-based curricula? Several mathematics standards-based studies at all school levels (Cramer, Post, & delMas, 2002; Flower, 1998; Goodrow, 1998a; Mokros, 2000) profess that students who use these curricula materials attain higher test scores on examinations, as well as on measures of conceptual understanding. These studies are concerned with the effects of curriculum on student achievement and fail to enlighten practicing teachers on the classroom cultures that existed for these achieving students. Current literature on standards-based curricula does not reveal the salient classroom features of what goes on in schools that have adopted these types of curriculum textbooks.

Purpose of the Study

The study explore modes of practice in implementing *Investigations in Number, Data and Space* [Investigations] (TERC, 1998) mathematics units by fifth grade teachers in an urban school district. *Investigations* is a K-5 standards-based curriculum whose objectives offer students connected and meaningful mathematical problems to promote in-depth thinking. The curriculum is designed to develop students' conceptual understanding and critical thinking skills, and it encourages the use of inquiry-based instruction. Mokros (2003) notes that the *Investigations* require students "to develop their own strategies for solving problems, compare

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

approaches with their peers, and engage in serious discussion about differences in their strategies and results" (p. 113). *Investigations* has the dual purpose of also communicating "mathematics content and pedagogy to teachers" (Tierney, 1998, p. I-1).

Implementing the *Investigations* requirements has created a dilemma since a majority of practicing teachers have learned, observed, practiced and taught traditional mathematics. Teachers are wrestling with shedding their old pedagogical beliefs, weighing the inquiry-based teaching strategies, mathematics content, and learning how to use *Investigations* (Putnam, 2003). The discrepancy between the implementers' prior experiences, National Council of Teachers of Mathematics standards and principles, and *Investigations'* objectives presented an important problem for study. How a teacher makes choices among the mathematical strategies in the textbook, plans a lesson, launches it, and sustains it were explored in this study.

Theoretical Framework

Inquiry-based instruction is multifaceted. It encompasses both subtle factors and non-subtle ones. This could be a teacher's competency in mathematics content or a teacher having a highly-structured class. This study postulates that a teacher facilitates inquiry-based instruction by planning out an activity, launching it, and creating a safe environment for exploration. This includes sharing and summarizing the activity either in a whole-class setting, or as collaborative small-group work, or between pairs of students. In organizing such groupings, each student is forced to rise above self by building on the contributions of other members. Both the teacher and students promote classroom discussion by asking probing questions, the 'why' and 'how' types of questions. Students influence one another, the teacher influences students, and students in turn stimulate the teacher's ways of thinking by their divergent thoughts. In the process, students come to understand mathematics, make connections, and are enabled to communicate or use the new knowledge.

This study attributes any practice not perceived as inquiry-based to lack of teacher content knowledge, teacher pedagogical knowledge, and teacher pedagogical beliefs. Studies embedded in these three areas also points to that too. First, teacher content knowledge refers to subject matter, in this case mathematics (Ball & Bass, 2000). Shulman (1987) says that content knowledge is "the knowledge, understanding, skill and disposition that are to be learned by school children" (p. 8-9). Fennema and Franke (1992) remark that teaching become difficult without understanding content knowledge. Second, pedagogical content knowledge is seen as knowledge for teaching (Ball and Bass, 2000) a specific content. Shulman (1987) states that pedagogical content knowledge "represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (p. 8). Third, teacher pedagogical beliefs do stem from prior school experiences (Brown & Borko, 1992), including experiences as a mathematics student, the influence of previous teachers or of teacher preparation programs, and prior teaching episodes. According to van den Berg (2002), teachers pedagogical beliefs are personal truths that "typically reflect the teacher's opinions regarding the processes of teaching and learning" (p. 579).

How well a teacher negotiates the content knowledge, the pedagogical knowledge, and pedagogical beliefs would make them effective at using inquiry-based strategies. Accordingly, the study employs negotiation of meanings in its theoretical considerations. This study views the instruction of mathematics as the negotiation of practices of school mathematics with the teacher as initiator. Negotiation in this study involves reasoning, interpreting, and making sense of mathematical meanings. Frid (1994) defines negotiation of meanings in mathematics classrooms as "specific mechanisms of classroom interactions by which teacher conjecture, criticize, explain, test and refine ideas and procedures …" (p. 271). This study attributes any practice perceived as inquiry-based to teacher's negotiation of meanings.

Method of Study and Data Collection

A qualitative case study research design was used to explore the teachers' emerging practice models of implementing Investigations' mathematics in urban fifth grade classrooms. Data were collected through open-ended interviews, classroom artifacts, audiotape and videotape of lessons, group meetings, lesson plans, lesson observations, and post-lesson conferences. Data were collected during fall 2003 from three fifth grade classrooms at Lake (pseudonym) Elementary School, with student ages ranging from 10-12 years. Lake is in a large city in a New York State school district.

During preliminary analysis of data encompassing all elementary schools in this district, purposive sampling was used to identify this research site and the study's participants. An assumption of the study was that if all the major extraneous school factors were the same and static then the methods teachers used in fifth grade would be similar. Thus of interest was a school known to have implemented the *Investigations* curriculum for over five years continuously. Lake Elementary School happened to be most suited to this study. The concept of inquiry-based instruction was not new in this school. Lake also had availability of *Investigations* materials, school administrators' support, and equal distribution of students in class size and in their academic abilities, and so on. All the teachers at Lake used *Investigations* materials; additionally, teachers were focused on instructional issues, a key aspect of this study. These teachers concentrated on instruction using the *Investigations* curriculum and formed learning communities to develop professionally through lesson study. Another assumption of the study was that a teacher who closely taught at least six observed lessons as communicated by *Investigations* was seen as effectively employing inquiry-based instruction in this study.

The study employs analytic induction in its data analysis (Bogdan & Biklen, 2003). For example, I analyzed the first five classroom observations by coding the field notes and listening to the tapes for recurrent themes and patterns. I further pursued those themes that emerged in-depth. This helped me to focus my attention on issues pertaining to inquiry-based instruction. Post-lesson conferences were held with each teacher to understand their perceptions of the *Investigations* curriculum. Examples of questions posed in this meeting were: "How would you teach this lesson next time?" "Tell me about the lesson." Also data were collected through in-depth interviewing. This is a tool for studying phenomena. In this study the phenomenon is 'inquiry-based instruction' and my interest was to understand the experience of the teachers and the meaning they make of the phenomenon. Once a month, I carried out a 30-minute interview with a participant, where they answered open-ended questions. Open-ended interview questions arose from the classroom observation notes and were connected to the purpose of this study except for the first interview. This is when I gathered demographic data about the teachers and familiarized myself with them to create a good rapport. All interviews, short conferences, etc were audio taped, transcribed and the data were analyzed to reflect the teachers' voices and narratives.

Analysis and Interpretations

Due to page limitation I will discuss only one of the analyses conducted. I used the components of inquiry-based instructions that were in the data to describe the emerging realities of inquiry-based instructional models. These were:

- listening;
- use of small groups;
- partners;
- responding;
- teacher-led whole class discussion;
- student-led whole class discussion;
- questioning e.g. "How many dimes in a dollar?";
- discourse;
- interaction;
- use of games;
- use of manipulatives;
- choice time activities;
- demonstration;
- students' work on "post-it";
- journal entry;
- calculators;
- charts; posters;
- problem solving; problem posing; "redirecting them"; independent work; lecture;
- discipline i.e. "What could you have done better?";
- and creation of a safe learning and teaching environment, among others.

In analyzing the data, I summarized the inquiry components to include the following categories: launching a lesson, sustaining a lesson, summarizing a lesson, discussions, small groups, questioning, student reflections, humor, play-and-learn, and one-on-one interaction of teacher with student. Play-and-learn categories consisted of choice times and exploring classroom activity in the *Investigations* units. These aspects of inquiry-based instruction are based on this study's purpose of: what are teachers' ways of planning, launching, exploring, engaging students, sustaining students' learning, and patterns of summarizing an *Investigations* activity?

I analyzed six sets of classroom observation data that were collected from late October 2003 to mid December 2003 in the three classes to quantify how the teachers used the above-mentioned components in their instruction. These data were collected at a time of the year when everyone was comfortable with my presence in the classroom. The classroom's sociomathematical norms and routines had been set and the teachers spent more time on mathematics instruction. I gave a score of one if a component was used on the observation day. A summary of this quantification can be seen in Table 1. A visual side-by-side bar chart of the Table 1 summary is given in Figure 1.

Inquiry Components	Zsa	Nora	Jesse
Launch Lesson (LL)	6	6	6
Sustain Lesson (SL)	2	6	6
Summarize Lesson (SuL)	1	2	6
Humor, play-and-earn/Explore Activity (EA)	5	3	6
Discussion (Ds)	3	4	6
Small Groups (SG)	5	0	6
High-level Questioning (HQ)	2	5	6
Student Reflection (i.e. journal) (SR)	0	2	4
One-to-one student-teacher Interaction (OI)	4	1	6

Table 1: Summary Table of Inquiry-Based Components

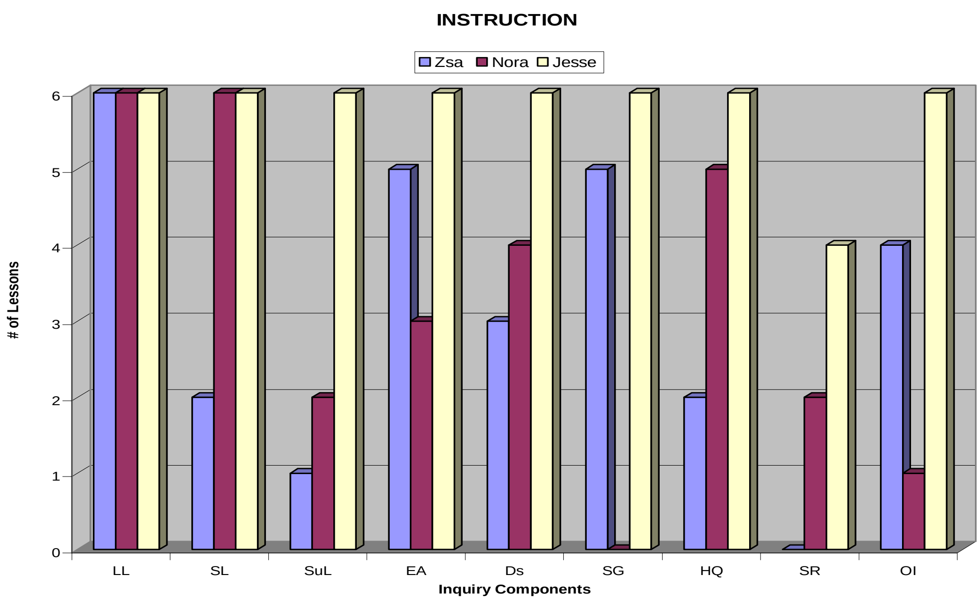

Figure 1: Bar chart of Inquiry-based Components

Table 1, figure 1 and other analyses lead to three models of practice. The instructions in all the three classrooms were dissimilar despite several factors being the same at Lake. These were labeled as the *partial inquiry-based* instruction, *traditional inquiry-based* instruction, and *inquiry-based* instruction. I label Zsa's practice as *partial* inquiry-based instruction model. I view her instruction as calling for more expert support to enable the inquiry components to become her day-to-day reflection-in and reflection-on practice. She understands the approaches implied in the *Investigations* curriculum, but she has not immersed herself in this type of instruction. She struggles not to fall back to traditional methods of instruction. She knows what she wants, but she has not observed enough of inquiry-based approaches to know all the components that have to be in place. I label Nora's practice as *traditional* inquiry-based instruction model. Her instruction was teacher-centered even though her key curriculum unit was *Investigations.* She told the students what to do and gave little opportunity for the students to construct knowledge. I label Jesse's practice as an *inquiry-based* instruction model. His model matches existing research perspectives on inquiry-based instruction and he also used the *Investigations* materials as communicated by its developers.

In implementing *Investigations* effectively, a teacher must facilitate the lesson launch, sustain exploration and summarize the lesson. A safe environment with humor is conducive for discussion. A culture of listening and responding to one another is easily created in such safe learning communities. Use of cooperative small groups, play, exploration, posing of high level questions, and reflection are key points. From this study, such an environment needs to be highly structured otherwise it would be chaotic. Jesse's instruction illustrates the most effective inquiry-based instruction, yet he is the one that had a highly structured learning environment.

Conclusions and Suggestions

Besides those studies conducted by curriculum developers, few studies to my knowledge have explored closely the various models of instructional strategies that practicing elementary school teachers use in implementing *Investigations* in fifth grade classrooms, which is the focus of this study. Ball (1996) observes that however clear and illustrative curricula materials are, they still fail to "provide guidance on the specifics of day-to-day, minute-to-minute practice" (p. 502); what English (2002) refers to as "issues of significance to the classroom" (p. 7).

In-depth competency in teacher content knowledge, teacher pedagogical knowledge, and teacher pedagogical beliefs enables a teacher to be versatile in using inquiry approaches. This causation is based on the assumption that all the major extraneous factors were the same and static for this study's participants. Despite, several factors being the same such as availability of *Investigations* materials, school administrators support, equal distribution of students in class size and in their academic abilities, and so on, the teachers' models of instruction varied. This is also in agreement with Remillard and Byrans (2004) study which found that teachers' orientations impacted how they used standards-based curriculum.

This study adds to the knowledge base, in that it focuses "on learners through focusing on teachers," which is one of the PMENA (2006) goals. Classroom teachers are the implementers of a curriculum and how they negotiate its meaning impacts learner's fluency and numeracy. This study assumed that inquiry-based instruction model as conveyed in *Investigations* materials would suit this study's participants especially the newly graduated teachers from preparatory programs. However, the study found that teachers in each classroom employed different models of inquiry-based instruction and the novice teacher (Nora) was more inclined to use traditional methods of pedagogy.

The study goes beyond looking at how teachers practice a standards-based curriculum. It found that the teacher preparation programs should align their courses to produce graduates who are capable to handle standards-based curriculum. Again, it found that teachers' collaboration in learning communities assisted in narrowing the gaps in their practice. Teacher's Zsa model of practice attests to this. She learned how to improve her instruction from peer collaborations and lesson study (a model of professional development) meetings. By the end of this study, she enacted *Investigations* materials as required.

This study recommends more concerted effort should be put to pursue the lesson study model of professional development because it holds the key to narrowing gaps in classroom instruction with a hope of leaving no child behind. In considering this teacher-led model of professional development, the issues that influence inquiry-based learning, such as structuring the environment, using students' thinking, and developing good questioning skills, should be addressed.

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REPRESENTATIONS OF MATHEMATICS TEACHER QUALITY IN A NATIONAL PROGRAM

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This research examines grantees' work to improve mathematics teacher quality in a nationally-funded program. The analysis employs qualitative methods using secondary source documents provided by 48 National Science Foundation Math and Science Partnership (NSF-MSP) grantees. Findings show that representations reported for mathematics teacher quality by the grantees mirror those used in previous research. Conditions used to influence mathematics teacher quality included research-based professional development, various roles for teacher leaders, and emerging collaboration between STEM and education faculty for the improvement of mathematics content knowledge for teachers at all levels.

The No Child Left Behind (NCLB) Act of 2001 and related changes to educational policies for teachers have escalated the focus on teacher quality and the need for more well-trained mathematics teachers in the United States. Teacher quality in mathematics has a significant impact on the teaching and learning process. While there is agreement that teacher quality matters, there is less agreement on the variables used to measure teacher quality characteristics (Rice, 2003). As research, policy, and public interests converge around this issue there is a growing impetus to seek answers that improve teacher quality. These interests have resulted in funding for national initiatives focusing on the quality of mathematics teachers. The purpose of this study was to examine the work conducted by grantees in one of these national programs.

Research on Teacher Quality

To establish a background against which to examine what represents and influences teacher quality in a national program, we first examined the literature to determine how these constructs are discussed in research. We identified six primary variables researchers have studied as representations of individual teacher quality: subject matter (content) knowledge; pedagogical knowledge; teacher behaviors, practices and beliefs; certification status; experience; and general ability. We discuss three of these in the paragraphs that follow.

Subject matter knowledge is a valued characteristic of mathematics teacher quality. Reviews of research indicate links between teachers' subject matter preparation and teacher effectiveness, although these results are not always clear (Rice, 2003; Wilson & Floden, 2003). Results of studies examining the relationship between teachers holding subject specific degrees and student achievement vary, although mathematics results are generally positive (Goldhaber & Brewer, 1997). Similarly, studies measuring teachers' subject matter knowledge using undergraduate or graduate coursework in the subject generally show a positive relationship with students' mathematics achievement. While the data suggests a generally positive relationship between coursework and student achievement, there is evidence of a curvilinear effect (Monk, 1994).

Teacher education research often examines teachers' pedagogical knowledge as evidence of teacher quality. These studies use measures such as degrees in education, educational coursework, and scores on exams measuring professional knowledge. Several studies indicate

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the positive effects of teachers' pedagogical knowledge (Ferguson & Womack, 1993). Generally, studies of teachers' pedagogical knowledge have found positive relationships between education training and teacher effectiveness (Darling-Hammond, 2000).

Teachers' behaviors, practices, and beliefs are important characteristics of mathematics teacher quality. Although the observation of teachers' behaviors and classroom practices provides a rich source of data, few large-scale studies have examined these practices. One such report (Weiss, Pasley, Smith, Banilower, & Heck, 2003) found that only 15 percent of observed mathematics lessons were categorized as high quality, while 27 percent and 59 percent were labeled medium and low quality respectively. Further results found that while teachers rarely make decisions about content, they often decide how to teach the content and those decisions are influenced by teachers' beliefs about mathematics, about pedagogy, and about their students.

Methods

The present study is one sub-study of the Math and Science Partnership Program Evaluation (MSP-PE). The National Science Foundation Math and Science Partnership (NSF-MSP) Program is a major research and development effort with grants awarded to partnerships among preK-12 schools and institutes of higher education. One goal of the program focuses on improving mathematics teacher quality. The following research questions guided this analysis: a) How do grantees represent characteristics of teacher quality in mathematics? and b) What conditions do grantees identify as influencing teacher quality characteristics?

The data sources in this study come from 48 grants awarded in three cohorts (FY2002-04) in three categories (Comprehensive, Targeted, and Institute awards). Grantees' reporting requirements called for them to submit Annual and Evaluation Reports describing the grants' yearly activities. These secondary documents were the source for the analysis. Data were obtained from documents available to the MSP-PE team between January 2005 and February 2006. Researchers analyzed 123 reports. The examination was conducted using qualitative methods for a document analysis of secondary data sources (Miles & Huberman, 1994). The unit of analysis was the individual grant. Researchers analyzed documents in three phases. Six readers used an analytic protocol to code information and write summaries in the first phase. During the second phase, two PhD level researchers read and coded all of the written summaries using open and axial coding to examine themes (Strauss & Corbin, 1998). At the end of this phase, researchers identified main categories with examples from the reports. During the third phase, researchers used the categories in a key-word search process for the purpose of categorical aggregation (Stake, 1995) using the search tool on Adobe Acrobat Reader. By the end of this phase, researchers had created documents with lists of categories, examples from grantees' reports, and frequencies of the main themes.

Results

The results are organized around two major themes: 1) how grantees represent characteristics of teacher quality in mathematics, and 2) the conditions they identify as influences on those characteristics. In this section, we discuss categories when their description was evident in at least 15 percent of the grants, showing the percentage of grants reporting a given category in parentheses. These percents are not meant to imply statistical relationships, but rather, to give the reader a sense of the proportion of grants reporting each theme and offer a "big picture" view of mathematics teacher quality in the program.

Representations that Characterize Teacher Quality

Characteristics of mathematics teachers described by grantees focus on subject knowledge, pedagogical knowledge, and behaviors, practices, and beliefs. The most common representation of teachers' subject knowledge was a score on a test of mathematics subject knowledge (63%). Twice as many projects used test scores to represent teachers' subject knowledge as any other representation. Additional representations of subject knowledge included improved student achievement (29%), teachers' subject preparation (including subject-specific degrees and courses taken in mathematics content) (23%), observations of the teacher that focused on subject knowledge (19%), and teachers' responses to surveys about subject knowledge (19%). Approximately the same number of grantees reported representations for pedagogical knowledge as subject knowledge, although the representations differed in type and frequency. Grantees reported responses on surveys as the most frequently used representation of pedagogical knowledge (52%), followed by observations of teaching (42%). Other representations of teachers' pedagogical knowledge included improved student achievement (27%), and teaching practices reported during interviews (25%). Unlike subject knowledge, where scores on tests were the most frequently used representation of teacher knowledge, scores on tests were the least likely representation used for pedagogical knowledge. However, several grantees used instruments that examined a combination of subject and pedagogical knowledge (i.e., Learning Mathematics for Teaching, Hill, Schilling, & Ball, 2004). The most common representation of mathematics teachers' behaviors, practices and beliefs was responses on a survey (58%). Other representations of teachers' behaviors, practices, and beliefs included observations of teaching (31%), and responses to interview questions (31%). Essentially these representations are how grantees operationalize teacher quality characteristics.

Conditions Reported as Influences on Teacher Quality Characteristics

Grantees report a variety of conditions that influence characteristics of individual teachers. All grantees identify Professional Development and Teacher Leadership as conditions influencing teacher quality characteristics (100%). Another commonly reported condition was Linking STEM Faculty with Teachers and Schools (48%).

Professional development. Professional development focused on courses, workshops, and other training activities. The most common statements about professional development focus on content and pedagogy, and they are commonly described as intertwined. Grantees use terminology such as pedagogical content knowledge (Shulman, 1986) and mathematical knowledge for teaching (Hill, Schilling, & Ball, 2004) to show these interrelationships. While the focus on subject knowledge is traditionally emphasized for high school teachers, grantees focus on subject knowledge for teachers at all grade levels. Professional development frequently uses curriculum materials (71%) and includes work with student assessment items (54%) (i.e., developing various methods of student assessment, developing test items, and interpreting test item data). Grantees incorporate the use of mathematics standards documents (52%) in an effort to understand the contents of the standards documents and align standards with instruction. Professional development seminars also focus on analyzing students' thinking using student products and videotaped episodes of students working (50%). Professional development uses teacher networks (48%) including peer observations and feedback, peer coaching, peer support structures, and study groups. Additional properties of professional development include the use of technology and other mathematics tools (46%), learning to conduct action research in one's own classroom (27%), and lesson and unit planning (23%).

Teacher leadership. All grantees describe some form of teacher leadership and a majority discuss formal teacher leader positions (94%). The largest responsibility of teacher leaders described by grantees was to provide professional development for other teachers (96%). To a lesser extent, teacher leaders engaged in aligning curriculum, selecting and reviewing curriculum, and designing curriculum (35%). About one-fourth of grantees report teacher leaders engaged in setting, sustaining, and achieving school or grant goals (23%). Because the work of a mathematics teacher leader reaches beyond the work of a mathematics teacher, training for leaders was reported in many grants (81%). The most common attributes of leadership training included development of subject knowledge (42%), leadership skills and dispositions (42%), and pedagogical strategies (42%). Leadership training included such topics as conflict management and strategies for leading change in a school setting. About one-third of grantees include standards and curriculum (31%), coaching and mentoring strategies (29%), and how to provide professional development (27%) as part of leadership training sessions. Grantees are building capacity by developing local mathematics teacher leadership expertise for professional development and teacher induction. Examples of their leadership roles include coaches, mentor teachers, lead teachers, department chairs, curriculum specialists, master teachers, and locally-based staff developers. Teacher leadership was described in all grade bands (elementary, middle, secondary). In some cases grantees utilize teacher leadership roles already in place in the school system, while other roles were constructed as part of the grant.

Linking STEM faculty with teachers. Almost half of grantees report linking disciplinary faculty in the fields of science, technology, engineering and mathematics (STEM) with K-12 mathematics teachers as a condition influencing teacher quality (48%). STEM faculty worked with education faculty, teachers, and teacher leaders to design, revise, and teach courses for teacher education programs, summer workshops, and in-service teacher programs (33%). Reports discuss STEM faculty serving in management roles, such as directing project activities (27%), and advisory or "expert" roles, including attending professional development sessions to provide on-site support (25%). The increased presence of STEM faculty in programs for mathematics teachers was reported as a means for increasing teachers' subject knowledge. In some grants it appeared that STEM faculty were engaged in the grant in name only. Courses taught by STEM faculty as part of the grant were sometimes the same ones taught before the grant began. STEM faculty "involvement" is often recorded in numbers of hours of participation. However, rather than being engaged in teaching or designing teacher workshops, STEM faculty may attend a workshop where they learn more about the grant itself. There are also reports that allude to concerns among STEM faculty and education faculty showing misunderstandings about each others' professions, philosophical differences on pedagogy, and resistance by STEM faculty (and/or their departments) to engage in education work.

Discussion

These results provide one view into the work of grantees in a nationally funded program focused on influencing mathematics teacher quality. Although the descriptive nature of grantees' reports was a limiting factor in the analysis, researchers believed the selection of what to include in the reports was indicative of what grantees found to be important. While certain aspects of reporting are required across the program, there is still great latitude in what the grantees are permitted to submit, as evidenced by the range in the length of the reports (29 to 707 pages). These findings illustrate how grantees represent teacher quality characteristics in their work and conditions they report as influences on those characteristics.

Improving Teacher Quality

Grantees' language on teacher knowledge emphasizes the importance of subject knowledge, similar to recent policy and professional organization statements. Their descriptions of representations used (i.e., scores on tests, surveys, observations) for teachers' subject and pedagogical knowledge closely align with variables used to measure teacher quality in research. The importance of subject and pedagogical knowledge as an influence on the quality of individual teachers is clearly based on research and policy statements (Monk, 1994; Wilson & Floden, 2003). It is clear from the findings that grantees have adopted research-based practices in the design of professional development (Loucks-Horsley, Hewson, Love, & Stiles, 1998). Their work includes core features (content knowledge, active learning, and coherence) and structural features (type of activity, duration, and collective participation) that have been shown to have a significant positive effects on teaching (Garet, Porter, Desimone, Birman, & Yoon, 2001). Findings in the present study show that grantees report a strong emphasis on subject preparation. They report active learning, the use of standards, and the analysis of student work. Grantees foster coherence and collective participation by using teacher networks in the same subject areas, grade levels, and schools. It seems clear from the reports that grantees are knowledgeable on the conditions shown to be effective in improving individual teacher quality characteristics.

Negotiating New Relationships

Almost half of grantees described the involvement of STEM faculty in education activities for teachers. The results show that linking STEM faculty with K-12 teachers and schools is used to bolster the content of teacher learning activities. STEM faculty are involved in planning and teaching courses and workshops and participating as content experts. Grantees' reports hint at a disconnect between some STEM and education faculty. In another report on the MSP, issues arose in one grant over differences of opinion about pedagogical strategies between STEM and education faculty (Zhang et al., 2006). Differences of opinion between STEM and education faculty on education issues are not isolated to the grants. In a publication of the American Mathematical Society, an opinion piece suggested taking the following approach to supporting the work of standardized testing: "If you have an opportunity to discuss K-12 pedagogy, pass. There are exceptions to this of course, but at the moment alienation is more likely than progress" (Quinn, 2005, p. 399). In contrast, other academic mathematicians have described their involvement with school mathematics and how essential it is that they be involved with K-12 education efforts (Bass, 2005). Bass highlights the importance of mathematicians developing an understanding of the work of K-12 mathematics so that they can see ways that their own mathematical knowledge can contribute to solutions for problems in mathematics education.

Traditional university reward structures for STEM faculty often hinder their involvement in mathematics education work. In the MSP Program, STEM and education faculty across the country are working together to improve mathematics education. These parallel efforts have the potential to influence the structure of future collaborative work in K-12 mathematics education. In addition to the teacher retention challenges faced by educators, reports show that the proportion of students earning degrees in STEM fields has declined, and that factors contributing to this decline include subpar teacher quality at the high school and college levels, among other factors (Ashby, 2006). These are interrelated challenges that face STEM and education faculty.

The Promise of Teacher Leadership

The findings suggest that grantees view teacher leadership as an important means of influencing teacher quality. While teacher leadership is a construct that has been examined in the literature for several decades (Rowan, 1990), recently, there has been increased interest in teacher leadership, including broader views of the construct, and its effects on teaching and learning (Spillane, Halverson, & Diamond, 2001). Much of the existing literature on teacher leadership focuses on formal roles of leadership, characteristics of teacher leaders, and conditions that facilitate teacher leadership development; less research focuses on the effects of teacher leadership, particularly on other teachers and students (York-Barr & Duke, 2004). Descriptions of teacher preparation and professional development programs which are, in part, intended to develop teacher leadership include three major foci: ongoing knowledge development of pedagogical issues; knowledge development of methods of school change; and knowledge and skill development of techniques for supporting colleagues' growth (York-Barr & Duke, 2004). These elements are evident in the reports with 81 percent of grants describing teacher leadership training which includes these key features.

Teacher leaders in the present study are viewed as sources of local outreach for the grant by assisting in the development of, facilitating the implementation of, and communicating the goals and activities of the grants and they serve. There is an underlying assumption in the reports that teacher leaders influence teacher quality in this more systematic way. For example, when teacher leaders with subject specific skills mentor new teachers in their schools who are teaching in the same field, new teachers may be more likely to be successful in their beginning years of teaching. In this example, the teacher leader has the potential to influence new teacher induction. Most of the existing research on the effects of teacher leadership has focused on the effects on teacher leaders themselves. Evidence of the effects of teacher leadership outside the individual leader is more unclear. The grants in the present study are in a unique position to contribute to this research.

Conclusion

Several important insights have emerged from this examination. The representation of characteristics of individual teacher quality for mathematics teachers and the conditions identified as influencing those characteristics, in particular, professional development, appear to be well defined by the grants and are consistent with research findings. There is collaboration among STEM and education faculty for the support of K-12 mathematics teaching improvements; however, institutional structures, evidence of STEM impact, and intensity of STEM faculty engagement, are still being sorted out as faculty negotiate new roles and relationships. Teacher leaders may play an important role in influencing the conditions that influence teacher quality, and there is much research to be done in this area. Documenting and disseminating the new knowledge gleaned in these initiatives is the key to ensuring that others will learn from grantees' experiences.

Endnotes

1. This research is part of the Math and Science Partnership Program Evaluation (MSP-PE), supported by Contract No. 0456995 from the National Science Foundation. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation (NSF).

2. This paper is an abridged version of a report submitted to the NSF for the MSP-PE June 2006 Quarterly Report.

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TEACHER PEER COACHING IN GRADE 3 AND 6 MATHEMATICS

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This study reports the results of research on the effects of peer coaching on two dimensions closely linked to student achievement: teachers' instructional practice and teacher beliefs about their capacity to impact student achievement. The study tracked 12 grade 3 and 6 teachers as they participated in a professional development program over a six-month period. The mode of in-service delivery consisted of peer coaching, workshops on standards-based teaching, and self-assessment. The study found that 1) teachers implemented peer coaching largely as intended. 2) They enacted key elements of standards-based mathematics teaching in their own classrooms.3) Teachers changed their practice in intended directions with regard to student-student interaction and tasks assigned to students.4) Effects on teacher practice can be attributed to a combination of peer coaching with content specific in-service sessions.

Research Objectives

The focus of this study was to measure the effects of peer coaching and related in-service on grade 3 and 6 teachers' instructional practice and their beliefs about their instructional capacity teaching mathematics.

Perspectives/Theoretical Framework

Our theory of teacher change (described in Ross & Bruce, in press) is based on a model of teacher self-assessment developed within the broader framework of social cognition theory (Bandura, 1997). In this conception, teacher willingness to experiment with instructional ideas, particularly techniques that are difficult to implement, depends on teachers' expectations about their ability to bring about student learning; i.e., teacher efficacy. Of the four sources of teacher efficacy information identified by Bandura (1997), the most powerful is mastery experience – teachers' judgments about being successful in the classroom. The primary data for such self-assessments are teacher perceptions of changes in student performance gleaned from student utterances, work on classroom assignments, homework, and formal assessments.

Teachers who anticipate that they will be successful set higher goals for themselves and their students. High expectations of success motivate classroom experimentation because teachers anticipate they will be able to overcome obstacles and experience the benefits of innovations. Teachers with high efficacy produce higher student achievement (Mascall, 2003; Muijs & Reynolds, 2001; Ross, 1992; Ross & Cousins, 1993; Watson, 1991), provided that teachers have access to powerful innovations. Teacher efficacy contributes to achievement because high efficacy teachers: use classroom management strategies that stimulate student autonomy; attend to the needs of low ability students; and, positively influence students' perceptions of their abilities (evidence reviewed in Ross, 1998).

Peer Coaching

Norms of privatized practice limit peer opportunities for influencing teacher self-assessments. Isolation can be overcome by creating professional school communities with shared values, collaborative decision making, and reflective dialogue (Louis & Marks, 1998). A

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structured approach for building such a community is peer coaching where pairs of teachers of similar experience and competence observe each other teach, negotiate improvement goals, develop strategies to implement goals, observe the revised teaching, and provide one another with feedback. Coaching has reported positive effects when the appropriate climate is developed (McLymont & da Costa, 1998). In a review of peer coaching literature, Greene (2004) found that teachers in peer coaching programs implemented new instructional strategies more than control group teachers, used the new strategies in more appropriate ways, had longer term retention of new strategies, and understood the purposes of instruction. Peer coaching increases teacher implementation of reform-based teaching practices and contributes to increases in teacher efficacy (Kohler, Ezell, & Paluselli, 1999; Licklider, 1995; Wineburg, 1995).

Peer input can influence teacher self-assessments in multiple ways. For example, peers can influence self-observations by directing teacher attention to particular dimensions of practice. Peer feedback can also influence teacher judgments about the degree of their goal attainment. Further, peers can influence teacher practice by suggesting and implementing specific strategies together. These opportunities for positive peer influences on teacher self-assessment involve recognizing teaching success (valid mastery experiences). Peers also have opportunities to influence teacher efficacy through three other sources of efficacy information proposed by Bandura (1997): social persuasion (persuading colleagues that they are capable of performing a task), vicarious experience (observing successful performances of a similarly capable teacher peer), and physiological and emotional cues (increasing positive feelings arising from teaching and connecting them to teaching ability or reducing negative feelings arising from teaching experiences).

Peer coaching is not universally successful however. For example, Perkins (1998) found that teachers had difficulty with communication skills when interacting with their peers in coaching settings. They asked few open-ended questions, paraphrased infrequently, and used limited facilitative probes. Busher (1994) reported a study in which teachers were randomly assigned to peer coaching and control groups. Training consisted of sessions on supportive skills, questioning, nonverbal communication, modes of learning and thinking skills. The treatment had no effect on instructional practice, most likely because there was no attempt to provide teachers with specific instructional skills. These findings suggest that an effective peer coaching program needs to combine training of the peer coaching process with training in curricular content.

Integration of in-service on peer coaching and mathematics instructional training

In this study, a four-session in-service series was designed to heighten and direct peer influences on instructional decisions with the goal of increasing teachers' implementation of standards-based mathematics teaching and enhancing their perceptions of their ability to enhance learning using a reform curriculum. The key challenges were reducing teacher isolation to make peer influence accessible (through peer coaching opportunities) and providing teachers with the conceptual and strategic tools to move toward mathematics reform implementation.

The in-service brought peers together and provided strategic tools to enable teachers to move toward a more constructive approach to mathematics teaching. The central tool was a rubric for mathematics teaching that focused teachers' peer observations and their improvement goals on dimensions of mathematics teaching of highest priority to subject experts. We developed, from a research synthesis (Ross, McDougall, Hogaboam-Gray, 2002) and NCTM policy statements (NCTM 1989; 1991; 2000), ten characteristics of standards-based mathematics teaching. The rubric was constructed from observations and interviews with teachers who ranged from traditional to innovative (McDougall, Lawson, Ross, MacLellan, Kajander, Scane, 2000; Ross, Hogaboam-Gray, McDougall, & Bruce, 2001; Ross, Hogaboam-Gray, & McDougall, 2003). For each of the 10 dimensions, we identified four levels, arranged in order of increasing fidelity to NCTM Standards. The validity of the hierarchy of levels was established by a panel of content experts (Ross & McDougall, 2003) and by a series of studies that tested the validity of a self-report survey and the related rubric (Ross et al., 2003).

Methods & Data Sources

The in-service program was based on the Professional Development Standards for Elementary Mathematics (Hill, 2004): teachers constructed mathematical meaning by engaging in tasks and content comparable to those undertaken by their students; the in-service focused on classroom practice (e.g., teachers examined examples of student work); teachers worked together, rather than individually, on in-service tasks; in-service presenters modeled the recommended instructional practices; the in-service illustrated how students learn mathematics; teachers participated in the design and delivery of the in-service. These standards directly contribute to teacher learning (Brandes & Erickson, 1998; Garet, Porter, Desimone, Birman, & Yoon, 2001; Loucks-Horsley & Matsumoto, 1999; Ross et al., 1998).

Participants were 12 grade 3-6 teachers reflecting a range of mathematics teaching from traditional to reform. Four pairs were grade 3 teachers; two pairs were grade 6. All were volunteers. Sources of data included:

(a) Teacher observations at the beginning and end of the project with regard to three sets of teaching strategies that were the focus of the in-service: selection of mathematical tasks, construction of mathematical knowledge, and support for student-student interaction. Five observers were trained using the Classroom Observation Guide which provides observers with a definition of the three dimensions of mathematics teaching and specific probes to guide the observer's collection of information. Observers recorded specific examples of teacher actions relevant to each dimension. The observer training sessions emphasized the importance of detailed descriptions of teacher practice, consistency in application of the observation template, and collecting sufficient information to make a rubric placement decision on the four point scale.

(b) Teachers completed an online assessment at the beginning and the end of the study. The assessment provided a global score representing commitment to standards-based teaching.

(c) Each teacher was observed by his/her peer on three occasions. Each pair compared peer observations to self-perceptions, negotiated improvement goals, devised strategies to implement goals, and provided feedback on instructional changes. Each teacher brought a summary of their peer coaching experience to the following in-service.

(d) Each teacher pair was interviewed at the end of the study. The interview guide focused on whether teachers perceived change to have occurred, the identification of specific examples of teacher and student activity that illustrated changes in practice, and teacher theories about which aspect(s) of the in-service contributed to the change. These interviews were transcribed verbatim.

(e) Three researchers recorded their observations of teacher responses to the in-service sessions in field notes that were compiled immediately after each session.

Analysis was qualitative, relying primarily upon pattern matching (Mark et al., 2000).

Results and Conclusions

The coaching reports indicated that in virtually all pairs in each of the three sessions, teachers observed their partner teaching mathematics; gave feedback to their partner on the lesson observed; obtained feedback from their partner on their own teaching; helped their partner set mathematics teaching goals; and, were given help on goal setting from their peer.

The main finding of the study is that teachers moved their mathematics teaching toward reform. The observational data summarized in Table 1 found that the 12 participants moved toward a more constructivist approach in the support they gave for student-student interaction (D8). In addition, teachers' assignments of student tasks were more likely to include rich problems that encouraged multiple solutions. Although there were no pretest to posttest changes in construction of knowledge (D5) during observations, teacher reports of encouraging students to construct their own meaning in mathematics class were clearly described in peer interviews.

Dimension of Mathematics Teaching	Pretest		Posttest	
	Mean	SD	Mean	SD
D5: Construction of Knowledge	2.92	.76	2.96	.66
D4: Tasks: Multiple Solutions	2.75	.87	3.08	.82
D4: Tasks: Multiple Representations	2.46	.66	2.46	.72
D8: Student-Student Interaction: Explicit Instruction	2.33	.78	2.85	.82
D8: Student-Student Interaction: Task Assignment	2.79	.89	3.60	.84
D8: Student-Student Interaction: Communication	2.21	.94	2.70	1.25

Table 1 Pre and Post Teacher Ratings (N=12)

Teachers attributed the improvements in their practice to peer coaching and to the information about mathematics teaching presented at the in-service sessions. Contrary to our expectation, it was not an either-or situation in which one factor was clearly more powerful than the other. The two core processes reinforced each other. Conceptually, 1) the peer coaching process awakened in teachers the need for change; 2) the workshop presentations provided explicit models of alternative practice; 3) the between session activities provided opportunities for experimentation, and 4) the debriefing conversations provided opportunities for teachers to establish ways to integrate new practices into their existing styles. These four stages correspond to the model of professional reflection in Ross and Regan (1993a). They argued that teacher change occurs through four processes embedded in professional reflection: 1) dissonance, 2) synthesis, 3) experimentation, and 4) integration.

The second key finding of the study is that the in-service had positive effects on teacher beliefs in their capacity as mathematics teachers. The initial response of some teachers to both the peer coaching process and the workshop presentation was depressed confidence. By in-service end most teachers reported that they felt more capable of teaching mathematics conceptually. Teacher interpretations of their effectiveness were elevated through several affirmations of their competence, such as recognizing that some of their existing practices were similar to those recommended by presenters, by receiving positive feedback from their partners, and by acquiring and successfully using new instructional strategies in their own classrooms.

In some instances peer coaching was more successful than previously attempted strategies for dissemination of teaching ideas. Jill reported that prior to the coaching session she had been trying for some time to persuade her partner, Janice, to adopt a specific strategy for mental mathematics that worked really well in Jill's class [Int-Jill]. It was only when Janice saw the method in action in Jill's class during the peer coaching session that she decided to use it in her own classroom. By the end of the project, Janice reported that she was using it on a regular basis [Int-Janice].

Teachers frequently reported that they were able to put their observations into immediate use. For example, Susan watched Karen teach a patterning activity, was deeply impressed ("I was in awe" [Int-Susan]) and then used the same lesson with her own students. Susan was particularly appreciative of the opportunity to observe an experienced peer because Susan was teaching grade 3 for only the second time—most of her experience was as a kindergarten teacher.

Teachers also cited student evidence (enthusiasm, quality of student discourse, effort seeking multiple solutions) for their claims of improved teaching performance. The evidence of increased mastery experiences was extensive and explicit. The in-service also provided teachers with vicarious experiences. By observing teachers like themselves successfully enacting standards-based teaching, teachers felt more capable.

A less anticipated third finding was that the opportunity to engage in peer coaching led participants to self-reflect more frequently and deeply. Participants reported that they normally have little opportunity to reflect on the success of lessons, beyond the private ruminations that occur "on the fly…as you are driving home" [Int-Janice]. In contrast the peer coaching process provided an opportunity for teachers to explicitly share their interpretations of lesson outcomes with a knowledgeable colleague who provided feedback. For example, Helen's observations of William using a new text resource led her to think about how she was using that same resource. Helen saw that William's implementation was more advanced than hers but she felt that she had incorporated some elements into her teaching. Helen concluded that although she was not "following it as strictly as" William, she was on the right track [Int.-Helen]. Simultaneously, William, as the observed, was questioning his own teaching: "I find myself questioning things that I am doing more and more...critically looking at the way I'm teaching and evaluating." [Int-William] Both Helen and William believed that self-questioning led to higher quality instruction.

Limits of the peer coaching relationship

Some teachers had difficulty making contact and sustaining conversations about their teaching [Field notes-S3]. A key impediment was that five of the six pairs involved cross-school groupings. This resulted in considerable travel to get to the partner's school. Difficulty in the debriefing component of peer coaching may have also been related to the anxiety some teachers felt about being observed. For example, Janice remembered asking herself during a peer observation lesson: "Why can't I understand what that student is saying? I bet Jill [the peer observer] knows what that student is saying." [Field notes-S2] Further, some peers were reluctant to suggest substantive changes unless their partner suggested it first.

Recommendations/Implications

The in-service had a positive effect on teachers, demonstrating that professional development combining peer coaching with carefully designed input on instructional strategies is a fruitful approach to the development of teacher capacity. We recommend that the procedures used in this study be used in subsequent in-service but we also think they could be strengthened in several ways: 1) Consider a whole school approach by moving in-service to the school. We believe in-service effects would be heightened if teachers worked in same grade pairs embedded within a school staff. A key part of this strategy would be to link the peer coaching process directly to the school plan. 2) Extend the treatment to five coaching sessions rather than three. Since the initial reaction to the peer coaching process for some teachers was a reduction in confidence, which subsequently rebounded, extending the number of coaching cycles would maximize teacher learning. 3) In subsequent sessions we suggest sharing more control with teachers by inviting them to self-select goals from among the ten dimensions in the rubric.

Relevance of Paper to PME-NA Goals

This paper addresses the core goal of deepening the psychological aspects of teaching and learning mathematics by connecting a key mathematics education problem (how to increase teacher implementation of standards-based mathematics teaching) to a psychological theory (social cognition) and to a key theme in psychology (teacher beliefs about their capacity).

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THE ROLE OF COVARIATIONAL REASONING IN LEARNING AND UNDERSTANDING EXPONENTIAL FUNCTIONS

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While past research supports that students experience difficulty developing deep understandings of exponential functions (Confrey & Smith, 1995), little research efforts have focused on how students come to understand this function family. Moreover, little is known about how in-service secondary mathematics and science teachers think and reason about exponential functions. The intent of this study is to describe the various ways in which high school mathematics and science teachers build their knowledge of exponential functions using covariational reasoning as a tool for understanding exponential behavior.

Understanding the concept of exponential functions and multiplicative rate of change is critical for students as they progress through mathematics. Not only are exponential growth and decay topics encountered in our everyday world, these functions are embedded in the sciences as well as mathematics, and they provide a model for representing multiplicative growth and decay patterns for real world phenomena. The National Council of Teachers of Mathematics Principles and Standards (NCTM, 2000) advocate for high school and college mathematics curricula to include the topic of exponential functions and emphasize the importance of developing this functional understanding and multiplicative behavior conceptually through the use of real world contexts. Research has shown that students experience difficulty developing a profound and robust understanding of exponential functions (Confrey & Smith, 1995; Weber, 2002). However, irrespective of these findings, more research is needed to investigate the process of coming to understand exponential functions. This paper will discuss the role of covariation in learning exponential functions and in the development of building a profound, flexible knowledge base of multiplicative structures. Findings from a study will be discussed on the various ways that high school mathematics and science teachers use covariation when performing exponential function tasks.

Theoretical Framing of the Study

Many researchers have investigated covariational notions using rate (Thompson, 1994a, 1994b), multiplicative structures of exponential functions (Confrey & Smith, 1994, 1995), quantities varying simultaneously (Saldanha & Thompson, 1998), and functions modeling dynamic events (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Beginning with Thompson's (1994a) study of images of rate, it is evident that his description of rate involves aspects of covariational reasoning and proportionality. Consider the case of constant speed in his example that illustrates speed as emerging out of the activity of coordinating distance and time. Total distance traveled can be imaged as accumulations of accrued intervals of time, which lead to the idea that "the total time of the trip is the same as the accrual of distance in relation to the accrual of time" (p. 235). The concept of rate emerges from the mental activity of covarying distance and time proportionally.

Additional research supports that students experience difficulty developing a profound and robust understanding of exponential functions (Confrey & Smith, 1994, 1995; Weber, 2002),

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

which could be connected to impoverished understandings of rate and multiplicative relationships. Many textbooks and mathematics curricula present this topic by first introducing formulas and rules of exponents while emphasizing conventional algorithmic methods devoid of context (Confrey & Smith, 1995). Confrey and Smith (1994) argued that this conventional correspondence approach is counterintuitive for students given the emphasis of curricular materials on creating a correspondence between x and y without understanding the progression of the output values in relation to the input values. Both Confrey and Smith advocate a more covariational approach to learning functions where students move flexibly from one output value to the next while coordinating with moves from corresponding input values.

Furthermore, Carlson et al. (2002) define covariational reasoning “to be the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 354). In this study, Carlson et al. proposes a covariation framework encompassing detailed mental actions that students exhibit as they engage in mathematical activities. As a result of this study, the covariation framework was extended to include a dimension that describes the mental actions of coordinating input and output values in the context of exponential functions. Certain questions were at the forefront of this analysis, such as what do the mental actions of reasoning covariationally look like in terms of exponential functions and to what extent do in-service high school mathematics and science teachers apply covariational reasoning when attempting to solve mathematics situations in the context of exponential functions. Carlson et al.’s framework was used to analyze data gathered from this study and attempts were made to describe the mental actions of reasoning covariationally within the context of exponential functions. The results of this study are summarized in this paper.

Methods

The subjects of this study were 15 in-service secondary mathematics and science teachers with a wide range of teaching experience, as well as mathematical ability. These teachers were participating in a graduate-level mathematics course, uniquely named the *Functions Course*, which was designed to promote the concept of functions (i.e. linear, quadratic, exponential, and trigonometric functions). Particular emphasis was directed to building a deeper understanding of rate of change through covariation for each of these function families. The researcher for this study was also one of the *Functions Course* instructors and curriculum designers for materials used in the course.

A 25-item *Precalculus Concept Assessment* (PCA) instrument was given both as a pre-test and post-test to all 15 mathematics and science teachers participating in the *Functions Course*. The assessment instrument was designed to capture subjects’ knowledge of functions in general and each item on the assessment is grounded in past research (Engelke, Oehrtman, & Carlson, 2005). PCA pre-test and post-test scores for these teachers were analyzed with the lens focused on three exponential function tasks. The results of these tasks will be discussed and two of these questions will be explored in detail (*Bacteria Question* and *Decreasing Question*).

In addition to the PCA analysis, videotaped clinical interview sessions were conducted to gain further insights on mathematics and science teachers’ ways of thinking about exponential functions. For this discussion, the focus will be on one interview question (*Half-Life Question*) that provided interesting insights into the role of covariation when reasoning through exponential tasks. We will see in the results section of this discussion that exponential decay proved to be a larger obstacle than was first expected.

These interviews were conducted after completing a 3-week unit on exponential functions in the *Functions Course*, but before the PCA post-test was given. Each interview session was transcribed and initially coded using open and axial coding to build potential categories. After potential categories were constructed, the analysis concentrated on teachers' use of reasoning covariationally through each of the responses to the questions above. Carlson et al.'s (2002) five general levels of covariational reasoning were used to help further code utterances and to characterize the mental actions involved in applying covariational reasoning when responding to exponential function tasks. The process of portraying each teacher's reasoning abilities on these tasks provided insights for extending the covariation framework to include a dimension that characterizes the reasoning abilities central to learning and understanding exponential functions.

Results Quantitative Analysis and Results

The results for the quantitative portion of the study (PCA analysis) revealed that both mathematics and science teachers have difficulty with function-related tasks. The mean score for the PCA administered to the subgroup of 15 teachers was 11.6 (out of 25) for the pre-test and 15.3 (out of 25) for the post-test. Focusing specifically on the three exponential tasks of the PCA, the mean score was 1.27 (out of 3) for the pre-test and 1.87 (out of 3) for the post-test. Itemized results of the scores for the exponential tasks are detailed in Table 1.

Exponential Tasks Correctly Completed (n = 15)						
	Bacteria Question Pre-test	Bacteria Question Post-test	Inverse Question Pre-test	Inverse Question Post-test	Decreasing Question Pre-test	Decreasing Question Post-test
Mathematics	3	5	3	4	6	5
Science	3	6	1	1	3	7
# Correct	6	11	4	5	9	12
% Correct	40.00%	73.33%	26.66%	33.33%	60.00%	80.00%

Table 1. PCA Data (Version H)

Each of the exponential tasks emphasizes various levels of general understanding of functions, as well as specific exponential behavior. The *Bacteria Question*, for example, highlights algebraic and contextual components of an exponential task:

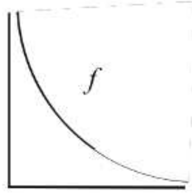
Bacteria Question: The model that describes the number of bacteria in a culture after t days has just been updated from $P(t) = 7(2)^t$ to $P(t) = 7(3)^t$. What implications can you draw from this information?

- The final number of bacteria is 3 times as much of the initial value instead of 2 times as much.
- The initial number of bacteria is 3 instead of 2.
- The number of bacteria triples every day instead of doubling every day.
- The growth rate of the bacteria in the culture is 30% per day instead of 20% per day.
- None of the above.

From the table above, only six out of 15 teachers were able to identify the correct response of 'c' during the pre-test implementation. Of the remaining nine teachers who answered incorrectly, four chose response 'a' as correctly describing the variation between $P(t) = 7(2)^t$ and $P(t) = 7(3)^t$. This finding reveals the difficulty with interpreting exponentiation within functions as a dynamic process where the new growth factor is raised to the input power (days) and the effect this process has on the final output number (number of bacteria). Perhaps it can be conjectured that the ability to reason more covariationally (i.e., the ability to coordinate two varying quantities while attending to the ways in which they change in relation to each other) using algebraic models could lead to clearer insights of the effects of the output values when changing the growth factor in an exponential function.

Post-test scores for the *Bacteria Question* improved significantly resulting in 11 out of 15 teachers responding correctly. Three of the remaining four teachers who answered incorrectly again chose response ‘a’ as the correct response. Two of these teachers (one mathematics and one science) had also chosen this incorrect response during the pre-test implementation, thus demonstrating their deep-seeded misconceptions despite interventions designed to promote accurate conceptions.

Decreasing Question: A function *f* is defined by the following graph. Which of the following describes the behavior of *f*?



I. As the value of x approaches 0, the value of f increases.
II. As the value of x increases, the value of f approaches 0.
III. As the value of x approaches 0, the value of f approaches 0.

a) I only b) II only c) III only d) I and II e) II and III

In reference to Table 1, nine out of 15 teachers were able to correctly identify I and II as the desired descriptions during the pre-test implementation. Of the remaining six teachers who answered incorrectly, two chose only description II while ignoring the activity *of f* as *x* decreases to zero. Perhaps this response could be contributed to an inability to view the graph from right to left, which leads to an inability to reason covariationally in the same direction. This issue raises the question as to whether we should be striving to build a multidirectional covariation ability where teachers and students can describe function behavior as *x* increases and as *x* decreases. It seems that this ability would prepare teachers and students for the concept of limit as approaching from the left or from the right. Incidentally, the other four teachers who answered this task incorrectly either did not answer the question at all or choose other distracters such as III only or II and III.

Consistent with the *Bacteria Question* results, the post-test scores for the *Decreasing Question* also improved significantly resulting in 12 out of 15 teachers responding correctly. The increase from 60% accuracy to 80% accuracy for this task demonstrates a stronger ability to reason through graphical behavior with an awareness of how one quantity is changing in relation to another, yet the ability to reason covariationally still eludes many teachers as they grapple with exponential tasks. The remaining three teachers (two mathematics and one science) who answered incorrectly each chose different distracters. Interestingly, all three of these teachers who answered this question incorrectly on the post-test actually answered correctly on the pretest.

Qualitative Analysis and Results

The results for this portion of the study reveal that teachers exhibit difficulty with using (or relying upon) covariation as a tool for building an understanding of exponential functions. Coordinating images of two quantities changing in tandem over time proves to be a weakness for many of the teachers in this study. Reasoning covariationally within the context of exponential growth appeared to be less difficult than attempting to reason through exponential decay behavior. As we will see in the following transcript excerpts from the *Half-Life Question*, teachers had difficulty in reasoning through exponential behavior between half-life intervals despite their understanding that the situation did indeed represent exponential decay:

> ***Half-Life Question:*** Suppose a radioactive substance is decaying exponentially so that there are 20 grams at 1:00 p.m. and 10 grams at 2:00 p.m. What would you expect the mass of the radioactive substance to be at the midpoint in this time period, at 1:30 p.m.?

This question involves an understanding of multiplicative rate of change for exponential decay contexts. Focusing on teachers' ways of reasoning and covariation offers a glimpse into how these teachers build conceptions of rate of change of exponential behavior. Surprisingly, three teachers during their explanations believed this problem did not provide enough information to say how many grams of the substance remained at 1:30 p.m. While two of these teachers ultimately changed their mind, one teacher settled with this reasoning as noted in the following interview transcript:

> ***Jada (Mathematics):*** So we started at twenty [begins to draw a linear graph], this is ten and this is zero and this is one o'clock and here is two o'clock. That is linear but I know it is not and I don't know if it is because I don't have enough data

I don't know where it was so I don't think I have enough data. Jada seems to be experiencing a mental tug-of-war between the incorrect graph she has drawn and the fact that she knows the situation is not linear; thus she settles on her "not enough information" conclusion. Jada's faulty reasoning stems from her initial decision to reason through the pattern of the decaying substance as being a constant decrement of 10 grams until zero is reached after the third time interval. Her belief that zero grams is obtained appears to have caused the conflicting conceptions. Jada did not attempt to consider the covarying quantities of time and grams in this half-life situation.

More linear reasoning was evident when seven of the 15 teachers initially offered 15 grams as the amount of substance remaining at 1:30 p.m. Another teacher, Piper, offers her explanation below which demonstrates an example of covariational reasoning captured by Carlson et al.'s MA1 (coordinating the value of one variable with changes in the other) despite her incorrect conclusion:

> ***Piper (Mathematics):*** It's probably at fifteen.
>
> ***Interviewer:*** How did you decide on fifteen?
>
> ***Piper:*** I decided because if I were doing this at one o'clock and two o'clock, if I did my in and out table…if I took twenty and ten here and one and a half would be here, I would take, there is a difference of ten here, so half of that would be five so it will be fifteen.

Minimal covariational reasoning was evident in Piper's response, yet her reasoning illustrates the lack of reasoning power when attempting to think through situations where reasoning covariationally can be a helpful tool. Another teacher, Dale, also guessed that 15 would be the amount at 1:30 p.m. but he was able to offer a glimpse into his reasoning of rate:

> ***Dale (Science):*** …we went from 20 grams to 10 grams. So at 3:00 it will be 5 and then at 12:00 if we back up we're going to be at 40. So we want to know what the grams are here at 1:30. So let's see every 60 minutes we are using ½. I guess I'll go with the 15.

Clearly, Dale understood that this situation illustrated half-life and he was successful in extrapolating the values of the past (12:00 p.m.) and future (3:00 p.m.). However, interpolating between known values, such as between 1:00 p.m. and 2:00 p.m., proved to be an obstacle for him. His misconception that exactly 15 grams would be left at 1:30 p.m. demonstrates linear reasoning even though he was aware of how many correct grams of the substance remained at each one-hour interval. His thinking of rate appeared to be more of a discrete, static image whilecoordinating the correct effect of each half-life time interval with amount of substance remaining. Despite his incorrect response, Dale's reasoning in this problem seems to point to Carlson et al.'s MA2 (coordinating the direction of change of one variable with changes in the other variable) given that he was able to double the amount of 20 grams to obtain the previous amount of 40 grams and also continue halving 10 grams to obtain the future amount of 5 grams. He is able to reason covariationally as the independent variable (time) increases, as well as when the independent variable decreases. This is evidenced by his calculation of the amount of grams available at 12:00 p.m. (deceased independent variable from given information) and 3:00 p.m. (increased independent variable from given information).

Evidence of more advanced covariational reasoning within the context of the *Half-Life Question* can be seen in the following transcript excerpts from another teacher, Claire:

> ***Claire (Science):*** Well if it was fifteen it would be linear, this is exponential. It is not gonna be fifteen because fifteen is gonna mean that every minute it loses the same exact amount, so at halfway through it would have lost half of its mass, so I would say it has got to lose more in the first, I would say it has got to be more than fifteen at one thirty. Just because it is exponential decay and that it has got to loose more at the beginning, it is decaying more at a faster rate at the beginning, it is decaying at the same rate but the total amount of substance is decreasing much faster at the beginning than it is at the end.

In this explanation, Claire offers a glimpse of her understanding of rate in the context of exponential decay. Based on Carlson et al.'s covariation framework, it appears that Claire's actions point to somewhere between MA3 and MA4 reasoning (coordinating the amount of change of one variable with changes in the other variable and coordinating the average rate of change of the function with uniform increments of change in the input variable). Reasoning at MA4, which specifically calls for coordinating the average rate of change for functions, cannot be determined because it is not clear if Claire views this situation as a function.

Discussion

While the results of this study were interpreted using Carlson et al.'s covariation framework, it appears that more refinement of this framework is necessary for capturing the reasoning abilities and mental actions specific to exponential functions. The results of this study will be used to further revise the exponential function unit in future implementations of the *Functions Course* to further promote additional covariational reasoning abilities and ideas of proportionality. Additional research will continue in order to further investigate the role of covariational reasoning in building a deep, robust understanding of exponential functions.

Conclusion

Consistent with Confrey and Smith's (1994; , 1995) assertions, the results of this study illuminate many cognitive obstacles apparent when attempting to solve exponential tasks. Even high school mathematics and science teachers who often teach this topic in their courses experience difficulty when reasoning through exponential situations, especially decay and half-life. Not only did teachers in this study struggle with ideas about finding midpoint values of half-lives as in the Half-Life Question, they also failed to connect how these values are proportionally related to previous and future values (i.e., the ratio of the midpoint amount to 20 grams is precisely the same as the ratio of 10 grams to the midpoint amount or in other words,

$$\frac{midpt\ grams}{20\ grams} = \frac{10\ grams}{midpt\ grams}$$

). No one in this study offered such an explanation nor realized that enough information was provided in the *Half-Life Question* to determine the exact amount of the substance present at 1:30 p.m.

Additional evidence demonstrated that covariational reasoning plays a minimal role when teachers grapple with exponential ideas, despite the powerful insights into understanding function phenomena provided by the ability to reason covariationally. It also appears that building a multidirectional covariation ability (e.g., interpret graphs from right to left as well as left to right) where teachers and students can describe function behavior as *x* increases and as *x* decreases could provide a powerful mechanism for increasing the ability to reason through exponential function behavior.

Acknowledgements

I would like to thank Marilyn Carlson for reviewing previous drafts of this manuscript. NSF grant #0412537 supported this work.

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IMPROVISATIONAL COACTIONS AND THE GROWTH OF COLLECTIVE MATHEMATICAL UNDERSTANDING

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In this paper we characterize and describe the growth of collective mathematical understanding as a process of improvisational coaction. Drawing on the theoretical work of Sawyer (2003) in improvisational jazz and theatre, we explain how mathematical understanding can be observed to emerge from the complex, improvisational ways that a group of learners work together mathematically. We also distinguish coaction from interaction to highlight the importance in improvisational flow of acting with the contributions of other group members, rather than merely acting on these, suggesting that the growth of collective mathematical understanding is a truly shared process.

Mathematical Understanding as a Dynamical Process

In discussing mathematical understanding, we are influenced by, and employ in our analysis of data, elements of the Pirie-Kieren Theory for the Dynamical Growth of Mathematical Understanding. This theory is well established and has been extensively presented at previous PME meetings and elsewhere (e.g., Pirie & Kieren, 1994, Martin, Towers, & Pirie, 2000). The theory characterises mathematical understanding as an on-going process in which a learner responds to the problem of reorganising his or her knowledge structures by continually revisiting existing understandings to generate 'thicker' understandings. Pirie and Kieren have termed this process 'folding back'. The theory considers understanding in terms of a set of embedded levels or modes of knowledge building activity. (Towers & Davis, 2002, p.318)

It is important to note that although the levels or layers of the model grow outwards from the local to the general, this does not imply that understanding grows in this way. Instead, growing mathematical understanding occurs through a complex movement backwards and forwards through the layers of understanding. Thus, mathematical understanding is not a static outcome or product, but instead is seen to emerge through the actions of the learners, moment by moment, as a process of mathematical engagement.

The theory also draws significantly on the notion of "images", meaning any ideas the learner may have about the topic, any "mental" representations, not just visual or pictorial ones. When Image Making, learners are engaging in specific activities aimed at helping them to develop particular ideas about a concept or topic. Image Making often involves the drawing of diagrams, working through specific examples or playing with numbers. By the Image Having stage the learners are no longer tied to actual activities, they are now able to carry with them a general mental plan for these specific activities and use it accordingly. This frees the mathematical activity of the learner from the need for particular actions or examples.

Collective Mathematical Understanding

Collaborative working, whether in small groups or as a whole class, and the associated practices of interaction and discussion, continue to be strongly advocated organisational strategies in the mathematics classroom. However, whilst it is now widely recognised that

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discussion is an essential part of developing mathematical understanding, and collective mathematical action is an area in which there has been significant research interest in recent years, we still know relatively little about how collective actions contribute to students' growth of understanding. Drawing on socio-cultural and distributed views of learning, a range of theories for characterising mathematical thinking as "an inherently social process" (Bowers & Nickerson, 2001, p.2) have emerged, and continue to be developed. (See, e.g., Cobb, 1999; Crawford, 1999; Lave, 1997; Saxe, 2002; Sfard & Kieran, 2001). Whilst important, and widely applicable, much of this existing work is not explicitly concerned with the growth of collective mathematical understanding as such, nor does it offer to the researcher or teacher a means to document this at a detailed level of emergence – something that the Pirie-Kieren Theory offers for individual learners. Thus, to more fully make sense of the observed growth of understanding, it becomes necessary to move beyond merely focusing on the individual learners, and to also look at what we term the "coactions" of the group, and the ways in which mathematical understanding emerges from these. Our research (Martin, Towers, & Pirie, 2006; Martin & Towers, 2003) therefore attempts to contribute to the field by systematically exploring the interplay between the individual and collective in the mathematics classroom through the development of a new theoretical perspective and through the analysis of the implications of the application of that perspective to classroom practice. In this paper we will focus on one element of our developing framework, that of coactions, and illustrate how this, together with elements from the Pirie-Kieren Theory, can be used to describe the growth of collective mathematical understanding as it is seen to emerge moment by moment.

Collective Mathematical Understanding as an Improvisational Process

In a recent paper (Martin, Towers, and Pirie, 2006) we offered the beginnings of a theoretical framework that proposed a way to look at, and account for, the growth of mathematical understanding at a collective level. We defined collective mathematical understanding as the kinds of mathematical actions and learning we may see occurring when a group of learners, of any size, work together on a piece of mathematics. More specifically, we suggested that by using the lens of improvisational theory, it was possible to observe acts of mathematical understanding that could not simply be located in the minds or actions of any one individual, but instead emerged from and existed in the interplay of the ideas of individuals, as these became woven together in shared action, as in an improvisational performance. We thus suggested that, as when listening to or watching an improvisational performance, in considering the growth of collective mathematical understanding the observer's attention should focus on the group as a whole, and not simply on what each of the participating individuals is contributing.

Coactions and the Growth of Mathematical Understanding

In elaborating the notion of creative, improvisational group performances, Sawyer (2003) talks of improvisational activity as being conceived of "as a jointly accomplished co-actional process" (p. 38). For us the use of the term coaction rather than interaction emphasises the notion of acting with the ideas and actions of others in a mutual, joint way. More precisely we use the term coaction as a means to describe a particular kind of mathematical action, one that whilst obviously in execution is still being carried out by an individual, is also dependent and contingent upon the actions of the others in the group. Thus, a coaction is a mathematical action that can only be meaningfully interpreted in light of, and with careful reference to, the interdependent actions of the others in the group. (Martin, Towers & Pirie, 2006, p.156)

Sawyer (2003), in talking about improvisational performance, suggests that when an idea is offered to the group, they can respond in a variety of ways. In particular, they collectively have the option of accepting the innovation (by working with it, building on it, making it "their own"), rejecting the innovation (by continuing the performance as if it had never occurred), or partially accepting the innovation (by selecting one aspect of it to build on, and ignoring the rest). This evaluative decision is a group effort, and cannot be identified clearly with any single individual. (Sawyer, 2003, p.92)

This notion places a responsibility on those who are positioned to respond to an offered action or innovative idea, as much as on the originator, and it is this "social process of evaluation" (p.92) wherein the group collectively determines whether, and how, the idea will be accepted into the emerging performance, that we suggest is the key to the emergence of collective mathematical understanding. We therefore see coaction as being a specific kind of interaction, but whereas interaction allows for reciprocal, complementary collaboration, without the requirement to be mutually building on the just offered action, coaction goes beyond this and requires mutual joint action. This is an important distinction in our work, as it must also be recognised that collective mathematical understanding is not an automatic or simple occurrence whenever two or more people are collaborating or working together. In such cases, what is observed may instead be a set of individual understandings occurring simultaneously, even though there is a high level of interaction. It is the nature and form of the collaboration which may (or may not) give rise to the growth of mathematical understanding at the collective level, and it is on this process that we focus in this paper, by offering some necessarily brief examples of coactions and explaining how collective mathematical understanding emerges from these.

