PME-NA XXVI

North American Chapter of the International Group
for the Psychology of Mathematics Education
October 2004
Toronto, Ontario, Canada

Volume 1

Editors:
Douglas E. McDougall
John A. Ross

Proceedings of the
Twenty-sixth Annual Meeting

North American Chapter of the International Group for the

Psychology of Mathematics Education

Volume 1

PME-NA XXVI
October 21 - 24, 2004
Toronto, Ontario, Canada

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Cite as:

McDougall, D.E & Ross, J. A. (Eds.). (2004). *Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

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Fax: 416-926-4744

ISBN 92-990027-0-3

Printed by Preney Print and Litho Inc., Windsor, Ontario

HISTORY AND AIMS OF THE PME GROUP

PME came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and the North American Chapter are:

(1) To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
(2) To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers;
(3) To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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2004	PME 28	Norway	

Preface

It is wonderful that the University of Toronto is able to host the 26th annual meeting of PME-NA. We are excited to invite our colleagues from Canada, United States, Mexico and other countries to join us in Toronto for the weekend of October 21 to 24, 2004.

The theme of this year's conference is *Building Bridges between Communities*. The conference will highlight examples of building connections between the communities of mathematics education such as mathematics teacher educators, mathematicians, teachers, students, and other stakeholders in mathematics education. The plenary speakers will address area in mathematics education where connections have been built between and among these communities.

We received over 300 proposals for paper sessions and 92 poster proposals for PME-NA 2004. We had a group of over 200 reviewers who volunteered their time to review between 3 and 10 papers. We reviewed the papers for proceedings by standardizing the formatting and references to leave the work of the authors virtually untouched. Special thanks to Lynn Kostuch for her work in reviewing the papers with assistance from Lisa Loughlin. Special thanks goes to Robert Fantilli for his work on the proceedings and program.

We wish to express our appreciation to the many people at the Ontario Institute for Studies in Education of the University of Toronto for their assistance in hosting this conference. Thanks to the PME-NA program committee, and Lara Cartmale, Dennis Thiessen, Bessie Giannikos, Roz Zulla of the Department of Curriclum, Teaching and Learning at OISE/UT.

Douglas E. McDougall
John A. Ross
September 2004

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MATHEMATICS, THE NOETIC AND THE AESTHETIC

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This paper seeks to build a bridge between the community of mathematics educators and the community of mathematicians by examining the nature of the mathematical aesthetic. Our sense of this aesthetic draws on the affinity between mathematics and the senses that the Pythagoreans appreciated and celebrated. We point to some of the important themes regarding this affinity that emerged in the twentieth century and discuss some of the new ways that the mathematical aesthetic plays out in the work, beliefs and values of modern mathematicians. Finally, we suggest some new directions in the psychology of mathematics education that our own work has encouraged.

Introduction

The main theme of this year's PME-NA conference—Building Connections Between Communities—has prompted us to turn our attention to the community of mathematicians, a cognate community that perhaps receives less attention than it might at PME-NA. In fact, one could argue that the "P" in *Psychology of Mathematics Education* has predominated over the "M." We are interested in taking this opportunity to nudge the "M" more into the limelight, by considering some of the psychological aspects of doing mathematics and being a mathematician that are seldom thought germane to problems in mathematics education. It is possible, and we would claim important, to undertake both research and scholarly inquiry within and about such a community (see, for instance, Sinclair, 2002, or Burton, 2004). [1]

In this paper, we have chosen to focus on concerns surrounding the notion of the mathematical aesthetic. We believe that, in addition to being of interest to those examining the psychological, sociological and philosophical aspects of mathematics teaching and learning, it also can and should inform certain topics that have more recently become prominent in mathematics education, including ones as diverse as embodied cognition, socio-mathematical norms and critical mathematics education.

We have recently edited a book entitled *Mathematics and the Aesthetic: New Approaches to an Ancient Affinity* (Sinclair, Pimm and Higginson, in press), in which contributors from various communities (including professional mathematicians, software programmers and philosophers, as well as mathematics education researchers) have explored connections between mathematics and the aesthetic. In this paper, we would like to point to some of the important themes that emerged, not only over the past century, but also within the chapters of this recent book. It has been written at a time of considerable resurgence in serious writing dealing with deeper relations between mathematics (and science) and 'the beautiful' (and at a time of apparently increased 'sexiness' of mathematics in popular books, movies and plays). We will also tentatively offer some remarks on potential future themes worthy of examination in this area in the coming century and, in particular, on their relation to mathematics, mathematics education, and the psychology of mathematics education.

The Noetic and the Aesthetic

Of all the disciplines taught in schools today, mathematics has perhaps the strongest roots to ancient philosophy and to the importance of knowing: a paper recalling or evoking that epistemological (rather than doxological, one based on *doxa*, 'opinions') connection could aptly be titled *Mathematics and the Noetic.* The Ancient Greeks made an epistemological distinction between the *aesthetic* and the *noetic*: whereas the aesthetic referred to the sensible (or perceivable) [2], the noetic referred to the reasonable (or conceivable), or the "purely intellectual" (as opposed to emotional or intuitive). (For more on this distinction, see the feisty book *The Baumgarten Corruption* by Robert Dixon, 1995,)

Such a title for a paper would seem far more conventional than the one we have actually chosen, in terms of how mathematics is customarily seen. But part of what our book's title and subtitle is asserting is mathematics' connection with an ancient perceived affinity with the senses. Even among the Ancient Greeks, some groups had different epistemological persuasions regarding mathematics, seeing it much more as an *aesthetic* discipline than a *noetic* one. The Pythagoreans, for instance, established and celebrated a fundamental affinity between the mathematical and the aesthetic. This affinity was nothing about surface charm or happy coincidences. It had deep roots, integral as it was to the world-view of the Pythagoreans, to their beliefs about knowledge and learning. It closely connected the raw world of sense and experience to the divine world of perfection and beauty. Number was the principle that governed all things, rather than being simply useful for counting or measuring—as modern minds might think, if indeed they stop to consider this omnipresent convenience at all. Through number, one could come to know the world, and through the harmonies found in numerical patterns and in geometrical forms, one could gain access to the clearest and most indubitable essence—the real.

The spiritual dimension of the Pythagoreans' views of mathematics and aesthetics may, for many, increase the slipperiness of the term *aesthetics*, whose modern usage often equates it to words such as 'beautiful.' So to bring some precision to the matter: for us aesthetic considerations concern *what* to attend to (the problems, elements, objects), *how* to attend to them (the means, principles, techniques, methods) and *why* they are worth attending to (in pursuit of the beautiful, the good, the right, the useful, the ideal, the perfect or, simply, the true). Note that such a specification applies equally well to mathematics as to art, historically the realm of much discussion of things aesthetic.

Two Centuries of the Mathematical Aesthetic

The richest (extant) discourse on the mathematical aesthetic prior to the twentieth century can be found in the writings of the Ancient Greeks. In many ways, more recent contributions to the literature revisit and draw on themes that were prominent over two thousand five hundred years ago. Only fragments of this discourse persisted from the Ancient Greeks to the beginning of the twentieth century, at which point an explosion of interest in the mathematical aesthetic took place, particularly around questions such as: *is mathematics an art or a science?* and *can criteria for mathematical beauty be identified*? We explore the literature on these questions in the next section, and then, in the following sub-section consider some of the new themes that seem to be emerging in the twenty-first century.

The Mathematical Aesthetic in the Twentieth Century

Though the eighteenth and nineteenth centuries were extremely fruitful in terms of mathematical discoveries and advances, it seems that mathematicians infrequently, as least in print, reflected on issues related to the mathematical aesthetic. This is not to say, however, that they did not think about or mention aesthetic values. Gauss's mathematical diary (see Gray, 1984), for example, contains many references to the beauty or elegance of his own mathematical ideas and discoveries. For instance, as a nineteen-year-old in 1796, Gauss wrote about a new proof obtained "all at once, from scratch, different, and not a little elegant" (p. 108). In another entry, this time made in 1800, he described his work on the arithmetic–geometric means as being "most beautifully bound together and increased infinitely" (p. 122) to the theory of transcendental quantities. In addition to beauty and elegance, Gauss made reference to aesthetic qualities such as a "charming theorem" (p. 125) and to a "most simple and expeditious method" (p. 124).

However, for some reason, the turn of the past century brought about a comparative flurry of interest in the nature of mathematics. In particular, there were concerted efforts to ascertain whether mathematics belonged more to the arts or to the sciences, from which it had not long ago been divorced (during the latter part of the nineteenth century, not least due to developments in connection with non-Euclidean geometry). It also marked the beginning of sustained inquiries into the development of mathematical knowledge and the extent to which it is fuelled by some aesthetic as well as utilitarian or logical considerations (which were usually seen as relatively distinct).

Finally, and early on in this flurry of activity, mathematicians became interested once more in the psychology of mathematical discovery. Some twentieth-century mathematical writers on the aesthetic turned to the central question of the extent to which affective responses and aesthetic sensibilities were involved in the process of mathematical creation. Their attention to the aesthetic was not as intense and all-encompassing as that of the Pythagoreans, but they each began, in their own way, to rekindle the embers of this ancient affinity. Here, we examine each of these themes in turn, tracing out, when possible, aspects of their historical developments.

The aesthetics of mathematical creation

In 1908, Henri Poincaré began to bring renewed attention to the aesthetic dimension of mathematical creation, but his focus was more pragmatic and markedly different from that of the ancient Greeks. He was most interested in probing the aesthetic influences that affect the process of mathematical discovery. This focus proved unlike that of many of the mathematicians who would follow him, who attended more to the aesthetic values or principles that exist in mathematical ideas or products (the discoveries themselves). By analysing the process of mathematical creation, Poincaré tried to show that mathematical invention depends upon the often sub-conscious choice and selection of 'beautiful' combinations of ideas, those best able to "charm this special sensibility that all mathematicians know" (1908/1956, p. 2048).

In his book *The Psychology of Invention in the Mathematical Field,* Jacques Hadamard (1945) proposed the first expansion of Poincaré's aesthetic heuristic theory, additionally claiming that aesthetic sensibilities often guide a mathematician's general choices about which line of investigation to pursue. He wrote specifically about the "sense of beauty" (p. 130) which can inform the mathematician that "such a direction of investigation is worth following; we feel that the question *in itself* deserves interest" (p. 127; *italics in original*). Hadamard also added to Poincaré's ideas on the role of the mathematical sub-conscious in mathematical thinking, locating the period in which it is most operative—the *incubation* period—within a larger theory of mathematical inquiry. Through both historical and empirical studies, he was able to gather some evidence to support his account from mathematicians such Pierre de Fermat, Evariste Galois, Bernhard Riemann, George Birkhoff, George Pólya and Norbert Wiener. Morris Kline (1953) subsequently pointed out that aesthetic concerns not only guide the direction of an investigation, but motivate the search for new proofs of theorems already correctly established but lacking in aesthetic appeal—by means of their ability to "woo and charm the intellect" (p. 470) of the mathematician. Kline took this aesthetic motivation as a definitive sign of the artistic nature of mathematics. Wolfgang Krull (1930/1987) illustrated how aesthetic preferences—such as a mathematician's desire for simple, symmetric structures—can seriously influence the further development of mathematics, as well as the derivation of new properties and the creation of new theories.

In his earlier attempt to define mathematics as the "classification and study of all possible patterns" (p. 12), Warwick Sawyer (1955) implied that the heuristic value of mathematical beauty stems from mathematicians' sensitivity to pattern and originates in their belief that *"where there is pattern there is significance"* (p. 36; *italics in original*). He went on to explain the heuristic value of this trust in pattern: "If in a mathematical work of any kind we find that a certain striking pattern recurs, it is always suggested that we should investigate why it occurs. It is bound to have some meaning, which we can grasp as an idea rather than as a collection of symbols." (p. 36) Sawyer might well have explained Poincaré's special aesthetic sensibility as a sensibility toward pattern, viewed broadly as any regularity that can be recognised by the mind. For him, the mathematician is not only able to recognise regularities and symmetries, but is also attuned to look for and respond to them with further investigation.

Poincaré's writing on the mathematical aesthetic, which was definitely excluding of most everyone, (more so than Sawyer's account), suggested that only the very creative mathematicians had access to this aesthetic guide. This claim may have provoked the "literary superstition" that Alfred North Whitehead (1926) mentioned, which views the aesthetic appreciation of mathematics as being a "monomania confined to a few eccentrics in each generation" (in Hardy, 1940, p. 85). Hardy quotes Lancelot Hogben (1940) "the aesthetic appeal of mathematics may be very real for a chosen few" (p. 86) and accuses him of echoing this "superstition." Indeed, Bertrand Russell's (1917) famous quotation, "Mathematics, rightly viewed, posses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture" (p. 57) does seem to suggest that mathematics exercises a coldly impersonal attraction, one not meant for normal individuals. As we shall see, Russell's frigid tastes are not the only ones that mathematics can satisfy. But this theme of the exclusivity of mathematical aesthetic judgement (concerning who is able to make them), to be found in the writings of Poincaré, Russell and Hardy, persists in the mathematics literature.

Armand Borel (1983) was faced with overcoming a different kind of exclusivity in his attempt to convey the nature of mathematics and the mathematical aesthetic to a wider audience, of both mathematicians and non-mathematicians. He began by arguing that the development of mathematics was "derived from, guided by, and judged according to aesthetic criteria" (p. 11), thereby astutely acknowledging both Poincaré's heuristic aesthetic and Hadamard's aesthetic of choice. [3] However, he then attempted to show how what may seem like the "pure and esoteric" aesthetics of mathematicians are actually bound up with "more earthly yardsticks" (p. 15), such as applicability and usefulness, values that Borel hoped non-mathematicians would find more recognisably mathematical than beauty or elegance.

Mathematics: an art or a science?

The mathematics literature has long been replete with questions about the nature of mathematics and its place in the plural world of the arts and sciences. While Gauss's claim that mathematics is the queen of the sciences has often been repeated, so has the claim that mathematics belongs more properly to the arts. The mathematician J. W. N. Sullivan made the latter argument in 1925, claiming that mathematics is the product of a free creative imagination, unconditioned by the external world. It is, he argued, just as 'subjective' as the other arts, even though it can be used to illuminate natural phenomena.

Moreover, Sullivan (1925/1956, p. 2020) claimed that mathematicians are impelled by the same incentives as artists, citing as evidence the fact that the "literature of mathematics is full of aesthetic terms" and that many mathematicians are "less interested in results than in the beauty of the methods" by which those results are found. His interest in the mathematical aesthetic experience, which he saw giving rise to the same satisfactions as the artistic experience, was distinct from Poincaré's focus on the mathematical aesthetic sensibility, which acts as a guide. Yet Sullivan saw neither art nor mathematics as existing to satisfy "aesthetic emotions": rather, he saw both art and mathematics as means by which "humans can rise to a complete self-consciousness" (p. 2021).

The philosopher of mathematics Thomas Tymoczko (1993) may well have pointed out the most operative difference between aesthetic judgements in mathematics and those at work in the arts. This is that the mathematics community does not have many (any?) 'mathematics critics' to parallel the strong role played by art critics in appreciating, interpreting and arguing about the aesthetic merit of artistic products.

Criteria for the mathematically beautiful

In 1940, G. H. Hardy published what became arguably the most widely-read inquiry into the mathematical aesthetic. Unlike either Poincaré or Hadamard, Hardy was primarily interested in defining mathematical beauty as it exists in mathematical products, particularly in proofs. He proposed a somewhat complex scheme that distinguished 'trivial' beauty—which can be found in chess—from 'important' beauty, which can only be found in serious mathematics. But, for Hardy, serious involved significant, which in turn required generality—scope or reach—and depth. *Generality* and *depth* are both difficult to define, but can, according to Hardy, be immediately recognised by those with a "high degree of mathematical sophistication" (p. 103). Such mathematicians will find a mathematical idea significant when it can be "connected, in a natural and illuminating way, with a large complex of other mathematical ideas" (p. 89). Hardy illustrated his notion of mathematical beauty with two examples: Euclid's proof of the infinity of primes and the Pythagorean proof of the irrationality of $\sqrt{2}$: These two proofs appear frequently in the literature as particularly fine examples of beautiful proofs (for example, see Dreyfus and Eisenberg, 1986, or King, 1992).

Having defined mathematical beauty in terms of significance and seriousness, Hardy went on to say that the triviality of ideas (such as those found in chess problems, but not in beautiful mathematics) "disturbs any more purely aesthetic judgement" (p. 113). Hardy proposed that purely aesthetic qualities are unexpectedness, inevitability and economy. Considerably later, Roger Penrose (1974) would add to Hardy's list the criterion of "unexpected simplicity" (p. 267). Hardy advanced a formalist perspective of mathematical beauty by only acknowledging responses to formal properties. For Hardy, and many others, formalism represents the dominant 'public aesthetic' of mathematics; if mathematics presents *any* aesthetically relevant qualities, these qualities *must* be formal in nature.

Shortly after Hardy's publication, François Le Lionnais (1948/1971) proposed a completely different, non-formalist way of approaching the problem of mathematical beauty—without making reference to either Hardy or Poincaré. Le Lionnais was not interested in the process-oriented aesthetic sensibilities that Poincaré was, but his scope was wider than Hardy's, including as it did various kinds of 'facts' and 'methods' as potential objects of mathematical beauty. Le Lionnais effectively enlarged the sphere of mathematical entities that can have aesthetic appeal, including not only entities such as definitions, shapes, proofs, solutions and theorems, which are appreciated after the fact, but also the various methods and processes used to work with mathematical entities, which can be appreciated *while* doing mathematics.

In addition, Le Lionnais emphatically drew attention to the subjectivity of aesthetic responses, by classifying mathematicians' orientations as either "classical" or "romantic", thus allowing for degrees of appreciation—banned by Hardy—according to personal preference. These categories represent two styles of human endeavour: on the one hand, a desire for equilibrium, harmony and order; and, on the other, a yearning for lack of balance, form obliteration and pathology. A very similar distinction is made by Freeman Dyson (1982), who distinguished between 'unifiers' and 'diversifiers', the former finding and cherishing symmetries, the latter enjoying the breaking them.

Le Lionnais's stance on the subjectivity of aesthetic responses did not do much to quell the belief, common among mathematicians especially, that most mathematicians agree on their aesthetic judgements. This common belief was fuelled in part by the exclusivity of Poincaré and Hardy, which seemed to imply that if your aesthetic judgement did not agree with that of a great mathematician, then you were simply not a great mathematician. It was also fuelled by the enormous discrepancies of taste and judgement found in the arts, which, by any mathematician's definition of subjectivity, dwarfed the differences identified in the mathematical world.

Jerry King (1992), like Hardy, presumed the supposed homogeneity of mathematicians' aesthetic response and further concluded that mathematicians work from some set of commonly-accepted aesthetic principles. Moreover, he assumed that mathematicians' judgements are not subjective, but instead depend solely upon the mathematics itself, making it possible to formulate decisive criteria. In his book *The Art of Mathematics,* King drew on aesthetic theories of philosophy and art criticism in order to articulate "a complete aesthetic theory of mathematics" (p. 157). Rather than expanding Hardy's list of factors that contribute to aesthetic appeal, King's primary goal was to identify general-level aesthetic criteria that would help distinguish 'good' mathematics from 'bad' (thereby conflating Hardy's distinction between the beautiful and the aesthetic). He thus proposed two definitive criteria: the principle of minimal completeness and the principle of maximal applicability. King illustrated both principles using the Pythagorean proof of the irrationality of $\sqrt{2}$. The principle of minimal completeness, in effect, functions as a super-class to Hardy's aesthetic qualities. However, King's principle of maximal applicability resonates more with Hardy's notions of significance, depth and generality.

Finally, David Wells's (1990) survey of contemporary mathematicians has most convincingly illuminated the subjectivity question. He asked the readers of *The Mathematical Intelligencer* to rate, on a scale of one to ten, twenty-four theorems according to their mathematical beauty. From the seventy-six responses, many from top mathematicians mostly from North America, he drew a number of inferences. First, mathematicians do not always agree on their aesthetic judgements—at least in terms of evaluating the beauty of theorems.

Wells identified many factors that contribute to the differences in judgement: field of interest; preferences for certain mathematical entities such as problems, proofs or theorems; past experiences or associations with particular theorems; even mood. He also pointed out that aesthetic judgements change over time: this was particularly evident in the rating of Euler's formula, which was historically considered "the most beautiful formula of mathematics" (p. 38), but is now, according to Wells's respondents at least, considered too obvious even to elicit an aesthetic response. The inferences made by Wells correspond to a contextualist view of aesthetic appreciation and are summed up by this respondent: "beauty, even in mathematics, depends upon historical and cultural contexts, and therefore tends to elude numerical interpretation" (p. 39).

Indeed, John von Neumann had already spoken of the phenomenon of mathematical 'styles' back in 1947, arguing that, it is "hardly possible to believe in the existence of an absolute, immutable concept of mathematical rigor, dissociated from all human experience" (p. 190). He used as evidence the changes in styles of mathematical proofs and fashionable areas of interest over the past two millennia.

One might wonder why these changes in style appear so much less dramatic than the ones found in the arts. Are the styles necessarily more confined in mathematics, owing to the handful of aesthetic commitments that ultimately define the discipline? Or does the study of mathematics attract a small enough number of like-minded people that aesthetic revolutionaries such as Pablo Picasso, Jackson Pollock or John Cage do not have mathematical equivalents?

The mathematical aesthetic: too "meta" for mathematicians?

It is striking to us that mathematicians often mention 'beauty,' yet there seems to be a relative dearth of further amplification. One might have expected those past mathematicians who thought in these terms to have been capable of developing ideas of, say harmony, proportion, fit, rhythm, etc, more precisely. To some extent, this is what Hardy (1940) tried to do, though only by connecting 'beauty' to other barely less opaque terms such as 'elegance,' 'depth,' 'seriousness' and 'significance'. We do see instances here and there, for instance with Alfred North Whitehead (on rhythm) or Warwick Sawyer (on pattern). But it may well be that many mathematicians simply do not consider this to be a serious enterprise, one worthy of their time and attention. Even Hardy (1940) expressed a sense that such reflection 'about' mathematics (offering a different sense of 'meta'-mathematical activity) is not really the preferred activity or even the very business of mathematicians.

> It is a melancholy experience for a professional mathematician to find himself writing about mathematics. The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done. (p. 61)

There seem to be some 'inevitable' combinations of aesthetic words that are mathematically invoked as if conjoined: for instance, beauty *and* elegance, perfection *and* beauty. 'Elegance', in particular, seems to have been co-opted by mathematicians in their rather restricted aesthetic language, as conveying a sense of both succinctness and sophistication. In ordinary parlance, 'elegance' might be seen as a classical, class-ridden term – not so much socio-economic 'class' perhaps as intellectual 'class' (though Bertrand Russell, for example, certainly partook of both). Of course, there must be a sociological proviso here – it was only very few (privileged) Greeks, and then for a long time a very few (privileged) other individuals, who could sit and think as opposed to practice or teach.

The Mathematical Aesthetic in the Twenty-First Century

> I was very young […] and I remember my father folded down the flaps of a cardboard box so that each was holding down the next. And I remember [...] being amazed, it was so perfect [...] my first experience of joy in abstract thought. (Sipser, 1986, p. 80)

We would like to focus here on three important, different themes we see starting to emerge in the current reflections on the mathematical aesthetic. These themes are not 'new,' in the sense that they do have strong historical antecedents. However, they have not been clearly singled out for attention by writers on mathematics. In the same way, we are not suggesting that the three twentieth-century elements identified in the previous sub-sections have run their course or somehow been resolved. On the contrary, they seem to have considerable mileage left in them, if in nothing else than their perennial success in focusing renewed attention on the core questions of the *what* and the *why* of contemporary claims about the nature of mathematical activity and mathematical knowledge.

The three themes we have identified all relate to potential sources of pleasure for the mathematician. We are well aware that the notion of aesthetics by itself is not synonymous with that of pleasure (or desire). For instance, many authors explicitly seek to identify the 'universal' in aesthetic aspects of mathematics, whereas the notion of pleasure is more easily situated within an individual. Nevertheless, it seems to us that there is a danger in construing aesthetics purely as a socio-cultural construction and consequently risk losing sight of the richness and vividness of individual experience that mathematicians report. A small instance is provided by the MIT professor Michael Sipser cited above – his word was "joy".

The three sites we propose be further explored are:

- the pleasure to be found in (seeking or achieving) distance and detachment;
- the pleasure in (striving after, finding a place to invest a desire for) certainty and perfection;
- the pleasure in melancholy.

Distance and detachment

Detachment, disinterest and 'aesthetic distance' arise over and over as important concepts – although ones difficult to define – in discussions about aesthetic experience and aesthetic judgements (e.g Kant's *Critique of Judgment*). These concepts have helped philosophers of art attempt explanations of how people can have aesthetic experiences prompted by, for instance, a dangerous and unpleasant fog at sea or a violent, ugly depiction of war. For Kant, disinterest, that is, separation from personal beliefs, passions and commitments, was essential to aesthetic judgement.

While a number of philosophers (see Dewey, 1934, and Polanyi, 1958, in particular) have argued that Kantian disinterested judgments are impossible for humans, some continue to argue that aesthetic judgements require some kind of distance, such as Bullough's (1963) *psychical distance* and Beardsley's (1982) *detached affect*. Furthermore, the non-utilitarian quality of art works (which was an early but sustained misinterpretation of Kant's notion of disinterest) continues to be a focus of discussions in the art world. This supposed non-utilitarian quality of art has been the basis on which some mathematicians have compared mathematics and the arts. (Recall Hardy's insistence on the "purity" of his work, on the fact that his discoveries had no practical applications.) Beyond that, however, almost none of the mathematicians who have been quoted have explicitly mentioned the detached or disinterested aspect of the mathematical experience.