Coactions in action: The emergence of collective mathematical understanding

We now turn to some extracts of data to illustrate both the improvisational character of the growth of collective mathematical understanding and how this can be seen to emerge through coactions. The examples we offer are drawn from a set of video data, collected with the aim of seeing cases of collective mathematical understanding, and with the purpose of facilitating this. We worked with a number of students preparing to be elementary school teachers. The students were invited, over a series of one-hour lunchtimes, to come and work on some mathematical problems, which, we hoped, as well as serving our research purpose, would also help them with their own mathematical and pedagogical knowledge. They were allowed to form their own groups of three or four, and to choose tasks from a booklet supplied, which contained nine tasks covering different areas of mathematics. In this paper we shall discuss one of the videotaped sessions, with three students known here as Mary, Shauna and Hilary. Our discussion will focus on the observed growth of mathematical understanding of the students, and will employ elements of the Pirie-Kieren Theory, with a specific focus on the ways in which it is coaction that leads us to characterise the growth as collective in nature. The group has chosen a task that requires them to label sixteen different triangles as equilateral, isosceles or scalene. The question asks them to do this by considering "side length properties". Each triangle is drawn on a 3 x 3 grid of dots.

Extract One: "None of them are equal…"

We join the group as they start the task. There is a short pause as they seem to consider where to start:

Shauna: It's all coming back to me

Hilary: I don't remember scalene or isosceles

S: Isosceles is this, okay? (drawing) where two are equal?
Michelle: Yeah
S: Equilateral is when they're all equal?
H: Hm hm
S: And scalene is?
M: they're all wonky?
H: This must be scalene
S: OK
H: When it has one, one sss…(pause)
M: One longer?
S: Isos, eq and scale. So the scale none of them are equal?

We see the three students begin the task at the Image Having layer. The initial statement of Shauna suggests that properties of triangles and their associated names are not new to them, and they have an existing understanding of these. However, none of the three students seems immediately able to simply recall and restate the definitions for the three different kinds of triangles, and instead what we see is individual students' definitions, posed mainly as questions, inviting the other students to accept and confirm these. For example, Shauna gives correct definitions of isosceles and equilateral, but looks for agreement from the others that her ideas are viable for the group. However, in the case of the scalene triangle, she is not sure (suggesting perhaps she either does not have or can not recall her image for this particular shape) and invites (and requires) the others to participate in what becomes a collective act of Image Having. No one student is able to simply offer a complete definition and instead the three students each offer what can be characterised as partial fragments of an image for the scalene triangle. Mary talks of it as being "wonky"; Hilary and Mary both develop the idea of "one longer", whilst Shauna extends this idea to the conception of "none of them [the sides] are equal". We suggest the students are mutually coacting on the ideas of each other, and building on what has been offered to attempt to collectively work together to have (or even re-have) a useable image. This Image Having is not located in the understandings of any one individual, but instead emerges from the way that the individual mathematical ideas are starting to intertwine, as the group collectively accepts ideas.

Extract Two: "Because of the way the pins are…"

Following this, they begin work on the task and start to label each of the triangles as equilateral, isosceles or scalene, using their definitions. However, they decide that in the case of some triangles they need to measure the sides to be able to determine their type. But, they do not have a ruler, and prior to the extract below have been trying to determine the type visually. What we now see is the emergence of a new approach for deciding what kind of triangle they are looking at. As they cannot accurately find the length of sides, they instead turn to examining the relationship between the dots (what the students call pins or dots) and the sides of the triangles:

H: I don't think there are any equilateral
M: Yeah, I was just going to say, I don't think any of them are equilateral
H: Because of the way the pins are actually...
M: …but..like that distance should be the same as that, if they're, wouldn't these dots all be equal distance
H: Yeah, but distance (pause) yeah that's what I'm thinking, this should be the same as this

M: So, this distance, this one would be then, yeah right, don't you think?
H: Yeah, but they're not..see look
S: (laughs)
M: This one right here. No, this one down here, where this is 'cos this is one peg away and this is one peg away
H: But that's not the same
M: Oh but..
H: OK, but if you have a square. See this is a square [*indicating the geoboard on which triangle 2 is drawn. See Figure 1*], so if you have a square. This is two and this is two and this is two and this is two. But that is not two [*indicating the hypotenuse of triangle 2*] and that's what the triangle is.
M: Yeah. So none of them are equal.

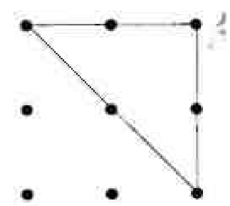

Figure 1. Triangle #2

In the above excerpt we see the students folding back to construct an image of length that is based on being able to compare the distance between different pairs of pins. Hilary begins by conjecturing that none of the triangles are equilateral "because of the way the pins are". Mary agrees, and this prompts the group to move from thinking explicitly about sides of triangles, and instead to work with the concept of length on a pin grid. In particular, we see Hilary and Mary coacting as they offer fragments of ideas that are picked up and developed by the other. Towards the end we see Hilary drawing on, and offering to the group, an image for the properties of a square and using this to establish that two pins that are vertical or horizontal neighbours are not the same distance apart as two that are diagonal neighbours. She links this to the triangle, and at this moment the thinking of the group is returned to the task in hand, and the image they have for characteristics of triangles. However, the image they now have for an equilateral triangle is thicker than that prior to folding back. Whereas initially their image only considered the visual or measured length of sides, it now also allows for the comparison of sides in terms of pins on the grid. At the very end of the extract we see Mary using this image to state that "none of them are equal" meaning that none of the triangles in the task are equilateral.

Extract Three: "An equilateral can't have a right angle in it"

Later in the session the students became confused about which distances were actually equal on the board, and were struggling to use their "pin based" image to confidently label the remaining triangles. We join them as they try to label the triangle shown in Figure 2.

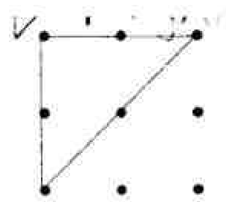

Figure 2. Triangle #9

M: Like if that's not an equilateral triangle then what is? (She is referring to triangle #9, See figure 2)
H: No, see to me, this one looks like its longer, shorter, shorter. To me an equilateral triangle is more like a yield sign.
M: But that's a right angle [indicating the right angle in triangle #9]. (pause). Does that have anything to do with it?
H: An equilateral can't have a right angle in it, I don't think.
M: No..it can't
S: It couldn't. Yeah, you're right. Perfect.
M: It's like sixty, sixty, sixty.
S: That helps.
H: (laughing)
S: It's all coming back to me!
M: Slowly but surely (all laughing)

Here we see the emergence of a focus on the interior angles of the triangle and a new 'rule' – the idea that "an equilateral triangle can't have a right angle in it." Although Mary initially suggests that the triangle is equilateral, this is an idea that is rejected by Hilary, who indicates how the triangle has two shorter sides and one that is longer. To illustrate her idea she offers a visual representation of her image – that an "equilateral triangle is more like a yield sign". Mary picks up on this, and notes that triangle # 9 has a right angle, though she isn't sure what this means. Hilary responds and the group collectively agrees, including Shauna, that an equilateral triangle cannot have a right angle, and should have three angles of sixty degrees each. This is a collective moment of Image Having for the group, where there is a sense of confidence now that they have now recalled what they know about triangles and together re-have an image for an equilateral triangle. Again, although individuals clearly contributed to this process, the image they now have cannot be simply attributed to any one student. Their image having was collective, involving a process of offered ideas being built upon. Each student offered parts of the image, which the group collectively interweaved to have something that was acceptable to them all. Using this new image, the three students confidently and correctly identified the remaining triangles in the task.

Discussion

In all of the short extracts above we see the three students collaborating and working together. Further, we suggest that their activity can usefully be seen in terms of coactions. There is a sense of unpredictability about the pathways their collective mathematical actions, and emerging understandings, will follow. In fact, the image they make and have for an equilateral triangle in terms of the size of interior angles is not one that is hinted at by the task instructions, which instead suggest working with the length of sides. Also, no one student is able to offer an image that is immediately viable to use in completing the task. Instead, what occurs throughout the session (and as illustrated in the extracts), is a continual process of the offering of ideas or "innovations" and the collective coacting on these. There is a sense in the extracts that no one student simply wants (or is able to) tell the others what to do, or to merely state a mathematical idea without expecting some response. Equally, those listening to the idea seem to accept their responsibility to do something with what is offered, and not merely receive it. The ways in which the group is able to interweave fragments of each individual's knowing, to allow a shared (rather than taken-as-shared) image to emerge from their coactions, is what enables their collective mathematical understanding to grow, and is what ultimately enables them to successfully complete the task.

In theoretical terms, and to emphasise the value of researchers' attention being oriented to coactions (rather than simply individual's statements or even interactions), we note that the students'actions can only be meaningfully interpreted in light of, and with careful reference to, the interdependent actions of the others in the group, and hence we propose the notion of mathematical coactions as a fruitful tool for enabling a more fine-grained analysis of the growth of collective mathematical understanding.

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PROFESSIONAL-DEVELOPMENT DESIGN: BUILDING ON CURRENT INSTRUCTIONAL PRACTICES TO ACHIEVE A PROFESSIONAL-DEVELOPMENT AGENDA

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Two different activities were designed and used in professional-development collaboration with middle school mathematics teachers where the goal was developing instructional practices centered in students' reasoning. The activities resulted in qualitatively different learning opportunities for teachers. We examine these activities with respect to their (1) pragmatic usefulness *to teachers' daily instruction, and (2)* legitimacy *within the teachers' institutional setting, as perceived by the teachers. Considerations of these dimensions of teachers' work can contribute to the effectiveness of professional-development designs that capitalize on teachers' current instructional practices.*

We use our[1] experiences from a five-year collaboration with a group of middle-school mathematics teachers to ground the discussion of building on mathematics teachers' current instructional practices to achieve a professional-development (PD) agenda. Researchers working with teachers argue for pragmatic importance of positioning teachers as professionals whose instructional decisions and ways of participation in PD sessions are reasonable from their perspective (Simon & Tzur, 1999). Many are committed to taking teachers' current instructional practices "as a valuable starting point, not as something to be replaced, but a useful platform on which to build" (McIntyre & Hagger, 1992, p. 271). Fulfilling such commitment is a nontrivial task, especially when teachers' current instructional practices differ significantly from those envisioned by PD designers. However, in PD settings where constitution of activities critically relies on interpretations of participants, teachers' instructional practices necessarily provide grounding for teachers' interpretations. This is because teachers' interpretations are informed by their views of teaching and learning developed in and through their own practice. Understanding of teachers' current practices is then critical for designing effective interventions.

To illustrate the complexity of building on teachers' current practices toward those envisioned we examine two attempts made by the research team at different points in our collaboration to support teacher learning. We focus on two activities that were each intended as a point of departure within a sequence designed to support teachers' development of instructional practices centered on students' reasoning. We illustrate that in both cases the teachers' interpretations of the intent of PD activities were grounded in their practices at that time. What differed were researchers' insights into and anticipation of specific teachers' interpretations. This difference was important because it had consequences for what the research team envisioned as feasible means of supporting teacher learning over extended periods of time.

We then revisit the researchers' design conjectures for the two activities and introduce a heuristic that, in retrospect, shed light on their differential viability in the PD setting. We examine closely two aspects of teachers' practice that highlight differences in conjectured *starting points* for the PD activities from teachers' perspectives – *pragmatic relevance* of the

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

issue under discussion to teachers' current classroom instruction, and its *legitimacy* within the institutional context of teachers' work. We propose that attention to these aspects when planning PD activities can enhance designers' capacity to capitalize on teachers' current instructional practices while pursuing PD agenda.

Theoretical Framework and Methodology.

The analysis is guided by a framework described by Cobb and colleagues (Cobb, McClain, Lamberg, & Dean, 2003) that coordinates individual teachers' learning with the development of collective practices of the Professional Teaching Community (PTC) as they are situated in the institutional setting of a school and school district. This framework was developed out of a practical need to account for teachers' learning in the social context of the PTC as it is enabled and constrained by the broader context of the institution.

The overarching goal for our collaboration with the group of teachers was to investigate how teachers' development of instructional practices that place students' reasoning in the center of planning and decision making can be supported. For that purpose we engaged in cycles of design and research where conjectures about the learning route of the teachers and the means of supporting it were continually tested and revised in the course of ongoing interactions. The methodology falls under the heading of a design experiment (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Following from Brown's characterization of design research, the collaboration with the teachers involved *engineering* the process of supporting teacher change. In this highly interventionist activity, decisions about how to proceed were constantly analyzed against the current activity of the teachers.

Method of Inquiry and Data Sources.

The data are taken from a collaboration with a group of 9-12 middle-school mathematics teachers that work in a large urban district in the southeast United States. The school district serves a 60% minority student population and is located in a state with a high-stakes accountability program. During each of the five years of our collaboration, we conducted six one-day work sessions and a three-day summer session. In the first 19 months of the collaboration, the group evolved to a professional teaching community (Wenger, 1998) with a joint enterprise, mutual engagement, and a shared repertoire (Dean, 2004).

A central principle guiding the analyses was to assume that teachers' participation in work sessions as well as instructional practices they develop in their classrooms are always reasonable and coherent from their perspective (Zhao, Visnovska, Cobb, & McClain, 2006). Our retrospective analysis of the data involved a method described by Cobb and Whitenack (1996), an adaptation of Glaser and Strauss' (1967) constant comparative method. The tentative and eminently revisable conjectures that were developed both prior to and while actually interacting with the teachers were continually tested and revised while working through the data chronologically, resulting in the formulation of claims or assertions that span the data set yet remain empirically grounded. In this paper, we focus on teachers' participation in several PD sessions in which researchers envisioned two types of activities to provide a viable starting point for teachers' further learning. We look at these sessions with hindsight, taking into account the retrospective analysis of the entire data corpus, which consists of videotapes of all work sessions accompanied by a set of field notes and copies of all teachers' work. The intent of this paper is not to present the analysis (for that purpose refer to Zhao et al., 2006) but rather to discuss distinctions highlighted by the analysis. Modified teaching sets (cf. Simon & Tzur, 1999) were also collected on each teacher at least twice a year. This entailed videotaping each teacher's class and then conducting follow-up audio taped interviews that focused on issues that emerged in the course of instruction.

Student Work

In the third year of our collaboration, we initially attempted to support teachers in focusing on their students' reasoning by engaging them in activities centered on analyzing their students' work. Student work has been characterized as an important means of supporting teachers in focusing on students' reasoning (e.g., Kazemi & Franke, 2004). We conjectured that through examinations and guided discussions of students' written solutions in PD sessions, opportunities would arise for the teachers to gain insight into the diversity of their students' reasoning that would be useful when they attempted to build on their students' solutions while conducting whole class discussions. In this way, we conjectured, the teachers would develop a need to capitalize on students' mathematical ideas in their instruction. Because of the teachers' daily experience with student work in their classrooms, we saw enhancing teachers' capacity of using student work in supporting their students' mathematical learning as an extension of their current instructional practices.

The specific questions that we posed in order to orient the teachers' analysis of their students' work and test our conjectures were as follows:

- What are the different solutions that you can identify from your students' work?
- How would you categorize students' solutions according to their levels of sophistication?
- How would you, as a teacher, build on these different solutions? Which solutions would you choose to talk about in class and why?

Our design conjectures around the use of student work proved to be unviable despite our detailed preparations. The teachers seemed to find the activity engaging and discussed their interpretations of the student work openly. Furthermore, most were able to discriminate between students' solutions in terms of levels of sophistication. However, the teachers' primary orientation was evaluative in that they assessed whether the instructional activity had been successful or not. This orientation that teachers took toward students' work was particularly evident when the researchers' question of "how are you going to build on students' different solutions?" received almost no response (and puzzled looks) from the teachers. The conversation within the work group came to a halt at this point in several consecutive PD sessions where the teachers discussed their students' work.

The teachers' orientation toward the use of students' work in PD session apparently deviated from our original expectation. This incident indicated that our design conjecture involving the use of students' work was ill-founded. Taking seriously that the activity, as constituted in the session, made sense from the teachers' point of view led us to realize that there was something about these teachers' classroom practices that we had yet to understand. We thus conducted an unscheduled series of teaching sets (Simon & Tzur, 1999) with all the participating teachers. The analysis of collected teaching sets revealed that within the teachers' current instructional practices, student work was primarily used as an evaluative tool of whether students "got it" or not. Teachers' instructional planning did not involve students' reasoning; rather, it was guided by objectives that had to be covered. Therefore, researchers' repeated attempts at renegotiating the purpose of looking at student work in terms of prospective instructional planning did not find grounding in teachers' experience.

This illustration is significant to our discussion. The use of student work as *envisioned* by the research team was not aligned with the teachers' views of student reasoning or the use of students' work in the context of their classroom instruction. For the teachers, the activity as *constituted* in the PD session was an extension of their classroom instructional experience albeit in ways that we did not expect. The research team's failure to support the teachers in focusing on students' reasoning resided in our failure to understand which interpretations of the designed PD task might be a viable extension of teachers' practices.

Student Motivation.

As the research team collected the teaching sets to understand teachers' unexpected orientation toward student work, we also sought to understand more viable foci for PD activities. We focused particularly on topics that the teachers were likely to consider problematic and worthy of further understanding. Additionally, when discussed and examined with our support, these topics should generate a *need* for teachers to look at instruction from students' perspective. The analysis of teaching sets revealed that the teachers perceived student motivation to be one of the most pressing issues in their classroom instruction (Zhao et al., 2006). The teachers viewed student motivation as intimately related to students' attention and engagement – issues with which they struggled in their daily instruction. They accounted for students' lack of attention and unwillingness to engage in instruction in terms of their *inherent* lack of motivation. Because the teachers felt motivation was out of their control (i.e., determined by factors outside of school) they viewed this issue as highly frustrating and problematic, preventing students from learning mathematics.

In order to capitalize on the teachers' instructional challenge of motivating students, the research team developed a conjectured trajectory of teachers' learning in which teachers' initial interest in student motivation could be supported to evolve into interest in student reasoning as a key instructional resource. First, we saw an opportunity to challenge the notion among the teachers that motivation is inherent, determined mainly by societal and cultural factors beyond the classroom (Zhao et al., 2006). We viewed this notion of motivation as problematic in that it deprived the teachers of opportunities to effectively teach children they saw as "unmotivated" and thus deprived these children of opportunities to learn. The research team conjectured that, in their planning and orchestrating of instruction, the teachers would benefit more from understanding student motivation as situational, reflecting whether the students could (or could not) see the relevance of *particular* classroom instructional activities in which they were required to participate. Second, we conjectured that once teachers developed a situational view of motivation they would consider building instruction on students' reasoning as an important aspect of math classroom that students find relevant and worthy of their engagement. In this way, adopting students' perspectives when examining issues of student motivation would have become a vehicle for supporting teachers in seeing students' reasoning as a key instructional resource.

To test the design conjectures we developed a sequence of PD activities. The teachers first discussed and examined their initial ideas about student motivation and instructional strategies that they used to encourage students' engagement in mathematical activities in their classrooms. We then presented the teachers with a case, developed from our prior NSF-funded research (Cobb, McClain, & Gravemeijer, 2003), in which the same group of middle-school students participated in two mathematics classes in the same school term but came to engage with the mathematics in sharply contrasting ways. Teachers were asked to analyze interview data in which the students described their obligations in each of the two mathematics classes and their valuations of those obligations. In their analysis, the teachers built on their own expertise and experiences with “motivated” and “unmotivated” students to evaluate the students’ participation in the two mathematics classes. To their surprise, the same students who appeared quite unmotivated in one class seemed highly motivated in the other. The surprising finding challenged the teachers’ notion that motivation is inherent to students and determined out-of-class. The teachers thus saw a need to understand what it was about the classroom instruction in one of the classes that helped to motivate the students.

In contrast to the previously discussed activity in which student work was the focus, this activity, as constituted in the session, reflected research conjectures. The researchers not only knew that student motivation was relevant to instruction from teachers’ perspective, we could also envision ways in which it was meaningful, from the teachers’ perspective, to approach the topic in ways that had face validity within their current instructional practices. Importantly, the preliminary analysis of the ensuing years of collaboration suggests that, by grounding the initial discussions in the context of student motivation, it was possible to support the teachers in changing significant aspects of their instructional practices in envisioned ways. Upon conclusion of the collaboration two years later, the teachers routinely adapted instructional materials that had proven effective elsewhere to their local setting. In doing so, they justified their decisions about including and modifying instructional activities by referring to the opportunities that the activities afforded for their students’ mathematical reasoning. This was an important development given that the teachers’ planning was previously primarily oriented by the objectives that they had to cover.

Revisiting Viability of Conjectured Starting Points.

Reflection on viability of design conjectures is a critical component in design research. It enables researchers to capitalize on contingencies that arise as the design unfolds (Cobb, Confrey et al., 2003). The fact that the research goals in the two conjectured learning trajectories were the same enabled us to look at differences in how the two interventions were initiated. In the remainder of this paper we discuss the identified distinctions in relevance of the conjectured starting points from the teachers’ perspective. We examine the conjectured starting points of the discussed trajectories along two complementary dimensions of teachers’ instructional reality. The first of these dimensions relates to teachers’ daily classroom experiences, while the second one concerns the institutional setting in which the teachers work. We outline two aspects of relevance of PD activities from teachers’ perspective – *pragmatic usefulness* of the issue under discussion to teachers’ current classroom instruction and its *legitimacy* within institutional context of teachers’ work. We use this distinction to discuss how teachers’ experiences from “inside” their classrooms, as well as those from “outside,” can be seen as shaping teachers’ interpretations of relevance of different PD activities to their instructional practice.

In the following discussion, we do not attempt to explain how teachers actually rationalized their engagement (or lack of it) in the designed PD activities. We suggest that the teachers were always engaged in activities that made the most sense from their perspective. They did not consider alternatives that would be, in their view, unreasonable. Our goal here is to hypothesize why one of the envisioned activities did not make sense from the teachers’ perspective while the other one did. We do so to enhance our ability to formulate more viable design conjectures.

Pragmatic usefulness.

Analysis of teaching sets revealed that, from the teachers' perspective, students' learning was a somewhat mysterious process where the same instruction leads to different learning outcomes with different students. It was therefore the students who should be responsible for their own learning and who should be held accountable to make the most out of the teacher's instruction. Teachers felt they had limited control of students' learning; nevertheless they were in charge of creating situations where it *could* happen. Thus, from the teachers' perspective, it was important that they (1) provide enough *opportunities* for their students to engage in mathematics outlined in state objectives, and (2) ensure that students pick up the opportunities and *engage* in mathematics instruction in desired ways (Zhao et al., 2006).

Against these findings, it became apparent why the teachers did not view the activity in which students' work would be used as a resource in planning as relevant to their classroom instruction. From the perspective of teachers who felt they had limited control over their students' learning, understanding students' reasoning did not seem most useful. Pragmatically useful issues that these teachers needed to consider when planning instruction were how to cover a specific objective in a way that would provide students with a variety of opportunities to engage with the mathematical idea, so that eventually the most students could "get it." Focusing on students' reasoning in planning instruction had unclear benefits to teachers and required compromising on what they saw at the time to be the best ways of planning and orchestrating instruction.

On the other hand, the teachers perceived issue of students' motivation as critical to their classroom instruction. They experienced difficulties in engaging their students in mathematics on a daily basis. Many of them developed explanations for this experience in terms of students' lack of motivation, which was predominantly determined by socioeconomic factors beyond their school. It is not surprising that these teachers saw activities that could help them address this critical aspect of their current practice as pragmatically useful.

Legitimacy.

In addition to teachers' classroom experiences on which they drew when interpreting meaning and intent of PD activities, we looked specifically at out-of-classroom activities within teachers' institutional setting as another resource for teachers' interpretations. The institutional setting in which the participating teachers worked can be generally described as a setting in which administration responds to the accountability pressures of state testing by attempting to monitor and assess teachers (Cobb, McClain, Lamberg et al., 2003). To understand issues that were constituted as a legitimate part of teachers' institutionally shaped jobs we analyzed school- and district-wide activities that were an inherent part of being a teacher in the district. We looked for issues that were treated as important in teachers' interactions with their principals, math coordinator, or one another.

Principal's visits in teachers' classrooms, faculty meetings, and district-organized PD serve as illustrations of activities in which teachers participated on a regular basis. During their routine (at times weekly) visits to teachers' classrooms, principals checked for teachers' coverage of the prescribed objectives, as well as for students' engagement. In doing so, they equated students' engagement with students' good behavior and appearance of attention to teacher's instruction. Among issues addressed at faculty meetings were pacification of misbehaving students and strategies for enhancing state test performance of borderline-failing students. District-wide PD for mathematics teachers focused on enhancing teachers' understanding of mathematical intent of newly adopted, reform-based instructional materials.

The contrast between opportunities created for teachers to reflect upon issues of student reasoning and student motivation as topics for professional discussion within their institution was apparent: Student reasoning as a resource for instructional planning was not a part of discussion in any school- or district-wide activities, and was not part of a vision for mathematics instruction in the district. Student motivation, on the other hand, was a persistent concern, as it presented a legitimate way to explain a lack of student attention during mathematics instruction and student misbehavior.

Discussion.

We characterized two aspects of design starting points that might impact viability of the designs in PD setting – pragmatic usefulness and legitimacy – by examining relevance of the two conjectured learning trajectories from teachers' perspective. Attempting to characterize the differences is important given that the two respective initial activities resulted in different opportunities for teachers' learning. The initial focus on student motivation allowed the research team to progress with the PD agenda to the extent that the teachers became proficient in anticipating students' reasoning when they planned for instruction. In contrast, the initial focus on student work did not result in comparable progress. However, the ways in which the PD activities were constituted in the sessions were grounded in the teachers' instructional practices in both cases. It was thus upon the research team to anticipate the viable extensions of teachers practices on which to build teachers' further learning.

Understanding key aspects of the setting in which student work did not prove to be a viable means of initiating teachers' focus on student reasoning is equally important. In juxtaposition with a wealth of studies in which student work was critical, this understanding could enhance theories of supporting changes in mathematics teachers' instructional practices.

Endnotes.

1. The analysis is part of a larger project focused on analyzing the commonalities and contrast between two sites. The collaborators are Paul Cobb, Kay McClain, Chrystal Dean, Teruni Lamberg, Qing Zhao, Jana Visnovska, Melissa Gresalfi, and Lori Tyler.

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HOW DO CHANGES IN TEACHER BEHAVIORS IMPACT THE LINKING OF REPRESENTATIONS AND GENERALIZATIONS IN STUDENTS?

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In the present study, we focus on the ways in which five urban middle school teachers interacted with their students in order to help them build representational fluency, with a particular emphasis on the ways in which the students moved toward increasingly abstract representations and generalizations. We document advances or changes in students' representational fluency, or facility with generalizations and abstractions, how teacher interventions impact this, and how the corresponding changes in students impacts teachers.

Introduction and Theoretical Framework

This paper addresses one component of a five year longitudinal study in which Rutgers University researchers partner with teachers, teacher educators, and administrators in the Newark Public Schools[1] in order to help students in grades kindergarten through eight develop a deeper and more meaningful understanding of mathematical concepts. In the current study, we focus on the ways in which teachers interact with students as they build, extend, link, and refine representations. In particular, we extend prior research (e.g. Schorr, 2004; Schorr & Lesh, 2003) in which the reciprocal relationship involved between teacher behaviors and actions on student behaviors and actions, and vice versa, is documented. This paper extends that work to include student development of representational fluency leading toward generalizations and abstractions.

Schorr, 2004; Schorr & Lesh, 2003; Schorr, Warner, Gearhart & Samuels, (in press) report that as teachers develop new knowledge, they notice new things about their students as well as their teaching practices. This, in turn, causes them to revise their approaches to teaching. As this happens, their students also change, thus resulting in further changes on the part of the teacher. Our central hypothesis is that when teachers encourage students to make sense of different types of representations (both their own and their peers) in a way that elicits meaning and sense making, students move toward increasingly abstract representations and generalizations. As this happens, teachers modify their behaviors, which elicits further changes in students' movement toward generalization and abstraction. Before continuing, we define, in a broad sense, the ways in which we use the words representation, abstraction, and generalization.

Representations: Broadly defined, the term representation "refers both to process and to product—in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself" (p. 67, NCTM, 2000). Some well recognized forms of representation include pictures, tables, graphs, diagrams, and strings of symbols. These forms of representation have a long established and highly stable place in school mathematics, and in many cases, are often taught for their own sake (NCTM, 2000; Kaput & Schorr, in press). Representations also take place in the context of spoken, kinesthetic, physical or cybernetic formats (Kaput & Schorr, in press), and can include mental representations, although it often happens that one makes use of physical materials in the process. Goldin & Shteingold (2001) offer a broad view of representation by describing it as sign or a configuration of signs, characters or objects. They note, “the important thing is that it can stand for (symbolize, depict, encode or represent) something other than itself” (p. 5). Speiser & Walter (1997) add a very important dimension in their description of a representation as “a presentation, perhaps to oneself, as part of an ongoing thought; or perhaps to others, as part of an emerging discourse.”

The development and use of various representations, whether verbal or written, are important for many reasons. For example, Lesh (1998) underscores the importance of representations that are well known to students by stating that students can build and use sophisticated constructs when these understandings are grounded in familiar modes of representation. Yackel (2002) shows how actions, diagrams and notation, as well as verbal statements, can function as an important component of argumentation. Kaput (1999) notes that when students use their own representations to reason and argue, link their own representations to abstract representations in an effort to justify ideas and use other representations to justify their generalizations, they develop a sense of ownership of the mathematical ideas.

Abstraction and Generalization: Abstraction and generalization are two of the most important aspects of mathematical thought (see Bochner, 1966, as cited in Kaput & Schorr, in press). Abstraction, which the NCTM Standards (2000) refers to “the stripping away by symbolization of some features of a problem that are not necessary for analysis, allowing the ‘naked symbols’ to be operated on easily. In many ways, this fact lies behind the power of mathematical applications and modeling” (p. 69, NCTM, 2000).

Kaput (1999) describes generalization as "deliberately extending the range of reasoning or communication beyond the case or cases considered, explicitly identifying and exposing commonality across cases, or lifting the reasoning or communication to a level where the focus is no longer on the cases or situation themselves but rather on the patterns, procedures, structures and the relationships across and among them” (p. 136). Mason (1998) notes that generality is at the heart of all mathematics and states, “Explicitly getting students to specialise from generalities, and to generalize from particular cases supports them in processes which are often left below the surface of awareness because they are so fundamental, so important” (p.3).

Teacher actions and behaviors leading toward abstraction and/or generalization: Mason (1998) notes that some types of teacher questions can lead students toward or away from generalizing. In particular, he notes that teacher questions can be used as springboards upon which students simply try to guess what is in the teacher’s mind while at other times, teachers can ask questions in a manner that genuinely supports students as they learn to analyze, generalize, and justify conclusions. Martino & Maher (1999) note that, “teacher questioning that is directed to probe for student justification of solutions has the effect of stimulating students to re-examine their original solution in an attempt to offer a more adequate explanation, justification and/or generalization” (p. 75). Questioning, however, is not the only mechanism by which teachers can help students to link representations, generalize or abstract ideas. Other researchers (e.g. Speiser & Walter, 1997; Warner & Schorr, 2004) note that when used appropriately, teachers can use students’ own representations to focus emergent explanations and justifications. One of the practical implications is that students need to be given the opportunity to construct their own representations of mathematical concepts and relationships, as well as the time and support needed to develop and use symbolic representations. Beyond that, Warner & Schorr, (2004); and Warner, Schorr, Gearhart & Samuels, (2005) note several other types of teacher behaviors that can impact students’ ability to link representations and move toward increasingly abstract representations and generalization. These include, for example, highlighting student ideas, encouraging students to build on their own and/or others' ideas, and setting up hypothetical problem situations based on an existing problem.

The research questions that will be addressed in this paper are: How do teachers interact with their students as they build, modify, and link representations and move toward abstraction and/or generalization; and, how do teacher actions impact student actions, and vice versa?

Methodology

Background

The research reported on in this study takes place in Newark, the largest urban school district in the state of New Jersey. It is based upon a pilot project in which the work of several middle school teachers was examined over the course of several years (Schorr, Warner, Gearhart, & Samuels, in press; Warner, Schorr, Gearhart & Samuels, 2005). The teachers who were involved in this study participated in professional development sessions with researchers both at the University and within the context of their own classrooms. During all aspects of the professional development, teachers had the opportunity to consider, amongst other things, mathematical ideas; classroom implementation strategies; student knowledge development; and, building a classroom culture in which proof, justification, sense making, and high cognitive demand are the norm (Stein, et al, 2000). Two key components of the professional development involved weekly meetings at a university and on-site support to the teachers as they implemented project activities in their classrooms (see Schorr et. al. in press, for a more complete description).

Subjects

The subjects for this research were five middle school teachers (teachers of grades 6, 7, or 8). In all instances, the teachers had participated in at least three months of professional development. See Table 1 for details[2]. The reader will notice that two teachers were relatively new to the project while the other three were involved for at least one full year.

Teacher	Total length of time teacher was involved in study	Grade level taught during the study	Duration of time the observations took place	# of actual minutes of videotape used for analysis[3]	# of sessions @ # of minutes each
Ms. A	3 months	6th	3 months	450 minutes	5 @ 90 min
Ms. J	7 months	7th	3 months	540 minutes	9 @ 60 min.
Mr. C	1.5 years	8th	3 months	810 minutes	9 @ 90 min.
Ms. E	2.5 years	7th & 8th	1.5 years	1260 minutes	9 @ 100 min. & 4 @ 90 min.
Mr. R	3 years	8th	6 months	960 minutes	5 @ 120 min. & 5 @ 90 min.

Table 1: Information for each Teacher

Data

The data that forms the basis for this study consists of at least three months of actual classroom sessions (University students videotaped the sessions). In each case, at least two video cameras captured different views of the teacher, students' group work, students' presentations, etc. In addition, the teachers were asked to provide written reflections immediately after each session. All student work was collected and descriptive field notes were compiled.

Analysis

The classroom episodes were analyzed (using observations, field notes and videotapes). We identified episodes by students' movement toward abstraction and/or generalization. These were then summarized, transcribed, and coded for instances of behaviors that appeared to impact the growth of student representations, abstractions, and generalizations. These were checked and verified by at least two researchers.

Results

The results will be presented according to several overarching themes that emerged from close analysis of the data. The themes represent some of the many types of interactions that took place between teachers and students regarding the development and use of representations, especially those that resulted in abstractions and generalizations. We note that while all teachers were unique in their development, there were many instances in which one or more exhibited similar types of behaviors. These are described in the sections that follow.

Overall reactions of teachers

In the early months of our observations, all five teachers allowed their students to solve complex problems in small groups. The two teachers (Ms. A & Ms. J) that had joined the project later on, appeared to be uncomfortable spending too much time on any one task—time that would be needed for students to move beyond their initial conceptualizations toward abstractions and generalizations. Ms. A actually noted that she had trouble understanding where the students' representations might lead. During the last month however, Ms. J became more comfortable with providing additional time for the students to explore a task, although she noted that she wasn't sure what to look for or to ask.

We also found, initially, that while the teachers allowed students to have extended periods of time to work on tasks and would go from group to group observing students and questioning them about their work, some of the teachers (Ms. E & Ms. J in particular) spent very little time with each group and tended to ask questions based upon their own thinking rather than on their student's. Over time, however, this decreased for all teachers. Further, the number of instances in which the teachers interrupted or interfered with a student's way of thinking decreased as well. Not coincidently, teachers appeared more able to follow students' ideas and build upon their thinking. In all cases there was an increase in teachers encouraging students to explain, question, use, justify, and build on their own or others' representations, and a corresponding increase in the students' willingness to actually talk about their ideas with their peers.

As the weeks progressed, Ms. A, Mr. C, Ms. E & Mr. R began to pose hypothetical problem situations (e.g. "What if…" scenarios) to the students as a way to stimulate movement to generalization. The students responded by posing their own hypothetical situations. The instances of this increased over the course of the observations (for a more complete description of what is meant by "raising hypothetical problem situations" see Warner, Coppolo & Davis, 2002). Ms. J. also began to raise hypothetical problem situations, but only at the end of the observation period. Several of her students began to raise hypothetical problem situations shortly thereafter. This appeared to contribute to students' movement toward abstraction and generalization in several cases for Ms. J & Ms. A and even more so for Ms. E, Mr. C & Mr. R.

Linking representations and the movement toward abstraction and generalization

A central tenant of the professional development involved encouraging teachers to build, extend, and link the representations that they had built for a particular idea. Early on in the study, several of the teachers expressed a desire to have their own students do the same. Mr. R, Mr. C and Ms. E all noted that by doing this, their students would most likely be able to formulate more succinct and convincing justifications. Early on in the observation period, Mr. C. noted: "I also like to investigate how students ideas and/or depictions of their ideas interrelate… I asked them, when all the groups were done presenting, was if they saw any relationships between each other's models, if so, what were they and if any one model would be sufficient to answer the question. If every student sees every other student's methodology and has it explained by the creator then solving this problem and other subsequent problems will become easier" (personal communication, March 24, 2005). In a similar vein, Ms. E writes:

> I had an aha moment today while watching … [videotapes of another class]. I think I know what I need to be doing when the students are starting to generalize….In the past, I thought if one student found the rule or the formula or the shortcut (whatever the students want to call it), that that student should present and everyone in the classroom will get it by listening from that one student. I realized … that the students do not internalize the rule if they do not find it themselves through exploration. The fact that only one or two students have generalized says to me that the class in general is not ready to see the formula. Thus, I should try to ask the students to "link the representations". This is what you [referring to the researchers] told me before, but like the students I have not internalized what you mean by that. The goal is not for one student to find a formula and to present it to the class, but for all the students to try to link different kinds of representations such as diagrams, reports, mathematical representations, and so on (personal communication, September 25, 2005).

As Mr. R, Mr. C & Ms. E discussed this idea with the researchers, and each other, there was an increase in their students linking representations and movement toward generalization.

Encouraging students to justify their generalizations

Early in the study, Mr. R wrote, "Looking back… I see now that a lot of my students did get it, but I didn't allow them to explore what they had....If I had allowed them to vocalize their solutions and come to some real ownership, then several months later when asked to express these ideas in a different context, they'd be able to apply it" (personal communication, May 24, 2004). In the earlier part of our observations, Mr. R, Mr. C & Ms. E began to encourage the students to continue to explore even after they developed a generalized solution to the problem. They also encouraged the students to create more than one type of generalization or express their generality in different formats (when appropriate). Mr. R and Mr. C also asked students to justify their generalizations or general statements and look at the relationships between them. At the end of the observation period, the students in Mr. R's classroom appeared to be dissatisfied unless they could generate multiple generalizations, and convince each other of the efficacy of each. Indeed, at a certain point in time, Mr. R. no longer had to instigate these discussions. The students spontaneously asked each other to explain and link their generalizations and justifications.

Allowing students to assume ownership of their ideas

We noticed that at the beginning of the observation period, students in all classes sought their teacher's approval for their ideas, questions and solutions. Mr. R and Ms. E dealt with this by encouraging the students to challenge, question, and direct their responses to each other. They shared that they often used cues determined by "reading" the body language, facial expressions and/or gestures of the students as a way to encourage them to talk directly to each other. Their students did talk directly to each other, but this did not happen consistently, except in the case of Mr. R, where by the middle of the observation period, his students consistently directed their comments to each other, without using him as an intermediary. Further, his students challenged each other to explain all representations and ideas, demanded that these be connected, and felt that they could and should try, when possible, to build generalized formats that connected solutions to the problem and hypothetical extensions of the problem. Mr. R. noted that, "the questions that we ask serve as models for the students. They begin to ask the same types of questions of each other as we do of them. Students are no longer satisfied with the answers to the problems that are posed, rather they seek proof and justification and question until they are satisfied. As students began to question each other with skill, tasks took on a whole new life." (personal communication, June 7, 2004). The other two teachers (who were newer to the project) did not encourage this to the same extent, and their students did not tend to "talk" directly to each other.

Finally, we note that as Lannin, Barker & Townshend (in press) point out, there can be great benefit in having students investigate their own errors as a way to deepen their understanding and ability to generalize. All five teachers attempted to use errors as a way to probe students, and they also tried to minimize the number of instances in which they actually "told" students that they were incorrect. Rather, they tried to let students investigate their errors by encouraging them to justify their solutions. All of the teachers noted the importance of not "telling" students the answer, but many had difficulty in actually doing this. There were many occasions, particularly at the beginning of the observations, where they used facial expressions, intonations, or body gestures, to let students know exactly what they were thinking, even when they did not directly tell students the answers. Once again, Mr. R. was unique in that by the middle of the observation period, his words, voice, posture, and facial expressions suggested neutrality.

Conclusion

Teachers in our study appeared to proceed through several steps as they helped students link representations and move toward abstraction and generalization. In the beginning, many did not see the advantage in having students engage in lengthy explorations wherein such activity could take place. Once they recognized the importance, they had difficulty in actually making it happen. Some would ask students questions that they felt would stimulate representational fluency, generalization, and abstraction, but upon reflection, noticed that their questions were often based on their own preconceived notions about what a "correct" representation should be. Some teachers used subtle cues (i.e. facial expressions), while others actually felt the need to more directly steer students in their problem solving process. As teachers encouraged students to, for instance, investigate hypothetical problem situations, link representations, etc., we found that the students actually started demanding this themselves. This, in turn, served as a feedback mechanism for the teachers, wherein they felt a sense of affirmation for their new practices. Ms. E, one of the teachers involved in the study, and an author of this paper, best sums it up by saying, "It is when the teacher models certain behaviors such as requiring students to justify their answers, pose hypothetical situations, provide opportunities for them to generalize and abstract representations, that students emulate them to the extent of even owning these behaviors after several times of being on the receiving end of the experience…the possibilities of exploring and learning mathematically challenging ideas are extended way beyond the finding of a generalization that is derived from the problem initially posed" (personal communication, January 29, 2006).

Endnotes

1. The material contained herein is based upon work supported by the U.S. National Science Foundation (NSF) under grant numbers 0138806 and ESI-0333753. Any opinions, findings and conclusions or recommendations are those of the authors and do not necessarily reflect the views of the NSF, Rutgers University or the Newark Public Schools.
2. Multiple camera views do not add to the number of minutes.
3. The number of minutes varies by school and year.

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EXPLORING THE RELATIONSHIP BETWEEN ACADEMIC SELF-EFFICACY AND MIDDLE SCHOOL STUDENTS' PERFORMANCE ON A HIGH-STAKES MATHEMATICS TEST

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This study investigates the relationship between middle school students' academic self-efficacy and performance on the TAKS high- stakes mathematics test. The baseline sample was (n=2,508) middle school students. Self-efficacy scores did not have a significant interaction with gender, but were found to decrease significantly (as do TAKS passing rates) as the grade level increased – a sobering trend. This study has major implications for pedagogy and curricular approaches and interventions.

This study aimed to explore the possible relationship between middle school students' academic self-efficacy and performance on a high stakes mathematics test. Researchers are beginning to recognize that both the cognitive and affective aspects of learning are present when students construct mathematical understandings. Affective influences are hard to measure, but clearly influence students' participation in mathematics study and careers. Research indicates that self-efficacy influences academic motivation (Pajares 1996; Schunk 1995). Self-efficacy is one's self-judgments of personal capabilities to initiate and successfully perform specified tasks at designated levels, expend greater effort and persevere in the face of adversity (Bandura 1988, 1986). Self-efficacy is grounded in a larger theoretical framework known as Social Cognitive Theory which supports human achievement as dependent on interactions between one's behaviors, personal factors and environmental conditions (Bandura 1986, 1997).The work of Bandura concerning self-efficacy indicates that students' perceptions of their abilities to perform tasks, greatly influences their success. Self-efficacy beliefs influence task choice, effort, persistence, resilience, and achievement (Bandura 1997; Schunk 1995). Compared with other students who doubt their learning capabilities, Pajares and Schunk find that those who feel efficacious for learning or performing a task participate more readily, work harder, persist longer when they encounter difficulties, and achieve at a higher level. The stronger the perceived self-efficacy, the higher the goal challenges people set for themselves and the firmer is their commitment to them (Bandura 1991). Ability is not a fixed attribute residing in one's behavioral repertoire. Rather it is a generative capability in which cognitive, social, motivational, and behavioral skills must be organized and effectively orchestrated to serve numerous purposes (Bandura 1993).

The measures used were the Middle School Self-Efficacy Scale (validated in Fouad, Smith, Enochs 1997) and the Texas Assessment of Knowledge and Skills (TAKS). Regression and two-way ANOVA were used for analysis.

Baseline data was collected from 1,148 students at Jackson Middle School and from 1,360 students at Waldo Middle School (names are pseudonyms for schools in El Paso County). Based on TAKS passing rates in each subject for all students (and for each ethnic student subgroup meeting minimum size), schools are monitored and classified as "academically unacceptable", "academically acceptable. One school is considered a "high performance" school and the other is a "low performance" school. Scores were matched on ethnicity and grade levels. A subset of

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

students from Jackson Middle School was selected to match individual scores on both instruments using a regression to determine the relationship between academic self-efficacy and math scores.

High self-efficacy scores were consistent with mean TAKS scores across both schools and among grade levels. TAKS scores and self-efficacy scores decreased across the middle school experience with both schools. However, there is a decrease in mathematics test scores as students progress through the middle school grades. This pattern is evident in both the lower socioeconomic and the higher socioeconomic middle school samples. There is a decrease in students' self-efficacy scores as they progress through the middle school experience across socioeconomic status variables. These patterns are consistent with both sample populations.

A two-way ANOVA indicated that self-efficacy scores were significantly influenced by the school attended: $F(1,790) = 5.181$, $p = .023$. There was not a significant influence from gender ($F(1, 790) = .049$, $p = .825$), nor was there significant interaction between school and gender ($F(1,790)=1.879$, $p = .171$). The positive correlation ($r = .215$) between individual students' TAKS and self-efficacy scores is highly statistically significant ($n = 406$, $p < .001$).

This study supports the PME goals by seeking to further a deeper understanding of major psychological aspects of learning and teaching mathematics. Additional related detail appears in Blake, Lesser, Perez, Fonseca, Jablonski, and Gallo (2006).

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UNDERSTANDING TEACHERS' USE OF THE TEACHER GUIDE AS A RESOURCE FOR MATHEMATICS INSTRUCTION

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Teachers may draw upon a variety of curriculum resources, including the teacher guide (TG), when planning for and during instruction. As it is designed to support teachers in making pedagogical decisions, teachers potentially vary in their use of the TG. Accordingly, this study examines how teachers use the TG as a resource to inform their planning and instructional decisions.

In response to national reform efforts (NCTM, 1989), some curriculum developers have designed mathematics curricula to help teachers integrate these reform ideas into their practice. Such curricula, however, place new demands on teachers and teacher learning. Accordingly, the designers of the Connected Mathematics Project (CMP), in particular, have developed materials with the specific intent to help both teachers and students learn mathematics, as well as to provide teachers with the necessary pedagogical support to help students learn. Little is known, however, about how teachers use these materials to inform their teaching decisions and actions. This study examines how four middle school mathematics teachers use the CMP teacher guide (TG) to inform their planning and instructional decisions.

There are a variety of factors that potentially influence teachers' use of the TG. Researchers, in particular, have pointed to experience as influencing the nature of teachers' decisions and actions in the classroom (Borko & Shavelson, 1990; Leinhardt & Greeno, 1986; Livingston & Borko, 1990). As mathematics teaching is inherently complex and uncertain, experienced teachers seem better able to manage such complexities and uncertainties as they arise in the course of their practice. Accordingly, this study examines four experienced (5+ yrs. teaching experience, 3+ yrs. curricular experience) teachers' use of the CMP TG in planning for and enacting mathematical tasks.

Ethnographic methods in combination with grounded theory methods were employed in this study. Four experienced 6th grade mathematics teachers who were currently using CMP volunteered to participate in this study. Individual teacher interviews and classroom observations constitute the data sources for this study. The interviews and observations were then used to inform the writing of case studies for each teacher, highlighting patterns and themes in terms of teachers' use of the TG in order to make comparisons across teachers.

Results and Discussion

Findings from this study indicate that a variety of factors mitigate experienced teachers' use of the TG in planning for and enacting tasks. For example, teachers' views of mathematics teaching influenced, in most instances, the content and ways in which teachers engaged students with the task. In addition, teachers' underlying conceptions of the CMP materials, their teaching experience, and their curricular experience (or curricular inexperience) in having previously taught the task impacted how teachers used the TG in planning for and enacting tasks. Other factors also emerged as influencing teachers' use of this curricular resource, though not as consistently or to the same extent. Teachers' perceived time constraints, for example, impacted their enactment of tasks, and hindered them from enacting the task as they had intended and often led to a decline in the level of classroom discourse and mathematical activity.

Looking across the four teachers' practice in light of their enactment of the different tasks, teachers' particular conceptions of and orientations towards mathematics teaching, in large part, explain the extent to which their enactment of different tasks reflected the ideas underlying CMP. For example, Alicia stated and displayed a more conventional orientation towards mathematics teaching, leading much of the class discussion and focusing students' work on practicing computations. Despite her use of the TG in planning for tasks, Alicia enacted the tasks in a more conventional fashion. Thus, it appears that Alicia's overall view of mathematics teaching has become somewhat cemented throughout her teaching career and seemed to have hindered her from planning for enacting CMP tasks in accordance with the underlying principles. Similarly, Tiffany exhibited a more conventional orientation towards mathematics teaching, despite her stated philosophy of mathematics teaching and professed agreement with the CMP approach to learning and teaching. Although she regularly used the TG in planning for tasks and largely planned to discuss the central ideas of the task, Tiffany resorted to more conventional teaching methods and focused on the procedural aspects of the task, thus constraining the learning opportunities afforded by the task. As with Alicia, Tiffany's views of mathematics teaching appear to have been continually reinforced over the course of her career, and these views seemed to prevent her from enacting the curriculum in accordance with its underlying principles.

Richard, on the other hand, exhibited a strong orientation towards non-conventional teaching practice. Despite his limited use of the TG, Richard largely enacted tasks in ways that reflected the ideas espoused in the curriculum. Richard's approach to mathematics teaching may also have been cemented over time, and despite the fact that his views differ from Alicia's and Tiffany's views, they strongly influence his implementation of the curriculum. Susan similarly exhibited a strong orientation towards non-conventional teaching practice. Unlike Richard, however, Susan regularly used the TG as a prescription for enacting tasks. Even when enacting old, familiar tasks, Susan's enactment largely reflected the TG suggestions as described in the LES section. Unlike Richard, Tiffany, and Alicia, Susan's use of the TG and her enactment of tasks seemed to be driven largely by the text of the TG itself and not her views of mathematics teaching.

In short, teachers' conceptions of mathematics teaching are readily apparent in the presence of extensive teaching experience. Teachers who do not have such extensive teaching experience consider the curriculum, particularly the TG, as more of a script for enacting tasks. However, even when less experienced teachers view the TG as a prescription for instruction, their views of mathematics teaching play a more prominent role in their enactment of tasks when faced with various teaching problems. Even for experienced teachers whose particular views of mathematics teaching are regularly apparent in their practice, their views also underlie how they manage teaching problems in the course of their work. Thus, it seems that neither teaching experience nor curricular experience impacts the ways in which teachers use the TG to manage teaching problems. Instead, teachers rely on their personal resources to manage teaching problems as they arise in their practice.

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COOPERATIVE LEARNING: A PERSPECTIVE FROM MATHEMATICS PROFESSORS' EYES

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This paper documents the experiences of four mathematics professors as they incorporated cooperative learning into their entry-level mathematics courses. Using an autoethnographic approach, the participants documented changes in how they used cooperative learning in their classes as well as how their beliefs and understanding about cooperative learning changed over the course of the study.

The traditional college mathematics classroom was dominated with teacher-centered teaching (Rogers, 2001) until recently. As various pedagogical approaches made their way into elementary and secondary schools, American universities began to investigate if these alternatives to lectures would be valuable in the post-secondary classroom. One such approach was cooperative learning. In the 1990s, publications and studies were published suggesting how cooperative learning could be used in the university classroom (Foyle, 1994; Johnson, Johnson, & Smith, 1991), and specifically in the university mathematics classroom (Dubinsky, Matthews, & Reynolds, 1997; Rogers, 2001). While these publications address various aspects of cooperative learning, including basic elements of effective cooperative learning, implementation, assessment, the role of the teacher, and classroom strategies, none of them address the experience of professors as they work to implement cooperative learning in their own classrooms. This paper documents the experiences of four university mathematics professors as they began to implement cooperative learning into their entry-level mathematics courses.

Methodology

Because the participants in the study were the researchers themselves in their own classrooms, and the data collected was from the self-reflections of the participants as they interacted in their own classrooms, it was natural for this study to be an autoethnography. An autoethnography is "an autobiographical genre of writing and research that displays multiple layers of consciousness, connecting the personal to the cultural. Back and forth autoethnographies gaze, first through an ethnographic wide angle lens, focusing outward on social and cultural aspects of their personal experience; then, they look inward, exposing a vulnerable self that is moved by and may move through, refract, and resist cultural interpretations." (Ellis & Bochner, 2000) Autoethnographies can give deep reflective accounts of one's personal experiences while immersed in a new situation.

Each of the four professors in this study chose an entry-level mathematics course in which to use cooperative learning as an integral aspect of the course design. In preparation of this, we read relevant articles and met as a group over the summer of 2005 to discuss these articles and reflect upon our concerns and curiosities. As the fall semester approached, we designed our courses to reflect various aspects of cooperative learning, including regular group interaction, interdependence on assessments, and group accountability.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Data Collection and Analysis

Before beginning our research on cooperative learning, each participant kept a journal to document initial feelings, beliefs and attitudes regarding cooperative learning. As the project progressed, we maintained these journals to document changes in our use of cooperative learning, along with our struggles, successes, and changes in beliefs. We also met as a group throughout the process to talk about our experiences and support each other during the process.

Each meeting was recorded so they could be later transcribed. The journals were maintained electronically and shared with the group at the end of the study. After two semesters of using cooperative learning in entry-level mathematics courses, these meeting transcriptions and journals were analyzed using constant comparative analysis. (Denzin & Lincoln, 2000)

Findings and Conclusions

Several trends in the data emerged after analyzing the data. The first was that each of the participants initially stated that they had used cooperative learning prior to the study. After reading formally about cooperative learning in preparation of the study, each of the participants made a distinction between "students working together" and "true cooperative learning." This distinction relied on whether students began to take responsibility of the learning of the members of their group. As the study progressed, the participants recognized noticeable differences between what they had called cooperative learning in the past and how they utilized cooperative learning during the study. However, each of the participants documented the challenge of adapting true cooperative learning and having students take responsibility for their own and their groups' success. A second theme was that each of the professors initially had doubts about fully immersing their class in a cooperative learning environment. Each questioned if some lessons were best left to individual learning, if some lessons were too easy to benefit from cooperative learning, or if some students would resist the process and disrupt the experience for the class. At the end of the study, each of the participants reiterated these concerns, documenting that some lessons were more natural for cooperative learning than others, and that struggles with individual students disrupting the cooperative process were common. The third noticeable trend was that each participant recognized the importance of a supportive group of other professors with which to study cooperative learning and to share their experiences and difficulties.

While cooperative learning has proven itself worthwhile in many different learning environments, little research has documented the experience of the professor. The stories of these four professors may be of benefit to others as they begin to experiment with cooperative learning in their classroom.

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"I LEARNED HOW TO…": RESULTS FROM THE TECHNOLOGY IN MATHEMATICS EDUCATION PROJECT

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This paper describes how the Technology in Mathematics Education Project impacted participating middle grades and secondary mathematics teachers' preparedness to teach via technology. Results indicated that the project positively impacted participants' in that regard. Accordingly, the methods employed in the TIME Project appear to be viable avenues for preparing middle level and secondary mathematics teachers to teach with technology.

Many teachers do not believe that they are well-prepared to infuse technology into their pedagogy. Accordingly, the Technology In Mathematics Education (TIME) Project was developed to help middle and secondary mathematics teachers enhance their knowledge of resources and methods for teaching mathematics via technology. It was believed that such learning would foster significant improvement in:

1. Participants' perceptions of their knowledge of a wide variety of technological resources (videos, software, calculators, Internet, etc.) and methods of using them to teach mathematics, and
2. The frequency with which participants incorporated technological tools into their mathematics instruction.

Hence, this paper relates to the conference theme of integrating theory and practice in that the paper concerns methods of preparing in- and pre-service teachers to infuse technology into their pedagogy, which is a practice that has been endorsed on theoretical and logical grounds. Further the instructional methods employed in the project are consistent with contemporary theories concerning best instructional practice. Accordingly, this paper highlights the effectiveness of efforts to use theoretically sound educational strategies to empower mathematics teachers to teach via technology.

Participants

Nineteen middle level or secondary mathematics teachers participated in the TIME Project. Four of the participants were certified (1) or seeking certification (3) in middle level mathematics, 1 was certified in elementary education but taught at the middle level (4th grade), and 14 were certified in secondary (7-12) mathematics. Eleven of the participants had at least 5 years of teaching experience, five had from 1 to 4 years of experience, and three participants were pre-service teachers and therefore, had no teaching experience. The participants represented 10 school districts, 3 rural districts and 7 urban districts.

The Context

Participants completed a course that focused on exploring resources and methods for teaching mathematics with technology. Eleven of the secondary participants completed a five-day course during the summer of 2003 and 8 middle level (4-8) teachers completed a 12 week course during the fall of 2003. Both courses entailed 35-40 contact hours and focused on

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methods of teaching mathematics via technology, but some activities were different in order to better meet the varying needs of the middle and secondary participants. Constructivism was used to inform the course instructors' instructional methods. Therefore, a premium was placed on engaging participants in activities through which they could explore a variety of mathematical topics and problems relevant to middle and/or secondary mathematics through a variety of technological resources. These methods allowed participants to become familiar with technological resources while experiencing the use of those resources from a student's perspective. The methods also afforded participants opportunities to explore how multiple resources could be used to teach a single topic. In addition, participants constructed lesson plans and technology infused activities that allowed for independent exploration. Writing was also incorporated into the curriculum by having participants explore and write critiques of technological resources, usually software. It was believed that such experiences would better prepare participants to teach via technology. Technological topics addressed in the course included using videos and multimedia programs to enhance instruction; evaluating resources; using resources such as *Geometer's Sketchpad*, graphing calculators and spreadsheets to solving problems, teach concepts, and link mathematics to realistic contexts. The course was structured to approach such topics by setting them in explorations of mathematical topics such as probability, patterns and sequences, linear regression, data representation, distance-rate-time problems, limits and mathematical modeling. Participants not only completed the course, but attended follow-up meetings intended to support participants' efforts to teach with technology. Support was provided both by providing participants with a forum in which to reflect on and discuss their efforts to infuse technology into their pedagogy and by exposing participants to additional activities and resources for teaching mathematics with technology.

Methods & Data Sources

Both quantitative and qualitative data were collected and analyzed, and results associated with both data types of will be addressed. The primary data collection instrument was a questionnaire developed by the investigator that contained both five-point Likert scale and open-response items. Participants could choose from a standard 5-point scale of responses ranging from strongly agree (5) to strongly disagree (1). The survey was completed on the first and last days of the course. Likert scale items focused on the frequency with which the participants integrated technology into their instruction, their perceptions of their knowledge of and preparedness to teach with various technological resources, and their perceptions of how the project impacted their ability to teach via technology. Chi-square tests and one-sample tests of population proportion (z-tests) were conducted to determine if significant differences across pre- and post-tests existed in the frequency of responses to Likert-scale items, with an alpha level of .05 used in all analyses. To help establish consistency between participants perceptions of their preparedness and their knowledge of resources and methods of teaching with technology, open-ended items asked participants to outline a lesson that incorporated the use of technology, to list technological resources that could be used to teach mathematics, and to explain both how the TIME Project impacted their ability to teach with technology and how the Project could be improved. Data from the open-ended questions on both pre- and posttests were examined to identify trends in the participants' responses, which was a very simple form of coding. One-time observations and interviews of three participants' efforts to use technology as a teaching tool were conducted to verify that participants were teaching with technology in manners that were consistent with the vision of the project.

Results & Implications

One hundred percent of the participants indicated that participation in the project enhanced their ability to teach with technology and 95.24% indicated that they would recommend the project to other teachers (One respondent was neutral on that issue.). A z-test revealed that participants reported that the project had a positive impact and would recommend the project to others at rates that are significantly greater than 50% ($p > 0.0001$ in both cases, $n = 19$). Two way chi-square analysis across pre- and post-course surveys of participants' responses also revealed significant differences in the frequency with which participants chose SA or A versus all other responses on the following, among other, items ($n = 19$).

- At this point in time, I am familiar with a wide variety of technological resources. ($p = 0.0009$)
- At this point in time, I am familiar with a wide variety of methods of teaching mathematics through the use of technological resources. ($p < 0.0001$)

The preceding results consistently implied that the TIME Project was positively impacting participants' knowledge of methods and resources for teaching middle level or secondary mathematics via technology.

Qualitative data also indicated that the TIME Project was attaining its goals. The following statements are representative participant responses to the question of, "How has participation in the TIME Project helped prepare you to use technology as a teaching tool?"

- I have learned methods of using spreadsheets and graphing calculators that I did fully not understand. (Fall Participant)
- It made me familiar with the TI-83 calculators, CBR & CBL's in my room. I can now find many technology lessons on the Internet. I can send email with attachments. (Fall Participant)
- I knew almost nothing about the TI-83, Sketchpad, Power Point, and spreadsheets. Now, I feel confident to use any of these in teaching a lesson. (Summer Participant)
- My knowledge of technology has been greatly increased. I feel more comfortable with some technological tools that in the past, I would have been afraid to touch. (Summer Participant)

Observations also indicated that participants were using technology as an instructional tool in ways that were compatible with the goals of the project. A seventh grade teacher, Paula helped her pupils enhance their knowledge of pie graphs, box and whisker plots, scatter plots and histograms by pairing students and having them input data into lists of TI-74 graphing calculators and then generate graphs. Cathy, a 7-12 instructor, had her 7th and 8th grade students complete a culminating project that she created for a unit on interest that required pupils to use spreadsheets to calculate compound interest on an investment.

Based on the findings noted herein, it appears that the TIME Project positively impacted participants' preparedness to teach via technology as well as the frequency with which they do so. Accordingly, it appears that a viable method for empowering middle level and secondary mathematics teachers to infuse technology into their pedagogy is to engage the teachers in activities in which a variety of technological resources are used to explore a variety of problems and topics that relate to the level at which the participants teach.

MATHEMATICS IN THE SCHOOL: PRACTICES IN SITUATION. A STUDY CARRIED OUT IN MEXICO CITY

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Ideas and expectations of 337 teachers of public primary schools in Mexico City related to students and their mathematics performance were studied. This investigation is part of a wider research carried out with 2 000 educators in the Federal District (FD) and three states of Mexico.

The Research: Approaching Reality

I. Teaching students in order to solve problems and to make decisions in different situations from their environment is one of the prime objectives of the national *curriculum* of mathematics in primary school. Nevertheless, the national and international assessments carried out in the country during the last six years show that student's basic education is far from managing (see for example, SEP, 2001; OECD, 2003, and TIMSS, 1995). II. The authorities have impelled actions to improve the quality of education. Among others one can mention: courses and materials for teachers; textbooks for students; supply schools of electronic and technological resources, and diverse stimuli programs for teachers and institutions. One could ask, why all those actions do not reflect on the results? III. Few studies of the educational system and the effects that actions, like the aforesaid, have in students' mathematics performance have been done in Mexico. Although, assessment of primary and secondary education have been carried out, it is until 2002, with the creation of the National Institute for the Assessment of Education (Instituto Nacional para la Evaluación de la Educación) (INEE, 2005) that an evaluation system of basic education at national level is designed and setup in order to understand educational achievement and to plan intervention strategies to enhance the quality of education. IV. Results from studies made in different countries show that teachers' beliefs about mathematics and their purpose in basic education influence their teaching, their teaching expectations and their students' learning (Thompson, 1999 and Gilian, Pehkonen & Torner, 2005).

Research Purposes

In this national context, in 2002, a research project was setup in the Federal District and three states of Mexico in order to obtain information about the factors that affect low performance in mathematics and are attributed by teachers to the students. To reach that goal, the attention was focused on the ideas that teachers have of good students in mathematics and those that are classified by them as low attainers.

Survey Design

Due to the fact that the working teams of researchers of the global study are distributed in states located in the center, south and northwest of Mexico, it was decided to make a survey using a paper and pencil questionnaire in order to control how the questions are posed and to apply it to a big number of teachers. The questionnaire was done in the following way: 1) questions were written and an internal validity process was carried out; 2) teachers from the State of Mexico were individually interviewed; 3) an opinion type survey based on a gradation scale

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was buildup; 4) a pilot study was carried out in the four regions of the country; and 5) the final version of the questionnaire was made. This questionnaire is composed by 12 groups of questions. In this paper groups (V, VI) and (IX, X) designed to characterize ideas about good students in mathematics and group (VII, VIII) about low attainers are considered.

The Study In The Federal District

The results of the so called IDANIS test (Diagnostic Exam for First Grader Secondary Students - Instrumento de Diagnóstico para Alumnos de Nuevo Ingreso a la Secundaria) were used to choose a representative sample of primary school teachers that work in the FD. The IDANIS test consists of three parts; nonetheless, only the results of the mathematical ability part assigned to each school were used, the numbers associated are contained in the interval (12.50, 88.33). This interval was divided in ten parts, which allowed the classification of the 2 260 FD public schools in classes. The sample size, 35 schools, was conditioned by the number of team members. The schools were chosen in the following way: 1) using proportional affix to determine the number of schools in the classes, and 2) the institutions of each class was selected randomly. Some classes had a small number of schools compared with others; for that reason, to analyze the data, the 10 classes were grouped in 4 strata: Ep+, Ep =, Ep-, and Ep--. This process determined the representative sample of teachers and its size.

Data Analyses

In the groups (V, VI) and (IX, X) part of the answers correspond to a gradation made by teachers according to levels of importance of characteristics associated to good mathematics students. In the other part of these groups of questions, teachers were requested to choose five of those characteristics and order them from more to less importance. Therefore, one of the analyses carried out with the data required the building up of two indices: Priority Index and Importance Index. For groups (VII, VIII) a scale of frequencies was used for statements related to low attainers behaviours. In the other part of this group of questions teachers were asked to choose five statements that could characterize better those students.

Some Results

I. Both, the Importance and the Priority indices allow a comparative interpretation of the data. With this view one can assert that teachers' choices revel a representation of students' mathematics performance they have buildup based on their teaching experience. II. Moreover, it was possible to identify that teachers adapt their professional knowledge to particular conditions of their working place and fit their expectations to what they believe is possible to obtain in their classroom; that is to say, practices in situation can be characterized for different strata, or to the accomplishments of the school as a whole. III. It is interesting to stand out that teachers' image of good mathematics students goes beyond to what the school can provide. IV. Some teachers recognize low attainers' behaviours that are successful and enable opportunities for them. In spite of this fact, negative aspects are enhanced without considering that more than not students are given tasks that exceed their capabilities, creating a vicious circle that confirms their poor image of those students.

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AN INVESTIGATION OF TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING: THE CASE OF ALGEBRAIC EQUATION SOLVING

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This presentation introduces a conceptual framework for examining teachers' mathematical knowledge for teaching procedures and related concepts, which includes two dimensions: Three forms of mathematical knowledge (subject-matter, learner conception, didactic representation) and Three aspects of mathematical procedures (basic algorithms, alternative algorithms, related concepts and procedures). It is applied to analyzing secondary school mathematics teachers' knowledge for teaching algebraic equation solving. Some preliminary data and results are discussed.

Background and Research Question

Solving algebraic equations is a central topic in traditional school algebra curricula. Although there have been extensive studies on students' understanding of equations and equation solving, few has been conducted with mathematics teachers. The knowledge that teachers employ for teaching equations solving becomes a particularly important issue for inquiry when the function-based approach has been reshaping school algebra curriculum, teaching, and learning in the past decade, and challenging the conventional, formal rule-based approach to equation solving.

This presentation provides a sketch of the presenter's ongoing dissertation research on secondary school mathematics teachers' knowledge for teaching with specific focus on algebraic equation solving. The main research question is: What kind of mathematical knowledge do secondary school mathematics teachers draw upon when handling fundamental problem situations related to the teaching and learning of algebraic equation solving?

Theoretical Framework

A key piece at display is a conceptual framework for examining teachers' mathematical knowledge for teaching procedures and related concepts. It is constructed based on several pieces of work: a summary of existing theories and conceptualizations of mathematics knowledge for teaching (e.g., Artigue, Assude, Grugeon & Lenfant, 2001; Ball & Bass, 2000; Burrill, Ferrini-Mundy, Senk & Chazan, 2004), a review of research on students' and teachers' understanding of equations and equation solving, an analysis of the ways equation solving is treated in selected algebra textbooks, as well as the presenter's own empirical experiences in working with mathematics teachers. The framework includes the following two dimensions:

Forms of Mathematical Knowledge

Subject Matter. Teachers' knowledge of the mathematical content as the subject matter of a scientific discipline and a course of study.