It does not require much digging, however, to appreciate the extent to which detachment acts as a fundamental value or mode in the creation of mathematics. Indeed, the very process of mathematisation can be specified in terms of its plurality of detachments; it is the detachment from specifics through generalisation, the detachment from real-world referents as well as from details that would define a particular situation, as well as the detachment from personal connections. Furthermore, the codifying and communication of mathematisations, which are often carried out through the writing of proofs, can be seen as an exercise in detachment: a proof distances itself from the situations and specific examples to which it applies, as well from the personal commitments and attractions that formed it. In fact, Nicolas Balacheff (1988) has written that the language of conceptual proof demands that the speaker "distance herself from the action and the processes of solution of the problem" (p. 217). In particular, this requires:

- a *decontextualisation*, giving up the actual object for the class of objects, independent of their particular circumstances
- a *depersonalisation*, detaching the action from the one who acted and of whom it must be independent;
- a *detemporalisation*, disengaging the operations from their actual time and duration: this process is fundamental to the passage from the world of actions to that of relations and operations. (pp. 217-218; *italics in original*)

One might wonder about the costs and consequences of this triple detachment: what happens when language is used as a tool for logical deductions rather than as a tool for communication? What happens with the mathematician writes so as "to conceal any sign that the author or the intended reader is a human being" (Davis and Hersh, 1980, p. 36). Gian-Carlo Rota (1996) draws out one consequence, namely, that mathematical proof becomes a form of "pretending", since the language of proof produces a striking gap between "the written version of a mathematical result and the discourse that is required in order to understand the same result" (p. 142). And, thus, understanding is compromised, or rather, exchanged for a kind of functional language, one where: "Clarity has been sacrificed to such shibboleths as consistency of notation, brevity of argument and the contrived linearity of inferential reasoning" (p. 142). Rota's notion of proof as detachment from human understanding, or at least it involving a concealment from that understanding, is related to the values held and promulgated in the mathematics community. After all, one could imagine a world of research mathematics where the journals were filled with proofs that illuminate rather than conceal or proofs that open up and explain rather than codify and hide.

However, as Alan Bishop (1991) has pointed out, mathematics is much more than a set of results, facts, methods and tools; the world of research mathematics is animated by a specific set of values that distinguish it from other fields of intellectual endeavour. Although Bishop does not focus on the aesthetic dimensions of the values he identifies, for us his work highlights the fundamentally aesthetic nature of the values that characterize mathematics and that determine the very shape of mathematical knowledge and practices.

The ideological value of mathematical rationalism shuns—and seeks to detach from—other forms of explanations, forms that betray the presence of their human creators: instances include trial-and-error pragmatism, rules of practice, traditional wisdom and reliance on images and inductive or analogic reasoning. Even outside mathematics, rationalising can mean seeking logical connections between ideas, thus overcoming the inconsistencies, disagreements or incongruities that may arise from personal interpretations of situations or ideas. Yet Yves Chevallard (1990) has argued that, 'Mathematics is a perfect example on which a *celebration of ambiguity* could be founded" (p. 8; *italics in original*).

Explaining, ironically enough, can also mean "explaining away": I can rationalise certain decisions or actions by calling on some external, logically attractive principle, one that frequently ignores the principles that tacitly guided my actions. Bishop writes that we are both guided by and uphold the values of rationalism:

> when we disprove a hypothesis, when we find a counter-example, when we pursue a line of reasoning to a 'logical conclusion' and find it is a contradiction to something known to be true, and when we reconcile an argument. (p. 63)

The aesthetic dimension of rationalism relates to the valuing of the completeness and wholeness that belong to a logical argument. Fuzziness, imprecision and loose ends are banished and are replaced, more and more, with Rota's 'pretend' proofs, which are optimised for consistency, containment and cohesion. The mathematician's desire for such aesthetic qualities may stem from a discomfort with graded truths: logical conclusions, like cohesion and consistency, it is claimed, do not admit degrees. Arguments are, it is claimed, on the one hand, either logical or not, and, on the other, either consistent or not: no middle ground.

The ideological partner of such rationalism is objectivism, which characterises a world-view dominated by pseudo-material objects. In particular, the kind of objectivism that drives mathematics, as Bishop points out, is one in which ideas can be given objective meanings, thereby enabling them to be dealt with as if they were objects, and as if they partook of an objective reality. The focus on objects detracts—and detaches—from that of process: it constantly insists on reifying behaviour into atomistic things or objects (such as turning the input–output process relating two variables into the reified object called a function); along the way, verbs turn into nouns.

The process of detachment may also enable certain forms of mathematical reasoning. In Sinclair's (2002) interview with mathematician Hendrik Lenstra, he explained the role of detachment in the work of algebraists:

> They somehow try to get a handle on their properties by taking a more distant attitude. That is why they introduce more abstract-sounding notions such as rings, ideals. They nevertheless end up being able to prove very concrete theorems about identities that exist or may not exist without really ever exhibiting them, just because they have this superior mechanism.

In addition, objectivism gives mathematics its foundationalist bent: that is, its tendency to search for the 'atoms' of theories and proofs, and to get to "the bottom of things". Theories that are built on a small set of fundamental axioms have an attractive simplicity – and perhaps a misleading one, as Whitehead's *dictum* "Seek simplicity, and mistrust it" implies. Objectifying process—dehumanising and decomplexifying it—betrays a quest for pure and safe ideas that can provide order, regularity and predictability. The mathematician does not necessarily want to dehumanise mathematical ideas; rather, the desire is to bracket and 'freeze' them. This requires extracting them from the complex flux of the human scene and perhaps setting them at a (safe) distance emotionally. To this point, Catherine Chevalley, writing of her father Claude one of the main members of Bourbaki, famously observed:

> If you look at the way my father worked, it seems that it was this which counted more than anything, this production of an object which, subsequently, became inert, in short dead. It could no longer be altered or transformed. This was, however, without a single negative connotation. Yet it should be said that my father was probably the only member of Bourbaki who saw mathematics as a means of putting objects to death for aesthetic reasons. (in Chouchan, 1995, pp. 37-38; *our translation*)

Emotional detachment features strongly in the famous words of Bertrand Russell (1917), already quoted in this book, about the "cold and austere" beauty of mathematics and its "stern perfection", without "the gorgeous trappings of painting or music". These sentiments imply that the aesthetic appeal of mathematics involves – and perhaps even necessitates – a detachment from the warmth of emotions and the imperfection of the senses. In fact, Russell went on to say:

> Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the actual world. (Russell, 1910, p. 73)

One has the impression that Russell saw mathematics ‘living’ outside the urges, impulses and vicissitudes of human life and that mathematical beauty has some kind of objective truth and existence.

Russell had not been the only mathematician to write in this way about mathematical beauty: it is a way of writing about mathematical experience, however, that Gian-Carlo Rota (1997) has trenchantly termed a “cop-out”. He writes: “mathematical beauty is the expression mathematicians have invented in order to obliquely admit the phenomenon of enlightenment while avoiding the fuzziness of this phenomenon” (p. 132). Rota suggests that mathematicians acknowledge a theorem’s beauty when they see how it “fits” in its place, how it sheds light around itself. A proof is beautiful not because it is ‘pure’ or detached from human emotions and needs, but because it gives away the secret of the theorem or leads the mathematician to perceive after the event the apparent inevitability of the statement being proved. Rota suggests that it is precisely this phenomenon of personalised enlightenment that keeps the mathematical enterprise alive, a phenomenon that possesses the fuzziness and subjectivity that Russell seemed to disdain.

In fact, within Rota’s sense of the mathematical aesthetic, the importance of understanding is elevated. And, the need to understand necessarily connects back to subjective human experience and needs and to the senses. Therefore, even though mathematicians prize detachment in many ways, several chapters in our book certainly reveal some of the important ways in which the body, the senses and earthly passions and feelings are important to mathematical understanding and, perhaps, constitute part of the aesthetic value of mathematical objects and experiences.

Certainty and perfection

It is now apparent that the concept of a universally accepted, infallible body of reasoning

– the majestic mathematics of 1800 and the pride of man
– is a grand illusion. Uncertainty and doubt concerning the future of mathematics have replaced the certainties and complacency of the past. (Kline, 1980, p. 6)

While the most devoted Platonist might some day (probably a weekend day, according to Davis and Hersh’s famous pronouncement) finally concede the uncertainty of mathematics, there would be no escaping the *feeling* of certainty one gets or the very human *desire* for certainty one expresses in doing mathematics. According to mathematician Brian Rotman, the fantasy of the mathematician involves a dream of certainty:

> The desire’s object is a pure, timeless unchanging discourse, where assertions proved stay proved forever (and must somehow always have been true), where all the questions are determinate, and all the answers totally certain. (in Walkerdine, 1988, pp. 187-188)

Mathematics is certainly one significant place where human beings invest their desire for certainty. As Robert Crais (2003, p. 8) asserted (albeit in quite another context), “The world was not certain; the only certainty was within you”.

We have already seen claims of mathematicians being motivated to do mathematics because it is beautiful, elegant or harmonious. But there is another common yet potentially problematic motivation: one hears mathematicians – and school students as well – explaining that they do mathematics because it is the primary or even the only place they find some degree of certainty in their lives; students experience mathematics as something stable (especially against their chaotic adolescent lives), consistent, reliable, an area of life where one can have certainty. A popular, troubled and not very mathematically-oriented student once surprised her teacher (Sinclair) by asking for “more polynomials to simplify.” Asked why she would possibly want to spend her time doing math homework, she responded: “I know how to do this. I follow the steps and I get it right.” Applying rules of simplification to polynomials might have been the one safe and certain part of her life. No one could disagree with her; and even if they did, she would know she was right. Simplifying polynomials allowed her to escape into an ordered world marked by calmness, stability, regularity, predictability and personal agency.

In this sub-section, we want to explore the implications of the quest for certainty in mathematics, as well as the aesthetic values of 'perfection' and 'order' that mathematics indulges. But, as before, there is also the question of costs, obvious or otherwise, of such a grail. As Sylvia Plath (1965), in her poem *The Munich Mannequins*, suggested:

> Perfection is terrible, it cannot have children.
> Cold as snow breath, it tamps the womb. (p. 74)

In an important sense, our discussion here extends the theme of distance and detachment developed in the previous sub-section. Certainty involves a tendency toward abstraction and away from the contingent events of the everyday world, while perfection too takes us away from the here and now of the actual into considerations of and comparisons among the possible.

Here, we want to make use of Wilhelm Worringer's particular use of the terms 'abstraction' and 'empathy' as described in Richard Padovan's (1999) book *Proportion*. 'Abstraction' marks a tendency: "to regard nature as elusive and perhaps ultimately unfathomable, and science and art as abstractions, artificial constructions that we hold up against nature in order in some sense to grasp it and command it." (p. 12) Padovan contrasts Worringer's sense of 'abstraction' with that of 'empathy', which marks: "the tendency to hold that, being ourselves part of nature, we have a natural affinity with it and an innate ability to know and understand it." (p. 12) The abstraction viewpoint sees mathematics as a human-made creation, a purely artificial construction, a system of conventional signs and the rules for manipulating them. Nature is thus something we can interpret though mathematics only because mathematics is a human creation. There is no need for the fallible human senses and no reference need be made to natural forms. An empathy viewpoint—also, a Pythagorean one—sees mathematics as being inherent in nature and distilled out of it by human reason, through human senses. The shifting appearances of things can be penetrated by mathematics, which can reveal the essential nature of imperfectly manifested phenomena.

Padovan sums up the paradox encapsulated by these two viewpoints:

> No knowledge is possible unless it comes first through the senses; but such knowledge is at best uncertain. The certainty of mathematics is due precisely to the fact that it is man-made, the uncertainty of nature to the fact that it is not. (p. 11)

Abstraction requires both dehumanisation and decontextualisation. It is powerful in mathematical and scientific work because it allows one to look for common features across local instances. Then, when we apply our abstract models back to the world and see how they fit, we should eventually get a better sense of how things work. But do we? Or are we, in fact, entranced by the certainty that *only* our abstract models are able to offer? More strikingly, is this related to Oliver Wendell Holmes's famous *dictum* that "certainty leads to violence" (see Menand, 2001). The danger alluded to by Holmes is that we begin living in a decontextualised world where we forget about the impact of our abstract models on the real and messy lives of human beings. And this is one way in which certainty can lead to violence.

In a radio interview with the Canadian Broadcasting Corporation reporter Michael Enright, twelve days following the events of September 11th, 2001, the respected Islamic scholar Hamza Yusuf Hanson described how "terrorists are identified" in the "Muslim world":

> from an early age they identify students that are very brilliant in mathematics and they direct them towards only studying the physical sciences to the neglect of what makes us

> human, which is humanity, is poetry, it's literature, as well as philosophy and theology, these things are absent now.

The connection here may seem exaggerated or, perhaps, out of context. Even so, certainty can lead to other disturbing things admittedly less alarming than violence, including anxiety and exclusion, but also perhaps to deep environmental, ecological, economic and political problems.

Exclusion has been a dominant theme in mathematics, even from "inside" the mathematics community. For example, mathematicians such as Frances Kirwin (in Arnold *et al.*, 1999) and researchers such as Margaret Murray have reported on female mathematicians' feelings of exclusion from and isolation in the mathematics community. These feelings that are based at least partly on the extreme abstraction demanded of them, the neglect of what Anneli Lax referred to as "the human stuff" (in Murray, 2000, p. 217).

The less mathematically-inclined often experience anxiety in addition to exclusion. Buerk (1983), for instance, described the mathematical anxiety of able and intellectually mature women who seeing mathematics as a discipline that is "rigid, removed, aloof, and without human ties" (p. 20), where "the wicked mathematician has all the answers in the back of the book" (p. 19). Even after having mathematical experiences designed to promote less dualistic conceptions of mathematical knowledge, these women continued to struggle with their relationships to mathematics. One woman commented, "I have to take it home with me and where is it going to fit into my life now?" (p. 23). This question underscored her empathic view of mathematics: despite better understanding and lowered anxiety, she still needed for mathematics to have human ties, to maintain a flexibility, connection and presence in her life.

In *The Essential Difference: Men, Women and the Extreme Male Brain*, the psychologist and autism specialist Simon Baron-Cohen (2003) argues that abstraction and empathy lie on a continuum that is connected to gender. His most convincing arguments relate to "extreme" examples of the male abstract brain, which he links to certain forms of autism and exemplifies using the mathematician Richard Borcherds – a Fields medal winner. Does leading-edge mathematics demand extreme abstraction and, if so, what are the costs and consequences for innately empathic human beings?

The melancholy disposition of the mathematical mind

Hardy (1940), for example, as quoted earlier, used the word 'melancholic' with respect to finding himself writing about mathematics rather than doing it. On the one hand, this seems to be the lot of anyone attempting to engage in critical commentary on mathematics (whether philosophers of, historians of, or members of other, less-accredited para-disciplines feeding off such a single-minded—'monomaniacal' in Whitehead's term—one). On the other, however, Hardy's comment introduces a word pregnant with overtones and resonance, one that has been used both in negative but also in more neutral connotation for over two millennia.

As far back as Aristotle, in his *Metaphysics*, we find speculations of the nature of the creative mind (and mathematics as one clear instance) and the state in which it prepares itself prior to and during such work. And as recently as November 2003, mathematician Ioan James (former Savilian professor of mathematics at the University of Oxford) has written an article entitled 'Autism in mathematicians' (picking up on the same theme as Baron-Cohen's illustration mentioned at the end of the previous sub-section). James is at pains to draw his retrospective net of famous 'autistic' mathematicians (identifying those with Asperger's syndrome in particular), especially of the past two centuries, very widely. Asperger, a Viennese physician, himself was struck by the fact that "they usually had some mathematical ability and tended to be successful in scientific and other professions where this was relevant" (p. 62) Is this perhaps a case of over-ordered, over-distanced individuals who nevertheless find in mathematics sufficient characteristics in common to support their own psychology? Or is there more of a two-way reinforcement, where mathematics could be different if undertaken by more other-psyche-ed individuals?

Others who have written about this topic include such widely divergent authors as Henry of Ghent in thirteenth-century Paris (whose evocative phrase heads this sub-section) and Benjamin Bloom (1985) in his extensive study of the personal and family history of North American prodigies in a variety of fields (including mathematics). Some of the personality traits Bloom identified in his mathematical subjects included a 'penchant for solitude' and a 'desire for precision', as well as what he termed a 'fundationalist' tendency, a desire to get to the bottom of things.

But it also includes psychoanalyst Jacques Lacan who is said to have challengingly observed that mathematicians "show cowardice towards matter, toward objects, toward the material world". Right at the end of *Psychoanalytic Politics*, in Sherry Turkle's (1981) book that provides a partial history of Lacanian psychoanalysis in France, she described Jacques Lacan's intense involvement with mathematics, especially knot theory. (Indeed, at the core of what turned out to be his final project before he died, Lacan was starting to explore the Paris asylum records of Georg Cantor.) She wrote:

> For Lacan, mathematics is not disembodied knowledge. It is constantly in touch with its roots in the unconscious. This contact has two consequences: first, that mathematical creativity draws on the unconscious, and second, that mathematics pays its debt by giving us a window back to the unconscious. […] But theories that use mathematical formulation are seen as "cold," "impersonal." Definitionally, something that is cold leaves out the warmth of the body. (p. 247)

Another psychology of mathematics education

Ten years ago, one of us (Pimm, 1994) wrote a book chapter with this title, encouraging a psychoanalytic turn in work in mathematics education. In it, certain mathematics education examples were given about mathematical origins. Freud asserted that psychoanalysis was part of psychology proper and that it as perhaps its very foundation. Foundational issues are of concern both to psychology and mathematics: mathematics education needs to come to terms with both.

One of the shared tasks then, as we see it, of mathematics education along with mathematics itself (and thus an overlapping goal of the two communities), is to deal with origins, both phylogenetically (i.e. within the species), as well as ontogenetically (i.e. within the individual): to develop a creation myth, if you like. (In passing, we recall Seidenberg's interesting historical papers from the 1960s in *Archive for History of Exact Sciences* and *Historia Mathematica* offering religious origins of counting and geometry.) In an appendix to this paper, we offer a few clusters of deliberately naïve questions, answers to which would serve to draw boundaries around the mathematical.

Michel Serres (1982), writing of the origins of mathematics, claimed it to be "an interminable discourse. That which speaks of an absent object, of an object that absents itself, inaccessibly" (p. 97), In our book, Dick Tahta (in press) has written an inspiring psychoanalytically informed chapter entitled 'Sensible objects': the adjective 'sensible' may suggest what can be perceived by the senses or what can be understood in the mind (the aesthetic thus strongly linked to the noetic). But, in the end, he argues, the nature of the mathematician's objects – like those of many others – remains a mystery. We would venture to suggest that neither psychology nor mathematics likes unresolved mysteries very much, even though it was one of the two sociological values that Bishop (1991) proposed: ‘openness’ and ‘mystery’. It is also worth recalling that much of mathematics’ early–and not so early–history has involved serious accusations of investment in magic, conjuring and sorcery.

We have predicted that one aspect of twentieth-first-century work on the mathematical aesthetic will be an ever-deeper analysis of the mathematical psyche, seeking out perhaps the impulses that leads some humans seek distance, certainty and melancholy, and why a goodly number select to place it so centrally within mathematics. If we look back at the three peculiar ‘pleasures’ or ‘desires’ of the mathematician that we have identified in the previous section, they all can be seen as aesthetics-related components of what historically has been termed ‘the mathematical mind’ (e.g. ‘*L’esprit de géométrie*’ in Pascal, though the term is much older than that). While the definiteness and apparent specificity (is there only one?) of this term would have given it a bad press in the later part of the twentieth century, we nonetheless feel it worthy of its re-presentation as an important area of study for mathematics education in the twenty-first.

Appendix

(a) Our first question has to do with the species-specific nature of mathematics: what in the realm of the mathematical are we willing to attribute to the animate but non-human (chimpanzees, birds, …)?

(b) The second concerns the human/machine interface: are we prepared to call something that inanimate machines are able to do ‘mathematics’?

(c) Our third question is *historical* (a *diachronic* concern): when do we wish to locate the beginning of mathematics and why? In light of this, what could be considered the earliest mathematical artifact (and how would we know)? What do different answers reveal about the investments of different proposers?

(d) The fourth refers to the chronological present (hence, a *synchronic* concern) and relates to the distribution of mathematics, mathematical awareness and sensibility within our culture, as well as globally across cultures. In other words, this relates to what we might call the *ethnomathematical* question: in what sense does everyone mathematise or ‘do’ mathematics?

(e) Our fifth and final question concerns the genesis of mathematics, not historically within cultures, but psychologically within the individual. Are all mentally functioning humans born with the possibility for mathematics? Does the potential for mathematisation become actualised, say, in a manner akin to language acquisition–that is, only with adequate external input? What aesthetic elements are necessary? What necessary adult awareness needs to be brought to bear on a young child, in order to draw attention to the very possibility of mathematics?

Notes

[1] In her Acknowledgements, Leone Burton (2004) thanks the seventy mathematicians whom she interviewed for “helping to create a bridge, even if not yet very robust, between mathematics and mathematics education” (p. xv).

[2] Many books on philosophy trace the origin of the word ‘aesthetics’ back to Alexander Baumgarten (1750/1961), specifically the Latin of his book title *Aesthetica*. However, because of the word’s etymological origin in the Greek verb *aisthanomain* meaning “to perceive” and *aisthesis*, meaning “sensorial perception”, the meaning of ‘aesthetics’ for us is firmly rooted in the senses by means of which we perceive.

[3] The Pythagoreans, who celebrated the Muses as “the keepers of the knowledge of harmony and the principles of the universe which allowed access to the ever-lasting gods” (see

Comte, 1994, p. 135), would have been delighted by the trust that scientists, and mathematicians, have come to place on this aesthetic muse.

References

Aristotle *Metaphysics* (Apostle, H., trans.), Bloomington, IN, Indiana University Press, 1966.

Arnold, V., Atiyah, M., Lax, P. and Mazur, B. (1999) *Mathematics: Frontiers and Perspectives*, Providence, RI, The American Mathematical Society.

Balacheff, N. (1988) 'Aspects of proof in pupils' practice of school mathematics', in Pimm, D. (ed.), *Mathematics, Teachers and Children*, London, Hodder and Stoughton, pp. 216-235.

Baron-Cohen, S. (2003) *The Essential Difference: Men, Women and the Extreme Male Brain*, Harmondsworth, Middlesex, Penguin Books.

Baumgarten, A. (1750/1961) *Aesthetica*, Hildesheim, G. Olms.

Beardsley, M. (1982) *The aesthetic point of view. Selected essays*. Ithaca, NY: Cornell University Press.

Bishop, A. (1991) *Mathematical Enculturation: a Cultural Perspective on Mathematics Education*, Dordrecht, Kluwer Academic Publishers.

Bloom, B. (1985) *Developing Talent in Young People*, New York, NY, Ballantine.

Borel, A. (1983) 'Mathematics: art and science', *The Mathematical Intelligencer* 5(4), 9-17.

Borwein, J. and Bailey, D. (2003) *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, Natick, MA, A. K. Peters, Ltd.

Buerk, D. (1983) 'An experience with some able women who avoid mathematics', *For the Learning of Mathematics* 3(2), 19-24.

Bullough, E. (1963) '"Psychical Distance" as a factor in art and aesthetic principle', in Levich, M. (ed.), *Aesthetics and the Philosophy of Criticism*, New York, NY, Random House, pp. 233-254.

Burton, L. (2004) *Mathematicians as Enquirers: Learning about Learning Mathematics,* Dordrecht, Kluwer Academic Publishers.

Chevallard, Y. (1990) 'On mathematics education and culture: critical afterthoughts', *Educational Studies in Mathematics* 21(1), 3-27.

Chouchan, M. (1995) *Nicolas Bourbaki: Faits et Légendes*, Argenteuil, Editions du Choix.

Comte, F. (1954) *The Wordsworth Dictionary of Mythology.* Ware. Hertfordshire, Wordsworth Editions, Ltd.

Crais, R. (2003) *The Last Detective*, New York, NY, Random House.

Davis, P. and Hersh, R. (1980) *The Mathematical Experience*, Boston, MA, Birkhaüser.

Dewey, J. (1934) *Art as Experience*, New York, NY, Perigree.

Dixon, R. (1995) *TheBaumgarten Corruption: from Sense to Nonsense in Art and Philosophy*, East Haven, CT, Pluto Press.

Dreyfus, T. and Eisenberg, T. (1986) 'On the aesthetics of mathematical thought', *For the Learning of Mathematics* 6(1), 2-10.

Dyson, F. (1982) 'Discussions', in Curtin, D. (ed.), *The Aesthetic Dimension of Science,* New York, NY: Philosophical Library, pp. 107-145.

Gray, J. (1984) 'A commentary on Gauss's mathematical diary, 1796–1814, with an English translation', *Expositiones Mathematicae* 2(2), 97-130.

Hadamard, J. (1945) *The Psychology of Invention in the Mathematical Field*, Princeton, NJ, Princeton University Press.

Hanson, H. (2001) Available from: www.ihyaproductions.com/articles/hyCBCtranscript.htm.

Hardy, G. (1940) *A Mathematician's Apology*, Cambridge, Cambs, Cambridge University Press.