Learner Conception. Teachers' knowledge of learners' mental representations of the content, including learners' typical pre-conceptions, misconceptions, errors, and learning trajectories.

Didactic Representation. Teachers' knowledge of the content as represented by instructional media and strategies, including sequencing of units and topics, various examples, metaphor, models, tasks, tools, and technologies used.

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Aspects of Mathematical Procedures

Basic algorithms. The algorithm(s) that are most general and precise for carrying out the procedure, and most efficient for most cases. They are usually taught as the standard or major algorithm(s) in school mathematics curricula

Alternative algorithms. Algorithms and strategies for the procedure that are less often or formally introduced in school mathematics curricula, or have lower generality, efficiency, or precision, compared to the basic algorithms

Related concepts and procedures. The concepts and procedures that are built directly upon, or are closely tied to, the given procedure

Research Instruments

The framework is applied to analyzing the mathematical procedure in focus: algebraic equation solving. Based on the second dimension of the framework, about a dozen of concepts and algorithms are identified and selected for the study. Three sets of research instruments are being developed: 1) A demographic survey on the participating teachers' academic background and algebra teaching experience. 2) A set of open-ended items on knowledge of algebraic equation solving, most of which embedded in teaching and learning contexts, and 3) A semi-structured interview protocol. Each open-end item includes several questions focusing on one specific aspect of equation solving processes, while covering all three forms of mathematical knowledge. The interview questions will be revised based on results from administering the other two sets of instrument.

Pilot Study and Preliminary Results

In a recent pilot study, a draft instrument which consists of 7 open-ended items was administered to 20 mathematics teachers from 5 high schools in the state of Michigan. The teachers have an average of 13 years of mathematics teaching experience, and particularly, an average of 9 years of algebra teacher experience. Data from the pilot shows that

1. These teachers have quite good understanding of the basic concepts and methods related to solving linear equations, for instance, the balancing method, equivalent linear equations, the connection among the graphical methods for solving linear equations of the forms ax + b = 0, ax + b = c, and ax + b = cx + d, and the difference between the graphical solutions to the equation ax + b = cx + d and to the system of equations $\begin{cases} y = ax + b \\ y = cx + d \end{cases}$.

2. In explaining concepts and processes, teachers may be good at using examples and representations (e.g., solving certain linear equations with algebraic tiles or scale balances) given in the textbooks, but may not have the habit of thinking about the applicability of alternative examples (e.g., what kind of linear equations cannot solved with algebraic tiles or scale balances). Also, teachers tend to develop routines for students to follow (e.g., a well-formulated three-step model for solving equations by undoing operations.

3. Not all teachers are clear about the subtle differences among some similar algorithms, e.g., the balancing method, the undoing method, and the method of transposition. An in-depth study need to be conducted through clinical interviews to generate details of teachers' understanding.

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THE EFFECTS OF A PROFESSIONAL DEVELOPMENT MODEL ON MATHEMATICS INSTRUCTION AND STUDENT ACHIEVEMENT

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To identify conditions that promote gains in the developmental and conceptual nature of mathematics instruction, a two-year professional development model was implemented and assessed. The model involved modeled lessons, reflective focus sessions, and workshops designed to promote consistent methods of teaching among grade levels Pre-K to 8. Analysis of pre and post data revealed gains in instructional design and student achievement.

Purpose

Cwikla (2003) states that the classroom is a learning environment for students in the same way as professional development activities are learning environments for teachers. To identify conditions that promote gains in the developmental and conceptual nature of mathematics instruction, a two-year professional development model was implemented and assessed.

Modeled lessons involved the teachers observing lessons in their classrooms as well as other teachers' classrooms that engaged students in constructivist learning environments. Focus sessions held after modeled lessons provided opportunities to reflect and share feedback. Workshops engaged teachers in constructivist learning environments and promoted awareness of the similarity of techniques modeled throughout the grade levels.

Theoretical Framework

Efforts to support conceptual learning environments rely heavily on teacher learning and professional development (Hill, 2004). Researchers find that effective professional development characteristics include collaborative active learning opportunities that focus on content and student learning, involve modeled pedagogical practices, provide opportunities for reflective practice, span over a period of time, and are systematically assessed (Cohen & Hill, 2001; Garet et al., 2001; Shifter & Fosnot, 1993).

Modes of Inquiry/Data Sources

To extend current research concerning the effects of professional development on instructional design and student achievement, the professional development model designed using the above mentioned recommendations was systematically assessed. Eighty-six teachers of grades Pre-K to 8 in ten low socioeconomic inner-city schools located in the metropolitan area of New York participated. The teachers were categorized into three grade level groups, namely Group 1 (teachers of grades Pre-K to 2), Group 2 (teachers of grades 3 to 5), and Group 3 (teachers of grades 6 to 8).

A pre/post survey was administered to the teachers that involved statements which gauged responses on a 5-point Likert scale. Statements concerned use of pedagogical practices that fostered conceptual understanding of mathematics. A narrative response question included on the survey asked the teachers to indicate the challenges they face while teaching mathematics. The teachers were also asked on the post survey to assess the features of the professional development model. The gathered data were recorded, and the pre and post overall mean

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responses to the statements were compared using a (2x3) repeated measures ANOVA to report the interaction between time (pre/post) and group. Narrative response questions were analyzed for patterns in responses.

Pre/post interviews were conducted with each teacher to provide for more in-depth investigation into the challenges of teaching mathematics as well as teacher reactions to the professional development model. The gathered data were transcribed and analyzed for patterns in responses. Each teacher was observed on three separate occasions while teaching mathematics both prior to and after participation in the professional development. An evaluation form was used that consisted of two parts, namely Part I (Instructional Approach) and Part II (Instructional Focus). Part I classified a lesson's instructional approach as either developmental or explanatory. Part II classified a lesson's instructional focus as either procedural or conceptual. The frequency of each type of instructional approach and focus among the groups of teachers were recorded. A chi-square analysis was conducted both prior to and after the professional development to determine the existence of relationships between grade level and instructional approach as well as instructional focus.

The Normal Curve Equivalent mathematics scores on the Terra Nova Standardized Exam achieved by the students prior to and after their teachers' participation in the professional development were compared using a paired samples *t* test to determine any significant gains in student achievement.

Results/Conclusions

The repeated measures ANOVA showed significant gains in pedagogical practices ($p<.05$) for each teacher group. Interviews and narrative responses revealed that the majority of all three groups of teachers viewed using manipulatives and an extensive curriculum as challenging aspects of developing conceptual understanding of mathematical topics. The majority of all groups viewed the professional development features as valuable steps towards relieving such challenges and supporting their efforts to create sound learning environments. A chi-square analysis revealed a relationship between the teachers' grade level group and their instructional approach ($chi^2 = 21.654$, df = 2, $p<.01$; $chi^2 = 22.966$, df = 2, $p<.01$, respectively) as well as their instructional focus ($chi^2 = 38.894$, df = 2, $p<.01$; $chi^2 = 37.494$, df = 2, $p<.01$, respectively) prior to and after participation in the initiative. The teachers of grades 3 to 5 were consistently more likely than the other teacher groups to implement lessons with an explanatory approach and a procedural focus prior to and after the professional development. However, a paired samples t-test indicated significant improvement in the teachers' instructional approach and focus after participation in the initiative ($p<.01$) with the teachers of grades 3 to 5 using a developmental approach and a conceptual focus more often. The comparison of Terra Nova standardized mathematics achievement scores revealed significant gains ($p<.01$) in student achievement upon completion of their teachers' professional development.

Relationship of Paper to Goals of PME-NA

The model and its effects are shared to serve school leaders and staff developers with effective action steps towards the continued implementation of reform based mathematics instruction. The described professional development model furthers a deeper and better understanding of the psychology of teaching and learning mathematics.

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ASSESSING THE EFFECT ON STUDENTS OF TEACHER PROFESSIONAL DEVELOPMENT ACTIVITIES

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We describe a professional development program with a focus on mathematics and mathematical pedagogy and connections to practice. Our analysis showed that the students of participants had significantly higher scores on a state-mandated test than a comparable group of students drawn from the same enrollment files. Further statistical analyses describe clearer connections between teacher professional development and student achievement.

As part of a reform effort aimed at improving student achievement by improving teacher quality, mathematics educators at an urban university designed a professional development (PD) program for elementary school teachers. The program was designed to encompass aspects of effective PD–long-term engagement with teachers, focus on mathematics and student thinking in mathematics, and connections to practice (Garet, Porter, Desimone, Birman, & Yoon, 2001; Sowder, Philipp, Armstrong, & Schappelle, 1998). Grossman, Wilson, and Shulman (1989) suggest teacher knowledge is a combination of mathematical knowledge and pedagogical content knowledge and connections must be made between PD and practice. One clear goal of PD programs is improved student performance and achievement.

Setting and Participants

Over a two-year span, this PD offers elementary school teachers the opportunity to re-examine the mathematics they teach to come to a deeper understanding of content and its connections to other mathematical concepts. The PD is offered as six units of university course credit in undergraduate mathematics courses–Number and Operations, Geometry, and Algebraic Thinking. Another six units is offered in graduate teacher education courses that focus on providing a language and lens for looking at classroom mathematics teaching and children's thinking about mathematics, with an emphasis on inquiry into practice.

This study compares scores on the mathematics portion of the California Standards test (CST), a state-mandated standards test, of students of teachers who participated in the university PD with a comparable group of students whose teachers did not participate. The students are in the entire range of socio-economic status, encompass multiple ethnicities, and a range of abilities. The study also included measures of teacher knowledge, teachers' self reports of changing practice and follow-up focus group interviews with participants. Here we report on improved student achievement.

Methodology

Approximately 90 teachers of grades 4–7 completed the 12 units in the year 2004. The treatment group consisted of students who were taught mathematics in the academic year 2004-2005 by a participant teacher who had finished coursework in the prior year. A control group was constructed by randomly drawing from the group of same-grade students from comparable schools (in terms of Academic Performance Index (API)) whose scores matched in the 2002-2003 academic year. (See Figure 1.) The control group had not had a participant teacher.

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Because of the need to match on comparable scores two years prior, our treatment group was a subset of the teachers' students. Finally, the standard error of proportion, a statistical analysis, was done to assure the treatment and control groups look sufficiently alike in terms of gender and ethnicity. We then compared the 2005 scores of the treatment and control groups.

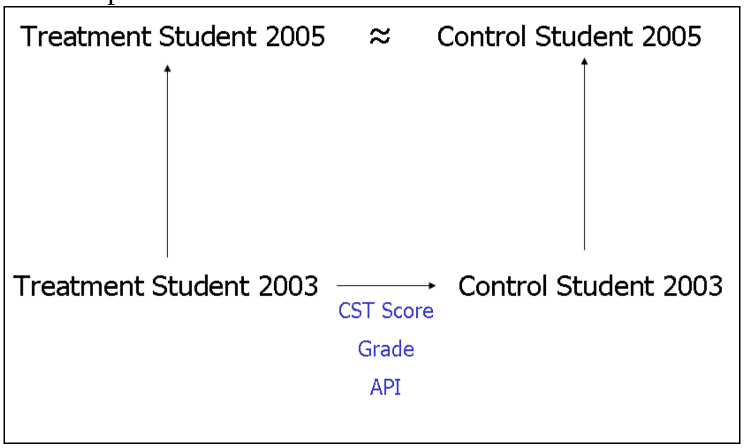

Figure 1. Illustration of pairing of students.

Results and Discussion

We conducted a paired t-test (□=.05) for a test of difference of means and report highly significant results (p= .01). Also, a correlation was found between API and difference in the mean scores. Students attending lower-API schools had greater differences.

For instructors, content test item analysis showed gains in content knowledge. In addition, 93% of the teachers in this cohort completing a survey reported that they had a better understanding of the mathematics they teach. One teacher wrote, "Yes, it's like learning to read. You can't pinpoint exactly when and where you finally learned how, but [you develop] a continuous connection and build up of concepts." In focus groups, two main themes emerged. The most enduring effects teachers reported as a result of their participation in our courses were a new belief that students can understand mathematics and an orientation toward teaching for understanding.

Our analysis suggests that a program that focuses on connections between mathematics, pedagogy, and practice can support teachers in helping students better meet the standards of mathematics content knowledge.

Acknowledgements

Preparation of this paper was supported by a grant with major funding by QUALCOMM Corporation. Any views expressed herein are those of the authors and do not represent the views of QUALCOMM.

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TEACHER'S SUGGESTIONS TO SUPPORT STUDENT CONSTRUCTED SOLUTION STRATEGIES

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This paper focuses on suggestions teachers use to support the development of multiple solution strategies by their students. Lessons from six middle school teachers were coded for "suggestions". These instances of suggestions were then categorized and frequencies were calculated. The most common types of suggestions identified were representations, procedures, and referent problems. Vignette analysis was then used to provide a window into the ways in which suggestions unfolded in each teacher's classroom.

Background and Focus Statement

The National Council of Teachers of Mathematics (2000) and the mathematics education community over the last fifteen to twenty years have called for changes in mathematics instruction. The role of the teacher is decidedly different within this new vision of mathematics. One difference is the way that teachers should choose appropriate tasks for students, assist students in the use of representations, comprehend student thinking and help students to justify, explain and connect the different solution strategies that occur in the classroom (NCTM, 2000; NRC, 2001). This study examines one of these new roles: the ways that teachers provide suggestions to support student constructed solution strategies.

Method

Participants

Six middle school teachers from two metropolitan school districts participated in this study. These teachers were a subset of teachers who participated in a professional development that focused on Supporting the Transition from Arithmetic to Algebraic Reasoning (STAAR)[1]. The teachers were chosen because of their expressed desire to promote and encourage diverse solution strategies in their classrooms.

Data Collection and Analysis

Each teacher was observed on at least two occasions. Observations were scheduled for lessons during which teachers anticipated using tasks that would result in multiple solution strategies. Each lesson was videotaped and semi-structured post-observation interviews were conducted. All interviews and lessons were transcribed and the transcriptions were coded for instances of teacher suggestions[2]. The suggestions were then categorized to identify the different types of suggestions used by the teachers. In conjunction with this process of coding a vignette analysis was also performed to add contextual richness to the different suggestion techniques identified.

Findings

Findings from this study include frequencies of suggestion techniques and episodic descriptions that illustrate teachers' implementation of these techniques. Ninety-two instances of suggestions were identified across fourteen different lessons. These 92 suggestions were then coded at a secondary level to identify different types of suggestion techniques. Three distinct types of suggestions were identified: 1) representations, 2) procedures, and 3) referent problems. Procedural and representational suggestions each occurred 27 times (29.3%). Referent problem suggestions occurred 22 times (23.9%)[3].

In addition to identifying and calculating the frequency of the three suggestion techniques, vignette analysis was used to identify characteristic episodes of the ways teachers make different types of suggestions. One common theme was the repetitive nature of the suggestions made during lessons with a large frequency of suggestions. These lessons would usually begin with the teacher guiding the students on possible representations (e.g., table, number line), procedures (e.g., factor, substitution) and/or referent problems (e.g., "this problem is similar to the snake problem"). The students would then work in small groups and the teachers would make the same suggestions about how to approach the problem that they made during the introduction. During a lesson intended to promote multiple solution strategies, the repetitive nature of the suggestions used by the teachers may indicate that they were struggling to support diverse solution strategies in their classrooms.

Conclusion

In order to meet the needs of reform oriented approaches to teaching mathematics, one important pedagogical strategy is to suggest and support students' construction of multiple solution strategies. This study uncovers three techniques that teachers used to support student thinking as they encouraged the development of students' diverse solution strategies. Further research is needed to examine in more detail the impact of these different types of suggestions on the development of solution strategies by students.

Relationship to PMENA

This research study is aligned with the conference theme of focus on learning or focus on teaching. This study examines the way in which teachers support student development of multiple solution strategies through the use of three techniques: referent problems, representations, and procedures.

Endnotes

1. The research reported in this paper is part of the collaborative project entitled, "Supporting the Transition from Arithmetic to Algebraic Reasoning." (http://algebra.colorado.edu). This material is based upon work supported by the National Science Foundation under Grant No. 0115609. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

2. The study reported here examines just one technique, that of providing suggestions to students, in order to support students solution strategies. For more information on the larger study refer to Pittman, 2006.

3. Sixteen of the suggestions were not classified as one of the three identified suggestion techniques.

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THE EFFECTS OF BELIEFS, ATTITUDE, AND PERSONALITY ON TEACHERS' IMPLEMENTATION OF STANDARDS-BASED MATHEMATICS INTO PRACTICE

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Elementary school teachers are participating in a two-year long professional development course in standards-based mathematics. The course includes components of (1) a lesson planning framework, (2) content (task) instruction, (3) modeling/mentoring, and (4) lesson study. Attitude surveys, classroom observations, and psychological measures are used to identify traits that affect teachers' ability to incorporate reform strategies into their practice.

Teachers in the U.S. claim to be aware of the National Council of Teachers of Mathematics *Standards*; however, very little evidence of such instruction is actually found when teachers are observed. Efforts to rectify this gap between perception and practice are generally addressed by short-term professional development (PD) courses. But as Schifter & Fosnot (1993) point out, "... significant and enduring change in the way teachers teach cannot be induced by a course of lectures, a handful of workshops, or even books...no matter how informative or persuasive. Instead, teacher development programs will have to dig deeper, furnishing their participants opportunities to construct for themselves more powerful, alternative understandings of learning, teaching, and disciplinary substance" (p. 23).

Perspective on Changing Teachers' Classroom Practice

Ma found that most practicing elementary teachers in the U.S. believed mathematics was "an arbitrary collection of facts and rules in which doing mathematics means following set procedures step-by-step to arrive at answers (1999, p. 123). Thus when teachers engage in PD that focuses on "understanding concepts," they often face uncomfortable confrontations with their existing attitudes as well as their own knowledge of mathematics.

Reconstructing beliefs is more complex than providing teachers with standards-based curricula and PD on implementation. Indeed, Remillard and Bryans (2004) found that a teacher's orientation towards a curriculum influences how he or she engages those materials in the classroom as much as the curriculum itself. Instead, teachers change more readily "in ecologically embedded settings of real classroom practices, real students, and real curricula - elements that teachers define as central to their profession" (Confrey, 2000, p. 100). Learning occurs when teachers are given the opportunity to reflect on and communicate about the mathematical thinking of their students (Franke et. al., 2001; Margolinas et. al., 2005). And, since teachers' beliefs can either support or constrain their students' learning, mathematics teacher educators need "to attend carefully to these beliefs" (Warfield et. al, 2005, p. 453).

Any change in practice is further affected by individual character traits. Research shows that factors such as psychological reactance, extroversion, independence, and self-control affect an individual's ability to change (Bartram, 1995). For instance, therapy clients who are high in reactance have low expectations for change and low therapy outcomes (Dowd & Wallbrown, 1993). Extrapolating from psychotherapy literature to education, it is probable that teachers who are high in reactance would be less likely to change teaching practices.

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Research Purpose

This research involves a long-term PD course in mathematics for elementary school teachers. The goal is to identify traits that affect teachers' ability to incorporate standards-based strategies into their practice, leading to conclusions about ways to facilitate change.

Participants, Methods, and Data Sources

In 2004, 40 teachers began a 2-year PD course, with 90 more beginning in 2006. Five school districts are participating. The PD has components of (1) a lesson planning framework incorporating essential characteristics of standards-based instruction, (2) content (task) instruction, (3) modeling/mentoring, and (4) lesson study.

Each participant takes the Integrating Mathematics and Pedagogy (IMAP) Beliefs Survey at the beginning and end of the course. In addition, teachers complete two personality measures: the Therapeutic Reactance Scale which gives scores on psychological reactance, and the 16 PF Questionnaire which measures normal adult personality dimensions (i.e. extraversion, anxiety, tough-mindedness, independence). Teachers complete self-assessments to rate their own progress at reforming their teaching practices. In addition, an outside evaluator makes formal observations of each teacher's classroom practice during regular mathematics lessons twice a year. The collaborative lessons associated with the course are also formally observed. Teachers' work and reflective journals are collected. Progress of students in participating teachers' classrooms is monitored via the state standardized multiple-choice core test as well as the BAM (MARS) problem-solving instrument.

Identical data is collected on teachers and students from control schools with similar demographics who are not participating in any mathematics PD.

Use of Results Related to PME-NA Goals

This study hopes to gain a deeper understanding of possible incremental changes in classroom practice that may occur during long-term in-service teacher PD courses with sensitivity to the social and emotional impact on teachers. These findings lead toward the end goal of improving student understanding: a focus on learners through a focus on teachers.

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IMPROVING MATHEMATICS ACHIEVEMENT ACROSS A HIGH POVERTY INNER CITY SCHOOL DISTRICT

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The initiative reported on herewith is designed to improve K-8 mathematics instruction in Newark, the largest city in NJ. The goal is to encourage the development of mathematical ideas in students, particularly those whose abilities and achievements often go unnoticed. Results thus far indicate that the students are achieving at significantly higher rates throughout the district.

Haberman, 1991; Ladson-Billings, 2003; Knapp et.al. 1995; and Ferguson, 2003, note that the poor achievement levels that have been documented regarding low income minority students may be due, at least in part, to differences in the ways in which they are taught. Haberman, 1991, talks about the "pedagogy of poverty" wherein he notes that while an observer of urban classrooms may see many different types of pedagogical practices taking place, there still remains a typical form that has become accepted as basic—one that is characterized by a directive, controlling pedagogy. The work in this study is based upon the premise that if teachers teach in ways that encourage the development of mathematical thinking and reasoning, students, particularly those whose abilities and achievements often go unnoticed in inner city settings in which more traditional approaches take place, can achieve at significantly higher levels. Such teaching practices include: having students share ideas and explanations; defend and justify solutions; solve challenging problem activities; grapple with complex ideas, probe each other for ideas, etc. (see Schorr and Lesh, 2003; and Schorr et. al for additional references). Consistent with this goal, the initiative that is the subject of this paper, the Newark Public Schools Systemic Initiative in Mathematics (NPSSIM), was designed to provide professional development for K-8 teachers to help them to encourage the development of mathematical ideas throughout the district. Based upon our research hypothesis, this should result in increased achievement for students across the district. It is to this issue that we address this paper as we share results of a longitudinal analysis of student assessment data that documents the positive impact that has occurred thus far.

Methods

The school district is the largest district in the state of New Jersey, and serves a population characterized by, amongst other things, high poverty, high student mobility and poor student achievement on local and state assessments (approximately 85% were classified as not competent or minimally competent in mathematics before implementation of the initiative), and 83% of the residents are African American or Hispanic (2000 U.S. Census).

The initiative began in 2000 and is deliberately planned to stimulate all learners (teachers, students, teacher educators, etc.) to refine, extend, test and share their evolving models for teaching over extended periods of time. All professional development aspects involve multi-

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layered interactions amongst students from local schools, students at the University, teachers, administrators, and researchers, who work together to consider mathematical content, pedagogical content knowledge, as they consider how students build representations, formulate justifications, and build understanding (see Schorr, Warner, Gearhart & Samuels, in press). All sessions use complex problem solving activities – for both students and teachers – that encourage answers that involve constructions or explanations that reveal aspects of the thought process. In addition, district and University researchers and/or mathematics specialists often accompany teachers as they implement ideas in the context of their own classrooms and discuss the mathematical ideas that may be elicited, implementation strategies, classroom culture, and maintaining high cognitive demand.

Results and Conclusions

While not presented here, qualitative analyses have been done to assess the nature and types of changes that have occurred in teachers' classroom practices (see Warner, Schorr, Gearhart & Samuels, 2005, Schorr, et.al, in press, for examples). Quantitative results thus far indicate that indeed, district students are achieving at significantly higher levels. For example, at the fourth grade level in 1999 (before implementation of the initiative), there was no statistically significant difference between schools who were participating in the program and those who were not. In subsequent years, the non-project schools did not improve at the same rate as schools involved in the Project. By 2002, the differences between the groups of schools were statistically significant, i.e., schools involved in the Project dramatically improved and schools not involved showed little improvement, if any. In 2004 at the request of both principals and teachers, all schools joined project NPSSIM. Further, in grade 4, initial equivalence/differences across groups can be measured (and accounted for) using students' reading/writing achievement scores. Reading/writing achievement scores have tended to be highly correlated to mathematics achievement scores in Newark, and are not likely to be affected by NPSSIM; therefore, they serve as an ideal covariate. In 2004, the rate of change in reading/writing scores (9.3% increase) for the New Jersey Assessment of Skills and Knowledge of grade 4 (NJASK4) from 2003 was not significantly different than the rate of change in mathematics scores (11.1% increase). In both 2005 and 2006, the rate of change in reading/writing achievement scores (0.7% decrease in 2005 and 2% decrease in 2006) for the NJASK 4 was significantly different than the rate of change in mathematics achievement scores (5.6% increase in 2005 and 3.8% increase in 2006). Considering the lack of improvement of reading/writing achievement scores and the significant gains made by mathematics achievement scores, this preliminary data suggests that the rise in standardized test scores can be attributed to the NPSSIM Project. In comparison to other school districts in the state of New Jersey, NPSSIM has had a significant impact on grade 4 and grade 8 NPS students, as measured by New Jersey's state-mandated NJASK4 (Grade 4) and GEPA (Grade 8) assessments.

The results gleaned thus far provide cause for great optimism. We contend that such projects can and do make a difference in the classroom practices of teachers, and in turn, influence the overall mathematical achievement for students throughout the district.

Endnotes

1. This work was supported in part by National Science Foundation [NSF] grants. The material contained herewith is based upon work supported by the U.S. National Science Foundation (NSF) under grant numbers

0138806 and ESI-0333753. Any opinions, findings and conclusions or recommendations are those of the authors and do not necessarily reflect the views of the NSF, Rutgers University or the Newark Public Schools.

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LEARNING FROM EACH OTHER: PILOT STUDY ON ELEMENTARY TEACHER PREPARATION IN MATHEMATICS ON THE US-MEXICO BORDER

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This paper will present a collaborative research project involving teacher preparation of math teacher educators on both sides of the US-Mexico border. The need to improve educational outcomes of students living on the US-Mexico border has become a mission for two neighboring universities across the US-Mexico border. Research about the internalization and globalization effects of mathematics education has opened the doors to new research paradigms (Atweh & Clarkson, 2002). Teacher education programs at both institutions are striving to make a difference in the lives of the future teachers and their pupils.

In keeping with the research based efforts to evaluate student learning outcomes, this study focuses on how mathematics education is addressed in teacher preparation programs and in actual classroom practice; and on the impact of the differences and similarities mathematics education may have on pupil learning along the U.S-Mexico border. By investigating how both teacher preparation programs address the teaching of skills and how culture influences teaching methods and strategies, researchers will discover how and whether cross-cultural differences may manifest themselves and how teachers can "capitalize on the performances students do exhibit" (Driscoll, 2005, pg. 243). Researchers will investigate the cultural influence of how teachers are prepared to teach the learning mathematics. Methods of data collection include focus groups, interviews, and observations in schools on both sides of the border.

Our unique border research setting allows for collaborative research, crossing not only physical borders but theoretical constructs as well. The constant crossing of borders of our students, pupils and faculty lends itself to the study of the relationships forged to build new paradigms for interactions and teacher preparation programs.

One of the institutions is located in a city of 600,000 at the western most tip of the state. We are in a different time zone (Mountain) than the rest of the state. In fact, we are closer to San Diego, California than we are to Houston, Texas. We are also on the border with Mexico. Our Mexican sister city, whose rapidly growing population now stands at 1.8 million, joins together to form the largest bi-national metropolitan area in the world.

Both institutions have experienced robust graduate program growth during the past ten years and are developing a binational research agenda. One of the key objectives of this development is to work with existing programs in recruiting a diverse student population, focusing particularly on the recruitment of students from Mexico. The Hispanic (Mexican-American) student population at one of the universities is composed of 70% US-citizen Hispanics, of whom 10% are Mexican citizens who cross the international boundary every day to attend school.

The Mexican counterpart's student population of 18,000 is similar to the other with a 18,919 student population. Recent political changes with reference to teacher education programs in Mexcio have impacted the this Mexican border university programs of Education. Over the last three years the education programs have become the fastest growing programs at this institution. The need to expand research opportunities to public universities to has impacted the institution's

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research capacity. Notably, this university has been recently identified as one of the highest ranked public institutions in Mexico.

The U.S. and México have clearly defined national standards for mathematics education. The teacher preparation program follows the National Council of Teachers of Mathematics Principles and Standards for School Mathematics (NCTM, 2000) to teach pre-service teachers math using a series of rich inquiry-based math investigations and in-depth discussions on selected topics. Group work and collaborative learning are embedded in class sessions. We try to reduce the approach where a teacher shows and tells students how to use certain math procedure without introducing its concept/meaning and foster conceptual understanding of the subject matter.

How these standards are being taught to new teachers and how teachers are actually implementing these practices in daily classroom activities is a focus of this study. Using the established national standards as benchmarks for comparison in both countries, the researchers will gain insight into how pupils are being taught mathematics in border schools. Current research is confirming that knowledge is constructed differently by each student, based on his or her cultural experience, family backgrounds, and learning styles (Wardle, 2004). Of critical importance is how teachers make choices to implement the curriculum, "based on acquired social patterns, ideas and values, including attitudes toward gender, race, ethnicity, language, religion, and social class" (pg. 180 in Wardel & Cruz-Janzen, 2004). Another issue that is of utmost interest to teacher educators at UTEP is the role that language plays in second language learners. In this border city a large majority of students are English language learners. The opportunity to observe students learning mathematics in their own country and in their own language may provide educators in the US insights that can help inform future practices. Often misunderstood is the notion that mathematics is a universal language. Yet research of the language of mathematics is described as precise, technical, and highly specialized by Cantoni-Harvey, 1987; Chamot & O'Malley, 1986, 1994; Dale & Cuevas, 1987 1992 (in Hernandez, 1997). Results of this pilot project aims to reveal some of these misperceptions and realities.

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STUDYING A CURRICULUM IMPLEMENTATION USING A COMMUNITIES OF PRACTICE PERSPECTIVE

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The publication of the 1989 NCTM Standards (NCTM, 1989) marked the launch of extensive efforts to reform mathematics teaching and learning. These efforts have included the development and publication of curricula which implicate constructivist instructional practices. Implementing reform curricula in a way that changes core teaching practices has proven to be a difficult endeavor (Spillane & Zeuli, 1999), especially so in urban settings, which are typically stressed in terms of teacher turnover, lack of material resources, and funding for professional development.

A number of researchers have noted the importance – if not necessity – of professional community in facilitating and sustaining teacher change towards constructivist-based pedagogy (Cobb, McClain, Lamberg, & Dean, 2003; Secada & Adajian, 1997; Stein, Silver, & Smith, 1998). In this study I use Wenger's (1998) three dimensions of community of practice (CoP) to analyze the extent to which core learning principles exist within the professional communities in my study. I focus on the learning principles of collaboration, reflection, recognition, and autonomy, which have been identified as characteristics of effective learning in communities of practice (Gee, 2003; Schon, 1983; Secada & Adajian, 1997; Wenger, 1998). This study describes characteristics of CoP's in an urban school system implementing the Connected Mathematics Project (CMP) (Lappan, Fey, Fitzgerald, Friel & Phillips, 1998) curriculum.

Theoretical framework

The framework builds from the notion of communities of practice (Wenger, 1998), which implies the existence of groups of people organized around shared activities. Wenger identifies three dimensions of communities of practice: mutual engagement, joint enterprise, and shared repertoire. Mutual engagement refers to the rich diversity of interactions which reflects the range of interests, motives, experiences, and characteristics within a community. Through these interactions, the community collectively negotiates meanings and norms for the actions of its participants, which Wenger characterizes as the joint enterprise of the community. The community's shared repertoire consists of ways of reasoning with tools and artifacts. For the purposes of this study, the focus of the CoP will be efforts to implement CMP. External resources or influences can be conceptualized in terms of focusing on the community of teachers within a school as its own entity but one that is influenced by district and state policies and mandates. The tools and artifacts will consist mainly of the CMP curriculum and policy documents related to the implementation of the curriculum.

The intersections of communities of practice are defined by boundary objects and brokers (Wenger, 1998). The main boundary object for this study will be the Rochester City School District's pacing chart for teaching CMP units. The pacing chart represents the negotiated meaning of mathematics learning of one community of practice (citywide mathematics leaders) but will be used as part of another's (teachers in a school building) shared repertoire. Brokers, who serve to bridge communities of practice, include the school's math specialist and administrator in charge of academics.

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Research focus

The study focused on characterizing the professional community for the 8th grade teachers in an urban mathematics department. The initial analysis focused on characterizing the community by the three dimensions articulated by Wenger (1998); that is, I tried to identify the nature and extent of mutual interactions, joint enterprise, and shared repertoire. The next step was to determine the extent to which the four learning principles were evident in the professional communities. A goal for the analysis was to situate the learning principles within the three dimensions of communities of practice.

Research Methods

I employed a case study methodology to investigate teachers in grade 8 at a school in one of the 'Big Five' New York state urban districts. This included mathematics certified teachers as well as special education teachers who assist in 8th grade mathematics classes. In addition to teachers, I also interviewed administrators in the building and the school district whose work influenced 8th grade mathematics teaching. I attended departmental meetings, observed classes, and collected documents related to the CMP implementation.

Results

The data collection and analysis were in preliminary stages at the time of this proposal. The initial analysis shows that there was little opportunity for meaningful collaboration, reflection, or recognition within the professional community of 8th grade mathematics teachers. The set of mutual interactions were limited mainly to required departmental meetings, and the joint enterprise appeared to deal with minimal or superficial aspects of implementation, such as the timeline for implementing investigations. The shared repertoire consisted mainly of the district's pacing charts and the written tasks in the curriculum. There were few signs that the professional community delved into deeper understanding of the tasks or philosophy of the curriculum.

Much of the teachers' formal mutual interactions consisted of departmental meetings focused on administrative or accountability concerns. There was virtually no discussion of strategies and experiences of teaching CMP at these meetings; that is, most of the joint enterprise of the teachers' professional community consisted of establishing timelines for covering curricular content and for administrative activities not directly related to the teaching of mathematics.

It was clear that artifacts related to accountability played a large role in how the teachers deliberated their enactments of the CMP curriculum. The district's pacing chart, based on the state assessment, dictated the particular investigations and length of time for investigations. This, by itself, would not preclude a more expansive joint enterprise; however, other considerations, such as classroom management and understanding the logistics of the curriculum, dominated the mutual interactions.

One resource that had the potential to develop the learning principles within the teachers' professional community was the district-wide professional development. However, this consisted of only a few opportunities and these did not ultimately nurture collaboration, reflection, recognition, or autonomy within the teachers' professional community.

The classroom observations suggested that teachers were struggling to implement CMP. In a number of classes, teachers struggled to highlight the main mathematical ideas embedded in the investigations. At times, the logistics of the investigations seemed to dominate the classroom interactions.

Discussions and implications

Although the preliminary results are not surprising, they point to some larger issues. If one accepts the premise that effective CoP's include the four learning principles, then the district's implementation of CMP would be unlikely to change core teaching practices. That is, the development of professional community, such as it was, did not facilitate collaborative efforts to understand the curriculum or how to effectively teach with it. There were few opportunities to learn the curriculum and even when these opportunities were present, it was clear that the teachers were not accustomed to interacting with each other in a meaningful way.

In order to effectively implement reform curricula, much attention will need to be paid to the development of CoP's that incorporate the four learning principles. The professional development literature has not sufficiently conceptualized how to grow communities of practitioners who effectively collaborate and reflect on their practice, who have the opportunity to observe and recognize effective reform teaching, and who can act autonomously to improve the mathematics learning of their students. This is especially true for urban school systems. Purely psychological accounts of teacher learning need to be complemented by a focus on the development of professional communities in order to scale up reform teaching practices.

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A FRAMEWORK FOR UNDERSTANDING TEACHERS' CURRICULUM INTEGRATION STRATEGIES

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Theoretical Background

Several studies (e.g., Drake & Sherin, 2006; Remillard & Bryans, 2004) have documented teachers' strategies for using and adapting reform-oriented curriculum materials. These studies have focused on teachers' interactions with *particular* sets of curriculum materials. However, there is another significant aspect to teachers' curriculum use that has been under-theorized in the literature – teachers' strategies for *integrating* reform-oriented curriculum materials with other instructional resources. There continues to be a need for more detailed information about how in-service teachers use and learn from curriculum materials, as well as how to support pre-service and in-service teachers in their use of these materials. In other words, there is a need for research-based ideas about how best to use and combine multiple sources of materials to support children's learning, rather than relying on only one set or type of resources. While curriculum developers may have previously had a vested interest in preventing teachers from using curriculum materials other than their own, many publishers are now providing their own guidelines for helping teachers with this process of integration. An example is the web site, http://investigations.scottforesman.com/jup.html that details three different strategies for teachers to use in integrating *Investigations in Number, Data, and Space* with the *Scott Foreman-Addison Wesley* textbook series.

A Framework for Understanding Teachers' Curriculum Integration Strategies

The framework described here was derived in large part through analysis of interview and observation data from twenty teachers piloting a reform-oriented curriculum in the early elementary grades. While these teachers were not asked directly about their strategies for curriculum integration, examples of curriculum integration were consistently identified as we coded for evidence of teachers' reading, evaluating, and adapting the curriculum. In working with the twenty teachers, I identified four major kinds of reasons teachers cited for integrating the new reform-oriented curriculum materials with other resources – 1) addressing the need for different *kinds* of activities (e.g., manipulatives, games), 2) addressing the needs of different *groups* of students (e.g., those struggling or needing challenge), 3) addressing the demands of standardized *tests* and other policy mandates, and 4) a desire to *maintain the use* of activities that had been successful in previous years. To be clear, these are strategies for curriculum integration that were being used and described by teachers without prompting or support from either researchers or curriculum developers. In the poster, excerpts of teachers' descriptions from each of these four categories of reasons will be provided.

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JOINING A STUDY OF CULTURE AND MATHEMATICAL THINKING BY EXAMINING AN INDIVIDUAL CHILD

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This study examined the extent to which teachers could learn to take into account children's out-of-school experiences in examining and adapting their mathematics teaching practice. Six teachers participated in a semester long professional study group which followed a protocol based on the Descriptive Review Process developed by Carini (Himley & Carini, 2000). The six teachers in the study, all from a single elementary school, each conducted a case study of a particular child from their respective classrooms. They explored this child's mathematical thinking in and out-of-school. Through activities which were built on the work of Moll and Civil (González, Andrade, Civil, & Moll, 2001; Moll, Amanti, Neff, & Gonzalez, 1992; Moll & Gonzalez, 2004), the teachers examined the students with particular attention to their mathematical thinking, abilities, and interests in multiple contexts within the school setting and at home. Each teacher observed the target child in the classroom, shadowed the student in the school for an entire day, and consulted with the students' mother who documented her child's experiences out of school with photographs.

Five of the six teachers moved away from considering the home of their student as a possible impediment to learning and came to view it as a support. Three of the teachers identified specific out-of-school mathematics activities, interests, or competencies of their target students but they did not act on this information in their classrooms. One teacher not only saw and commented on these experiences, she also planned instruction to take advantage of it. Although five of the teachers began and ended the study focusing on in-school mathematics, all the teachers developed considerable specific knowledge of the mathematical performance of their target students. Four of the teachers established significant relationships with the student and took specific action in order to engage further with the target children. In addition, the teachers made and implemented plans for changing practice which were based in considerations of student thinking in mathematics. The results of this study demonstrate that professional development which focuses teachers' attention on individual learners and their in and out-of-school experiences can, at least in some cases, support teachers in reflecting on changing practice in ways that attend to the particular strengths and needs of individual children.

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PROFESSIONAL LEARNING COMMUNITY: POWER OF COLLABORATION BETWEEN URBAN MATHEMATICS TEACHERS AND UNIVERSITY PROFESSORS

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Grounded in action research as an approach for implementation of standards-based practice in mathematics education, we established a professional learning community (PLC), the Urban Teacher-Researcher Collaborative (UTRC). Our view is that a professional learning community is a group of people who share a common interest in a topic or area, a particular form of discourse about their phenomena, tools and sense-making approaches for building collaborative knowledge, and valued activities (Fulton & Riel, 2004). Engaged in this study as a community of inquiry were eight middle and secondary mathematics teachers from two urban school districts, recognized as 'high need districts', and three university faculty.

Our PLC model is designed out of common efforts and goals. In particular, we focus on the ways in which our collaboration influenced and impacted the teaching of middle and secondary mathematics teachers as well as university professors. The UTRC began with the idea that the main benefit for teachers would be effective implementation of a new curriculum through action research. While the implicit gain for university faculty would be in learning and implementing the knowledge acquired through this process in teacher preparation programs. Ultimately, both believed they would benefit from working together to address their responsibilities in effecting positive changes in student learning. Our approach of engagement was guided by a model of collaborative professional development (Rosaen, 1998), which included seven key practices: (1) talked about teaching; (2) shared in planning, teaching and reflecting on practice; (3) conducted classroom observations; (4) engaged in developing pedagogical skills; (5) immersed ourselves in studying curriculum design, communities of practice, reflective practice and action research; (6) conducted research; and (7) communicated with a wider audience through presentations and publications.

We used the Social Theory of Learning (SLT) as the framework to guide our research and interactions as co-researchers (Wenger, 1999). Our research question is: *In what ways are teachers and university professors impacted through collaboration in an urban PLC?* In accordance with SLT, we analyzed our narratives through the lenses of meaning, practice, community, and identity. In the presentation, the process of analysis and excerpts from the narratives will also be examined. Analysis of the impact of the collaboration shows that all components of the STL resonated with the members of our PLC. However, the greatest impact was in "community" and "meaning." The data analysis validates our reflections on the interaction and comfort level of the members of our PLC with respect to sharing ideas and developing meaning and identity. The PLC members were fluent in describing the impact of the collaboration on their pedagogical knowledge, which was transformed through participation.

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MODEL FOR IMPROVING THE TEACHING AND LEARNING OF MATHEMATICS IN SCHOOLS: THE IMPACT OF SCHOOL LEADERS

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This poster extends earlier work on teacher development and describes a successfully implemented model for improving the teaching and learning of mathematics in schools. It demonstrates how teachers, classrooms and schools can change toward implementing a more thoughtful curriculum in mathematics. This view sees learning as arising in a problem-solving context in which students are engaged in investigations in mathematics that give them an opportunity to explore patterns, make conjectures about their character, test hypotheses for their effectiveness in problem solving, and reflect on the formulation of the concept for use in new situations (Maher, 1988). The view that will be described makes the partnership of the principal and the teachers central. It demands for teachers and teacher-leaders a high order of mathematical and pedagogical competence. The model we propose provides a set of interrelated experiences for teachers, which enables them to develop a philosophic perspective on mathematics instruction (Maher, Alston, & Landis, 1986). Our presentation will describe certain substantial changes in teachers and classrooms that can be achieved in schools when certain conditions are in place. These conditions include, but are not limited to, the following:

For teachers and school leaders (principal):

- *Provide* released time for workshops (long-term and connected) to work together on open-ended *math* problems
- Develop rubrics *and then use them* to analyze problem solutions
- *Schedule time for teachers to* visit classrooms of "master" teachers
- Videotape classroom lessons (new and revised) *and then* study videotapes *individually and collectively*
- *Build a master schedule that includes* common planning time for teachers and principal to discuss/plan math units as a team *and* reflect on videotape segments of children doing mathematics

For school leaders (principals):

- Build master schedule, maximizing length of time for math period
- Provide for teacher release time to informally visit other classrooms
- Creatively find budget sources to provide teachers with necessary tools and materials, including math literature, manipulatives, technology, etc.
- Encourage the display of student math work/projects throughout the building
- Communicate with parents about the benefits of active, thoughtful mathematics
- Schedule evening sessions for teachers, parents and children to engage in thoughtful math activities
- Support teachers to share their work locally, regionally, statewide and nationally

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TEACHERS' CHANGES IN THE FIRST YEAR OF MATHEMATICS PROFESSIONAL DEVELOPMENT: MATHEMATICAL DISCOURSE

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Intensive mathematics professional development incorporated the four teaching principles of Bransford, Brown, and Cocking (2000), being assessment-centered, community-centered, knowledge-centered, and learner-centered. Guskey's (2000) model was used to evaluate the program. In this model, each step constrains the success of the next step. Participants reacted positively to the institutes, their learning was significant on paper-and-pencil measures, and there was substantial, although sometimes uneven, organizational support at the participating sites. These elements suggested participants might be able to use their new knowledge.

Through observations, written self-evaluations and semi-structured interviews, I sought to determine how and whether classroom teaching had changed. Observations were coded for evidence of each of the four principles. Then I compared observation data with teachers' self-reports (written and oral). Teachers had made the most progress being community-centered, focusing specifically on using cooperative pairs or groups and developing the associated classroom management strategies. The most successful teachers went further and developed sociomathematical norms (Yackel & Cobb, 1996) in which students were expected to justify and make sense of their answers. Teachers positively appraised the changes and felt the classroom management was comparable to traditional teacher-centered lessons.

An analysis of the professional development revealed characteristics that may have led to this element being more successful than others. An analysis of the teachers' classrooms and interviews revealed differences between successful and less successful teachers in terms of perceptions of the content.

Our program utilized this data in designing the second-year institutes. The application of the four principles to organize professional development and to analyze teaching has implications for other programs.

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EFFECTS OF IN-SERVICE FOR MIDDLE SCHOOL TEACHERS USING MULTIPLE INSTRUMENTS RELATED TO KNOWLEDGE AND BELIEFS

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It is well documented that teachers' content knowledge of mathematics is crucial for improving the quality of instruction in classrooms (An et al, 2004; Hill and Ball, 2004). Influencing teachers' beliefs and values may also be essential to changing teachers' classroom practices (Stipek et al, 2001). Hill and Ball feel that teachers can deepen their mathematics knowledge for elementary school teaching in the context of a single professional development program, and that a feature of successful programs is to foreground mathematical content (2004).

This study examined whether changes in teachers' knowledge and beliefs were measurable after an eight month period, and which measures were most useful. Fourty Canadian seventh grade teachers received three days of professionally delivered Number and Operation inservice training and about half also took one or two online courses for teachers. Multiple measures were used in a pretest and posttest format, and two showed significant changes. The CKT-M Middle School Form A (Hill, Schilling & Ball, 2005) showed change in Number and Operation (the strand in which training was provided) but not in the other strands. The beliefs portion of the Perceptions of Math (POM) Survey (Kajander, 2005) showed an increase in valuing conceptual learning, and a decrease in valuing procedural learning.

These results indicate that mathematics content training can show change but is likely needed in all mathematics strands, as improvement was noted only in the strand addressed. Beliefs about the nature of mathematics itself also showed a shift to valuing conceptual learning.

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THREE MODELS OF IMPLEMENTING *INVESTIGATIONS IN NUMBER, DATA, AND SPACE*: A STANDARDS-BASED MATHEMATICS CURRICULUM

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The poster displays three models of practicing standards-based curriculum in an effort to add to the mathematics education knowledge base. It explores elementary teachers' modes of practice in implementing *Investigations in Number, Data, and Space* [Investigation] (TERC, 1998) units. *Investigation* is a K-5 standards-based curriculum whose objectives offer students, connected and meaningful mathematical problems to promote in-depth thinking.

A qualitative case study research design was used to study teachers' emerging practice models of implementing Investigation mathematics in urban fifth grade classes. Data were collected through open-ended interviews, classroom artifacts, audiotape and videotape of lessons, group meetings, lesson plans, lesson observations, and post-lesson conferences in three classrooms. An assumption of the study was that a teacher who closely taught at least six observed lessons, as conveyed by *Investigation* was seen as effectively employing inquiry-based instruction.

This study found that teachers in each of the three observed classrooms employed different models of inquiry-based instruction. These were labeled as the (i) *traditional inquiry-based* instruction, (ii) *partial inquiry-based* instruction, and (iii) *inquiry-based* instruction.

Traditional inquiry-based model is a practice in which the teacher consistently supports student learning and construction of meanings. The students do not have an opportunity to explore an activity and make meaning of it on their own. For instance, if any activity is assigned for exploration, it is often weighed down with hints of what should be done. Good questions are occasionally posed to students but little wait time is provided before the teacher reveals everything. *Partial inquiry-based* model is a practice where the teacher is transitioning from traditional to inquiry-based paradigm. The teacher allows for cooperative learning but does not know how to manage it successfully. The teacher strives to be a facilitator, but at times when the unforeseen arises, the teacher falls back to the familiar traditional instruction. That is when the teacher delivers knowledge and does not give students an opportunity to share their thinking. *Inquiry-based* model is a practice characterized with high level of mathematical conversation among students, and the teacher is a facilitator. Students are fully engaged in a task where they take responsibility, and ownership of the activity. The teacher is seen as effectively addressing all classroom issues such as management, interactions, motivation, group work, discipline, humor, student or teacher questioning, reflection, and active listening.

Remillard and Byrans' (2004) study found that teachers' orientations impacted how they used standards-based curriculum. Mathematics educators should design professional development courses to aid all standards-based practitioners to become effective at inquiry-based teaching and learning.

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"WHAT DOES AN EMBARGO HAVE TO DO WITH A LLAMA? I'M SO CONFUSED..."

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Many mathematics teachers of Spanish-speaking, English language learners (ELLs) in the United States are ill prepared to identify and meet the unique needs of their students (Khisty, 1995). Because most of these secondary mathematics teachers have never had to learn mathematics content in a non-English speaking classroom, they do not have first-hand knowledge of how ELLs engage with mathematics instruction or textbooks. For example, though aware of the need to teach mathematical vocabulary (via word walls and the like), teachers often do not fully grasp that to access mathematics problems, ELLs must also make meaning from everyday language expressions, word order, and the mortar that connects distinct language chunks into a coherent text (Fillmore & Valadez, 1986).

Though some professional development activities already exist that speak generally to the linguistic and affective needs of ELLs, very few are specific to mathematics and ELLs at the secondary level. Of those few, almost none discuss the mathematics register (Halliday, 1974; Cuevas, 1986) or how multilingual students often shift back and forth between different language registers when doing mathematics (Moschkovich, 2000). Therefore, a professional development activity was designed and facilitated to enable participants to encounter and feel, albeit briefly and temporarily, just some of the experiences and emotions that ELLs have on a daily basis in mathematics classrooms. Asked to solve one mathematics problem in a Spanish-only environment, participants engaged with activity tasks that were intentionally created and sequenced to optimally parse out different linguistic and cultural aspects during the simulation and highlight their respective pedagogical implications in the concluding whole group debriefing.

The paper itself is in five sections. The first section introduces the need for such an activity and its study, the second documents the activity's tasks and their facilitation, the third explains the methodology used to document the insights gained by participants during and after the activity, the fourth shares those insights, and the fifth proposes next research steps.

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About the author

Dr. Lager is an Assistant Professor in Mathematics Education at the University of California, Santa Barbara. His research foci include the preparation of pre-service English language learner (ELL) mathematics teachers, the professional development of in-service ELL mathematics teachers, and the improvement of the comprehensibility of mathematics instruction, curricula, and assessment.

DIFFERENTIATED INSTRUCTION FOR COGNITIVELY DIVERSE LEARNERS

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As educators, we struggle with finding (instructional strategies) ways to meet students' interest, abilities, and learning styles. We tended to ignore students' cognitive abilities and fail to encourage them to reach their full potential in mathematics. In this regard, Bartolomé (1996) points out that the instructional methods used with subordinated(1) students are narrow and mechanical. She argues instead for the infusion of a humanizing pedagogy that respects and uses the multiple perspectives, histories and intelligences of students as an integral part of pedagogical practices. Instruction may be differentiated in content, process, or product according to the students' readiness, interests, or learning style. Differentiated Instruction is an organized yet flexible way of proactively adjusting teaching and learning to meet students where they are and help all students achieve maximum growth as learners (Tomlinson, 1999). We can no longer use the pedagogy of the dominant culture to reach the students teachers have marginalized for years.

The purpose of this paper is to identify different instructional strategies to use for cognitively diverse(2) learners among African American students. These instructional strategies must include critical pedagogy, specifically, culturally relevant pedagogy and humanizing pedagogy. Our research question is, "what kinds of instructional strategies are used in bridging the gap of different content knowledge found among African American students in middle school mathematics classrooms?"

In attempting to answer this question, we conducted a professional development workshop for teachers at a school in a local school district on how to implement differentiated instruction in their classrooms. From the mathematics teachers that participated in the workshop, we identified two teachers from which to observe their classrooms as they implement differentiated instruction. A total of six lesson periods were used for instruction during this project. The teachers and investigators met to plan the lessons used for this project. The different levels of differentiated lessons will be displaced during the poster presentation. During classroom implementation with the researchers present, teachers were encouraged to recognize and analyze students' interpretations and thoughts about the types of problems presented at different cognitive level. Independently, the teachers reflected and revised their previously planned lessons based on the students' feedback, and then shared their new ideas and thoughts with us.

Interviews were conducted before and after instructions with the two teachers and six of their students (one from three differentiated levels in each class). The purpose of the interviews was to gain a deeper understanding into how the teachers' content knowledge influence them in designing standards-based instructions and utilizing concept maps and the students thinking about these lessons.

Our data sources consist of (a) the teachers' curriculum concept, (b) transcripts from pre- and post- semi structured interviews of teachers and students, (c) the students' work on the individual classroom activities at the different cognitive levels, (d) notes of classroom activities, (e) field notes taken while working with teachers during planning times, (f) transcript of the teachers planning times, and (g) the reflection of the teachers on their work.

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It is our aim that our findings would reveal positive impact on student achievement in mathematics due to the attention given to the different cognitive abilities present among African American students in the two teachers' classrooms when differentiated instruction is implemented. Detailed results, conclusion, and implications for teachers and teacher educators will be presented during research poster presentation.

Endnotes

1. Subordinated refers to cultural groups that are historically, political, socially, and economically disempowered in the greater society.
2. Cognitively diverse learners refer to students with different content knowledge within a single classroom.

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ISSUES OF LANGUAGE: INSIGHTS FROM MIDDLE SCHOOL TEACHERS' PARTICIPATION IN A MATHEMATICS LESSON IN CHINESE

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Teachers of English Language Learners participated in a mathematics lesson taught in Chinese, first with traditional lecture followed by a session infused with multiple representations. This event set the stage for a discussion about students learning mathematics through a second language. Teachers' insights on the experience encouraged a cognizant sensitivity and a critical level of examination of their current practices

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TWO ELEMENTARY MATHEMATICS TEACHERS' JOURNEYS THROUGH INTEGRATING MANIPULATIVES AND TECHNOLOGY INTO HER CLASSROOM

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Research studies on the enactment of mathematical tasks (Doyle, 1988; Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996), reform-based mathematics curricula (Kim & Stein, 2006; Remillard, 2005;) and technology-rich problem solving units (Cognition and Technology Group at Vanderbilt [CTGV], 1997) indicate that teachers face a myriad of barriers as they attempt to implement reform-based mathematical activities. Further, when teachers encounter these issues they teach in a didactic, teacher-centered manner. While the use of mathematical tasks, hands-on activities and technology has the potential to impact student learning, teachers must be given more support integrating these resources into the classroom.

Research Design

This paper shares the findings of two case studies of two elementary school teachers who participated in a professional development program. The program was designed to support their enactment of mathematical tasks, technology and manipulatives. This paper shares data about the research question, How are teachers' use of manipulatives and technology influenced by participation in a professional development program? During the study, teachers were videotaped when they intended to use the instructional practices that were emphasized during the professional development. Semi-structured interviews were also conducted after each interview to examine teachers' intended practices (what they planned to do) and espoused practices (what they thought they did). The videos were analyzed for evidence that supported teachers' use of specific instructional practices or instances in which specific instructional practices.

Findings

Keisha. Keisha, a fourth grade teacher, views herself as a teacher who is "different" and "non-traditional." Keisha allows her students to use both manipulatives and technology, but in both cases the emphasis is on using the tool for the sake of using it, not to help students learn the relevant mathematical content. In one instance her students used tangrams to examine geometric transformations. However, Keisha did not bring any mathematics into the activity, so the lesson was merely a time for students to play with tangram pieces.

Selena. Initially, Selena intended and espoused that she was posing meaningful tasks to her students, since she was giving them problems and allowing them to use manipulatives. However, Selena gave students explicit procedures to follow with the manipulatives, which denigrated opportunities for students to develop their problem solving skills. As the study continued, Selena started to get away from the procedures and began posing more open-ended tasks along with the use of manipulatives, such as base-10 blocks and centimeter tiles. While Selena enacted the emphasized instructional practices more frequently towards the end of the study, when she attempted to use technology, Selena returned to a didactic approach, in which she had students follow a strict procedure to find their answers.

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Conclusion

This poster will present more data and further findings about each participant's integration of technology and manipulatives into their mathematics classroom.

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TEACHER EDUCATION – PRESERVICE

ASPECTS OF PRESERVICE TEACHERS' UNDERSTANDINGS OF THE PURPOSES OF MATHEMATICAL PROOF

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This qualitative study on preservice teachers found that preservice teachers are both sensitive to role of the intended audience and struggle with the trade-offs between rigor and explanation in proof writing. Also discussed here is that preservice teachers may mean different things when they say they are "convinced" by a mathematical argument. The results of this study have implications for the training of preservice teachers.

The importance of proof to the practice of mathematics scarcely needs to be supported since reading and writing proofs occupies much of the time and energy of practicing mathematicians. NCTM (2000) recognizes the centrality of proof in school mathematics as well. In spite of its importance to mathematics and the attention given to it by NCTM, many students at all levels find proof writing very difficult. The research literature documents many of the barriers to consistent, successful construction of mathematical proofs among high school and college students.

Some researchers (Knuth, 2002a; Lowenthal & Eisenberg, 1992) have suggested that teachers too suffer from some of the same inabilities and misconceptions as students. Knuth reported that a third of his participants believed that counterexamples to proven theorems were possible. In addition, most of his participants "genuinely did not seem to understand (or, at the very least, did not seem to be confident in their understanding of) the generality of a proven statement" (p. 389) and as a group, his participants could not reliably distinguish between valid and invalid arguments. Selden and Selden (2003) reported that undergraduate mathematics students have difficulty distinguishing between correct and incorrect proofs. They presented undergraduate mathematics students with a series of proofs and the students in their study correctly validated or invalidated them only 81% of the time. Lowenthal and Eisenberg reported that after practicing teachers had derived some number theory facts, some indicated that before they could be considered as facts, they would have to be proved by mathematical induction. This over-reliance on proof by mathematical induction is interesting and may be an indicator of the proof schemes being employed by these teachers. The broadest categories in Harel and Souder's (1998) taxonomy of students' proof schemes are the *empirical* proof schemes, the *externally based* proof schemes, and the *analytic* proof schemes. Students employing an empirical proof scheme would both accept and submit to others arguments based on either drawings or numerical examples. Students employing an externally based proof scheme would both submit and accept arguments based on factors other than the correctness of a proof, such as (1) the form of an argument, (2) the authority of the book or teacher, or (3) the formal manipulation of symbols without comprehending the meaning. Students who understand the meaning of symbols and are comfortable manipulating them without regard to their meaning or are comfortable building from axioms and definitions are employing an analytic scheme. The teachers in Lowenthal and Eisenberg's study who withheld their validation of the derivation of a fact until it had been checked by mathematical induction may have been employing an externally based proof scheme.

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The concerns of mathematics teachers as both *teachers* and *knowers* of mathematics are important. Knuth (2002a) investigated teachers' understandings of proof as knowers of mathematics. Knuth (2002b) investigated teachers' understandings of proof as *teachers* of mathematics and suggests that some mathematics teachers believe that proof is a topic of study rather than a tool for studying mathematics and that it is not suitable for the majority of high school mathematics students. His participants indicated that roles for proof in school included the development logical thinking skills, displaying student thinking, and communicating mathematics, and establishing results. Understanding not only what practicing teachers but also what preservice teachers know and understand regarding both the mathematical and the pedagogical aspects of proof is important when considering how to better prepare them to meet the expectations of recent reform documents. This report is part of a larger study of preservice teachers understandings of the purposes of proof and reports on the specific questions of (a) How do preservice teachers perceive the role of communication when writing proofs? and (b) What value do preservice teachers place on proofs that explain relationships?

Methods and Data Sources

Ten mathematics or mathematics education majors were recruited from upper division geometry courses at two colleges with preservice teacher preparation programs. Twenty-one tasks were designed to probe participants' perceptions of mathematical rigor in relation to three purposes of proof – explanation, communication, and conviction. Participants were interviewed twice either singly or paired. During the first interview, I asked participants open-ended questions about their mathematical histories and about their perceptions of the purposes of mathematical proof both in the discipline of mathematics and in mathematics education. The second interview was task-based (Goldin, 1999). Participants were shown a series of mathematical arguments that varied in rigor, clarity, and explanatory power and participants were asked to discuss whether or not the arguments were clear, convincing, and valid, and if they could or would submit them to their professors. All interviews were audiotaped and later transcribed. I used two coding schemes (internal and external) in the analysis of data. My external codes included the purposes of proof (verification, explanation, communication), the role of rigor, the role of diagrams, and to what purposes proofs had been used in their mathematics courses. Internal codes that emerged from the data included writing to the intended audience, the value of explanation in proofs, and the differences between abstract mathematics courses and the mathematics courses that preceded them.

Two of the ten participants will be discussed in detail below. Marilyn and Ken were both well reasoned and articulate and the views they expressed are important for different reasons. In many respects, Marilyn's responses are typical of the responses of the group. In several instances, she expressed much more clearly the stated views of the other participants. Ken's responses on the other hand were atypical and frequently contrary to the responses of the group. Both Marilyn and Ken had been out of school for a few years and were returning to pursue a master's degree in mathematics education. Marilyn had a BS in mathematics. Ken had a BS in business and after returning to school had completed the mathematics course requirements for a BS in mathematics as well.

Results

The results of the analysis yielded three main findings. First, preservice teachers are sensitive to the role of the intended readers (or audience) of proofs. Second, preservice teachers place a value on proofs that explain over proofs that merely verify. Finally, preservice teachers accept that proof by mathematical induction is a valid form of argumentation even though they don't find these arguments personally convincing.

All of the participants expressed concern for the intended readers of a proof. These concerns included writing proofs that are both understandable and convincing to the uninitiated reader, and writing proofs that would receive high marks from professors. For Marilyn clarity and communication were very important considerations when writing proofs. She stated that the proof entitled *Summation* (see below) and a proof by mathematical induction for the same claim were both valid but that they were best written to different audiences. "If I'm showing something in class, for students of a lower skill level, I would go with [*Summation*]" but she did not think that she could turn in *Summation* to her math professors who, she believed would want something more rigorous.

Summation

Claim: $1+2+3+\cdots+n=\frac{n}{2}(n+1)$

Proof: If we add all the numbers from 1 to n twice as in the following fashion, the sum is twice the sum from 1 to n. So:

$$\begin{array}{ccccccccc} & 1 & + & 2 & + & 3 & + \cdots + & n \\ + & n & + & (n-1) & + & (n-2) & + \quad\; + & 1 \\ \hline (n+1) & & + & (n+1) & + & (n+1) & + \cdots + & (n+1) \end{array}$$

So twice the sum from 1 to n is $n(n+1)$.

Therefore $1+2+3+\cdots+n=\frac{n}{2}(n+1)$.

The proof *Parity of Zero*, "If zero were odd, then 0 and 1 would be two odd numbers in a row. Even and odd numbers alternate. So 0 must be even" (NCTM, 2000, p.59), was written by a first grader to establish that zero is an even number. No respondent believed that this proof was sufficient to establish the claim that zero is even but when they were informed that the proof was written by a first grader, most indicated that it was an acceptable or even a "pretty hot" chain of reasoning for a first grader. Marilyn went so far as to say that this argument constituted a valid proof at the first grade level and that it might be an acceptable argument up as far as high school.

Marilyn's concerns for the reader are not merely to write with less formalism to immature audiences, they include writing sometimes with less rigor to mathematically elite audiences as well. The proof *Hand Waving*, "…assuming it is true for n=k, then it is *easy to show* that it holds for n=k+1…" was deemed insufficient by most participants as well. Marilyn however, suggested a context in which this proof would be acceptable. She would not turn *Hand Waving* in to a professor unless

[It was] part of a larger work. [If that was the case] I would feel safe leaving out the algebra to make it more clear...It depends on who your audience is. If it's something that you are turning in as like a senior project or a master's project, your audience is pretty educated people so you can say, it is easy to show that it holds for a proof. And by the time you've gotten there, you're allowed to do that.

She did not believe that the proof was complete or valid as written but she believed that there was a time and a place for "proofs" of that nature and provided not only the context but also a rationale for submitting such an abridgement. For Marilyn, making an argument clearer by decreasing the level of rigor was sometimes allowable.

All participants were convinced by proofs by mathematical induction but they seemed to be convinced based on a superficial belief in the form of argumentation rather than on a rich understanding of the argument. When the proofs *Summation* and its mathematical induction counterpart were presented to the participants, it became apparent why they found this argument convincing. One student reported, "I think [the proof by induction] is more convincing...because I know mathematical induction." Marilyn reported that she found the induction more convincing "because I've been indoctrinated that induction is a valid form of proof." Another declared that the inductive proof was valid "because I've learned that it's valid...It's gotten me good grades on tests," and another reported that proofs by mathematical induction are convincing "probably because I have faith in the method. Without having faith in the method, it's a little harder to make that leap that assuming it's valid for k, showing it's valid for k+1." It seems these students wrote and accepted proofs by mathematical induction based on an externally based proof scheme – that is they believed that mathematical induction is a valid form of argumentation rather than a clear and personally convincing argument. In Ken's opinion, "because you can memorize this and regurgitate it and get credit for it [without] a true understanding" only students who really understood mathematical induction should be permitted to in hand such proofs. It is difficult to say what these students meant when they said that proofs by mathematical induction are convincing. They both accepted and submitted arguments of this form. They exchanged these arguments like a mathematical currency, expecting that these mathematical communications would be understood and accepted by a reader but found these arguments convincing only because they had been "indoctrinated" into the method of mathematical induction, not necessarily because they found the arguments personally convincing.

In analyzing the interview data, I found that communication and conviction as different aspects of proofs were almost impossible to separate. It was difficult to determine what students meant when they used the word *convincing*. For example, one student reported that while she *believes* analytic proofs, "I feel like you could use an analysis proof to prove things that aren't true" and that "I'll go along with it but I might make it a life's goal to disprove it." This participant seems to be employing an externally based proof scheme because she accepts and even writes epsilon/delta proofs but isn't truly invested in the method. Marilyn reported that proofs by exhaustion are "not as rigorous... not as convincing" as other arguments. Both these students accepted these forms of argumentation as valid but found them unconvincing on a personal level and seemed to be indicating that they believed that certain forms of valid argumentation are less convincing than others. Other participants expressed the opposite view that certain proofs were convincing but not valid. Six said that they could be convinced by a proof they did not understand if it was in a book, and all found proof by mathematical induction convincing although as discussed above, this seemed to be based superficially on the form of argumentation. These examples reveal that the students mean different things when they say they are convinced. It can mean that the student has come to believe on a level of personal meaning, or that the student is capable of suspending his/her disbelief, or that the student affirms that the argument is correct, or that the student sees no reason that the statement cannot be true. There may be many other meanings as well.

While some of the participants indicated that they would resort to working examples to persuade others if necessary, none of them reported that they found examples or diagrams convincing arguments. Externally based proof schemes however, were common among the participants. Six of the ten participants were willing to accept on authority a proof in a textbook. Marilyn, for example, indicated that she would accept a statement even if she did not understand the proof

[If] it was published. Someone checked it. Someone out there has gone through that proof. I'd want to go through and see if I can put in the steps in between to explain to myself where that came from. If I weren't able to do that, I'd probably believe it anyway. It's in a book.

The expressed belief is that *somebody* understands the proof and has lent their credibility to it. That is – if a proof has been reviewed and accepted by those who publish textbooks, then the theorem is probably true and can tentatively be trusted. In this respect, the behaviors of these participants are not all that different from those of practicing mathematicians.