Hogben, L. (1940) *Mathematics for the Million*, New York, NY, Norton.
James, I. (2003) 'Autism in mathematicians', *The Mathematical Intelligencer* 25(4), 62-65.
King, J. (1992) *The Art of Mathematics*, New York, NY, Plenum Press.
Kline, M. (1953) *Mathematics in Western Culture*, London, George Allen and Unwin.
Kline, M. (1980) *Mathematics: the Loss of Certainty*, Oxford, Oxon, Oxford University Press.
Krull, W. (1930/1987) 'The aesthetic viewpoint in mathematics', *The Mathematical Intelligencer* 9(1), 48-52.
Le Lionnais, F. (1948/1971) 'Beauty in mathematics', in Le Lionnais, F. (ed.), *Great Currents of Mathematical Thought*, New York, NY, Dover, pp. 121-158.
Menand, L. (2001) *The Metaphysical Club*, New York, NY, Farrar, Straus and Giroux
Murray, M. (2000) *Women Becoming Mathematicians: Creating a Professional Identity in Post-World War II America*, Cambridge, MA, MIT Press.
Padovan, R. (1999) *Proportion: Science, Philosophy, Architecture*, New York, NY, Routledge
Penrose, R. (1974) 'The role of aesthetic in pure and applied mathematical research', *The Institute of Mathematics and its Applications* 7/8(10), 266-271.
Pimm, D. (1994) 'Another psychology of mathematics education', in Ernest, P. (ed.), *Constructing Mathematical Knowledge: Epistemology and Mathematical Education*, London, Falmer press, pp. 111-124.
Plath, S. (1965) *Ariel*, London, Faber and Faber.
Poincaré, H. (1908/1956) 'Mathematical creation', in Newman, J. (ed.), *The World of Mathematics*, New York, NY, Simon and Schuster, vol. 4, pp. 2041-2050.
Polanyi, M. (1958) *Personal Knowledge: Towards a Post-Critical Philosophy*, New York, NY, Harper and Row.
Rota, G.-C. (1997) *Indiscrete Thoughts*, Boston, MA, Birkhäuser.
Russell, B. (1917) *Mysticism and Logic*, Garden City, NY, Doubleday Anchor Books.
Sawyer, W. (1955) *Prelude to Mathematics*, Harmondsworth, Middlesex, Pelican.
Serres, M. (1982) *Hermes: Literature, Science, Philosophy*, London, The Johns Hopkins University Press.
Sinclair, N. (2002) *Mindful of Beauty: the Roles of the Aesthetic in the Learning and Doing of Mathematics*, Unpublished doctoral dissertation, Kingston, ON, Queen's University.
Sinclair, N., Pimm, D. and Higginson, W. (eds) (in press) *Mathematics and the Aesthetic: New Approaches to an Ancient Affinity*, New York, NY, Springer-Verlag.
Sipser, M. (1986) 'The Reporter column', *Technology Review* 89(1), 78-80.
Sullivan, J. (1925/1956) 'Mathematics as an art', in Newman, J. (ed.), *The World of Mathematics*, New York, NY, Simon and Schuster, vol. 3, pp. 2015-2021.
Tahta, D. (in press) 'Sensible objects', in Sinclair, N., Pimm, D. and Higginson, W. (eds), *Mathematics and the Aesthetic: New Approaches to an Ancient Affinity*, New York, NY, Springer-Verlag.
Turkle, S. (1981) *Psychoanalytic Politics: Freud's French Revolution*, Cambridge, MA, MIT Press.
Tymoczko, T. (1993) 'Value judgements in mathematics: can we treat mathematics as an art?', in White, A. (ed.), *Essays in Humanistic Mathematics*, Washington, DC, The Mathematical Association of America, pp. 67-77.
Walkerdine, V. (1988) *The Mastery of Reason*, London, Routledge.
Wells, D. (1990) 'Are these the most beautiful?', *The Mathematical Intelligencer* 12(3), 37-41.
Whitehead, A. (1926) *Science and the Modern World*, New York, NY, Macmillan.

EXPERIMENTAL TEACHING AS A WAY OF BUILDING BRIDGES

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This paper centers on a case study of successful learning through extended exploration of one task. The research subjects are six arts undergraduates, in an experimental mathematics course. These students had broad latitude to make choices, to explore the consequences, and to help shape discussion and instruction. In the present case, we examine how one student builds, reasons from, and reshapes representations of motion so that they make sense in terms of her experience. Analysis centers on her choices, motivations, reasoning, and presentations, to give some indication of the full group's collaboration. These six students, who avoided mathematics in the past, demonstrate striking motivation, insight and ability, as they build toward central mathematical ideas and practices in ways that are both personal and public.

Introduction

This paper centers on the choices learners make, and what these choices might suggest to us about what they value. The learners here will be six undergraduates, four in dance and two in industrial design. The setting is a general education mathematics course (Honors 250, at Brigham Young University) in mathematical modeling, which I taught in the fall of 2002 as part of a research project. I was interested especially in learning how these six students reason and present their thinking about change and motion, and I had an axe to grind.

I thought that students in the arts, especially, might come to understand the world through movement, either their own motion in space, or through the process of making drawings. Such experience, and the capabilities that such experience might make available, it would seem, could help with learning certain kinds of mathematics. Nonetheless, arts students like these six have typically avoided mathematics, sometimes with a passion.

With these ideas in mind, I put together a sequence of extended task investigations that I hoped might give these students opportunities to build important concepts in the course of explorations and discussions that they would have a major part in shaping. From the choices that they made, and how they justified their choices, I hoped to gain a clearer understanding of the ways in which they worked with past experience, and especially of what they valued. In particular, I wanted to understand as clearly as I could what these students might come to value in the mathematics that they addressed, and why.

If we think of classroom interactions as a field (a locus of personal and social action) we can try to bring that field into a temporary focus by tracing carefully the work and thinking of one individual, as she moves and acts within the field. Here we follow one student, Krista, for about an hour. Krista, a modern dance student, grew up in Nanaimo, BC. From childhood a highly gifted dancer, Krista has recently emerged as an impressive and original choreographer. In high school, she was also a kickboxing champion, which might seem surprising given her small size until you come to know her.

This presentation represents collective effort. The analysis presented here, joint work with my colleague Chuck Walter and my undergraduate assistant Marin Bradshaw, will appear in greater detail, with more attention to context and social interaction, elsewhere. Marin, with Tiffini Glaze, ran cameras, captured and edited many hours of video, and helped energetically with several levels of analysis. Members of my research practicum contributed further insights and ideas, especially Janet Walter and Malina Duff.

Experimental teaching

Consider for a moment how a mathematics class might bridge between the different worlds inhabited by mathematicians and artists. Here, for simplicity, we concentrate on dance. From a dancer's point of view, good dance ought to make sense and accomplish something. Nonetheless, that something may be difficult to put in words. We seem to experience what good dance can accomplish as a kind of discovery or recognition, perhaps during rehearsal or performance, or perhaps after. In particular, just going through a sequence of motions, without sense or purpose, even with impressive skill, is not enough to make good dance.

Thus, I thought, in order to convince a dancer, good mathematics might have to work in the same way. In other words, to be recognized as good, a given piece of mathematics should also make sense and accomplish something of importance to its audience. Here, perhaps in contrast to dance, the accomplished purpose might be simpler to express in ordinary language, as Sara, another dance student in this project, demonstrated in her writing (Speiser & Walter, 2003; Speiser, Walter & Glaze, 2004).

For our research group, as we explored the thinking of a group of students who were also working artists, even quite familiar mathematics could begin to look at least a little new or strange. This sense of novelty or strangeness, we felt, might afford a useful vantage point to make certain aspects of our own understanding, that we may easily take for granted (like the medium of water for a fish) available for detailed consideration. Hence I chose to work with students outside the sciences, and strove to establish, jointly with my students, a culture of mutual, task-centered exploration and experiment. This is the sense I have in mind when I denote the work reported here *experimental teaching*. In particular, I worked and planned quite consciously to learn new ways of thinking from my students, and to follow their concerns and interests. Perhaps, working in this way, I might learn something new about familiar mathematics.

To gain better understanding of my students' work and thinking, I chose tasks I hoped would lead to rich, extended explorations. Further, I sought to limit my own interventions mainly to proposing tasks, facilitating student discourse, and to making potentially useful tools available. (Such tools here included rulers, Cuisenaire rods, markers, paper, several basic definitions and a handful of core ideas the students had likely seen before, such as using slopes to picture ratios and using areas to picture products.) I also tried to limit my own questions. Now and then I asked a question simply to gain a better understanding for myself of what a student did or thought. And very rarely, I might ask a question to trigger a potential renegotiation of a student's interpretation of the task at hand. (I'll say more about this kind of question later.)

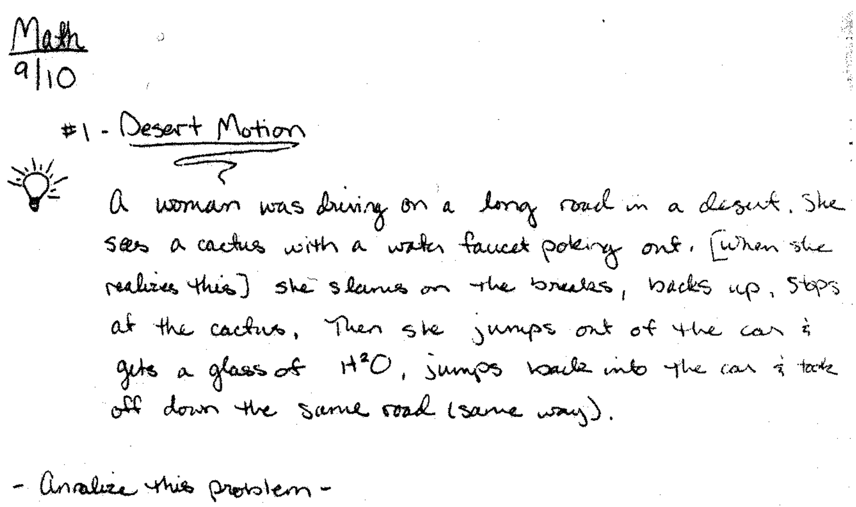
These rough rules for my activity remained in force throughout the students' exploration of a given task, which in practice might require several three-hour class periods. I should add, however, that these rules were loosened when completed task solutions were "unpacked" in subsequent discussion. At such times I often helped my students to compare their own discoveries, notations and language use to more "official" terms, results and usages. These often turned out, as we shall see below, to be quite similar.

While my students bridged from dance to mathematics, the research team soon found a range of opportunities to look at mathematics in new ways, based in large part on student discoveries in progress. The approach we took to learn about our students' thinking entailed (a) strong focus on the selection and design of tasks, (b) strong emphasis on analysis of student presentations, and (c) a research approach (a variant of grounded theory) where frameworks for analysis can develop through close study of emerging data, with instructional decisions (including the selection and design of further tasks) based on the unfolding data and analyses. I will now illustrate this approach in more detail, with Krista's work and thinking as a partial focus.

The task

Krista's presentations, in the main data segment to be analyzed below, reflect both her own thinking and the movement of ideas within the group of six. The group began, on September 10, by working on a task designed several years earlier by diSessa and several coworkers in Berkeley, the *desert motion* (diSessa & Sherin, 2000; Sherin, 2000). I proposed this task in words, as a kind of story problem. To be precise, the task is anchored to the story of a journey. Student discussion quickly gave the task the shape that it assumed for all of us. The desert motion task was designed deliberately to offer learners opportunities to invent, criticize, refine, and reinvent representations of motion collectively, based on what turned out to be a rich and useful repertoire of prior representational capabilities. I expected my six student subjects, as artists, to arrive with perhaps even greater capabilities, so I chose the desert motion to begin our work together.

In Figure 1 we see the task as reproduced on the first page of Sara's journal. This version is perhaps the clearest student presentation of the task as it was understood and solved over the next few sessions.


Math
9/10

#1 - Desert Motion

A woman was driving on a long road in a desert. She sees a cactus with a water faucet poking out. [When she realizes this] she slams on the breaks, backs up, stops at the cactus. Then she jumps out of the car & gets a glass of H^2O, jumps back into the car & takes off down the same road (same way).

- Analize this problem -

Figure 1. Sara's version of *desert motion*.

In class, I gave the last line slightly differently, as follows. (Quoted transcript segments here begin with the initial time code for the tape segment at hand.)

06:45

RS. Now mathematics, partly, is about how you represent things. This story has a lot of information. How would you, how would you represent this?

Krista [After three seconds, smiles, holds up a notebook page]. Like *this*.

Krista stepped directly to the whiteboard, opened a marker, and began to draw a series of cartoon-like illustrations, which Sara would soon continue (Figure 2). These drawings would first be refined and later drastically reshaped, to make further details of the car's motion readable.

Figure 2. Krista (seated, left), with the drawings she began.

Sara (standing, right) continues the sequence.

A week later, on September 17, all six students, indeed, had progressed to more detailed, more refined and in some ways more standard presentations.

The centrality of presentations

In the work discussed below, Krista will think out loud, often literally on her feet, building public presentations as she goes along. As a starting point for analysis, and to illustrate the way our research group has come to work with student presentations, I should emphasize that each such presentation often can be shown to serve two purposes (Speiser, Walter & Maher, 2003): to present part of a learner's thinking to herself, and to provide a basis for communicating information and ideas to others. On the one hand, Krista often shares her thinking, in the making, as she strives to clarify her thinking for herself. On the other hand, her presentations can be shown to link her own ideas, in detail, to ideas already in circulation, often to refine and summarize collective understanding. In this way, for Krista, mathematics merges with performance.

Drawings as in Figure 2 soon became more elaborate, to present time series where a running person, with blowing hair and vivid facial expressions, would soon replace the car. Here (to suggest a new interpretation of the task) I asked if some way might be found to convey distances traveled and perhaps to give some sense of time. At this point, in a rare intervention, I offered Cuisenaire rods as tools, without suggesting how they might be used. For Ali (short for Alison), a design student, it was natural to work on paper or with concrete objects. Her construction (on September 17, one week after Figure 2) appears in Figure 3 along with several sketches. Ali used arrows to present first forward, then backward, and then forward motion. Green rods present the cactus, while two horizontal rods indicate the driver's excursion to and from the cactus while the car stood still. We should note that height here, for the arrows, seems to correlate with speed. As Ali presents, Krista, standing at her shoulder, triggers a brief discussion of possible alternatives.

14:49

Krista. I think that the longer, cool it, the faster, faster like the speed is higher too, but if you look at the length, that means because they're going faster, it travels in…

Sara. Right, the longer rod shows...

Krista. …in the time, you could even like, you know, break it up. So there's, you know, have a thing here [points to an arrow]…

Sara. …more distance of travel.

Ali. You could put like the rods vertically showing like the actual, like taking for each square equals a certain amount of time or whatever, and then, it could be speed and distance, or something. Right.

Krista. Like it makes sense, cause like they're going faster they cover more area.

Ali. Yeah.

The arrow construction represents a group of several contrasting models developed by the students here, in which motion is made to correspond to movement back and forth along a horizontal line. Such linear models seem natural to many learners, who often prefer them to more standard graphs (Speiser & Walter, 1994, 1996; Speiser, Walter & Maher, 2003). Nonetheless, Ali's suggestion to place rods vertically would also seem to point toward standard graphs.

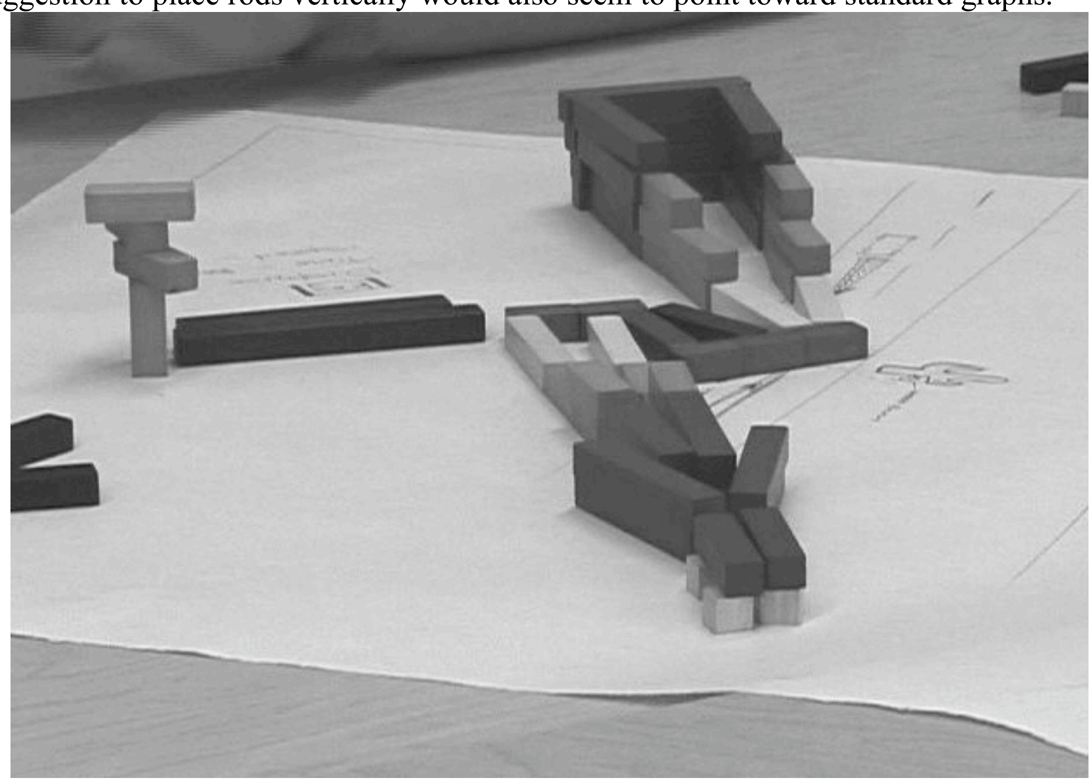


Figure 3. Ali's arrows.

Our attention focuses on fine details of presentations here, because these students *reasoned* from the presentations that they built. In particular, their presentations reflect goal-directed choices. In this way, what we learn about our students' choices might enable us to reconstruct some of their goals and values. Ali's arrows, for example, may help her to draw conclusions about the car's position as it varies with respect to time, and to communicate her reasoning to others. It also represents a visual performance, where colors and arrows can direct attention to a sequence of implied events. Further, based on later conversation, Ali and others saw her model as a starting point for further development. From their presentations, students can reason and make further choices. The need to understand the reasons for their choices leads directly to specific choices we must make about how to make our own sense of the extensive data gathered as the students worked.

Analytical perspectives

To motivate the next discussion, here are several questions that I sought to pose directly to the data. How do these students make sense of relationships between speed, distance, and time? How do they reason about change and motion? How do they present their thinking to themselves and others? In this experiment, I could not know in advance what students would bring with them to the given tasks, either in terms of their experience or in terms of their intents and dispositions. Hence the research team sought a methodology that could afford us ways to attend flexibly to learners' strengths and motivations, and to respond, in the spirit of experimental teaching, based on our own emerging curiosity and understanding.

We chose a variant of grounded theory, based on a kind of open coding (Strauss, 1987). We built memos first in our video editor, identifying chosen clips for later study, and recording briefly what we saw in them in an initial viewing. Then as we transcribed, we built what soon turned out, upon reflection, to be two further layers of memos, in much finer detail, organized by time codes, clip by clip.

The first layer consists of memos about data right at hand. These memos address more or less directly the initial questions posed above. As we built answers to these questions, a second layer of memos soon emerged. We have come to view these as memos about theory. How might we understand the sense that learners' presentations make, the purposes they serve, and the choices that have led to them? In this way, a second set of research questions surfaced in the process of analysis, and led to further close inspection of the data.

To illustrate both layers in practice, here is a very brief example, based on one line of the transcript of a conversation (September 17) not long after as Ali's arrow presentation.

Figure 4. Krista puts two rods together.

Developing her earlier comment, Krista has just explained that when a rod is held vertically, its width can stand for a unit of time (perhaps a second or an hour), while the length can measure speed. The same rod, horizontally, for her, represents simply the distance traveled, seen as the product of rate and time. These remarks emerge from Layer 1. Now she puts two rods together, first horizontally, then vertically.

> 22:25
>
> *Krista*. [Holds 2 black rods horizontally] You would be traveling 14 miles [places the rods together vertically, Figure 4] in two seconds, uh, of seven miles per hour.

We can now proceed in Layer 2. Krista has proposed a vertical representation where rods, in effect, present a graph of speed coded as *height*. Her physical actions in the transcript line above show how to pass from a horizontal (linear) presentation to the new vertical one. Thinking aloud, Krista seems imprecise with speed units (Layer 1), but her actions with the rods suggest her motivation (Layer 2): to connect speed, time and distance, as presented in two different but linked models. We built this second layer of interpretation in response to the unfolding action of the class. Its content seems largely theoretical in that this layer's memos emphasize how we might understand how learners make sense, reason, and make goal-directed choices. In this way, I see experimental teaching as one way of building theory.[1]

Data and Analysis

To introduce the central data segment, it might help to review some background. On September 17, a week before the data to be discussed below, Krista had built two models of the desert motion, using Cuisenaire rods. The first (see Figure 5) is a linear model, a sequence of rods placed end-to-end, flat on the table. In this model, each given rod indicates a distance traveled (through its length) in one unit of time. Krista has pointed out already that each rod could thus also be read as showing us a *speed* (distance per unit time). We shall call this linear array the *line model*. It presents the total distance traveled (both forward and backward) as a sequence of incremental segments.[2]

In the second model, the same collection of rods, in the same order, has been set up vertically, like slats in a fence. (This fence also appears in Figure 4.) Based on Krista's prior explanation for the special case of two equal rods, each rod here should be understood to represent the car's average speed for each successive unit interval of time. Such speeds now will vary. Hence, as Krista explained, the width of each rod (one centimeter) always represents one unit of time, while its length presents the speed, which now changes as the car's motion proceeds. Krista explicitly identifies the rod's area (more precisely, the area of the side facing the viewer) as the product length × width, which, as she explains, equals the distance traveled, given by the corresponding product speed × time. This presentation, called the *fence model*, appears in the foreground of Figure 5. Note that the intervals without rods stand (as Krista made explicit) for intervals of time in which the car was motionless.

Figure 5. Krista's line and fence models (September 17).

One week later, on September 24, following a suggestion (teacher intervention) to consider *velocity* as well as speed, so that the rate of change for distance with respect to time would become negative for backward motion, Krista built a third model, based on a grid of one-centimeter squares that she had drawn by hand. On this grid, Krista built a variation on the fence model, where rods corresponding to backward motion extend downward from a horizontal axis. This model, shown in Figures 6 and 7 below, will be denoted *rods on paper*. In effect, the rods, this time, approximate a standard graph of velocity with respect to time, with axes shown and units carefully identified.

In our central data segment, just below, Krista connects this model to the first two, offers a fourth model (a variation of the line model) and then explains how distance in the line model can be made to correspond to area for the rods on paper.

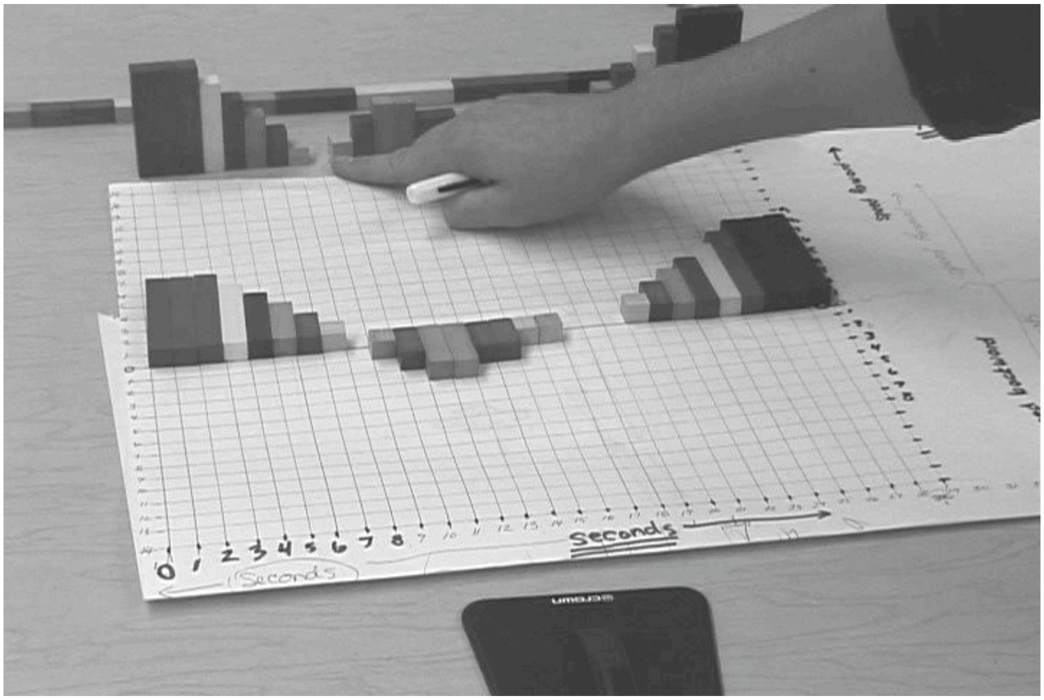


Figure 6. Krista points to her fence model, with rods on paper in foreground (September 24).

31:50

Krista. This [points to fence model, Figure 6] is my original model right here and each width of the block equals seconds. The height equaled how fast it is going. So therefore with having the height and having the time, or with having the speed and the time, or the rate and the time, you can figure out the distance.

In earlier discussions, Krista had emphasized that if one knew two of the three quantities, distance, speed and time, one could find the third. Further, as we have seen, she interpreted the relation

$$\text{distance} = \text{rate} \times \text{time}$$

explicitly in terms of surface area, in the context of the fence and rods on paper models. Next, Krista will connect her fence model to a narrative of the car's motion, reading from left to right. In effect, Krista provides her listeners a way to read the fence model (and by implication other models) as a presentation of the motion of the car.

32:28

Krista. So, like at so many… at this many seconds, just an example [points to left end of the fence model, three dark green rods], they're traveling this speed, and then [moves to the right and points to shorter rods] they slow down, slow down, slow down. And when there's no speed at all [points to the gap after the first group of rods] that's where there's a blank, that's where there's no speed there. And this [points to the next group of rods, tracing upward with her finger] they're reversing, they can't go as fast [points to relatively short rods that represent speeds for this segment of the motion] as they can when they're reversing.

Krista continues, interpreting the entire length of the fence in a similar way. Because the fence model presents speeds rather than velocities, the line model presents the total distance traveled, regardless of whether the motion was forward or back. Next, she connects the fence model to her linear model, where she simply lined up copies of all the rods in the fence. Here for the first time, she mentions the rods on paper model. As described above, the rods on paper are the same as in the fence, except that when the car moves in reverse, the rods on paper will extend *below* the horizontal axis rather than above the table.