While some participants stated that the admissibility (or validity) of arguments is based partially on the mathematical level of the reader, and to a lesser extent the on the mathematical level of the writer, Ken assigned admissibility to the arguments in the proof items based in part on the sense they made and the insights they showed. In other words, Ken evaluated proofs on how well they explained mathematical relationships or tied mathematics to real life situations. In his comments below it is clear that he believed that a good explanation is a proof and that explanation should sometimes be given as much consideration when writing proofs as careful mathematical reasoning.

The proof *Transitivity,* "A dollar can be made up entirely of coins of a single denomination and any dollar value can be made up entirely of one-dollar bills," was an analogical proof of the claim that divisibility is transitive. Only Ken and one other participant believed that it was valid. The one student believed that if the terms "dollar" and "coin" were carefully defined, then any holes in the proof would then be taken care of but Ken believed this argument was a proof just because of the sense it made. He claimed that the proof was "about as clear cut as any proof I have ever read...[and] when in doubt, I'll go in my pocket and get out my change and prove it to the students." In his view, this proof was not merely valid at some lower level, or only for certain audiences "if [my professor] didn't give me credit, I'd be confused." His expressed belief was that the insights it provided should be appreciated and the argument acceptable at *any* level.

Ken believed the proof of the *Intermediate Value Theorem*, "When you go from the northern hemisphere to the southern hemisphere, you have to cross the equator somewhere" (Davis, 1993, p. 337), was valid and that:

> If a student gave me that proof I would accept it and write excellent, you know, that's using your brain. Rather than just memorizing the theorem and regurgitating it on the paper, you're applying it. And that's what we go to school for, to learn how to apply things and use them in life, or that's what we're supposed to be in school for...I probably wouldn't [turn this in to my analysis professor] just because I would be concerned that it would be too simplistic but I can't see how I wouldn't get credit for it because…you're just symbolizing...It's just substitution really. I think it would be accepted.

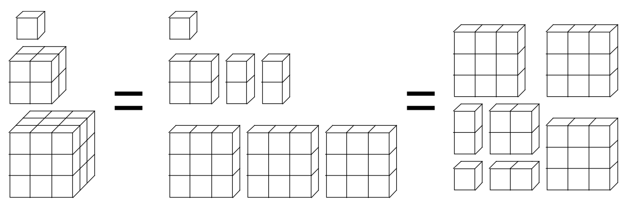
For Ken, this argument if not general, could be generalized by substitution. He believed that this proof is good for explaining the theorem to uninitiated audiences. He also believed that it shows insight into the theorem and while it might be too simplistic a proof for some, it should receive full marks at both the high school and college levels of mathematics instruction.

Not only would Ken accept the proof *Sum of Cubes* (Nelson, 1993, p. 86), (see below) from students but he insisted that his teachers should accept it from him as well "because not only are you explaining what's going on but you're given a visual as well…I'm going to say yes, they should accept this damn it! They should accept it. I question whether they would or not." The picture in the proof is just the demonstration of the special case of n=3 but he insisted "Even though it's finite picture here, it's simple enough to take this out to n numbers. I think it's very clear." He recognized not only that a theorem cannot be proved by looking at a special case but also that this form of visual argumentation was not universally accepted. Even so, the proof was valid nonetheless and should be accepted as a mathematical proof at the college level.

Sum of Cubes

Claim: $\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$

Proof:



In all, there were four proof tasks in which Ken was either the sole participant or one of only two participants that believed the proofs were valid. The proofs in question (*Intermediate Value Theorem*, *Transitivity*, *Sum of Cubes*, and a proof by Venn diagram) differ significantly from what most would consider standard mathematical writing. In each case, the argument had sacrificed rigor in favor of explanation. Ken's ideas, while unorthodox are not naïve. Judging by the mathematical coursework of the participants, Ken had had more extensive training in mathematics than most of the other participants. He understood why others might not accept these arguments but still believed that he could make a case for accepting proofs such as *Intermediate Value Theorem*, *Transitivity*, and *Sum of Cubes* from students and for submitting them to his teachers as well.

Discussion and Conclusion

When writing to a low level reader such as a grade school or even a high school student, Marilyn and most of the other participants wanted to be clear and convincing, but when writing to a high level reader such as their professors, they wanted to write proofs that they perceived to be more rigorous. Marilyn's belief that a proof by mathematical induction was more rigorous than the proof *Summation* and that she would feel safer submitting the induction to a teacher was representative of the group. All participants in this study both accepted and submitted proofs by mathematical induction but their superficial "faith in the method" indicates that they were at times employing an externally based proof scheme.

Ken on the other hand had a way of thinking about proof that ran counter to the rest of the group. He preferred arguments that explained relationships or showed insight even if they fell short of what he knew was accepted mathematical practice. He would accept explanatory arguments from others and argues that he should be allowed to submit them at any level of schooling as well. It should be noted that throughout both interviews, Ken interpreted questions in light of teaching high school mathematics more consistently than any other participant and in his opinion, proofs in high school mathematics should explain why a statement is true, not merely verify that it is, but when he states that he should be able to turn in some of these proofs to his professors he indicates that proofs that give good explanations or show some special insight should be acceptable at all levels.

Preservice teachers' ideas about what it takes to write a proof depend on many things including the purpose for which it is written and the reader for whom it is intended. As students of mathematics, they are accustomed to seeing proofs that they don't fully understand and so may have to employ externally based proof schemes such as accepting the word of an authority or borrowing a ready-made proof format such as mathematical induction without fully understanding its meaning. As persons intending to one day teach mathematics, they are sensitive to the needs of an imagined reader and try to fashion arguments intended to meet those needs. To understand the reasons for students' failure to produce valid proofs, we first need to understand what proof schemes they employ and how they negotiate between arguments that are rigorous and arguments that explain, convince, or clearly communicate as they construct their ideas of mathematical proof.

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AFFECT FACTORS: CASE OF A PEDAGOGICAL APPROACH FOR PROSPECTIVE TEACHERS

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The paper reports on a comparative study investigating the effects of a pedagogical approach (ICFB) on perception, attitude, and confidence in the context of mathematics. Three sections of a mathematics content course for prospective teachers participated in the study. The section with the pedagogical approach showed the most number of students changing their opinion on many of the 16 Likert scale statements toward a more positive view. The section with the traditional approach however showed no statistically significant positive change from pre- to post-survey.

Students' concerns about mathematics can significantly affect their ability to learn and understand the subject. Furthermore, their anxieties and attitudes may greatly affect how they perceive their own mathematical competence (Hopko, Ashcraft, Gute, Ruggiero, and Lewis, 1998; Mandler, 1989; McDonald, 1989). For instance, students may perceive mathematics as an incomprehensible set of abstract procedures and methods to follow, not being aware that there are reasons underlying these methods. Frustration due to the lack of in-depth understanding may discourage students from pursuing studies in mathematical sciences (Hopko and et al., 1998).

These characteristics are commonly observed among pre-service teachers too. Many researchers reported findings indicating that majority of prospective teachers in the United States show high anxiety and negative perceptions toward mathematics (Ambrose, 2004; Battista, 1994). Future teachers see mathematics as a highly abstract subject requiring rote memorization of procedures, symbols and formulas. Furthermore many have a lack of belief in the importance of mathematics, especially at the EC-4 level. Although pre-service teachers certainly have the ability to learn mathematics, their lack of self-confidence and the fear of failure distract them from learning, and adversely affect their performance in the classroom. Clearly, we must deal with the affect factors so that our prospective teachers may fully benefit from their experiences in learning mathematics (Ambrose, 2004; Battista, 1994; Fennema, 1989; McDonald, 1989). Affect factors range from emotions to attitudes to beliefs. There have been reports discussing the role of active learning environments in reshaping students' perceptions and emotions about mathematics (Fennema, 1989; McDonald, 1989).

Pedagogical Approach to Teaching and Learning Mathematics

A set of mediating activities as part of a pedagogical approach, named *An Integrated, Collaborative, Field-Based (ICFB) Approach*, to teaching and learning mathematics has come about in order to address some aspects of the attitude and perception factors our pre-service teachers seem to hold, and consequently enhance their mathematical knowledge. The ICFB approach includes activities that are developed to support various components of a cyclic process of *Learn, Develop, Practice, Reflect* and *Teach*. The approach is modified from another approach that has been implemented into a block offered for prospective middle school teachers. The ICFB approach is implemented into a mathematics-focused block of courses offered for elementary pre-service teachers.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

There exist two block structures at the University: Bock I is a block with a focus on mathematics, and Block II focuses on science. Block I consists of three courses; pedagogy, mathematics methods and content mathematics offered as a cohort. That is, pre-service teachers take the three courses in their designated block sections as a cohort. In the past, even though these courses were still offered in a block section and taken as a cohort, they were taught in isolation with little to no collaboration between them. The mathematics content course was taught traditionally. The block structure also entails a field-based internship component. While taking the three courses, teachers get their internship at area elementary schools attending classes two days and interning three days a week. Figure 1 outlines the activities of ICFB and the involvement level of the three courses. For instance, the arrow between the math content and the methods boxes with “LEARN/PRACTICE” indicates that the learning of mathematical concepts and the practice of micro-lessons mainly occur as a result of a collaboration between the mathematics content and methodology courses. It also implies the existence of common assignments and requirements. Furthermore, the bubble with “Micro-TEACHING” indicates that all three courses involve in the micro-teaching process. It also indicates that mathematics is considered as the core subject for each block I course.

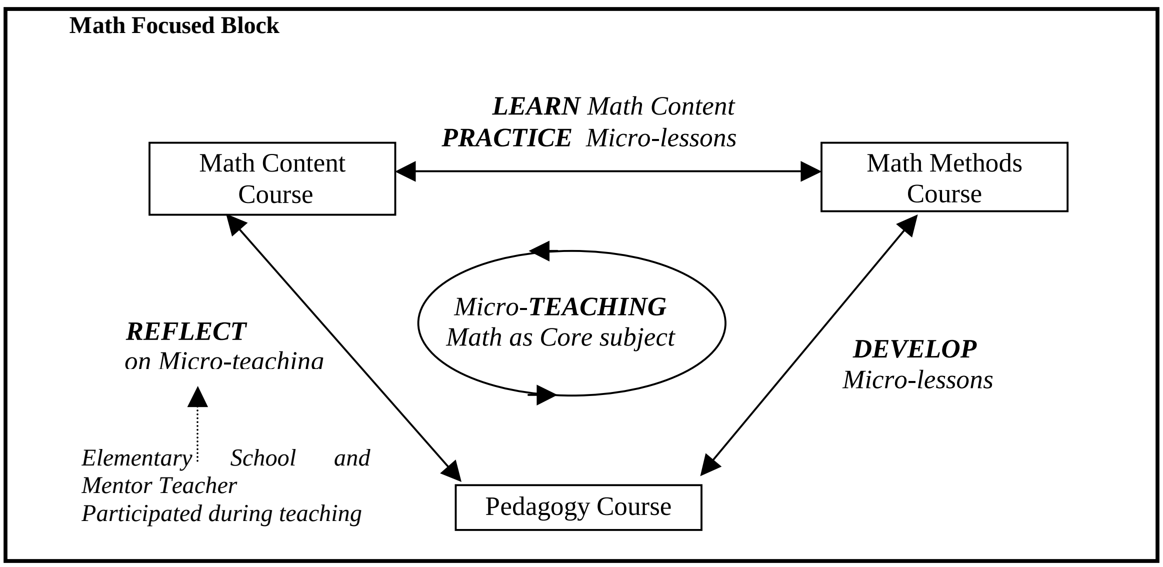

Figure 1. Diagram of a Pedagogical Approach to Teaching and Learning Mathematics.

The approach demands that students first learn mathematics content, and next develop lessons on mathematics topics relevant to EC-4 teaching. Working in an active, collaborative and inquiry-based learning environment, they proceed to practice their micro-lessons during the mathematics methods and content classes at the same time reflecting on their progress and experiences during the pedagogy and mathematics content hours. Using designated elementary classrooms, students teach the lessons with the support of in-service teachers whose classrooms are visited. This process is repeated about four times a semester. A crucial component of the approach is that students receive timely and continual constructive feedback on their work from their classmates, the block faculty, and in-service teachers through out the semester.

Methodology

The paper reports on the findings of a comparative study between the three groups of prospective teachers on their perceptions and attitude toward mathematics. Data consisting of pre- and post-surveys were collected from three sections of a mathematics content course required for pre-service teachers at a four year midsize southwest University in the United States. All three sections covered the same mathematics topics, and required similar assignments.

Table 1. Three prospective teacher groups.

Groups	04	05	03
Year	2004	2005	2003
Sample Size	21	28	41
The same content coverage	√	√	√
The same content instructor	√	√	
ICFB approach implemented	√		
In Field-Based Block	√		
Social, constructive, active learning	√	√	

Table 1 outlines the primary components of the three groups. Group 04 was subjected to the ICFB approach implemented in a section of block I courses during the spring 2004 semester. The groups 05 and 03 took sections that were offered as stand alone during spring 2005 and fall 2003 respectively. That is, the two groups were not integrated with mathematics methods, pedagogy or any other potential mathematics or education courses. 04, 05 and 03 consisted primarily of English speaking Hispanic students (approximately 85%). There were no male students in 04. In addition, everyone in this group was specializing in elementary education. 03 had three males, and 05 had two. In average ten students in groups 05 and 03 were specializing at the middle school level in either science or mathematics.

S1	Math is simply a bunch of procedures to follow
S2	Math is a tool used to solve problems and/or find solutions
S3	Math is difficult
S4	In order to do math you need to think and use logic and reasoning
S5	I fear math
S6	I become frustrated with math
S7	I do not understand math
S8	I like math
S9	I do not like math
S10	From this course, I expect to improve/improved my own math skills and abilities
S11	From this course, I expect to learn/learned how to teach math
S12	From this course, I expect to learn/learned how to make math fun for my future students
S13	From this course, I expect to become/became more comfortable/confident with my abilities in math
S14	I believe I can learn and understand math
S15	I am looking forward to teaching math
S16	I have the ability to learn new tasks.

Table 2. Sixteen statements from pre- and post-surveys.

The results reported in the paper come from a study investigating many dimensions of the ICFB approach through both qualitative and quantitative means. The quantitative analysis of 16 Likert scale statements relating to perception, emotion and confidence are discussed in this paper. See table 2 for the 16 statements given in both pre- and post-surveys. Statements were ranked on a 5 point scale of 1=strongly disagree to 5=strongly agree. One-way ANOVA with α=0.05 and α=0.1 significance levels is applied in order to document differences in pre- to post-survey responses within groups, and between groups. Furthermore, a Spearman-Brown Split-Half reliability measure (Garrett and Woodworth, 1967) is applied to pre- and post-survey statements similar in content to test for the consistency of student responses. The measure ranges from .70 to .90.

		04		05		03	
Statement		Pre	Post	Pre	Post	Pre	Post
S1	SD	1.19	0.89	1.19	1.05	1.12	1.17
	Mean	3.50	3.20	2.59	2.26	2.81	3.13
S2	SD	0.61	0.60	0.75	0.65	0.80	1.04
	Mean	4.50	4.16	4.25	4.35	4.37	4.27
S3	SD	0.72	0.81	1.18	1.18	1.21	1.08
	Mean	4.25	3.65	3.14	3.13	3.56	3.42
S4	SD	0.59	0.72	0.51	0.67	0.74	0.67
	Mean	4.35	4.25	4.46	4.52	4.40	4.37
S5	SD	1.03	0.89	1.28	1.20	1.28	1.21
	Mean	3.70	3.20	2.37	2.52	2.83	2.75
S6	SD	0.86	0.88	1.29	1.33	1.28	1.24
	Mean	4.00	3.65	2.96	2.96	3.28	2.97
S7	SD	0.94	0.83	1.03	0.88	1.18	0.88
	Mean	3.45	2.90	2.43	2.04	2.66	2.41
S8	SD	0.95	1.00	0.96	0.97	1.03	0.91
	Mean	2.80	2.76	3.50	3.87	3.51	3.56
S9	SD	0.90	0.89	1.20	1.10	1.23	1.10
	Mean	3.58	3.24	2.54	2.13	2.56	2.46
S10	SD	0.50	0.48	0.57	0.94	0.55	0.94
	Mean	4.63	3.86	4.39	4.17	4.56	3.54
S11	SD	0.47	0.83	0.74	1.04	0.81	1.30
	Mean	4.70	3.60	4.39	3.91	4.44	2.95
S12	SD	0.44	0.96	0.74	0.98	0.74	1.22
	Mean	4.75	3.71	4.46	4.04	4.56	3.03
S13	SD	0.44	0.73	0.58	0.95	0.51	1.13
	Mean	4.75	3.67	4.50	4.00	4.71	3.31
S14	SD	0.45	0.50	None	None	1.05	0.84
	Mean	4.74	3.95	None	None	2.71	3.95
S15	SD	1.00	0.75	1.03	1.08	0.94	1.00
	Mean	3.95	3.52	3.79	3.91	3.98	3.58
S16	SD	0.51	0.54	0.49	0.42	0.63	0.64
	Mean	4.55	4.24	4.36	4.22	4.40	4.33

Table 3. The pre- and post- survey SD and Mean Scores.

Results

There has been a considerable difference observed among the groups. Compared to the other two groups, the group that was exposed to the ICFB approach (04) differed on many statements showing significantly positive changes from pre- to post-survey. Table 3 provides the means and standard deviations (SD) of the scores, and table 4 provides the p-values, from one-way ANOVA, for seven statements that showed notable changes by at least one group. In this section, we report the findings for the particular statements providing pairwise comparisons of the groups. We finalize the paper with the conclusion section including a summary of the results.

Groups 04 and 05 showed a notable decrease on statement 1, **"***Math is simply a bunch of procedures to follow,*" as opposed to an increase 03 displayed on the mean scores of its students' opinions from pre-survey to post-survey. The post-survey means of groups 04 and 05 showed a significant difference whereas the difference between 04 and 03 was not statistically significant (see tables 3 and 4). One should keep in mind that the pre-survey mean scores for the groups also differed significantly with the exception of the comparison of the groups 05 and 03. The two groups did not have a statistical difference on their pre-survey responses. Group 04 had a pre-mean 3.5, and 05 had 2.59 while 03's was 2.81. The most notable difference among the three groups on the statement 1 is that as opposed to the opinions of the groups 04 and 05, increasingly more students in the group 03 considered mathematics as a bunch of procedures to follow from pre-survey to post-survey. Recall that 03 differed in its traditional lecture style whereas the other two groups implemented a collaborative, active and inquiry-based learning.

	Between Group Comparison						Within Group Comparison		
	04-03		04-05		05-03		04	05	03
	Pre	Post	Pre	Post	Pre	Post	Pre-Post	Pre-Post	Pre-Post
S1	0.03*	0.8	0.01*	0.003*	0.45	0.005*	0.3	0.3	0.2
S3	0.02*	0.4	0.001*	0.1**	0.15	0.33	0.01*	0.9	0.5
S5	0.01*	0.1**	<0.001*	0.04*	0.15	0.47	0.1**	0.6	0.7
S6	0.02*	0.03*	0.04*	0.05*	0.33	0.96	0.2	0.9	0.2
S7	0.01*	0.03*	0.05*	0.002*	0.4	0.11	0.05*	0.1**	0.2
S8	0.01*	0.003*	0.002*	0.001*	0.96	0.21	0.9	0.1**	0.8
S16	0.5	0.3	0.1**	0.8	0.76	0.46	0.06**	0.2	0.6

* Statistically significant at α=.05. **Statistically significant at α=0.1.

Table 4. p-values for between and within group comparison of seven statements from one-way ANOVA.

All three groups showed a decrease on statement 3, **"***Math is difficult,*" from pre- to post-survey. Group 04 is the only group showing a statistically significant decrease on the number of students agreeing with the statement. The pre-mean of the group 04 is 4.25 whereas its post-mean is 3.65. One however should point that the pre-means of the other two groups are notably lower than that of 04. This difference on the pre-survey may be attributed to the fact that the groups 05 and 03 had students specializing in middle school science or mathematics contrary to group 04 which was all female and specializing at the elementary level.

Groups 04 and 03 showed a decrease on statement 5, "*I fear math,*" and 05 showed a negligible increase from the pre-mean of 2.37 to post-mean of 2.52. As it was the case on statement 3, group 04 is the only group showing a significant (at α=0.1) change on this statement. Compared to the number in the pre-survey, a significantly more number of students in 04 disagreed with the statement in the post-survey.

None of the groups showed a significant change on their opinion of the statement, "*I become frustrated with math,"* from pre- to post-survey. 04 and 03 however displayed a notable decrease on the number of students agreeing with the statement, and students in 05 did not change their opinion. Comparing the three groups, the pairs 04-03 and 04-05 differed significantly on their opinion both in pre-survey and post-survey.

Groups 04 and 05 showed statistically significant changes on the statement, **"***I do not understand math,"* from the means of 3.45 and 2.43 in the pre-survey to the post-means of 2.90 and 2.04 respectively. The group 03, on the other hand, showed a non-significant decrease indicating that many of its students did not change their judgment about the statement.

Groups 04 and 03 did not seem to have very many students changing their opinion on liking mathematics (Statement 8). 05 on the other hand had notably more students changing their opinion favoring an agreement with the statement, "*I like math."* Group 05's post-pre mean difference is 0.37 which is significant at the α=0.1 level.

On statement 16, "*I have the ability to learn new tasks,*" all three groups reflect similar opinions in both pre- and post-surveys. That is, the post-mean difference between the groups is not significant. Even though it is not statistically significant, group 04 starts with a higher mean in pre-survey, and shows the highest decrease from pre- to post-survey. Our observation and conversation with the students indicate that this decrease may be due to the fact that 04 was given an opportunity to teach mathematics topics in actual elementary classrooms leading to experiences and realization of the amount and depth of knowledge needed for effective teaching. Experiencing the importance of mathematics in EC-4, and realizing the inadequate mathematics knowledge held at the time of actual teaching, many of 04 students began to realize that "*learning new tasks*" is not about recalling facts but about gaining deeper understanding. This resulted in self-criticism of their knowledge leading to the feeling of inability to learn new tasks. Recall that 04 is the only group with significant number of its students disagreeing with the first statement, **"***Math is simply a bunch of procedures to follow"* at the end of the semester. Thus, learning did not seem to mean recalling facts any more for significantly many of 04's students.

Conclusions

In this paper, we reported quantitative findings of a comparative study investigating the effects of a pedagogical approach (ICFB) on perception, attitude, and confidence in the context of mathematics. Only the *ICFB* group (04) showed notable changes on many of the statements. On the statements addressing negative feelings about mathematics, 04 had significantly fewer numbers of students indicating agreement. Group 04 also showed an increase in the number of students indicating confidence in their ability to understand mathematics on statements 3 and 7 having a post-pre mean difference of -0.55 on S7. Furthermore, many students in 04 changed their opinion on mathematics being a bunch of procedure to follow showing a notable decrease on its mean from pre- to post survey. Group 05 also showed a notable decrease on this aspect of mathematics. Group 03 however had increasingly more students considering mathematics as a bunch of procedures to follow. This may be attributed to the content delivery difference between this group and the other two groups. 03 implemented mainly a lecture mode as opposed to the active collaborative learning 04 and 05 implemented. As the case with group 04, increasingly more students in 05 also disagreed with the statement *"I do not understand math,"* indicating an increase in their confidence level. Here, we should however note that all three groups showed a notable decrease in their means from pre- to post-survey on the course expectation statements (S10-S13). This decline is statistically significant at $\alpha=0.1$ for the statements S11-S13.

In summary, contrary to no significant positive change observed in 03, the groups 04 and 05 showed notable positive changes on their opinion from pre-survey to post-survey. The ICFB group showed a statistically significant opinion change on four statements (S3, S5, S7 and S16) addressing emotion and confidence. Group 05 showed significant changes on two statements (S7 and S8). The findings then may be interpreted as that for teachers to gain positive changes in their emotion, perception and confidence, one may need to implement an approach similar to ICFB. However, we should note that one needs to interpret the results cautiously since the sample sizes were small, and the groups differed on some of the factors. For instance, the three groups' pre-survey scores on many statements differed significantly, and 03 had a different instructor than the instructor who taught the other two groups 04 and 05.

The implications of the findings for the teaching of mathematics might be that an approach similar to ICFB can lead to changes in pre-service teachers' perceptions and attitudes toward mathematics, and as a result lead to an increase in teachers' confidence in their ability to learn and think mathematically (Fennema, 1989; McDonald, 1989).

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RECONCILING BELIEFS WITH THEORY AND PRACTICE: A PRE-SERVICE TEACHER'S DILEMMA

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Even and Tirosh (2002) suggested that research is needed to further our understanding of how to help secondary pre-service teachers gain the skills and knowledge necessary to advance student cognition. This study examined whether a secondary pre-service teacher's reflection on the types of questions that he asked and his students' responses would persuade him to ask questions that elicited students' mathematical reasoning. Findings of this study indicate that an individual's implicit beliefs may hinder the adoption of new practices even when presented with contradictory evidence.

Even and Tirosh (2002) acknowledge the challenge of preparing secondary mathematics teachers in their discussion of the current state and future directions of research on teacher education. They summarize research findings to describe three different types of knowledge that teachers need to advance students toward more sophisticated mathematical thinking. First, teachers need knowledge about frameworks that describe student cognition to interpret students' responses. Second, teachers need knowledge about learning theories to develop different forms of mathematical knowledge. Third, teachers need knowledge about the role of classroom culture to develop students' mathematical thinking. Even and Tirosh suggest that while research has advanced our understanding of the complex nature of teacher knowledge in general, we do not yet understand how to help prospective teachers gain the skills and knowledge necessary to advance student cognition nor whether adopting the current advocated classroom culture in the secondary mathematics classroom facilitates students' learning.

Review of 56 research reports on elementary teachers from the Proceedings of PME-NA (2002-2004) indicate that while the process of change is difficult, some elementary teachers are enacting new practices. In general, these reports suggest that helping teachers interpret students' responses, experiment with new discourse patterns, and engage with researchers as partners supports the adoption of reform teaching recommendations. During the same time period, research reports on secondary mathematics teachers are fewer (19) and indicate that teacher educators use many professional development strategies that supported change for elementary teachers.

However, secondary mathematics teachers are different from elementary teachers in many ways and helping them recognize how their conceptions of teaching and learning impacts the use reform curriculum can be challenging (De Geest, Watson, & Prestage, 2003; Gutierrez, 2002; Olson & Kirtley, 2005). De Geest, Watson, and Prestage found that the mathematical proficiency of low-performing students was increased by nine secondary teachers who concentrated on developing key mathematical ideas and ways to think about them. Olson and Kirtley found that after a secondary teacher experienced cognitive dissonance learning and thinking about mathematics, she became committed to reform mathematics recommendations and changed her teaching practices.

These findings suggest that preparing secondary teachers to be committed to increasing the achievement of low-performing students requires experiences that are safe environments for

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

experimentation and prompt cognitive dissonance. This study analyzed the teaching practices of a secondary pre-service teacher, using a framework of question types, and his beliefs about students' potential to achieve in mathematics. Specifically, we wondered whether a pre-service teacher would change his commitment to the attainment of more sophisticated thinking by low-performing students when the supervising teacher educator focused his attention on the types of questions asked of two different student populations.

Theoretical Perspective

The research question focused our attention on the psychological processes of a pre-service teacher as he considered how his actions influenced students' opportunities to learn. From the perspective of symbolic interactionism, interactions between people are indications of their personally constructed meaning (Becker & McCall, 1990; Denzin, 1992). Communication is thought to be a symbolic process that consists of an ensemble of social practices (including language, intonation, gestures, and written symbolic representations) that portray an individual's private construction of knowledge. Thus, an individual's interactions can be analyzed and interpreted to indicate this constructed knowledge. The character and new uses of verbal language by an individual during social interactions indicates the assimilation of new ideas (Kumpulainen & Mutanmen, 2000).

Proulx, Kieran, & Bednarz (2004) characterized pre-service teachers' classroom discourse as (a) explanations that are technical and focus on procedures with precise vocabulary; (b) attempts to develop a concept but over-generalize a specific example; (c) explanations that follow prescribed steps; (d) reformulation and revoicing of students' ideas to create a culture of mathematical reflection. In this study, we used these characterizations to analyse a pre-service teacher's classroom discourse and create learning trajectories to advance his practice.

Method

This study used a single case-study design to investigate how a pre-service teacher used information about the types of questions that he asked students in two different geometry classes. Previously, Charles (pseudonym) completed his mathematics methods course with Hartter which included a two-week clinical experience in a secondary mathematics classroom. This study began as Charles assumed responsibility for remedial and grade-level geometry classes. Hartter observed him teaching three times during his semester-long student teaching assignment (beginning, midway, and end). Each observation included a pre-lesson conference in which Charles identified the lesson's objectives and how he would assess student understanding. During the post-lesson conference, he identified what (a) surprised him, (b) went well, (c) he would change, and (d) he knew about the students' understanding of the concept.

Hartter used an activity-reflective cycle (Simon, Tzur, Heintz, Smith, & Kinzel, 1999) to engage Charles in a sequence of cyclical activities designed to develop his questioning techniques. The cycle consisted of five stages in which Hartter (a) assessed Charles' knowledge of questioning through observation, (b) cited instances where students exhibited mathematical thinking and suggested types of questions that could be asked, (c) created a learning trajectory to help Charles realize that students could think about mathematical ideas during class discussions, (d) selected activities to help Charles identify and use conceptual questions, and (e) supported his reflection by providing specific feedback. Four data sources included (a) written reflections and lesson plans constructed by Charles, (b) field notes of conversations between Charles and Hartter, and (c) detailed field notes of classroom observations. These data were analyzed using a time-ordered conceptual matrix (Miles & Huberman, 1994) to describe his professional growth as he implemented the suggested activities and reflected on the intersection between his actions and students' responses.

Results and Discussion

Charles taught two geometry classes, one described as "remedial" and the other as a regular tenth-grade geometry class. Initially, he relied on the textbook as an authoritative guide that dictated the sequence of topics, discussion guide, and problem source for both geometry classes (observation, March 29, 2005). Hartter noted during an observation of Charles, "Was he afraid that the students won't respond or that he won't recognize the correct response? In the regular tenth-grade geometry section, he seemed to ask a few more questions seeking explanation or clarification, such as: "How did you get that?" However, there was still very little student-student interaction in this class [regular section]." (field notes, March 29, 2005). We characterized the questions that Charles posed as managing and clarifying questions that focused students' attention on solving the problem following his demonstrated procedure. His discourse followed a pattern in which he initiated a question, listened to a student response, and evaluated whether the response was correct. This discourse pattern has been described by many researchers and was characterized by teachers (Kumpulainen & Mutanen, 2000) as an initiate-respond-evaluate pattern (IRE).

Hartter wondered why this discourse pattern dominated his interactions after the extensive discussions on questioning in the methods course. Pre-service teachers had practiced developing questions that probed students' mathematical thinking and pushed students to create generalizations from their observation of patterns. In class, Charles had created very insightful questions designed to help students articulate their observations: "Describe the difference between the area and circumference of a circle." and "When is it appropriate to use the Pythagorean Theorem?" (after an investigation involving a variety of triangles).

Analysis of Charles' reflections about surprises during the lesson and students' understanding about the concepts addressed revealed an assumption about his students. He wrote, "The first class was a remedial class; therefore, they just don't respond to questions very well." This statement indicated a belief that remedial students are unable to answer questions that require mathematical reasoning. When Hartter provided evidence that the type of questions and discourse pattern for the regular tenth-grade geometry class was similar, Charles responded, "Maybe I could get better responses from them [regular students], but open-ended questions should be saved for out-of-class projects." These responses indicated two beliefs. First, the focus of questioning during class should be to help students learn specific procedures for solving problems. Second, questions eliciting students' thinking are reserved for high-performing students outside of class discussions.

Questions by teachers that direct students toward an expected response are described as the 'Topaze effect'(Brousseau, 1992). Proulx, Kieran, and Bednarz (2004) also found that pre-service teachers may use this type of questioning to help students understand a mathematical concept. In their study, Bertrand provided a context and asked students to write a linear equation that reflected the time it takes workers to pick strawberries. In contrast, Charles did not provide a context to develop a mathematical idea. He wrote "Simplify $\sqrt{75}$" on the board.

Charles: How could we break this down?
Student: 25 times 3
Charles What do we know about $\sqrt{25}$?

Student: 5

Charles: Can we simplify √3?

Student: No.

Charles: (Wrote on the board √75 = √25 √3 = 5√3) See how we got this? (no response) Make sense? (no response from class and Charles went on to the next example).

Clearly, Charles used directive questions that led students to a correct solution. He repeatedly asked, "Does anyone remember how to do this?" which was also met with a silence. These silences were uncomfortable and he chose to go on to another example hoping that the students would learn the procedures through repetition. When asked what students understood about simplifying radicals he responded, "I think that they knew how to do it because they answered my questions." Hartter probed, "What about the silences?" Charles replied, "They didn't have any questions." Charles concluded that his students could respond only to factual questions that led them to expected answers. His experiences in teaching reinforced his belief that his students could not reason mathematically.

Hartter provided evidence from both classes that students did use mathematical reasoning. For example, in the remedial class Charles drew two similar right triangles on the board. The legs of the larger triangle were 3 inches and x. The corresponding legs of the smaller triangle were 1 inch and 2 inches.

Charles: Can anybody think of a proportion that I can set up to solve for x?

Student: 1/3 x = 2

Charles: That's not a proportion (wrote 1/3 = x /2).

In this interaction we interpreted the student's response to indicate his understanding of scale factor between the two triangles. Specifically, the student recognized that the smaller leg was 1/3 the length of the larger corresponding leg. When Hartter provided this example as evidence of student's mathematical reasoning, Charles commented, "Well, I was looking for a proportion and I did not want to confuse the students." Then, Charles shifted in his chair and was silent for several moments before commenting, "I guess they can [think mathematically]." (field notes, April 26, 2004).

The discomfort displayed by Charles indicated a level of cognitive dissonance as he realized that his students were able to reason mathematically even though his questioning did not encourage it. He recognized that his practices in the two classes suggested a belief that not all students can learn mathematics, a contradiction to discussions held during the methods course. Initially, we theorized that Charles would reconsider his questioning techniques in both classes and change the types of questions that he asked after the cognitive dissonance he experienced in his post-lesson discussions with Hartter.

During subsequent observations, it was apparent that Charles was much more comfortable in the second classroom. He used the same lesson for both classes and drew a right triangle on the board with the hypotenuse as the base. An altitude was drawn from the right angle to the hypotenuse. In the remedial class, he told the students which triangles were similar and how to set up a proportion to find the missing pieces. One student offered the proportion z/10 = 16/z as a method for finding side z. Charles wrote 10/ z = z /16 on the board. The student asked, "Will my way give you the wrong answer?" Rather than accepting the student's suggested proportion as one possible strategy, Charles ignored the question and solved his own proportion. In the second class, he began by asking students to consider which triangles could be compared. Then he prompted them to consider, "What are some different ways to find the missing pieces?" Various solution strategies were proposed. One student questioned another student's suggested proportion and Charles responded, "Either way, it's fine." His questions in the remedial class continued to be factual in nature with few requests for explanation or reasoning.

Unlike Kirtley, who experienced cognitive dissonance while deepening her knowledge of mathematics content (Olson & Kirtley, 2005), Charles experienced cognitive dissonance while reflecting on his pedagogy. Kirtley reconceptualized her notions of teaching and learning and began to focus her questions to elicit students' mathematical thinking for making instructional decisions. Cognitive dissonance allowed Kirtley to discuss her beliefs and eventually accept the data as she changed her theoretical stance on teaching and learning mathematics. Perhaps the context in which teachers experience cognitive dissonance influences their response to it. Kirtley privately experienced discomfort and addressed it by publicly asking questions and personally reflecting on her own beliefs. Charles experienced cognitive dissonance in a public forum as he reflecting on his practice with Hartter. Kirtley and Charles reacted to cognitive dissonance in different ways. Their responses exemplify three of seven ways that individuals react when presented with anomalous data (Chinn & Brewer, 1993).

Chinn and Brewer (1993) characterized these responses as (a) seek to ignore the data, (b) reject it, (c) exclude it, (d) hold it in abeyance, (e) reinterpret it while retaining theory, (f) reinterpret it and make peripheral changes, or (g) accept the data and change the theory. Charles experienced significant cognitive dissonance while Hartter challenged him to consider how his actions influenced students' opportunities to learn in the two geometry classes. In the remedial class, Charles rejected the data and held onto his belief that remedial students are unable to use mathematical reasoning to solve problems. In the regular geometry class, Charles reinterpreted the data and made peripheral changes. He began to encourage alternative solutions and asked more questions that prompted reflection. However, he still believed that student investigations of mathematical ideas and open-ended questions were not appropriate for in-class activities. Thus, he interpreted the data to indicate that students in the regular geometry class could create multiple solution strategies but that they were incapable of using mathematical reasoning to solve non-routine problems.

In summary, this study illustrates how one individual responded to cognitive dissonance in very different ways and suggests that helping secondary pre-service teachers use reform recommendations is indeed challenging. We found that cognitive dissonance does not necessarily provide a context in which pre-service teachers can examine their practices from a new perspective. De Geest, Watson, and Prestage (2003) suggest that pre-service teachers include in their lesson plans tasks in which students create and solve their own examples. We theorize that this strategy would help teachers examine new data, accept it as legitimate, and change their beliefs about whether students can think mathematically. We also believe that asking students to create their own examples would provide teachers with data that is less threatening than having an outside individual provide student data. Olson (2004) found that evoking teachers' curiosity also provided a non-threatening context in which teachers could collect data about their students' mathematical thinking which may help them reinterpret their theories about student learning.

Even & Tirosh (2002) encourage teacher educators to go beyond implementing one or two strategies that may help pre-service teachers develop questioning skills. They challenge teacher educators to build theory that guides practice. Additional research is needed to characterize different situations that prompt cognitive dissonance and pre-service teachers' responses to dissonance. We suggest that Chinn and Brewer's (1993) framework of responses to anomalous data may be a useful tool to understand how cognitive dissonance can foster change in beliefs and practice.

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PEDAGOGICAL CONCEPTS AS GOALS FOR TEACHER EDUCATION: TOWARDS AN AGENDA FOR RESEARCH IN TEACHER DEVELOPMENT

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What conceptions of mathematics learning and teaching might contribute to increased mathematics teacher effectiveness? I argue that identification of goals for mathematics teacher education is critical to both effective teacher education and productive research on teacher development. Based on empirical and theoretical work in the context of three major research projects, I propose a set of pedagogical concepts for consideration as goals for teacher education. These concepts are proposed both because they are important to mathematics teaching and because they are generally not part of the pedagogical understanding of teachers in the US that we have studied. Successful fostering of these pedagogical concepts through teacher education will depend on research investigating teacher development of these concepts.

Underlying Assumptions

The arguments advanced in this theoretical article derive from both a social and a cognitive perspective. The use of these perspectives is pragmatic rather than the result of epistemological commitments. In addition, the arguments are based on two assumptions: 1. Aspects of the knowledge base in mathematics education are critical content (goals) for mathematics teacher education (1). 2. Identification of goals for mathematics teacher education is critical to both effective teacher education and productive research on teacher education.

Teacher education efforts, including those that are the context for research on teacher education, can be sorted into two categories: those with process goals only and those that have content and process goals. Highlighting the former category are programs that derive from the Japanese lesson study model (e.g., Yoshida, 1999)) and programs focused on teacher inquiry or teacher research (e.g., Dana, & Yendol-Silva, 2003). The basis of these programs is that the engagement of teachers in inquiry-based, reflective practices combined with appropriate support and communication structures can enable the ongoing professional development of mathematics teachers. These programs, which have demonstrated significant ongoing benefit for teachers of mathematics, are not focused on the learning of particular pedagogical principles (other than learning of the inquiry and communication processes).

The second category of teacher education efforts involves courses for teachers in which teacher educators plan for teacher learning of particular mathematics education concepts, skills, and dispositions. Although teacher education courses are often criticized as removed from practice and unresponsive to the needs and interests of the teachers involved, these negatives are not inherent properties of such an approach.

An assumption underlying this article (#1 above) is that there are understandings of mathematics learning and teaching that are important for teachers to develop. Therefore, although lesson study and teacher inquiry are important and useful, they are not sufficient. Courses that are designed to promote powerful ideas about learning and teaching are needed as well. This assumption (in conjunction with assumption #2 above) leads to the question, What pedagogical understandings would be useful foci for mathematics teacher education?

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

In this article, I focus on teacher education that aims to promote teacher learning of particular aspects of the knowledge base on mathematics teaching; I discuss potential goals for teacher education of this type.

Current Articulation of Goals for Teacher Learning

Currently, the identification of goals for teacher education courses is largely a part of teacher educators' practices and not the focus of theoretical and empirical reports. A perusal of articles in the Journal of Mathematics Teacher Education since its inception shows a scarcity of discourse on this subject. Some articles focus on process goals such as developing reflective practitioners (e.g., McDuffie, 2004). Hiebert, Morris, and Glass (2003) focused on learning to teach from practice. Within this broad objective, they identified specific requisite dispositions and skills.

Literature that focuses more on specific learning includes reports of fostering teachers' understanding of students thinking (e.g., Crespo, 2000). Schifter, Bastable, and Russell (e.g., 1999) developed materials targeted at developing knowledge of students thinking as they learn particular mathematics and teacher reflection on related teacher interventions. The Cognitively Guided Instruction Project (Carpenter, Fennema, Franke, Levi, & Empson, 1999) focused on providing research-based information on students' solution strategies to teachers.

The Current State of the Knowledge Base in Mathematics Teaching

Many countries of the world have been involved in a reform in mathematics education over the last 15-20 years. The formal start of the reform in the United States is recognized to be the publication of the Standards (National Council Teachers of Mathematics, 1989). The reform has generally been an effort to focus mathematics instruction on conceptual learning, mathematical thinking, communication, and problem solving for all students. These goals for instruction have led to a decreased acceptance of direct instruction (teacher telling and showing) as the primary mode of teaching. Mathematics educators have replaced direct instruction with a set of reform strategies, such as the use of collaborative group problem solving, whole class discussions, manipulatives, software environments, calculators, and probing questions. Lacking are models of teaching -- frameworks for guiding the fostering of students' mathematical conceptions. As a result, teachers' use of reform curricula and strategies is often unprincipled and ineffective.

In many locales, there is neither a consensus model of teaching, nor a recognized set of alternative models. Rather teaching is implicitly defined by the curricula, the reform strategies, and the consensus "don'ts" (e.g., teacher telling, showing, giving answers) (2). The lack of clearly articulated, established models of teaching handicaps teacher education and research on teacher education. Without such models the goals of teacher education are at best under-defined. Teacher education tends to be directed towards broad skills (asking probing questions, focusing on students' thinking, writing lesson plans) as opposed to the development of particular pedagogical principles. In the next section, I identify potential goals for teacher education based on our emerging framework on mathematics learning and teaching.

Identifying Key Conceptions for Mathematics Teaching

Through three major research projects on teacher development grounded in the research literature, my colleagues and I have identified pedagogical concepts that seem to be important for high-quality mathematics teaching. This work has been interwoven with theoretical work on mathematics conceptual learning and teaching (Simon, Tzur, Heinz & Kinzel, 2004; Simon & Tzur, 2004; Tzur, & Simon, 2004). The pedagogical concepts that we have identified derive from the perspectives represented by this theoretical work. In this article, I identify key pedagogical concepts that derived from our work and the work of others in order to discuss goals for teacher education and agendas for research on teacher development. I make no attempt to provide an exhaustive list of concepts; rather I raise a subset for consideration.

Briefly, our theoretical work involves both social and cognitive perspectives. We use social perspectives to account for the norms that are negotiated in the classroom (McNeal & Simon, 2000) that afford and constrain the learning and communication in the classroom. We use a cognitive perspective to describe how new knowledge is developed from extant knowledge, particularly Piaget's constructs of assimilation and reflective abstraction.

Following are brief discussions of a set of five pedagogical concepts that are important to consider because of their impact on mathematics teaching and because we have found them to be generally lacking among mathematics teachers in US classrooms. Most of these concepts are overlapping and interrelated. Each of these concepts deserves extensive discussion. In lieu of space in this short article, the reader is referred to articles related to each of the concepts.

- Negotiation of classroom norms. The notion that classroom norms are negotiated, not imposed (McNeal & Simon, 2000; Yackel & Cobb, 1996), allows teachers to be conscious of their contribution to the constitution of classroom norms. This understanding of their role allows teachers to engage intentionally in the negotiation of norms that support rich mathematical classroom learning. Although mathematics researchers introduced the construct ten years ago, it has generally not been an explicit goal for teacher education.
- Assimilation. An understanding of assimilation is essential for teachers to understand the determinants of what students perceive and understand and to focus on the resources students bring to learning situations. Cobb, Yackel, & Wood (1992) described a representational view of mind to characterize educators' lack of understanding of assimilation. Our study of teachers involved in the reform (Simon, Tzur, Heinz, Kinzel, & Smith, 2000) highlighted the distinction between teachers with perception-based perspectives (lacking a concept of assimilation) and those with conception-based perspectives. Understanding assimilation affords better anticipation of student responses to lessons and teacher reflection as to why lessons were unsuccessful. It allows teachers to question assumptions that students' perceptions/experience are the same as the teachers'.
- What it means to develop a new mathematical operation. Teachers struggle with what it means for students to develop a new operation, for example multiplication. Teachers tend to teach about multiplication to students who have no concept of multiplication to learn about. Missing is the idea that the term "multiplication" must label for the student a commonality (abstraction) that they perceive in their actions in particular situations. It is only when students observe that what I did in this problem about the cost of 5 candy bars is "the same" as what I did in this problem about 7 boxes of pencils, that they have something to label as multiplication – that commonality. This perception of commonality builds on the learner's anticipation of the activity needed and the effect of that activity. This pedagogical concept is based on the concept of assimilation and the concept of learning through activity discussed next..

- Learning through activity. Teachers often focus on the dialogic aspects of teaching. Classroom discussions and small group conversations can be an important part of the learning process. However, teachers need to be able to do more than encourage participation in discussions of mathematical problems. It is helpful if teachers can think about how learners learn through their own goal-directed activity (Simon, et al, 2004; von Glasersfeld, 1995). Students' goals influence what they attend to. Their activity and reflection afford them a way to extend current conceptions and create new ones. Teacher understanding of learning through activity can contribute to effective selection, sequencing, and modification of mathematical tasks.
- Reflective abstraction versus empirical learning. Learning of mathematical concepts is not an empirical learning process (Simon, in press; 2003); rather it is a result of reflective abstraction. An empirical learning process is an inductive process through which learners discover patterns by observing a set of inputs and related outputs. Through an empirical process, learners learn that a pattern exists. The phenomenon that underlies the pattern remains a black box to the learner. Reflective abstraction, according to Piaget (2001), is the process by which higher-level mental structures are developed from lower-level structures, a coordination of actions leading to a new conception. He described it as having two phases, a projection phase in which the actions at one level become the objects of reflection at the next and a reflection phase in which a reorganization takes place. Reflective abstraction develops anticipation of the logical necessity of a mathematical relationship. Teacher awareness of this distinction helps them make students' abstracting the central focus of instruction, rather than pattern noticing.

The five concepts identified in this section represent only a part of the knowledge needed for teaching. They represent concepts that emerged in our work as important and needed by current teachers. They provide examples of what might be meant by key pedagogical concepts and should provoke discussion of this particular set of concepts.

An Agenda for Research on Mathematics Teacher Development

Research on mathematics teacher development can be enhanced by the articulation of clear goals for teacher learning, goals that can help to define what counts as successful learning. Teaching experiments with teachers (Simon, 2000) can be structured around a clear set of learning goals.

Although we have worked with and studied a number of fine, reform-oriented teachers, the teachers have generally not demonstrated an understanding of the concepts identified above. There is a need for research that can inform efforts to engender teacher learning with respect to these concepts. For each concept we can ask the questions:

- To what extent can teachers at different stages of professional development come to understand this concept?
- What is the process of development for each concept and how can development of the concept be fostered?
- How are concepts related in terms of prerequisite concepts and co-developing concepts?

The identification of pedagogical concepts that can serve as goals of teacher education is the first step in establishing and enacting a research agenda on mathematics teacher development.

Endnotes

1. We use "teacher education" to include both pre-service and in-service education unless otherwise specified.

2. Perhaps the most clearly articulated principled approach to mathematics instruction is Realistic Mathematics Education (RME) in the Netherlands (Gravemeijer, 1994). Its principles deal primarily with curriculum development, but they can be seen as providing a framework for mathematics teaching as well.

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PEDAGOGY THAT MAKES (NUMBER) SENSE: A CLASSROOM TEACHING EXPERIMENT AROUND MENTAL MATH

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We report on a classroom teaching experiment around number sensible mental math in a semester-long content course for preservice elementary teachers. We designed, implemented, and revised an instructional sequence aimed at students' development of number sense with regard to mental math. The data corpus included: a number sense test administered pre and post, interviews with 13 students pre and post, students' written work, and the instructor's journal. Analysis of the data suggests that students did develop greater number sense as a result of their participation in classroom activities. Particular pedagogical innovations, such as those involving the use of models for reasoning, seem to have supported students' development of number sense. Results can inform mathematics teaching at various levels.

The development of number sense in students is a widely accepted goal of mathematics instruction (NCTM, 2000). Good number sense is especially essential for elementary teachers. Without it, they are ill-equipped to make sense and take advantage of children's often unorthodox but very number sensible solution strategies. Mental math ability is considered a hallmark of number sense (Sowder, 1992). Much work has been done with the aim of identifying the characteristics exhibited and strategies used by individuals who are skilled at mental math (cf. Reys, Rybolt, Bestgen, & Wyatt, 1980, 1982; Hope & Sherrill, 1987; Markivits & Sowder, 1994). Of note is *flexibility* in thinking about numbers and operations (Sowder, 1992).

In two classes of preservice elementary teachers enrolled in a mathematics course focused on Number & Operations, we conducted a classroom teaching experiment (Cobb, 2000) in which the instructor attempted to foster students' development of number sense with regard to mental math. In previous courses, the instructor had found that such development had not occurred. How could one design a class that supported the development of number sense? We undertook a classroom teaching experiment in the paradigm of Design Research (Stephan, 2003) to answer this question. The first author developed a hypothetical learning trajectory (HLT) aimed at students developing the characteristics of skilled mental calculators and estimators. We found that students developed greater number sense with regard to mental math. In addition, the actual learning trajectory that resulted can inform future pedagogy.

We report here on the results of the classroom teaching experiment. This work was part of a larger study, which constituted the first author's Master's thesis**.** In this paper, we focus specifically on the role of models in supporting students' development of number sense.

Theoretical Perspective

Our theoretical orientation can be characterized as sociocultural. Students' individual mathematical activity is recognized as taking place in a social context, while the social environment of the classroom is constituted by collective mathematical activity. As such, the instructor concerned himself with the negotiation of norms and practices (Cobb, 2000).

Reys & Yang (1998) state that "[n]umber sense refers to a person's general understanding of number and operations" and "includes the ability and inclination to use this understanding in

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

flexible ways to make mathematics judgments and to develop useful strategies for handling numbers and operations" (p. 226). The perspective on number sense that we take is rooted in Greeno (1991)'s metaphor of situated knowing in a conceptual domain. Thus, the instructor focused on providing students with experiences that would enrich their ability to navigate that domain. According to Greeno (1991), in the environment/model view, "the main capability that we want students to acquire involves constructing and reasoning within models" (p. 212). The use of models became an important aspect of the classroom activity around mental math.

Setting

This study was conducted with undergraduates at a large, urban university in the United States. The participants were preservice elementary teachers enrolled in two sections of a first semester mathematics course, belonging to a four-course sequence. Of the 50 students who agreed to participate in the study, 42 were female. The first author was the teacher of the course. He had taught it for two prior semesters. Basic course topics included quantitative reasoning, place value, meanings for operations, and number sensible mental math. The instructor decided not to treat mental math as an isolated curricular unit but to integrate *authentic* mental math *activity* (Brown, Collins, & Duguid, 1989) throughout the curriculum.

Hypothetical Learning Trajectory

The formulation of the HLT was informed by a review of the literature around number sense and mental math, together with the first author's previous experience teaching the course. We designed an instructional sequence with the goal that students would develop greater number sense with regard to mental math. The HLT was envisioned in terms of three layers. First, the course content was expected to support students' understanding of the mathematics behind particular mental calculative strategies. Second, discrete tasks were devised as a means of assessing students' abilities and the availability to them of various strategies, as well as of providing individual feedback. Students' responses to these tasks would then inform the design of subsequent tasks. Third, mental math activity would be an integral part of problem solving and would provide the occasion for *reflective discourse* around students' strategies (Cobb, Boufi, McClain, & Whitenack, 1997).

The HLT can be briefly articulated in terms of the following sequence of conjectured outcomes:

1) Students recognize opportunities for mental math, both inside and outside the classroom.
2) Students make sense of place value and, as a result, the standard addition and subtraction algorithms.
3) Students make sense of meanings for the operations and consider the use of new mental calculative strategies that build on their understanding.
4) Students confront and make sense of unorthodox strategies and alternative algorithms that are radically different from the ones they know.
5) Students recognize the difference between the use of standard algorithms and tools, such as the empty number line.
6) Students develop their own number sensible mental calculative strategies.

Note that the planned instructional sequence for the HLT was atypical. Typically, an HLT applies to an isolated unit in a curriculum (cf. Gravemeijer, Bowers, & Stephan, 2003; Simon, 1995). In our case, the aspects of classroom instruction that related to number sensible mental math represented a strain of activity that ran through several curricular units.

Data

The data sources drew from classroom events, written artifacts, and individual interviews. Specifically, the data corpus consisted of the following:

a) The instructor's journal, which included accounts of classroom events, as well as rationales for the teaching modifications made during the semester;
b) Students' written work, which included responses to mental math tasks, both in-class and take-home, as well as responses to exam questions;
c) Transcripts of early- and late-semester clinical interviews with 13 students;
d) An adapted version of the Number Sense Rating Scale (Hsu, Yang, & Li, 2001), used as a quantitative measure of number sense, which was administered to students at both the beginning and end of the semester.

Methodology

The general design of this study was that of Design Research, which is characterized by the reflective relationship between classroom-based research and instructional design encompassed in the Design Cycle (Stephan, 2003). As such, data analysis involved three distinct phases:

Phase 1. During the course of the semester, the instructor engaged in formative analysis, in which the instructional sequence was revised in accordance with his interpretations of classroom events and written records of student thinking.

Phase 2. At the end of the semester, data from the number sense test and individual interviews was analyzed in order to assess the effect of the program of instruction on students' number sense. Interviews were structured and task-based (Goldin, 2000). Students were asked to solve one-step story problems mentally and to describe their thinking. Analysis of interviews was interpretive (Clement, 2000), seeking to identify the variety of mental calculative strategies students had employed.

Phase 3. Having noted students' improved number sense, the authors conjectured that certain features of the classroom activity had been particularly significant in supporting that development. These were then analyzed in terms of relevant theoretical constructs.

Results

Significant, selected results are presented here in terms of the three phases of analysis:

Phase 1. The instructor's interpretations of classroom events led to alterations to the instructional approach. A very significant alteration came about in the course of a particular teaching episode. The instructor made immediate innovations to address a local learning goal. Subsequently, aspects of the instruction related to mental math were altered as a result of the instructor's reflections on the episode.

The Teaching Episode spanned four class meetings. On the first day, four interpretations, or distinct meanings, for multiplication were discussed. (Rectangular array/area is one such interpretation.) On the second day, the instructor selected a homework problem for discussion, the solution to which required computation of 26 x 26. As was typical of the integration of mental math activity into the classroom instruction, the instructor asked students to compute this product mentally. A few students shared their solution strategies, which were discussed amongst the class. These suggestions (e.g. 20 x 20 + 6 x 6) seemed to point to a lack of understanding of the origins of partial products in multi-digit multiplication. Although only a few students made such suggestions, no student managed to refute any one of them in sense-making fashion. The instructor suggested making use of a meaning for multiplication and made drawings of rectangles segmented place-value-wise. Students accepted this application of rectangular area and were then able to decide on correct and incorrect solutions. However, the instructor was dissatisfied with this outcome. Students had not made sense of the matter themselves. They had not thought meaningfully about multiplication.

The outcome of the next day was similar. Students' answers to an estimation question again suggested that they were not thinking in terms of partial products. Again, the instructor made sketches of rectangles to help students settle their questions. These were guided more by students' suggestions than had been the case the day before. Still, however, students had not seemed to think meaningfully about multiplication on their own. They would need to understand the origins of partial products in order to reason about mental multiplication strategies. This became the local learning goal.

The instructor designed a Geometer's Sketchpad sketch and a short lesson around it. The sketch was a dynamic representation of a rectangle, segmented place-value-wise, with the areas of the partial rectangles shown. The lesson involved students being asked a sequence of challenging questions related to estimation of products. Students' conjectures were confirmed or refuted either by the sketch, by drawings of rectangles, or by students' arguments. In this context, students began to reason with rectangular area as a *model for* products (Gravemeijer, Bowers, & Stephan, 2003).

Results on a midterm question connected to the sketch lesson were exceptional. Students seemed to have made an important connection between rectangular area and partial products, as well as acquired powerful tools for estimating products. This episode precipitated an important alteration to the greater instructional sequence. The instructor recognized that connections between mainstream course content and applications to mental math were nontrivial. He would need to address the process by which explicit connections could be made between the two. Models came to be emphasized as a means to that end. Rectangular area, in particular, represented an unanticipated tool, which became central to the collective activity around mental multiplication strategies. They empty number line had been used similarly for reasoning about mental addition and subtraction strategies.

Classroom discourse around students' strategies after the Teaching Episode emphasized reasoning with models. For example, commonly seen applications of additive distributivity were characterized in terms of breaking up a rectangle, usually place-value-wise. Figure 1 depicts an example of the strategy that students called "Break up, then make up." In this example, the product of 15 and 24 is represented as a 15-by-24 unit rectangle. Initially, the value of this product is unknown. By breaking up the rectangle conveniently, it is shown to consist of two readily known products, the sum of which gives the total product.

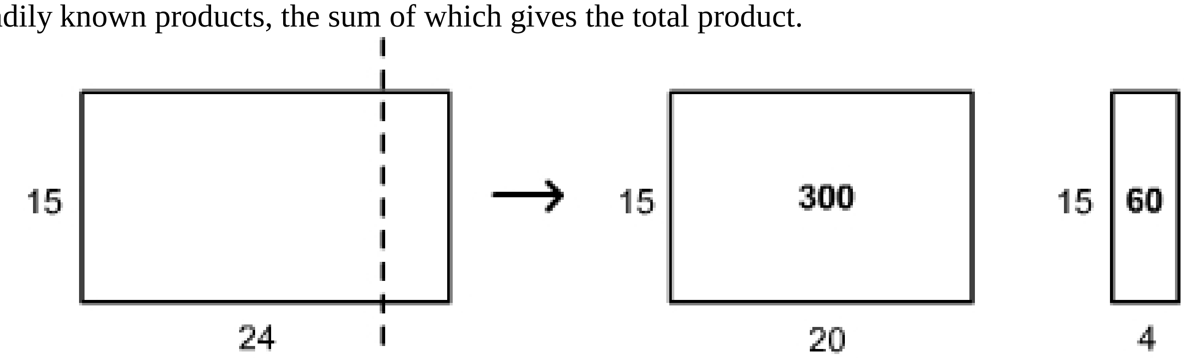

Figure 1. "Break up, then make up."

As the use of models is the focus of this paper, we only mention significant results with regard to the other two layers of the instructional approach. The discrete tasks were deemphasized due to lack of practicality. Mental math activity evolved over the course of the semester. The instructor sought from the beginning to engage students in reflective discourse. Mental math activity seemed lacking until the practice of naming strategies was introduced. Naming facilitated reflective discourse.

Phase 2. Though not the focus of this paper, we note that we found strong evidence in the individual interviews, as well as the number sense test, that students developed significantly greater number sense as a result of their participation in classroom activities. Interview subjects' strategies for mental computation of sums, differences, and products were categorized via constant comparative analysis (Creswell, 1998). Six strategies were seen for mental addition, eight for subtraction, and eight for multiplication. In first interviews, most subjects used only one or two distinct strategies for each of the operations. In second interviews, 12 of 13 subjects used three or more addition strategies, 12 of 13 used three or more subtraction strategies, and 10 of 13 used three or more multiplication strategies.

Markivits & Sowder (1994) categorized their subjects' strategies for mental computation of sums, differences, and products in terms of the degree to which each departed from the mental analogue of the standard algorithm (MASA). In this scheme, *Standard* refers to the MASA for a given operation, *Transition* refers to a method that is still tied to the standard algorithm but differs from it, *Nonstandard with no reformulation* refers to a method that is free from the standard algorithm but does not change the given numbers or operation, and *Nonstandard with reformulation* refers to a method in which the problem is altered to make the computation easier. For our purposes, the above taxonomy was used as an organizing framework. Interview subjects' strategies were categorized as *Standard* (S), *Transition* (T), *Nonstandard with no reformulation* (N), or *Nonstandard with reformulation* (N w/R). This allowed for subjects' strategies to be described in terms of number sensibility. Figure 1 shows the frequency of use of strategies from each category in first versus second interviews.

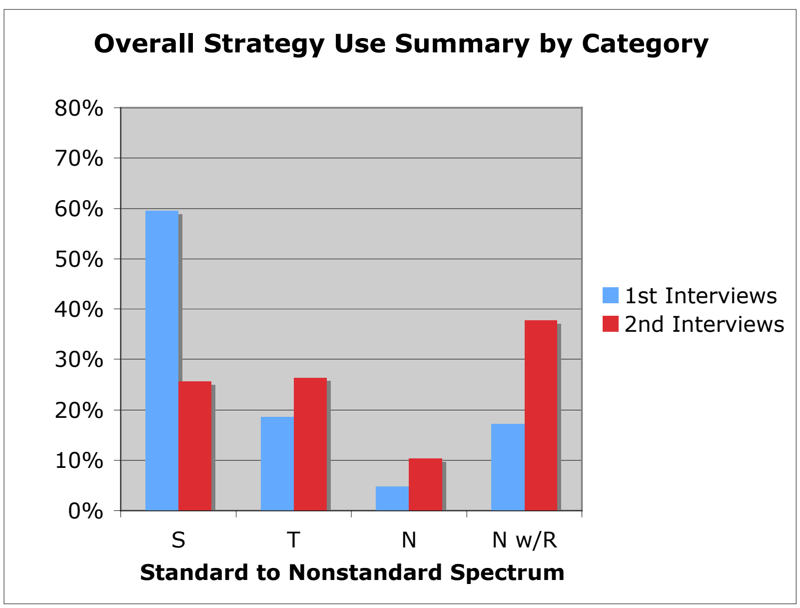

Figure 2. Overall Strategy Use Summary, Pre vs. Post

For each operation, there was a large decrease in the frequency that the MASA was used, accompanied by an increase in use of alternative strategies. Thus, given story problems that called for mental addition, subtraction, and multiplication, subjects exhibited greater flexibility by making use of a wider variety of strategies in second interviews than they had in first interviews. Furthermore, strategies used in second interviews were more number sensible. This is apparent in the movement we see along the spectrum from *Standard* to *Nonstandard*. It is compelling evidence for change in the direction of number sense that *Standard* methods were most common in first interviews, while *Nonstandard with reformulation* became most common in second interviews.

Students also showed significant increases in number sense as measured by the adapted Number Sense Rating Scale. Average scores for the early section increased from 61% to 73%, pretest to posttest. Average scores for the later section increased from 66% to 77%. A total of 48 students took the number sense test both times it was administered They were treated as one group in determining statistical significance. A paired *t*-test was used for a difference of means. Results were statistically significant ($p < 0.005$).

Phase 3. In post-hoc analysis of the instructional sequence, the authors conjectured that the innovations of naming and the use of models had been keys to students' development of number sense. Analysis showed that the classroom discourse around mental math was indicative of reflective discourse and that the practice of naming facilitated *vertical mathematizing* (Freudenthal, 1991). Cobb, Boufi, McClain, & Whitenack (1997) claim that students' participation in reflective discourse "*constitutes conditions for the possibility of mathematical learning*" (p. 264).

The use of models also seemed to be a key to the success of the instructional sequence. Although the use of the empty number line and rectangular area evolved differently, both can be said to have transitioned from a *model of* students' informal activity to a *model for* more formal mathematical reasoning (Gravemeijer, Bowers, & Stephan, 2003). The *model of* to *model for* transition is conjectured to support students' increasingly sophisticated mathematical reasoning. Our use of the empty number line and of rectangular area facilitated students' reasoning more formally about shared mental calculative strategies. In this way, it seems to have supported their development of number sense with regard to mental math.

Continuing the Design Research cycle, the actual learning trajectory that was charted during the classroom teaching experiment informed the construction of a new HLT for the following semester. The new instructional sequence incorporated the practices of naming and the use of models from the start.

Conclusion

The development of number sense in students is an important aim of mathematics instruction. Essential to this goal is that teachers, themselves, have good number sense. In this work, we begin to answer the question of how an instructor can support preservice teachers' development of number sense with regard to mental math. Furthermore, the key practices that emerged in this study can be incorporated into instruction of elementary school students. Analyses such as these can benefit teachers, curriculum developers, and teacher educators. It is also significant that the integration of authentic mental math activity into an existing curriculum supported students' development of number sense without any of the course content being sacrificed.

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CONNECTED REGISTERS FOR GEOMETRY: LEARNING TO GENERALIZE

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We are investigating the coordination of a dynamic geometry environment (DGE) with a traditional hand tool environment (HTE) for construction of 2D geometric objects as a basis for teaching an undergraduate, elementary teacher education course on geometric reasoning. This is a preliminary report of our project encompassing three foci: (1) describing the adaptive elaboration of a teaching trajectory for students' geometric reasoning and understanding by examining generalization within a DGE, (2) describing connectedness among different modes of representation for 2D geometry, and (3) describing the development of argumentation through construction tasks in various media. This report addresses the first focus topic.

Our theoretical outlook depends closely on Duval's (1998) model of various registers for cognition. We envision students' modes of representation for geometric concepts including registers for dynamic visual processes on a computer, for discursive (verbal) accounts of objects, for imagined or drawn images and for symbolic notational accounts. Thus, learning proceeds by an interaction among registers, through reflective abstraction, as the learner attends to an activity-effect cycle mediating progress toward a goal (Simon & Tzur, 2004). To account for the learning of geometric concepts, we adapt the phases of a learning process (van Hiele, 1986): (1) anticipate a context for a concept in discussion, (2) engage several representations in specified tasks to organize and integrate sub-concepts, (3) explicitly notice geometrical abstractions (structure) through reflection on task-effect relations, (4) meet conventional, formal language and symbols indicating structure and look across contexts at examples. Further, we rely on the theory of levels of thinking--recognition, analysis, ordering, deduction and rigor—to describe students' levels of reasoning about geometric concepts. Lastly, we characterize mathematical learning of generalized concepts as a gradual integration of verbal definitional knowledge with imagistic, contextual history of one's experiences (Arshavsky & Goldenberg, 2005; Vinner & Dreyfus, 1989).

We ask the question: in what ways do students develop generalized argumentation and justification for 2D geometric concepts? We believe they learn to make explicit, generalized arguments as they learn to generate and coordinate extensive, comprehensive sequences of examples, especially by addressing boundary cases of relations or objects. We predicted that juxtaposing DGE and HTE tasks would allow a teacher to prompt students to reflect on an activity and its effect within a context, and within a collection of relevant activities, thereby emphasizing the abstract properties of objects and relations.

Our methodology follows from our need to characterize learning within natural instructional contexts. We are engaged in a Teacher Development Experiment (Simon, 2000) as a means of testing our hypotheses related to students' ways of learning to generalize within a cycle of tasks, discussion, and assessment. Further, we set out to characterize growth in understanding as it relates to task deployment, requiring the coordination of teaching and research. We have collected data throughout a 15-week course with 28 college students in a general education course on geometric reasoning. The data corpus includes videotaped records of bi-weekly classroom discussions, two rounds of case study interviews with five students, written and drawn

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artifacts for case study students, and observational notes by a classroom observer. Our analysis is based on a history of the 36 tasks used as the curricular basis of an experimental section of the course. Here we present one central finding based on our concurrent analysis, with a retrospective analysis of the project to follow: we focus on generalization and justified reasoning to trace students' learning of geometric reasoning.