33:06

Krista. So, [gestures across the rods on paper] doing this with these blocks, if you want to find the, exactly how far the car has gone, um, like how much gas it's used or something, how far it's really gone, all you have to do is just lay it sideways [takes several rods from the left end of fence model and places them consecutively on the table to construct a segment of the line model] and put the model together. Because it doesn't travel anywhere where there's no speed. So that's just showing an example, just putting all these in the exact same way, just laying them flat.

Here two points might be observed. First, Krista points out, in the line model, that rods can be joined together at places that correspond to gaps in the fence (or rods on paper) model because no distance had been traveled while the car was standing still. In this way she emphasizes that spaces in the first model (which present intervals of time as horizontal distance) should not translate to distance in the second model, which presents segments of road distance as horizontal distance). Second, Krista emphasizes that what she shows us here is merely an example, hence that she intends to explicate a general insight or approach through one specific presentation.

But, we may now ask, what happens when the car goes backwards? In this case Krista directs attention firmly to the rods on paper.

33:51

Krista. So with doing this with the velocity [points to rods below the time axis], so there's seconds [points successively to the numerals 0 through 8 at lower left of grid, see Figure 7] and then speed forward [points upward on the grid] and speed backwards [points downward on grid]. So they're going at six, whatever six is [indicating 6cm of height above her horizontal axis], however many miles per hour, then deceleration [indicates first gap between rods], they're not traveling anywhere, for this amount of time, then decelerates [indicated next group of rods, below time axis, for negative velocities] up to, say 3, whatever speed that represents, and then [next gap] the car's not traveling anywhere… [She continues, to complete the narrative using the rods on paper].

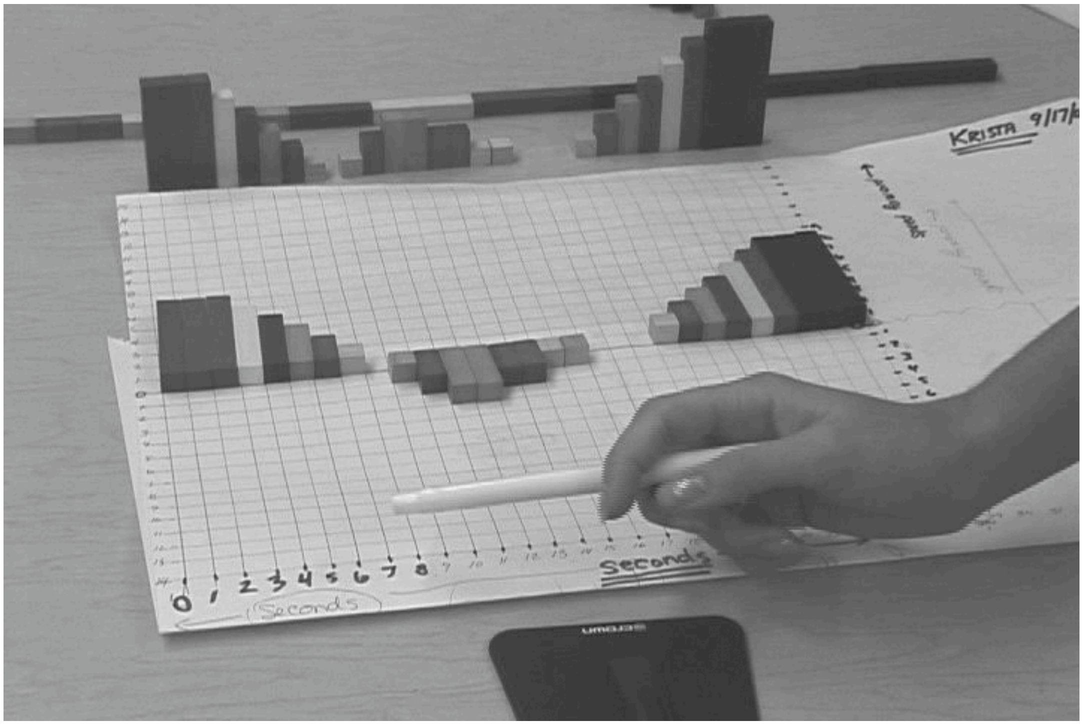


Figure 7. Krista points to her time scale at the lower left (September 24).

Taking stock so far, we see how Krista maps areas on her graph (counted positively, as absolute distances) to corresponding segments on her line model for total distance traveled. In this way she builds an isomorphism between both models, where area in one will correspond to distance in the other.

Next, Krista makes a choice that leads her to propose a variation on her horizontal line model. In earlier class discussion, we had noted that odometers in cars (in practice) only count distance when a car is moving forward. Hence Krista now opts to show how her models might correspond to an odometer. We include this segment to mainly suggest how flexibly Krista has come work with models.

> 34:47
>
> *Krista*. So that you would still be traveling, traveling all that space using that much gas [indicates the line model] but just going forward, you could take that out [takes the segment corresponding to reverse motion out of the line and places it some distance to the left, so that the backward motion, in effect, can be subtracted from the total distance traveled] and put these [the remaining forward rods] together [slides them together], they go together, and that's how far you've actually gone forward.

By now, Krista has four models in play: her original linear model for total distance traveled, her fence model for speed in terms of time, the rods on paper model for velocity in terms of time, and her modified linear model for the distance driven forward. She has linked them all, at least roughly, in her previous discussion. Now she prepares to summarize. Krista's reasoning, so far, has been to connect her presentations to the desert motion narrative or to each other, in the latter case by setting up explicit, detailed correspondences between her models. Along the way, she has indicated carefully how distances, rates and times should be interpreted in each presentation, contrasting similarities and differences.

> 00:35:16
>
> *Krista*: And then because…

Lainie: [Points to rods on graph] I like this one a lot.

Krista: … because you have both [line model and the rods on the graph] you can figure out the distance, how far you've gone. Because you have the rate and the time. Because distance equals rate times time. So therefore like this whole kind of surface kind of thing [points first to her initial group of rods, then to the whole model] could show, you know, in effect, how far [sweeps one hand along the line model] it actually traveled. And this [points to rods on graph] shows more the time frame.

In this summary, Krista's reasoning converges on how "surface" (which we interpret to mean area, based on transcript data from the week before) in the fence and graph models corresponds to distance, based on her earlier discussion of the formula for distance traveled as the product rate × time. Further, Krista address here our central question about choice: for she has compared the advantages and drawbacks of her models, in detail, based on different purposes that they might serve. Further, she seems to treat her models as flexible designs, in that they can be redesigned or reshaped to serve a *variety* of purposes.

Now we can connect this analytic narrative of Krista's presentations to the memos that have led to it. So far, the discussion has mainly stressed the issues organized in Layer 1: presentations, reasoning and conceptual grounding. We next turn to several issues raised in Layer 2. Krista's presentations solve the desert motion problem by illustrating a *general* approach that uses linear and graphic models to present motion in terms of rate and distance. (Linear models emphasize the latter; graphic models emphasize the former.) No single model fully solves the desert motion task (or more generally any such task) as Krista now understands it. Hence different partial models, each emphasizing different aspects, must be linked. We have explicated Krista's choices by examining the reasons she has given for establishing or rejecting models to accomplish given purposes. As we have seen, she anchors her choices to examples that she hopes will illustrate the range of purposes she sees.

Finally (completing Layer 2), we can explore how we might understand what Krista's choices may have *done* for her. In this connection, the analysis above suggests that building a variety of models (while explicitly, concurrently connecting them) may serve two central purposes for Krista: (a) to allow her to present a given change or motion from a variety of perspectives, once she has recognized that no single presentation, taken by itself, fully embodies what she knows; (b) to suggest, based on considered evidence, that what she has come to know seems to *depend fundamentally* on that variety, in the sense that each model or presentation, to be clearly understood (whether by her or someone else), needs to be connected and compared, in detail, to significant alternatives.

That a variety of carefully linked models may be *necessary* to present the desert motion is a statement about *knowledge*, with potential implications (again, perhaps for theory) about how people might come to gain it.

Discussion

Recall the wider field that our close look at Krista, potentially, might help to focus: the complex collaboration of six students and one teacher. Along the way I've tried at least to hint at the variety of ideas in play among the students, and how new presentations would evolve, in clusters, as the learners (with very little help from me) progressively rethought and reshaped their understanding of desert motion task. In particular, as shown clearly in the data here, the students' presentations became increasingly numerical and analytic, although no statement of the desert motion task ever offered specific distances or speeds. In this development, Krista and Sara seemed to lead the way, based partly I think on their prior sense of what kinds of information mathematics ought to treat. For Sara in particular, mathematics, at some level, had to work with (in her words, *crunch*) numbers. For Krista also, numbers never seemed far below the surface, as in her explicit work with units, products and proportions. Nonetheless, Krista, like Sara, focuses primarily, as we have seen, on how sense can be made, in other words, on what needs to be *understood* about a given problem.

I would now suggest that for these students, understanding seems to come by *sensing, noticing, or finding* something in the process of a kind of *search*. In other words, they need to have an axe to grind. On September 17, just at the beginning of the session, Krista shared impressions about scientific problem solving. She seemed concerned especially with what a person does when normal ways of thinking fail to help her to make progress.

> 02:00
>
> *Krista*. I don't know, because maybe some people, like I just don't know about engineers or anything, but maybe just, well maybe, I don't know, because maybe people who are more left-brained thinking, just think different ways if they just maybe get out of how they normally think and you know maybe try to come up with different ideas. Like Einstein, he was thinking about traveling at the speed of, well, light, or something, and seeing himself in a mirror wondering if he could see his reflection and that's how he came to some of his stuff. And so all of that time was just you know sitting imagining, in his imagination. He said also, using your senses, all your other senses, like can help a lot with problem solving. Like, cause most of us are usually, see everything, like use multiple things with their visual senses, maybe like tactile, smelling, hearing, other kinds.

In the first sentence, Krista seems to stand on her side of bridge connecting art to science. She sees good science as creative, and by this she means, I think, that the best science *entails* breaking free of prior ways of thinking. The crux, of course, is how one does this. Here she cites the case of Einstein: we proceed by means of thought experiments, in other words through carefully considered acts of the imagination. Einstein, who imagines looking at a mirror moving at the speed of light, deliberately provokes a conflict with his prior thinking. How such conflicts come to be resolved depends perhaps on how one chooses to *reshape* one's prior understanding. In this way, Krista's thinking about science might resemble that of Freeman Dyson (1995), who considers scientists as rebels.

Nonetheless, I suspect strongly that the main point for Krista might be in the *imaginative act* itself: the way, in our imagination, we can build up a concrete situation that can be explored with understandings gained from sensory experience in a variety of contexts. Here again, I think we see an emphasis on choice. Indeed, in the work we saw Krista do later the same day, as when she re-imagined Cuisenaire rods so that their length and thickness measured, respectively, speed and time, Krista worked not only visually, but also, through the way she moved the rods, kinesthetically: her own motion led from one interpretation of the objects she was holding to another, thus forming a bridge from one experiential context to another. Based on this re-imagining, this bridge between experiential contexts, I think that Krista *saw* how to connect information that she had encoded in the fence model (in front of her in Figure 4) to the information coded in her line models. While Krista focused her discussion on her objects' interchanging meanings, I suspect strongly that she needed to *hold* the objects in her hands, not simply to communicate, but also to bring key senses (touch and movement awareness) into play.

For Walter Freeman (2000a, 2000b), a neuroscientist, brains process *meanings* rather than raw information. "Brains obtain information about the world through the consequences of their own embodied actions. The information thus obtained is used in constructing meaning and is then discarded." (Freeman, 2000b, p. 93) Thus, for Freeman, information does not appear in memory as an invariant body of facts, but instead depends profoundly on context and contextual experience. For example, Freeman, based in part upon his own research, considers how the brain identifies a smell: the information that we search in memory does not take the form of a molecule catalog, but rather a dense variety of contextualized experiences where each given molecule appeared. For Freeman (as for James, Dewey, Gibson and their followers) contexts emerge as we pursue specific goals and purposes. For Freeman, representations (which we call presentations here) are built *externally* as tools to facilitate the communication of meaning.

> I also infer that an organism that constructs and transmits representations cannot know their meanings until their sensory consequences have been delivered to its own limbic system. More generally a poet, painter, or scientist cannot know the full meaning of his or her creation until after the act has been registered as an act of the self, nor until the listeners and viewers have responded with reciprocal representations of their own, each with a meaning unique to the recipient. (Freeman, 2000b, p. 98).

For Krista, too, meaning seems closely tied to context and personal intent. As we saw, she carefully explained a range of possible, linked presentations of the desert motion, so that in a given context, with a particular intent, a useful presentation might be chosen. In this way Krista, a dancer and choreographer, makes sense of her mathematics, with the desert motion as her main example at the point that we observe her here. In particular, she sees the mathematics that she builds as serving, potentially, a variety of purposes.

The general ideas that Krista explains (ways of presentation and connection that will hold for many kinds of change and motion) are given, both to her and to her listeners, on the basis of a single case example (the desert motion), interpreted through drawings, rod models and (through the rods on paper) graphs. The models, built to serve a range of different purposes, are made sense of by a *theory* that Krista has built with them, in which different readings of a given graph or set of rods can be coordinated through an understanding of important underlying mathematics. That theory, that coordination, is also tied to quite specific contexts, where the rods are used as tools, to present different but closely connected meanings that Krista has assigned to them. Given the neuroscience, these meanings, I suspect, should correspond quite closely to what Krista has built up in memory. We next turn to Krista's listeners, and to their roles in the development of sense and meaning.

Conclusion

In this final section, I would like to stand first on the science side of Krista's bridge. From here, her presentations and her reasoning connect to key ideas in the development of calculus. Her fence and rods on paper could easily evolve (although we didn't fully take them there) to Riemann sums, and her connection between area and length could anchor (perhaps later) a clear understanding of the Fundamental Theorem of Calculus and even help to motivate its proof. In Krista's context, the ideas seem very natural, although I didn't find them so when I first learned about them as a student many years ago.

Looking back at the analysis presented here, it might seem that Krista and the other students moved in a direct line from cartoon-like presentations to vivid intimations of important mathematical ideas. However, I should warn you now that that any such impression might be quite misleading. The interactions between students, and indeed each student's journey through those interactions, reveal much greater complexity, and perhaps greater depth, than this carefully selected presentation has surely made available. In particular I wish that I could share more of the analysis that my undergraduate assistant (now law student) Marin Bradshaw has begun, that reveals how much Krista would move around the room, watching, questioning, learning from, and reaching out with interest and support to other students. In this way, Krista's participation seems very much in keeping with Freeman's inference, quoted above, that presentations, by their very nature, may reach their full significance only when they function as tools for building and conveying meanings to be shared with others. Krista is not alone. Tiffini Glaze (2004) has just completed an analysis of Sara's questions, to demonstrate not only how Sara built important meanings for herself through rigorous, persistent questioning, but also, like Krista, she helped others, through her interactions with them, to think and work more deeply on their own. Further, I regret that I have hardly touched on the imaginative, richly conceptual contributions by both design students, whose presentations here and later in the course had striking influence, especially for Sara.

These blind spots, where we could easily lose contact with the interactive field and its complexity by focusing too closely on one person, nonetheless seem necessary. They result from goal-directed choices on my part. I chose to share one case example in some detail because I hope that we can learn from Krista, as she offers us a chance to understand some of the life and thinking of the group of students that she worked with. I chose initially to collaborate with students in the arts because these students have chosen, as artists in the making, to center their working lives on building and communicating meaning. Hence I wanted you to have this chance to hear their voices, watch them move, and sense clearly their potential. Indeed, these students might exemplify, as anchors for new ways to think and work, the potential of many, many learners that we have not reached so far, and also suggest our own potential to connect with them.

Endnotes

1. Our work therefore reflects a long tradition in psychology that includes, for example, Dewey (1896), J.J. Gibson (1979), Siegler (1996), and, in neuroscience, Freeman (2000a, 2000b). We also owe much to diSessa (2000) and his coworkers, who have emphasized persuasively the strengths that learners bring with them, partly by means of the carefully considered task design we work with here. Further, our emphasis on presentations as a basis of analysis reflects strong influence from cognitive science, especially by way of R. B. Davis (1984).

2.We might view this line model as a variant of Ali's arrows (Figure 3), but here the motion goes in one direction only.

References

Davis, R. B. (1984). *Learning mathematics: A cognitive science approach to mathematics education.* Norwood, NJ: Ablex.

Dewey, J. (1896). The reflex arc concept in psychology. *Psychological Review,* 3, 357-370.

diSessa, A. A. (2000). *Changing minds.* Cambridge, MA: MIT press.

diSessa, A. A., & Sherin, B. L. (2000). Meta-representation: An introduction. *Journal of Mathematical Behavior,* 19, 385-398.

Dyson. F. (1995). The scientist as rebel. In J. Cornwall (Ed.), *Nature's imagination* (pp. 1-11). Oxford: Oxford University Press.

Freeman, W. J. (2000a). *How brains make up their minds.* New York: Columbia University Press.

Freeman, W. J. (2000b). A neurobiological interpretation of semiotics: Meaning, representation, and information. *Information Sciences,* 124, 93-102.

Gibson J. J. (1979). *The ecological approach to visual perception.* Boston: Houghton Mifflin.

Glaze, T. C. (2004). Reasoning about motion: A case study. Unpublished master's thesis, Brigham Young University, Provo, Utah, USA.

Sherin, B. L. (2000). How students invent representations of motion: A genetic account. *Journal of Mathematical Behavior,* 19, 399-441.

Siegler, R. S. (1996). *Emerging minds: The process of change in children's thinking.* New York: Oxford University Press.

Speiser, B., & Walter, C. (1994). Catwalk: First-semester calculus. *Journal of Mathematical Behavior,* 13, 135-152.

Speiser, B., & Walter, C. (1996). Second catwalk: Narrative, context, and embodiment. *Journal of Mathematical Behavior,* 15, 351-371.

Speiser, R., & Walter, C. (2003).). Getting at the mathematics: Sara's story. In N. A. Pateman, B. J. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 2003 joint meeting of PME and PMENA,* Vol. 4, (pp. 245-249). Honolulu, HI: University of Hawai'i.

Speiser, R., Walter, C., & Glaze, T. C. (in press). Getting at the mathematics: Sara's story. Educational Studies in Mathematics.

Speiser, B. Walter, C., & Maher, C. A. (2003). Representing motion: An experiment in learning. *Journal of Mathematical Behavior,* 22, 1-35.

Strauss, A. (1987). *Qualitative analysis for social scientists.* Cambridge, UK: Cambridge University Press.

LOCAL THEORETICAL MODELS IN ALGEBRA LEARNING: A MEETING POINT IN MATHEMATICS EDUCATION

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The need of interpreting unanticipated phenomena, arising from studies on the transition from arithmetic to algebra in the 80´s, gave rise to the long-term research program Acquiring Algebraic Language. *The theoretical formulation in this program is characterized by its condition of local elaboration linked to specific phenomena under study, which makes it possible to probe the nature of such phenomena in a way that bordering disciplines meet face to face with mathematics education. Through an example about operating the unknown, the paper discusses how this multidisciplinary theoretical job brings together the work of specialists who have analyzed algebra language from the standpoint of linguistics, semiotics and pragmatics. Implications for algebra teaching practice, when working with this theoretical perspective, are also discussed.*

Introduction

Mathematics education is a discipline located half way between the exact sciences and social and humanistic sciences. Moreover, its multi-disciplinary nature has made attempts at characterizing it an enormous challenge. Works devoted to meeting this challenge head on have contributed to defining the tasks, methods and theoretical foundations of mathematics education (see, for example: Freudenthal, 1973; Sierpinska & Kilpatrick, 1998; Lerman, Xu & Tsatsaroni, 2002; Boaler, 2002, among many others), yet in all cases there is an acknowledgment of the fact that the disciplinary boundaries it shares with converging areas of knowledge are very vague, particularly as regards to mathematics itself (see Goldin, 2003; Dörfler, 2003). This difficulty in determining disciplinary limits is usually aggravated in work aimed at interpreting observations made during empirical studies, especially if it is a matter of theoretically interpreting findings or results that were unanticipated (and perhaps not anticipatable) from the observation design stage. In many of the above-mentioned cases, one tends to turn to the theoretical frameworks derived from those other bordering disciplines, such as psychology, history, epistemology, semiotics or sociology, but the general frameworks of such disciplines do not always respond to the analysis needs of what is observed. This paper deals with just such phenomena, as reported by studies on the transition from arithmetic to algebraic thought in the 80s, and which led to a long-term research program that proposed development of theoretical elements that would make it possible to fine tune the very analysis of such phenomena.

One of the special features of the theoretical formulation in the research program *La Adquisición del Lenguaje Algebraico* (*Acquiring Algebraic Language*) [1] undertaken in the later 80s, is that although the point of departure is a general notion, that of a *Mathematical Sign System*, its very nature of local elaboration makes it possible to delve deeply and, hence, to generate new knowledge on the matter under study. In local treatment under the theoretical light in question, the perspective of *Local Theoretical Models* (developed by E. Filloy in the 90s) opens avenues among the multiple components that usually make the research problems up, instead of approaching them from a partial perspective. Indeed each local model contemplates the study of cognitive, formal mathematical competency, teaching and communications aspects.

It is precisely this comprehensive approach that I would like to emphasize here, given that it raises the possibility of substantively contributing to highly focused research, within multiple disciplines and, what is better yet, because the scope of such contributions depends upon significant exchanges between specialists and communities related to those disciplinary fields. The paper goes on to present one of the didactic phenomena that motivated this theoretical development, to then illustrate its usage by way of a few examples. While the final sections of the paper discuss the implications for disciplinary confluences, from a theoretical standpoint, and the possible repercussions in the area of mathematics teaching, from a practical standpoint.

The Polysemy of X: Manifestation of the First Didactic Cut

"From the outside-in" type of theoretical interpretations have been explored in the Mexican project *Acquiring Algebraic Language*, in which the phenomenon of the *polysemy of x* was reported on at a time when the general theoretical frameworks of mathematics education failed to provide sufficient elements for it to be described, let alone explained. The *polysemy of X* consists of the spontaneous reading that teenagers tend to engage in with equations of the type: $X + X/4 = 6 + X/4$, by saying that "this X (the first term on the left hand side) is equal to 6 and that these two (the Xs that appear in the terms X/4 on both sides) can have any value, but that both must have the same value" (Filloy & Rojano, 1989). The item intended to make it clear in clinical interviews situations that one is in the presence of "term to term equalization" resolution strategies, which make it possible to identify the identical terms (X/4) and to infer the equality between the remaining terms (X and 6). At that time, as has already been mentioned, not only were there no theoretical elements to analyze the children's favorite response (*polysemics*), but even the appropriate vocabulary needed to refer to it failed to exist. Thus terms were borrowed from semiotics in order to refer to the fact that the first step in the equalization ("this X is 6") comes from a reading made in the semantic field, to which restricted equations belong, in which case the X is an unknown, while the second step of the equalization ("these two can have any value") is derived from a reading made in the semantic field of tautological equations or algebraic equivalences and, in such case, the X is a general number. Hence the expression the *polysemy of X*, since it is a matter of allocating meanings derived from different semantic fields of algebraic language to one and the same symbol within one single algebraic formulation.

A specialized vocabulary would be needed just to engage in a mere description of the phenomenon, but in the process of delving into the origin of polysemous readings other needs arose, such as speaking to the cognitive nature inherent in tending to make this type of reading in algebra, as well as the possible impact of prior learning (in the field of pre-algebra) and the role that could be played by this type of interpretation in conceptualizations the likes of algebraic equality, mathematical unknowns and in learning formal methods for solving linear equations. A clinical interview undertaken with 12 and 13 year old students, who had not been taught how to use algebraic methods to solve such equations and who demonstrated that they engaged in a polysemous reading of the equations, provided data that made it possible to answer the research problem's underlying questions: Do the characteristics of the students' spontaneous solution of linear equations, in which operating the element represented is necessary (for instance, in which the unknown appears on both sides of the equation), attest to the presence of a didactic cut point in the transition from arithmetic thought to algebraic thought? (Filloy & Rojano, 1989). In the case of the different modes of linear equations used in the study, the answer was yes; in the particular case of the equations that led to polysemous readings, the answer is yes as well, and the arguments based on detailed descriptions of those readings can be found in the reports published for several years starting in 1984 (Filloy & Rojano 1984, 1985a, 1985b, 1991; Filloy, 1991; Rojano, 1986, 1988).

Amongst other things, the *polysemy of X* showed the need for an in-depth re-conceptualization of equality in mathematics that went well beyond its arithmetic meaning (as C. Kieran had already pointed out (Kieran, 1980)), thus in turn enabling conceptualization and manipulation of unknowns in the syntactic field of algebra. Attempts at drawing students' focus away from their polysemous reading, by restating to such students formulation word problems that corresponded to equations of the X + 5 = X + X type led nowhere, because although they were well willing to accept that all occurrences of X had the same referent within the context of the problem, once the question at the syntactic level was asked "what is the value of X?", the response was once again "this X (the first on the right side) is 5 and these two (the first on the left hand side and the second on the right) can have any value". The persistent manner in which they focused on certain ways of reading algebraic expressions led to discussions of intermediate language strata between arithmetic and algebra, as well as of cognitive tendencies during periods of transition leading to algebraic thought. E. Filloy expressed the need to be very specific when dealing with phenomena of this nature, in which the general theoretical frameworks prevented deep analysis (Filloy, 1999), and proposed development of local models. Said theoretical development has over time made it possible to study the evolution toward algebraic thought in teenagers from a perspective that considers algebra as a language and in which competent usage thereof is preceded by abstraction processes found in language strata that come before that of the *Mathematical Sign System of Algebra*. Then a brief reference is made to this theoretical proposal, to then go on to illustrate its application in the case of a *second didactic cut*: when there is a need to operate something unknown, and when said unknown is represented in terms of another unknown quantity.

Local Theoretical Models and Mathematical Sign Systems

E. Filloy defines *Local Theoretical Models* (LTMs) according to the following four characteristics: (for an extensive presentation, see Kieran & Filloy, 1989; Filloy et al, in process): 1) an LTM consists of a set of assumptions about a concept or system; 2) an LTM describes a type of object or system by attributing an internal structure to it, which when taken as reference will explain several of the object's or system's properties; 3) an LTM is considered an approximation, which is useful for certain purposes; 4) an LTM is often formulated and developed based on an analogy between the object or system that is described and another different object or system. The author goes on to mention a series of differences that exist between model and theory so as to make it very clear why he emphasizes the term *model* within the perspective he adopts.

In the foregoing light, symbolic algebra is considered a language and there exists the interest, as has been previously stated, in also studying the relationship between the latter language and prior or intermediate language levels, thus incorporating the notion of a *Mathematical Sign System* (MSS) in the broad sense of the term (Filloy, 1999, Puig, 1994). Consequently MSSs are sign systems in which a socially agreed possibility exists to generate *signic* functions. As a result, the definition has room enough for including cases in which functional relations (*signic*) have been established for use of didactic devices within a teaching situation and in which usage thereof may be intentionally temporary. In other words, also considered or included are sign systems or sign system strata produced by students in order to give meaning to what is presented to them within a teaching model, even when said systems are governed by correspondences that have not been socially established, but that are rather idiosyncratic.

The notion of MSS plays an essential role in (locally) defining the components that make up the LTM and that deal with different types of: 1) *teaching models*; 2) *models for cognitive processes*; 3) *formal competency models*, which simulates the performance of an ideal user's of a SMM; and 4) *communication models*, in order to describe the rules of communicative competency, text formation and decodification, and contextual and circumstantial disambiguation.

The Second Didactic Cut, Under the Light of LTMs

Simply put, a *didactic cut* in the transition from arithmetic to algebraic thought consists of the manifestations of students, who are facing tasks of an algebraic nature for the first time, as regards the need to build new meanings and new senses for arithmetic objects and operations, with the added special characteristic that such newly built meanings and senses necessarily presuppose a *break with* arithmetic. The study *Operación de la Incógnita* (*Operating the Unknown*)[2] provides evidence of this type of manifestation, at the time when students first face the task of solving equations in which an unknown must necessarily be operated, for instance equations of the AX + B = CX + D type (for a more detailed description, see Filloy & Rojano 1984, 1989). The *polysemy of X* described in preceding paragraphs is precisely one of those manifestations that we could call the *first didactic cut*. In this case, if algebraic equations are to be interpreted, new meanings for equality and unknowns must absolutely be built, and operations dealing with these mathematical objects must be endowed with new sense.

The analysis carried out in the study for the first cut led to surmising the presence of other breaks, in particular terms conjecture arose regarding the manifestation of *a second cut* at the time when students are first faced with unknowns and when the latter are represented in terms of another unknown. A corresponding algebraic task is the solving of systems that contain two linear equations with two unknowns, that is, systems of the type:

$$
\begin{aligned}
Ax + By &= C \\
Dx + Ey &= F
\end{aligned}
$$

(A, B, C, D and F are known numbers, x & y are the unknowns), whose solution eventually requires to manipulate expressions like:

$$y = (C - Ax)/B$$

where one of the unknowns (y) has a representation in terms of the other unknown (x).

Expansion of the study *Operación de la Incógnita* (*Operating the Unknown*), so as to delve deeper into this new cut point, produced data that are recently being studied from the perspective of *Local Theoretical Models* (Filloy, Rojano & Solares, 2003, in press). In order to accomplish the foregoing, components of the specific LTM have been elaborated and new data have been compiled and analyzed (Solares, 2003, 2004). Said elaborations are briefly described below in order to have examples at hand that illustrate the potential inherent in this theoretical perspective, but primarily to that they can serve as a basis to, in the next section, re-touch upon the issue of disciplinary boundaries and of LTMs as a meeting point for different areas of knowledge and for the corresponding specialist and teacher communities.

The Formal Competencies Model

The *Formal Competencies Model* of a *Local Theoretical Model* belongs to the realm of formal mathematics and its formulation consists of defining a *Mathematical Sign System* that enables decodification of texts produced within a problem learning situation (Filloy, 1999, Solares, 2002a, 2002b). For the case at hand, that of operating an unknown when said unknown is represented in terms of another unknown, it is a matter of defining the MSS of symbolic algebra in such a way that it is possible to analyze student responses in terms of the different contexts and semantic fields of algebra, as well as through different language strata located between natural language or the language of arithmetic and that of algebra. The mathematical actions that are interesting to describe under said terms are *algebraic equalization* and *algebraic substitution*, simply because both are the basis for the two solution methods for equation systems (2 x 2). With these requirements in mind a description of the *Formal Competencies Model* encompasses three different perspectives, namely: from the perspective of *syntax*, from that of *semantics*, and that of *pragmatics* (Solares, 2002a). In describing the model's *syntactic* aspects, D. Kirshner's (Kirshner, 1987) description of *algebraic syntax* was used; while for the semantic description, J.P. Drouhard's analysis of the *significances of algebraic writings* (Drouhard, 1992) was used; and finally, for the *pragmatic* description (in process), the plan is to resort to (amongst other things) an historical analysis of the evolution of algebra's MSS (see Solares, 2002a).

The Syntactic Perspective

As in Kirshner, who uses *generative and transformational grammar* to generate simple algebraic expressions and carry out transformations upon them, in the *Formal Competencies Model*, competent users produce a version of *superficial forms* of algebraic expressions. In *transformational grammar*, the transformation of said expressions lies in the corresponding *deep forms*, which reveal the structure of the productions in terms of the operations that make them up and their hierarchy. Using the *generative and transformational grammar* of algebra's MSS, Kirshner's work has expanded to production of equations and systems of linear equations and to the transformations that lead to their solution (Solares, 2002a), thus obtaining a syntactic and structural analysis of the transformations as though they were carried out by a competent algebra user. That is to say, modeling the formal competency to work with two unknown quantities, one expressed in terms of the other (Fig. 1 depicts the formal competency model for this particular case, using the generative and transformational grammar symbology used by Kirshner).

Transformation of equalization :

$$\mathbf{X = (Y\, O_1\, \omega)\ O_2\, \theta} \quad \wedge \quad \mathbf{X = (Y\, O_3\, \xi)\ O_4\, \tau},$$

$$\leftrightarrow \mathbf{(Y\, O_1\, \omega)\ O_2\, \theta = (Y\, O_3\, \xi)\ O_4\, \tau}.$$

where $\mathbf{O_2}$ y $\mathbf{O_4}$ are additions or subtractions; $\mathbf{O_1}$ y $\mathbf{O_3}$ are multiplications or divisions; $\boldsymbol{\omega}$, $\boldsymbol{\theta}$, $\boldsymbol{\xi}$ y $\boldsymbol{\tau}$ are real numbers ; and $\boldsymbol{\omega}$ y $\boldsymbol{\xi}$ are not simultaneously e qual zero .

Transformation of algebraic substitution :

$$\mathbf{X = (Y\, O_1\, \omega)\ O_2\, \theta} \quad \wedge \quad \mathbf{(X\, O_3\, \tau)\, O_4\, (Y\, O_5\, \xi) = \zeta}$$

$$\leftrightarrow \mathbf{(((Y\, O_1\, \omega)\ O_2\, \theta)\, O_3\, \tau)\, O_4\, (Y\, O_5\, \xi) = \zeta}.$$

where $\mathbf{O_2}$ y $\mathbf{O_4}$ are additions or subtractions; $\mathbf{O_1}$, $\mathbf{O_3}$ y are multiplications or divisions ; $\boldsymbol{\theta}$, $\boldsymbol{\omega}$, $\boldsymbol{\tau}$, $\boldsymbol{\xi}$ y $\boldsymbol{\zeta}$ are real numbers ; and $\boldsymbol{\omega}$, $\boldsymbol{\tau}$ y $\boldsymbol{\xi}$ are not simultaneously e qual zero.

Fig. 1

Semantic Perspective

This entails describing the significance of algebraic expressions and transformations that arise from their competent use, in other words, within the *formal competencies model*. In order to accomplish this, the notion of *significance* as developed by Drouhard and its four associated aspects are used: *reference*, which corresponds to the *algebraic evaluation function*; *sense*, which results from the *set of transformations* applicable to the expression; *interpretation*, which corresponds to the different readings made of the expression within the different *contexts* in which it may appear (such as the theory of numbers, analytical geometry, etc.); and *connotation*, which falls into *psychological significance* (which in turn depends on each individual) (Drouhard, 1992).

The aspects that Drouhard distinguishes as regards to the significance of algebraic expressions correspond to the significance given by a competent user. Whereas for purposes of the LTM presently under discussion, said distinction is extended to the analysis of pupil productions when tackling new algebraic problems, and said pupil productions are usually located at intermediate levels when representing and manipulating an unknown that is given in terms of another unknown.

Pragmatics perspective

The formal competencies model becomes complete when symbolic algebra is described from the *pragmatics* perspective and the research program includes analysis of the use of *substitution* in the historical evolution of algebraic language in trying to find out which strata of MSS incorporate *algebraic substitution* (Solares, 2002a). The research program includes analysis of ancient texts, such as the following:

- The *Arithmetic of Diophantus* (Book I), in which expanding upon L. Radford's analysis, one can say that Diophantus applied the equalization method to solve equation systems, as in problem 26 of Book I (Radford, 2001);
- The *Abbacus Books*, particularly the book by *Fibonacci* (Boncompagni, 1857), as well as previous analysis of these books (Filloy & Rojano, 1984). The language used in the *Abbacus Books* can be allocated at an intermediate stratum of the *algebraic system of signs*;
- *J. De Nemore's De Numeris Datis* (Hughes, 1981), which is a treatise on systems of quadratic equations written in a pre-symbolic stratum, but in which a type of substitution is used, in fact calculus' *algorithm substitution* is used to solve the equation systems;
- *Viète's Analytical Art* (Witmer, 1983), which marks the birth of symbolic algebra;
- *Stevin's Arithmetic* (Stevin, 1634), in which, according to Paradís and Malet, the first formal algebraic substitution is carried out (Paradís & Malet, 1989).

The Model for Cognitive Processes

In the study *Operating the Unknown* [2], which deals with the *first didactical cut*, a series of cognitive tendencies were identified and characterized, and they have since served to analyze student responses to their first point of contact with operating unknowns. The following are some of the cognitive tendencies: (1) conferring intermediate senses; (2) returning to more concrete situations upon occurrence of an analysis situation; (3) focusing on readings made in language strata that do not enable solving the problem situation; (4) the presence of inhibitory mechanisms; (5) the presence of obstructions arising from the influence of semantics on syntax, and vice-versa; (6) the need to confer senses to the networks of ever more abstract actions until they become operations (Filloy, 1991). By the way, the *polysemy of x* is a number 3 type tendency, because a polysemous reading of an unknown prevents students from recognizing that the equation is a restricted equality, and from searching for a single value for X.

Cognitive Process Model in the Second Cut

The entire set of cognitive tendency categories from the first study constitutes a model for LTM cognitive processes in the study of the *second cut* point, in which 12 students were interviewed. All of the students had been taught to solve linear equations, in which the unknown has to be operated, and its corresponding word problems. None of the students had yet been introduced to solving linear equation systems. The items list was divided in two sections: word problems and syntactic tasks. Figure 2 shows the syntactic items.

S.1	$x + 2 = 4$ $x + y = 8$	**S.6**	$14 + x = 37$ $4 - y = 28$	**S.11**	$3x + 4 = 22$ $4x + 2y = 34$	**S.16**	$y - 6 = 3x + 20$ $5y - 4x = 64$
S.2	$x + y = 10$ $x - y = 4$	**S.7**	$45 - x = 17$ $x + y = 41$	**S.12**	$3 \times (8 + x) = 6$ $2x + y = 23$	**S.17**	$3x + 8y = 84$ $8x + 3y = 59$
S.3	$x + y = 9$ $2x + 3y = 23$	**S.8**	$x + y = 60$ $3x = 171$	**S.13**	$2 \times (3 - x) = 6$ $4x + 3y = 12$	**S.18**	$4x - 3 = y$ $6x = y - 7$
S.4	$x + y = 12$ $5x - 6 = y$	**S.9**	$2 \times (x + 6) = 84$ $x + y = 104$	**S.14**	$4 \times (3 - x) = 4$ $x + y = 13$	**S.19**	$3x - 2 = y$ $5x = y + 8$
S.5	$x + 33 = 48$ $x + y = 73$	**S.10**	$4 \times (x - 8) = 72$ $x + y = 17$	**S.15**	$x - y = 1$ $x + y = 5$		

Fig. 2

As can be seen, although the first 14 items on the list have an increasing degree of complexity, all can be solved without requiring *algebraic substitution*, as *numerical substitution* alone is enough. As such in this section the students were asked to find the value of X and of Y, using their own means to solve the problems. The spontaneous strategies used by the students manifested cognitive tendencies the likes of (2) *returning to more concrete situations upon the occurrence of an analysis situation*, given that in all cases they resorted to a trial and error strategy. The following extracts, taken from the interviews of two girls, illustrate this tendency:

SL.3. $\boldsymbol{x+y=9}$ $\boldsymbol{2x+3y=23}$	L writes: $\boldsymbol{0+9}$ $\boldsymbol{1+8}$ $\boldsymbol{2+7}$ $\boldsymbol{3+6}$ $\boldsymbol{4+5}$ $\boldsymbol{5+4}$ Crosses the last line and then points out line by line, starting at the first one. She stops at " ***4 + 5***" and writes: ***4*** ***5*** ***8*** ***15*** ***23*** **L:** The numbers are ***4*** and ***5***. **Interviewer:** ***4*** and ***5***? **L:** Yes. First thing I did was to obtain the possible additions which gave ***9*** as a result and then, by "trial and error": ***2*** multiplied by ***1*** is ***2*** (points out " ***1 + 8***") for checking if they lead to the correct amount.	*Observations: L interprets these two equations as "linked", that is, as equations in which "x" and "y" have the same value in both equations. L writes the different ways for obtaining 9 through the addition of two positive integers and then performs the operations upon the unknowns indicated in the second equation until she finds those values for the unknowns, with which she obtains 23.*

Fig. 3

These spontaneous readings and strategies can obstruct learning of general solution methods such as is demonstrated in the following case:

SMt.17. $4x - 3 = y$ $6x = y - 7$	Mt wants to obtain the value of 'y'. So, she transforms the proposed system into: $4x - 3 = y$ $6x + 7 = y$ but instead of performing the equalization of the two expressions, she looks for the solution through the "trial and error" method using only the positive integers. Mt: In here ($4x - 3 = y$) says that four times 'x' minus three equals 'y'. And here ($6x + 7 = y$) says that six 'x'... plus seven, equals 'y'. This ($6x + 7 = y$) has to be bigger than this ($4x - 3 = y$).	*Observations: Mt is able to solve one-unknown equations, regardless of the numerical domains of the operated numbers or of the solutions or complexity of the equations' algebraic structure.* *Besides, she uses comparison in the case of equation systems derived from verbal problems in which both equations have the same unknown solved and the solutions are positive integers.*

Fig. 4

Other results obtained based on the interviews indicate that in the process of learning general methods to solve equation systems, amongst other things developing new significances for equality is needed. The foregoing becomes increasingly clear as the children progress along the list of items to be solved. In the section on Teaching Models, this idea is dealt with in greater detail.

Teaching Models for Solving Two-Unknown Equation Systems

From the analysis performed at the formal level, the following *didactical route* for introducing the general methods was adopted –coming from the previously acquired competencies for solving one-unknown linear equations: (1) reduction of the two-unknown and two-equation system to a one-unknown equation by applying comparison or substitution; (2) solution of the one-unknown equation applying the previously learned syntax; (3) substitution of the numerical value found in one of the two equations; and (4) solution of the equation through application of the previously learned syntax. Depending on whether the comparison or substitution is applied in step (3), this route leads to classical *equalization* and *substitution* methods for solution of equation systems (2 x 2).

EXAMPLE :

(i) $x - y = 1$ (ii) $x + y = 5$ **First Route** Reduction, from (i): $x - y = 1$, then $x = 1 + y$ Substitution in (ii): $(1 + y) + y = 5$ Solving for y: $1 + 2y = 5$, then $2y = 4$, therefore $y = 2$ Numerical Substitution in (i): $x - 2 = 1$ Solving for x: $x = 3$	**Second Route** Reduction from (i) and (ii): $x = 1 + y$ & $x = 5 - y$ Comparison of the two expressions for x: $1 + y = 5 - y$ Solving for y: $2y = 5 - 1$, then $2y = 4$, therefore $y = 2$ Numerical substitution in either (i) or (ii), let´s say in (ii): $x + 2 = 5$ Solving for x: $x = 5 - 2$, therefore $x = 3$

School Methods for Solving Linear Equation Systems

Linear system (2 x 2) solution methods by *equalization* and by *substitution* may be introduced by following the didactic path described previously and assume, for the first case, equalization of two chains of operations or two manners of calculating the value of one of the unknowns ($x = 1 + y$ & $x = 5 - y$, in the Second Route of the example above) and for the second, the need to substitute the representation of one of the unknowns in terms of the other in one of the equations of the system ($(1 + y) + y = 5$, in the First Route of the same example). There is a tendency to take for granted that the significance of that equalization, which is algebraic, is generated as of the significances attributed to the numerical equalization. There is likewise the tendency to assume that the algebraic substitution is an extension of the numerical substitution. Nonetheless results of the study on the *second cut* report that only students who are able to make sense of the actions that involve these solution methods (the transitivity of the equalization, among others) are in turn able to generate new *significances* for the algebraic equality, as required by operation of the unknown at this second level of representation (Filloy, Rojano & Solares, 2003).

The results referred to here, interpreted within the framework of the LTM prepared ex profeso for this study, made it possible to decipher the *significances* and the *senses* of the chains of actions involved in the school methods, as well as the obstacles in learning them, and all based on the *formal competencies* model and the model for *cognitive processes* (see the examples in section IV.2.1 of this papers, entitled *model for cognitive processes*). Hence the implications for the field of teaching are described as the mathematical, semiotics and cognitive perspectives, through the integrating axis of the *MSS* and with a level of detail that would have been very difficult to achieve from more general frameworks, even where they to come from the field of semiotics or psychology.

Models in Technology Learning Environments (The spreadsheet method)

To date reference has only been made to the syntactic level of unknown operations at a second level of representation; in other words within a more abstract MSS stratum of algebra, than that of the stratum in which it is only represented by literal symbols. Nevertheless, in the field of teaching support from the referents of a word problem for introduction of manipulative algebra issues is absolutely fundamental. Even the *second cut* study incorporates a section with word problems involving two unknown quantities (Filloy, Rojano & Solares, in press). This part of the study includes the classic methods: the *Cartesian Method*, which assumes a translation from natural language into the algebraic code and the arithmetic method or *Method of Successive Analytic Inferences*. In addition, a third method is included, the *Method by Successive Analytic Explorations,* which is located at an intermediate stratum between the *MSSs* corresponding to the latter two (see Filloy & Rubio, 1993). The latter begins by assuming a numerical value (arbitrary) for an unknown and the analysis process of the problem formulation is carried out in terms of numerical relations without having to deal from the beginning with incorporating the unknown into the analysis. Thus the relationship between the data and the unknowns becomes explicit, eventually leading to writing such relationships in algebra's *MSS*.

The results that arose in this part of the study, as well as its theoretical perspective of the corresponding LTM, represent inputs that can be taken advantage of by revisiting the data obtained in the Anglo-Mexican *Spreadsheets Algebra Project* [3], which was undertaken in the 90s. The project showed that it is feasible for pre-algebra students aged 9 through 12 to solve, with the aid of a spreadsheet, word problems involving 2, 3 and 4 unknown quantities. One explanation given for this fact is that said computer environment helps students in the analysis phase and in translating the word problem into a language that is similar to that of algebra, since it enables them to bear in mind the symbol referents in the problem during the analysis phase. A second reason is that students do not have to master symbolic-algebraic manipulation in order to solve linear equation systems, because in that phase the solution is numerical and is obtained by means of automated calculation procedures (Rojano & Sutherland, 1991, Rojano & Sutherland 2001; Sutherland & Rojano, 1993). Then an example is provided in order to illustrate the spreadsheets method used in the Anglo-Mexican project; the algebraic method is described also, so as to show the syntactic mastery requirements for its application.

THE PARTY PROBLEM (a simple case):

420 people attended a cocktail party; the number of men was twice the number of women. How many women and how many men went to the party?

Algebraic Method:

If

x= # of women

y= # of men

Then y = 2x

And x + y = 420

After solving this system of equations, it is found that: x = 140 and y = 280, which is the solution to the problem.

Spreadsheet Method:

Identify the unknown quantity (or quantities) as well as the problem data. Suppose that, that which is unknown, is known, and allocate an arbitrary value to one of the unknown quantities, for example the number of women. This number is then introduced into one of the cells. In the neighbouring cells, introduce the corresponding formulas for the number of men and the total amount of people who attended, as shown in the following diagram:

	A	B	C	D
1	Nº of women	Nº of men	Total Nº of people	
2	100	200	300	
3				

Let's suppose that 100 women

=A2*2

=A2+B2

Fig. 5

Note that these formulae shall include the name of the cell of one of the unknown quantities.

The presupposed value is then changed until the number in the cell relating to the total number of people corresponds to the problem data (420). The following diagram shows the moment in which this value is obtained and, as a result, the correct values for unknown quantities.

	A	B	C	D
1	Nº of women	Nº of men	Total Nº of people	
2	140	280	420	
3				
4				

Fig. 6

The spreadsheet method can help in the analysis of the problem's text by recording the steps of this analysis in a system of representations, in which natural language (column labels) is used along with numerical language and an algebra-like symbolic language. Consequently, the analysis process, which consists of clearly stating the relationship between elements of the problem (data and unknown quantities), uses all of these languages:

- Natural language allows the presence of referents which provide the context of the problem;
- Formulas allow relationships between data and unknown quantities to be expressed and, more importantly, allow functional relationships between unknown quantities to be expressed;
- The supposition of a specific value for one of the unknown quantities allows the analysis and symbolization process to be undertaken, through the use of a known number instead of an unknown quantity.

The numerical variation of the assumed value for one of the unknown quantities incorporates one of the intuitive methods most frequently used by students, the method of trial and refinement.

In the case of the PARTY PROBLEM, the spreadsheet method is illustrated using a simple case. However, this method can be used for the solution of problems of different levels of complexity. For example, problems where the relationships between unknown quantities are more complex or problems which include a larger number of unknown quantities, as shown in the following example:

SWEETS PROBLEM:

500 sweets are to be shared between three groups of children. The second group receives 20 more sweets than the first group and the third group receives three times as many sweets as the second group. How many sweets does each group receive?

Algebraic method:

If

x= # of sweets for the first group

y= # of sweets for the second group

z= # of sweets for the third group

then

$y=x+20$

$z=3y$

$x+y+z=500$

Solution:

x=84, y=104, z=312

Vary the number in cell A2 until 500 (specified in the problem) appears in cell D2

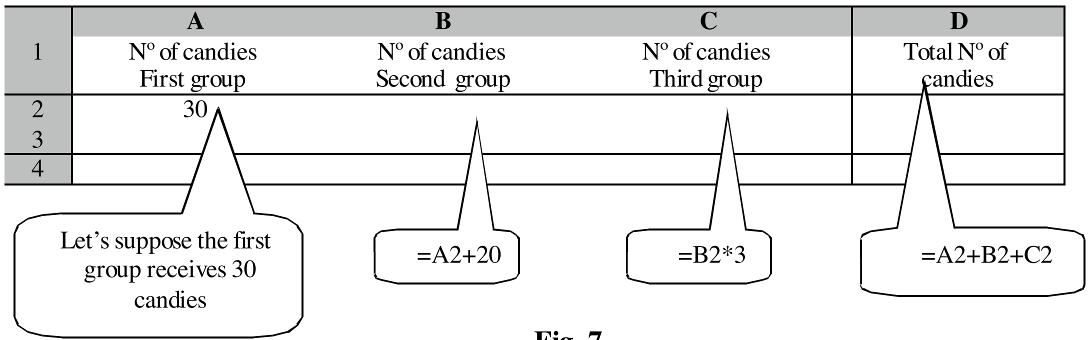

Fig. 7

Although the majority of the results reported in this project suggest that the *spreadsheets* method may make the tasks of solving problems that have more than one unknown accessible for very young students because it is not paradigmatic, to solve said method is nothing other than a didactic device used to place students on the road towards the *Cartesian Method*. And this situation raises questions, such as at what point can the two methods come together? Does the presence of the referents, which is very useful during the translation to spreadsheet symbolism phase, end by being an obstacle in developing the syntactic skills needed to solve the corresponding equation system? Or does the dependence upon a numerical means of solving the problem anchor students to the field of what N. Balacheff calls symbolic arithmetic? (Balacheff, 2001).

The foregoing questions have been broached by resorting to an analysis scheme proposed by (Puig & Cerdán 1990) in order to classify word problems as either arithmetic or algebraic, depending on whether the translation process leads to a chain of operations that only involves data or if said process leads to a chain of operations involving an unknown quantity. In other words, it depends on whether or not the process leads to an equation . The authors use two general methods as their analysis tools: *the method of analysis and synthesis* and the *Cartesian method*. By applying this scheme, the results of the Anglo-Mexican Project suggest that translations from natural language into the spreadsheets code, as undertaken by the students, has several characteristics in common with the translation into the algebraic code used in the *Cartesian method* (Rojano, 2002; Rojano & Sutherland, 2001). Nonetheless, regardless of whether these processes are of an algebraic nature or if they fall into the field of *symbolic arithmetic*, some researchers still consider them valid provided a full breaking away from arithmetic is not shown (Balacheff, 2001). As such the issue of things "algebraic" in children's solution processes using spreadsheets can now be touched upon from the standpoint of *LTMs*, and particularly from the point of view of the components of teaching models and cognitive processes. At the end of the day, this is a matter of analyzing both the inter-relationships between the variety of MSS' that come into play in both solution methods (Cartesian and Spreadsheets), as well as the cognitive tendencies that arise during their use.