Our findings indicate the level of geometric thinking exhibited by most of the students at the outset of the experiment, dominating students' discourse, as analysis or ordering (van Hiele, 1986). Early in the course, students exhibited a view of justification that employed special cases--usually prototypical, standard images--to address and resolve tasks, lacking logical steps from the particular to the general. For example, the first task in the course consisted of placing a bridge at the best location along a river to give the shortest path between two towns separated by the river (purposefully non-specific). Students collaborated for several days and were prompted for more general solutions, yet their solutions and drawings were constrained by prototypical references to congruent triangles or in symmetrical ways. As a second example of this early level of generalization, students responded to a challenge to justify the formula for finding the area of a triangle (1/2*b*h) by drawing a square and creating a diagonal to show the square consists of two triangles (3rd week). One student, Mack, argued that the square case was more clear than a quadrilateral case while others argued that a rectangle would be best; they argued that an explanation should use a familiar case. During the third week, the class began an alternating pattern of using both a DGE and the HTE to address tasks. Students began hearing and using the phrase "unbreakable sketch" to describe constructed objects in the DGE that preserved intended properties for an object. By the end of the third week however, students began using the word general and the phrase special case as ways of challenging or evaluating claims made by fellow students. Mickey suggested that a construction of a right triangle would not be "general enough" if it relied on specific lengths of segments (3, 4 and 5 units).

By the fourth week, a vocal majority of students began using argumentation that we characterize as ordering or deductive in level. For example, on February 7, students drew at the board to talk about an HTE and then used a DGE to explain a construction of a bisected angle. One student, Hessie, had suggested a sequence that would only work in a special case. Three other students intervened as they found ways to follow her sequence of verbal instructions, yet not produce a bisected angle. They used the phrases, "only a special case", and "not generalized" as they offered other ways to work toward the construction. They justified their constructions by claiming that they would be unbreakable in the DGE. We found a pattern of increasing generalization associated with the use of the term "unbreakable" as a measure of a construction in a DGE. We found students gaining connected, integrated images and definitions for concepts as they reflected on actions across both environments.

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ENHANCING ELEMENTARY PRESERVICE TEACHERS' UNDERSTANDING OF VARIATION IN A PROBABILITY CONTEXT

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The purpose of this paper is to report on research aimed at elementary preservice teachers' understanding of variation. Other research has already begun to illuminate precollege student thinking about variation in several contexts, such as sampling, data and graphs, and probability situations (e.g. Reading & Shaughnessy, 2004; Watson & Moritz, 1999; Shaughnessy & Ciancetta, 2002). However, as the picture begins to get painted about how precollege students reason statistically, the research on how teachers reason about variation remains thin. In particular, there is a paucity of research about how preservice teachers think about variation, or variability in data. Therefore, doctoral research was undertaken to explore the following research question: How do elementary preservice teachers' responses concerning variation in a probability context compare from before to after an instructional intervention? After describing the conceptual framework and methodology for the study, the results will next be presented, followed by further discussion.

Conceptual framework

Three key aspects of understanding variation that governed the overall study focused on how students were expecting, displaying, and interpreting variation. In dealing with expectations, students need an opportunity prior to conducting statistical investigations to express both what they expect and why. With displays of data, students need to create their own graphs to either highlight or disguise variation, depending on the context of the situation. They also need to evaluate displays and compare distributions in ways that take an aggregate view of data, considering shape and spread in addition to centers (Shaughnessy, Ciancetta, Best, & Canada, 2004). From discussions about probabilistic and statistical situations, students' interpretations of variation emerge as they speculate on both causes and effects of variation and also on ways of influencing variation and expectations.

Methodology

The thirty subjects in the study of EPSTs (24 women, 6 men) were enrolled in a ten-week preservice course at a university in the northwestern United States designed to give prospective teachers a hands-on, activity-based mathematics foundation in geometry and probability and statistics. During the first week of the course, prior to instruction in probability and statistics, subjects took an in-class survey (called a PreSurvey) designed to elicit their understanding on a range of questions about sampling, data and graphs, and probability. The probability question (PreSurvey Q7c) that relates to the current paper concerned six sets of fifty flips of a fair coin. For each of the six sets, students were asked how many times out of the fifty flips the coin might land heads-up. They were also asked why they had chosen the numbers they did. Following the PreSurveys but prior to the class instruction on probability and statistics, individual interviews were conducted with ten subjects to allow further probing of their thinking. After instructional interventions took place in class, a similar PostSurvey question (PostSurvey Q1c) was asked

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concerning six sets of fifty spins of a fair half-black and half-white spinner. For each of the six sets, students were asked how many times out of the fifty spins the pointer might land on black, and also why they had made the choices they did. Finally, after the PostSurveys the same students who had been earlier interviewed were interviewed once again.

Results

Both parts of the probability question (what students expected and why) were taken into consideration for coding purposes, primarily to retain consistency with an analogous rubric derived for a similar question asked in a sampling context (Shaughnessy et. al., 2004). The rubric places a higher value on responses that integrate proportional reasoning as well as variation. The codes and class results for this subquestion are presented in Table 1.

Code Level	Description of Category	Number of Students (Pre)	Number of Students (Post)
L3	Appropriate choice & Explanation explicitly involves proportional reasoning as well as variation	2 (7.4%)	9 (31.0%)
L2	Appropriate choice & Explanation reflects proportional reasoning or notions of spread	10 (37.0%)	15 (51.7%)
L1	Appropriate choice & Explanation left blank or lacks any specific reasons relating to details of the distribution	4 (14.8%)	3 (10.3%)
L0	Inappropriate choice (Regardless of Explanation) W(ide) = Range > 19, N(arrow) = Range < 2, H(igh) = Choices > 24, L(ow) = Choices < 26	11 (40.7%)	2 (6.9%)

Table 1: Results for PreSurvey Q7c & PostSurvey Q1c

Only inappropriate choices for listing what was expected (or blank answers) were coded at Level 0. Deciding what would constitute an appropriate choice for the results on six sets of flips or spins involves making a judgment call, and the subcodes used for this subquestion question help identify inappropriate choices as (W)ide, (N)arrow, (H)igh or (L)ow. Of the 30 students enrolled in the class, 27 were in attendance to complete the PreSurvey and 29 completed the PostSurvey.

Conclusion

If a goal is for teachers to provide students with authentic, inquiry-based tasks meant to develop children's reasoning about variation, then a natural step in achieving this goal is to improve teacher training courses. Thus, by discerning components of preservice teachers' reasoning, teacher educators can better design university experiences that promote an understanding of variation for preservice teachers, as well as an understanding on how precollege students come to learn this topic. As research in the field of statistics education advances, one goal is that teacher education can improve not only the subject matter knowledge of EPSTs, but also the pedagogical content knowledge of teaching about variation. Steps toward improved pedagogical content knowledge can certainly be informed by recent research about how precollege students learn. Meanwhile, steps toward improved subject matter knowledge can be informed by a consideration of what are the conceptions of variation held by preservice teachers as they enter university programs. Collective discourse in the class, bolstered by activities and simulations targeted at eliciting conceptions of variation and developing these concepts, hold promise as ways of building EPSTs knowledge while also reflecting the kinds of practice they themselves will want to demonstrate in their own classrooms.

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LEARNING HOW TO USE MATHEMATICS CURRICULUM MATERIALS IN CONTENT AND METHODS COURSES

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Learning how to use mathematics curriculum materials to create opportunities for students to learn is, arguably, an important part of the work of teaching. In this session, the author will discuss elementary preservice teachers' conceptions of curriculum materials in the context of both a mathematics content course and methods course.

Efforts to reform the ways that mathematics is taught and learned in classrooms across the country also have implications for mathematics teacher education. As teacher education programs aim to develop teachers' knowledge of mathematics and their knowledge of students as learners, these programs "…should [also] develop teachers' knowledge of and ability to use and evaluate instructional materials and resources…[in order] to use these resources effectively in their instruction" (NCTM, 1989, p. 151). While using mathematics curriculum materials (MCM) effectively is arguably an important part of teachers' work, it is an aspect of practice that is often overlooked in teacher education programs.

In order to design mathematics methods and content courses to better help preservice teachers use MCM effectively, it is first necessary to understand preservice teachers' conceptions of these materials and how they see themselves as using these materials in their future work as teachers. Teachers' use of MCM, and ultimately their teaching practice, is shaped by their particular conceptions of how mathematics should be taught and learned (Manouchehri & Goodman, 1998; Thompson, 1984). Written in response to recent reforms, innovative MCM, in particular, present new modes of learning and instruction, and thus, place new demands on teachers (NCTM, 1989). Such curricula, however, are sometimes an affront to teachers' own beliefs about what it means to teach mathematics (Manouchehri & Goodman, 1998). This fact is potentially exacerbated in the case of preservice teachers who have little, if any, knowledge of curriculum materials and who have spent limited time in the classroom formulating their own views of mathematics teaching. This situation raises concerns that preservice teachers can use these materials with an inattention to the actual content and nature of the tasks, activities, and pedagogical suggestions contained in these resources.

This study focuses on how elementary preservice teachers think about and consider ways of using MCM in the classroom. Preservice teachers' conceptions of three curriculum activities designed to help them learn to use MCM were tracked across their mathematics content and methods courses. These activities included determining the overall goal of a textbook lesson, evaluating different mathematical definitions in textbooks, and analyzing a textbook lesson with careful attention to the tasks, examples, problem contexts, and mathematical representations.

Participants from this study were drawn from a group of elementary preservice teachers who were enrolled in an intensive, one-year Master's and certification program at a large, Midwestern university. Fifteen students volunteered to participate in this study. Students' class notebooks and individual interviews comprise the data sources for this study. Students were interviewed once after the first semester content course, and again after the second semester methods course.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Results and Discussion

Students' conceptions of MCM changed in relationship to the three curriculum activities over the course of the study. First, preservice teachers' conceptions of what constitutes MCM fell into three primary categories: textual materials (e.g., textbooks, student notebooks); non-textual materials (e.g., paper, pencils); and manipulatives (e.g., Base 10 blocks, Unifix cubes). While few students considered MCM to be exclusively one of these three categories, a majority of students considered MCM to be a combination of all three categories. Notably, the number of students citing some combination of textual materials and manipulatives increased across both courses.

Students also held strong, yet diverse, conceptions of how MCM can be used. Some students saw MCM as materials that children use to learn mathematics while other students viewed MCM as tools that are used to support teachers' decisions. By the end of the study, all fifteen students viewed MCM as materials that support teachers' decisions. This shift in students' conceptions seems to indicate a move to more teacher-based conceptions of MCM. An interesting issue that arose during the course of the study was that students began to formulate particular views of how teachers use MCM, which fell into three categories: scripted use; modified or adaptive use; limited or narrow use (e.g., Remillard, 2004). Though these views of MCM use were not apparent at the beginning of the study, a majority of students adhered to a modified or adaptive view of use by the end of the study.

Overall, the first and second curriculum activities seemed to have little effect on students' conceptions, with only six students and four students, respectively, who even mentioned the activities as helping them learn to use MCM. In contrast to the first two curriculum activities, twelve students said that the third curriculum activity was very helpful to their learning how to use MCM. In short, it is unclear to what extent students' changing conceptions can be attributed to the three curriculum activities. Moreover, the findings do not indicate the extent to which students were able to use MCM skillfully. Nevertheless, mathematics content and methods courses do appear able to provide students with some conceptions of MCMs in order to use them in skillful ways. It seems unreasonable, however, to think that the administration of three curriculum activities will equip students with the necessary skills to enable them to use MCM effectively, as evidenced in students' comments about the different curriculum activities.

For this reason, we need a more cohesive framework that integrates MCM into preservice coursework to a greater extent, and that includes several key components. First, content and methods courses should expose students to different mathematics curricula, and provide opportunities for students to learn about and familiarize themselves with the potential resources that are available to them. Second, students should have opportunities to select, develop, and/or adapt mathematical tasks and appropriate instructional strategies in ways that maintain the task complexity (Stein et al., 2000). Finally, students need to learn to use manipulatives that support and scaffold children's learning, as opposed to superficially and seemingly making mathematics fun and applicable to children's everyday lives (Moyer, 2001; Stein & Bovalino, 2001).

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CONSTANCY AND CHANGE IN PROSPECTIVE TEACHERS' CONCEPTIONS OF AND ATTITUDES TOWARD MATHEMATICS

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This study explored the change and lack of change of prospective elementary school teachers' conceptions of and attitudes toward mathematics during the time the subjects were enrolled in a "teaching mathematics" course. The research involved 34 students who completed a Mathematics Inventory, a "What Is Mathematics?" journal entry, and a concept map of mathematics at both the beginning and end of the semester. Four students were also interviewed on two different occasions. A framework was developed to synthesize, analyze, and organize the data. Results showed a shift in positioning in the students' relation to mathematics, an increased awareness of the usefulness of mathematics and the processes used in mathematics, and improved attitudes toward mathematics and teaching mathematics.

The study of teachers' conceptions of and attitudes toward mathematics has been an important part of research in the field of mathematics education for many years. This study specifically explores the role a "teaching mathematics" course might play in changing these conceptions and attitudes of prospective elementary school teachers. Following is a brief summary of this study. (A full report can be found in Tuft, 2005.)

Description of the Study

Rationale and Questions

Many have suggested that two factors that influence how teachers teach mathematics are their conceptions of mathematics and their attitude toward it (Dossey, 1992; Ernest, 1991; Thompson, 1992). The literature is also replete with reports of research that conclude that many prospective elementary school teachers have negative attitudes toward mathematics (see for example, Becker, 1986) and conceptions of what mathematics is that differ from the view of mathematics espoused by The National Council of Teachers of Mathematics (2000). It would seem, therefore, that part of the aim of "teaching mathematics" courses should be to improve these students' attitudes toward mathematics and change their conceptions of mathematics. But, does this happen?

To answer that question, I designed a study to investigate the specific questions of what a group of preservice elementary school teachers' conceptions of and attitudes toward mathematics were at both the beginning and the end of the semester in which they were enrolled in a "teaching mathematics" course. The other question that guided the study was what factors influenced whether these conceptions and attitudes changed or did not change.

Methodology

Subjects and Situation

The subjects for this study were the 34 elementary education majors I taught in a "teaching mathematics" course at a large midwestern university.

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Data Sources and Analysis

Four data sources informed the questions of this study: a Mathematics Inventory which included both Likert-type questions and open-ended questions, a “What Is Mathematics?” journal entry, a mathematics concept map, and interviews of four focus students. The students completed each of these data sources at both the beginning and end of the semester. The Likert-type questions were analyzed by using a matched-pairs t-test for independent means, and the other data were synthesized and organized qualitatively in various data displays with quantitative descriptors (see Tuft, 2005).

Development of a Framework of Mathematics

I originally expected to report the findings by determining categories that would best describe the students in relation to their conceptions of mathematics such as those suggested by Ernest (1991) or Cooney, Shealy, and Arvold (1998). However, as I began examining the data, I concluded that, for this study, labels would not adequately describe the many facets of the students’ conceptions and attitudes. Since most of the data sources were open ended, many of the students’ responses did not fit neatly into predetermined categories. I also found that the same participant could fall into different categories depending on the data source.

As a result, I developed a framework to describe these conceptions and attitudes. This framework played a crucial role in the synthesizing, analyzing, and reporting of this study. It allowed me to organize the description of different facets of the students’ conceptions of and attitudes toward mathematics. It also allowed me to code virtually every statement in the students’ writings and every item in their concept maps as well as categorize every item in the Mathematics Inventory. (For a full description of this framework, see Tuft, 2006.)

Discussion of the Findings of this Study

Major Findings, Implications, and Contributions

There were several noteworthy and significant findings that emerged from an analysis of this study. One of these findings is that the students shifted their position in relation to mathematics from that of an experienced student to that of a prospective teacher. Other findings were that the students’ awareness of the usefulness of mathematics and the processes used in doing mathematics was increased. The findings also indicated a more positive attitude toward mathematics and a more positive attitude toward teaching mathematics.

This research provides some implications for mathematics teacher education. It indicates that students’ shift in positioning in relation to mathematics can serve as a vehicle for changing their conceptions of and attitudes toward mathematic. It also provides insight into areas where change is more likely to occur. This study also makes contributions to the field of educational research such as a study that shows there can be change, a new framework for looking at conceptions of mathematics, ideas for using concept maps as a data source, and understanding the significance of the shift in positioning.

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EXAMPLES THAT CHANGE MINDS

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In this report we introduce the notion of a pivotal example – *an example that changes an individual's mind or way of operation, an example that induces conceptual change. We present two episodes of interaction between a teacher and a learner and highlight the role of examples in these interactions.*

> Swans are white. This is because all the swans I saw in pictures, photographs, lakes and zoos were white. Even the gray ugly duckling turned into a gorgeous white swan. And then, in my recent visit to Australia, I saw a black swan. This was a pivotal example.

Conceptual change and cognitive conflict

The term "conceptual change" is used to characterize "the kind of learning required when the new information to be learned comes in conflict with the learners' prior knowledge usually acquired on the basis of everyday experiences" (Vosniadou and Lieven, 2004, p. 445). Conducting research with prospective teachers, rather than with young learners, our perspective on the notion of cognitive change highlights the importance of a conflict – a cognitive conflict – between information and experience. However, the information does not have to be "new", but "newly realized" or "newly attended to", and the experience may come from prior learning opportunities rather than from everyday engagement.

A cognitive conflict is invoked when a learner is faced with contradiction or inconsistency in his or her ideas. It is important to mention that learners may posses conflicting ideas, and co-existence of these ideas may not be acknowledged and thus will not create a dissonance. However, inconsistency of ideas presents a *potential conflict*, it will become a *cognitive conflict* only when explicitly invoked, usually in an instructional situation. Implementing a cognitive conflict approach has been reported in studies on a variety of topics, such as division (Tirosh and Graeber, 1991), or sampling and chance in statistics (Watson, 2002).

When errors arise from some misconception, it is appropriate to expose the conflict and help the learner to achieve a resolution (Bell, 1993). However, while there is some understanding how a cognitive conflict can be exposed, once a potential conflict is recognized, there is little knowledge on how to help students in resolving the conflict. In this report we introduce the notion of "bridging/pivotal example" as a possible means towards conflict resolution.

Pivotal and Bridging examples

The central role of examples in teaching and learning mathematics has been long acknowledged. In particular, counterexamples may help learners' readjust their perceptions or beliefs about the nature of mathematical objects. Further, the role of counterexamples has been acknowledged and discussed in creating a cognitive conflict (Klymchuk, 2001; Peled and Zaslavsky, 1997). However, counterexamples may not be sufficient for a conflict resolution. As teachers, we are to seek *strategic examples* that will serve as *pivotal examples* or *bridging*

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examples for the learner. *Pivotal examples* create a turning point in the learner's cognitive perception or in problem solving approaches; such examples may introduce a conflict or may resolve it. In other words, pivotal examples are examples that help learners in achieving a conceptual change. When a pivotal example assists in conflict resolution we refer to it as a *pivotal-bridging example*, or simply *bridging example*, that is, an example that serves as a bridge from learner's naïve conceptions towards appropriate mathematical conceptions.

Episode 1: Prime numbers:

Setting: Clinical interview in a research project on learning elementary number theory.

Int: So you started to check whether 437 was prime

Selina: Yes it is, because it's two prime numbers [437 was calculated as 19 x 23], of course it is, because two prime numbers multiplied by each other are prime, (pause).

Int: Is 15 a prime number?

We will discuss the conflict that is invoked with the interviewer's choice of example and Selina's pathway towards conflict resolution. We will show that while 15 is a "pivotal example" instrumental in invoking the conflict, it is insufficient for resolving the conflict.

Episode 2: Comparing fractions

Setting: Methods course for prospective elementary school teachers.

After a thorough classroom discussion on a variety of ways to compare fractions, Tanya approached the instructor and introduced a "different strategy":

Tanya: You simply take away the top from the bottom and see what is larger. Where the number is larger, the fraction is smaller, like 2/7 and 3/7, 5 is greater than 4, so 2/7 is smaller.

In the following conversation with the instructor examples of 1/2 and 2/4, 5/6 and 6/7, 9/10 and 91/100 were presented to the scrutiny of Tanya's method. Having faced a counterexample, Tanya's immediate tendency was to amend her strategy, to reduce its scope of applicability, rather than to abandon it. We will discuss Tanya's struggle with disconfirming evidence, the conflict that she faced and her reluctance towards conceptual change.

Discussion

We note that the notion of bridging/pivotal example is learner-dependent, that is, a strategic example that is helpful for one learner may not be helpful to another. Further, in some cases a 'critical mass' of examples may be necessary to serve as a pivot or a bridge.

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THE CHALLENGES OF INFUSING EQUITY ISSUES IN MATHEMATICS METHODS COURSES

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Improving the preparedness of preservice teachers to teach mathematics and address equity issues begins with teacher educators' ability to struggle with these issues themselves in their mathematics methods courses. "Teaching equity will not only empower beginning teachers, it will also begin to offer more strength to the overall shift in the acceptance and understanding of societal equity issues" (Kelly, 2002, p. 39). This is especially important with the changing demographics of our public schools. Given that public schools are becoming more diverse, preservice teachers need to be better prepared to teach students from a variety of backgrounds.

The NCTM Research Committee (2005) suggests that equity as a legitimate object of study for mathematics educators can potentially move the field into new and significant directions. This poster presentation will add to the emerging literature by examining the beliefs and practices of teacher educators as they infuse equity issues in their mathematics methods courses. The presentation is designed for teacher educators, professional developers, pre-service and in-service teachers, and administrators. It will allow audience members to better understand how equity issues may be addressed in mathematics education courses.

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CONTEXTUALIZING MATHEMATICAL TASKS TO ADDRESS EQUITY IN PRESERVICE MATHEMATICS EDUCATION

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By the end of their teacher education programs, many preservice secondary mathematics teachers (PSMTs) lack a deep conceptual understanding of a variety of mathematics concepts which they will be expected to teach (Even & Tirosh, 1995), despite completing extensive coursework. Furthermore, many preservice teachers, who continue to comprise a fairly homogeneous group of White, female, lower- or middle-class and provincial persons, exhibit beliefs that may counter goals for educational equity, while the student population is becoming more diverse (Sleeter, 2001). Attaining equity in mathematics education requires an examination of expectations and beliefs about students who have traditionally underperformed (Allexsaht-Snider & Hart, 2001). The aim of this study is to examine one PSMT's expectations of students as well as his openness to issues of equity in mathematics education as a result of his engagement in mathematical tasks contextualized by issues of equity.

The research questions are: (1) What is the participant's conception of equitable mathematics education and his expectation for poor students and students of color? (2) What is the PSMT's reaction to mathematical tasks addressing equity-related issues as they are employed in a secondary mathematics methods course? (3) How does contextualizing these tasks within equity issues impact PSMTs' willingness to engage equity issues within a methods course?

Data sources include a survey, three 30-45 minute semi-structured interviews, and student solutions and reflections related to the mathematical tasks. The tasks employed in this study are designed to allow students to investigate mathematics concepts they will be expected to teach, and they are contextualized by equity related issues. The participant is a senior-level undergraduate student enrolled in a secondary methods course at a large university. Data are coded in order to determine emergent themes and undergo constant comparative analysis (Strauss, 1987).

This study begins to add to our understanding of how PSMTs view traditionally disadvantaged students and how employing mathematical activities which address equity might respond to PSMTs' resistance to equity issues. Insights on ways to integrate equity issues into methods courses are provided.

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STUDENT TEACHERS' CONCEPTIONS OF PROOF AND FACILITATION OF ARGUMENTATION IN SECONDARY MATHEMATICS CLASSROOMS

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Prospective secondary mathematics teachers encounter proof in university mathematics courses and are expected to prove, justify, and communicate in their secondary mathematics classrooms. Krummheuer's (1995) description of argumentation captures the essence of a relationship among proving, justifying and communicating. Yackel (2002) suggests several functions a teacher might serve in the development of collective argumentation, such as negotiating classroom norms, providing support for students as they interact to develop arguments, and supplying data, warrants, or backings that are omitted from students' arguments or are implicit in students' statements.

Drawing on the work of Krummheuer, Yackel, and others in argumentation, and research on proof and proving (e.g., Knuth, 2002; Weber, 2001), this study considers the relationship between a teacher's conceptions of proof and his or her facilitation of classroom argumentation. In particular, this study addresses the following questions:

- How do prospective secondary maths teachers support claims, data, warrants, and backings as elements of argumentation in secondary mathematics classrooms?
- What characterizes the relationship between the argumentation observed in a particular classroom and the prospective secondary mathematics teacher's conception of proof and justification?

Major data sources include interviews and observations of three student teachers during their student teaching experience. Student teacher interviews addressed their conceptions of proof and expectations for students' explanation and justification, while observations focused on their facilitation of argumentation in classrooms. Mentor teachers were interviewed to describe the classroom environments into which the student teachers had been placed.

The analysis of the various data sources includes characterizing the participants' conceptions of proof along three continua: ability to prove and analyze proofs, affective perception of proof, and perception of the purpose and need for proof. The analysis of argumentation uses Toulmin's (1964) components of argumentation: claim, data, warrant, and backing. These components are diagramed and attributed to students or student teacher. The student teacher's actions in each episode of argumentation are analyzed to determine how they change or support the argumentation. Analysis of data from this study is ongoing.

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THE MATHEMATICS ENDORSEMENT RESEARCH GROUP (MERG)

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There is an abundance of literature on the need for highly developed, specialized content knowledge for teaching elementary mathematics (i.e., How People Learn and Knowing and Learning Mathematics for Teaching, National Research Council; Adding it Up, Mathematics Learning Study Committee). There is also a large body of research on the important role beliefs play in teaching mathematics (Thompson, 1992; Cooney & Sealey, 1997; Tőerner & Pehkonen, 1999). Study of the relationship of these factors to teacher development is the purpose of the Mathematics Endorsement Research Group (MERG). This poster session shares results from the first year of a four year research project analyzing change in and comparison of the beliefs and mathematics content knowledge of two groups of undergraduate preservice elementary teachers.

MERG is the outcome of a recently mandated four-course mathematics sequence required by the Board of Regents in the State of Georgia for undergraduate elementary teachers. Previously, early childhood students were required to take two mathematics methods courses in early childhood and two mathematics courses in the mathematics department. With the new mandate, requirements changed to four mathematics courses and one mathematics methods course. Members of the mathematics education faculty in early childhood were curious about the impact of these changes on beliefs and content knowledge of preservice teachers.

The project will follow four cohorts of students (n = 139) who matriculate through the old program and four cohorts of students (n = 180) who matriculate through the newly mandated program. Data on beliefs will be collected for each cohort upon entry into the program, at each transition point between semesters, and at the end of the program. Content knowledge data and demographic information for each cohort will be collected at the end of the program.

As of Spring, 2006, the mathematics teaching efficacy survey (Enochs, Smith & Huinker, 2000) and the beliefs about math pedagogy survey (Peterson, Fennema, Carpenter & Loef, 1989) were administered four times to two old program cohorts (n = 65). Four subscales within the surveys were used: self-efficacy (SE), outcome expectancy (OE), children construct their own math knowledge (CONST), and math teaching should facilitate children's construction of math knowledge (FACIL). Mathematics content knowledge for each group was measured after student teaching using the Learning Mathematics for Teaching Instrument (Hill, Schilling & Ball, 2004).

Using demographic data of age, race and high school background we asked: *What are the predictors of initial mathematics teaching beliefs? What are the predictors of change in those beliefs*? In addition, we asked: I*s there is a relationship between teachers' beliefs and the specialized content knowledge necessary for teaching elementary mathematics?* We also analyzed the effect of grade level placement and socio-economic status (SES) of the student-teaching school on change in beliefs about teaching mathematics.

Based on hierarchical linear modeling (HLM), we found that math teaching content knowledge was related to initial SE, CONST, and FACIL beliefs. None of the demographic variables were related to initial beliefs. Once content knowledge was included in the HLM models to explain initial differences in these beliefs, it was not related to change in beliefs. Age was a significant predictor of change in beliefs for all subscales with students older than 23 years showing slightly more growth in beliefs than students between 18 and 23 years. Finally, SES of the student-teaching school predicted change in SE and FACIL. Student-teachers in low, medium and high SES schools differed in the change in their beliefs with those in high SES schools showing more significant change towards standard's based beliefs.

PUBLIC MATH METHODS LESSONS PRESENTED AT CONFERENCES: JAPANESE LESSON STUDY IN OUR UNIVERSITY CLASSROOMS

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Seven professors from five universities formed a research team to implement lesson study. The literature suggests that lesson study can facilitate greater reflection and more focused conversations about teaching and learning than is often realized with other types of professional development (Lewis, 2002). Specific and authentic conversations about management, student learning, the impact of significant and subtle changes in lesson design are often the result of lesson study. The authors found similar benefits in their own lesson study and presented it as a public research lesson at a national math and science education conference. This allowed conference attendees to join the lesson study process and challenged the culture of academic conferences. While many sessions at conferences have interactive elements, it is difficult to engineer a salient shared experience as a focus of the interaction. The result was a successful interactive session with authentic participant ownership. The attendees became a part of the lesson study team and contributed to the process.

To begin the process, lesson study team members set goals to help preservice teachers (PST's) better understand the value of allowing children to invent their own strategies, realize the deep understanding of mathematics that they need to appropriately analyze and facilitate discussion about children's strategies, and ultimately to become more skilled at implementing problem-based mathematics instruction. Prior to the conference session, we completed four cycles of the lesson study at three campuses. At the conference, a local class of PST's was brought in and the lesson was taught a fifth time as a public lesson. Session participants observed while collecting additional data about the students' responses to the lesson. A discussion followed the lesson and included issues about the lesson study process, problem-based mathematics teaching, children's two digit multiplication strategies, and preparing teachers to facilitate children's discussion about their invented strategies.

Participants at the session were introduced to the lesson study process (Stigler & Hiebert, 1999). Participants *collaborated* with the lesson study teachers for the afternoon, providing valuable feedback and reflection on the lesson. They observed the lesson in *real time*, rather than listening to lesson presenters retell their experiences. Participants focused their observations on *students' thinking and actions*, rather than a focus on how well the teacher was teaching. The structure of the session allowed participants to contribute to the *knowledge base* and deepen their own understanding of teaching and mathematics. For example, one participant said, "I felt like I was finally able to just listen to the students' thinking. In my own classes it's too hard, but here I got to listen to the students as they discussed the problem." Another participant commented that being a part of the lesson team for the afternoon was "thought-provoking and invigorating". The authors recommend including public lessons at teacher education conferences.

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BRIEF RELATIONAL MATHEMATICS COUNSELING APPROACH TO SUPPORT STUDENTS TAKING INTRODUCTORY COLLEGE MATHEMATICS

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At the small 4-year liberal arts colleges in the northeast where I have worked in mathematics support and instruction (and across the U.S.), too many students withdraw from or fail introductory mathematics courses required for their general core quantitative reasoning requirement, or for their major. Not all have issues of poor preparation in mathematics though many do. Affective problems—beliefs, attitudes and emotions, (McLeod, 1992, 1997) have also been found to figure significantly in student achievement. Research in mathematics education has focused on understanding and reforming students' preparation in elementary through high school. Understanding links between students' mathematics cognition and their affect has also been a focus of research (Buxton, 1991; Skemp, 1987). I found, however, little research into combining mathematics counseling and constructivist tutoring, while students were engaged in a college course, to identify and remedy their mathematics blockages.

I developed a brief relational assessment and counseling approach (cf. Mitchell, 1988) that I integrated with cognitive constructivist tutoring to help students identify counterproductive relational patterns with self, teachers, and mathematics, change their minds about their mathematics selves, change identified behaviors, become reattached to mathematics, and succeed in their course. As my doctoral project, I piloted this approach with students in an introductory statistics in psychology class at a small U.S. university in the northeast. I discovered that students' experiences and level of prior mathematics preparation, relative to the current course, interacted with their sense of mathematics self (expressed in their state of mathematics self-esteem) to determine their membership in one of three broad categories of student: Category I: well-prepared students with sound mathematics self-esteem; Category II: adequately prepared students with undermined mathematics self-esteem, and; Category III: underprepared students with low mathematics self-esteem.

Effective assessment and tutoring required a change from the traditional focus on the student, to a focus on the student-tutor dyad and the interrelationship. As the tutor-counselor I needed to attend to and make explicit to the student the transference and countertransference dynamic between us. My approach helped students in all three categories. In particular, it helped Category II students repair their mathematics selves, restore a damaged attachment to mathematics, and succeed. It also helped Category III students who were willing to commit to the process, to repair their mathematics selves and develop their attachment to mathematics.

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ANALYSIS OF EFFECTS OF TABLET PC TECHNOLOGY IN MATH EDUCATION OF FUTURE TEACHERS

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The concept of a “digital divide” separating those with access to computers and communications technology from those without is simplistic. Research (Peslak, 2005) shows that computers per students and total number of computers in a school significantly effects student learning, but surprisingly there is a negative impact of this metric on standardized reading and math scores. Another study (Warschauer, 2005) shows that students from a higher socioeconomic status are more likely to use computers for experimentation and critical inquiry, while students from a lower socioeconomic status usually engage in less challenging drills. To benefit from computers teachers should be familiar with the available software and should be able to create math activities that guide students to higher order thinking.

The main focus of our research is the study of the impact of Tablet PC technology on mathematical content pre-service teachers. We also consider other dimensions involved, i.e., "instrumental" dimension (“taking into account that a student using a tool to do mathematics develops knowledge on the tool together with mathematical knowledge” (Lagrange, 2005)).

Future teachers enrolled in math, math methods courses, and internships at local elementary schools were participants of this study: treatment group (15 students that regularly met in a professional development school and used Tablet PCs) and control group (23 students who were enrolled in the same courses with the same instructors, but met at different times and location and did not use Tablet PCs). We statistically compared the effectiveness of our technology-enhanced method for teaching mathematics. This comparison was based on the results of two distinct items: Final Exam given at the end of a four month learning period and students’ Final Grade (cumulative grade based on all the investigations throughout the semester). Our analysis shows that the treatment group achieved significantly higher mean scores than the control group. These higher mean scores imply that the treatment group acquired greater understanding of math content when compared to the control group. This result can be directly contributed to the effective implementation of the Tablet PC technology in the math and math methods courses.

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CULTURALLY DIVERSE MATHEMATICS TEACHER CANDIDATES AS LEARNERS AND AS TEACHERS

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Culturally diverse pre-service teachers of mathematics are more vulnerable than other teacher candidates. They often have strong accent or can be described as visible minorities (in the Canadian school system). While Andrew, Cobb and Giampietro (2005) show that for "acceptable, good, very good, and outstanding teachers, there is no significant correlation between verbal scores and expert assessment of teacher effectiveness" (p. 343), such teacher candidates are often labelled by their associate teachers as low in verbal abilities and therefore less competent as teachers because "it is difficult for students to understand them."

Coming from other cultural and educational backgrounds, such teacher candidates may experience difficulty in following curriculum instructions about emphasising communication in mathematics classroom. Furthermore, similar to Costa et al. findings (2005), mathematics textbooks or curriculum materials may be written in language not sensitive for users (students and teachers) whose first language is not English.

However, attracting teacher candidates from diverse backgrounds (in terms of social class, ethnicity and primary language as defined by Au, 1993) is recognized as important, since there are many students in Canadian schools with such backgrounds.

Participants in this study were three pre-service mathematics high school teachers enrolled in the program at the Faculty of Education of the medium size Canadian university. Data were partially collected through onsite observations during their practice teaching in schools and individual interviews. The researcher observed the participants' body language, gestures, use of "teacher voice," selection of mathematics exercises, artefacts used during teaching mathematics and communication with the students. Data sources pertaining to the study were analyzed using methods of discourse and content analysis in order to find common themes and trends pertaining to the research questions.

Preliminary data analysis showed that participants mostly found difficult to deal with issues of power and respect, which were different from what they experienced as teachers in their countries of origin. Also, they did not see much value in asking questions in class or organizing group work compared to drill and practice and assigning individual homework. This research will help shaping methods for working with mathematics teacher candidates from diverse cultural backgrounds.

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TURNING TO MATHEMATICS: UNDERSTANDING THE VARIATION IN PRESERVICE TEACHERS' PERCEPTIONS OF THEIR MATHEMATICS TRAINING

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The purpose of this study was to better understand the variation in preservice secondary mathematics teachers' (PSMTs') perceptions of their content preparation and how this mathematics instruction served as a resource in their process of learning to teach. PSMTs undergo rigorous education in mathematics, often completing coursework nearly equivalent to a mathematics major. Studies have demonstrated that the value teachers place on this coursework is variable (Goulding, Hatch. & Rodd, 2003), with some espousing great worth and others near irrelevance. Having documented a range of perceptions among a cohort of PSMTs (Hodge & Staples, 2005; Staples & Hodge, 2006), we sought to understand the variation among the PSMTs' perceptions and how they were making sense of their experiences in their content courses in relation to their future work as teachers.

Eight focal students were selected from a cohort of 16 PSMTs at a large, public university. All PSMTs had completed 37-45 units of mathematics and standard education courses. We used mathematics grade point averages to select eight focal PSMTs to represent the cohort range (2.4–4.0). Data collected for the case PSMTs included two semi-structured interviews prior to and after student teaching (~1 hour each); two surveys on PSMTs' perceptions of their mathematics coursework and feelings of preparedness; a set of mathematics problems; two concept maps; and academic transcripts. For half of the cases, a third interview was conducted during student teaching that was coupled with a classroom observation by the researchers. Data analysis consisted of multiple passes through the data corpus and followed standard qualitative and quantitative techniques.

Analysis revealed the variation across the PSMTs' perceptions of their experiences in mathematics classes was closely linked to their visions of the type of learning environment they were working to instantiate in their classrooms. The PSMTs' "visions" shaped what they attended to and drew from their content courses. Consequently, their assessments of the value of a particular course, as well as the role it played in supporting their teaching work, were influenced by their visions. These cases were examined in-depth, juxtaposing the PSMTs' experiences, and exploring implications for PSMT learning and teacher education program design.

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PRESERVICE MATHEMATICS TEACHERS' PERCEPTIONS OF MATHEMATICAL DISCOURSE

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Mathematics reform efforts emphasize the need for teachers to engage students in mathematical discussions. Discussions can support conceptual understanding and promote students' participation in valued mathematical practices such as communicating and reasoning (NCTM, 2000). However, not all dialogue is created equal. Some classrooms evince high levels of student participation but demand little of students in terms of cognitive press (Nathan & Knuth, 2003). Others have a more rigorous focus and allow for exploration and extension of ideas (Kazemi & Stipek, 2001). One challenge for the educators is to support preservice teachers in recognizing and valuing "high quality" dialogue as they work to make sense of their new roles as reform teachers. Given the continued prevalence of traditional modes of instruction (Stigler & Heibert, 1999), teacher education may be a critical intervention point to help shape future teachers' understanding of productive dialogue.

This poster presents results from an investigation of preservice teachers' perceptions of mathematical discourse and its relationship to student learning. Subjects included elementary and secondary preservice teachers enrolled in courses taught by the authors. The advanced methods' group (master's level) had completed student teaching. The second group was enrolled in their first mathematics methods course, allowing for comparison over time. Subjects read and responded to two excerpts of classroom discourse on fractions. One excerpt (Ms. C's class), from Kazemi & Stipek (2001), provided an example of "high press" discourse. The second excerpt (Ms. R's class), from Truxaw (2004), provided an example of "low press" or univocal discourse. We were interested in what the teachers noticed and how they derived their evaluative judgments of the dialogues in relation to student learning.

The advanced methods teachers gave overwhelmingly positive evaluations of the Ms. C excerpt, but were split in their evaluations of the Ms. R excerpt. Thus while the teachers articulated the value of reform-oriented discourse, nearly half still valued many aspects of the low- press discourse and found the exchanges productive for student learning. Evaluative comments clustered around themes of the mathematical focus of the discussion, levels of participation, quality of student thinking, and perceived affective support provided to the students. Comparisons between the elementary and secondary teachers, as well as between groups entering and finishing the program, will also be presented. Implications include the need for teacher educators to focus not only on the ways in which reform-oriented classrooms support student learning, but also the limitations of more univocal discourse, as preservice teachers seem to be able to hold simultaneously both forms of discourse as productive.

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TEACHER KNOWLEDGE

STRIVING FOR EQUITY IN MATHEMATICS EDUCATION: LEARNING TO TEACH MATHEMATICS FOR SOCIAL JUSTICE

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This study followed 8 secondary mathematics teachers as they collaboratively designed, implemented, observed, revised and re-taught mathematics lessons for social justice, tracing the conversations teachers had around what it means to teach mathematics for social justice and the challenges that they recognized that they faced in implementing these ideas into practice. Analyses revealed that although teachers' conversations of teaching mathematics for social justice were well-articulated and consistent with the literature on teaching for social justice, instantiating these ideas into practice proved difficult. Particularly, how to balance and integrate the social justice and mathematical pedagogical goals was challenging. Results suggest implications for professional development in this area.

Education is intricately linked to differential patterns of economic, political, and social power structures in society that serve to perpetuate inequity and injustice in both schools and society (Kozol, 2005). The goal of increasing equity in mathematics education entails connecting schooling to these larger sociopolitical contexts of society (Gutstein, 2006) and shifting from preparing students to live within the world, as it currently exists, to preparing students to restructure those systems for the purpose of removing obstacles that women, minorities, the poor, and others experience (Secada, 1989). Mathematics education can play a role in this endeavor, serving as "a vehicle through which to accomplish this change" (Gutstein, 2006, p.13, emphasis in original), specifically in the form of mathematics teaching for social justice. Like all mathematics teachers, those employing social justice pedagogies recognize the necessity of mathematical knowledge and include mathematics-specific goals for their students (Frankenstein, 1995; Gutstein, 2006). Concurrently, they engage students in using mathematics to critically analyze their world, empowering students to take action in an effort to promote a socially just society (Frankenstein, 1995; Skovsmose, 1994). Little research exists, however, that examines mathematics teachers learning to teach for social justice, a necessary step in understanding the entailments of teaching mathematics for social justice. This research investigated secondary mathematics teachers' conversations around learning to teach mathematics for social justice as they developed, implemented, observed, revised and re-taught mathematics lessons for social justice. In particular, this study examined the following research questions: "How do teachers' conversations around teaching mathematics for social justice evolve through participation in the graduate course?" and "What challenges do teachers recognize that they face in teaching mathematics for social justice?" This report will present selected results from this investigation.

Theoretical Perspectives

This research draws upon situated, socio-cultural perspectives of teacher learning (Lave & Wenger, 1991). Socio-cultural theories of teacher learning center on the concept of learning as situated social practice, which includes discourse, social interaction and participation structures. This shifts the focus to people jointly engaged in mutual enterprise, with a shared repertoire of

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actions, discourses and tools (Wenger, 1999). Teacher learning, then, is influenced not only by personal orientations, but also by teachers' interactions within various social communities.

Methods

The context. To address the research questions, a graduate course was designed to provide secondary mathematics teachers an opportunity to explore their conceptions of teaching mathematics for social justice through assigned readings and written reflections, and through the collective development of a mathematics lesson incorporating social justice goals. Eight teachers participated in the graduate course, which met 15 times for 2 ½ hours per session. Of these eight teachers, seven were employed as mathematics teachers in one of four comprehensive high schools and the eighth was a licensed science teacher employed as a full-time substitute teacher, regularly substituting in mathematics classrooms. For scheduling purposes, these teachers were split into two groups of four. In Group 1 (Pat, Gerry, Chris, and Jamie), all four teachers were White, two were female and two were male, and their teaching experience ranged from 4 to 17 years. In Group 2 (Ann, Roxy, Dana, and Holly), all of the teachers were White and female, and their teaching experience ranged from 6 to 16 years. The central activity for the first part of the course was discussion and analysis of readings focused on teaching for social justice in general and teaching mathematics for social justice specifically. To situate their study of these readings in the activities of mathematics teaching, the group also examined sample mathematics lessons and mathematics teaching cases. Verbal and written reflection prompts were provided in each session to focus the discussions and to support teachers in clarifying their conceptions of teaching for social justice. The central activity of each seminar in the second part of the course was the design, implementation, observation, revision, and re-teaching of a mathematics lesson for social justice.

Data analysis. The primary source of data for this study was teachers' discussions during the graduate course. All discussions were audio-taped and transcribed. Additionally, pre- and post-seminar interviews were conducted with all participants to help understand how teachers' conversations around teaching mathematics for social justice evolved. Teachers' lesson plans and written reflections were also collected. A grounded theory methodology was employed (Strauss & Corbin, 1990) to identify recurring themes in the data. I began first by compiling a list of general framing codes drawn from my research questions, including teachers' conceptions of teaching mathematics for social justice and challenges in teaching mathematics for social justice. Next, I coded all data and the emergence of additional codes occurred through multiple passes of the entire data set; four passes were required before categories began to stabilize. The coding scheme aimed to characterize the nature and content of teachers' comments.

Results

Analysis of teachers' conversations around teaching mathematics for social justice revealed that in Group 1, three of the four teachers had narrower conceptions of teaching mathematics for social justice that became more elaborated over the duration of the course. In the beginning of the course, Pat, Gerry, and Jamie's conversations suggested that teaching mathematics for social justice was about relating mathematics to all cultures or relating math to society, with no explicit mention of students looking critically at how societal issues connect to their experiences or of students acting upon their world in order to transform it. For example, Pat suggested that teaching mathematics for social justice was "for the purpose of opening their eyes to different aspects of our society and what's going on in the world and relating that to maybe mathematics."

Pat's notion of "opening [students'] eyes" is similar to Jamie's notion that teaching mathematics for social justice would "enlighten [students] in a way that's maybe different from just kind of like teaching the topic of mathematics." Chris, on the other hand, suggested that it means students "use math as a tool for dismantling systems of oppression. It entails teaching ways that math can analyze and address societally-constructed inequalities."

Almost immediately after the course began, Jamie, Pat, and Gerry's conversations around teaching mathematics for social justice broadened to include notions of students taking action and confronting inequities in society. These conceptions remained consistent throughout the course. Jamie, for example, maintained that teaching mathematics for social justice included students' recognizing "that social injustices do exist" where, "once there's awareness then there can be analysis and actions that follow those things." Similarly, Chris remarked that teaching math for social justice "address[es] the social justice issues that [students] might run into in their lives. They'll use [math] to examine racism, classism, sexism, as it pops up in their lives and in the larger society," where students consider "what could develop with the knowledge [they] gained."

In Group 2, teachers seemed to come in with fairly well-articulated and previously developed conceptions of teaching mathematics for social justice that remained consistent throughout the course. Four themes emerged: teaching mathematics for social justice meant (a) confronting the gate-keeping role that mathematics education traditionally holds, (b) taking action, (c) raising students' awareness, and (d) using mathematics as a tool to analyze and understand issues in society. These four themes reappeared at each stage of the data collection process, expressed by at least three of the four teachers at one time or another. In reference to confronting the gate-keeping role of math education, teachers suggested that they "want to help students make themselves ready so that they can pass through those gates" (Roxy), arguing that "since math functions as 'gatekeeper' to many other opportunities, to teach for social justice must include students' developing mathematical power…" (Ann). At the same time, teaching mathematics for social justice also means "students start seeing that they could use math to make an argument to change something about society" (Dana) and with such teaching "…we're trying to make them go through that painful journey to become aware" (Holly) and "start to build consciousness of what the inequities are" (Roxy). Finally, teachers in group 2 saw teaching mathematics for social justice in terms of the utility of mathematics, stating that students "analyze [issues] mathematically" (Dana), "[seeing] like this is how you would actually use it in a real world setting" (Dana) and "students [would] step away from the lesson with a new outlook on how math can be an effective tool in their lives – it empowers them to solve critical, close-to-home problems" (Holly).

Across both groups, teachers' conversations that emerged were similar to the central tenets of teaching mathematics for social justice addressed in the literature: teachers' saw teaching mathematics for social justice as students gaining awareness of social issues through critically examining their world, challenging students to the point that they would feel empowered to take action and transform their world.

Analyses around the second research question revealed that instantiating ideas of teaching mathematics for social justice into practice proved difficult, particularly in integrating the social justice and mathematical goals of the lesson. Each group tended to focus more on one or the other, rather than both simultaneously. The results for this section also include descriptions of the lesson design, implementation and revision process for each group to set the stage for understanding teachers' conversations and for interpreting the results.

In Group 1, teachers chose to focus more on the social justice goals of their lesson, agreeing that mathematics did not have to be tied in all the time. The teachers seemed content with this decision because this was their first time teaching mathematics for social justice. For instance, just after the first implementation of the lesson, Jamie said, "…we're trying to get used to including these social justice goals...And if it takes a small step mathematically that's fine for me but I think we should try to incorporate more ideas of the social justice." Specifically, Group 1 designed a lesson around the topic of prison populations and school achievement. The initial lesson design focused on students extracting necessary information from data presented to them in order to use mathematics to gain awareness of the costs associated with schools and with prisons. Using the data, students calculated the cost of a student for one day and of a prisoner for one day and discussed possibilities for why these costs might be similar or different. Upon lesson implementation, students did not get as far as the teachers expected, and no whole class discussion took place. In the debriefing session, teachers' comments focused on the fact that it was a good lesson; they just didn't have time to get through it all. I concurred that they had not met any social justice goals that day and that this needed to be a focus of their revisions.

The group decided to add a component to the lesson that emphasized the relationship between prisons and schools, not just the costs, feeling that such a lesson would address a social justice goal. The revised lesson plan began as before with the calculation of the cost of a prisoner and of a student for one day, but reducing the number of questions asked. Next, the teachers decided to present students with a graph of local GPA data by race (White and Asian students had higher GPAs than African American, Hispanic, and SE Asian students) and the following quote: "More than 6 in every 10 persons held in correctional facilities were Black or Hispanic. Of all inmates: 48% of inmates were Black, non-Hispanic; 36% White, non-Hispanic; 14% Hispanic; 1% Native American; and 1 % Asian/Pacific Islander." Once these data were presented, teachers planned to ask students whether there was a connection between the two, trying to bring out the teachers' perceived relationship between prisons and schools.

Teachers implemented this lesson the following week, and since I had been out of town, this was the first time I saw their lesson revisions. After sharing their reflections on the lesson implementation, which were again mostly positive comments about the lesson implementation, I asked the group to clarify what the goals were for the new part of the lesson. Their goal was that they wanted students to see the connection between academic achievement and the population in prisons. Jamie said that one student's response that "maybe we should be spending more money in schools so that these people are better educated so that they don't go to prison" reflected what they hoped students would get out of the lesson. I then asked the teachers what some of the student comments were at this point in the lesson. Gerry reflected that one student said, "A lot of people are doing bad in school and they're going to jail." Based on my field notes, what the student actually said, which I shared with the group, was that "Black people are doing bad in school and Black people are in prison." As teachers started reading through student responses to the lesson's question, "Is there a connection?" Gerry noted that students were not reacting the way they had expected. Jamie was particularly surprised, as evidenced by the following responses to students' written comments: Student 1 Response: Yes, because since some African American students are not as wealthy as others, they don't think that school is important so they will skip and maybe commit a crime. Jamie: "…pretty negative…it is more directed toward the idea of the individual rather than the schools." Student 2 Response: Yes, Black people do the worst in school and more black people are in jail. Jamie: "…so yeah, these are responses I wouldn't expect." Taking into consideration these unanticipated student responses, the teachers revised the lesson's goals and the lesson plan itself. Instead of asking students, "Is there a connection?" the teachers created two multiple-choice questions, one for each piece of information presented, asking students what might explain the disparities. The goal of this section was to have students recognize that the racial disparities were not about individual student choice; rather, they are connected to issues of institutional racism and White privilege.

In contrast, Group 2 teachers initially weaved both mathematics and social justice into their lesson plan design, having students use proportional reasoning to develop an understanding of the discrepancy between minimum wage and living wage. The lesson began with a pre-homework assignment that had students calculate basic measures of central tendency for data and reason about which measure was most appropriate. The next day, the lesson design began by asking students to use local data to determine the average cost of housing for one person in Lakeview. Next, students calculated how much someone making minimum wage (working 40 hours per week) could afford to pay in monthly rent. Students then calculated, using the average cost of housing for one person previously found, what hourly wage this person would need to make to afford that housing. In other words, teachers asked students to calculate the living wage. The subsequent class discussion centered on the discrepancy between minimum wage and living wage and students' brainstormed possible solutions that might alleviate it. The goal in doing this was two-fold. First, teachers hoped students would get beyond thinking of this issue as a personal, cautionary tale about the need to do well in mathematics to make more than minimum wage. Rather, teachers wanted students to see beyond themselves, recognizing that no one should be working full time and not making a living wage. Second, teachers wanted students to see themselves as agents of positive social change, motivated to take action to address the economic injustices that result from disparities between minimum wage and living wage.

In the first lesson implementation, students did not have much time for the discussion about the discrepancy between living and minimum wage, in part because the pre-homework had to be completed in class. The teacher who taught the lesson noted, "what didn't happen, and it's because I didn't get to have that discussion, was we didn't have the impact of, okay, what are we going to do? This isn't right." Upon reflection on the lesson implementation, the teachers made only minor changes in wording to clarify what they were asking students to do. During the next three weeks, the other three teachers implemented the lesson in varying classroom contexts (e.g. 90-minute class periods and regular and Honors Geometry classes in contrast to the first lesson, which was an Algebra class), and the pre-homework was completed by students as planned. Again, none of the teachers felt like they got to enough, or any, of the social justice discussion anticipated for the end of the lesson. Ann reflected on the first two lesson implementations that "I guess one thought that I had in watching both these classes and then in thinking about me is that that's where our comfort zone is. Is in going over this kind of stuff and talking about the mathematics…" The teacher who taught the lesson the fourth time similarly reflected that as a group, "none of us was able to escape the lure of multiple solution strategies to the same problem, unpacking the mathematical proof in student work, and displaying more than one student solution to a problem," so "we never finished the piece of a living wage is important, not everyone has a living wage, how could we solve that problem mathematically?"

To reiterate, these results were presented in the context of the lesson design, implementation, and revision process to illuminate what issues arose for teachers as they tried to both instantiate their ideas of teaching mathematics for social justice into practice and as they worked to balance and integrate the mathematical and social justice goals of the lesson.

Discussion

The goal of this study was to investigate teachers' conversations around teaching mathematics for social justice to understand better the processes involved for teachers learning to teach mathematics for social justice. This section presents a synthesis of my conclusions about the results for each of the research questions of this study.

Teachers' evolving conversations. Teachers' early conversations in Group 1 indicated that three of the teachers had narrower conceptions that became more elaborated and consistent over the duration of the course. The fourth teacher, and the teachers in Group 2, seemed to come in with fairly substantive understandings of teaching mathematics for social justice that remained consistent. This may have been because the teacher in Group 1 participated in a previous professional development course focused on equity, which included discussions of teaching mathematics for social justice, and because three of the Group 2 teachers were participating in concurrent graduate work. Across both groups, the conversations that emerged around math teaching for social justice were similar to the central tenets addressed in the literature. An important component in the literature that was not mentioned by teachers, however, is that teaching mathematics for social justice also includes the goal of students learning mathematics. Rather, teachers' conversations suggested that teaching mathematics for social justice was easier the first time if they accessed students' existing mathematics knowledge. It is possible for students to use mathematics to examine complex social issues with or without developing new mathematical knowledge (Gutstein, 2006), but the omission of building awareness and building mathematical knowledge is worth noting. Additionally, teachers' conversations around teaching mathematics for social justice seem to suggest that students first learn or know the mathematics and then use that mathematics to learn about and analyze a social issue. This is reflected in the fact that the lessons they designed did just that. Moreover, the idea that students should learn math first, then apply it to understand social issues, implies that teachers' conceived of teaching mathematics for social justice as something "added on" to a curriculum, rather than something that might be an integral part of the curriculum. The focus in this graduate course on designing only one lesson may have facilitated this notion of teaching mathematics for social justice as an "add-on" curricular piece.

Balancing mathematics and social justice. Each of the groups dealt with the issue of balancing the math and social justice goals differently. Group 1 teachers focused primarily on the social justice component of the lesson, and Group 2 teachers, in the lesson implementation stage, focused on the mathematics and did not reach the intended social goals. Group 1's focus on the social justice goal of the lesson, however, contributed to some significant issues related to mathematics teaching and learning. First, this focus may have contributed to teachers' over-simplification of the data. With their data, for example, one cannot simply say that the explanation for disparities in GPA values according to race is institutional racism in the schools, as a number of other factors affect students' GPA levels. Additionally, it doesn't seem that the teachers ever thought their data was problematic. Even when students responded in many unanticipated ways, rather than finding data that might support the goals of the lesson, teachers created multiple-choice questions to lead the class discussion in a particular way, almost attempting to force students to the "right" conclusion. Not only is this problematic from a social justice standpoint, in that accepting only one view does not constitute an examination of a complex social issue, but structuring the lesson in this way could facilitate students' development of misconceptions about data analysis.

Group 2's focus on the mathematics during the lesson implementation stage suggests that the teachers may have been more uncomfortable addressing social justice issues with their students than they realized. Additionally, perhaps teachers felt that since they had planned the lesson initially for one class day that this was the timeframe within which they had to work. Thus, adjusting the lesson to facilitate getting to the social justice component would mean sacrificing some of the mathematical discussion. These teachers were there to teach students mathematics, so in the end, the mathematics "trumped" the social justice (Roxy).

My analysis of these data suggest that one important component of teachers learning to teach mathematics for social justice is an understanding of how to select and use appropriate data to examine complex social issues. To support teachers in the selection of appropriate data, teachers could first examine social justice issues, and then engage in the identification of appropriate data, asking questions such as, "What mathematics would I need to understand and examine this issue?" "What data do I need to support this conclusion, and why?" In learning to teach mathematics for social justice, teachers also struggled with integrating the mathematics with the social justice. The fact that the teachers in both groups each taught different mathematics subjects may have hindered them in the process of designing a lesson to match existing curricular goals, and hence integrate the social justice with the mathematics. Thus, to better support teachers in this integration, teacher groups could be content-specific. Additionally, the focus on one lesson may have prompted teachers to think of teaching mathematics for social justice as lessons to be inserted into a curriculum. As such, changing the grain size from one lesson to a focus on continued integration throughout the school year, in various ways, may be more appropriate.

Teaching mathematics for social justice is not a matter of designing and implementing individual lessons, but is a process that pervades every aspect of the classroom. The challenges that arose for these teachers in balancing mathematical and social justice goals in lessons should not be viewed as a reason not to engage in this pedagogy, but instead as a natural part of the process of learning to teach mathematics for social justice. This study begins to speak to how teachers might begin to engage in the practice of critical pedagogy and how teacher educators might help teachers do so. Continued research in this area will help us understand the significance of supporting teachers in integrating mathematical and social justice goals, in terms of their teaching, student learning, and moving toward social justice.

Acknowledgements

The material in this paper is based in part on work supported by the National Science Foundation under Grant No. ESI-0119732 to the Diversity in Mathematics Education Center for Learning and Teaching.

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THE ROLE OF CHALLENGING MATHEMATICS CONTENT

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The study examined the role of challenging content in the NSF Math and Science Partnership program. Within the grant program, individual grant programs have taken on some topic areas and not others. The program evaluation, of which this study is a part, investigated the types of content included in grants and how grants have interpreted challenging content.

Purposes and Objectives

What is challenging mathematics content? Who decides? Recent reports such as the TIMSS studies and the NAEP results reveal that the United States continues to struggle with mathematics achievement (Ferrini-Mundy & Schmidt, 2005; Hiebert et al., 2003; Lindquist, 2001; Stein, Smith, Henningsen, & Silver, 2000). One reason cited is the lack of challenging content and the differences in curriculum across states and school districts. According to a study produced by the National Council of Teachers of Mathematics and the Association of State Supervisors of Mathematics, content standards vary significantly between states by grade level (Lott & Nishimura, 2005). Many interested constituencies (educators, mathematicians, parents, educational organizations, and school boards) have their own interpretations and standards for what mathematics students should learn. Finding and implementing a single definition of challenging content remains an elusive challenge. What is challenging for students varies by setting and interpretation. The goal of this study is first to analyze what content is being addressed by different grants in an NSF program (in particular content courses for teachers) through teacher development programs and second to consider how that content represents different interpretations of challenging in the K-12 setting.

Background and Context

This study is situated within the program evaluation of the NSF Math and Science Partnership (MSP) Program[1]. The MSP program is a grant program to fund large-scale grants for 3 to 5 years. The program includes over $500 million in total funding. The MSP grants address five key features identified by NSF in the initial requests for proposals: challenging curriculum, teacher quality and diversity, partnerships, evidence-based design, and student achievement. The grants include math or science content, collaboration with K-12 schools and districts, collaboration with disciplinary faculty (e.g., mathematics, science) and education faculty, and span a wide range of grade levels and content areas. Within the MSP Program, there are curriculum initiatives that are designed to encourage the implementation of challenging content so all students have access to science, technology, engineering and mathematics (STEM).The focus here is on challenging content and curriculum in MSP and specifically mathematics content. Presented here are a few interpretations of challenging content and content within teacher professional development that is receiving attention in MSP projects.

Addressing teachers' knowledge of content and curriculum has been a focus of teacher development since Shulman's paper on the different types of knowledge for teaching (1986). Related to the research emphasis on teachers' knowledge is the theory that impacting teachers' content knowledge will impact students' learning of mathematics. While it may be obvious that

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teachers' should have deep knowledge of mathematics, it is not clear what mathematics they need to learn or how they should learn mathematics for teaching. It is also not clear how learning mathematics then impacts teachers' practice. In MSP grant initiatives, there may be more focus on teacher development rather than direct work with students. Efforts to effect teachers' knowledge of mathematics are intended to have a trickle-down impact on students' learning.

Within studies of curriculum, teachers' role in implementing curriculum and the teacher's mathematical knowledge have an effect on the nature of the classroom use. (e.g., Mokros, 2003; e.g., Remillard, 2000; Sarama, Clements, & Henry, 1998) The standards-based NSF-funded curricula such as *Connected Mathematics* have been incorporated into work by grants in the MSP program, and there are implications from the studies of curricula for teacher development. Namely, that teacher development plays a critical role in students' learning with the materials, and teacher development is a critical component to the successful implementation of a reform-based curriculum.

Theoretical Perspective

Capturing the role of challenging content appears elusive within the MSP program given the large scale, the diversity, and the complexity of each grant. A common measure of whether students have learned the mathematics is standardized achievement scores on state tests. However, challenging content can go beyond what is contained on statewide-standardized tests. Advanced Placement courses, cutting edge content (e.g., nanotechnology) and other content at the outer limits of what students can learn across K-12 can also be considered challenging. At other level, challenging content can also be challenging in terms of content an individual student should learn. This is the difference between external measures of challenge such as standards and tests developed by states or national organizations and individual measures of challenge related to an individual student's abilities and knowledge.

The perspective in this study is drawn from a modeling perspective (Lesh, Doerr, Carmona, & Hjalmarson, 2003) that has been applied to this study of content and curriculum. The language of modeling and the model as structure have been used to organize the analysis of the diverse set of MSP grant initiatives. When models are used to describe students' mathematical thinking, they allow for multiple ways of thinking, representations, and conceptual systems generalizable beyond the local setting in order to categorize and classify student thinking (e.g., Carmona-Dominguez, 2004). The model for curriculum includes three parts: a representation developed for a purpose with underlying conceptual systems. Examples of models for challenging content are courses developed for teachers. Such courses seek to address particular content (e.g., algebra) with a purpose (e.g., develop middle school teachers' understanding of algebraic structures) and an underlying conceptual system or theory (e.g., development of pedagogical content knowledge). Though courses are developed for local conditions, they have common characteristics to analyze across grants. A model for content then includes why the content has been included, how the content was presented, and what theory underlies the development of the content within the course.

Due to the diversity of perspectives and purposes within the MSP grants related to challenging curriculum, the modeling perspective helps to organize a diverse set of initiatives and characterize grant activity related to challenging content. This paper focuses on descriptions of content courses for teachers as one representation of the content viewed as important by the grants. The underlying purpose in most of the courses is to develop teachers' content and pedagogical knowledge. The representations used for this study are the course descriptions available in project reports and websites. The conceptual systems include theories about teaching and learning mathematics as well as methods for helping teachers develop their teaching practice.

Methods and Evidence

Data analysis at this stage has been primarily focused on annual reports and evaluation reports submitted to NSF with some supplementary data being collected from project websites. Document analysis (Miles & Huberman, 1994; Patton, 1990) is being used with such sources in order to gather a comprehensive picture of the scope and scale of the projects' mathematics curriculum efforts. The evaluation has been primarily qualitative (Greene, 1998) due to the types of data available; however quantitative data analysis may be possible at later stages. The focus of the analysis is on the development of categories and descriptors that can be used to analyze mathematics content efforts in the program as a whole. In addition, such descriptors should be used to analyze the progress of the grant projects over time. "Grants" is used throughout this paper to refer to a grant as a whole. "Project" is used to refer to project within a grant. Due to the size and complexity of the grants, there are often multiple, simultaneous, parallel projects occurring within one grant. To avoid mischaracterizing the nature of a grant and ignoring equally important, distinct activities, I make the distinction between "projects" and "grants". For example, a grant may have one group developing content courses for teachers and another group developing learning units for students. Some of the content of the modules and courses may be new to K-12 (e.g., engineering) and some may rely on content historically within K-12 (e.g., algebra, chemistry).

The data analysis is proceeding in two stages using two major sources of data. The two data sources are the project proposals and the annual reports (including evaluation reports as a subset of this data). In some cases, projects have designed websites with additional materials related to mathematics curriculum efforts or placed additional materials on MSP-Net (the online collaboration environment for MSP and supported by an MSP project). However, external dissemination of materials used in courses or developed for students is limited as are the publications (e.g., journal articles) at this early stage in the MSP projects. We are utilizing document analysis forms (Miles & Huberman, 1994; Patton, 1990) to code and summarize information from the data sources.

Specifically for the teacher content course data, course descriptions were downloaded from publicly available websites developed by individual MSP grants. Based on information on websites gathered between August 2005 and February 2006, 30 grants listed at least one content course for teachers and 140 course descriptions were available (note that some descriptions described multiple courses). The data set represents a snapshot of activity during a period of time in the duration of the grants. An average of 4 courses were listed per grant with a range of one course to 26 courses. Projects update their websites periodically with some updates occurring more frequently and depending on course offering time and duration (e.g., summer or academic year, one session or multi-session courses). Additionally, not all grants post course descriptions on websites or include them in project reports.

Mathematical Content

For the purposes of MSP, each grant is determining for itself what mathematics is important and relevant for their project. In some cases, they are drawing on state standards as the definition of challenging content. In other cases, they are carrying out professional development focused on

NSF-funded curriculum materials (e.g., Connected Mathematics) in order to encourage and facilitate their use in K-12 classrooms. Regarding both groups of projects, another question we can use to classify the projects is whether they are introducing new content to the curriculum (e.g., introduction of new content such as nanotechnology in learning units), increasing the challenge of the curriculum (e.g., revision of the state standards for mathematics and MSP activities related to the revision) or working within existing standards and expectations while developing teachers' content knowledge. Finally, the projects may have a particular content area emphasis (e.g., algebra or geometry). In some cases, this results from an intentional design and proposal decision (e.g., a project targeted on algebra). In other cases, the content emphasis may emerge over the life of the project.

Broadly, the projects tend to fall into two groups. The first are projects working within existing content in K-12 and the second are projects that are developing new content. Working with existing standards and curriculum is a different type of activity than pushing for change in content and curriculum. Both activities represent "challenging courses and curriculum" but the goals, evidence, and objectives related to designing new content are different than the goals, evidence, and objectives for working within existing frameworks and materials. The development of nanotechnology units is new content for K-12 and an emerging field of engineering and research. Materials do not exist for the presentation of such content in K-12 classrooms and teachers will likely have limited knowledge of the field. In contrast, a project working within a traditional topic such as Algebra may have less work to do in terms of finding resources for professional development or for implementation with students.

As large categories, grants often follow the organization of content used in the *Principles and Standards for School Mathematics* (National Council for Teachers of Mathematics, 2000) including algebra, geometry, number and operations, measurement, and data analysis and probability. However, there is also focus on calculus concepts within some grants. There seems to be more emphasis on algebra and geometry than on other content areas at this point. Further data analysis will examine what aspects of these content areas are being emphasized as well as the grade levels. One question (when grants are closer to completion) will be why certain content was excluded. A second question to investigate is why grants placed content in a particular order. Was there a theoretical or research-based reason or was it a question of opportunity (e.g., the availability of instructors with the relevant expertise)?

Content Courses for Teachers

Most grants are offering course work and professional development opportunities for teachers. For content courses, one theory is that deepening teachers' content knowledge (and specifically their pedagogical content knowledge for mathematics teaching) will increase the quality of mathematics education in the classroom and the mathematics learning of students. The goal of the study of the course descriptions was to determine how grants had interpreted what should be contained in a course for teachers. There was particular interest in what mathematics was the focus of professional development efforts as well as when courses were blending content and pedagogy.