The components of LTMs and Disciplinary Boundaries

With this walk through the history of the development of a research agenda on acquiring algebraic language, including one of its derivations to the application of the use of technology learning environments, the idea had been to show a particular manner wherein there is a point at which bordering disciplines meet face to face with mathematics education. The framework of that convergence is of a theoretical nature, in which one of the traits, that of local development (ex profeso for a series of phenomena that is presented in the transition towards algebraic thought), often puts at the limit the possibilities of such disciplines to advance our knowledge as regards the different aspects of the phenomena studied. In the example described here, the linguistics, semiotics and mathematical inputs used to prepare some of the LTM components were not incorporated in their raw state; helping hands were taken from the work of specialists who had broached the none too trivial task of analyzing the algebraic language from the standpoint of those particular disciplines. In this case, the boundary lines between said disciplinary fields and mathematical education are covered by that particular specialized work. Likewise, elaborations underway, such as that of the formal competencies model, from the point of view of the pragmatics of language, are based on prior work on the history of algebra that analyze the uses of algebraic language versions in ancient texts which correspond to stages prior to the emergence of symbolic algebra in the 16^{th} C. Hence the *LTM* of the *second cut* may be considered a point at which linguistics, semiotics, pragmatics, history and algebra actually meet.

Algebra as a Language: Connecting Communities

The disciplinary confluence of Local Theoretical Models should not be seen as an exercise of extreme eclecticism or as puzzle that needs to be put together. The theoretical axis of consistency is represented by the notion of a Mathematical Sign System, which makes it possible to maintain the formal, historical, "in use" and school versions of mathematics (algebra, in our case) at the core of the research work. Whereas the idea of using this underlying notion arises from the view of treating algebra like a language, this view is shared by many researchers who have expressed their analysis, thoughts and perspectives, not only through formal scientific publications, but also in the close interaction possible in international symposia and seminars (see for example: Arzarello,1993; Bell,1996; Drouhard,1992; Filloy,1990; Kaput,1993; Kirshner,1989,1990,2001; Lins 2001 Mason,1989; Pimm,1995; Radford,2000). In particular, the Algebra Working Group held substantial and heated discussions for several years on the issue within the framework of the PME meetings (international and North American). Furthermore the seminar on semiotics and algebra held on Fridays at the Department of Mathematics Education at Cinvestav-Mexico, has incurred in new variants with the arrival of young researchers and their joining ongoing study of the issue. All of this prolonged and intensive interaction amongst algebra scholars has represented another important factor in enabling the theoretical conception expressed here to be a meeting point of neighboring fields of knowledge that extends well beyond a mere eclectic use of their theoretical elements.

Final Remarks: Theory and Practice

Focusing in on theoretical analysis in the *second cut point* also reveals another type of connection: the link between theory and practice. Detailed development of the *formal competencies* model component, as well as knowledge of cognitive tendencies that flourish in the form of *reading levels* or the interpretation of linear equations in different strata of algebra's MSS, all led to the didactic route used in the clinical interview for the respective study. On the one hand said route anticipates the difficulties faced by students in manipulating unknowns in the *second cut*, in view of the structural syntactic characteristics of linear systems and of developing the sense required to undertake their transformations; these two conditioning factors are revealed by the syntactic and semantic components of the formal competencies model. While on the other, this trajectory suggests linear system modalities which, if presented gradually, place students in the position of needing to advance conceptually in algebraic *equalization* and *substitution*. The foregoing demonstrates how theoretical work on the formal and cognitive components of LTMs becomes one of the basic inputs (directly usable) for formulating a *teaching model*, which is actually located in the field of practice.

The task of bridging the gap between theory and practice shall only be complete when the figure and role of teachers is incorporated into theoretical treatises. As regards the case at hand, that is to say LTMs for the *second cut*, the task will have been dealt with when a local communication model has been developed, a model that encompasses the text formation and decodification processes that are triggered during teacher-student interactions. That is precisely when the elements will be available for the teacher -just as Balacheff indicates- to play a role in enabling students to recognize the *cut point*, an experience that simply cannot be done away with. In other words, the teacher will be able to play the role of master of the rupture (Balacheff, 2001) in the algebra lesson.

Endotes

[1] *Adquisición del Lenguaje Algebraico (Acquiring Algebraic Language)* is a research program, conducted by Eugenio Filloy and Teresa Rojano at the Centre of Research and Advanced Studies (Cinvestav) in Mexico, since 1980, which intends to probe the learning processes of algebra, when the latter is consider as a language.

[2] The "non-operation on the unknown", which might seem to be related to the presymbolic character of ancient algebra texts (belonging to pre-Vietan stage) led to the formulation, in the field of mathematics education, of conjectures as to the presence of "didactic cuts" in the processes of transition from arithmetic to algebraic thought. *Operación de la Incógnita (Operating the Unknown)* is the clinical study that aimed to confirm such conjectures at the ontological level. Eugenio Filloy and Teresa Rojano undertook this study in the early 80´s at Cinvestav – Mexico.

[3] Project funded by the National Program for Mathematics Teachers Training, The National Council for Science and Technology (Grant No. 139-S9201) in Mexico, and The British Council. Part of this work derives from the ESRC funded project "The gap between arithmetic and algebraic thinking" (Grant No. R000232132) in England.

References

Arzarello F.(1993). Analysing Algebraic Thinking. In R. Sutherland (Ed). Algebraic Processes and the Role of Symbolism *Proceedings of the ESRC Working Conference*. London, UK: Institute of Education London University.

Balacheff, N. (2001). Symbolic Arithmetic vs Algebra the Core of a Didactical Dilemma. Postcript. In Sutherland, R.; Rojano, T.; Lins, R.; & Bell, A. (Eds.) *Perspectives on School Algebra* (pp. 249-260). Dordretch, The Netherlands: Kluwer Academic Publishers.

Bell, A.(1996). Algebraic Thought and the Role of a Manipulable Symbolic Language. In Bednarz, N.; Kieran,C.& Lee,L. (Eds.) *Approaches to Algebra* (pp.115-136). Netherlands: Kluwer Publischers.

Bolaer, J. (2002). Exploring the Nature of Mathematical Activity: Using Theory, Research and "Working Hypotheses" to Broaden Conceptions of Mathematics Knowing. *Educational Studies in Mathematics* 51(1-2), 3-21.

Boncompagni, B. (Ed.) (1857). *Scritti di Leonardo Pisano matematico del secolo decimoterzo. I. Il Liber Abbaci di Leonardo Pisano*. Roma: Tipografia delle Scienze Matematiche e Fisiche.

Dörfler, W. (2003). Mathematics and Mathematics Education: Content and People, Relation and Difference. *Educational Studies in Mathematics* 54(2-3), 147-170.

Drouhard, J. P. (1992). *Les Ecritures Symboliques de l'Algèbre Elémentaire*. Unpublished

doctoral disertation, Université Denis Diderot, París 7, France.
Filloy, E. (1990). PME Algebra Research. A Working Perspective. In Booker G. et al (Eds.) *Proceedings of the Fourteenth Annual Conference for the Psycology of Mathematics Education,* Vol. 1 (pp.PII1-PII33) Morelos, México.
Filloy, E. (1991). Cognitive Tendencies and Abstraction Processes in Algebra Learning. In F. Furinghetti (Ed.) *Proceedings of the Fifteenth Annual Conference for the Psychology of Mathematics Education*, *Vol. 2* (pp. 48-55). Assisi, Italy: Università di Genova.
Filloy, E. (1999). *Aspectos teóricos del álgebra educativa*. México: Grupo Editorial Iberoamericana.
Filloy, E. & Rojano, T. (1984). From an arithmetical to an algebraic thought (a clinical study with 12-13 year olds). *Proceedings of the Sixth Meeting for the Psychology of Mathematics Education-North American Chapter.* (pp.51-56) Wisconsin, USA: Wisconsin University.
Filloy, E. & Rojano, T. (1985a). Obtructions to the acquisition of elementary algebraic concepts and teaching strategies. *Proceeding of the Psychology of Mathematics Education, International Group,* (pp.154-158) Utrech, Holanda.
Filloy, E. & Rojano, T. (1985b). Operating the unknown and models of teaching (a clinical study with 12-13 year olds with a high proficiency in pre-algebra). *Proceedings of the seventh Annual Meeting for the Psychology of Mathematics Education - North American Chapter*.(pp. 75-79).Ohio, USA: Ohio State University.
Filloy, E. & Rojano, T. (1989). Solving Equations: The trandition from Arithetic to Algebra. *For the Learning of Mathematics,* 9 (2), 12-25.
Filloy, E. & Rojano, T. (1991). Translating from natural language to the mathematical system of algebraic signs and viceversa: a clinic study with children in the pre.algebraic stage. *Proceedings of the Psychology of Mathematics Education, North American Chapter,* (pp.29-35). Virginia Tech, USA.
Filloy, E., Rojano, T. & Puig, L. (in process). *Theoretical Aspects in School Algebra*.
Filloy, E., Rojano, T. & Solares, A. (2003). Two Meanings of the Equal Sign and Senses of Substitution and Comparison Methods. In N. A. Pateman, B. Dogherty & J. Zilliox (Eds.) *Proceedings of the Twenty-seventh Annual Conference for the Psychology of Mathematics Education*, *Vol. 4* (pp. 223-229). Honolulu, Hawaii, USA: CRDG, College of Education, University of Hawai'i.
Filloy, E.; Rojano, T. & Solares, A. (in press). Arithmetic/Algebraic Problem-Solving and the Representation of Two Unknown Quantities. *Proceedings of the Twenty-eighth Annual Conference for the Psychology of Mathematics Education*, Bergen, Norway.
Filloy, E. & Rubio, G. (1993). Didactic Models, Cognition and Competence in the Solution of Arithmetic & Algebra Word Problems. In H. Ichiei, S. Keiichi and L. Fou-Lai (Eds.) *Proceedings of the Seventeenth Annual Conference for the Psychology of Mathematics Education, Vol. 1* (pp. 154-161). Tsukuba, Ibakari, Japan: University of Tsukuba.
Freudenthal, H. (1973). *Mathematics as an educational task*. Dordretch, The Netherlands: Kluwer Academic Publishers.
Goldin, G. (2003). Developing Complex Understandings: On the Relation of Mathematics Education Research to Mathematics. *Educational Studies in Mathematics* 54(2-3), 171-202.
Hughes, B. (Ed.) (1981). *Jordanus de Nemore. De numeris datis.* Berkeley, CA: University of California Press.
Kaput,J. (1993). Multiple interpreted activity structures. In R. Sutherland (Ed). Algebraic Processes and the Role of Symbolism *Proceedings of the ESRC Working Conference*.

London, UK: Institute of Education London University.
Kieran, C. (1980). The interpretation of the equal sign: Symbol for an equivalence relation vs. an operator symbol. In R. Karplus (Ed.), *Proceedings of the Fourth Conference of the International Group for the Psychology of Mathematics Education* (pp. 163-169). Berkeley, California, USA: University of California.
Kieran, C. & Filloy, E. (1989). El aprendizaje del álgebra escolar desde una perspectiva psicológica. *Enseñanza de las Ciencias*, Vol. 7, 229-240.
Kirshner, D. (1987) *The grammar of symbolic elementary algebra*. Unpublished doctoral dissertation, University of British Columbia, Vancouver, Canada.
Kirshner, D. (1989). The Visual Syntaxis of Algebra. *Journal for Research in Mathematics Education* 20(3), 276-287.
Kirshner, D. (1990).Acquisition of algebraic grammar. In Booker G., Cobb P. & Kieran C.(Eds.), *Proceedings of the Fourteenth Annual Conference for the Psycology of Mathematics Education,* Morelos, México.
Kirshner, D. (2001). The Structural Algebra Option Revisited. In Sutherland, R.; Rojano, T.; Lins, R.; & Bell, A. (Eds.) *Perspectives on School Algebra* (pp.83-98). Dordretch, The Netherlands: Kluwer Academic Publishers.
Lerman, S., Xu, G. & Tsatsaroni, A. (2002). Devveloping Theories of Mathematics Education Research: The ESM Story. *Educational Studies in Mathematics* 51(1-2), 23-40.
Lins, R.(2001) The Production of Meaning for *Algebra*: a perspective Based on a Theoretical Model of Semantic Fields. In Sutherland, R.; Rojano, T.; Lins, R.; & Bell, A. (Eds.) *Perspectives on School Algebra* (pp. 37-60). Dordretch, The Netherlands: Kluwer Academic Publishers.
Mason, J. (1989). Mathematical Abstraction as the Result of a Delicate Shift of Attention. *For the Learning of Mathematics,*9(2), 2-8.
Paradís, J. & Malet, A. (1989). *La génesis del álgebra simbólica. Vol. 1: Los orígenes del álgebra: de los árabes al Renacimiento*. Barcelona: PPU.
Pimm, D. (1995). *Symbols and meanings in school mathematics.* London: Routledge.
Puig, L. 1994. El De numeris datis de Jordanus Nemorarius como sistema matemático de signos. *Mathesis, 10*, 47-92.
Puig, L. & Cerdán, F. (1990). Acerca del Carácter Aritmético o Algebraico de los problemas verbales. *Memorias del II Simposio Internacional sobre Investigación en Educación Matemática* (pp. 35-48), Cuernavaca, Morelos, México: PNFAPM-Universidad de Morelos.
Radford, L. (2000). Signs and Meanings in Student´s Emergent Algebraic Thinking: A Semiotic Analysis, *Educational Studies in Mathematics,* 42(237-268).
Radford, L. (2001) The Historical Origins of Algebraic Thinking. In Sutherland, R.; Rojano, T.; Lins, R.; & Bell, A. (Eds.) *Perspectives on School Algebra* (pp.13-36). Dordretch, The Netherlands: Kluwer Academic Publishers.
Rojano, T. (1986) Learning and Usage ef Algebraic Syntax, its Semantic Aspects: case studies with 12-13 years olds. *Proceedings of the Psychology of Mathematics Education – North American Chapter* (pp.121-126).USA: Michigan State University.
Rojano, T. (1988) Teaching equation solving: A teaching experiment with a concrete-syntactic approach. *Proceedings of the Psychology of Mathematics Education - North American Chapter*. (pp. 150-156). USA: University of Northern Illinois.
Rojano, T. (2002). The potential of Spreadsheets in the Learning of Algebra. *The International Journal of Educational Policy Research and Practice*, (2), 91-106.

Rojano, T. & Sutherland, R. (1991) Symbolizing and solving algebra word problems: the potential of a spreadsheet environment. *Proceedings of the Fifteenth Annual Conference for the Psychology of Mathematics Education*, *Vol. 2* (pp. 207-213). Assisi, Italy: Università di Genova.

Rojano, T. & Sutherland, R. (2001) Arithmetic World Algebra World. In *"The Future of the Teaching and Learning of Algebra" Chick, H.; Stacey, K; Vincent, J. & Vincent J.* (Eds.), *Proceedings of the 12th ICMI Study Conference* (pp.515-522). Melbourne, Australia: The University of Melbourne.

Sierpinska, A. & Kilpatrick, J.(Eds.).(1998). *Mathematics Education as a Research Domain: A Search for Identity*, Great Britain: Kluwer Academic Publishers.

Solares, A. (2001). School algebra as a language: the firts elements of algebraic syntax and semantics. In Speiser, R. et al (eds.) *Proceedings of the Twentytird Annual Conference for the Psychology of Mathematics Education-North American Chapter, Vol. 1* (pp. 135-136). Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Solares, A. (2002a) Algebraic Substitution. Syntax and Significance Extension. In Mewborn, D. et al (eds.) *Proceedings of the Twenty-fourth Annual Conference for the Psychology of Mathematics Education- North American Chapter, Vol. 1* (pp. 417-424). Columbus, OH: ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Solares, A. (2002b). *La sustitución algebraica. Extensión de significados y transformaciones*. Unpublished pre-doctoral dissertation, CINVESTAV, México.

Solares, A. (2003). *La sustitución algebraica. Extensión de significados y transformaciones: entrevistas clínicas*. Unpublished research report I, CINVESTAV, México.

Solares, A. (2004). *La sustitución algebraica. Extensión de significados y transformaciones: análisis de entrevistas clínicas*. Unpublished research report II, CINVESTAV, México.

Stevin, S. (1634). *Les ouvres mathematiques de Simón Stevin*. Leyden: Girard.

Sutherland, R. & Rojano, T. (1993). A Spreadsheet Approach to Solving Algebra Problems. *The Journal of Mathematical Behavior*, 12, 4, p. 353-383.

Witmer, T. R. (Ed.) (1983). François Viète. *The Analytic Art*. Kent, OH: The Kent State University Press.

Working Groups

COMPLEXITY OF LEARNING TO REASON PROBABILISTICALLY

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Nature and Topic of the Working Session

This Working Group was formed at PME-NA 20 (Maher, Speiser, Friel, & Konold, 1998) and has convened annually at PME-NA each of the past six years (see Maher & Speiser, 1999; 2001; 2002; Speiser, 2000; Stohl & Tarr, 2003). During the joint meeting of PME-NA 25 and PME 27 in 2003 (Hawaii, USA), we expanded our working group to include many more international researchers across 11 different countries. Through shared research, rich and engaging conversations, and analysis of instructional tasks, we continually seek to understand how students learn to reason probabilistically.

Aims of the Working Session

There are several critical aims that guide our work together. In particular, we are examining: (1) mathematical and psychological underpinnings that foster or hinder students' probabilistic reasoning, (2) the influence of experiments and simulations in the building of ideas by learners, particularly with emerging technology tools, (3) learners' interactions with and reasoning about data-based tasks, representations, models, socially situated arguments and generalizations, (4) the development of reasoning across grades, with learners of different cultures, ages, and social backgrounds, and (5) the interplay of statistical and probabilistic reasoning and the complex role of key concepts such as sample spaces and data distributions. Through our work, we have stimulated collaborations across universities and plan to engage in and support additional research related to the complexity of learning to reason probabilistically. Future research will seek to include the development of statistical notions that promote robust stochastical understanding among students and teachers.

Planned Activities

At PME 27, members of the Working Group decided to create a listserv to promote follow-up contact and subsequent discussion through the use of e-mail. All participants attending the Working Group sessions were promised a copy of the Probability Explorer (Stohl, 1999-2002) software for research and instructional use with both students and teachers; participants were given access to the program through its web site. Through this listserv, we suggested several tasks that can be used with a variety of students (including middle school, high school, preservice and inservice teachers) in a variety of contexts (one-on-one [interview], whole class, and small group). It was agreed that several participants use a single task with students and collect data in an agreed upon format (e.g., video, audio). One problem task, Schoolopoly, which participants have found promising is described below. This task was originally part of a research study conducted by Stohl and Tarr (2002).

The Problem Task

Suppose your school is planning to create a board game modeled on the classic game of Monopolytm. The game is to be called Schoolopoly and, like Monopolytm, will be played with dice. Because many copies of the game will be sold, several companies are competing for the contract to supply dice for Schoolopoly. Several companies, however, have been accused of making poor quality dice. These companies are to be avoided since players of Schoolopoly need to know that the dice they're using are actually "fair." Each dice company has provided a sample die for analysis.

Each dice company has provided a sample die for analysis. You will be assigned one company to investigate. The company you're assigned may or may not have produced a fair die. Probability Explorer has been set up to model simulated rolls of their dice.

Working with a partner, investigate whether the die sent to you by the company is fair. Collect data to infer whether all six outcomes are equally likely or not, and answer the following questions:

1. Would you recommend that dice be purchased from the company you investigated?
2. What compelling evidence do you have that the die you tested is fair or unfair?
3. Use your data to estimate the probability of each outcome of the die you tested.
4. Estimate the weights associated with each event, 1-6.

Several participants are planning to pose the Schoolopoly task (or derivation of it) to students and teachers in a variety of contexts and with a variety of available tools (e.g., software, loaded die, etc). Data will be collected in the form of videos and paper-and-pencil work. This data, collected across contexts, cultures, and ages, will serve as a common data set for our continued work at PME 28 (Norway) and PME-NA 26 (Toronto). In particular, during our sessions, we plan to collaboratively analyze videotape data of students' probabilistic reasoning on a technology-based task by using several different theoretical perspectives. From this analysis, we seek to generate additional authentic tasks that are appropriate to elicit and extend students' probabilistic reasoning into a broader perspective that includes statistical reasoning. Members of the Working Group may use these tasks in future research.

We are planning to maintain this working session group in both organizations so that international collaborations can continue. Several participants will be attending both PME 28 and PME-NA 26 in order to allow for consistency and communication across groups. It is hoped that our analysis of students' work on this task will lead to a set of papers that describe our work. These papers could be part of a monograph, journal special issue, and many joint presentations at future conferences. In the weeks immediately following PME 27, several members of the Working Group worked collaboratively and submitted a proposal to speak at the Research Presession of the Annual Meeting of the National Council of Teachers of Mathematics. The proposal was accepted and a series of papers are to be presented at the Research Presession of the 82nd Annual Meeting held in Philadelphia, Pennsylvania, USA in April, 2004.

GENDER AND MATHEMATICS: MOVING TOWARD NEW SPACES

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Introduction

This year at the 2004 PME-NA XXVI sessions in Toronto, the Gender and Mathematics Working Group members regroup to begin moving our work into new spaces. In these sessions we explore ways in which we can more deeply examine the relationship between gender and mathematics in our work, and do so with reflection upon international perspectives, connected work in gender and technology, and critical perspectives on pervasive, recurring questions about the place for gender work in mathematics education. In this paper, we begin with a brief history of our work, mention the issues in PME-NA that are a connection to our work, and describe further the ways in which our work is a continuation of previous Gender and Mathematics Working Group and where that work is intended to lead us.

History of the PME-NA Gender and Mathematics Working Group

Members of our Gender and Mathematics Working Group have maintained a continuous commitment to scholarship fed by our involvement in the group since 1998. We initiated our discussions with a review of the scholarship around gender and mathematics, defining research strands, needs and absences, and committing to the integration of our work. We also defined future directions. Eventually, we generated a visual representation of the field as we conceptualized it, where we acknowledged and represented the complex dimensions of the elements with which we were trying to work (Damarin & Erchick, 1999; Erchick, Condron & Appelbaum, 2000).

In the years following our first session we met at each PME-NA meeting, where we shared papers, redefined immediate and long-range goals, solicited feedback from the more open PME-NA audience through discussion groups, and focused and refocused our work. We formed peer groups, critiqued participants' developing papers and discussed the papers as a whole body of work. Some members of the Gender and Mathematics group elected to conduct research exploring practice; others chose to theorize or to apply research. What all members had in common was a commitment to research that was, as suggested by Fennema and Hart (1994), feminist and qualitative in nature.

From the early days of our work together, a guiding agenda for our working group has been the development of a monograph on gender and mathematics. At working group sessions at the annual PME-NA meetings we organized themes and developed a structure for our work. Through the Gender and Mathematics Working Group monograph project, begun by pursuing inquiry around the absences in the research on gender and mathematics, the working group participants have committed themselves to an interpretation of the field of gender and mathematics as complex and nonlinear. We have chosen to investigate the absences we encounter with a respect for the reflective voices of the researchers, teachers, students, women and girls who contribute to the work. In the papers and processes of this project, we worked consistently to respect the structure and voices that emerge.

We organized around multiple perspectives that included those of the researcher, teacher, and student; history, critical theory, and feminism; and methodological, self-reflective and empirical standpoints. And we have pulled together the work of 8 scholars for our monograph project, a project that is near completion at the time of the writing of this proceedings paper. We are prepared to move forward on a next agenda, and developing that agenda, as discussed below, will be a primary goal of this years' PME-NA Gender and Mathematics Working Group sessions.

Issues in the Psychology of Mathematics Education to be a Focus of Our Work

Researching around topics that would allow us to impact classroom practice has been a goal of the Gender and Mathematics Working Group since its inception. Inherent on that goal is direct connection to the PME goals to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof; and to promote and stimulate interdisciplinary research, with the cooperation of psychologists, mathematicians, and mathematics teachers. We also find that with technology an increasingly present and important component of the mathematics classroom, we wish to pursue the inclusion of technology-related issue in our work. Learning from research on gender and technology is expected to contribute to our work in new and exciting ways.

A third goal of PME, to promote international contacts and the exchange of scientific information in the psychology of mathematics education, is one to which we are committed in terms of further and broadening growth and exchange of ideas. As we redefine new directions for the Gender and Mathematics Working Group, we are committed to an increased inclusion of international perspectives on gender. Thus, our plan for this year's session and our agenda for this coming year together address that goal. We expect the development of this component will allow us to develop collegial connections and integrate diverse perspectives into our research agendas.

Our work is further connected to the PMENA XXVI conference theme of "Building Connections Between Communities." Both our overarching goal of bringing research on gender and mathematics into the classroom and having that research emerge out of the practice of teaching are inherent elements of the concept of making connection between communities. Additionally, the Gender and Mathematics Working Group commitment to broaden the scope of our perspectives to include international experiences also enriches the concept of connection between communities.

Plan for Active Engagement of Participants

As has been true each time that we have met, the Gender and Mathematics Working Group remains committed to an initiative that depends upon participant voices for direction and support. As always, the sessions we conduct this year are intended to be active with discussion, decision making, and work activities. In this years' sessions, we begin with introductions and a short synthesis of the work to date, as well as updates on current projects.

One of two major components of the sessions is a series of three panel discussions on areas of research we feel we need to examine more fully in order to inform our scholarly directions. The content of these panels is discussed below. Following each panel will be a short discussion, with the intention of capturing salient points for the second major component of the sessions. That second component is comprised of work sessions centered on each of the panel topics, again explained below.