The content courses fall into three major categories. The first are courses purely focused on mathematics content (including process content and history of mathematics). The second category of courses is strictly focused on pedagogical strategies and issues (e.g., assessment, differentiation strategies). Such course descriptions do not have mathematics learning as a central focus (if it is mentioned at all). The third group is courses that blend content and pedagogical knowledge. Such courses blend learning about mathematics with learning about mathematics teaching and learning by students. They are along the lines of recommendations that teachers' knowledge of mathematics is different than mathematics for other professions (Hill, Rowan, & Ball, 2005). The course descriptions vary in terms of emphasis on content or pedagogy. Determining how much mathematics enters the course content is not possible simply from the description. Establishing a continuum or a more fine-grained categorization of blended content and pedagogy courses is a goal for further research.

Based on the course descriptions available, most of the courses cover algebra, geometry or statistics (including data analysis and probability). Courses were directed primarily toward elementary or middle school teaching. Neither of these results is surprising as a focus of standards documents and initiatives is on the increased learning of algebra and how algebraic ideas are developed throughout elementary and middle school. The emphasis at elementary and middle school mirrors the overall focus of the MSP grants where fewer grants have targeted high school. Geometry courses may have an emphasis on the use of technology tools such as Geometer's Sketchpad ® in the classroom to provide a dynamic view of geometry and develop spatial and geometric reasoning further. A few courses focus on content new to or outside the typical K-12 subject matter including mathematical modeling, applications in industry, and mathematics history.

Another distinction for content courses is the format of the course. An initial distinction among courses offered by the grants is in the administration of the course by either an institution of higher education or a school district. Related to this distinction is whether "course" refers to a workshop (lasting a few days), a summer workshop (lasting a few weeks), a semester-long course or year-long series of workshops. The nature of teachers' learning in each context is different and the duration may impact the content and nature of the course. The development of content courses also occurs for both pre-service and in-service teachers as well as for district or school level mathematics specialists. Further analysis and data regarding the nature of the courses (e.g., syllabi or other materials) is necessary to continue the analysis of these distinctions.

Finally, courses may also utilize standards-based curricular materials as part of course content (e.g., *Connected Mathematics* units and modules). The use of the materials may depend on whether the school districts in the partnership have adopted the curricula. The course may be organized around the development of content knowledge in connection with a particular curriculum in order to facilitate teachers' use of the new curriculum by helping them understand the structure of the curriculum. Related to existing materials, grants may also employ materials developed by the grant or support teacher development of curriculum (e.g., by developing materials in workshops and providing ways to share them across sites on a website).

The impact of content courses for teachers on students' learning is still under evaluation and investigation. As many of the grants are still in their first two years, data about student achievement related to the content courses is still not available due to the time required for courses to be developed, for teachers to complete the courses and for teachers to develop their teaching practice following courses. As with the student achievement results, studies of teacher practice after a course (or series of courses) is still underway by the grants. Overall, the use of content courses is widespread throughout the program, but varied in purpose, content and intent.

Conclusion

The study of teacher content courses is one slice of the role of mathematics content in the portfolio of MSP grants. The content courses cover the K-12 grade level spectrum of content, include new content (e.g., mathematical modeling), focus on developing content across multiple grade levels (e.g., algebra), and seek to develop teachers pedagogical content knowledge for teaching. The purposes and formats for the courses vary across grants, but there is consistency in the perceived need for them within endeavors to encourage challenging content and curriculum. However, the grants are still in progress and evaluations of the impacts on teaching and learning are still in progress.

Mathematics content and curriculum play varying roles within the portfolio of MSP grants. The projects are designing curriculum, evaluating curriculum, and conducting teacher professional development efforts related to content and curriculum that should all lead to increased student achievement in mathematics. Projects are introducing new content, developing teachers' content knowledge, and working with state standards and assessments to evaluate their impact. Further investigation will examine the role different purposes for content play in a project. More specifically, how does content drive (or not) grant activities? What are the implications for a focus on different content areas? For definitions of challenging content, it is then critical to ask how challenge plays a role in content decisions in a project. What is challenging for one setting may not be challenging for another. "Challenging" may mean introducing new content or ensuring all students have equal opportunity to content in order to reduce achievement gaps. The second definition is an equity question based on the need for challenging curriculum for all students.

Authors' notes

1 This research is part of the Math and Science Partnership Program Evaluation (MSP-PE), supported by Contract No. 0456995 from the National Science Foundation. The MSP-PE is led by COSMOS Corporation and departments at Vanderbilt University and Brown University, as well as the Mathematics Education Center at George Mason University. A fourth organization, The McKenzie Group, is coordinating advisory board activities, assuring the autonomy and integrity of the external peer review work. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation or other participating organizations.

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FUNCTION COMPOSITION AS COMBINING TRANSFORMATIONS: LESSONS LEARNED FROM THE FIRST ITERATION OF AN INSTRUCTIONAL EXPERIMENT

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We report on an instructional experiment designed to support K-12 teachers conceiving of the composition of linear functions as combining transformations in a context involving an imaginary elevator. We discuss the design and implementation of instruction in terms of a two-phase cycle involving the formulation of an initial and then a revised local instructional theory for the concept of function composition. Our empirical results are embedded within this discussion and presented in two interrelated parts. We highlight teachers' thinking regarding addition of integers in relation to the elevator context, their thinking regarding formalizing the composition operation in relation to this context, and we articulate our principled efforts to leverage the former in support of the latter.

Introduction

The reform-based Grade 3 *Investigations Series* curricular unit *Up and Down the Number Line* (Tierney, Shulman-Weinberg, & Nemirovsky, 1995) entails a context (an imaginary elevator) designed to support children's thinking of integers as transformations and addition (implicitly) as composition. We report on our study that employed this unit as a point of departure for an instructional sequence designed to support K-12 teachers' understanding of function composition.

Our instructional sequence adapted and extended aspects of the *Investigations* unit to develop the concept of function composition in the elevator context. Our aim was to help teachers develop coherent conceptions of function composition - a concept that is not well represented in the research literature but with which college students apparently have difficulty (Engelke, Oehrtman, & Carlson, 2005).

We first describe the setting for the implementation of the instructional sequence and elaborate initial conjectures about students' conceptions and learning (Simon, 1995). We then provide selected findings from our retrospective analysis of the implementation of the instructional sequence. We discuss these findings with an eye toward revising our initial conjectures. Finally we describe the revised local instructional theory and our plan for further elaboration of it in order to support the development of the concept of function composition as combining transformations.

Methodology

Local Instructional Theory

Although we expect our project to result in the production of an instructional sequence, our overall goal is to produce something more generalizable than a specific sequence of instructional activities – a local instructional theory. The purpose of the local (content specific) instructional theory is to provide a rationale for the instructional activities that draws on the researchers' models of students' emerging and developing conceptions in relation to their engagement with

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designed instruction. Local instructional theories feature three key ingredients, the description of which is adapted from Gravemeijer (1998):

- Identification of students' informal knowledge and strategies on which the instruction can be built.
- Design principles for instructional activities that can be used to evoke these kinds of informal knowledge and strategies.
- Design principles for instructional activities that can capitalize on these informal understandings in order to meet the goals of instruction.

We use this description as a framework for our research. Our results are presented in two sections. First, we describe the teachers' (our students) informal ways of thinking about integer addition in the elevator context. We identify those ways of thinking that seemed to provide a starting point for developing the concept of function composition, and we discuss aspects of the instructional sequence that helped to evoke these ways of thinking (or seemed to evoke less productive ways of thinking). We then describe our efforts to capitalize on the teachers' ways of thinking about integer addition in the elevator context in order to develop the notion of function composition. We also describe the teachers' ways of thinking about function composition that emerged as they interacted with the instructional tasks. A retrospective analysis of this first cycle of design and implementation guided the development of a revised instructional sequence (and conjectures about the anticipated learning) on the way to developing a local instructional theory.

Setting and Participants

We engaged 4 cohorts of K-12 teachers with the sequence of instructional tasks over four 2-hour-long sessions occurring on consecutive days. The sequence was part of a course designed for a three-week summer residential institute for K-12 mathematics teachers. Instruction generally featured an inquiry-based approach to concept development. The tasks were designed to evoke the participants' informal understandings and strategies and to leverage these as the foundation for the development of the more formal or conventional mathematics. Activities typically began by having teachers first consider a problem or issue in private and then asking them to share their thinking with a partner in anticipation of small group interactions and whole class discussions. Participants were encouraged to compare and contrast ideas, to question ideas and explanations, and to offer or ask for elaborations of ideas. These classroom interactions were captured with two video cameras. The written work of individual teachers was digitally photographed, as were the posters created by each group.

Preliminary Local Instructional Theory

Step 1: Thinking of Adding Integers as Combining Changes (The Elevator Context).

The exploration of the concept of function composition was grounded in the *Investigations* curricular unit already mentioned (Tierney, Shulman-Weinberg, & Nemirovsky, 1995). This unit develops the idea of net change in a context involving an elevator in an imaginary skyscraper that extends infinitely in both vertical directions. The elevator's push buttons are labeled with integers representing *changes* (magnitude and direction) in position rather than positions (floors of the building). Thus, pressing a particular button can be seen as making the elevator move that many floors up or down the skyscraper, depending on the sign of the button's numeral. The curricular unit employs this context to emphasize thinking of integers as transformations (changes) in position and the chaining together of such transformations to obtain a net change in the elevator's position.

Our plan was to have teachers first conceptualize the operation of adding integers as the operation of combining changes. The rationale for starting the sequence in this way was two-fold. First, this way of thinking about integers involves implicitly thinking of integers as functions. We expected this way of thinking to support both the transition to formulating the changes as functions and the eventual formulation of function composition as a way to combine changes. Second, we expected that the teachers' understanding of addition as a binary operation (one that takes two integers as inputs and produces an integer as an output) would support their thinking of function composition as an operation that takes two functions as inputs and produces a function as an output.

Step 2: Thinking of Changes as Functions

The second part of our instructional plan was to have teachers formulate each change (or change button) in the elevator context as a function. This was to be done by first asking the teachers to articulate the relationship between a starting floor, a change button, and the resulting ending floor (i.e. START + CHANGE = END). The next step was to introduce function notation as a way to capture this relationship (for a specific button). For example, the change button, +2, can be associated with the function $f_{+2}(S) = S + 2$. Our intent was for the teachers to associate this notation with 1) the change (process) associated with a given change button and 2) the number (object) used to label the button. Thus, the goal was to support the teachers' ability to conceive of functions as both processes (that can be combined to form other processes) and objects (that can be combined to form other objects).

Step 3: Thinking of Composing Functions as a Way to Combine Changes

Our overall goal of instruction was to have the teachers develop the idea of function composition as the linking of processes in order to produce another process. The final step of our instructional plan was to engage the teachers in thinking about how to formulate a function associated with a combination of two change buttons. Our strategy entailed having teachers think of the combination of changes as a two-step process in which the result of the first step (the ending floor after pressing the first button) is seen as the starting point for the second step (the starting floor before pressing the second button). We conjectured that this line of reasoning would support their thinking of substituting the formal rule for the first function (say, $f_{+3}(S) = S + 3$) into the rule for the second function (say, $f_{+2}(S) = S + 2$) in order to produce the new function rule, $f_{+5}(S) = S + 5$, associated with the combination of the two buttons. This intended line of reasoning is expressed by the following string of equalities: $f_{+2}(f_{+3}(S)) = f_{+2}(S + 3) = (S + 3) + 2 = S + 5 = f_{+5}(S)$.

Going into the implementation phase, we were aware that it is possible to produce a formula for such a combination simply by combining two changes to obtain a single change and then writing a function rule for this change. This approach does not involving thinking of function composition in the way we intended because it does not employ two function rules to produce the new function. Instead, the two changes are used to find the new change, which is then used to produce the new function. (This distinction will become clearer when we discuss our results.) In order to focus the teachers' attention on the process of composing functions as we intended, we asked them to describe a mathematical way to combine the two function rules in order to obtain the new function.

In sum, our overall goal in this stage of the instructional sequence was to have the teachers develop the notion of function composition (and the associated symbolic operation of substitution) as a way to formalize the process of linking changes. The remainder of the paper will focus on the first and third steps of our instructional plan.

Results Part 1: Ways of Thinking about Adding Integers in the Elevator Context

As a first task, we asked the teachers to describe what it meant to add two integers in the elevator context. This task turned out to be quite challenging for them: They tended to impose their existing view of integers as positions on the number line onto the curricular unit. Most described an integer exclusively as representing a floor of the building and not as a change in position. This interpretation had a dramatic impact on the teachers' ability to make sense of addition in the elevator context. From our perspective, the process of successively pushing two change buttons corresponds to adding two integers in the elevator context and so addition can be seen (at least implicitly) as the composition of functions. This way of thinking provides a good point of departure for developing the notion of function composition. It is also reasonable to think of integer addition in this context as combining (adding) a change and a position to obtain a new position. This way of thinking supports conceiving of a specific button as a function, but does not provide an informal way to think about function composition in the context. However, many of the teachers struggled to develop a different interpretation - one in which *both* addends were positions. The teachers were not able to generate coherent interpretations of this type. While they interpreted integers as floor positions, they associated the operation of addition with the process of moving from one floor to another (see Alice's response below). We coded the teachers' responses to capture the apparent structure of their ways of thinking about adding integers in the elevator context. The three most common ways of thinking are presented below:

Way of Thinking	*Example Response*
Integers are Floors: Addends are both floors, addition is loosely associated with moving from one floor to another.	*Alice:* Adding two integers is like riding the elevator up from a start to an ending floor where the integers are the floor positions.
Adding a Change to a Position: The start floor and the change are the addends. The sum is the ending floor.	*Erika:* If you add the net change to the starting floor, you will get your ending floor.
Combining Changes: Two changes are combined resulting in a net change.	**Pam:** ***It is the combination of 2 movements of the elevator. It is the result after the elevator has gone through 2 movements.***

Discussion Part 1: Implications for a Revised Local Instructional Theory

Our analysis suggests that a modification of the starting point for the instructional sequence is in order. In retrospect, the focus on integers and integer addition seemed to be unhelpful because it appeared to evoke the teachers' conceptions of integers as positions *not* changes in position. However the change elevator context itself did turn out to be a productive one for developing the notion of function composition. Thus, we conjecture that it might be more productive to focus teachers' attention initially on combining changes in the elevator context and not on interpreting the meaning of integer addition in this context. One possible argument against such a revision is that the teachers' conceptions of integer addition as an operation would not be tapped to support the development of function composition as an operation. However, it seems likely that 1) the teachers can think of the combination of change buttons as an operation and that this could support the development of function composition as a mathematical operation and 2) the teachers might still spontaneously draw on their understanding of integer addition even without our explicit attention to the fact that this system is isomorphic to the integers under addition.

Results Part 2: Ways of Thinking about Composing Functions in the Elevator Context

The final task situated within the elevator context involved representing each elevator change button as a linear function and thinking about how to create such a representation for a combination of two change buttons. After the teachers had some experience representing individual change buttons as functions, we asked them to consider the two-button changes given by the functions $f_{-5}(x) = x - 5$ and $f_{+3}(x) = x + 3$ and to write a single function rule to express the combination of these two changes. Because the study was conducted in a classroom setting, for the most part it was not possible to track individual students' learning as they worked on these tasks. However, by considering snapshots of different individuals' ways of thinking we are able to identify steps along a possible learning trajectory.

Combining the Changes and then Constructing the Function

Denise created the function for the combination of changes by first combining these changes to get a net change and then writing the function associated with this change:

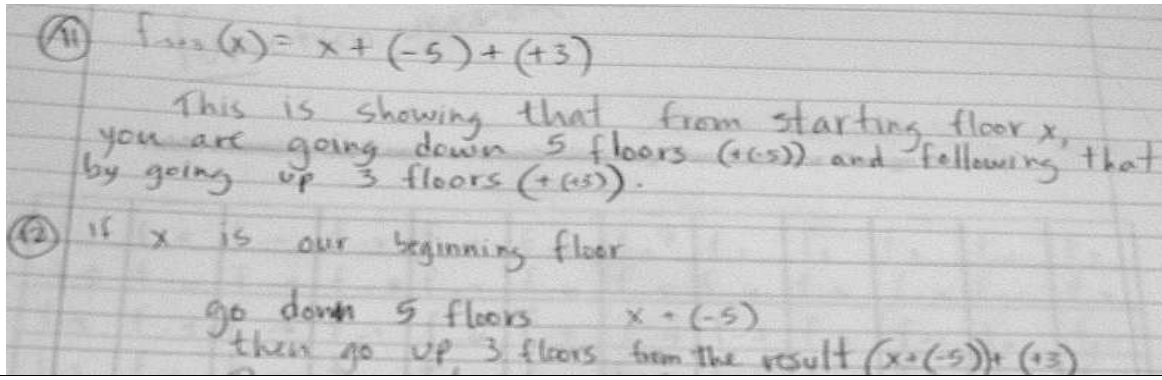
Denise: So it went down by 5 floors […] it started at any floor, and then came back up 3 […] and the net change between where you start and where you end is minus 2. So no matter what floor you start on, your net change is always going be -2.

Denise does not use an algebraic procedure to combine the function rules. Instead, she proceeds by combining the transformations and then writing a function for the resulting transformation, effectively bypassing the symbolic process of substituting the rule for the first function into the second function. It is also important to note that Denise's way of thinking appears to be deeply embedded in the elevator context. It seems that she is not reasoning about functions at all (at least explicitly), but is reasoning about changes in this context and then translating her results later into function notation.

Augmenting the First Function with the Second Change

Jim's first attempt involved adding the two function rules: $(x – 5) + (x + 2)$. He realized that he should not have $2x$ in his result and then reconsidered the task. He went on to develop an approach that was more symbolic than Denise's. However, like Denise, his reasoning was more about combining changes than combining functions.

Jim's Written Work

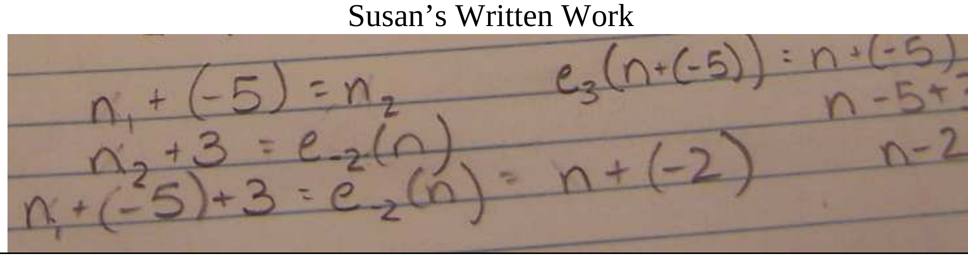


Jim combined the two functions, but not by substituting the output of one into the other. In response to the task of creating a function for the combination of the two buttons, he essentially created a new function from scratch, adding the first change and then the second to the starting floor *x* (see Figure above). In response to the task of describing a procedure for combining the two functions rules, Jim used a two-step process. He first thought of the rule for the first function, $x + (-5)$ as expressing the result of the first part of the trip, and then he considered the result of going up an additional 3 floors from this result, expressing symbolically as $(x + (-5)) + (+3)$. We note that this way of thinking is subtly different from substituting the output of one function into the other – it involves seeing the first function rule as expressing the result of the first part of the trip, and then applying the transformation associated with the second function rule to this result.

Using Notational Devices to Facilitate Substituting

An elementary teacher, Susan, introduced subscripts to deal with the fact that the starting floor for the second part of the trip was not the same as the starting point for the first part of the trip. This allowed her to write an expression, $n_1 + (-5) = n_2$, that expressed the fact that this second starting floor was the ending floor of the first part of the trip. She then was able to use this expression to make a substitution that resulted in the desired function rule. After doing so, she was able to do the substitution without using this notational device.

Susan's Written Work



Discussion Part 2: Implications for a Revised Local Instructional Theory

The approaches that Jim and Susan developed to deal with the tasks are especially relevant for our revision of the local instructional theory. Our data suggests that these two students were not drawing on prior procedural understanding of function composition – in fact, the evidence suggests that they were not aware this was function composition until a later juncture in the instructional sequence when this term was introduced. Note that, like Jim, Susan first attempted to combine the two functions by adding their rules.

Since Jim and Susan's reasoning was apparently based on the elevator context, we can learn something about how the symbolic procedure of substituting can emerge for students as they work in this context. From Jim's response, we see that it is possible to develop a symbolic procedure in this case that does not explicitly involve substitution. Because the functions here are of the form, $f(x) = x + b$, it is possible to easily compose them without substitution. One merely augments one of the functions by adding the change part (the "$+b$") of the other function. This suggests that it may be useful to introduce an additional function type that does not allow this approach.

From Susan's response we see that it may be difficult to think of the rule for the first function as an input for the second function. Note that a student needs to realize that this expression can stand for a floor (the ending floor) and not just a transformation associated with the function. They then need to combine this realization with a construal of this ending floor as the starting floor for the second part of the trip. We contend that construing this duality (Gray and Tall, 1994) entails significant conceptual complexity and coordination. Susan was able to introduce a notational device to help her manage this complexity. The use of subscripts allowed her to encapsulate the rule for the first function and think of it as an input of the second function. Later she was able to set this notational device aside and perform the substitution in the more standard way.

Revisions to Preliminary Local Instruction Theory / Directions for Further Research

Our findings suggest a number of possible refinements of the instructional theory. First, we found that the teachers had difficulty setting aside their tendency to think of integers exclusively as positions, even in a context designed to support thinking of integers as transformations. This finding suggested a need to rethink the point of departure for the instructional sequence. In preparation for further research to develop our instructional theory, we have designed a computer micro-world that aims to ground students' thinking in the dynamic elevator context. In this micro-world, the dominant feature is the idea of change. The goal is that as a student works with this micro-world, it will become clear that the elevator buttons represent changes and that these can be combined to generate other changes.

After the teachers became comfortable thinking in terms of combining changes in the elevator context, we found that they were able to leverage their ideas in different ways to think about function composition. These different ways of thinking may suggest plausible signposts on the way to developing a rich understanding of function composition. Two particularly important ways of thinking are expressed by the approaches of Jim and Susan. Jim's approach makes it clear that students can resort to symbolic procedures other than substitution to compose the kind of functions associated with the change buttons in the elevator context. One possible approach would be to introduce multiplier buttons into the elevator context (e.g. the "×4" button would take the elevator 4 times as far from the 0 floor). This kind of button would give a function of the type, $f(x) = ax$. It is much more difficult to compose this kind of function with one of the type, $f(x) = x + b$, without performing a substitution. Susan's approach suggest that it is important to make sure that students can see the rule of one of these functions as also representing a floor – the ending floor of the first part of the trip and the starting floor of the second part of the trip. Her approach also suggests a way to assist students in handling the complexity involved with thinking flexibly about, and working with, the rule of a function when composing two functions.

Introducing a notational device to help a student think of this expression as a floor may support their ability to link the two processes and compose the two functions.

As we continue to work to develop the local instructional theory, the next step will be to conduct a series of teaching experiments (Steffe & Thompson, 2000) in order to more carefully elaborate a path by which students can develop the concept of function composition as a way to formalize the process of combining transformations.

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EXPECTATIONS VS. REALITY OF THE USE OF MATHEMATICS TEXTBOOKS IN ELEMENTARY SCHOOLS

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Textbooks are use to convey the national curriculum. After observing and interviewing Mexican teachers on their use of mathematics textbooks, we realized that the majority did not fulfill the expectation educational authorities had. Main factors are the lack of mathematical knowledge, lack of understanding of both the educational approach proposed by the Ministry and the rationale the textbook authors had when writing and designing the books. Research on this area is scarce and necessary.

Introduction

Textbooks are one of the resources most used by teachers in classrooms (Freeman et al, 1989; Robitaille et al., 1989; Robitaille, 1995; Stodolsky, 1988; Moren, 1999; Schmidth et al, 1997; Pepin et al, 2001; Boaler, 1997), as well as one of the most important means of conveying mathematics. Even considering the growing use of new technologies, a survey conducted in the United Kingdom revealed that about 90% of schools argued that a set of "good textbooks" was indispensable and effective to raise educational standards, while only a half of the schools judged the appropriate use of new technologies as valuable. *(Schoolbook Spending*, 2002). Textbooks in Mexico, where this research took part, are free and compulsory for all children in elementary school, which reveals its importance in the National Educational System.

In broad terms, a textbook serves two basic purposes: conveying educational reforms and/or curriculum (Amit and Freid, 2002), and providing support to teachers in conveying knowledge, organizing their classes and material, as well as being the source of activities and drills. In either case, there is scant research substantiating the achievement of such purposes.

In the international sphere there has been very few research efforts addressing teachers' understanding and use of mathematics textbooks. For the most part, such research has been based on questionnaires, and not on observational work on the use of textbooks.

In England, Moren (1999) observed how teachers use textbooks. Her findings reveal that: a) each teacher made use of the same material in his or her own way, a fact that contradicts the notion that books determine or impose a particular practice; and b) a single teacher was found to use the same material in different ways with different groups, primarily because of the characteristics of each group. Ball and Feiman-Nemse (1998) found that although teachers were taught that the textbook is a source of activities to access knowledge, in practice they used it as a class organizer and guideline. This happened because teachers were unaware of the contents to be taught, because inexperienced teachers lacked the self confidence to design their own lessons, and because of school authorities' stress on the use of the textbook.

On the other hand, if a textbook writer is going to use research results in order to write about each topic, research will be focused on the topic itself and not on how the topic should be addressed within a textbook and how the teacher is going to use that textbook. Furthermore, designers and publishers do not make decisions based on textbook design research, and very rarely do they interact with the authors themselves (Evans et al, 1987). Ginsberg, Klein, and

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Starkey (1998) reviewed the textbook production and dissemination process in the United States, and reported the complicated dynamics between researchers, government, professionals, publishers, authors, and teachers, and in consequence, the use of the textbook in a classroom becomes the least heeded factor when it comes to decision-making.

In the case of Mexico, every year, the Ministry of Education (SEP) publishes a series of free books by grades and subject matters, which are distributed to all elementary schools in the country. This is one of the primary means of representing and conveying the proposed curriculum.

The mathematics textbook has 5 units, each one with around 17 lessons. Each unit has lessons of all seven areas that articulate the curriculum: Number, Measurement, Geometry, Data analysis, Statistics and Probability, and Proportionality. There is different amount of lessons for each area, and the lesson sequence relating the areas is not evident.

Textbooks are based on a problem solving approach, as a "driving force that promotes mathematics learning and students capacity to think" and states "the need to start off from activities in which students use mathematics as a tool to solve problems; activities in which previous knowledge and informal procedures can be call for in order to solve mathematics problems; activities based on concrete experiences that encourage teamwork" (SEP, 1993, p. 15). Students should have the necessary tools to construct their own knowledge through problematic situation.

The scheme proposed to use the textbook is to use its lessons as a starting point when giving a class. Each lesson has a problematic situation to be address, and does not explain or review the mathematical concepts needed to solve the situation, nor does it have exercises which will reinforce the ability or concept learned. Children are expected to 'do' mathematics for the learning process to take place.

The Ministry of education has not been able to convey how teachers should use the textbook in order to meet the target (Santos et al, 2004) thus leaving the responsibility on the teacher itself. By conducting an ethnographic survey, Carvajal (2001) verified that first-grade teachers modify the mathematics textbook and, as a result, the book took on different perspectives and scopes, according to each teacher's personal background. Ruiz (2003) interviewed 29 primary school teachers in order to get their opinions on the free textbooks supplied by the government, and also to find out whether teachers were aware of the underlying approach of the math textbooks (based mainly on problem solving). Sixty per cent of responding teachers reported being unaware of the approach; the remainder knew the approach but reported having difficulties putting it into practice. Regarding textbook content, the responding teachers pointed out that, for the most part, it presented a degree of difficulty for them; most of them recognized they ignored how to use the books in the classroom, because their rationale was not easy to understand.

Starting Point

A teacher's interpretation of the curriculum through textbooks and the use he/she makes of the material is closely related to multiple categories of knowledge that enable a teacher to, first interpret, and then mediate the curriculum in their practice by using the book as a means to achieve curricular targets.

According to Shulman (1987), teachers transform curricular contents according to their own epistemological conceptions. Transforming the meaning of the curriculum results in using the textbook in a personalized manner, and not necessarily meeting the expectations of the educational authorities.

The present study is part of a research effort intended to find out how teachers use math textbooks throughout the primary school years, in order to propose teacher-support material, as well as potential changes to existing textbooks. The first stage, as reported here, seeks to determine the ways teachers use textbooks and clarify why they do so.

Method

To evaluate how Mexican teachers understand and use National Primary Math Textbooks, information was collected from four sources:

- No-participant observation of 12 primary school teachers during a school year. Five one week visits to their classroom where done, in order to register how teachers used the mathematics textbook. An observation guide was designed in order to realize a qualitative analysis of the data collected.
- Interviews to the same 12 teachers at the end of the year were carried out in order to know their opinion on the textbook, positive and negative aspects of the textbook itself and the way they used it with the students. Results of the interviews were codified in order to complement the information gathered through the observations.
- A survey to 400 elementary teachers was carried out in order to know: the use they gave the textbook (to introduce a topic, to exercise, to reaffirm knowledge, to asses, etc); the level of difficulty they considered the textbook had; the amount of time needed to accomplish the textbook lessons; the clarity of activities and instructions; the shortcomings and assets the textbook has; etc. A non parametric analysis using chi square was done.
- Opinion poll. Results from the above sources showed that the 6th grade textbook had its drawbacks. Fourteen teachers and 21 students were asked to rank textbook lessons in order of difficulty, in order analyze if there is any sort of consensus on topics or lessons' difficulty.

Findings

Observation and interview (12 teachers)

Two teacher profiles were outlined from the way they used the textbook (Figure 1). The way they use it does not fulfill the authority's scheme of using the textbook.

Ten out of the twelve teachers could be describe with one of these two profiles. Only two teachers used the textbook as expected by authorities, that means, 1) using the problematic situation of each lesson as a staring point for each topic they had to address, 2) encouraging students to discuss and work in teams, 3) analyze the process and not only focusing on the answers, and 4) helping students construct their content knowledge through the activities.

Survey (400 teachers)

Primary school can be divided in three cycles: 1st cycle, first and second grade; 2nd cycle, third and fourth grade; and 3rd cycle, fifth and sixth grade.

When asked about the length of each lesson of the textbook (Table 1), teachers of the 3rd cycle in comparison to the other cycles, considered them inadequate because they were very long (x^2=10.39, significant level=0.034)

Use of textbook	Profile 1	Profile 2
Frequency of use	Three or more times a week	Two to three times a week
How they use it	Guide students throughout the lesson, reading it all together out loud. Answers are given for each activity only to rectify it they are right or wrong. No process analysis. No discussion.	Explains what the activity is about. Reviews or explains the concepts needed to solve the activities.
Purpose of working with the textbook	Complete textbook lessons in order to fulfill authority's demands. They teach mathematics with other printed material	As an activity source. To decide the order of the topics to teach.
Clarity on the content and goals of the lessons	No clarity	No clarity
Clarity of activities and instructions	No clarity, they need to be explained to students	No clarity, they need to be explained to students
Difficulty of lessons	In general is the adequate level of difficulty, but often there are activities they have to skip because the teacher does not understand it.	Level of difficulty higher than what students can do.

Figure 1. Observation and Interview results

	Adequate	Long	Short
1st cycle	63.3%	11.9%	24.8%
2nd cycle	65.8%	13.7%	20.5%
3rd cycle	52.4%	26.7%	21.0%

Table 1: Percentage of teachers that consider the lessons adequate, long, short.

Teachers need at least four hours of work on each lesson, because the design of the latter does not provides the background information required to address the lesson or the drills necessary for students to practice what they have learned, or a closing to confirm the learning

With regard to how clear teachers think the instructions to follow the activities are given in the textbook (Table 2), again teachers of the 3rd cycle find them more difficult than the rest of their colleagues (x^2=5.185, significant level=0.07)

	Clear	Not clear
1st cycle	78.5%	21.5%
2nd cycle	75.0%	25.0%
3rd cycle	65.0%	35.0%

Table 2. Percentage of teachers that think the instructions are clear or not

Table 3 shows that teachers of the 3rd cycle report that the difficulty level of the lessons is higher than what their students are able to understand, not being the same case for their colleagues (x^2=49.349, significant level=0.00).

	Adequate	Lower	Higher
1st cycle	78.7%	10.2%	11.1%
2nd cycle	70.6%	7.3%	22.0%
3rd cycle	45.6%	1.9%	52.4%

Table 3. Level of difficulty of the textbook lessons in relation to what their students are able to understand.

The results of the survey revealed that 5th- and 6th-grade teachers exhibit a certain degree of mistrust in the use of the textbook, primarily for the following reasons: a) being unsure of the objectives and contents dealt with in each lesson, because of their high degree of difficulty; b) not understanding and consequently being unable to deal with the approach required to use the textbooks, and c) having difficulty to resolve the proposed problems and activities.

Opinion Poll (14 teachers and 21 students)

Because the level of the lessons is one of the main difficulties teachers reported in connection with not being able to use the textbook as expected by the authorities, an opinion poll on the 6th-grade textbook was carried out to evaluate the level of difficulty teachers and students consider the textbook lessons had. The textbook contains 87 lessons.

Students and teachers used a scale from 1 to 5 (1 very easy – 5 very difficult) to evaluate how difficult each lesson was with regard to understanding it. Teachers also evaluated each lesson considering the level of difficulty when teaching it. Lessons were classified into: easy, 'normal', and difficult.

Results showed that 50 lessons out of 87 were considered easy to understand by students and easy to understand and teach by teachers.

Eighteen lessons out of 87 were considered difficult to understand only by teachers' outlook but not by the students.

All teachers and students agreed that 19 out of 87 lessons were difficult to understand and teachers thought as well they were difficult to teach.

In table 4 it can be noted that the majority of lessons that work with fractions and measurement are regarded as difficult to understand by teachers and students. Students and teachers have problems with half of the lessons that deal with proportionality and data analysis.

Content / Area	% of lessons reported as difficult
Fractions (as part of Numbers)	88
Measurement	75
Decimals (as part of Numbers)	50
Proportionality	50
Data Analysis	18
Numbers	17
Probability	12
Geometry	9

Table 4. Percentage of lessons (Sixth grade mathematics textbook) reported as difficult by teachers or students

Conclusions

The factors preventing teachers from using the textbook as it was conceived by the author and the education authorities are as follows:

1. Difficulty in mastering math contents, which might reveal either that math teachers are deficiently trained, as they are expected to optimally manage the contents they are meant to teach, or that the book does not fit the realities of Mexican teachers and students.

2. One of the main factors why teachers do not fulfill the authority's expectation on how to use the textbook is the rationale of it. The textbook is based on problem solving activities. Each lesson has a problematic situation to be addressed, and does not explain or review the mathematical concepts needed to solve the situation, nor does it have exercises which will reinforce the ability or concept learned. As well, the solving problem approach conveyed in the textbook, causes each lesson to comprise various math contents, an issue that confuses teachers and makes them miss the objective of the lesson. This approach to teaching and learning mathematics is not understood by the majority of the teachers. Teachers need to know what exactly are they teaching and what specific mathematical content their children are going to learn.

3. The discrepancy existing between the teacher's mastery and use of the book, and the conception of the latter by the educational authority reveals a need for both players to establish better communications supported by field research work.

Discussion

It is already stated that the success in using a textbook depends on teachers training. The lack of teachers' expertise might make textbooks themselves not be the ideal medium to promote meaningful learning. Thus, the fact that teachers do not fulfill educational authorities and authors' expectations when using the National Mathematics Textbook is not surprising.

The rationale of the textbook is that mathematics is meant to be meaningfully learned through learning activities. Based on Ausubel's theory of learning, activities can range in a spectrum from being rote learned to being meaningfully learned; and the way to present the information to be learned can be of three styles: receptive learning, guided discovery learning, and autonomous discovery learning. Novak (1977) allocated different learning activities in a matrix based on Ausubel's theory. Textbooks are found on the extreme of activities leading to reception learning, i.e. the regularities to be learned and their conceptual labels are presented explicitly to the learner. Textbooks presentations are also found to be almost in the middle between rote and meaningful learning, but tending more to the rote end. This is no surprise and it would be difficult for it to be otherwise. The author's intention might be to structure the textbook lessons in order to guide to student to discovery learning, but, since the relationship between the learner and the textbook is static, the author's influence on the material attempts to be meaningful by relying on generalizations of what is considered meaningful for students.

It seems author's write textbooks putting themselves in the teachers' role, expecting the activities they design to fulfill the educational approach they support as they themselves would carry them out. If this is the case, this mode of conceiving textbooks has not proven to be successful. Research on how teachers use textbooks should be the backbone of authors and authorities rationale for writing textbooks.

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HOW DOES PEDAGOGICAL CONTENT KNOWLEDGE EMERGE FROM CLASSROOM TEACHING?

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This research traces two years of one sixth-grade teacher's developing pedagogical content knowledge defined as coupled student thinking and teacher assistance Discourses. Systemic Functional Linguistics coding is applied to a chronological string of student-teacher interactions about ratio. This provides detailed linguistic evidence that teachers develop PCK from their teaching, an often spoken claim that is not well documented or understood.

This research traces two years of one sixth-grade teacher's developing pedagogical content knowledge defined as coupled student thinking and teacher assistance Discourses. Systemic Functional Linguistics coding is applied to a chronological string of student-teacher interactions about ratio. This provides detailed linguistic evidence that teachers develop PCK from their teaching, an often spoken claim that is not well documented or understood.

Introduction and Theoretical Framework

This research will provide detailed linguistic evidence to support the commonly stated but not well documented claim that teachers develop pedagogical content knowledge (PCK) in the act of teaching. Some evidence supporting this claim indicates that unexpected student ideas trigger teachers to reevaluate their pedagogy (Sherin, 2002), and a teacher who recognized student thinking in video from her classroom used more effective assistance in subsequent teaching (Seymour & Lehrer, resubmitted). Systemic Functional Linguistic (SFL) (Halliday, 1978; Lemke, 1990) is particularly useful for tracing PCK development because it systematically documents how people make meaning by coordinating three separate aspects of language integral to classrooms; the content, the interpersonal relationships, and the continuity of discussions across time. This fine-grained analysis is intended to enhance the trustworthiness of previously used coarser-grained discourse analyses, and also better understand linguistically how PCK emerged across classroom conversations.

To provide detailed convincing evidence that PCK developed from classroom teaching, SFL coding is used to trace the linguistic features of chronological strings of teacher-student conversations. The participants are Ms. Gold (all participants names are pseudonyms) and two cohorts of her sixth-grade students who were engaged in a design experiment (Cobb, 2003) with an innovative curriculum unit uses multiple representations to ground understanding of slope as a ratio. This string of conversations was developed based on a model of PCK as interanimation (Bakhtin, 1981) between two Discourses (Gee, 1999) recognizable to the teacher. In this model (Seymour & Lehrer, resubmitted) a teacher with PCK recognizes the ways in which students talk (and act) during mathematical activity, and can recognize/predict teacher's talk (and actions) effective for orchestrating students' thinking toward accurate mathematical understanding. This previous research focused on orchestrating student understanding of slope. The current string of conversations capture Ms. Gold interacting with students who are thinking about ratios in two distinctly different ways called *between* and *within* (Lamon, 1994; Lehrer *et al.*, 2000) across

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different representations of ratio. These representations include mathematically similar rectangle cutouts, equations, coordinate graphs, function tables, and slope (Seymour & Boester, submitted). The overall goal is to coordinate the student thinking and teacher knowledge research emanating from this design experiment, and bolster it with a more detailed linguistically-based body of evidence to better support the claim that Ms. Gold's PCK did indeed emerge from classroom interactions. SFL analysis can also provide a better description of how this happened.

Method

Participants

The primary research participant, Ms. Gold had multiple advanced degrees and 16 years of experience at both the elementary and middle school levels. She won several teaching awards during this time, and participated in several university-affiliated research projects investigating teaching and learning in subject areas including mathematics and science which she published. In each cohort of students of approximately 20 students the socio-economic status spanned the range from homeless to upper-middle class, the ethnicity was mixed (Caucasian, Hmong, African American, Asian), and some received special education services.

Classroom procedures

Teaching routines consisted of summarizing previously learning, laying out the task for the day, and sharing findings from the task in full class discussions._During the first year, teaching varied in length from one to three hours for 30 days across 10 weeks beginning in March. During the second year, sessions varied from one to two hours for 14 days during three weeks beginning in May. This two-fold increase in instructional efficiency is often characteristic of sequential design studies (Cobb et al., 2003).

Data

Data sources for both years include video-taped planning sessions, field notes, videotaped classroom instruction (with debriefing interviews when possible), the teacher's journal, and a video-stimulated structured interview during which the teacher viewed video episodes from her classroom. The video camera always followed the teacher. During the first year of the study, each class lesson was videotaped. During the second year, six lessons were videotaped twice at the beginning, twice in the middle, and twice at the end of the unit. The analysis here draws upon all of these sources but focuses on the interviews and SFL analyses of the videotaped instruction.

Video-stimulated interview collection and analysis

During each interview, Ms. Gold viewed the same episodes using software that enabled her to view a transcript of the classroom conversation and the video simultaneously. The interviewer (JS) interrupted each episode at points immediately preceding assistance was rendered to students by Ms. Gold. At each point, the interviewer asked: (a) At this point, what do you believe the student(s) in the clip were thinking?, and (b) Today, what would you do to assist this student? Ms. Gold was interviewed three times during the second year, twice before she began teaching the unit and once after instruction was completed.

This analysis focuses on Ms. Gold's reactions to one of five video episodes viewed in each interview. The student in the clip understood slope as a repeating pattern of steps, for example, rise 4 and run 1 *between* points on a line. This student's understanding was challenged with a fraction: rise 2 and run ½ . This did not appear on the surface to fit the expected rise 4, run 1 pattern (although it too is a 4:1 ratio). Ms. Gold attempts to help the student understand using several different representations. The student understands when Ms. Gold asks her to measure to find the ratio *within* the sides of a single 1 ½ × 6 rectangle cutout (6 is four 1 ½ units).

The interviews were coded for themes that the teacher discussed when viewing this video episode each time. These thematic codes were used to generate a collection of videoclips in Transana 1 from both years of classroom videotape.

SFL Analyses

A longitudinal collection of clips focusing on between and within ratios were selected. SFL was used to separately code and then coordinate three different aspects of meaning in context: the disciplinary content, the relationships among people, and the continuity of meaning over time. Each of these three categories have multiple aspects that are coded linguistically to build a coherent picture of the meanings Ms. Gold built over time through her interaction with her students. The analysis of the disciplinary content are called ideational and includes coding of different types of processes (verbs), nouns, circumstances, and logical connections. The relationship analysis is called the interpersonal analyses and includes coding of the modality, intonation, mood, evaluation and attitude. The continuity is called the textual analyses and includes coding cohesion, conjunction, clause-combining, thematic development, and nominalization. These three aspects build a coordinated detailed picture of the meaning the teacher is making in and across her interactions with students.

Results and Discussion

Preliminary results illustrate that despite knowing about between and within strategies beforehand, Ms. Gold had to go through a process to come to understand that the rectangle cutouts were pivotal for connecting students understanding of the ratio, and interpreting between and within ratio strategies as different. In addition, Ms. Gold's learning continued across both years of instruction. In the first year, she appeared to be frustrated and focused on students' ideas about ratios and equivalent fractions. Later in the first year, and during the second year Ms. Gold can be characterized as encouraging and interested in how students used between and within ratio ideas to explain translations among the representations (Lesh *et al.*, 2003). Results in the paper will trace this evolution linguistically across the two years focusing on how the PCK developed through changes in the content (equivalent fractions to between and within ratios), mood (frustrated to interested), and continuity (topics to threaded discussions of between and within ratios in different representations) of the classroom discussions. Understanding this process is essential to validating that PCK does emerge from teaching, and uncovering the ways of talking that could help build the PCK that help teachers navigate the sea of student ideas.

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REFORMING MATHEMATICS TEACHING AND LEARNING IN ELEMENTARY SCHOOLS: TEACHER LEARNING, STUDENT LEARNING, AND THE ROLE OF COMMUNITIES OF PRACTICE

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The Boston Public Schools is engaged in a system-wide effort to strengthen mathematics teaching and learning. This short oral report uses the communities of practice framework to examine teacher collaboration and learning within schools across the district. Preliminary findings suggest that communities of practice that include engagement, imagination, and alignment contribute to communities of practice that support teacher learning.

Objectives and Purposes

The Boston Public Schools is currently engaged in a system-wide effort to strengthen mathematics teaching and learning K-12. This effort includes the adoption of standards-based curricula, ongoing professional development for teachers and administrators; formative assessments district-wide; and school-based support from mathematics coaches. In addition, mathematics leadership development is underway in every elementary school. These efforts, supported by an NSF-funded Urban System Initiative project and an NSF-funded Teacher Retention and Renewal project focusing on *leading while learning*, reflect many of the research findings related to teacher learning and the process of mathematics education reform (e.g., Ball, 1997; Ball, 2002; Louks-Horsely, 1997; Schifter, 2001). It is becoming increasingly clear that mathematics education reform does not take place one teacher at a time. Rather, it is of critical importance that schools have in place structures that support teacher collaboration and learning. We are finding that the *communities of practice* framework is a useful tool for examining how mathematics teaching and learning is strengthened within the context of these structures that support teacher collaboration.

Theoretical Framework

According to Wenger (1998), communities of practice have three dimensions that give them coherence: (1) participants are engaged in actions whose meanings they negotiate with one another over time; (2) they share a joint enterprise through which they create relations of mutual accountability; and (3) they develop a repertoire, or a shared set of resources that serve as internal reference points. As Wenger (1998) notes, communities of practice are not in and of themselves beneficial or harmful forces in our lives; but they are nevertheless significant. Wenger suggests the following three modes of belonging fuel the development of a community of learners: *engagement*, where individuals are actively involved in a mutual process of negotiating meaning, forming trajectories that together unfold into histories of practice; *imagination*, in which participants disengage from the time and space of their regular endeavors to form new images of the possible; and *alignment*, in which activities and energy are coordinated to contribute to broader enterprises in a coherent fashion. All of these need to be in the proper balance if they are to create a rich context for learning. Our goal is to use this framework, focusing on the three modes of belonging, to examine teacher collaboration and opportunities for teacher learning.

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Data and Methods

In our research, case study methodologies are being employed to investigate interactions that take place during opportunities for teacher collaboration using the context of school-based and cross-school-based classroom visits, where teams of teachers and their mathematics coaches visit designated classrooms in order to observe mathematics lessons, with structured opportunities before and after these visits to discuss the mathematics at play in a lesson and how students engaged in that mathematics. This context is one that provides an opportunity to focus on teacher learning through a focus on student learning. Data sources are qualitative, using an interpretative approach, assuming that “interpretation is an act of imagination and logic” that entails “perceiving importance, order, and form in what one is learning” (Peshkin, 2000, p.9). Data sources include formal and informal interviews with teachers and mathematics coaches along with any accompanying correspondence; observations of previsit and postvisit discussions as well as observations of the classrooms visited; and artifacts designed to support mathematics teaching and learning within the context of the classroom visit (e.g., the elementary math curriculum, district mathematics assessments, the scope and sequence pacing guide). These data were collected using regularly maintained field notes. Any ideas, actions, and interpretations that appeared significant were recorded.

Results and Implications and the Goals of PME-NA

Preliminary results from the examination and analysis of cases of teacher collaboration during structured visits to each other’s classrooms with a focus on the examination of student learning suggest the three modes of belonging—engagement, imagination, and alignment—contribute to the creation of communities of practice that support teachers’ opportunities to learn. These preliminary results, because they allow us to reflect on the nature and quality of these modes of belonging, allow us to consider how communities of practice that support teacher learning might be constituted and supported. What is learned from these findings has implications for how opportunities for teacher collaboration and learning, grounded in reflection on student learning, are structured within schools and districts.

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DEVELOPING KNOWLEDGE AND BELIEFS FOR TEACHING: FOCUSING ON CHILDREN'S MATHEMATICAL THINKING

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Preservice elementary teachers' mathematical knowledge and beliefs about teaching and the learning of mathematics can be developed by focusing on how children learn and think about mathematics in content courses. A study using a Likert survey, with control and treatment groups, found significant differences in prospective teachers' beliefs and knowledge using an approach that focused on children's mathematical thinking.

Preservice elementary teachers' mathematical knowledge and beliefs about teaching and the learning of mathematics can be developed by focusing on how children learn and think about mathematics in content courses. This mathematical knowledge entails an understanding of the conceptual nature of procedures such as standard algorithms—understanding why they work, making sense of children's unique self generated algorithms for computation, and having the mathematical knowledge to explain procedures and concepts (Feikes & Schwingendorf, 2004). Teachers' beliefs about how children learn mathematics and how to teach children mathematics impacts teaching and might also be more fully developed by focusing on how children learn mathematics. A goal of the NSF supported (DUE 0341217) Connecting Mathematics for Elementary Teachers (CMET) project is to develop the beliefs and knowledge of preservice teachers in mathematics content courses. This approach is unique in that these are typically freshmen level mathematical content courses rather than professional development with practicing teachers (Hill, Rowan & Ball, 2005).

CMET attempts to help preservice elementary teachers connect the mathematics they are learning in content courses with how children learn and think about mathematics thus tying research on children's learning of mathematics with practice. To this end, a supplement was developed that parallels the typical mathematics content course topics. The intent in helping preservice teachers make these types of connections is that they will both improve their own understanding of mathematics and eventually improve their future teaching of mathematics to children. The CMET materials primarily consist of descriptions, written for prospective elementary teachers, on how children think about, misunderstand, and come to understand mathematics. These descriptions are based on current research, and some of the connections include: how children come to know number, addition as a counting activity, and the importance of concept image in understanding geometry.

Methods

Evaluation of the CMET project is ongoing and considers multiple areas besides the focus of this paper including: teacher self-efficacy, teaching, and parents' use of these materials. For the study, 168 Likert survey questions were developed to correspond with the typical content in the mathematical content courses for elementary teachers. An analysis of the most popular textbooks for these courses was done to determine the content of CMET and the survey questions. The Likert questions were developed by two team members and two different members suggested revisions and verified the reversed worded questions. The control group

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consisted of 301 students who were given the survey the semester prior to the use of the CMET materials. The same survey was given to 249 students who had used CMET materials. In addition, instructors completed on-line surveys and students were interviewed. Because each university in the study organized the mathematics content courses differently, a separate questionnaire from the 168 beginning questions was developed for each course at each institution. The surveys ranged from 33-44 questions depending upon the content being taught in each course. The CMET materials were piloted at five sites, all in the Midwest. Control group data was obtained from three sites; all sites provided treatment data.

Results

The following preliminary analysis compares the data from one course with the same course the previous semester (treatment versus control). Later analyses will aggregate the knowledge data across institutions and make cross comparisons with results from the beliefs, self-efficacy, instructor, and interview data. The Likert items possible responses ranged from Strongly Agree to Strongly Disagree, responses were given numerical values accordingly from 5 to 1. The **negatively worded questions are in bold** and the scores have been reversed for these questions. For all reported data higher scores up to 5 would be closest to our theoretical position or may be an indication of the beliefs and knowledge that we believe are most important. Questions that have significant differences between the control and treatment groups using a simple t-test are presented in the following table.

Question	Control Group			Treatment Group		
	Mean	SD	(n)	Mean	SD	n
If children can count, they understand the concept of number.	3.41	1.125	(56)	4.14	.363	(14)
Children best learn the addition facts through extensive drill and rote memorization.	2.75	1.066	(56)	3.57	.938	(14)
Children do not need to understand the mathematics behind the standard algorithms.	3.46	.934	(56)	4.00	.555	(14)
To find the percent of a percent, one can add the percents.	2.91	.880	(56)	3.57	.852	(14)
Children who cannot divide can find the mean.	2.27	1.086	(44)	2.93	.829	(16)

Significant at p =.05

Discussion and Conclusion

These limited results suggest that prospective elementary teachers can develop the mathematical knowledge and beliefs for teaching by focusing on how children learn and think about mathematics in content courses for elementary teachers. Other analyses indicate that using knowledge of children's mathematical thinking may also influence students' self-efficacy. More significantly, this approach may also improve students' future teaching of mathematics to children.

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UNIVERSITY MATHEMATICS PROFESSORS' PERCEPTIONS OF PRE-SERVICE SECONDARY MATHEMATICS CONTENT PREPARATION

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The purpose of this study was to better understand both the beliefs and perceptions of mathematics professors who teach content courses populated in part by pre-service secondary mathematics teachers (PSMTs). Interviews revealed that the professors deemed strong content knowledge necessary for teaching. However, the professors did not envision a role for themselves to help PSMTs value this training for their teaching.

Researchers and policy makers (e.g., Ball, 2003; Conference Board of the Mathematical Science, 2001) are giving increased attention to the content education of pre-service secondary mathematic teachers (PSMTs) and the critical role it plays in their development as effective teachers. The purpose of this study was to better understand both the beliefs and perceptions of university mathematics professors who teach content courses populated in part by PSMTs. In particular, we sought to determine the mathematical skills, understandings and dispositions that professors believed necessary for PSMTs and their views on how PSMTs' mathematics content preparation supported the PSMTs' future classroom work.

Previous research, conducted at the same university, investigated the role PSMTs ascribed to their advanced mathematical coursework (Staples & Hodge, 2006). Despite the fact that most PSMTs experienced the same professors and coursework, and that most professed to value this coursework, the PSMTs' perceptions of the role their mathematics coursework played in their student teaching varied greatly. Thus, we were also interested in comparing the mathematicians' perceptions to those of the PSMTs.

Data sources and modes of inquiry

Seven mathematics professors at a large, Midwestern university participated in the study. The professors selected had all (a) taught one or more upper-division mathematics courses in which pre-service teachers typically are enrolled within the past two years and (b) taught for more than one year at this university. Each participant took part in an hour-long semi-structured interview with one or two of the researchers to help us gain insight into the mathematicians' perceptions regarding appropriate mathematics preparation for future secondary mathematics teachers. These interviews were audiotaped.

Data analysis was conducted using standard methods of qualitative research. We coded all interviews to bring out emergent themes in the professors' perspectives across the interviews. To ensure reliability, two researchers independently coded each interview. Coding discrepancies were resolved through group discussions, and we revisited the tapes when there were questions about the interpretation of an interview. This analysis allowed us to develop each professor's stance in relation to our research questions as well as the overall stance of the professors, the latter of which is in the focus of this paper.

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Results

In general, mathematicians felt that a strong background in mathematics as well as the ability to use this preparation was necessary for teaching secondary mathematics. The professors seemed to embody similar visions of the role of a secondary mathematics teacher, viewing it as requiring the skills of proof, problem solving, and abstraction. However, 3 of the 7 professors reported uncertainty regarding the mathematical demands and specific tasks required of mathematics teachers in the secondary schools. Although the professors' understandings of the secondary schools varied, their perceptions of the mathematics coursework required for PSMTs were similar.

The mathematicians reported that the preparation for undergraduate mathematics majors was equally appropriate for future secondary instructors, and the program at their university was designed to ensure that PSMTs developed a broad base of mathematical knowledge, comprising essential components such as geometry, algebra (linear and abstract) and statistics. They felt that courses not required for a mathematics major, such as history of mathematics, physics and non-Euclidean geometry, could also contribute positively to PSMTs' teaching, but all were reluctant to replace courses in the current curriculum. These same "elective" courses had been noted by PSMTs as particularly valuable for their education and future work (Staples & Hodge, 2006).

Most professors also agreed that PSMTs should value and use their higher-level mathematics coursework in their teaching. They also felt flexible thinking was critical for good teaching. However, some of the professors did not view themselves as having a role in helping PSMTs discover this value of mathematics for teaching or develop this flexibility.

Implications and conclusions

Prior studies (Staples & Hodge, 2006; Kehle et al., 2005; Goulding et al., 2003) have demonstrated that many PSMTs do not inherently value their higher-level mathematics education. If we assume PSMTs should place value on their extensive mathematics coursework, it begs the question as to whose responsibility it is to help them understand this value. Perhaps mathematics and mathematics education departments might collaborate to create course offerings that help PSMTs make explicit connections between higher-level mathematics and that which they will be teaching. An awareness of one another's goals and contributions may help these departments collaborate to best meet the needs of PSMTs.

As mathematics educators and researchers continue to strive to improve the quality of PSMTs' undergraduate education, continued attention to its mathematical components is needed. By further examining the facets of knowledge mathematics professors deem useful for PSMTs, mathematics educators may be able to design courses that develop these facets simultaneously or ensure that content courses are meeting more of PSMTs' educational needs. Mathematics professors who share responsibility for the education of PSMTs might familiarize themselves with the unique requirements, constraints, and challenges of teaching in secondary schools. Once this familiarity is obtained, professors may also be able to recognize that their teaching can make a difference in the quality of secondary mathematic teachers who are in a sense "keepers of their own field [mathematics]".

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A SEMINAR ON TEACHING COLLEGE ALGEBRA: UNDERSTANDING THE PEDAGOGICAL EXPERIENCES OF PRE-SERVICE SECONDARY MATHEAMTICS TEACHERS

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Researchers are debating how pre-service secondary mathematics teachers (PSMTs) can best be provided with experiences that will prepare them to teach mathematics at the secondary level (Floden & Meniketti, 2005). A course which offers PSMTs the unique experience of teaching a college algebra course in conjunction with a tri-weekly seminar has the potential to address some of these issues. A case study investigates the potential mathematical and pedagogical experiences for PSMTs from the novel opportunities offered in this course. Data collection will include multiple modes of inquiry: (1) questionnaires, (2) interviews, and (3) field notes.

The undergraduate education of secondary mathematics teachers, both their education and mathematics coursework, is under examination (e.g., Conference Board of the Mathematical Sciences, 2001; Darling-Hammond, 2000; Goldhaber & Brewer, 2000, 2001; Monk, 1994; National Research Council, 2001). Specifically, the education coursework is intended to educate teachers in pedagogy (Darling-Hammond, 2000), and the higher-level mathematics coursework is intended to help teachers learn the mathematics deemed necessary for teaching secondary mathematics (Floden & Meniketti, 2005). However, some studies have concluded that the separation of pre-service secondary mathematics teachers' (PSMTs') pedagogical and content instruction hinders their development into successful secondary mathematics teachers (e.g., Ball & Bass, 2000; Ball, Lubienski & Mewborn, 2001; Ma, 1999; Shulman, 1986). Furthermore, from a situative perspective, this separation of content and pedagogy limits the authentic experiences teachers are provided in their undergraduate education (Wenger, 1998). Currently researchers are debating how PSMTs can best be provided with experiences (e.g., mathematics coursework, education coursework, field experiences) that will help them to become better prepared, mathematically and pedagogically, to teach mathematics at the secondary level (Floden & Meniketti, 2005).

There exists a course at a large, Midwestern university that may have the potential to address the inadequacies in PSMTs' educational development. This course offers PSMTs the unique experience of teaching a college algebra and trigonometry course (secondary mathematics material) that is supported by a mathematics faculty member in the form of a tri-weekly seminar that runs concurrently with their teaching of the course. In this short oral presentation I will reveal preliminary results from an in-depth case study regarding changes in PSMTs' participation in the seminar course related to the cognitively defined construct of pedagogical content knowledge, with respect to the content basis for the course (algebra and trigonometry).

These results will be obtained from multiple modes of inquiry: (1) questionnaires, (2) interviews with the students and the course instructor, and (3) field notes from regular observations of the seminar course. Data from these sources will be analyzed using processes of coding to reveal emergent themes in participation that arose related to the construct of pedagogical content knowledge.

This short oral presentation will serve as a chance for the mathematics education research community to begin to understand the atypical mathematical and pedagogical learning opportunities for pre-service secondary mathematics teachers (PSMTs) that are taking place in a seminar course on teaching college algebra at a large, Midwestern university. It is important that the findings from this research are shared with the mathematics education community, since the results could shed light on nuanced ideas relating to the currently debated undergraduate education of pre-service secondary mathematics teacher.

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FACTORS EFFECTING MIDDLE SCHOOL TEACHERS' CHOICE OF AUTHORITATIVE DISCOURSE IN TEACHING MATHEMATICS

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This research considers factors influencing teachers' mathematical compass for deciding on mathematical directions in classroom teaching. Particularly examined in this research are the factors that create a compelling draw for mathematics teachers to adhere to the more traditional features of textbook-prescribed curriculum. We further examine how deferral to the textbook might be transformed through collaborative professional teacher development.

Authoritative and internally persuasive discourse

Our interpretive lens is framed by the theoretical assertions of Bakhtin. According to Bakhtin (1935/1981) the negotiation between self and other, between teacher and curriculum, teacher and student, teacher and policy, and so forth, is one of dialogue between those discourses that are *authoritative* and those that are *internally persuasive*. Authoritative discourses, according to Bakhtin are dogmatic discourses eliciting adherence in often a subversive and compelling way. The adherence to authoritative discourses sees teachers' choices simply parroted and minimizes teachers' personal expression and decision making.

In contrast, internally persuasive discourses are, according to Bakhtin (1935/1981), conceptually reflective of autonomous thought and action. Internally persuasive discourse represents a dialectic relationship between 'our' discourse and that of others. Developing an internally persuasive discourse is a process of distinguishing between our own and someone else's discourse – it is a process of developing an individual consciousness, of 'ideological becoming,' to use Bakhtin's term. In our study we (1) identify factors that affect the acquiescence to an authoritative discourse, and (2) examine other factors that might influence a more autonomous internally persuasive discourse.

Our conceptual framework of 'discourse' draws from Gee (1999) who describes "D"iscourse (Gee's uppercase of "D") as "socially accepted associations among ways of using language, of thinking, valuing acting, and interacting, in the "right" places and at the "right times with the "right objects" (p. 17). In contrast, "d"iscourse (lower case "d") is restricted, according to Gee, to "languages in use or stretches of language (like conversations or stories)" (p. 17). We examine asynchronous (i.e. the reading of a text or context) interpretation of discourse in our examination of the mathematics textbook as an authoritative discourse (consistent with Gee's "D"iscourse) (Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005).

Method

This research was conducted in a single academic year, in two fifth grade classrooms. Data in the form of taped transcribed recordings and researcher field notes was collected during in-class observations of mathematics classes in a variety of mathematics strands: numeracy, algebra, measurement and data management and probability.

Data was also collected from 4 focus groups held throughout the year, which included teachers and researchers. Our subsequent content analysis adhered to the "stage model of qualitative content analysis" defined by Berg (2004, p. 286). This model articulates a process that begins with the determination of "sociological constructs" (p. 286) as preliminary categories or themes emerging directly from the analysis of the data. The data is then sorted according to these categories or themes. The emergent categories or themes are then analyzed according to relevant theories or related research with the intent of establishing explanatory potential. These themes are listed below as factors that influence authoritative discourse.

Factors influencing authoritative discourse

Our analysis of the data suggests that there are potentially five different factors influencing the extent to which a teacher's mathematical compass defers to the textbook as an authoritative discourse. These are: (1) mathematics textbook as a privileged discourse, (2) curricular demands and time constraints, (3) parental pressures, (4) teachers' mathematical identities, and (5) students' privileging of mathematical discourses.

Developing a mathematical 'compass'

Some changes in the pedagogical practices of the classroom teachers were observed. We found that the interaction and collaboration of the research project team (teachers and researchers) encouraged the teachers to rethink some of their mathematics teaching and their own mathematical experiences that may be re-enacted in their own classrooms. Collaborative professional teacher development seemed to amplify internally persuasive discourse however, the teachers' mathematical compasses continued be off course, so to speak. Consequently, their teaching decisions and pedagogical directions are mapped by the sanctioned textbook as the ultimate authority.

Kang and Kilpatrick (1992) suggest that "the effective use of mathematics textbooks . . . *depends* [our emphasis] on the mathematics teacher's epistemological vigilance" (p. 6). Epistemological vigilance makes unequivocal requisite of a deep understanding of mathematics and pedagogy. Our findings suggest that the potential to enable more autonomous internally persuasive discourses through collaborative professional teacher development serves to unravel persistent authoritative discourse whilst simultaneously facilitating a deeper understanding of mathematics for teachers.

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THROUGH THE LOOKING GLASS: PERSPECTIVES ON THE EVOLUTION OF LEARNING COMMUNITIES THROUGH THE LENS OF INTERSUBJECTIVITY

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This paper illustrates the application of a theoretical framework that has as its aim to understand mathematics teachers' professional interactions so as to better support them. The interactions are analyzed under the framework of intersubjectivity as defined by Steffe and Thompson (2000) and Thompson (2000). For this paper, we illustrate the application of our intersubjectivity framework by focusing on the events in one Reflecting on Practice (RPS) session involving four high school teachers who are trying to re-conceptualize their teaching of the idea of function.

Theoretical Framework

Intersubjectivity is a state of dynamic equilibrium among participants in a conversation in which each person sees no reason to believe that others think differently than he or she presumes they do (Thompson, 2000). The idea of intersubjectivity, then, is not about consensus or agreement.

Instead, it is a pattern of interactions among participants' intended and attributed meanings that, for the moment, do not alert them to rethink either their own meanings or their understandings of others' meanings.

A state of intersubjectivity can be sent into mild ("ruffled") or severe ("punctured") disequilibrium in several ways, two of which are: (1) one or more participants detects that someone does not mean or believe what they had presumed she means or believes; (2) participants detect that the interactions of intended and construed meanings are incompatible, but they cannot locate the source of incompatibility. Equilibrium can be re-established in several ways, including: (1) individual teachers rethink their own meanings so that they become compatible with their new understandings of others' meanings; (2) individual teachers rethink their understandings of others' meanings so that they are more compatible with their own; (3) both (1) and (2); (4) individual teachers accept the disequilibrium as a persistent state.

Method

The RPS session discussed in this paper was one of 15 sessions that took place over the Fall 2005 semester of the Teachers Promoting Change Collaboratively (TPC2) project at ASU. Each session was guided by a facilitator and was videotaped. Various members of the research project then analyze the videos. We use our framework to analyze RPSs in three interleaved phases: (1) understanding each teachers' basis in meaning; (2) examining interactions among meanings and their repercussions; (3) considering how we might perturb the group's intersubjectivity to move it to states that are more closely aligned with our goals. At the first phase of analysis the discourse of the sessions is used to create a map of teachers' meanings and the interplay among teachers' meanings. Statements made by the teachers, their reactions to other teachers' statements and written artifacts are used to construct our hypotheses about their meanings. The second phase focuses on the interactions that dynamically reveal the meanings behind statements teachers

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make and reveal the assumptions held by other members of the RPS. Finally, discourse surrounding perturbations of the intersubjectivity reveal incompatibilities in teachers' understandings of each others' meanings; these perturbations also create an opportunities for teachers in the RPSs to re-evaluate their held meanings in relation to their images of how others understand them.

Example (Teachers' 8th RPS)

The teachers, at the behest of a facilitator, discussed a research article on the concept of function (Carlson & Oehrtman, 2005; hereafter "C&O"). The teachers appeared at the outset to be in a state of intersubjectivity with regard to the meaning of "being a function". A ruffle in the group's intersubjectivity began with a teacher, Kathy, expressing concern with her understanding of what that concept of function entailed. The other teachers in the RPS attempted to reassure Kathy that she did in fact understand the function concept because she emphasized the "one input, one output" relationship. That is, their understanding of Kathy's understanding was quite compatible with their own and their understanding of C&O. Kathy, however, was aware of an incompatibility, but she could not locate its source. The site of the incompatibility was C&O's statement that x+y=4 does not represent a function, whereas Kathy claimed that it did represent a function—any time you substitute a value for x, you get precisely one value for y. The other teachers said that this is true, but "x+y=4" represents a function only if you rewrite it as "x+f(x)=4". Kathy objected, saying this was only a matter of notation, writing f(x) instead of y.

We, as observers, detected that Kathy did, indeed, have a meaning for function that was incompatible with the others' meanings. Her meaning for function entailed the restriction that one only substitutes values for "x", no matter the proposition's form. With this restriction, "x+y=4" does indeed represent a function. The other teachers presumed that, without an agreement on what variables are defined implicitly as functions of other variables, the statement "x+y=4" is ambiguous—either variable could be defined in terms of the other. But they presumed that Kathy understood this, when she did not. She presumed they held the same restriction on "x" as she did, and they did not. The group's intersubjectivity was ruffled by what the group took as a puzzle, Kathy's claim of not understanding the concept of function when the others thought she did. It was later punctured when she claimed to not understand the idea of covariation when, again, based on her description of covariation they thought she did. Our presentation will explicate these interchanges and their effects on the group's lack of progress in regard to drawing implications for classroom instruction. It will also point out how the facilitator's insensitivity to the interpretations at play disable him from intervening productively to help teachers renegotiate their meanings. Our presentation will also provide an example of how, in later RPSs, the facilitator's increased sensitivity allowed him to intervene in subsequent sessions in ways that made renegotiation of meanings an explicit topic of discussion.

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DEVELOPING REFLECTIVE PRACTITIONERS: A CASE STUDY OF PRESERVICE ELEMENTARY MATH TEACHERS' LESSON STUDY

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Preservice teachers need the opportunity to engage in learning that bridges theory and practice. With the Japanese lesson study, we show how teachers made sense of the theory of the five strands of mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001). The study found that through collaborative planning, teaching, assessing student learning, and reflecting, the teachers integrated the theory into their practice.

Perspectives

In order for teachers to successfully engage in the profession of teaching, two types of broad knowledge are required, namely theoretical and practical. The two may seem disconnected to those just starting to learn the craft of teaching. Rarely do preservice teachers gain the opportunity to engage in deep learning that bridges theory and practice. Yet, most seasoned teachers would agree that the melding of the two is crucial. Rarely do preservice teachers get the chance to participate in an inquiry process that allows them to work closely with others while focusing on the enhancement of their own skills and understanding of subject matter. It is important that teacher education programs provide such meaningful learning experiences for their preservice teachers. A primary focus should be on the development of reflective practitioners (Dewey, 1904).

Lesson study is one such way to help create an inquiry stance in teachers. Lesson study is a form of collaboration-based teacher professional development that originated in Japan (Lewis and Tsuchida, 1998). In lesson study, teachers collaboratively (1) set a goal for their student learning, (2) plan a lesson, (3) teach a lesson while being observed, and (4) discuss student learning with the data collected during the lesson. This professional development tool ties both theoretical and practical learning together in a most authentic way - *through teaching*.

Theoretical Framework

Understanding student thinking is what binds different parts of the lesson study process. By experiencing how students learn mathematics, teachers learn how to bring about mathematical proficiency in their students. *Adding It Up* (2001) proposes five strands of mathematical proficiency: *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning*, and *productive disposition* that "are interwoven and interdependent[1]" (p. 116). These strands may sound important in theory, but preservice teachers lack experiences to readily see how these aspects are expressed in their teaching and in their students' learning. Lesson study provides experiences that support the building of such connections. The purpose of this case study is to describe how four elementary preservice teachers in a teacher education program make sense of the theory and integrate the five strands into their practice through participation in lesson study.

Methods

The four focus preservice teachers were part of a group of twenty, all enrolled in a teacher education program at a major research institution in the western United States. The data were

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collected in the elementary math methods courses that were structured around lesson study. Several forms of data were collected: (1) field notes of the lesson study planning meetings, (2) iterations of teachers' collaborative lesson plans, (3) materials (curricula and worksheets) used to plan lessons, (4) student artifacts (pre- and post-assessments), and (5) teachers' reflections.

Results and Discussion

By following the four elementary preservice teachers through the process of lesson study on the topic of multiplication, the value of this professional development tool was revealed in four vital components: planning, teaching, assessing student learning, and reflecting. We found that teachers integrated the two types of knowledge (theory and practice). Many changes in teacher beliefs and practice were identified as being the result of lesson study participation. Broadly speaking, lesson study enabled teachers to view the mathematical proficiencies as interconnected. The teacher discourse that resulted during the lesson debrief was of great value, as was the examination of student work.

In terms of *conceptual understanding*, the teachers displayed several grand shifts in thinking. First, teachers experienced a transformation in their conceptual understanding of multiplication. Second, lesson study showed teachers the value of using student misconceptions as a window into students' thinking to plan effective lessons. Third, the process allowed them to grapple with the conflict between procedure and concept, ultimately resolving the goal of the lesson to be conceptual understanding. Finally, lesson study underscored the need to use assessment to inform future instruction, rather than as a teacher evaluation tool.

Teachers' conceptions of *procedural fluency* also shifted over time. This occurred in two ways: (1) Teachers more fully understood the importance of the unit plan and how lessons should build from conceptual understanding to procedural fluency; and (2) Teachers came to understand the reasons for using a variety of strategies to solve a multiplication problem.

Teachers' ideas of *strategic competence* and *adaptive reasoning* also changed as a result of lesson study. Although the teachers had initially planned to have a problem-based lesson, they truly appreciated its value at the end of the process. Teachers saw their role as the facilitator in a student-centered classroom. Lesson study also highlighted the importance of using the lesson time to identify trends in student thinking. Thus, teachers came to value student discourse after witnessing the students identify efficient strategies as a community.

In regards to *productive disposition*, teachers started to acknowledge the power of student motivation in the facilitation of learning. They also recognized the need for students to be placed in an environment of discomfort for true learning to occur.

Lesson study helped the teachers to see that students are active participants in their own learning. It also enabled them to see teaching as a highly complex process requiring thought, planning, and reflection.

Endnotes

1. Conceptual understanding speaks to the ability of students to go beyond memorization of facts and deeply connect to the underlying mathematical concept. Procedural fluency is the efficient and accurate use of a specific algorithm, while strategic competence emphasizes problem-solving. Adaptive reasoning is used when identifying the utility of a particular approach to solving a problem. Finally, a productive disposition, or healthy attitude, is necessary in order to see math as useful.

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TEACHERS AS LEARNERS: A STUDY OF LEARNING TO TEACH MATHEMATICS FROM THE PERSPECTIVE OF THE BEGINNING TEACHER

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Studies on beginning teachers have focused on induction programs or districts, ignored the role of subject matter in learning to teach, and disregarded the voice of novice teachers. This paper adopts the perspective of two beginning teachers to explore the challenges of learning to teach mathematics and how its complexity makes subject-matter specific induction desirable and essential.

Purpose/Theoretical Framework

In spite of national concern for the teaching of high quality mathematics to all students, we understand very little of how to support the on-going development of content knowledge for beginning teachers. When studies have focused on this issue, they have assumed the perspective of induction programs or districts and have not included the voice of novice teachers. Additionally, literature on beginning teachers often ignores the role of subject matter in the learning to teach experience (Britton, Raizen, Paine, & Huntley, 2000). This paper is an attempt to address these two issues by focusing on the cases of two beginning teachers and exploring the challenges of learning to teach mathematics and how its complexity makes subject-matter specific induction both desirable and essential.

Our analysis of the concerns of beginning teachers draws upon Lampert's (2001) model of the three-prong problem space. Lampert proposes that the work of teaching requires establishing and simultaneously maintaining three relationships: (1) the relationship between teacher and students, (2) the relationship between teacher and content, and (3) the relationship students have with the mathematical content. Learning to teach mathematics then requires more than effectively managing a classroom and delivering content knowledge. New mathematics teachers are faced with the enormous challenge of learning how to manage each of the points of the three-pronged problem space as well as the relationships between them.

Methods/ Data Sources

This project is part of a larger NSF-funded study of content-specific induction programs from across the United States. We selected six sites with content-rich induction for secondary teachers. Within each site, we chose six or more new teachers to interview and observe on 3-4 occasions during the 2003-04 and 2004-05 school years. Data collection methods included interviews with program directors, coaches and mentors, school principals, and new teachers; observations of program events and new teachers in their classrooms. Data analysis involved collective reading of data and emerging analytic memos. Based on these memos, two teacher's cases were selected for further analysis. For each teacher, a series of memos were written that sought to present their concerns, worries, strengths, and struggles. Through our analysis, we identified subject specific needs that were voiced by these two teachers and corroborated by voices of other new teachers in the larger study.

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Findings and Implications

The two beginning teachers, Helen and Ona, are, according to their mentors, “good teachers.” Each works in a different context and with the support of a well-regarded (but different) induction program. The conditions under which they work could be considered supportive, and each comes to teaching with strong content preparation in mathematics. Ona has a wealth of school- and district- based resources to support her; Helen works in a far more challenging school environment, but has a school-based mentor. In spite of these supports, each teacher has articulated and enacted the need for additional support around three areas: understanding students, their mathematical thinking, and the family/community contexts; understanding the role of the mathematics teacher; and constructing and using productive learning tasks. When mapped onto Lampert’s three-prong problem space, two of these areas, understanding role and the creation and use of tasks, focus on the same prong, managing the relationship between students and content. In this paper, we will elaborate upon these two areas.

Much of the support for beginning teachers offers advice for managing students (e.g. Wong & Wong, 1998). This advice is typically generic, implying that what is to be taught is not an important factor in thinking about student behavior in classrooms. However, we found that both Helen and Ona had significant concerns about their role in managing students’ relationship to content. For example, Ona wrestled with her role of teacher as content provider. She felt the need to have answers to student’s mathematical questions and to mediate students’ interactions with content, yet she also wanted to support students in their desire to wrestle with mathematical challenges. Mapping Ona’s concerns about her role on Lampert’s three-prong problem space illustrates (1) how generic advice about managing students is insufficient for supporting Ona in resolving her concerns and (2) how Ona’s concerns might be addressed by helping Ona reframe her understanding of her teaching role.

From Helen we learn how difficult it can be for new teachers to create lessons and tasks that “cover what they [students] need to know” to perform well on standardized assessments, while also providing students with opportunities to engage with mathematical ideas and create important connections between them. In the struggle to provide appropriate learning opportunities to support her students’ conceptual understandings of slope, Helen, like Ona, finds herself trying to manage the relationship to content her students are negotiating. This work is particularly challenging for any teacher, and new teachers in particular, because it exists at the intersection of students, contexts, content, and the teacher role.

No one would deny that beginning teachers face many challenges. However, frequently new teachers are provided with some basic classroom management tips and an orientation to their schools and then left to their own devices (Feiman-Nemser & Parker, 1993; Wang, 2001). This is particularly true for beginning teachers who have adequate classroom management skills and can keep students “under control” and out of the principal’s office. Our study demonstrates that even strong new teachers still have needs, and these needs are directly connected to mathematical content. We urge districts and induction programs to strongly consider the ways in which all new mathematics teachers could benefit from additional content-specific support.

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FRACTION MULTIPLICATION: TEACHER AND STUDENT UNDERSTANDING AND INTERPRETATION IN A REFORM-BASED CLASSROOM

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The data used for this poster is part of the CoSTAR (Coordinating students' and teachers' algebra reasoning) project. The fundamental goal of CoSTAR is to gain access to and analyze teachers' and students' understandings of shared classroom interactions, and the teaching and learning that results. In particular, the project coordinates analyses of taken-as-shared classroom problem-solving practices with individual teachers' and students' understandings of those practices. Our strategy is to videotape classroom interactions and to pursue the sense that teachers and students make of those interactions during subsequent videotaped interviews. Thus, the project examines the sense that students make of their opportunities to learn and teachers' sensitivity to the core learning issues for their students.

The poster presents an analysis of lesson, teacher interview, and student interview data on fraction multiplication from the *Bits and Pieces II* unit in the Connected Mathematics Project (CMP) materials. The notion of three-levels-of-units forms the theoretical framework for the study (Steffe, 1993, 2003). I look at the teacher's understanding of three-levels-of-units, her flexibility within the three-levels-of-units domain as demonstrated by her teaching of the multiplication of fractions section, her understanding of student learning, her ability to support student thinking with three-levels-of-units, as well as students' understanding and sense-making of her explanations. I used classroom videos, teacher-interview videos, and three sets of student-interview videos in this analysis.

The poster focuses on the teacher's use of the number line as a form of representation when teaching multiplication of fractions. It also focuses on some students' use of the number line in their solution of fraction multiplication problems. The poster will discuss a variety of interesting teaching scenarios, teacher-student interactions, and student learning around the topic of fraction multiplication.

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FROM WEAKNESS TO RICHNESS: THE CASE OF BRENDA

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Mathematical Proficiency as described in Adding It Up: Helping Children Learn Mathematics (National Research Council [NRC], 2001, p. 116) was used as a way to interpret Brenda's mathematical knowledge. Mathematical proficiency, according to the NRC, has five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Reflective thinking was viewed from Dewey's (1933) perspective. Reflective thinking occurs in two phases. First, a person recognizes a situation as problematic and then searches for solutions to the problem. The demand for a conclusion is the definitive characteristic of reflective thinking.

The data sources included artifacts from Brenda's content and methods coursework and student teaching, her reflections about teaching, university supervisor observations and notes, and a qualitative survey. Inductive analysis (Patton, 1990) was used to interpret the data.

Although other factors were found to affect Brenda's lesson design and teaching, her mathematical proficiency and ability to think reflectively are the focus of this poster. Even though Brenda lacks procedural fluency and strategic competence, she values and continually seeks to develop conceptual understanding, adaptive reasoning, and a productive disposition. Brenda values aspects of mathematical proficiency that allowed her to develop high quality, inquiry-based instruction. She believes that in order to know mathematics, one must see mathematics as a logically connected web of facts and concepts and the most important thing is to understand the connections. Once you do that, you can move around in the subject, use your own judgment, recover from errors, decide for yourself what needs to be done, and solve more difficult problems. In the long run, understanding the principles is also more efficient than memorizing formulas and recipes (survey).

Brenda used reflective thinking as she designed her lessons. She viewed learning mathematics as problematic, and searched for ways to make sense of mathematics. She was able to examine her own learning and apply her analysis to her lesson design. She used student feedback to view the content through students' eyes and improve her instruction.

This research suggests that some aspects of mathematical proficiency may be more important than others with respect to teachers' abilities to develop high quality mathematics lessons. Further research could provide insights into the relative importance of each strand of mathematical proficiency and reflective thinking as they relate to teachers' lesson planning.

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INTUITIVE PROBABILITY IN ACTION: A CASE IN ELEMENTARY NUMBER THEORY

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Studies on learning number theory have paid attention to students' understanding and recognition of primes (Zazkis & Campbell, 1996, Zazkis & Liljedahl, 2004). Researchers observed that students' possessed the belief that a very large composite number should be divisible by a small prime. For example, students concluded that a (large) number was prime after checking its divisibility by a trivial number of small primes. This report echoes an interest in identifying sources that influence students' use of prime numbers; however, a subjective probabilistic framework was used as a lens in addressing these issues of interest.

Representativeness is a heuristic that is used to determine the probability that a particular object (A) belongs to a given set (B). Tversky & Kahneman (1974) found that probabilities are evaluated by the degree to which A would resemble B. The probability that A originates from B, or that B generates A, is high when the resemblance is strong and low when the resemblance is weak.

One of the tasks, administered in a clinical interview setting, invited students to simplify the fraction 448188/586092. The representativeness heuristic was witnessed in students' choices of primes, as possible factors. The "stereotypical" list of primes includes 2,3,5,7,11. Tversky & Kahneman (1974) showed that using the representativeness heuristic to evaluate probability leads to insensitivity to prior probability of outcomes. Students were aware that there were more primes greater than 11, than those less than 11. However, this fact was not taken into consideration in an attempt to reduce the given fraction. Numbers like 13 and 17 did not conform to the students' image of the stereotypical primes and were not taken into consideration. A second bias of the representativeness heuristic, the illusion of validity, states that as redundant input continues the accuracy of prediction decreases while a simultaneous confidence about the prediction is gained. This bias was also witnessed in the report.

Using the framework of subjective probability provides further insight into participants' responses and into implicit reasoning that guides their decision making.

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PRESERVICE SECONDARY SCHOOL MATHEMATICS TEACHERS' KNOWLEDGE OF TRIGONOMETRY: COFUNCTIONS

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In Adding It Up, a major report from the National Research Council (NRC), the authors concluded that three major components of mathematics teachers' knowledge are necessary for effective mathematics teaching: knowledge of mathematics, knowledge of students, and knowledge of pedagogy (Kilpatrick, Swafford, & Findell, 2001). This poster reports results of a study of pre-service secondary school mathematics teachers' knowledge of trigonometry. The study took careful account of the accumulated data and theories of teacher knowledge that point to the complexity of knowing (Ball, Bass, & Hill, 2004; Ball, Lubienski, & Mewborn, 2001; Dossey, 1992; Even, 1990; Fennema & Franke, 1992; Glasersfeld, 1996; Hiebert et al. 1997; Hiebert & Carpenter, 1992; Koehler & Grouws, 1992; Leinhardt & Smith, 1985; Ma, 1999; Shulman, 1986, 1987).

In phase 1 of the study, 14 pre-service secondary school mathematics teachers (participants) at a large university in the Midwest of the United States completed two concept maps from emic and etic perspectives, two card-sorting activities, and a test of trigonometric knowledge (TTK). In phase 2, five of the 14 participants partook in two interviews. Results from the study indicated, as a group, the participants' knowledge of trigonometry was uneven and that several fundamental ideas of trigonometry were poorly understood. In particular, their knowledge of periodicity, radian measure, co-functions, reciprocal functions, 1-1 functions, inverse trigonometric functions, identities, and sinusoids lacked depth. The findings support a conclusion that pre-service teachers' knowledge of school mathematics may not be sufficiently robust to support meaningful instruction on some key trigonometric ideas.

The focus of this poster presentation is pre-service teachers' conceptions and conceptual organization of co-functions, and the interferences presented by other notions such as inverse trigonometric functions and reciprocal trigonometric functions. The aforementioned study revealed that, as a group, the participants' knowledge of co-functions was particularly limited. The concept maps and the interviews showed that the participants possessed weak understanding of the meaning of the prefix co in the following co-function pairs (sine – cosine; tangent – cotangent; secant – cosecant). For example, 10 of the 14 participants used connectives (linkages) such as inverse, reciprocal, and co-functions to relate the co-function pairs in the concept maps; indicating that they confused inverse, reciprocal, and co-function as equivalent ideas.

A focus on understanding co-functions is reasonable given the flexibility, versatility and adaptability that such understanding can afford in problem solving situations. Knowledge of co-functions is helpful in simplifying trigonometric expressions to yield equivalent, yet simpler, expressions that facilitate writing proofs and enhance the process of resolving problems. The participants' limited understanding of co-functions inhibited their flexibility in resolving trigonometric questions involving analysis of inverse trigonometric functions and their properties, triangle resolution, and proofs. Therefore, this poster raises questions about what can

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and should be done at both the high school level and at the post-secondary level to help pre-service teachers grapple with such fundamental notions as co-functions. The criticality of high school experience in trigonometry cannot be overemphasized because pre-service teachers' opportunity to learn fundamental trigonometric concepts occurs in high school.

TEACHING AS LEARNING: MATHEMATICS GRADUATE STUDENTS' DEVELOPMENT OF KNOWLEDGE OF STUDENT THINKING ABOUT LIMITS

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We examined how mathematics graduate student teaching assistants (TAs) gained knowledge of student thinking. Research on K-12 teacher professional development (PD) provides insights into teachers' practices and relationships of those practices to student learning. In particular, knowledge of student thinking shapes teachers' instruction (Franke et al, 1998). Improving teachers' knowledge of student thinking can improve student outcomes (Fennema et al., 1996). For some teachers, the ways they interact with students are "generative" of new knowledge in that they create opportunities to learn more about student thinking (Franke et al, 2001). We extend this K-12 work to the undergraduate level by examining college teachers' knowledge of student thinking about limits. For mathematics TAs, who receive minimal PD, "learning while teaching" may represent their main method of learning about student thinking. We studied two groups: current/former TAs with some experiences that routinely provided them with access to student thinking through facilitating collaborative group work (CL group) and those whose teaching experiences were more traditional (lecturing, presenting problems, answering questions; TR group). Findings indicate that differences in teaching experiences correspond to differences in knowledge of student thinking.

Each of the 18 current and former mathematics doctoral students was interviewed individually using tasks modeled after research on student thinking about limit. Participants solved the tasks and described strategies they believed students would use and difficulties they anticipated students might encounter. Analysis focused on cataloging knowledge of strategies and difficulties graduate students had for each task as well as cross-task analysis to more generally examine the depth and breath of their knowledge. We also compared participants' knowledge to findings from research on student thinking for the topics.

Knowledge of student thinking differed substantially between the two groups. Participants in the CL group described in detail many methods students might use. TR group participants were typically unable to describe more than one strategy (the one they used to solve the task). Both groups generated lists of potential student difficulties, however, the nature of those difficulties differed. For example, many in the TR group anticipated only procedural errors while CL participants anticipated those difficulties as well as many of the common misconceptions detailed in the research literature. Since the two groups of participants did not differ significantly in other ways, we attribute these differences to variation in types of teaching experiences they had. We conclude that the two groups differed in the extent to which their teaching experiences were generative of new knowledge of student thinking.

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A CASE STUDY OF A TEACHER'S EVOLVING PRACTICES IN SUPPORTING STUDENTS' MATHEMATICS AND LITERACY DEVELOPMENT

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Significant changes in secondary mathematics curricula involving more opportunities for students and teachers to mathematize situations through talk, texts, stories, pictures, charts and diagrams have arisen from the National Council of Teachers of Mathematics (NCTM) *Standards* (NCTM, 2000) and several curriculum projects funded by the National Science Foundation. These changes pose great challenges to secondary mathematics teachers who are generally underprepared to mediate the intersections between mathematics and literacy (Muth, 1993), and even greater challenges to teachers and students in urban settings, where achievement in literacy and mathematics often lags behind achievement of students in other settings (Schoenbach, Greenleaf, Cziko, & Hurwitz, 1999).

In this poster, I address the question: How do one high school mathematics teacher's practices evolve as she seeks to understand and support her students' mathematics and literacy development through reform-based mathematics curricular materials?

The theoretical perspective of sociocultural research on mathematics and literacy frames this interdisciplinary research. Recent research has attended to the complex intersections of adolescent learners, texts, and contexts (Hinchman & Young, 2001). Literacy has come to be seen as multifaceted, involving reading, writing, speaking, listening, and other performative acts—all taking place in certain social settings for certain purposes (Hicks, 1995/1996).

Like other domains of study, mathematics classes at the secondary level require teachers and students to use various kinds of literacies and to participate in various discourse communities specific to the domain (Hinchman & Young, 2001; Hinchman & Zalewski, 2000). Recent studies have used a sociocultural frame (Atweh, 1993; Borasi & Siegel, 2000; Lerman, 2001; Sturtevant, Duling, & Hall, 2001) since it accounts for aspects of learning mathematics in complex classroom contexts that a focus on thinking processes alone may not. Understanding how teachers' practices evolve as they strive to teach in ways that engage students in communicative practices is the overarching goal of this research study.

We are using the methodology of the multi-tiered teaching experiment (Lesh & Kelly, 1999), which allows us to collect and interpret data at the researcher level, the teacher level and at the student level. Our research team is comprised of university-based researchers in mathematics education and literacy education, mathematics teachers, and their school administrators, including both principals and other instructional leaders, in a mid-sized urban district in the northeastern United States.

This paper, specifically, draws on data collected as classroom observations, planning and de-briefing meetings, interviews, and bi-weekly study group meetings with the mathematics teachers at a high school. Our analysis yielded a story of one teacher's evolving practices in supporting her students' mathematics and literacy development. The story begins with this teacher recognizing that her students were not meeting the literacy demands of the textbook, and moves through iterations of this teacher's learning about literacy informing her practices, and her practices informing her literacy learning. I offer several illustrations of the evolving practices of

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this teacher: (a) “templates”, and (b) activity structure for engaging students in oral communication.

ALTERNATIVE TEACHING STRATEGIES' AFFECT ON INSERVICE SECONDARY MATHEMATICS TEACHERS' ATTITUDES AND BELIEFS ABOUT MATHEMATICS PEDAGOGY

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Research indicates that teachers who experience reform-based methods of mathematics instruction not only develop conceptual understanding but also effectively change their beliefs about the way mathematics is learned and taught (Chapman, 1999; Crespo, 2000; Quinn, 1997; Schoenfeld, 2000; Timmerman, 2003). Teachers who get exposure to learning mathematics through cooperative learning small group study not only form a complete understanding of the mathematics but also believe that cooperative learning is an effective mode of instruction (Quinn, 1997; Timmerman, 2003).

A group of sixteen inservice middle school (n=3) and secondary (n=13) mathematics teachers participated in a yearlong professional development program focused on mathematical problem solving (MPS). The integration of mathematical content and alternative forms of pedagogy were central to the professional development program. Data collected throughout the program included interviews, journal entries, class work, pre- and post-measures of baseline skills, pedagogical strategies and a problem solving.

We focus on the analysis of four questions from the pre- and post- survey results to determine several factors regarding teacher confidence, attitudes and beliefs about using cooperative learning groups. Findings suggest that mathematics professional development programs aiming to impact teachers' use of cooperative learning in the classroom must also address related issues of time management and availability of lesson materials to facilitate translation to instructional practice.

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TEACHER, KNOWLEDGE, AND MATHEMATICS TEACHING

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Introduction

Teachers need appropriate experiences and materials from which to build new models of instruction, learning and assessment. Researchers generally agree that teachers of Mathematics need adequate training and teaching experience in order for them to construct a deeper understanding of the mathematical concepts they are expected to teach. Additionally, various studies show the need to study how mathematical concepts are understood and used in day to day learning by students. (Carpenter & Lehrer, 1999; Schorr, Maher, & Davis, 1997; Janvier, 1996; Cobb, Wood, Yackel, & McNeal, 1993). Research studies have shown that the personal beliefs and level of mathematical knowledge that teachers of Mathematics possess strongly influence their method of instruction. (Ball, 1990). These knowledge and belief systems are generally acquired prior to actual classroom experience, and held through years of teaching. In this study we specifically focused on the role of content knowledge in helping middle grade teachers of Mathematics to develop standard-based activities, particularly problem solving activities, through the use of concept maps.

Objective of the study

The objectives of the study were as follows: (1) to examine the nature and scope of the mathematical knowledge of middle grade teachers of Mathematics; (2) to study the way they use their knowledge to develop reform –oriented instruction; and (3) to analyze how our findings might be used to help teachers to develop their professional skills.

Perspectives

The framework that was chosen to examine the role that the content knowledge of teachers plays in helping them to develop standard-based instruction is the Mathematics Teaching Cycle [MTC] (Simon, 1997). As a conceptual framework, the MTC "describes the relationships among teacher's knowledge, goals for students, anticipation of student learning, planning and interaction with students" (Simon, 1997, p. 76). Simon (1997) explains that changes in the learning trajectory are based on interactions with students, which impacts teacher's knowledge, thus impacting goals, plans, and/or hypothesis of the teacher in a cyclical fashion. According to Schon (1983), changes in teacher's knowledge impacting the hypothetical learning trajectory [HLT] might occur during a lesson, not just between lessons, particularly if the teacher is reflecting while teaching.

Methodology

Our experimental subjects were three middle grade mathematics teachers selected from one of the public schools in Atlanta. Before using them as case studies, we met with them for a week, from 9 a.m. to 3 p. m. daily, to discuss underlying concepts and skills, to map concepts and to identify performance standards and important mathematical ideas that were embedded within rich mathematical problems. Additionally, the teachers had two weeks of professional training from Atlanta Public School Mathematics Coordinator to help them understand the underlying

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concepts and skills, design concepts maps to illustrate their ideas, and execute reform oriented instruction. Furthermore, we analyzed the teachers' focused attention on their students' mathematical thinking for the purpose of uncovering how this is used in designing their lessons.

A total of four lessons were used for instruction during this project. The teachers used the national, state, and school standards to document the types of concepts represented in each lesson. After sharing their own ideas and representations during planning times, they then used these concepts in their own classrooms. During classroom implementation with the researchers present, teachers were encouraged to recognize and analyze students' interpretations and thoughts about the types of problems presented. Independently, the teachers reflected and revised their own concept maps and shared their new ideas and thoughts with us and their colleagues. The teachers (along with other participants not reported in this project) critically analyzed each other's mappings. This helped them to see how their colleagues developed mathematical ideas, and it enabled them to discuss their students' understanding of mathematical concepts and the implications of their findings on their teaching.

Interviews were conducted before and after instructions with the three teachers and six students (two from each class). The purpose of the interviews was to gain a deeper understanding into how the teachers' content knowledge influence them in designing standards-based instructions and utilizing concept maps and the students thinking about these lessons

Research Goals

We wanted to find answers to the following questions: (a) how do our chosen subjects (the three teachers) identify connected concepts and skills in a given problem when developing concept maps? And (b) how do they endeavor to teach their students mathematical concepts that are beyond what they already know? Accordingly, we focused on how the teachers understood the underlying concepts and skills found in selected problems and how they use concept maps to illustrate their ideas and design curriculum in order to execute reform-oriented instruction. We also examined how they use the feedback they get from their students to revise and refine their instruction.

Data Sources

The following were the data sources for the research:

1. The teachers' curriculum concept maps;
2. Transcripts from pre- and post- semi structured interviews of students and teachers;
3. The students' work on the individual classroom activities;
4. Notes of classroom activities;
5. Field notes taken while working with teachers during planning times;
6. Transcript of the teachers planning times; and
7. The reflection of the teachers on their work.

Our collection of data followed the model that Ball and Lampert (1999) used. By collecting multiple perspectives on classroom practice, namely those of the researcher, teachers, and students, and the perspective gained from audiotapes, a rich collection of data emerged that revealed the complexity of standard-based instruction.

Findings

The study revealed that the teachers benefited tremendously from using concept maps to illustrate underlying problem and skills needed to solve particular problems. It added to their understanding of the middle school curriculum they teach. It helped them to see the interrelationship between mathematical skills and concepts. The depth of thought between the skills and concepts and how they might play out in the classroom was evident. Teachers considered new ways of teaching and learning while collaborating during planning times to discuss content and pedagogy.

Also, results show that as teachers listened to students' ideas and documented their thinking, they were able to progressively make sense of student work, make better pedagogical decisions based on their analysis of the problem, and created more detailed concept maps both globally and locally.

Moreover, teachers were able to construct and provide students appropriate problem sets or assessments that more accurately reinforced the problems done in class. As the teachers gained a deeper understanding of their curriculum, they were also able to explain and justify their curriculum goals and their alignment with textbooks and state standards in a variety of innovative ways. It appeared that the teachers' content knowledge, pedagogical knowledge, and knowledge of student's thinking deepened simultaneously. Their concept maps and documentation of student thinking served as conceptual tools that aided in their growth.

Conclusion

By gathering accounts of these middle grades teachers' practice, we developed an understanding of their development as mathematics teachers and added to the larger body of knowledge on mathematics teacher development in work done by Simon and Tzur (1999). Also, by using middle grades teachers, we gain experience that would be useful in the training of pre-service mathematics teachers.

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UNDERSTANDING THE IMPACT OF A SCHOOL – UNIVERSITY MATHEMATICS REFORM PARTNERSHIP IN HIGH-RISK URBAN SCHOOLS

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The Syracuse City School District (SCSD) is the 5th largest school district in New York State, serving more than 22,000 students from diverse cultural and socio-economic backgrounds. The University-District partnership has the goals of deepening the mathematical knowledge of elementary and middle school teachers including those in Special Education; establishing teaching practices supporting all children's learning of worthwhile mathematics; improving students' scores on standardized tests, and closing the achievement gap between diverse groups of students both within a school and across the district. The partnership is based on four projects: A Math Science Partnership project, *Beyond Access to Mathematics Achievement (BAMA)*, a Teacher/Leader Quality Partnership project, *Using Assessment and Supportive Technology to Strengthen Mathematical Learning and Teaching*, A New York State Wallace/Gates Foundation Leadership project, and a Title IID *Enhancing Education through Technology* project. The partnership also draws the expertise of the technology development center, the *Living SchoolBook (lsb.syr.edu)*.

To meet the goals of the partnership, the partnership targets 300 teachers each year for 60 hours of professional development based on their choice of experience. The key structural characteristics are: prolonged contact, a combined professional development model type, site-based, embedded coaching, availability of follow-up support, and continuous assessment. Activities used content specific materials, an inquiry approach, collaborative groupings, and established learning communities (Loucks-Horsley, Hewson, Love, and Stiles, 1998). From the data, support networks for teachers to move forward in change, and support networks for school change have emerged. Data is analyzed looking at the nature of both formal and informal relationships between participants and between participants and events, in order to interpret and understand this complex system. Findings on the relationships needed for capacity building and sustainability are discussed.

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FROM STANDARDS TO LEARNERS: HOW MUCH INTENDED CURRICULUM ARE STUDENTS EXPERIENCING?

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The word "curriculum" is used widely among teachers, researchers, and policy makers with a variety of meanings (Gehrke, Knapp & Sirotnik, 1992; Porter & Smithson, 2001; Remillard, 2005). We investigated the relationship between intended curriculum and implemented curriculum by analyzing state standards, textbook lessons, and lessons taught. We found the existence of an "extra curriculum," which we defined as textbook lessons not aligned to the intended curriculum. This work was funded by a grant from the Office of Educational Research and Improvement (OERI), U.S. Department of Education (# R303T010735).

We identified the intended curriculum to be the Minnesota Academic Standards in Mathematics (2003), the intended-written curriculum to be district adopted textbook lessons aligning to the standards, and collected data from five 7th grade teachers in Minnesota. Each teacher completed a Table of Contents Diary to self-report which lessons were implemented during one complete school year. Our first analysis examined how three textbooks (two textbooks were used by multiple teachers in one district) aligned with the state standards. Both MATHThematics Book 2 and Glencoe Course 2 textbooks aligned with 96% of the intended curriculum, while Glencoe Course 1 aligned with 79%.

Our second analysis determined the intended-written curriculum, or the percentage of lessons in each textbook that aligned to the intended curriculum. We found MATHThematics Book 2 to have 55% lesson alignment, while Glencoe Course 2 had 49% and Glencoe Course 1 had 48%. Approximately 50% of the lessons in each textbook did not align with at least one state standard—the extra curriculum. Our third analysis determined the percentage of intended-written curriculum, as well as extra curriculum, which was implemented. We found both teachers using MATHThematics Book 2 implemented 46% of the intended-written curriculum and 32% of the extra curriculum. The teacher using Glencoe Course 2 implemented 39% of the intended-written curriculum and 28% of the extra curriculum. Finally, the two teachers using Glencoe Course 1 textbooks implemented 32% and 27% of the intended-written curriculum and 30% and 27% of the extra curriculum, respectively.

From our results, we determined that students in these five teachers' classrooms were not experiencing the entire intended curriculum (based on the district adopted textbooks and their implementation). More research is needed to determine if this is true for a larger population, as well as the effects of limited and inconsistent implementation of intended and extra curricula.

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TEACHERS' MATHEMATICAL AND PEDAGOGICAL AWARENESS IN CALCULUS TEACHING

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The focus of the study

Teachers' mathematical and pedagogical knowledge have received increased research attention in recent years (Ball & Bass, 2000). However, most work in this area has focused on primary or early secondary education. Our study attempts to explore teachers' mathematical and pedagogical awareness in higher secondary education and most specifically calculus teaching and the concept of derivative. It needs to be noted that although there is a large amount of research on calculus education this looks into students' learning and not the actual teaching practices and the way that this affects students understanding learning. In particular, the main research questions that this study aims to answer are: a) what is the nature of teachers' mathematical knowledge concerning derivative? b) what are the teachers' pedagogical practices and views about teaching and learning calculus? and c) what kind of interrelationships can be identified between teachers' mathematical and pedagogical activity.

Theoretical background

The notion of teacher knowledge has been recognized as an increasingly complex phenomenon (Cooney, 1999). A number of studies have attempted to describe this knowledge and it seems that there is some consensus in regard to three of its most important elements: mathematical knowledge, knowledge of students and knowledge of mathematical pedagogy (Lappan & Lubienski, 1994; Even & Tirosh, 1995). Different labels have been used to refer to these elements such as subject matter knowledge, pedagogical knowledge, pedagogical content knowledge (Shulman, 1986), knowledge about mathematics (Ball, 1991), or mathematical know-how (Boaler, 2003). Ball, Lubienski and Mewborn (2001) emphasize the need to investigate how teachers' mathematical understanding affects their practice.They suggest that this should be investigated through the observations and analysis of actual teaching. Mason (1998) elaborates further the notion of teacher knowledge and talks about awareness in action, in discipline and in counsel both in mathematics and in mathematics teaching. In addition to this, mathematical and pedagogical knowledge constitutes not only knowing- that, knowing- how, knowing –why but also knowing to act and knowing to act in the moment (Mason and Spence, 1999). Although research on teachers' knowledge of mathematics as it is extrapolated through actual teaching practice is gaining ground, examples from the area of advanced mathematical thinking are very limited. This is in striking contrast to the large amount of research to calculus education concerning students' learning.

Methodology

The study is a qualitative research within an interpretative framework. The data was collected from three different schools in Cyprus. The data is comprised by classroom observations, informal discussions before and after teaching and audio-taped semi-structured interviews with

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each teacher after the school visits (their duration was about one hour). The researchers observed and took field notes from three teaching sessions conducted by each of the nine teachers. Field notes were taken by two researchers and summaries were constructed immediately after each observation from a combination of the researchers' field notes. The summaries included a general description of the lesson and important issues that emerged. Specific examples from the field notes were given as evidence to the identified issues. The interviews focused on a) teachers' experience concerning learning and teaching mathematics b) teachers' views about teaching and learning mathematics in general and calculus and derivative in particular and c) teachers' interpretations of specific pedagogical actions that were identified during the observations.

The data collected was analyzed systematically based on the grounded theory approach (Strauss & Corbin, 1998). The analysis of the summaries aimed at identifying elements of teachers' knowledge as they emerged from their practice according to Shulman's (1986) three categories: subject matter knowledge, pedagogical knowledge and pedagogical content knowledge. These categories comprised the general framework for the analysis but they started to get a special meaning for the specific content area under investigation. The analysis of the transcribed interviews was initially done vertically for each particular teacher and then horizontally across the nine teachers in order to identify general patterns and relations among the different elements of knowledge.

Results

In this paper, we focus on two teachers who were not pleased from their teaching of calculus in high school but they could not see any other alternatives. Their teaching was mostly teacher-centered while students' participation was limited to the performance of routine exercises or to their responses to teachers' close questions. Below, we briefly discuss how they introduced the concept of the tangent of a curve and we attempt to identify mathematical and pedagogical aspects of their knowledge both from the observations and the interviews. The teachers started with the tangent of a circle and discussed its critical property, that it has only one common point with the circle. The teachers gave two examples of curves that exemplified the inappropriateness of the above geometrical property as a definition for the tangent of a curve. However, the image of the tangent of the circle seemed to dominate students' responses. The teachers did not seem to build on these responses and they continued the lesson by introducing on the board the formal definition of the tangent of a curve. In this part of the lesson the teachers presented the new concept while the students did not actively participate. In some cases where the teachers asked some questions to encourage students' participation, the students seemed to have difficulties to make sense of the situation. A question that emerges is "why the teachers did not develop ways to face students' difficulties and how this is related to their mathematical and pedagogical knowledge?". A first analysis of the interviews allowed us to approach this question and offer possible interpretations.

Teachers' mathematical knowledge about the relationship between the tangent of a circle and the tangent of a curve seemed to be rather fragmented. For example, teacher A stated that "the concept of the tangent in a circle is not the same concept as in a curve…in a curve it is the tangent to a certain point". Teacher B was wondering about this relation not only in the final interview but also in the informal discussion before teaching: "Can we give a global definition for the tangent of a curve like in the case of circle?... I looked for a definition, as we say this is… but I did not find one in the textbooks..". Teachers had difficulties in seeing the tangent of a circle as a special case of the tangent of a curve because they expected to find a "general" definition for the curve of the same type as the one for the circle. In the case of the tangent of a circle they recognized a global characteristic property- exactly one common point – while in the general case of the curve such global characteristic does not exist. This fragmented nature of teachers' mathematical knowledge was also identified through their explanations why teaching the concept of tangent is essential for students' mathematical development. The main reason they offered was that it would help them in their future exams. The fact that the two teachers could not identify epistemological differences and commonalities between the tangent of circle and curve, and they could not also see the importance of the concept of tangent in mathematics and in science, is possibly an indication that their mathematical awareness had not reached the level of awareness in discipline.

In terms of teachers' pedagogical awareness both teachers seemed to realize that students had difficulties in understanding the particular concept: "Four or five students understood it. The others cannot understand… but these (concepts) need to be taught even for those students."(teacher A). However, even when they were asked to give specific reasons for these difficulties they mostly described a number of external factors (eg. the curriculum, lack of interest, tests, private lessons) and sometimes their own teaching: "they might have not understood yet the concept of tangent but I had not particularly analyzed it in my teaching."(teacher B). We could argue that their knowledge about students' mathematical understanding of the concept of tangent remained at a superficial level. They needed to integrate their practical knowledge with the theoretical- research based knowledge of Mathematics Education in order to be able to develop interpretive tools for teaching and learning.

Coming back to our initial questions about teacher knowledge and its role on teacher's practices, the first findings indicate that both have an effect on actual teaching as the teacher has to go deeply in mathematical and pedagogical aspects of her teaching in order to take "effective" decisions.

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TECHNOLOGY

UNDERSTANDING THE RELATION BETWEEN ACCUMULATION AND ITS RATE OF CHANGE IN A COMPUTATIONAL ENVIRONMENT THROUGH SIMULATION OF DYNAMIC SITUATIONS

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Some difficulties which are often present when learning calculus using paper and pencil is understanding that the accumulation of a quantity is closely related to its rate of change. To help students overcome these difficulties and understand these relations we designed a program which simulates the inflow and outflow of water in a tank. The study documents the behaviors of two students who were exposed to these dynamics situations.

Introduction

What kind of dynamics situations can we design to promote the understanding of calculus concepts? In the *Principles and Standard for School Mathematics* (NCTM,2000) point out "Technology should not be used as a replacement for basic understanding and intuitions; rather, it can and should be used to foster those understandings and intuitions."(p.25). Considering this recommendation our research shows how the technology can be used as a means to favor these understandings and intuitions in calculus contexts. The main aim of this study is to document the potential of dynamic situations to develop the basic intuitions that support the understanding of the essential calculus concepts, for example, the relation between the accumulation of a quantity and its rate of change (The fundamental theorem of calculus). The questions which served as a guide for this research were: at what level does the dynamics setting promote intuitions or reasoning to support the understanding of the relation between the accumulation of a quantity and its rate of change?, what type of connections do the students establish between the *ordinate* of the accumulation function and its rate of change?, at what level are the students able to identity that the *ordinate* of the accumulation function in each point represents the area below the graph of its rate of change?; what type of difficulties do the students show when interacting with dynamic situations?

We must point out that this investigation is continuation of a study (Estrada,2005) presented in the XXVII- PMENA. Now this research puts emphasis on the relation between the accumulation of a quantity and its rate of change.

Conceptual Framework

The basic intuitions, reasoning or thinking considered necessary to understand the more complex ideas of calculus, for example the fundamental theorem of calculus, was addressed by Thompson (1994). The author interpret this theorem as being an intrinsic relation between the process of accumulation of a quantity and its rate of change. In the same vein, Carlson *et al* (2001) designed curricular subjects to develop covariational reasoning and its role in acquiring the concepts of limit and accumulation. One of these tasks shows a graph which represents the rate of change of volume of water entering a container and ask questions related to the accumulation of the water in the container; but this task was presented in a static context (paper and pencil). Kaput *et al* (1999, 2002) also studied this relation between accumulation and rate of

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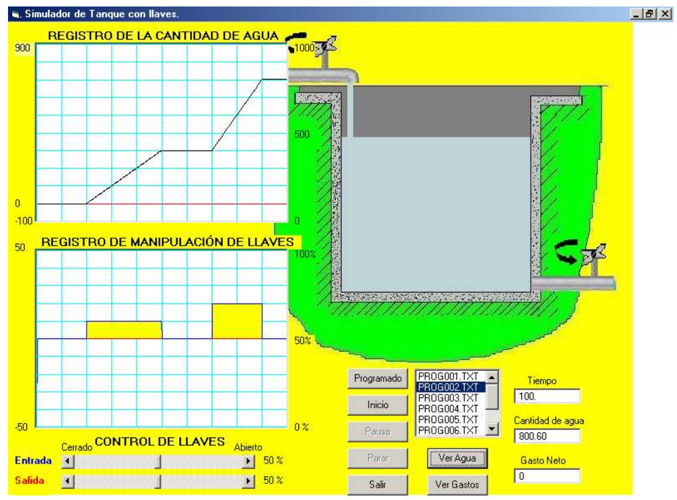
change by designing a software (Simcalc). In this environment, an elevator moves at a constant speed represented by a function defined by broken line forming steps and then to determine the location of the elevator. These ideas served us to design a program (Simulator) which imitates different dynamic situations in the filling and/or draining of water in a tank.

Methodology

This research followed the qualitative approach. We chose a pair of first semester engineering students from a public university who had already taken a course in differential calculus. The pair were videotaped working on tasks in one session of 2.5 hours. The investigators did not intervene, that is, the two students worked alone. The program is described below.



The program simulates the inflow and/or outflow of water in a tank with a capacity of 1000 liters. There are two taps; one controls the inflow and the other the outflow. The amount of inflow and/or outflow is controlled by two bars (taps) which are shown in the screen as CONTROL TAPS. For example, when the bar that controls the inflow is moving (open) y then is left fixed at 70 % it means that the flow of water is entering in the tank at 70 liters/sec regardless of the position of the other *tap*. If a moment later the tap that controls the outflow is opened in 50% (each bar has the percentage indicated at the right) that means, outflow is at the rate of 50 liters/sec, therefore the amount of water that entered into the tank is 20 liters/sec. In this context, the first derivative is represented by the *net consumption* (= inflow *minus* outflow). This difference seen as a *variable* is very important because it represents the behaviors of the volume of water in the tank. There are two windows on the left hand side of the screen. The top represents the REGISTRY OF QUANTITY OF WATER; here appears a graph representing the quantity of water in the tank as the tap is manipulated. At the bottom window titled REGISTRY OF TAP HANDLING , there is also a graph that represents the *net consumption.* This two graphs are generated simultaneously . It is very important to observe that below the graph of *net consumption* we see shaded areas (yellow on the screen). The vertical axes in the top window represents the quantity of water in the tank for each 100 liters. The zero in this scale marks the initial quantity of water in the tank. In the lower window the vertical scale represents the percentage (rate of change) the tap was opened. Both horizontal scales mark the time for each 10 seconds. It is important to state that the outflow tap remains fixed during the dynamic event, this permitting using the horizontal line (out flow tap) as the axes for reference to observe the *net consumption* .

The *simulator* has seven keys: programmed, start, pause/continue, stop, end, see water and consumption.

To interact with the simulator we also designed nine written activities which were scaled from easy to conceptual difficulties. Before beginning the experience the students were given a period to become familiar with the program. As an illustration we present below one of the activities used.

Activity 6.

Run the program **PROG005.TXT** that appears in the program windows by a “click” on the button marked PROGRAM. Observe carefully what is happening and answer the following questions, ifyou wish to reexamine the situation in order to observe in greater detail the characteristics of the task you may run the program again.

- 6.1 Give a description of what occurs during the dynamic event
- 6.2 Based on the graph that appears in the lower window answer the following questions and explain you reasons:
  What quantity of water accumulated between t = 10 and t = 40 ?
  What is the value of the net consumption at t = 30 ?
  What is the value of the *net consumption* of the water at t=70 ?
  What is the quantity of water in the tank at t=50 ?
  What happens to the quantity of water in the tank between t=50 and t=90 ?
  How much water is there is the tank at the end of the dynamic event ?
- 6.4 Draw a graph which represents the *net consumption* of the water in the tank during the dynamic event
- 6.5 Once you have finished, “click” the key SEE NET CONSUMPTION compare the graph which appears with the graph you drew. Based on this comparison explain what you observed
- 6.6 Now analyze the graphs which appear (on the register of Handling of taps and Quantity of Water) and explain if you find any relation between the shaded areas in the lower graph and the upper graph, what is it ?

When the students run one of the programs chosen, the simulator only showed only one graph, for example, the graph for the water quantity or the graph for handling the taps, which will be referred to as R_1 and R_2 respectively. In the above task, the PROGRAM 005 only the graph of the quantity of water (R_1: accumulation) appeared. The first task of students was to give a verbal description of what happened with the net consumption based on R_1 , answer some questions and draw the graph for R_2 . When the students finished the task they were asked to click the key NET CONSUMPTION a graph appeared, thus the students were able to receive feedback comparing this graph with they had drew. Finally, they were asked to identify the relation between both graphs. For the next task they were given the graph for handling the tap (R_2) and the task was similar to the above, but they were asked to draw the graph for R_1 and also to answer a series of questions. Generally speaking, there were two types of activities; one involved situations were the net consumption (rate of change) was constant (positive or negative) during certain intervals, which were represented in R_2 as a straight horizontal line and shaded areas below these graphs were in yellow if net consumption was positive and green if negative. The second one showed net consumption as not constant, others were a combination, that constant or not constant. These were represented in R_2 by increasing or decreasing straight lines.

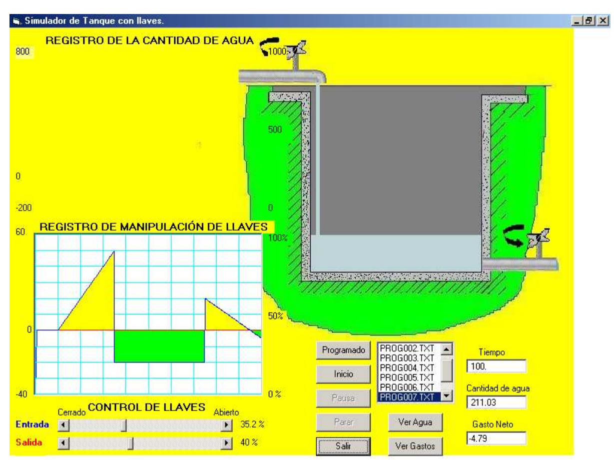
Discussion of Results

When analyzing the information particular attention was given the most outstanding behaviors (forms of thinking or reasoning, understanding connections, difficulties or patterns) shown by the pair. In the tasks of the first type (positive, constant or zero net consumption at an interval) the explanations given by students were as follows:

- Pupil 1: “What are we going to see ? They are the same and nothing is happening and time is passing, no net consumption, consumption is zero, blast, what happened. It increased 10%, yes, look they were the same and it went up 10%”
- Pupil 2: “it went down, well the consumption is going to be the same at zero, but the quantity of water went up
- Pupil 1: (run the program again) “I want to see when there is a change”

The pair noted the essential characteristics of the dynamic event and were able to establish a connection between R_1 and R_2 . However, they were not able, at this phase, to see the areas generated below the graphs for net consumption and its relation with R_1 . In order to make them aware about these relations further on, we asked “what does the *ordinate* represent in the graph for the quantity of water in t=70”. The answer was “the increase in the quantity of water, because the volume is represented by the area” (pupil 2). However, at this stage the students still could not clearly distinguish between the quantity of water in the tank (= initial quantity + accumulated quantity) and accumulation (= what was added to initial amount). Here was a tendency to base their answers on the numerical data supplied in the window for time and the quantity of water which appeared on the screen and not to base the answer on the graph in R_1 . When the students were asked expressively “What relations can you identify between the graphs in R_2 and R_1 ? They gave the following answers “here you moved the tap for an interval and then you left it, that is why the graph is the same, lets say no peaks, no movement”. Note that these comments refer to what happened in the dynamic event and not the relations between R_2 and R_1 , for example, be aware of the slope of R_1 and associated it with net consumption R_2 . In addition the students noticed global aspects of the graphs.

In later activities, the students were confronted with situations that involved positive and negative constant net consumption alternately with consumption equaling a zero. These activities helped the students to establish global connections between negative and positive net consumption graphs with increasing or decreasing straight line in R_1 . During this stage the students were able to attend the shaded areas and it relation to the quantity of water (“the shaded zone is water entering”, pupil 1) or “it is the area below the curve” (pupil 2). However, they still observed global aspects and not specific properties of the graphs. Neither were they able to see the graphs in R_1 as lineal functions in broken lines but as “triangles”. Here the students made some comments which recalled some concepts they had seen in their calculus course: “well here it is positive, it is as the derivative in calculus that graph (R_1), if you could get the derivative it would be positive, therefore the inflow (net consumption) is positive …in this interval it would be negative; therefore, the consumption is negative”(pupil 1). Nevertheless, these ideas were not used to understand the tasks. The final activities included a combination of constant , not constant, positive and/or negative net consumption. Here is an activity worked by the

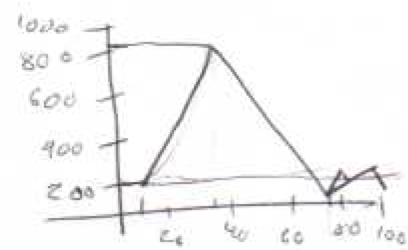


students.

As we can see, on the screen only a the graph that represent the handling of the taps (R_2) appears. We requested a description of what happened during the time lapse of the dynamic event and to draw a graph which represents the quantity of water (R_1). The most outstanding behaviors in the description given by the pupils are as follow:

- Pupil 2 "it goes up 20 it goes up a hell of a lot …its not going…the consumption is speeding up"
- Pupil 1: "hey, this is tough"

Note that students identified a relevant characteristic here which were not present in the initial activities. It seems that the pupils were able to see that the net consumption increased with certain speed and therefore, the accumulation of water in R_1 was not seen as being represented by an increasing straight line, if this were the case, it would means that the consumption was constant in this interval. But lets hear what the students say: "this is crazy it goes until zero" (pupil 2), calculate the green areas and I will do the yellow"(pupil 1). They realize that there is something new here: "generally speaking we calculate the consumption because its easy, but here it speeds up because the consumption is not constant, here we see its going up…if it could have been other graph the consumption would have been constant, but here no, it went up as fast as pace" (pupil 2). To solve this problem pupil 1 insisted on calculating the areas. But before this, pupil 2 insisted on reviewing the situation ("lets check what is being asked"). This allowed them to clarify what happened, for example, pupil 2 said "from 10 to 35 the consumption was not constant…it was changing therefore,…" the pupil 1 finishes the idea "it had a constant increase", pupil 2 " yes, until it reached 35 seconds then it abruptly went down until it became negative, from –20 during 40 seconds. To resume, the students were able to provide an adequate description of the relevant aspects of the dynamic situation, but were not able to draw the graph representing the quantity of water were the consumption was not constant. Here, while calculating the areas, they were almost able to establish relations between the shaded areas and the accumulated volume of water, but then changed their minds: "well, we get the area of the triangle which are equal, *ah no*, it is directly proportional to the volume of water which is entering". In order to draw the graph they calculated the respective areas of the "triangles", and this was the graph they draw.



In spite of being aware of the relevant characteristics in this new situation ("net consumption was never constant, it kept going up fast") they drew an increasing straight line in the first interval, contradicting what they said before.

Conclusions

In spite of the difficulties shown by the students in this last activity, from the evidences gathered we can state that dynamics situations have potential to promote the students making connections between the accumulation of a quantity and its rate of change. The research carried out in this phase shows that it is important to design activities which would be of aid to students to take the steps which they were not able to take in the last task. At the same time we should include more activities to strengthen the relations between accumulation and its rate of change, which are not very robust in the students.

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THE DEVELOPMENT OF MATHEMATICAL UNDERSTANDING IN CLASSROOMS WITH A COMPUTER

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We describe some of the results of a study that has as its main objective to find out the changes that are propitiated in the learning and teaching of math by the use of a computer and a projector inside the classrooms of elementary schools. This educational and research project is sponsored by the Ministry of Education of Mexico. For this purpose 120 activities were designed. The study showed that the mode of the teacher's instruction is a very critical factor in the whole learning and teaching process. However, this article will concentrate on describing the progress of the students' conceptual understanding. We will show that the students acquire visual images of the content matter treated in the activities which are helpful to guide their thinking, in some cases, noticing specific properties and making generalizations.

Introduction

In this paper we describe an educational project that has the purpose of using computers inside the math classrooms of elementary schools in Mexico. To do this effectively, a parallel research project consisting of a didactical experiment was conducted to evaluate the students' progress and teachers' influence. We will describe here some of the results of this study.

For the last ten years, the Ministry of Education of Mexico has being sponsoring, a national program to teach math and science with technologies at the secondary level (Mochon and Rojano, 1999; Mochon, 2001). This more recent project is an extension to the elementary schools, with some modifications. In this project, a single computer with a projector is used inside the normal classroom. The work is done cooperatively by the whole group of students to enrich the learning process. This makes the teacher's role even more important as a mediator.

The educational project

The teaching practices in elementary schools tend to give a lot of emphasis to the procedural, mechanical aspect of math. Also, the teaching method is based on the teacher as an "explainer". Our didactical proposal tries to change this, centering the learning process on the students and paying more attention to their conceptual development and their thinking.

For this purpose we designed a total of 120 activities, using the programming language Java. To illustrate the type of activities developed, Figure 1 shows an activity in which a block has to be weighed in several ways by fractional weights. These are dragged from the right side and placed on the right plate of the scale. The block changes when the "Start" button is clicked.

The different elements that this project introduces are: 1) A computer and a projector to show ideas dynamically and interactively. 2) The activities have a strong conceptual component. Each one has a specific content, but allows many diverse explorations. 3) A pedagogical model centered on the students, allowing them to reflect and interchange ideas.

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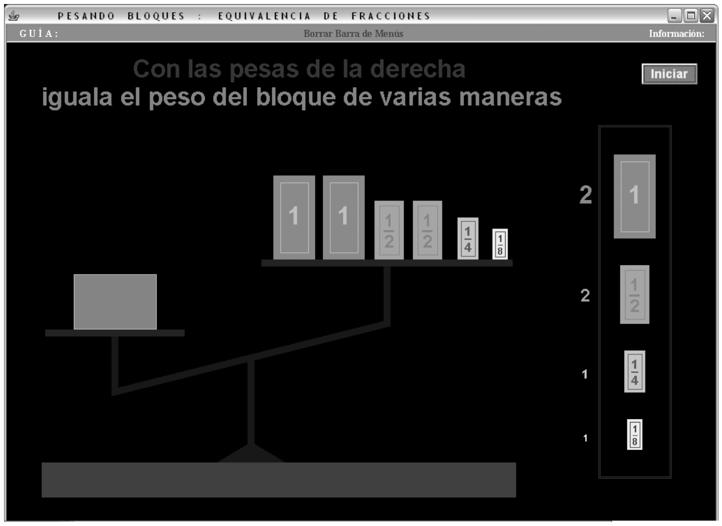

Figure 1: An activity of the fraction set developed for the project.

The research project

A series of studies are being conducted not only to evaluate the materials developed, but also to observe the changes in students' learning, the interaction between them and the teachers' acceptance and ways of working with them.

Theoretical Framework

There are four general aspects we are focusing on in our studies. 1) The use of the technology and the applicability of the activities designed. 2) The students' cognitive advances in the different topics. 3) The interaction and discussion between students generated by this pedagogical model. 4) The role of the teacher and his adaptation to this new form of teaching.

These aspects are strongly interconnected, so we will have to be aware of the effect of each one on the others. For example, the mode of teaching can influence the advance of the students, their interaction, etc.

For the second aspect, we will base our analysis on the categories of growth in mathematical understanding formulated by Pirie and Kieren (1994). As the authors mentioned, these can actually be seen by an observer. Although they give a list of eight different levels, we will describe briefly below the first five, since we do not expect students of elementary schools to achieve the last three:

Primitive knowing. It is the starting point. What the student knows and can do initially (we could add: *assuming the concept has not previously formed*).

Image making. The student does something to get a particular notion. Need to act on objects to form an image.

Image having. The image is formed and there is no need to do something or act on objects.

Property noticing. Can use or combine aspects of previous images to construct specific properties.

Formalizing. Moves into general or abstract statements, identifying common features.

In an article by Warner and Schorr (2004), these five categories were used to analyze student to student interactions that contributed to the development of their ideas. These middle school students were prompted to explain their thinking and justify their solutions. This article shows that this learning strategy is helpful to improve their conceptions about the topic discussed. In our own research we would like to observe, through this framework, the students' conceptual changes produced by our technological-didactical model.

The third aspect about interaction of students will be analyzed through a sociocultural approach based on the works of Vygotsky (1978) and Wertsch (1991). The book by Newman, Griffin and Cole (1995) describes some of the notions of Vygotsky's theory adapted to the school environment.

For the fourth aspect, we will base our analysis on several similar frameworks. In an article about cognitively guided instruction, Carpenter et al (2000) stresses the importance of the teacher's knowledge about the mathematical thinking of children. The authors identify four levels of teachers' beliefs that correlate with their mode of instruction.

Another study by Jacobs and Ambrose (2003) looked into how interviews applied by teachers to their students can improve instruction, developing their questioning skills. These authors proposed a classification of the different modes of teachers' interaction during an interview. The list of these categories is as follows. A) Directive. B) Observational. C) Explorative. D) Responsive. We believe that these categories can be use to analyze the mode of instruction of teachers within our project and with this, define the teachers' interactions that are more appropriate to work with the activities designed. Other similar studies in this line of research are Moyer and Milewicz (2002), Crespo and Nicol (2003) and Haydar (2003).

Methodology

With these different frameworks in mind (for each of the aspects that we would like to study), we planned several didactical experiments, working inside normal classrooms with some of the activities designed. For this, an elementary, middle economic class school (in the north of Mexico City) was chosen and each of four research assistants selected a grade level, a topic and ten activities from this topic to work with ("Car" chose second grade and Additive problems plus Decimal system. "Let" chose third grade and Geometry. "Ili" chose forth grade and Fractions. "Vic" chose fifth grade and Mental calculation and estimation.) These four assistants centered their observations primarily on the "students' cognitive advances" aspect of their particular topic.

Two stages of the studies were planned. In the first one, the same four research assistants would test the activities in a classroom as teachers. In the second one, the teachers of those groups will use the activities and the research assistance will become only observers. In each of the two stages and for each of the four topics, the method would consist of several steps as described below.

- An initial evaluation of the topic based on interviews of five students. For this, a guiding questionnaire was designed, containing the notions that will be touched upon during the experimental teaching.
- An initial interview with each of the four teachers to find out their beliefs about using computers in the classroom.
- A didactical experiment within the classrooms, testing some of the activities designed. This consisted of eight sessions. In the first two, the students got acquainted with the mouse through activities of the Start set. In the other six, the students worked on the activities of that particular topic (one or two per session).
- A final evaluation of the topic based on interviews with the same five students. The initial questionnaire was used with some small modifications.

- A final interview with each of the four teachers to find out the changes in their beliefs.

All these interviews and classroom sessions were audio taped (a few of these were also video taped, mostly for future presentations of the project). An additional research assistant was always present during all the sessions to write down important observations.

Results

The first stage of these studies was carried out between April and June 2005. The second stage took place during the months of October and November 2005. At this point, we can give the results of the first stage and some partial results of the second stage (we expect to have a more complete picture of the results by March 2006).

In this paper we will concentrate on the results of the students' development in their mathematical understanding. However, since the dynamics of the activities and therefore, the students' advancement greatly depended on the teacher's mode of working, we describe briefly first the teachers' behaviors found and the interaction between students.

One of the four teachers was classified as "Explorative" and "Responsive". In this case, we observed that the activities were conducted very efficiently and that the students progressed in their interpretations and conceptions of the topic. The other three teachers were classified as "Observational" or "Directive". In these cases, the students became restless more often and their progress was less effective.

Also the interaction and discussion between students, was influenced by the teacher's mode of working. However, in all the sessions, the students demonstrated great interest in the activities and we observed in general, an increase in the interchange of ideas and discussions.

Due to the amount of the data collected in each of the four didactical experiments, we will describe here only our findings on the students' cognitive advances of one of the teachers (the first one described above who worked on the topic of fractions) and concentrate on selected items and answers of the questionnaire. We must stress that the experiment time was very short (six sessions) and therefore, we did not expect significant advances on the children's conceptualizations. However, we were looking for some indications of progress and its characteristics (we are planning to make a follow up of this research during a full year).

The questionnaire for the initial and final students' interviews consisted of six tasks of the specific topic. The notions contained were: fractions as part-whole, as measure and as ratio, partition and equivalence within these contents and estimation of fractions.

The first task of the interview requested the student to give the fraction represented by the shaded area in each of the two squares shown in Figure 2. We will describe below the answers of three of the five students interviewed.

Fernando *–in the initial interview–* expressed the left shaded region as "two fourths". *–In the final interview–* he expressed that "each is one fourth and together they are two fourths or one half". We noticed here an advance to the level of "**Image having**".

Agustin *–in the initial interview–* divided the shaded regions of the left square with a horizontal line and stated that the top portion "was one eighth plus the bottom piece could be in total one ninth" (here we can see the misconception of: bigger piece corresponds to bigger denominator). For the right square, he initially called each shaded piece as "one half" but later he indicated that it was one eight since it is half of one fourth. *–In the final interview–* Agustin, for the left square stated: "two fourths because each triangle fits four times and there are two which is also a half". We observe here an advance to the level of "**Image having**".

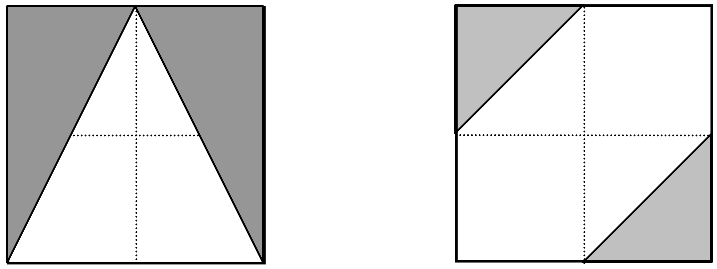

Figure 2: Diagram of task 1 of the questionnaire for the interview.

Ian *–in the initial interview–* expressed the left shaded area as “a half” and for the right square he mentioned that each shaded piece “is a third since it is smaller than a fourth” (here again we see the same kind of misconception). *–In the final interview–* Ian not only solved correctly the two parts but stated, referring to the names of the fractions stated that “the bigger the number is, the smaller the fraction”. This demonstrates an advance up to the level of “**Formalizing**”.

The second question of the interview asked to weigh in different ways, packages of 2 and 3½ kilograms with weights of 1, ½ and ¼ kg. *–In the initial interview–* the students employed mostly weights of a single type (“½ and ½ and ½ …”) and used extensively diagrams and drawing to show their answers. Agustin, for example, drew 4 weights of ¼ and one of ½ to represent 2 kg. When asked to explain, he said: “4 of ¼ make a kilo plus ½ make 2 but since they are halves, it would be two and a half”. His way of adding is: “one and one half is two … halves”. *–In the final interview–* all the students were able to do more combinations and with different weights. In particular, Agustin constructed 3½ kg as follows: “I put two halves, one integer and four fourths. We already have three and it is missing a half”. When asked about the different representations, one of them said that “they are the same quantity but in different forms”. This shows that they achieved the level of “**Image having**” and in some cases the one of “**Property noticing**”.

The figure of the fourth task of the questionnaire showed 2 or 3 whole oranges and 3 halves. The student was asked to divide this set between two children. *–In the initial interview–* three students divided each piece in two and expressed their answer like: “to each one, I give half of an orange, half of another and three times the half of a half”. Two other students cut one half orange to obtain two fourths. When they saw this result (“fourths”), they divided the next half orange into four pieces to obtain the “same” type of fraction! (This shows a mix-up with the unit). *–In the final interview–* we observed different procedures all showing a significant advance in the conceptualization of fractions.

Two students joined two halves to obtain in total 4 and a half oranges and stated: “each gets two oranges and half of this half which is a fourth”. This can be characterized as “**Property noticing**”. However, it is interesting that they expressed the result as “each gets two fourths”. Later on, Ian said: “it is not like I said because two fourths is only half an orange and they got more, but I don’t know how to join the quantity, each gets two oranges and another fourth of an orange”. This shows that symbolic expressions like “two and a half”, “two and a fourth” might not have the expected meaning to students.

The figure of the fifth task of the questionnaire showed 20 small faces. The student was requested to find 2/5 of this set. *–In the initial interview–* Fernando, cut the set in 4 parts and said: “I think that 1/5 is half of ¼ because it is smaller. Since it is two fifths then it is equal to one fourth”. Two other students signaled a fifth as five elements. Ian said: “each whole has five fifths but I don't know how to cut it". *–In the final interview–* all students were clear that they have to divide the set into 5 pieces: "I have to cut this in 5 parts to obtain fifths". This partition, which caused difficulties to all of them, was accomplished by trial and error: "if we take five for each group there is not enough for five groups; if three in each group, we have 5 left over; if we give four to each we have all". Agustin reasoned as follows: "I counted 20 in total, for five equal parts I remember that 5 times 4 is 20, so I enclosed four by four and got the five groups". Here we observed an advance to the category of "**Image having**", and in the case of Agustin, he showed signs of "**Property noticing**".