Panel I: How Research on Globalization and on Gender from International Perspectives

can Inform Work on Gender and Mathematics

This panel, led by Joanne Rossi Becker from San José State University, addresses gender from an international perspective, with the panelists each presenting a paper to introduce the topic. Becker reports on the sessions on gender held at the ICME 10 conference in Copenhagen in July 2004. Papers reviewed cover the areas of learning of mathematics, attitudes, and participation, and aim to take into account the way cultural, economic and other background factors influence the formation of female and male differences in these variables. Becker provides a global synthesis of research on gender and mathematics and provides a framework that might enable us to assess the current status of research methodologies currently in use. This framework is intended to lead to discussion of whether some new perspectives are needed to make further progress in our investigations of issues of gender and mathematics from a global perspective.

Binaya Subedi, from The Ohio State University at Newark, examines the state of global inequalities and how gender, race and class shape the unequal distribution of wealth. Subedi analyzes data from the UN and World Bank and its implications for contemporary globalization, emphasizing the complex ways in which inequalities impede the mathematics education of girls in rural Nepal. What we can learn from such studies is intended to inform gender and mathematics work of the participants of the gender and mathematics working group.

Olof Bjorg Steinthorsdottir, out of the University of North Carolina-Chapel Hill, focuses on gender and pedagogy in international contexts. She discusses the value of Culturally Relevant Pedagogy as a possible framework for examining gender and mathematics teaching in different countries. Culturally Relevant Pedagogy employs the argument that students' cultures and backgrounds are important when thinking about successful teaching. Gender is culturally constructed and therefore, for girls to be successful in mathematics, it is important to look a the role of culture in the practice of teaching. Considering cultural differences across nationalities and other international boundaries is an important step in understanding gender in mathematics, and would serve well the scholarship of the Gender and Mathematics Working Group.

Questions grounding the panel include:

- What perspectives are used to investigate gender and mathematics in different countries?
- How would new perspectives allow us to un/re/think gender as it pertains to the teaching and learning of mathematics?
- What new methodologies would enable us to investigate difficult and unresolved issues concerning gender and mathematics?
- Given the range of cultures that exist internationally, can culturally relevant pedagogy, as developed in the US, help us to identify "good" and equitable teaching across national boundaries?
- What is "good" teaching in gender-equitable classrooms?
- What is the significance of considering social change in the global context?

Panel II: Framing Issues on Gender, Mathematics and Computer Technologies

Led by Suzanne K. Damarin from The Ohio State University, this panel recognizes that the educational relationships between mathematics, computing and information technologies, and gender are rich and complex. Mathematics is central to computing, mathematics is "the language of computers," and indeed, it was through mathematics that computers entered the educational domain. Also, computers have been brought to bear on mathematics teaching and learning through a number of innovative efforts including *Logo*, geometry "environments" such as *Geometric Supposer* and *Cabri Geometry*, symbol manipulators popularized in mathematics classes as graphing calculators, calculus systems such as *Mathematica*, and various instructional software packages. While these innovations have influenced thinking about and development of mathematics education research and theory, their impact on everyday mathematics teaching has been less farreaching, limited in part by the demands for "basic proficiency" as a goal of schooling. None-the-less, computers and related technologies have found places in mathematics classrooms across the country and the world.

Simultaneously with the growth of computing in education, concern over the relationships between gender and mathematics has grown, and an extensive research literature has emerged. As computer-related technologies have found a place in schooling and other educational settings, gender equity has also emerged as a factor related to computing. One effect of the latter has been an effort on the part of computer-equity researchers to separate computers and their uses from mathematics in the hopes that this distancing would contribute to more acceptance of computing as an activity for young women who are disengaged from mathematics for one reason or another.

In 2004, more than a quarter century after computing technology was introduced into mathematics classrooms, researchers in the area of gender and mathematics find ourselves in a peculiar situation. Computing is clearly related to mathematics; relationships between gender and mathematics are increasingly well understood, though inequities and enigmas remain; gender has been shown to be related to computing in complex ways which typically favor males. Yet, although a few studies have been conducted, we have accumulated little insight, research, or understanding of the "three-way interaction" of gender, mathematics, and computer-related technologies in educational settings.

It is the purpose of this panel to begin a new initiative into understanding the interactions of gender, computing, and mathematics. Toward this end, panel members address the following questions (among others of their choosing):

- What *do* we know about the interactions of gender, math, and computing?
- What research and findings on gender and mathematics have implications for research questions regarding educational computing in mathematics contexts?
- What research and findings on gender and computing have implications for questions and research regarding gender and mathematics?
- What (if any) are the implications of the developments of "software for girls" and "websites for women" which are largely free of mathematics for women's and girls' experiencing of mathematics and technology?
- What are the implications of cyberfeminist and other postmodern approaches to the study of gender and computing for the study of mathematics in technology rich contexts?
- Are there contradictions implicit in this three way interaction? For example, does gender equity with respect to computing require acceptance of gender differences in mathematics and a separation of computing from mathematics

Panel III: Gender as a Scholarly "Problem" in Mathematics Education

This panel, led by Peter Appelbaum from Arcadia University, and including Abbe Herzig from SUNY at Albany and Diana B. Erchick from The Ohio State University at Newark, address the concept of gender as a scholarly problem in mathematics education. More specifically, this panel addresses a variety of questions related to gender and mathematics education, originating in a critical perspective on gender as a "problem" of the field in the first place. The first question for each panel member is how and why one might want to address gender and mathematics as an issue in mathematics education. Early work in this area posited gender as a problem for mathematics education; later work positioned mathematics as a problem for gender in education. More recently, scholars working in gender and mathematics education have noted various ways in which there is a "problem" of maintaining gender as a focus for research in light of seemingly more pressing "problems" and issues. Still others have noted how attention to gender as a category may itself contribute to a sense that gender is a problem to be solved; in this respect an important question is how research can attend to gender without contributing further to discourses that position girls (and others) as weaker and less capable. A response has been for gender scholarship to be a model of research well done, including and problematizing gender but also other categories of analysis such as class, race, sexual preference and orientation, ethnicity, and so on.

The scholars on this panel ask themselves: What is behind assumptions that the "problem" is solved? What do statistics really tell us about gender and mathematics? How does one's scholarly agenda focus on gender in environments that do not value such perspectives? How can one maintain a focus and value on gender while recognizing the complex connections to race, class, sexuality, language, age, nationality, and other categories?

Taking gender, mathematics, pedagogy, or other categories as "problems,"

- What exactly "is" the "problem"?
- What if the problem were solved? what would that "look like"? i.e., what indicators would we look for to establish that the problem is solved?
- Could it be that the problem could never be solved because the nature of this sort of "problem" has built into it that it could never be solved? What would that mean?
- Finally, if other research seems more urgent, what does this tell us about the values and priorities, et cetera, of the profession?
- If, in the end, we simply want to focus on "good pedagogy" for all students, how does research in gender and mathematics education contribute to our notions of "good pedagogy," and what would be the implications of such pedagogy for gender?

As mentioned earlier in this paper, following each panel is a short work time for the working group members to question and synthesize each panel's contributions, to capture salient points for further discussion. The completion of all three panels brings us to the second major component of the Gender and Mathematics Working Group sessions. At this point, the task of the working group members is to organize around the topics of the panels, to discussion, and contribute to the conversation, and to develop directions for research given the input of the panels. These suggestion are to be shared with the whole group. Subgroups are expected to study, plan for independent work for the coming year, and determine additional work sessions, both electronically and through other professional organization meetings such as IGPME, NCTM, and AERA.

Anticipated Follow-up Activities

As in the past, we will maintain electronic communications in the following year as we work on the projects decided upon in the sessions of PME-NA XXVI. Additionally, as discussed earlier, we intend to work collaboratively through other professional meetings, such as IGPME, NCTM and AERA. This work is in the early stages of developing a new project, overlapping the continuing monograph project of prior meetings.

Building Upon Previous Gender and Mathematics Working Group Initiatives

When Suzanne Damarin and Diana Erchick started this project in 1998, an early result of the working group sessions was a graphic, cited above, that revealed two conceptions determined by the scholars working within the group. One determination of the group was that the structure of our examination of the scholarly work of gender and mathematics was nonlinear and very complex. The other determination of the group was that there were absences in the field of study, and it would be part of our mission as a members of the working group to pursue scholarly inquiry in directions that would begin to contribute to the field in the areas of those absences. Our monograph project satisfies a part of that agenda. That project continues and is nearing completion.

Reflection upon that culminating project now reveals more absences, all of which are foundational is our decision on panel topics in this year's sessions.

Closing

In pursuing inquiry around Gender and Mathematics, the PME-NA Gender and Mathematics Working Group participants have committed themselves to an interpretation of the field of gender and mathematics as complex and nonlinear. We have also chosen to investigate the absences we encounter with a respect for the reflective voices of the researchers, teachers, students, women and girls who contribute to the work. In the papers and processes of this project, we work consistently to respect the structure and voices that emerge. Original absences apparent in 1998 have grounded our work since then. Newly emergent absences now ground our new directions, and our commitment to addressing absences in the field continues.

References

Damarin, S. K. & Erchick, D. B. (1999). Gender and Mathematics Working Group: Explorations of Absences in Research. In F. Hitt & M. Santos (Eds.), Proceedings of the Twenty First Meeting of the Psychology of Mathematics Education ñ NA XXI, 1, Cuernvaca, Morelos, Mexico, 170-175.

Erchick, D. B., Condron, L., & Appelbaum, P. (2000). Gender and Mathematics Working Group: Emergent Themes. In M. L. Fern·ndez (Ed.), Proceedings of the Twenty Second Meeting of the Psychology of Mathematics Education ñ NA XXII, Tucson, AZ,1, 53-57.

Fennema E. & Hart, L. E. (1994). Gender and the JRME. *Journal for Research in Mathematics Education,* 25,(6), 648-659.

GEOMETRY AND TECHNOLOGY WORKING GROUP

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Working Group History

The working group on Geometry and Technology has met since the PME-NA conference in 1998. The objectives were to explore the technology environment, student perspectives, and teacher perspectives. Since this initial year, we have discussed the student in an environment created and caused by micro worlds (PME-NA XXI), preservice teachers' understanding about the role of proof in mathematics and the impact of dynamic geometry software on their understanding of proof and proof-writing skills (PME-NA XXII), and the role of software in moving students from conjectures about drawings to the theoretical work of proof (PME-NA XXIII).

In 2001, the group leaders developed a format that would require participants to actively work on instruments and discuss potential student responses that would be meaningful to research. They presented two tasks from the Balanced Assessment Project and used *Rethinking Proof* (de Villiers, 1999) as a model for developing new performance tasks in research.

In 2002, the group focused on an underlying theme in de Villiers' work, the idea of making sense of geometry through verification, explanation, discovery, and systematization. A key reading for this working group session was "Sense Making: Changing the Game Played in the Typical Classroom" (Flewelling, 2002). Participants shared and created sense-making tasks for various levels of students and discussed the implications such tasks/activities have for research. The group also discussed how the tasks/activities might differ in design for preservice and inservice teachers. We plan to report further on these activities in the proceedings paper. Those present in 2002 voted to suspend meeting in 2003 and resume the working group in 2004.

Plans for PME-NA XXVI

In order to build on the work we have done in the past on sense-making activities and proof, the organizers now propose that the focus be on analysis of student work in a dynamic geometry environment—in particular in Geometer's Sketchpad (GSP). In preparation for these activities, past working group participants will be contacted via email and invited to participate in a pilot study designed to begin to answer the following research questions:

1. In what ways does a dynamic software environment facilitate the assessment of student understanding?

2. In what ways does this assessment differ from the ways such understanding can be assessed in a non-technology environment?

Each willing past participant will receive an array of GSP tasks and be asked to use them with students (at various school and university levels), collect their work, and bring this data in electronic form to the working group.

Working group sessions will be spent doing these activities:

- Discussing theoretical frameworks from the literature (Laborde, 2001; Laborde & Laborde, 1995; van Hiele, 1986) that can guide research on student understanding in a technology environment.
- Analyzing data using ideas from the theoretical discussions.
- Discussing implications of the preliminary findings of this analysis.
- Articulating and refining researchable questions on the use of technological environments to assess students' understanding of geometry concepts.

Anticipated Follow-up Activities

We view these activities as essentially making up a pilot study for researching the use of a technology environment to assess students' understanding of geometry concepts. Working group activities will end with proposals for conducting further research in these areas. Time will be given for small groups to discuss their plans for this collaborative research over the coming year. Leaders will be chosen for each of these groups. These leaders will be committed to organizing the next years' research and bringing research data and results to next years' working group.

References

de Villiers, M. (1999). *Rethinking proof*. Emeryville, CA: Key Curriculum Press.

Flewelling, G. (2002). Sense making: Changing the game played in the typical classroom. *Australian Mathematics Teacher Journal, 58*(1), 8-16.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning, 6*, 283-317.

Laborde, C., & Laborde, J. M. (1995). What about a learning environment where Euclidean concepts are manipulated with a mouse? In A. A. diSessa, C. Hoyles & R. Noss (Eds.), *Computers and exploratory learning* (pp. 241-262). Germany: Springer-Verlag.

van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Orlando: Academic Press.

KNOWLEDGE OF ALGEBRA FOR TEACHING

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Common wisdom suggests that students' knowledge is influenced by teachers' knowledge. For more than three decades, researchers have been trying to identify critical components of mathematics teachers' knowledge (see, for example, the reviews by Begle, 1972; Fennema & Franke, 1992; Ball, Lubienski & Mewborn, 2001). Shulman (1986) suggested that there are at least three components of knowledge for teaching: subject-matter content knowledge, curricular knowledge, and pedagogical content knowledge. Ma (1999) describes in detail the profound understanding of fundamental mathematics among some teachers of elementary school mathematics. The highly structured knowledge packages these teachers have seem to exhibit characteristics of all three types of knowledge Shulman hypothesized. Ball & Bass (2000a, 2000b) suggest that mathematical knowledge for teaching is different from the mathematical knowledge used by other specialists, just as the mathematical knowledge for engineering is different from the mathematical knowledge needed for chemistry.

Until recently most of the work about knowledge for teaching mathematics was confined to the lower grades. Recent work at the secondary level includes the development of a course about high school mathematics from an advanced standpoint (Usiskin, 2000; Usiskin, Peressini, Marchisotto & Stanley, 2002) and the development of a provisional framework for teaching algebra by Kahan, Cooper & Bathea (2003). The RAND Mathematics Study Panel (2003) also notes the need for clarification of the knowledge demands of teaching, and singles out algebra as a key area for further research.

The project, A Study of Algebra Knowledge for Teaching at the Secondary Level, was funded by the National Science Foundation to examine the mathematical knowledge needed for teaching algebra and to develop a framework to describe algebra knowledge for teaching. By “algebra”, we refer to both algebra as a strand in school mathematics and algebra as a mathematics course. To start to develop a framework, project personnel first reviewed the research on pedagogical content knowledge (e.g. Shulman, 1986, 1987; Wilson et al., 1987), knowledge for teaching mathematics at elementary school (e.g.,Ball, 1990; Ball & Bass, 2000a, 2000b; Ball & Hill, 2003), and the learning and teaching algebra (e.g. Fey, 1989; Usiskin, 1988; Wagner & Kieran, 1989; Kieran, 1992; Bednarz et al., 1996; Nathan and Koedinger, 2000), and on their own experiences teaching algebra. From this review a provisional framework was proposed for "algebra knowledge for teaching." The central components of that framework, which will be elaborated on in during the session, are as follows:

1. Substantive Knowledge of Mathematics
    a. decompressing (unpacking)
    b. trimming
2. Knowledge of the Nature and Practice of Mathematics
    a. Knowing conventions (in language, notation, etc.)
    b. Coordinating mathematical and everyday language
    c. Constructing and judging arguments
3. Knowledge of Connections
    a. Understanding intersections
    b. Translating and Coordinating representations

As the provisional framework was being developed other project personnel conducted several empirical studies, including interviews with teachers and curriculum developers, analyses of videotapes of algebra lessons, and analyses of high school algebra textbooks. To focus and unify the work of the research, all empirical studies examined the same topics in algebra. The topics chosen are central mathematically, appear across various conceptions of school algebra, pose difficulties for students to learn, raise questions that suggest teachers might need additional knowledge of students or algebra for teaching, and are the subject of some research. Two topics seem to fit these criteria: (1) variables, expressions, and equations and (2) linearity. The research team consists of members from departments of Teacher Education, Mathematics, Counseling and Psychology, Curriculum, and Measurement and Quantitative Analysis; the methodology and analyses reflect the diverse perspectives on approach and process. Analyses and syntheses of the various empirical studies are currently underway as is the compilation of a data-base of examples tied to the framework.

Issues that arose during the work

The study was designed to begin with an initial framework as described above. This initial framework was then informed by the empirical work of the researchers and modified and adjusted according to our findings. This approach created a tension for the researchers, making it difficult to focus the empirical work with a nebulous framework, and difficult to revisit the framework with data that was not always specifically tied to the framework. A second issue that arose was the lack of substantive work done in this area at the secondary level in contrast to that at the primary level and the tendency to think the same about secondary teacher knowledge as about elementary teacher knowledge, when in fact, the situations are clearly different with respect to the amount and depth of content knowledge teachers are expected to have and a difference in constructing the work of teaching at the two levels. A third issue was how to clearly traverse the path between research on interesting theoretical aspects that emerged and research clearly linked to helping inform the practice of those teaching teachers about the algebra knowledge needed for teaching. A fourth issue was finding language to describe the categories within the framework in a way that would be clear to those who might benefit from the study, even to the difference in stating the problem -algebra knowledge for teaching or knowledge of algebra for teaching. And a central issue is consideration of the proposed framework as a model, how it might contribute to the discussion about content knowledge, pedagogical content knowledge, and mathematical knowledge needed for teaching at the secondary level, and the role of the examples in this discussion.

Session Plan

This would be the first meeting of a discussion group on the topic, mathematical knowledge for teaching algebra. The session would consist of a brief overview of the project by the researchers, including its goal and a brief description of the methodology and some results of the four empirical studies: interviews with teachers, analysis of videotapes of algebra lessons, analyses of algebra textbooks, interviews with developers of those algebra textbooks. (45 minutes), after which the audience would divide into four groups, each of which would address one data collection aspect of the project (30 minutes). Each group will be led by one of the researchers who will provide additional examples from that study and lead a discussion of the algebra knowledge examples of the algebra knowledge needed for teaching involved in those examples. The session will conclude with a presentation to the whole group (20 minutes) by one of the principal investigators about the revised framework that was developed. The session would conclude with a whole group discussion of the issues identified by the researchers as well as those suggested by the participants in light of their discussion of the examples in their small groups (25 minutes).

Anticipated follow up activities include posting of papers related to the project on a project website along with notes from the session; a targeted on-line discussion related to the framework and the examples; an open forum for examples offered by participants with a discussion of how they fit into the framework, and a proposed session at the next PME building on the work of the project and the discussion that follows this session.

Implications for theory

Offering a framework to organize the mathematical knowledge needed for teaching for secondary teachers can help the community conceptualize avenues to think about what teachers should know about algebra for teaching and to design experiences into both pre-service and in-service teacher education programs to bring this knowledge to the forefront. Indeed, the provisional framework as written above suggests ways of thinking more broadly about all mathematical topics and in each case, the component of the framework can be interpreted specifically in terms of knowledge of algebra.

Implications for practice

The findings of the project can help us gain a deeper and better understanding of teaching and learning mathematics, in particular algebra, at the secondary level. We contend that the converging lines of work on pedagogical content knowledge and mathematical knowledge for teaching are rich and intriguing, with potential implications for more coordinated research, for improving the design of pre-service and professional development opportunities for teachers, and for communicating more effectively with mathematicians, mathematics teacher educators, and policy makers about the nuances of teachers' subject matter knowledge.

LEARNING AND TEACHING WITH PROOF: A "PROOF STORY" ACROSS THE GRADES: BEGINNING A CONVERSATION ON THE LEARNING OF PROOF IN GRADES K-16

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This working group explores the topic of proof as a common thread across the grades K-16. The group aims to bring together people with a focus on the learning and teaching of proof to provide a forum to begin to identify the common themes and researchable questions and to establish an agenda for future work. It aims to foster a collaboration between people with a focus on K-12 or college level mathematics. Specifically, participants in this group will focus on two issues that will form the basis of the subsequent work of the group: (a) identifying the current status of research on proof and generating specific goals for the group, and (b) generating a common line of research questions regarding the learning and teaching of proof that have the potential of involving intense collaborative work by groups of mathematics educators and practitioners at the K-16 levels.

"The concept of proof is one which not only pervades work in mathematics but is also involved in all situations where conclusions are to be reached and decisions to be made. Mathematics has a unique contribution to make in the development of this concept, and […] this concept may well serve to unify the mathematical experiences of the pupil"
Harold P. Fawcett (1938)

In the sixty-five years since the statement above was written, the assumptions about proof as *a logical argument that one makes to justify a claim and to convince oneself and others*, and its role in mathematics and mathematics education, have not changed. Mathematicians and mathematics educators agree on the importance of proof in mathematics and the necessity for students to develop both the understanding of concepts related to proof and the skills to read and write proofs. As such, there have been calls for the learning of proof to become a central goal of mathematics (AMATYC, 1995; MAA, 2000; NCTM, 2000; RAND Mathematics Study Panel, 2002; Royal Statistical Society, 1995). As such, it is crucial that we provide ways for students to think about proof throughout their education in mathematics, beginning in the elementary grades and extending through their post-secondary experience.

We see a larger need to detail a connected K-16 "story" of mathematical proof. The *Principles and Standards for School Mathematics* (NCTM, 2000) has emphasized that proof should be a part of all pre-college students' mathematical experiences in order to deepen and extend learning and to democratize access to these ideas to a broader population of students. Such a story would include understanding how the forms of proof, including the nature of argumentation and justification as well as what counts as proof, evolve chronologically and cognitively and how content and instruction can support this. Building a "proof story" across grades K-12 would inform how to integrate mathematical proof in instruction at all levels, thus building a habit of mind in students that would support their transition to higher mathematical thinking at the tertiary level.

An important step in this direction is to build a strong, research-based framework to characterize mathematical proof across different cognitive domains and to detail how content and instruction can support the integration of mathematical proof into students' experiences. This requires the building of a broad research effort on the learning, teaching and assessing of proof across grades. Currently, a number of researchers are focusing on proof at different age/grade levels, using a variety of methodological perspectives. The outcomes of these efforts are now beginning to appear in journals and in conferences such as PME (e.g., Blanton, Stylianou & David 2003; Harel & Sowder, 1998; Herbst, 2002; Knuth, et al., 2002; Raman, 2003; Weber, 2001; Yackel, 2001). But, while these studies are providing a significant contribution to our understanding of students' capacity for proof and some of the impediments to that understanding, they are often happening in isolation and do not capture the evolution of student learning over significant periods of time nor the interplay between different domains of learning. The ability to read and do proofs in mathematics is a complex one that depends on a wide expanse of beliefs, knowledge, and cognitive skills and that is uniquely shaped by the social realm in which learning occurs. It is not at all clear, however, which of these factors are the most salient for students nor how these factors interact with one another (Moore, 1994). Even less is known as to how students overcome difficulties throughout their study of mathematics and how college mathematics students' conception of mathematical proof progresses over a long period of time. Indeed, the development of one's capacity for proof is a long-term effort (hence, we characterize it here as a K-16 endeavor), and understanding this development will require us to coordinate efforts and perspectives..

This working group aims to bring together people with a focus on the learning and teaching of proof to provide a forum to begin to identify the common themes and researchable questions and to establish an agenda for future work. It aims to foster a collaboration between people with a focus on K-12 or college level mathematics. Specifically, participants in this group will focus on two issues that will form the basis of the subsequent work of the group:

(1) Identifying the current status of research on proof and generating specific goals for the group.
(2) Generating a common line of research questions regarding the learning and teaching of proof that have the potential of involving intense collaborative work by groups of mathematics educators and practitioners at the K-16 levels.

Each of the two issues is explained further below.

Group Goals and Future Directions

Several researchers (many of whom are active members of PME-NA) are currently involved in research on proof. Our goal is to bring this group together to share their visions and goals, as well as give short presentations on the status of their current work on proof. Subsequently, the group will be asked to discuss the ways in which these independent studies can inform each other and form a continuum of research.

Generating Research Questions

The main body of the existing studies on proof has focused on one side of the story, namely, the learning of proof. Researchers have sought to identify the characteristics of student-constructed proofs, including misconceptions and difficulties that students face at different levels of mathematics instruction. There have been few efforts to study the instruction of proof. The group will be asked to examine these current trends in the study of proof and to discuss ways in which we can expand our research foci to form an overall view of the topic of proof in mathematics education, and, at the same time to search for a common focus in our efforts.

General Direction

It is our goal to assist the members of the group to collaborate in order to generate and share research on the development of student learning of proof and the teaching of proof. It is important that we use the time in this working group session to organize and prioritize future work for the group. The session leaders will take the responsibility to summarize the discussions of the group and provide focused directions for the future. We have already secured funding from the NSF that will allow us to continue the work of the group throughout the year through an electronic discussion forum to form the basis for future meetings of the group (PME-NA 2005). For more information on this project please visit our website: www.theproofproject.org

References

American Mathematical Association of Two-Year Colleges (1995). Crossroads in mathematics: Standards for introductory college mathematics before calculus. Memphis, TN: AMATYC.

Blanton, M. , Stylianou, D. A. & David, M. M. (2003). The nature of scaffolding in undergraduate students' transition to mathematical proof. In the Proceedings of the 27th Annual Meeting for the International Group for the Psychology of Mathematics Education. (vol. 2, pp. 113-120), Honolulu, Hawaii.

Committee on the Undergraduate Program in Mathematics (2000). Curriculum Guide: Mathematical Association of America. http://www.maa.org/news/cupm_text.html.

Fawcett, H. (1938). *The nature of proof. Thirteenth yearbook of the National Council of Teachers of Mathematics.* New York: Bureau of Publications, Teachers College, Columbia University.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from an exploratory study. In A. H. Schoenfeld, J. Kaput & E. Dubinsky (Eds.) Research in College Mathematics Education III (pp. 234-283). Providence, RI: AMS.