The sixth question of the interview asked to color 8/10 of a rectangular bar representing the unit (two rectangular bars were given in the figure). *–In the initial interview–* three students just divided the rectangle into 8 pieces ignoring the denominator. Other two students used the second bar because "there are too many pieces". *–In the final interview–* all the students stated that they need 8 of ten parts which is less than the whole bar. The difficulty appeared in the partition. Some try to fit the ten parts with a "good eye" and shaded 8 of them. Ian divided the bar into 4 pieces and then reduced the size to get a fifth. He stated: "more or less I looked at the size of one fourth to get five and then I divided each into halves to get ten. Then I counted eight". Here again we can observe a significant advance to the levels of "**Image having**" and "**Property noticing**".

During the didactical experiment of six sessions, we observed continuously that the children moved in and out of the three levels: "**Image making**", "**Image having**" and "**Property noticing**". For the sake of completeness we will give a few examples where the children showed to be situated in the fourth level "**Property noticing**" and even in the fifth "**Formalizing**".

In the first session, the students worked with a screen that had in its center a square (the unit). Into it, the students could drag fractional pieces of 1/2, 1/4 and 1/8 in the shape of triangles, rectangles and squares. For example, a student said that "two fractions, even that they don't have the same shape, are equal because of the number of times that fit into the unit". Another student stated about an arrow constructed inside the square unit that: "It is a half because the square is one fourth and each triangle is an eight and the two are one fourth plus the other fourth makes a half".

In the second session, the students worked with the screen of the scale shown at the beginning of this article. For a block that weighed 3 kg, one of them said: "It is also equal to twelve fourths since one kilogram has four, plus four, plus four makes twelve". For a block of four and a half kg, somebody else said: "An equal fraction would be nine halves since in 4 kilograms there are 8 halves, plus the other one makes nine".

In the fifth session, the objective was to construct equivalent fractions on two bars in the screen by dragging fractional pieces. One student remarked that: "There are three sixths in one half because six can be divided into two to get three".

In the last session the students have to estimate the size of fractions. For the fraction 7/8, one said: "It is close to the unit because it missing only one eighth and it is a small piece before". For the fraction 6/4, another one said: "It is more than one because four fourths make one integer and the other two make another half".

Conclusions

We observed in general that the students' interaction improved as well as their cognitive development by acquiring images that guide their thinking. In the final interviews the students referred to the images of the activities when giving their answers. Their explanations showed that they have advance to higher levels of the Pirie and Kieren's classification.

We believe that this project, not only can improve what it is learnt by students in several aspects, but at the same time, it can develop communication and expression skills of students and teachers. One important advantage of this teaching method is that it helps the teacher to pay more attention to the thinking process of the students instead of simply looking at the final results. Another very significant benefit is that the evaluation of students becomes an integral part of the teaching process. In fact, the teacher observes directly and constantly the students' thinking, strategies and difficulties and can use this knowledge to improve their conceptions.

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THE CULTURE OF MATHEMATICS CLASSROOMS WITH THE USE OF ENCICLOMEDIA, A NATIONAL PROGRAMME

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In this paper we report on differences on mathematical classroom cultures where computer programmes from Enciclomedia, a Mexican national project, are used. Classroom cultures are characterised through the following aspects of students' behaviour: Active/Passive, Attentive/Inattentive, Working with others/Working individually, Freedom/Constraint, Giving correct answers/ Formulating explanations, Understanding/ Remembering. Results show that changes in classroom cultures are shaped by the kinds of digital resources used. Mathematical learning was observed in relation to the use of a programme that gives students' freedom to explore and which successfully promotes asking questions, reflecting and formulating explanations. Another programme, which restricts students' activities, reinforces already existing tendencies of giving answers automatically.

Introduction

Enciclomedia is a large-scale Mexican project that has been devised with the purpose of enriching primary school teaching and learning by working with computers in the classrooms. An electronic version of the mandatory textbooks that are used in all primary schools in Mexico is being enhanced with links to computer tools designed to help teachers with the teaching of all subjects. As members of the Mathematics group in Enciclomedia, we create resources and strategies which can help teachers and students in their teaching and learning of mathematical concepts. An additional and extremely important part of our work is to investigate how students learn mathematics as they use the computer tools that Enciclomedia provides them with.

One way of approaching the way in which students learn with an innovative tool consists in following, by means of careful observation, the interactions between teacher and students, amongst students and with the resources from Enciclomedia. Patterns in interactions constitute what we call classroom cultures. The purpose of this paper is to report on differences observed in mathematical classroom cultures as students interacted with two different programmes from Enciclomedia.

Some Ideas about the Learning of Mathematics

Our theoretical ideas about mathematics learning are based on enactivism, a theory of knowing which considers learning as adequate or effective action in a given context (Maturana and Varela, 1992). Learning occurs when individuals interact with each other, changing their behaviour in a similar way. In a particular context or location, the participants create together the conditions that will allow actions to be adequate. As members of a particular community interact with each other, patterns of behaviour are created; constituting a classroom culture (see Maturana and Varela, 1992). With these ideas in mind is that we are interested in investigating the way in which patterns in effective behaviour emerge in mathematics classrooms as teachers and students use Enciclomedia.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Learning mathematics with computer tools

From an enactivist perspective, the use of computer tools is part of human living experience since 'such technologies are entwined in the practices used by humans to represent and negotiate cultural experience' (Davis *et. al.*, 2000, p. 170). Tools, as material devices and/or symbolic systems, are considered to be mediators of human activity. They constitute an important part of learning, because their use shapes the processes of knowledge construction and of conceptualization (Rabardel, 1999). When tools are incorporated into students' activities they become instruments, which are mixed entities that include both tools and the ways these are used. Instruments are not merely auxiliary components or neutral elements in the teaching of mathematics; they shape students' actions. Every tool generates a space for action, and at the same time it poses on users certain restrictions. This makes possible the emergence of new kinds of actions.

When students and teachers work with the tools provided by Enciclomedia, their behaviour inevitably changes. Investigating the way in which students' actions are shaped when they use the programmes we create is a crucial part of the process of development of the tools themselves.

Some Ideas about Methodology

The choice of methods used in our investigation of mathematics learning is also inspired by the enactivist approach. 'Enactivism, as a methodology [is] a theory for learning about learning' (Reid, 1996, p. 205). Research is considered to be a way of learning, and therefore researchers are seen as individuals developing their learning in a particular context. The interdependence of context and researchers makes the research process a flexible and dynamic one. Research does not occur in a linear fashion; rather, it is seen as a recursive process of asking questions. The work reported in this paper is only the first part of a complex process of interaction and development of ideas. The methods we have started using to investigate mathematics learning will change in the future according to what we observe in the classrooms and to the feedback we receive from colleagues.

With the purpose of researching the learning of mathematics with Enciclomedia, we contacted a school in Mexico City where we worked with two Year 5 and two Year 6 groups of about 25 students each (aged 11-13). Two of us visited the classrooms at a time and our role was that of participant observers. When digital technologies are used, these change the way students and teachers interact with each other and therefore particular classroom cultures emerge. Teachers and students worked with the same interactive programme, associated to a particular textbook chapter, during several sessions. This allowed us to observe how behaviour changed gradually.

In order to register the characteristics and the development of the classroom cultures, we carried out detailed observations of students' and teachers' actions. We recorded whole group discussions as well as interactions that occur between two or three students and/or between students and teachers or researchers. So far we have video taped the teacher and different pairs of students on every session.

Teachers' actions are extremely important in that they shape the classroom culture in particular ways: for example, teachers decide what computer programmes will be used in the classrooms, and they encourage some of the students' actions while they reject others, they also decide when an explanation or a discussion is needed. Teachers can also create relationships between formal and informal mathematical actions. Their behaviour influences deeply the dynamics of the interactions in the classrooms. In this paper, we focus on students' actions;

however, we acknowledge the fact that these are inevitably influenced by those of the teachers. Teachers' behaviour will be reported and analysed in more detail elsewhere.

With the intention of monitoring students' actions in the classrooms, two different observation sheets were used during each one of the lessons we observed. The first instrument intends to register students' activities in the classroom in a general way, and includes the following aspects of students' behaviour: *Active/Passive, Attentive/Inattentive, Working with others/Working individually, Freedom/Constraint, Giving correct answers/ Formulating explanations, Understanding/Remembering.* These aspects had emerged in a previous study in which they had been helpful in analysing students' mathematical actions (Lozano, 2004). We decided to start investigating our classroom cultures by looking at these categories, keeping in mind that some of them might turn out to be irrelevant, while we might need to add others.

A second instrument was used to keep records of those actions that can be considered mathematical, especially the ones related to the mathematical concepts in the textbooks' chapters that were addressed during the lessons. This observation sheet includes the following headings: *Initial mathematical behaviour* (which refers to students' actions related to mathematics during the whole group introductory discussion at the beginning of the lesson), *Mathematical actions* (those observed during the rest of the lesson, which are related to the mathematical concept(s) in the textbooks' chapter) and *Other mathematical actions* (they do not explicitly address concepts in that chapter). Particular incidents, where mathematical behaviour is observed, were written at length under each heading. In addition, we have kept records of students' work with paper and pencil. Acting mathematically does not necessarily mean, to us, solving a problem in a conventional 'correct' manner. We collectively decide on what is mathematical by having discussions in which we talk about our notes, our transcripts from the audio tapes, and about what we observe on the videos. To support our interpretations about mathematical actions, we also read the literature on the teaching and learning of the different areas or mathematical concepts which are being explicitly addressed in each lesson. We use the textbooks to identify these concepts, and to learn about the purpose of the chapters in them. We are working on the development of criteria for identifying mathematical actions, which are not fixed but ever-changing.

Both observation sheets were filled in by those two researchers acting as participant-observers in the classrooms. These records, together with the audio and video tapes were analysed during joint sessions in which the three authors of this paper participated.

Results and Discussion

After a great number of classroom observations, which we carried out during a whole school year, we found that, when interactive programmes from Enciclomedia are used, certain patterns of behaviour emerge. These patterns of behaviour, however, vary according to the resources used during the lessons. Based on the data we obtained during classroom observations and on the records we kept from the observation sheets, we can say that:

- Students were active when they worked with the digital resources from Enciclomedia. They constantly interacted with the programmes and with their peers. This behaviour was observed during whole sessions. It was difficult to organise whole-group discussions as students were often absorbed in their work with the digital resources.
- Once a whole-group discussion was organised, students were eager to participate and most were attentive.

- Students sometimes got distracted when, working with an interactive programme, they could not solve a problem after many attempts.
- Individual work was frequent when students were working with activities from the textbook; when they start exploring the problem with the interactive programme; and when their solutions are giving them unexpected feedback (due to incorrect answers). Students appear to work in groups more frequently once they have an understanding of the problem.
- Students and often want to explain or show things to the teacher and researchers.
- Sometimes students' explanations include phrases such as 'that is the way we were taught' 'that is how the formula goes' which indicate memorisation. Particular patterns of behaviour were observed when certain programmes were used in the classrooms. In what follows we report the results from the use of two programmes: 'Perimarea' and 'The Balance':

Perimarea

Perimarea is a programme in which students are asked to find the area and/or the perimeter of different geometrical shapes which are shown on a grid.

Students write numerical answers and immediate feedback is given in different ways. The programme tells the users whether their answer is correct or whether they have 'too many' or 'too little' units or square units. (See Figure 1).

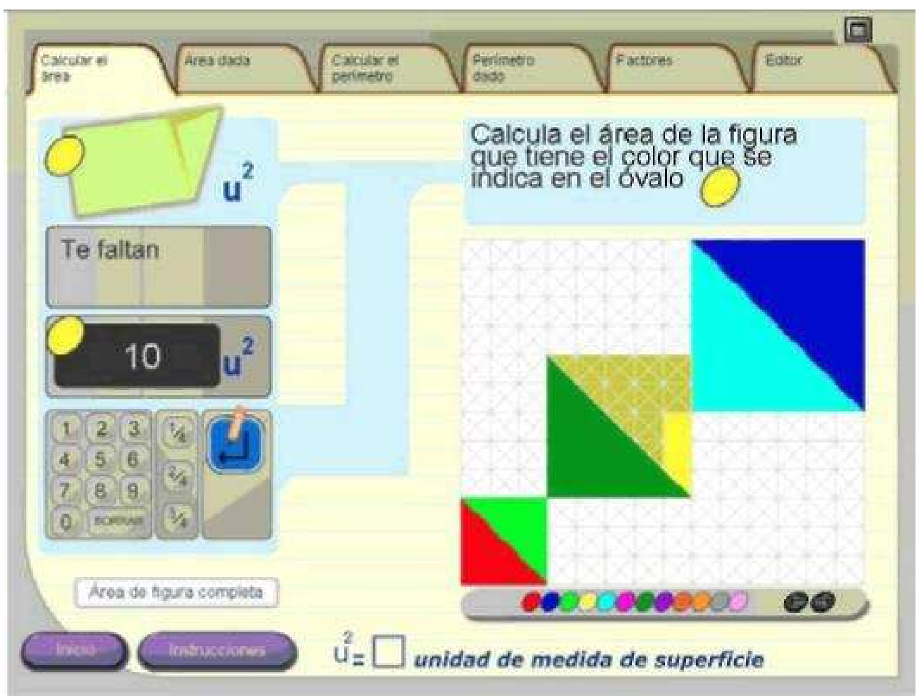


Figure 1. Perimarea

In addition, visual feedback is given by means of shades that show, on the geometrical shape, the area or the perimeter which reflect the students' answer. (See Figure 2).

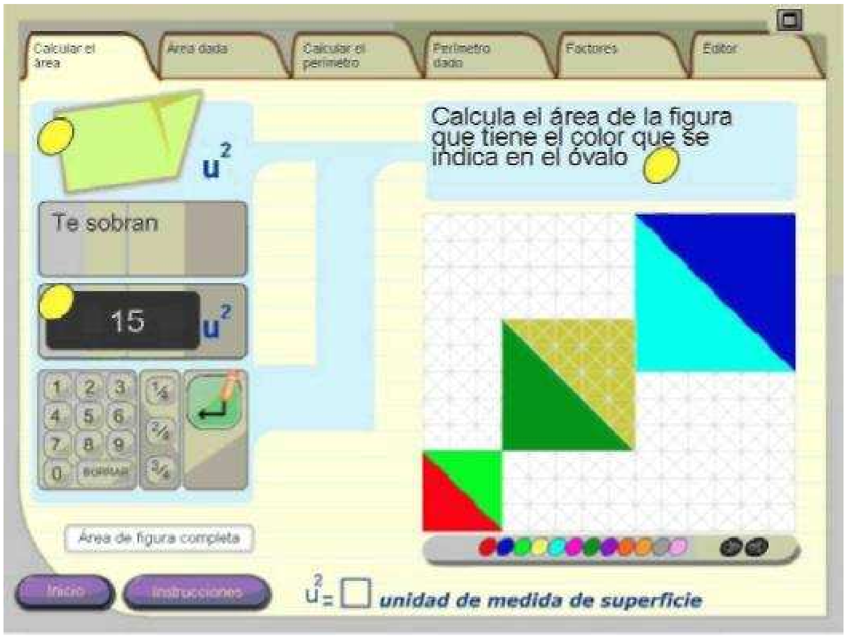


Figure 2. Activity with Perimarea

When students have worked with the programme "Perimarea", we have observed that:

- The main strategy was the use of a 'trial and error' approach. Initially, students worked with natural numbers, following the feedback given by programme and adding or subtracting accordingly, until they got the correct answer. When they noticed that the correct answer was not a whole number, students clicked on one of the three fractions that are shown on the numerical keybord from the programme (1/4, ½, ¾) to complete their response. Choosing an adequate fraction was done randomly. When questioned, students did not show awareness of the relationship that exists between these numbers and the gometrical shapes shown by the programme. Feedback given by Perimarea seemed to give further emphasis to students' already existing strategies of working towards correct answers without reflection. Perimarea did not generate the need to refine or change such strategies or to look for efficient ways of calculating the area or the perimeter of geometrical shapes.
- Changes in students' initial conceptions on area and perimeter were not observed. Some of these initial ideas, in relation to area were the following:

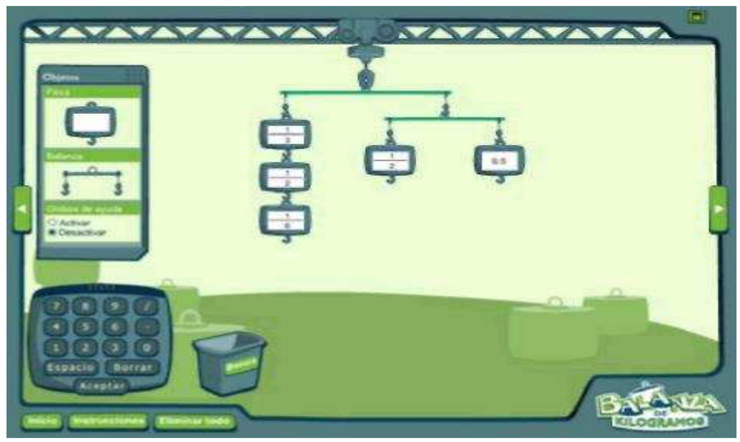
  51 - The area is the center of the shape.
  52 - It is the inside of the shape.
  53 - It is the background, what is not part of the edge of the shape.

  Some students obtained the area of the shapes by counting the number of squares in them.

  However, when, after a few sessions of working with the programme, students were asked about their ideas about the concept of area, they responded in the same way they did before they used the programme.

The activities provided by the programme Perimarea did not generate in students the need to develop their initial conceptual ideas or change their problem-solving strategies in relation to area and perimeter. Students had some knowledge about the conventional formulae used to find the area and the perimeter of geometrical shapes such as triangles, squares and rectangles. They did not relate, however, that previous knowledge to their ‘tying random numbers’ or ‘counting squares’ activities. Adequate behaviour included very limited activities and therefore we conclude that mathematical learning did not take place. We believe this was because feedback from Perimarea restricted students’ actions and did not encourage them to act reflectively. Activities done through the programme do not allow for exploration and did not provoke in students changes in their mathematical behaviour.

The Balance



With the interactive programme The Balance users can create balances with different numbers of weights and on different levels. On each weight, natural numbers, fractions and decimal numbers can be written.

The programme indicates, in real time, visually and with sounds, whether the balance is in equilibrium or not, according to the values which are assigned to the scales.Figure 1. The Balance

When working with the Balance students were asked, to equilibrate a balance that showed a weight of 1 ½ kg on one arm and two blank weights on another arm (See Figure 2). In relation to this and other similar activities we observed that:

- Students asked questions such as ‘what happens if we write 1 and ½? Can we use decimal numbers? What about 3/6 and 1/6?

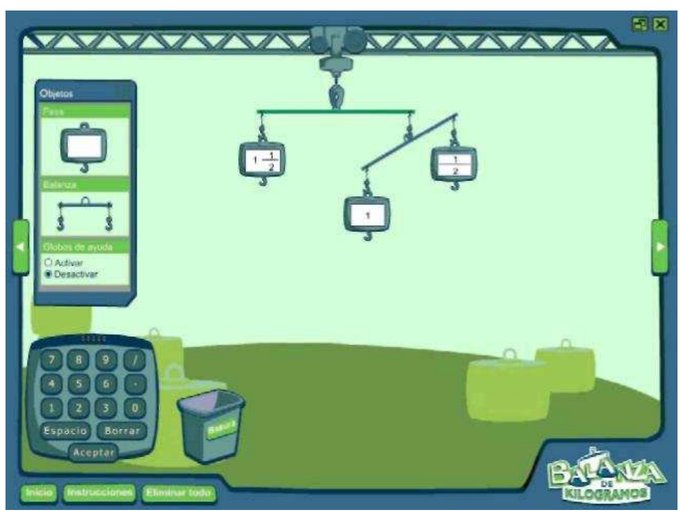


Figure 2. Activity with The Balance

- Students’ initial attempts included the use of ‘trial and error’ strategies aimed at getting correct answers. With time, however, strategies were refined and explanations were formulated, as students explored with different numbers. The following is an example taken from the video transcripts. Students are trying to equilibrate a balance which has a 2 ½ on one arm and, on the other arm, two 'empty' weights:

51 Let's see, which number will make it even?
52 This number (1/2) cannot be … because this would end up being heavier. We need a bigger number… (They do some sums)
53 We need big numbers, like 2 and ½
S5 Yes, but listen… we need to add up these two so that we get this, but these tow have to be the same, so that this side is also okay.

- As they used The Balance, students worked with fractions in different ways. For example, they added, subtracted and divided fractions in the process of equilibrating the weights in the programme. They used concepts such as 'equivalent fractions' and their work included the use of mixed numbers and decimal numbers.
- Sophisticated *explanations*, that involved complex mathematical ideas, were given by some students. These explanations often included the use of previous knowledge from students. Examples of these explanations are:

52 You have a pizza, and you divide it in 4 parts, we will keep 3 and eat one…whatever. Now, we want to half this… We can't do it, we divide each bit in 2 parts…so we have eights… we have three eights.
S1 Divide by 2, and then by 2 again, 8/5 divided by 2 is 4/5, if I half 4/5 I get 2/5'.
53 We just divide 3/2 by two (Writes 3^2^3^*)*

- Mathematical algorithms were discussed whenever there was a need for them to be used. Different non-conventional strategies were initially proposed but students were able to modify their initial intuitive ideas in order to make them more efficient. Adequate behaviour included conventional mathematical actions.
- Students used The Balance to work with problems from the textbooks, however, the programme was used when working with a variety of problems, including those posed by students themselves.
- We observed students using The Balance for several sessions. In this way, we were able to observe how some of the students' initial ideas about fractions changed. In the beginning, it was very common to hear students say that, for example, ¼ is greater than ½ because 4 is greater than 2. With time, however, students were able to compare fractions and decimal numbers adequately, even without the use of the Balance.

Mathematical learning occurred, and we believe this was partly due, on the one hand, to the fact that the programme gives immediate and useful feedback to the students, thus inviting them to reflect on their answers, and on the other, because it provides students with freedom to explore with different situations and to experiment with different strategies.

Some Conclusions and Directions for Future Research

Several materials have been developed for Enciclomedia. The decisions taken in their design have been based on specific criteria developed in the curriculum for different concepts, and on results of the research literature on mathematics education about those concepts. Most materials developed are designed to enhance interaction and reflection, however the observations in the classroom show evidence of important differences regarding the nature of students' mathematical actions as they use the different programmes.

So far, we have found that some programmes in Enciclomedia by *restricting* students' activities and options for answers, reinforce the students' tendency to try out responses without giving much reflection to them. Others, seem to invite students to act mathematically, using concepts form the textbooks in a variety of ways.

Different patterns of behaviour emerge when different programmes are used, that is, changes in the classroom cultures occur. More investigation is needed to explore, in more detail, the nature of those changes. In particular, research is needed to find out the way in which students interact with the tools and how they become instruments in that process of interaction. Our research instruments will need to be refined, so that the impact of the programmes on students' mathematical actions can be recorded in more detail.

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PRE-SERVICE TEACHERS USE OF TECHNOLOGY AS A PSEUDO-COLLABORATOR IN AN OPEN-RESPONSE, TASK-BASED ENVIRONMENT

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A research team in Australia argues that, under certain conditions, calculators can function as exploration partners for students (Goos, et al. 2000). We argue that, under very similar conditions, calculators can also be used in a collaborative task-based exploration setting to increase the Zone of Proximal Development of a group of students, thereby increasing the base knowledge of individuals in the learning group. Interaction between the calculator and the student becomes almost a conversation, suggesting that the calculator borders on becoming a collaborator. We use the term pseudo-collaborator for the role that the graphing calculator can play as it lacks several of the characteristics of a regular collaborator. Nevertheless, the calculator can aid in the assimilation of new base knowledge that would be more difficult to gain without it.

Objectives

This paper will provide evidence and explanation of how technology can aid in the acquisition of new mathematical knowledge in a task-based collaborative environment. We use the term pseudo-collaborator for the role that technology can play in the collaborative group. We argue that in this type of situation, students learn more mathematics faster than they would without the technology.

Perspective

Preparation of teachers must allow them to engage in mathematical sense-making and reflection through collaborative inquiry of challenging mathematics (NCTM, 2000, Zaslavsky & Leikin, 2004). As they collaboratively explore rich, open-response tasks, learners use questioning, reasoning, organizing, justification, and refutation to build meaning and to embrace a more robust conception of mathematics (Zaslavsky & Leikin, 2004). An environment where pre-service teachers are free to choose which tools (i.e. physical manipulatives, graphing calculators) they will use in their exploration further expands their abilities to engage in sense-making.

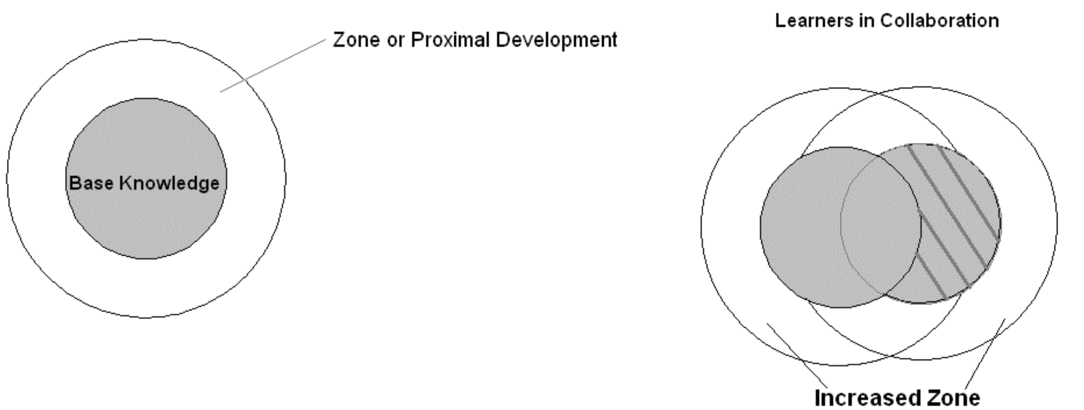
Electronic technologies—calculators and computers—are essential tools for teaching, learning, and doing mathematics. They furnish visual images of mathematical ideas, they facilitate organizing and analyzing data, and they compute efficiently and accurately…When technological tools are available, students can focus on decision making, reflection, reasoning, and problem solving. (NCTM, 2000; p. 24)

Zone of Proximal Development

Leon Vygostsky theorized that learners have a repertoire of knowledge which provides a foundation with which to build new learning. He argued that there are limits to both a learner's base knowledge and the range of new learning one can comprehend. For example, a small child with a limited understanding of numbers cannot comprehend complex algebra even if it were explained in detail. Vygotsky defined the area between what a learner already knows and

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

understands and what they are capable of doing, the Zone of Proximal Development (ZPD). (Rieber & Carton, 1987) This can be thought of as two circles: one representing the knowledge base and the other representing the ZPD. Any new knowledge which lies outside the ZPD is learned with great difficulty, if at all. Only knowledge within the ZPD can be assimilated into the actual knowledge case. Thus, “What lies in the Zone of Proximal Development at one stage is realized and moves to the level of actual development at a second. In other words, what the [learner] is able to do in collaboration today, he will be able to do independently tomorrow” (Rieber & Carton 1987, p. 211).



Vygotsky also proposed that the individual ZPD of learners working collaboratively is the combination of both.

What collaboration contributes to the child’s performance is restricted to limits which are determined by the state of his development and his intellectual potential. In collaboration, the child turns out to be stronger and more able than in independent work. He advances in terms of the level of intellectual difficulties he is able [to] face (Rieber & Carton, 1987, 209)

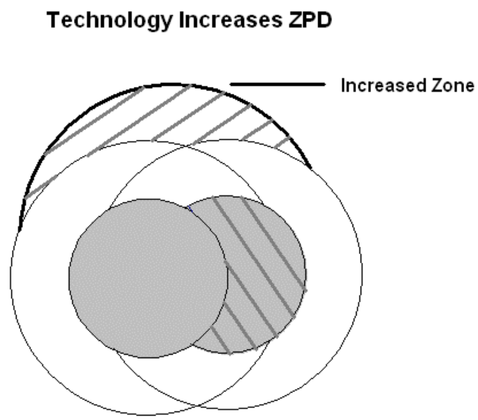
Essentially, the ZPD of a learner working collaboratively is larger than the learner working individually.

Technology as a partner

It is also well-documented that technology can be used as a tool to increase learning. Goos, et al. (2000), assert that technology, especially calculators, in some instances can be used by the learner as an exploration partner.

Here a rapport has developed between the user and the technological device – which may even be addressed in human terms. A graphics calculator, for example, becomes a friend to go exploring with rather than merely a producer of results. The user is still in control, but there is appreciation of the fact that calculator generated outcomes cannot be blindly accepted but need to be judged against mathematical criteria. Exploration, for example, in a graphical work, lead to situations where the output needs to be checked against the known properties of related graphical forms. It is possible for the calculator to be misleading, and a feature of its use in this mode is the way in which the respective authorities of mathematics and technology are balanced (Goos, 2000 p. 312).

In this paper, we continue this line of thought and argue that technology, in particular a graphing calculator, can serve as a pseudo-collaborator to increase the ZPD of a group of learners and thereby increase the base knowledge of individuals in the learning group. We use the term pseudo-collaborator for the role that the graphing calculator can serve because, although technology can contribute in a community of inquiry in much the same way as a collaborator, it lacks several of the characteristics of regular collaborators; i.e. the ability to ask questions, make value judgments etc. Nevertheless, technology can aid in the assimilation of new base knowledge that would not be possible with out it.



Setting

Our research covered two sections of a course in mathematics task-development and assessment designed specifically for preservice-secondary and -middle school mathematics teachers. Both sections, one a spring term and the other a fall semester, contained only students who had declared mathematics education as a major or a minor. The spring term class met for two hours once a week for six weeks and the fall semester class met for one hour a week for fourteen weeks. Each class consisted of about 24 students seated, at most, six to a table. Most of the students were in the latter half of the four-year undergraduate program.

In the summer course, a focus group was chosen from the four self-selected table groups because it was a particularly vocal group during collaborations. In the fall course, we chose the focus table before the students came in on the first day, so the focus group chose to sit at the focus table. In both cases, the same focus group was followed throughout the length of the course.

In class, the students collaboratively explored a series of rich and challenging open-response tasks. At the conclusion of each task, each table presented their explorations and their results to the class as a whole. The syllabus required students to have a graphing calculator although they were never told how or when to use it. The classroom itself had several white boards, graph paper, rulers, markers, transparencies, and other tools available for student use as they deemed appropriate.

Research Questions

1. How is technology used by pre-service teachers in an open-response, task-based environment?

2. How can technology be used to increase Zones of Proximal Development in a collaborative open-response, task-based environment?

Method

The collection of data consisted of audio or video taping the same focus group during each class period and collecting their completed work submitted for grading. The tapes for each day were then recorded on a server and viewed by multiple researchers. One researcher transcribed and time-coded each tape and another researcher verified the transcript and time codes.

The methodology of grounded theory was used in the analysis of audio and video data, wherein data was analyzed and reanalyzed until a theory, backed by strong evidence, was built to explain what happened in the class (Strauss and Corbin, 1998). Audio and video data were described, transcribed, and analyzed several times individually and in collaboration. Analyses were then triangulated with collected work.

Data Sources

In our research, we studied two particular tasks that each class explored. Each focus group spent about three hours in class exploring each task. The first task, the Timpanogos Cave Trail Task, was given as a project design request to the students in letter form from a fictional agency, as follows:

Dear Student,

We are trying to plan a new stroller and wheelchair accessible trail to the Timpanogos Cave. Your Teacher, Dr. Gerson, told us that you might be able to help us. We are placing the base camp at a place where the mountain is fairly smooth and evenly steep. We have enough room to make switchbacks up to 200 yards long, but they may be shorter than that. The difference in elevation between the cave and the base camp is 1100 feet. In order to be wheelchair accessible, the trail must have no more than 5% grade. We have found that a fit person can walk about 2.4 miles per hour pushing a baby in a stroller on a 5% grade. We would like you to design a trail and model a person climbing the trail. How long will the trail be? How long will it take to climb it? What else can you tell us that would be useful to us? If you need any further information you can contact us through your teacher.

Thank you in advance for your help.

Sincerely

Rock McCave

The Committee for Greater Access

Data Example: Timpanogos Cave Trail Task, Spring Term 2005

In the creation of a model, the group designed a trail on paper that zigzagged up the mountain. They calculated the length of the trail as well as the time it would take a person, walking, to reach the cave entrance. They then struggled to model their trail on a graphing calculator. Mary decided that the model, using parametric equations, should be able to represent the position of a person walking up the trail at any given time. Derrick's initial idea involved parametric equations using a periodic function such as sine or cosine. The group realized that the cosine function did not exactly model the straightness apparent in their model on paper. In collaboration, the group sought for a model that would more closely fit the design of their trail. In response, Mike proposed another idea to make the trail straight by including the function

$(-1)^n$ as part of their equation. He hoped that this would alternate the slope of the model from positive to negative and back.

Derrick	But then, how do we get it so it bounces back and forth? That's the, that's the number one question.
Ida	We could just design our trail to be curved
Mike	Um, if you want to, if you want just a switchback can't you do to the negative to the, negative one to the power n? [(-1)^n] … That goes positive negative positive negative.
Mary	Oh yeah, that is a… an oscilli
Derrick	So it would be, so what's that do on the calculator?

Table 1.

Mike suggested that incorporating $(-1)^n$ in the parametric equations for their model would change the slope from positive to negative to positive and so on in such a fashion that one set of parametric equations would be sufficient to model the entire trail. With Derrick's question, the students turned to their calculators. The students became distracted for several minutes, at the end of which, Derrick shared his graphing attempt.

Derrick	Ok this is, this is what I got
Mike	Did you do that $(-1)^n$ thing?
Derrick	Yeah, this is what happened
Mike	Oh, that's piecewise

Table 2.

Mike was surprised by the graph he saw on his calculator. The other members of the group continued to discuss other aspects of the task while Mike worked independently to make sense of the graph of $(-1)^n$ on the calculator. After several minutes, Mike realized one of the problems with his idea was that the graph was not at all continuous.

Mike	It's not continuous
Mary	It wouldn't show up on my graph
Mike	I don't, no, it's not continuous
Mary	It's just points
Mike	It's piecewise
Mary	But it's a function though. How can it be a function?

Table 3.

After another brief interplay, Mike exclaimed, "It's not going to work, this was a waste of my time." He had finally realized that his idea would not model the trail as he had hoped. Mike discovered that the $(-1)^n$ idea that he had hoped would yield either 1 or (-1) for any value of n did not always yield real numbers. With this new knowledge, he came to the conclusion that his function was "useless."

Mike	I just find it hard to belief that that is correct… So this function, this function is useless isn't it?

Table 4.

As the group members worked independently and collaborated with each other, they used their calculators to make sense of the function $(-1)^n$ and ultimately reject it as a possible model for the trail. This is a brief example where the calculator acted as a pseudo-collaborator. The calculator not only increased the group's ability to thoughtfully explore ideas but broadened the scope of ideas they were able to explore. Thus, the calculator increased the ZPD of the group, which served to broaden their base knowledge of mathematics.

Data Example: Placenticeras Shell Task, Fall Semester 2005

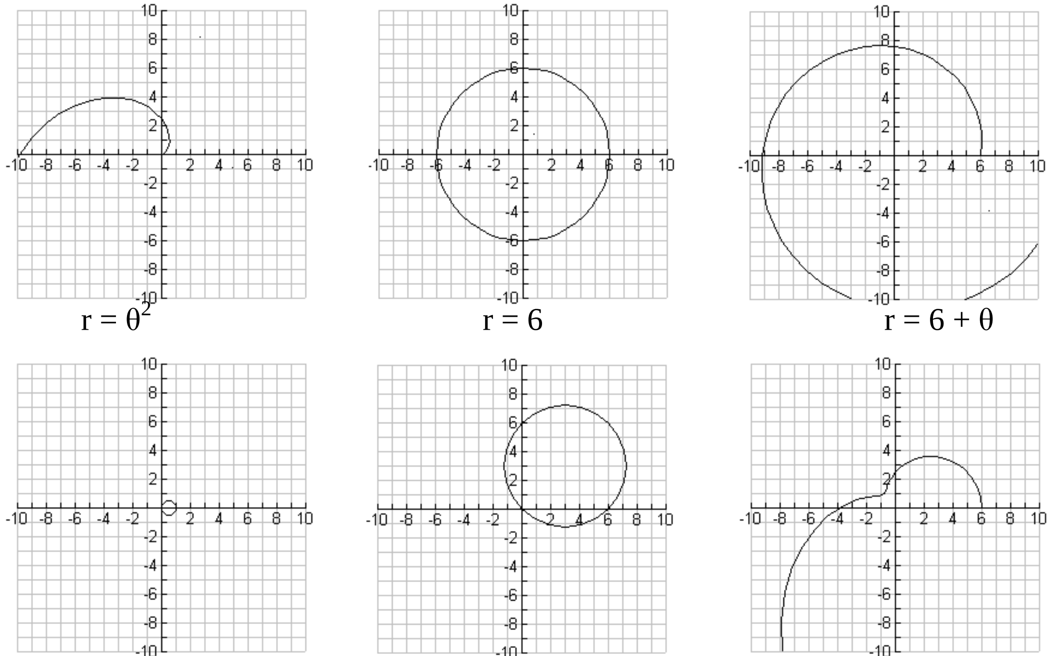
In the second task, the Placenticeras Task, students were each given a picture of a Placenticeras, a spiral fossil, and these instructions:

Placenticeras: You have been given a copy of a Placenticeras fossil, an ammonite that fell to the bottom of a shallow sea 170 million years ago, found near Glendive, Montana. The shell has been enlarged by a factor of 3.5 in order to make the shell structure more visible. Carefully locate the center of the shell and as accurately as possible draw a set of axes. Find a way to represent the spiral as a function.

Figure 1.

The group worked together in an attempt to create their own version of the spiral. It seemed to be understood that the final model would be created on the group's graphing calculators. They discussed the options of using either polar or parametric equations, finally deciding on polar equations. One student in particular, Taylor, used his calculator as a pseudo-collaborator to add to his limited knowledge of polar equations. He used his discoveries to find a way to construct a spiral that would model the placenticeras shell.

At one point in his exploration, Taylor explored polar equations by entering a series of equations into his calculator. To build his knowledge, Taylor entered an equation into his calculator, viewed the graph for several moments, modified the equation, and viewed the results again. While building some initial knowledge of polar equations, Taylor followed this pattern and created a series of six equations. It can be seen from the figure that not all of these graphs have the appearance of a spiral. However, Taylor gained an understanding of how to manipulate polar equations.



$r = \theta^2$ $r = 6$ $r = 6 + \theta$

$r = \cos(\theta)$ $r = 6\cos(\theta) + 6\sin(\theta)$ $r = 6\cos(\theta) + \theta^2$

Figure 2.

Taylor gained a sense of how to manipulate polar equations from this exploration. He and his calculator collaborated to build this knowledge. In essence, Taylor put together a question in the form of an equation, asked it to the calculator, and created a new question with the results. The calculator served as a pseudo-collaborator in this process. It did not ask questions nor give suggestions but it did give responses to questions that Taylor asked. By using the calculator in this way throughout the class, Taylor was able to develop a spiral that very accurately modeled the spiral given to him.

Conclusions

In both of these examples, technology played an intricate part in the development of new knowledge for the students using it. As a pseudo-collaborator, the technology added to the breadth of learning the student achieved as it provided a place for the students to investigate ideas and to test assumptions. Often these results prompted the students to ask additional questions. This led to discovery that arguably would not have been attained without the presence of the technology. The technology literally functioned as part of the collaborative team. It was not accepted as the authority but almost as an individual who could provide information that aided in discovery.

With this in mind, we would like to point out some of the commonalities between the situations where we observed technology acting as a pseudo-collaborator. First, obviously the technology needs to be in the hands of the students. An example from a teacher may serve as a teaching tool but without the students actually interacting with the technology, it cannot take on the role of a pseudo-collaborator. Second, a student must be familiar with the technology and its capabilities. The more familiar a student is with a particular technology, the more a student will learn through using it as a pseudo-collaborator. Third, prudent task selection increases the chance that technology can act as a pseudo-collaborator. Tasks must be open-ended and exploration-based in order to promote the use of technology as a pseudo-collaborator. Fourth, dynamic technologies are more likely to function as pseudo-collaborators. Dynamic capabilities enable students to engage in more discovery learning and exploration which heighten the chance of the technology becoming a pseudo-collaborator.

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THE IMPACT OF VIRTUAL MANIPULATIVES ON STUDENT ACHIEVEMENT IN INTEGER ADDITION AND SUBTRACTION

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The study investigated the impact of virtual manipulatives on student achievement in integer addition and subtraction. Participants were 99 sixth-grade students in six mathematics classes. Results showed that students made significant achievement gains in both integer addition and subtraction using three different virtual manipulatives. The general conclusion is that the virtual manipulative environments supported students' learning of these concepts.

The purpose of this study was to investigate the impact of virtual manipulatives designed for integer addition and subtraction instruction on student achievement and to examine the effects of features available in virtual manipulative models. If used properly, technology allows the creation of a more student-centered learning environment that can promote greater understanding of mathematical concepts (Norum, Grabinger, & Duffield, 1999). Instructional media can be classified in two major categories: dynamic versus static media and interactive versus static media (Kaput, 1992). Dynamic media make it possible for notational objects to change; interactive media allow for a much stronger constraint and support structure than other types of media (Kaput, 1992). These features allow the learner to manipulate objects, observe changes, and make connections. Linked representations facilitate the transition from the physical to the abstract—a key area in the development of mathematical concepts (Kaput, 1992). Computer-based environments, such as virtual manipulatives, support these features.

While research on the effectiveness of virtual manipulatives is still in its preliminary stages, a recent review of the literature indicates that students using virtual manipulatives either alone or in combination with physical manipulatives demonstrate significant gains in mathematics achievement and understanding (Moyer, Niezgoda, & Stanley, 2005; Reimer & Moyer, 2005; Smith, 1995; Suh, 2005). Further, teachers involved in studies using computer-based and virtual manipulatives report that students appear to be more engaged, on task, and motivated than when using physical manipulatives (Drickey, 2000). Due to the accessibility of virtual manipulatives and the potential of these tools to impact student achievement, the question is less whether to use virtual manipulatives in mathematics instruction and more of when and how to use virtual manipulatives appropriately and effectively. Research must begin to explore the features of virtual manipulatives that have the most impact on learning as well as the best methods for taking advantage of those features. This study was designed to contribute to that process by examining the impact of using different virtual manipulatives on student achievement in integer addition and subtraction.

The participants in this study were 99 sixth-grade students in six mathematics classes in two middle schools in the same public school system. The age of participants ranged from 11 to 12 years. This study used a quasi-experimental pretest-posttest design. In the design six classes were randomly assigned to one of three virtual manipulative treatment groups: Virtual Integer Chips, Virtual Integer Chips with Context, and Virtual Number Line. Each group received instruction in both integer addition and integer subtraction using one of three virtual manipulative treatments.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Data were collected using integer addition and subtraction pre- and posttests and task-based interviews conducted with a subset of randomly selected students from each treatment condition. Analysis procedures examined differences in student achievement based on the results of the pretest and posttest measures. Overall findings revealed that students in each of the three virtual manipulative treatment groups made significant pretest to posttest gains ($p < .01$) on both integer addition and integer subtraction concepts with effect sizes ranging from medium (lowest Cohen's $d = .63$) to large (highest Cohen's $d = 1.83$). An analysis of the most difficult subtraction items showed that students made significant gains ($p < .01$) in all treatment groups with large effect sizes (lowest Cohen's $d = 1.26$). Analysis of differences in the posttest scores among the three treatment groups indicated that students' posttest performance on integer addition and subtraction items were similar. When the groups were examined by student ability, analysis revealed no significant main effects for treatment nor was there evidence of an interaction between these two variables on either the addition or subtraction tests. Qualitative analysis of students' work in the task-based interviews revealed that students were able to work with integers using various representational forms (symbols, words, and pictures). In addition, students used various forms to facilitate, explain, and self-evaluate their work on integer addition and subtraction tasks.

The results of this study indicate that these virtual manipulative environments supported students' learning of integer addition and subtraction concepts. The virtual manipulatives were similar in that they employed several instructional design features proven to be effective across various media (Mayer, 2003). They differed in that they presented different models for integers and differed based on a few specific features. However, an analysis of differences in the posttest scores indicated that students' posttest performance on integer addition and subtraction items did not significantly differ among the three treatment groups. The general conclusion based on these results is that the virtual manipulative environments supported students' learning of integer addition and subtraction concepts. In addition, specific features shared by the three virtual manipulatives used in this study, including dynamic linked representations, interactive capabilities, multiple representations, and immediate feedback, appeared to be most important in supporting learning, and, in particular, enhanced student learning of the most difficult subtraction items.

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TEACHING INNOVATIONS FOR PROBLEMS INVOLVING RATES IN CALCULUS

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Related rates problems in first semester calculus are a source of difficulty for many students. These problems require students to be able to visualize the problem situation and attend to the nature of the changing quantities. I have developed a sequence of teaching activities that employs a computer program designed to foster the students' exploration of related rates problems in a covariational context. I investigated the impact of these activities on students' abilities to understand and solve rate of change and related rates problems. I present the results of the first two activities which focus on rate of change here.

Background

Little research has been published on how students understand and solve related rates problems in first semester calculus. The research to date suggests that students have a procedural approach to solving related rates problems (Martin, 2000; White & Mitchelmore, 1996). When solving a related rates problem, students tend to focus on using an algorithm that essentially consists of the following steps: draw a diagram, choose a geometric formula, differentiate it, substitute in values, and solve (Engelke, 2004). A student may need to engage in covariational reasoning to construct a mental model that accurately reflects the problem situation and that may be manipulated understand how the problem situation works (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Engelke, 2004; Saldanha & Thompson, 1998).

The Study

I conducted a teaching experiment consisting of six teaching sessions with a group of three students from my calculus class in the Fall 2005 semester. The participants were chosen from a group of volunteers and met for these teaching episodes outside of the regular class sessions. They did not attend the regular class periods in which related rates were taught to the complement of the class. The students were paid for each session they attended, and each teaching episode was videotaped and transcribed for analysis.

Results

In the first session of the teaching experiment, the students used a custom computer program to investigate the average rate of change and instantaneous rate of change for some common geometric problem situations. For example, the students were asked to consider the following problem: Suppose we have a plane that is flying over a RADAR tower, TA, and is on course to pass over a second RADAR tower, TB. Let u be the distance between the plane and TA, and let v be the distance between the plane and TB. What is the rate of change of u in relation to v? The computer program allowed the students to have a visual representation that may be manipulated to observe what happens as they make the plane move. The students decided that it would be helpful to have time given so that they may compute average velocities: $\frac{\Delta u}{\Delta t}$ and $\frac{\Delta v}{\Delta t}$. To allow the students to do this, another version of the plane problem was opened in the computer program that allows the students to observe what happens to each variable, including time, as

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they moved the plane and allows them to generate a table of values for the variables. After computing the average velocities, the students thought that they could relate u and v by flipping and multiplying to cancel out the Δt's.

The concept of rate as one quantity as opposed to the ratio of two independent quantities appeared to be difficult for students to grasp. Throughout the first and second sessions the students struggled with whether the delta t's really cancel in the above plane problem. Ali chose to open the second teaching session with the question: "I know I can relate u to time and v to time but I don't know how to relate them to each other." Amy and Ben referred back to the previous meeting saying that you just "flip and multiply" to cancel out the Δt's, suggesting that they may not have internalized the notion of rate.

In the second teaching session, during a discussion of the chain rule, the students began to think about rate as one quantity versus two. The students argued about whether they could really cancel the deltas when multiplying rates. This shift in thinking allowed the students to begin relating rates in other situations. This exploration allowed the students to choose time as a common variable through which they may relate variables, a common practice in related rates problems which were the focus of subsequent teaching sessions.

Conclusion

Time as a variable was student generated in these problem situations. Students' interactions with the computer program likely cultivated the students' thinking about how each variable changes across time and may have helped them internalize the notion of rate. The ability to imagine each variable as it changes across time and as a function of time may also foster students' understanding of the chain rule and its application to related rates problems. The data suggests that the use of the computer program to visualize problem situations and measure quantities can aid students' development of mental models in future problem situations and their understanding of the concept of rate.

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HIGH SCHOOL MATHEMATICS TEACHERS' USE OF MULTIPLE REPRESENTATIONS WHEN TEACHING FUNCTIONS IN GRAPHING CALCULATOR ENVIRONMENTS

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This study explored how teachers plan their lessons to accommodate multiple representations and how graphing calculators affect the teachers' approaches to teaching functions. Participants were four high school mathematics teachers, while data were collected via task-based interviews and classroom observations. Results showed that calculators influenced the teachers' selection and design of instructional tasks while the tasks mediated calculator usage.

Purpose of the Study

The concept of function has been widely recognized as being foundational to school mathematics and mathematics in general (Romberg, Carpenter & Fennema, 1993). Research has shown that graphing calculators can improve students' conceptual understanding of functions by allowing the students to explore the various representations of a function (Penglase & Arnold, 1996). The National Council of Teachers of Mathematics (NCTM, 1989, 2000) advocate a curriculum based on multiple representations, arguing that by encouraging students to incorporate many different types of representations into their sense-making, the students will become more capable of solving mathematical problems and understanding underlying concepts. In this paper, we explore how high school mathematics teachers use multiple representations when teaching functions in graphing calculator environments. We pay special interest to how teachers plan their lessons to accommodate multiple representations of functions when teaching with graphing calculators and how the calculators in turn influence the teachers' approaches to teaching functions. We also seek to explore the effect of the teaching strategies and instructional tasks on the ways in which graphing calculators are used.

Perspectives and Guiding Frameworks

This study draws on a theoretical framework developed by Salomon, Perkins, and Globerson (1991) for studying the interaction between technology and the user. In this framework, Salomon et al. distinguish between two sets of principal effects that arise when works in partnership with a technology tool, namely (1) principal effects with the technology and (2) principal effects of the technology. For purposes of clarity, we refer to the first set as planned effects, and the second set as emergent effects. The work of Goos, Galbraith, Renshaw, and Geiger (2003), which provides metaphors for studying the interaction between calculator and user, is closely related to this partnership framework and so we draw parallels to the metaphors when discussing some of the principal effects. Characteristics of planned effects include elaborate planning (laying out the specifics concerning how the calculator will be used), executing the plan (using the calculator in the desired ways), and interpreting the results. The teacher here predetermines exactly when it will be appropriate to turn to the calculator in the course of a lesson and in what ways this should be done

Emergent principal effects on the other hand are characterized by spontaneity, that is, effects that the teacher does not intentionally plan for. These effects are then retained and may be

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applied to other related but not calculator dependent mathematical activities (Jones, 1993). Using the metaphor of technology as partner, Goos et al. (2003) describe “cognitive re-organization effects” (p. 79) as those characterized by using technology to explore new tasks or new approaches to existing tasks and to mediate mathematical discussion in the classroom between students and teacher or between small groups of students. We contend that for meaningful principal effects of technology to arise in a classroom, the teacher must be willing to allow his or her students to explore new situations with the calculators and guide the students into discussions that will help them make sense of their findings. This study I investigated how principal effects that are planned for and those that emerge are manifested in secondary mathematics classrooms where graphing calculators are used.

Methods, Data Sources, and Analysis

Participants in this study were four high school mathematics teachers drawn from three high schools in a medium-sized city school district in northeastern United States. Data were collected through semi-structured and task-based interviews (Goldin, 1999) as well as classroom observations. The interview questions were divided into four major categories, namely (1) planning (what are the key things that teachers consider as they prepare to teach lessons on functions especially when they intend to use graphing calculators?), (2) sources of teaching tasks (where do teachers get their teaching activities/tasks and how do they use these tasks, i. e. do they modify them or not and what are the reasons for this?), (3) function representations (teachers presented with various tasks and asked to respond to the tasks as well as speculate on how their students might respond to those tasks), and (4) issues related to calculator usage.

Categories (1) and (2) helped us develop some insights into how teachers envision a lesson on functions in which graphing calculators are used and what outcomes they might expect, thus shedding some light on the planned principal effects. Categories (3) and (4) helped shed some light on the teachers’ choices of representation in various situations, the kind of partnerships these teachers had developed with graphing calculators, and the kind of expectations the teachers held for their students when using graphing calculators. This was important to this study since the tasks provided a common ground for all the four teachers given that no two teachers taught the same lesson.

During classroom observations we took note of both the teachers’ and the students’ interactions with graphing calculators, paying special attention to how the teachers facilitated the interaction between students and calculators. In this regard, we examined the kind of instructions the teachers gave to their students, the actions the students took and the questions they asked their teachers as well as their peers, and how the teachers responded to the students’ questions. All these helped provide data that would later be analyzed for emergent effects of technology.

Data were analyzed in two phases. In phase I, we carried out a microanalysis of the interview data for all the teachers, identifying broad theme statements from the interviews based on dominant phrases in the teachers’ responses to items under the categories of planning and sources of tasks and also on the actions they took while attending to items under the categories of function representations and issues related to calculator usage. In phase II, we analyzed the data from classroom observations against the statements generated above. We tried to identify situations from the classes that could support these statements (or sometimes challenge them). We then refined the statements into three major themes, namely (a) teaching strategies, (b) types of instructional tasks, and (c) representational forms that emerge

Results

While classroom organization varied from teacher to teacher, all teachers seemed to value involving students in decision making regarding calculator use. This would range from asking students to suggest what to do in order to get started with the calculator with respect to given information, to asking students to suggest how to modify various calculator menus in order to achieve various desired results. Often times the teachers encouraged students to share their work with the whole class using the calculator overhead projection unit. It was also common for teachers to ask probing as well as clarification questions. Occasionally the teachers would ask questions requiring students to compare solutions obtained using different representations and explain the differences if any. The teachers would also challenge students to interpret calculator results in the context of the problem situation and communicate their understanding of the calculator results to their peers.

Although in most cases the teachers seemed to balance among the various representations, equations and graphs seemed to dominate more than tables. Most instructional tasks made specific reference to either an equation (18 of 41) for which a graph would be drawn and various explorations done on it, or a graph (13 of 41) on which various explorations would be done. Only 10 of 41 tasks specified use of tables. In cases involving word problems, it was common to see equations being generated then graphs drawn.

Our analysis indicated that the choices for instructional tasks and teaching strategies are not unique to particular teachers; what seems to be unique however, is the pattern of representation forms that the teachers use. While some teachers will prefer to move from equation to graph and possibly to tables, others prefer going from equation to tables then graphs.

The first step towards developing intelligent partnerships with technologies is for the user to be able to plan on how to use the tool, execute the plan, and interpret the results. Results of this study indicate that teachers can help their students towards this end by guiding them to actively participate in the process of working with calculators either in small groups or as individuals. The teachers in this study had plans on how they wanted their students to use the calculators in the classroom, but they often times gave the students a chance to suggest their own approaches first.

The second step towards forming intelligent partnerships with technologies is for the user to gain new insights that can be transferred to other situations where the technology tool is not necessarily used. The teachers in this study tried to help their students towards this end by requiring them to interpret their solutions to real life situations and also to explain their answers to their peers. This would ensure that the student develop some kind of ownership to the knowledge they were acquiring and hence be in a better position to retain it beyond the classroom.

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THE EFFECTS OF UNDERGRADUATE MATHEMATICS COURSE REDESIGN ON STUDENT ACHIEVEMENT

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Purpose

While the traditional lecture dominates college and university classrooms, research shows that students need to do more than just listen. Much has been written about the need for active learning in postsecondary classrooms (Sutherland and Bonwell 1996; Chickering and Gamson 1991; McKeachie, Pintrich, Lin, and Smith 1987). A student who is *actively* involved in the learning process (rather than sitting in a room *passively* listening while an instructor lectures) will have improved learning and retention of that knowledge. Including instructional technology in this paradigm has led to even more success in improving student learning. In order to implement such a shift in the learning paradigm in our large enrollment mathematics classes, a redesign of how these courses were taught had to be undertaken. The College Algebra and Precalculus courses were redesigned based on the mathematics replacement model (Twigg, 2003). This model replaces traditional lectures with a variety of learning resources such as interactive software that encourages active learning, prompts ongoing assessment, and provides individualized assistance. There are significant failure rates in College Algebra and Precalculus. Redesigning these courses has allowed us to take advantage of computer technology available on campus and to provide students with options of choosing their best learning conditions and with opportunities to enhance their learning using resources beyond boundaries of time and space. Our goals were to significantly reduce the DWF rate in College Algebra and Precalculus and to better prepare students for subsequent courses in mathematics.

Theoretical Framework

The major focus of this study is to investigate the effects of the technology-rich environment of the redesigned course on students' learning and retention of mathematical concepts as they progress into successive mathematics courses, in particular, Calculus. It was determined that a guiding philosophy was needed to suggest principled changes in the curriculum and effective uses of technology as part of these changes (Forman & Pufall, 1988). Bruner's constructivist theory is the framework that guided these curriculum changes.

Constructivism is a theory of cognitive growth and learning. According to Bruner (1960), one fundamental idea of constructivism is that students actively construct their own knowledge. Students assimilate new information to simple, pre-existing notions, and modify their understanding in light of new data. Educational applications of constructivism exist in creating curricula that match, but also challenge, students' understanding, fostering further growth and development of the mind. Learning must be interactive (Cobb, 1994). The technology used in the redesigned course allows the students to assemble and modify their ideas, access and study information. The instructor engages the students by helping to organize and assist them as they take the initiative in their own self-directed explorations, instead of directing their learning autocratically.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Methods of Inquiry

The redesigned courses were College Algebra and Precalculus, both introductory courses enrolling over 2000 students each year in 41 sections. This study follows a quasi-experimental design because the participants cannot be randomized. The treatment group contains students who have completed an entire cycle through the redesigned College Algebra and Precalculus courses and are now in Calculus. As a part of the redesigned courses, these students divided their time equally between a classroom and The Mathematics Interactive Learning Environment (The MILE), a technology-driven facility that provided an array of interactive materials and activities. The control group consists of students previously enrolled in the traditional lecture-driven College Algebra and Precalculus courses and are now also in Calculus.

We will employ the following assessment techniques suggested by Peter Ewell, Senior Associate at the National Center for Higher Education Management Systems (Twigg, 2003): matched examinations, student work samples, behavioral tracking, and interviews. Both groups will complete a pre- and posttest designed to assess changes in their content knowledge. During previous semesters, the control group attended the regular classroom meetings in which the instructors primarily use the lecture method. The treatment group's intervention employed numerous classroom and web-based activities (available in The MILE).While working in the MILE, students worked one-on-one with instructors, graduate research assistants, and peer tutors.

The course redesign appears to be successful, but we now have the charge of thoroughly evaluating the redesign for several effects. We are currently analyzing data in order to: (*i*) investigate the variables that affect student learning in the successive courses and; (*ii)* analyzing the students' perceptions of their own preparedness for subsequent mathematics courses. Below, these steps are described in more detail.

i. Investigate the variables that affect student learning in the redesigned courses;

To answer the question "Did they really learn?" we will employ the following assessment techniques suggested by Peter Ewell, Senior Associate at the National Center for Higher Education Management Systems (Twigg, 2003):

Common Examinations: This refers to a final examination with selected common items that is administered to students to allow us to analyze instructor and student effects.

Student Work Samples: We will select a few examples of work that students complete as a part of the course. Once a reasonable sample (n=20 or so) from each class is assembled, the pieces can be cross-scored by a reading team using a scoring guide to look at things like communications ability, mastery of particular areas of knowledge, and so on. We will compare the two groups' performance on tasks in Calculus, based on their previous preparation.

Behavioral Tracking: This approach relies on following students who were enrolled in parallel sections (innovative and traditional) through student records to see what happened to them later. Several dimensions of behavior are especially useful to look at here, including:

- Course completion rates;
- Program completion/graduation;
- Grade performance in subsequent courses for which College Algebra and Precalculus are prerequisites.

ii. Research the students' perceptions of their own preparedness for subsequent mathematics courses;

The team will conduct interviews of students who have completed an entire cycle through the redesigned College Algebra and Precalculus courses and are now in Calculus. The purpose of these interviews will be to investigate how these students perceive how the redesigned courses prepared them for Calculus. We will also collect data on the students' performance in Calculus. We will collect data on the DWF rates and compare them to the past years in these courses.

Results and Conclusions

The data for this study is still being collected. The researchers will combine qualitative and quantitative methods to develop the instruments for data collection in future semesters. The results of the research studies will be used for revision of initially developed materials, development of new materials and for assessment of the success of the whole program in general. The researchers will determine the impact of the redesigned, student-centered learning environment on student achievement in successive courses.

This study is aligned with the goals of PME-NA to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics. There are opportunities for further studies on topics such as students' understanding of specific algebraic concepts, appropriate and effective technology use in the mathematics classroom, improvement of instruction and undergraduate mathematics education.

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WAYS OF REASONING AND TYPES OF PROOFS THAT MATHEMATICS TEACHERS SHOW IN TECHNOLOGY-ENHANCED INSTRUCTION

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In this report we document and analyze the ways of reasoning and types of proofs employed by high school mathematics teachers to validate conjectures and to justify procedures in order to solve problems, which emerge when they work in a dynamic geometry environment.

Proof is a fundamental activity in mathematical practice (Hanna, 2000; Weber, 2001), also is a key element in school mathematics (National Council of Teachers of Mathematics, 2000). However, research in mathematics education evidences that students have serious difficulties in understanding and presenting deductive proofs (Harel and Sowder, 1998; Schoenfeld, 1985).

Why do students experience difficulties in constructing mathematical proofs? One reason may be related to the way used to introduce aspects of proofs in mathematical instruction. Balacheff, 2000 argues that instructional methods rely on asking students to imitate their teachers behaviors.

In this context, we consider that in order for students to identify the proving activity as a central in their mathematical experiences, teachers need to have solid understanding of what the concept of proof entails (Stylianides, 2005), not only deductive proof, but also the use of arguments and justifications in general. In this context, it is necessary to carry out research studies that provide information to design instructional strategies and activities that encourage students to use distinct types of mathematical proofs, as well as observing dimensions and aspects that characterize them, this implies to focus on learners through studying teachers' behaviors.

Objectives

The goal of this study was identify the proof schemes (Harel y Sowder, 1998), showed by high school mathematics teachers, when they pose and solve problems using a dynamic geometry software (Cabri Géométre). This is, we were interested in documenting the rationality in which justification processes are based on, and to characterize the ways that teachers used certain type of reasoning when they employed dynamic software as an important part of the activity of doing mathematics.

Theoretical Perspective

The theoretical perspective of this work is based on the construccion of proof schemes (Harel & Sowder, 1998), and theory of problem solving (Polya, 1945; Schoenfeld, 1985). We use Harel and Sowder's taxonomy to explain types of convincing process used by students to construct their proofs (Harel & Sowder, 1998, p. 241) and because the appearance of those schemes is in accordance with the cognitive continuity among the discovery of mathematical relations, conjectures formulation and proof construction.

Alatorre, S., Cortina, J.L., Sáiz, M., and Méndez, A.(Eds) (2006). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Mérida, México: Universidad Pedagógica Nacional.

Method

Participants in this study were seven high school mathematics teachers (two males and five females), graduate students of mathematics education in México. None of the teachers had used Cabri before participating in this study but they had studied geometry at the high school or at university level.

Starting with either open geometric situation (e.g. simple dynamic configurations) or open problem, teachers were asked to pose problems using Cabri, solved them and justify their observations, as well as solutions procedures. These activities took place during fifteen sessions, ninety minutes each.

Data Sources

The data sources consisted in the Cabri electronic files of the activities developed by each teacher, in weekly written reports, a final report in which the participants were asked to put in writing their conjectures or theorems, and videotaped interviews.

Results

Main results of this work were that teachers often used in consistent way several proof schemes, mainly perceptual proof schemes and inductive proof schemes; likewise, the software supported significantly the use of transformational proof schemes and constructive proof schemes, which are related with the use of heuristics such as “considering a partial solution”, “working backwards” and “taking the problem as solved”.

Intuitive axiomatic proof schemes were also used in a consistent way, though with significant differences among teachers. Besides, it was identified a low performance in the written formulation of conjectures as conditional sentences; as well as a tendency to associating intuitive-axiomatic proof schemes with conviction on the truth of mathematical facts, leaving aside, apparently, the importance of empirical proof schemes.

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ATTITUDES, MATHEMATICS ACHIEVEMENT AND COMPUTERS: INITIAL PHASE OF A LONGITUDINAL STUDY

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A longitudinal study is being carried out with two groups that started first grade of secondary school in the Telesecundaria System: an experimental group that uses the spreadsheet as part of the mathematics class, and a control group that does not work with it. These two groups will be studied through out the three years of secondary school.

In this first phase of the study, both groups were given a mathematics test, as well as a Likert scale (AMMEC: Ursini et al, 2004) in order to measure attitudes toward mathematics. Attitudes were studied under the tripartite model which states that attitudes are conformed by affective, cognitive and behavioral components (Hernández and Gómez-Chacón, 1997; Ruffel et al, 1998).

Results show that the mathematics level and the attitude towards mathematics are slightly different in both groups: one that uses the spreadsheet in mathematics class and one that doesn't.

In the mathematics test, an average percentage of correct responses were observed slightly higher in the group that works with the spreadsheet than that of the group that does not.

Through a correlation test, a slight correlation was found between the mathematics score and the attitude towards mathematics.

Data was also gathered through interviews, which show that students that use computers in mathematics class are more self-confident when doing mathematics.

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GENERATIVE APPROACH TO THE DESIGN OF VIRTUAL MANIPULATIVES

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Virtual manipulatives (Moyer et al., 2000) are an alternative approach to the use of manipulatives to represent abstract concepts in mathematics. Currently, there are several Internet sources that contain virtual manipulatives, such as Illuminations at NCTM website and the National Library of Virtual Manipulatives at Utah State University website. There are also a considerable number of researches showing positive students' achievement results using virtual manipulatives in the classroom (Moyer, 2005; Moyer et al., 2005; Moyer & Bolyard, 2002; Reimer & Moyer 2005; Suh et al., 2005). A difficulty with virtual manipulatives is that there are not yet clear guidelines in the literature to design and develop these innovative tools.

In this poster presentation, I show the design of three different virtual manipulatives to help students to construct the concept of part-whole representation of rational numbers (Kieren, 1976). These virtual manipulatives follow the generative learning theory proposed by Merlin Wittrock (1974a, 1974b). Basically, the generation process points out that students need to generate two different types of relationships: first, among the different parts of the information that are being perceived; and second, between the new information and the learner's prior knowledge.

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AN INSTRUCTIONAL MODEL FOR LEARNING THE CONCEPT OF FRACTIONS WITH VIRTUAL MANIPULATIVES

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Recent research on virtual manipulatives has shown the utility of these new technological tools to construct mathematical knowledge (Moyer, 2005; Moyer & Boyard, 2002; Moyer, Bolyard, & Spikell, 2002; Moyer, Niezgoda & Stanley, 2005; Reimer & Moyer, 2005; Suh, Moyer & Heo, 2005). But, as Baroody (1989) stated, "simply using manipulatives does not guarantee meaningful learning" (p. 4). In addition, multimedia instructional messages, presented in virtual manipulatives, have also been a source of recent research from the educational psychology field (Mayer, 2001, 2005). Using these two concepts based on generative learning theory (Wittrock, 1974a, 1974b), we suggest that the SOI (Selecting, Organizing, Integrating) instructional model (Mayer, 1989, 1999) would help students to use virtual manipulatives to learn the fraction concept. These activities are motivated by the curriculum, teaching, learning, and technology principles stated by the National Council of Teachers of Mathematics (NTCM, 2000).

Based on different constructs (Kieren, 1976; Behr, Lesh, Post & Silver, 1983), "personalities" (Behr, Harel, Post & Lesh, 1992), or practicing representations (Greeno & Hall, 1997) of fraction concept, this study is focused on part-whole representation. As stated by Behr and Post (1988), "the part-whole notion of rational numbers is fundamental to the other interpretations" (p.192). For the current poster presentation, we use virtual manipulatives from the National Library of Virtual Manipulatives (http://nlvm.usu.edu/en/nav/index.html) to show three developed activities based on each step of the SOI model and the cognitive theory of multimedia learning. These activities are: generate notes to select the main information, generate summaries to organize the selected information, and generate examples to integrate the organized information with students' previous knowledge.

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