Herbst, P. G. (2002). Enabling students to make connections while proving: The work of a teacher creating public memory. Proceedings of the 24th Annual Meeting for the Psychology of Mathematics Education - North America (Vol. 4, pp. 1681-1692). Athens, GA.

Knuth, E., Choppin, J., Slaughter, M., & Sutherland, J. (2002). Mapping the conceptual terrain of middle school students' competencies in justifying and proving. In Proceedings of the 24th Annual Meeting for the Psychology of Mathematics Education - North America (Vol. 4, pp. 1693-1700). Athens, GA.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics,* 27, 249-266.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics. Reston,* VA: NCTM.

Raman, M. (2003). Key Ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics,* 52, 3, 319-325.

RAND Mathematics Study Panel (2002). Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education. (http://www.rand.org/multi/achievementforall/math/).

Royal Statistical Society (1995). *Tackling the mathematics problem.* London Mathematical Society, Institute of Mathematics and its Applications. London: UK.

Weber, K. (2001) Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics,* 48 (1), 101-119.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In Proceedings of the 25th Annual Meeting of the International Group for the Psychology of

Mathematics Education (Vol. 1, pp. 9-24). Utrecht, The Netherlands: The Freudenthal Institute.

MATHEMATICS TEACHING ASSISTANT PREPARATION AND DEVELOPMENT RESEARCH

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Teaching assistants (TAs) play vital roles in the mathematics education of undergraduates and may go on to become professors of mathematics. From the K-12 literature, it is clear that patterns of teaching practice, as well as beliefs about teaching and learning, form early in teachers' careers. Yet, while there is a large body of research about K-12 teachers, researchers are only beginning to consider the development of TAs. This discussion group exists to foster collaboration between K-12 and undergraduate mathematics educators in framing and carrying out research into the nature of TA experiences and professional development. This meeting of the research group will be devoted to discussion of participants' research projects at various stages of development. Participants will provide feedback on research in the planning, data collections, data analysis, and reporting stages. These discussions will serve as the basis for the group's goals of building a community of researchers interested in TA issues, the analysis of similarities and differences with K-12 mathematics education, and the development of an agenda for future work.

History

The Mathematics Teaching Assistant Preparation and Development Research discussion group met during one previous PME-NA conference (in 2002). At that augural meeting, time was divided between two activities. First, participants shared backgrounds, involvement with, and interest in teaching assistant (TA) issues. This activity was designed to further one of the group's major goals of forming a community of scholars and assisting people in forging connections with colleagues in the field. During our second session, participants discussed issues and potential research directions in the organizers' conference proceedings paper. In addition to furthering community development by engaging in substantive discussion, this activity provided organizers with insight into the areas of most interest to participants. This discussion also served as the beginning effort toward identifying key research issues and beginning the process of forming a research agenda to which all group participants can contribute.

Since the last meeting, organizers have engaged in several activities building on what was begun in 2002. The organizers expanded on and published the conference proceedings paper (Speer, Gutmann, & Murphy (in press)). An email listserv of people interested in TA issues was established, including participants in the discussion group as well as others encountered over the years at TA-related events.

Issues in the psychology of mathematics education to be focus of the discussion group

Broadly speaking, the group's work concentrates on issues of teacher development and practices. More specifically, research centers on mathematics TAs and factors that shape their teaching and their learning to teach. The group's work has a broad focus in the psychology of mathematics education, from a variety of theoretical and methodological perspectives, reflecting the diversity of its participants. Rather than concentrating on a single issue or a particular perspective, this group exists to serve the needs of its members and to provide a forum for discussion and collaboration on research into TA preparation and development from the varied perspectives of the participants. One of the developing aims of the group, however, is to generate and pursue a coherent agenda for research that builds on existing research on TAs and connects to K-12 educational research.

Plan for engagement at 2004 meeting and how proposed work builds on prior group activities

Included below are five short descriptions of projects at different stages of development. Two projects are in the planning and theoretical conceptulization stage, two others are in progress with some data collection completed, and one is entering the publication stage. Working group discussion time will be divided in three parts for the dual purpose of considering how these specific projects may be advanced and for building on the ideas from the previous meeting about the development of an overall research agenda. During the first segment, projects summarized below will be presented in greater depth. In break-out sections, time will be devoted to discussions of data, assessment of projects as contributions to the field, and to consideration of how projects might be expanded upon or advanced. Finally, participants will outline important and attainable goals for a research agenda and discuss ways to encourage collaboration and to ensure multiple methodologies are employed so that research findings will be widely acceptable.

Eric Hsu (San Francisco State University): Planning stage

I'm planning to do a study of how TAs form communities of practice and, in particular, how online communications mediate and influence this formation. I want to revisit the original work of Lave and Wenger (1991) to mine it for important aspects community structure that I believe have been unfairly overlooked. I have developed the outline of a theory and methodology. I am interested in talking to other people to see if these ideas make sense in other settings, such as Ph.D. institutions.

Karen Marrongelle (Portland State University): Planning stage

The goal of the proposed project is to develop a model of professional development for TAs, where teaching is improved through the collection and analysis of classroom data. TAs will teach one section of calculus and participate, with colleagues teaching the same course, in a process of curriculum and teaching improvement.

The philosophical orientation of the proposed project is that beginning teachers need time to learn to teach rather than be expected to master the complexity of teaching during a short time (Hiebert, Glass, & Morrow, 2003). Studying their own classroom practices affords teachers one of the richest environments in which to learn to teach effectively (Ball & Cohen, 1999; Clark, 2001). Beginning teachers need support to navigate the complexities of teaching, particularly how to use student thinking in their instructional planning.

The following research question will be investigated: What types of activities provide effective professional development for TAs planning to teach calculus? I plan to collect data through surveys, interviews of individual participants, and observations of participants' classrooms.

Tim Gutmann (University of New England): Data analysis stage

An important aspect of new research into the professional lives of TAs is the extent to which existing results related to preservice and beginning K-12 teachers can guide our understanding of TA experiences, processes, and professional development needs. First year experiences of teachers have been well documented and investigated. Because the TA experience has significant similarities, it is important to understand how existing conclusions about beginning teachers might be appropriate and might be inappropriate for TAs.

Preliminary interview data was collected from TAs beginning Ph.D. programs in Mathematics. Interview data are related to two questions: reasons for entering graduate school and comparative ideas about teaching in different fields. Data include novice TAs talking about their mathematical biographies, their early interest in mathematics and their early thoughts about studying mathematics juxtaposed against their early thoughts about teaching mathematics. Additional interview are being planned and those data should allow rich descriptions of the backgrounds and ambitions beginning TAs bring with them to graduate programs and how their personal beliefs about mathematics and teaching might influence their work as TAs.

Maria Terrell (Cornell University): Data analysis stage

As part of our NSF supported "Good Questions" project we collected data from 11 TAs who chose to use, or declined the use of materials designed to help foster an active classroom learning experience in teaching calculus. The data include TA reported assessments of why they chose to use the materials, how the materials and the pedagogical approach affected they way they spend time in class, we may also learn something about whether these methods and materials affected TAs understanding of how students learn mathematics.

Help is needed in figuring out what to do with data collected through surveys and interviews of the TAs, and with their students' exam performance and responses to surveys about their learning experiences. Assistance is also needed in identifying relevant literature and related work.

April Hoffmeister (The University of Memphis): Publication stage

I have four portraits of TAs and their experiences of teaching undergraduate students. The TAs were teaching either college algebra or calculus in the Merit Workshop at the time of the study. I would like to share these portraits and eventually submit them for publication.

In addition, I looked at the college algebra TAs and Merit Workshop TAs as a group. I met separately with each group once at week for one semester. I have results on these group interactions. I would also like to share my analysis of this research and submit it for publication.

Anticipated follow-up activities

Compared to the number of school teachers and preservice teachers who might serve as research informants, the number of TAs available at any one site is often small. Further, each university has its own professional requirements and professional development programs for TAs. As a result, validity of research results in this field will require the collaboration of professionals across institutions, even across types of institutions. This working group aims to help interested researchers form partnerships that will lead to collegially-accepted valuable contributions to the field.

For future conference meetings, the projects summarized here should be advanced, either through collaborative analysis of data or through organized collection of further data with updates ready to present in 2005. Furthermore, it is hoped that in 2005 individuals participating in this year's sessions will have findings to present that begin to address major themes and questions in the research agenda to be developed at this conference.

References

Ball, D. L. & Cohen, D. (1996). Reform by the book: What is – or might be – the role of curriculum materials in teacher learning and instructional reform. *Educational Researcher 25*(9), 6-8.

Clark, C. M. (Ed.). (2001). *Talking shop: Authentic conversation and teacher learning*. New York: Teachers College Press.

Hiebert, J., Glass, B., & Morrow, A. (in press). Learning to learn to teach: An 'experiment' model for teaching and teacher preparation in mathematics. *Journal of Mathematics Teacher Education*.

Lave, J., & Wenger, E. (1991). Situated Learning: Legitimate Peripheral Participation. New York, NY: Cambridge University Press.

Speer, Gutmann, & Murphy (in press). Mathematics Teaching Assistant Preparation and Development. *Journal of College Teaching.*

MODELS AND MODELING

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The Models and Modeling Working Group at PME-NA has successfully continued its work since 1999. The purpose of this Working Group is to discuss and enrich different views in which models are used in the learning of mathematics and applied science. That is, models are considered conceptual and representational tools that allow us to better understand how students, teachers, researchers, and other educators learn, develop, and apply relevant mathematical concepts. To this workshop we would like to invite participants to begin or continue the development of the greatly needed communities of researchers and practitioners to expand our focus of research on the ways in which models are used in Problem Solving, Curriculum Development, Student Development, Teacher Development, Assessment, and Research Design.

In this workshop, we will continue to reflect on *a models and modeling perspective* to understand how students and teachers learn and reason about real life situations encountered in a mathematics and science classroom. We will discuss the idea of a model as a conceptual system that is expressed by using external representational media, and that is used to construct, describe, or explain the behaviors of other systems. We will reflect on the characteristics that are elicited, including the complexity, dynamic, and iterative features of model-development. We will consider the types of models that students, teachers, and researchers develop (explicitly) to construct, describe, or explain mathematically significant systems that they encounter in their everyday experiences, as these models are elicited through the use of model-eliciting activities (Lesh, Hoover, Hole, Kelly, & Post, 2000). During the workshop we will continue to explore these aspects of learning, teaching, and research by continuing our work in smaller groups focusing in: Student Development, Teacher Development, Curriculum Development, Problem Solving, and a strong emphasis on Research Design and Assessment Design.

A models and modeling perspective has proven to be a rich context for research and development. During past workshops, we have discussed and continued to work on innovative designs for research and assessment that can help answer questions involving the understanding of complex situations that are dynamic and iterative. There are several characteristics that need to be sustained by the types of research design needed. These include:

First, it is important to radically increase the relevance of research to practice, involving many levels and types of participants (students, teachers, researchers, curriculum designers, policy makers, and others) (Lesh & Kelly, 2000). Second, it is necessary to understand that the educational phenomena that are researched are complex systems, in the sense that they are dynamic, interacting, self-regulating, and continually adapting. Third, it is necessary for educational decision-makers to rely on reports that involve more than simple-minded uni-dimensional reductions of the complex systems that characterize the thinking of students, teachers, and researchers. Recent advances in mathematics and other scientific fields have made available the use of technologies that are capable of using graphic, dynamic, and interactive multimedia displays to generate simple (but not simple minded) descriptions of complex systems (for example, weather, systems, traffic patterns, biological systems, dynamic and rapidly evolving economic systems, to name a few) (Lesh & Lamon, 1993). And fourth, research is about knowledge development; and not all knowledge is reducible to a list of tested hypotheses and answered questions. In particular, in mathematics and science education, the outcome products that are needed from our research often focus on the development of models (or other types of conceptual tools) for construction, description, or explanation of complex systems. Thus, distinctions need to be made between: (a) model development studies and model testing studies; (b) hypothesis generating studies and hypothesis testing studies; and (c) studies aimed at identifying productive questions versus those aimed at answering questions that practitioners already consider to be priorities.

From these assumptions, many participants from the Models and Modeling Working Group have been working on a research design first described by Collins (1990) and Brown (1992) called *Design Studies*. This type of research design explicitly focuses on the development of constructs and conceptual systems used by students, teachers, researchers, and other educators. Principles applying to Design Research, the types of research questions it allows to answer, appropriate methodologies involved in the design of these types of studies, and examples of Design Research Studies are some of the discussion topics that will be considered in our working sessions.

The Models and Modeling Working Group at PME-NA Toronto

At PME-NA Toronto, our sessions will consist of a series of brief 5-minute "elevator speeches" (so called because they're similar to what a researcher might be able to say about "the essence of his or her work" if asked about it on an elevator) followed by discussions related to the following issues: Student Development, Teacher Development, Curriculum Development, and Problem Solving. More particularly, for this year we would like to extend our work by placing an emphasis on Design Research Studies (Collins, 1990; Brown, 1992) as a framework for Research Design and Assessment Design.

As in past Working Groups, speakers will give highlights of their research, and then smaller groups will be formed according to participants interests to discuss these issues more in depth, as well as to outline a plan of action for future collaboration for those who are interested in continuing their work through out the year.

Some accomplishments of the Models and Modeling Working Group

Some of the publications and other accomplishments of the participants of this working group include the book *Beyond Constructivist: A Models & Modeling Perspective on Mathematics Teaching, Learning, and Problems Solving* (Lesh & Doerr, 2003). This book including chapters written by many of the participants of this working group, where the authors give a fuller description of a Models and Modeling Perspective.

A special issue on *Mathematical Thinking and Learning: An International Journal* edited by Lyn English explicitly dedicated to a Models and Modeling Perspective, as a theoretical perspective (Lesh & Lehrer, 2003; Lesh, Doerr, Carmona, & Hjalmarson, 2003), and how it applies to student (Petrosino, Lehrer, & Schauble, 2003), teacher (Schorr & Koellner-Clark, 2003) and problem solving (Lesh & Harel, 2003).

The *Handbook of Research Design in Mathematics and Science Education* (Kelly & Lesh, 2000) describes a variety of innovative research designs that have been developed by mathematics and science educators to investigate interactions among the developing knowledge and abilities of students, teachers, and others who influence activities in mathematics and science classrooms. This handbook has helped in setting the foundations to identify several characteristics that distinguish the type of research design needed to answer they types of questions we are most interested in, which at the same time, lead to the need of new research designs in mathematics and science education.

A Models and Modeling perspective has proven to be rich context for research and development. Nevertheless, we have found the need to innovative research designs that can better help us answer the types of questions we are mostly interested in. A research design that has proven to be very useful for conducting research from a Models and Modeling Perspective are *design experiments* or *design research studies* (Collins, 1990; Brown, 1992). One of the works in progress of many participants of this working group is the development of a book on this type of research design, and how it can be used to conduct useful research to better understand students', teachers', researchers', and other educators' development of relevant mathematical ideas. Not only will the new book focus on design research methodologies, but it will also describe on new types of dynamic and iterative assessments that are especially useful in design research –where rapid multi-dimensional feedback is needed about the behaviors of complex, dynamic, interacting, and continually adapting systems.

References

Brown, A. L. (1992). Design Experiments: Theoretical and Methodological Challenges in Creating Complex Interventions in Classroom Settings. *The Journal of the Learning Sciences, 2*(2), 141-178.

Collins, A. (1990). *Toward a design science of education*. New York, NY: Center for Technology in Education.

Kelly, A. E., & Lesh, R. A. (2000). *Handbook of research design in mathematics and science education*. Mahwah, N.J.: Lawrence Erlbaum.

Lesh, R., Doerr, H. M., Carmona, G., & Hjalmarson, M. (2003). Beyond Constructivism. *Mathematical Thinking and Learning: An International Journal, 5*(2/3), 211-234.

Lesh, R., & Harel, G. (2003). Problem Solving, Modeling, and Local Conceptual Development . *Mathematical Thinking and Learning: An International Journal, 5*(2/3), 157-190.

Lesh, R., Hoover, M., Hole, B., Kelly, E., & Post, T. (2000). *Principles for developing thought-revealing activities for students and teachers*. Mahaway, NJ: Lawrence Erlbaum.

Lesh, R., & Lehrer, R. (2003). Models and Modeling Perspectives on the Development of Students and Teachers. *Mathematical Thinking and Learning: An International Journal, 5*(2/3), 109-130.

Lesh, R. A., & Kelly, E. A. (2000). Multitiered Teaching Experiments. In R. A. Lesh & E. A. Kelly (Eds.), *Handbook of Research Design in Mathematics and Science Education*. Mahawah, NJ: Lawrence Erlbaum.

Lesh, R. A., & Lamon, S. J. (1992). *Assessment of authentic performance in school mathematics*. Washington, D.C.: American Association for the Advancement of Science.

Petrosino, A.J., Lehrer, R., & Schauble, L. (2003). Structuring Error and Experimental Variation as Distribution in the Fourth Grade. *Mathematical Thinking and Learning: An International Journal, 5*(2/3), 131-156.

Schorr, R. & Koellner-Clark, K. (2003). Using a Modeling Approach to Analyze the Ways in Which Teachers Consider New Ways to Teach Mathematics. *Mathematical Thinking and Learning: An International Journal, 5*(2/3), 191-210.

PROCEDURAL AND CONCEPTUAL KNOWLEDGE IN MATHEMATICS

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A Teaching-Research report on the Procedural-Conceptual divide
1) Written Conceptual Thought in Remedial Mathematics: The case of decimals
2) Written Thought that reveals Pitfalls in understanding the Limit Concept in Calculus

We present the results of a teaching-research projects that continue the study of the relationship between procedural and conceptual knowledge in the mathematical classroom began in (Rittle et.al.,2002) in both projects the students were adults at a community college in New York City. Both experiments involved conceptual orientated writing exercises that were developed and integrated with instruction of procedural knowledge. In the first experiment the relationship between these thought processes was assessed in the first context of the students' ability to assimilate procedural knowledge and solve word problems. In the second experiment writing was used to shed light on student's difficulties translating definitions(phrases) from a textbook into an understanding of the limit concept.

In many remedial mathematics courses, a large extent of the textbook and classroom presentation begins with a brief review of definitions and then focuses on computational modeling of procedural knowledge. In this 'traditional' curriculum, concept development is viewed as arising from computational proficiency with relevant procedures. "Knowledge of structures is regarded as meta-knowledge, growing out of arithmetical procedural proficiency, and has been developed as an extension of arithmetical knowledge." (Morris,1999) In like manner, instruction in calculus often emphasizes procedural knowledge grounded in algebra. Aspinwell and Miller express this view when they state: "students regard computation as the essential outcome of calculus and thus end their study of calculus with little conceptual understanding." (Aspinwell et al., 1997)

The perception that, use of this traditional approach encourages rote learning of procedural knowledge and leads to weak understanding of the relationships between such knowledge and the objects they act upon has lead to reform efforts that focus on underlying conceptual knowledge. Reform efforts that introduce conceptual instruction in mathematics have been extensively studied with mixed results. Nesher (1986) reviewing such studies in arithmetic concludes there is no evidence of a, "relationship between success in algorithmic performance vs. success in 2 understanding." Such reform efforts have frequently been founded on the hypothesis that, student's conceptual knowledge will necessarily increase their procedural proficiency, the so called "simultaneous action view." (Haapasalo, and Kadijevich , 2000) It is perhaps not surprising that writing which for Vygotsky (1986) epitomized conceptual thought has likewise been extensively studied with mixed results. This is evidenced by Powell and Lopez (1989) who issue a challenge, to reform efforts that use writing, to establish evidence on the role of writing in promoting, "conceptual development or increased mathematical maturity."

Theoretical considerations

One developmental model that can be used to justify the "traditional" approach to learning mathematics with its emphasis on procedural knowledge has been labeled the "dynamic action view." (Byrnes et. al., 1991), (Haapasalo et. al., 2000) In this model an individual learns by applying procedural knowledge to an existing conceptual foundation, then increasing proficiency with procedural knowledge opens the door to expanding upon one's conceptual knowledge. The notion that there are stages of development in mathematics and learners typically go through a procedural orientated phase before they can effectively use their conceptual knowledge is studied in (Davis, et. al.,2000) and is based upon Piaget's understanding of how procedural knowledge can be integrated or "assimilated" into one's conceptual schema, "the heart of the process involves assimilating the new material into appropriate knowledge networks or structures." (Heibert and Lefevre, 1987)

As mathematics instructors, using the tradition curriculum with its focus on computational modeling of procedural knowledge we had noticed that, although many of the students do well enough to pass during the semester they perform poorly on the final exam. (equally true in prealgebra or calculus) Acting on the Vygotskian hypothesis that, the conscious thought engendered in the writing process will assist the individual in assimilating their procedural knowledge into their schema and thus lead to retention, writing was introduced into the classroom pedagogy.

As researchers, we had two goals the first was, to assess the benefits of written mathematical thought during the semester in the assimilation (retention) of procedural knowledge and problemsolving skills on the final exam. The second goal was to extend the results of our work in (Rittle- Johnson et. al., 2002) in which we hypothesized that, "throughout development, conceptual and procedural knowledge influence one another in mutually supportive and integrated ways." In 3 particular we reflect upon whether the relationship between procedural and conceptual knowledge is marked by iterations of first one and then the other type of knowledge or whether procedural and conceptual knowledge influence one another continually throughout the developmental process.

Project - Writing in Pre-Algebra

While the classroom instruction involved extensive use of computational modeling, serious attention was given to the underlying conceptual knowledge that the students would need to solve word problems and answer writing questions. Data was gleamed from initial diagnostic exams, partial exams (with written as well as computational mathematics) and a departmental final exam.

The results of this study confirm that initial conceptual knowledge does indeed tend to dominate initial procedural knowledge in determining student's proficiency on subsequent partial exams. Furthermore, procedural knowledge on the partial exam is more important that written conceptual knowledge on the partial exam in determining students proficiency with procedural knowledge on the final exam (retention of procedural knowledge). Thus in this case the iterative or "dynamic view" can effectively be used to uphold the traditional method of classroom instruction. However, we note that, as the number of student included in this study grow each semester the role that written conceptual thought had in predicting the student's retention of procedural knowledge grew and became statistically significant (well beyond the 0.05 level) in relationship to the role of computational proficiency.

In retention of problem-solving skills, our results indicate that, the traditional method does not prepare students nearly as well as it did with retention of procedural knowledge and can clearly be helped by written conceptual thought. In this case the optimal environment for learning clearly involved coordination of procedural and conceptual knowledge during the semester. We close in noting that these results do not imply the simultaneous view that, a focus on conceptual knowledge will result in procedural proficiency, but they do provide evidence against the converse or dynamic action view.

Project on Conceptual knowledge in Calculus

In this experiment written conceptual thought was integrated or coordinated with computational instruction to better understand student's difficulties understanding calculus 4 concepts. The difficulties with coordinating terminology and instruction into conceptual understanding are presented. In particular, we note student's written responses that highlight how textbook phrases which intend to bring the limit concept of calculus within student's reach can instead create more confusion.

Several more progressive textbooks had used an interesting phrase describing the limit of a sequence: The number L is the limit of a sequence a_n if we can find terms of the sequence as close to L as we wish for n sufficiently large." The phrase is most probably thought as best conveying the idea that "L is the limit if for any epsilon larger than zero there exist N such that for any n>N, we have $| a_n – L|<e$ " without the formal meaning of the concept. However, students' essays indicate that the phrase causes probably exactly the same amount of confusion as it wants to avoid by that "conceptualizing".

> For n sufficiently large? If one would be speaking to an individual without much math experience, that individual would still be lost…It is hard to explain what for n sufficiently large is…a better explanation in English could have been, the larger n becomes, the closer to zero the value becomes.

A analysis of that phrase reveals high level of ambiguity which, unfortunately, allows to subvert the original structure of the definition into incorrect verbal interpretation. We will submit students written responses for the discussion as to what is the proper manner of conceptualizing calculus.

References

Aspinwell, L. and Miller, D. (1997). Students' positive reliance on writing as a process to learn first semester calculus. *Journal of Institutional Psychology*, 24, 253-261.

Baker, W., Czarnocha, B. (2002). "Written Metacognition and Procedural Knowledge," *Proceedings of the 2nd International Conference on the Teaching of Mathematics,* University of Crete, Hersonissos Crete, Greece, 1-6 July 2002.

Byrnes, J., Wasik, B. (1991) "Role of Conceptual knowledge in Mathematical Procedural Learning," *Developmental Psychology*, Vol 27, no. 5, 777-786.

Davis, G., Gray, E., Simpson, A., Tall, D. and Thomas, M. (2000). What is the object of the encapsulation of a process. *Journal of Mathematical Behavior* 18(2),223-241.

Morris,A. (1999). "Developing Concepts of Mathematical Structure:Pre-Arithmetic Reasoning Versus Extended Arithmetic Reasoning," Focus on Learning Problems in Mathematics, 21(1),44-71.

Haapasalo, L., Kadijevich, D. (2000). Two Types of Mathematical Knowledge and Their Relation. *Journal für Mathematikdidatik* **21**(2), 139-157. 5

Hiebert, J. Lefevre, P. (1987) Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis, ed. Hiebert,J. Conceptual and Procedural Knowledge: The Case of Mathematics. Hillsdale, NJ:Erlbaum.

Powell, A.R., Lopez, J.A. (1989). Writing as a vehicle to learn mathematics. in P. Connolly and T. Viardi (Eds.). *Writing to Learn Mathematics* (pp.157-177) New York: Teachers College Press.

Rittle-Johnson, B., Kalchman M., Czarnocha, B., Baker, W. (2002) An Integrated Approach to the Procedural/Conceptual Debate, (Eds.) Mewborn, D., Sztajn, P., White, D., Wiegel, H., Bryant, R., Nooney K., PME-N A XXIV, Athens, Georgia, October 2002.

Nesher, P. (1986) Are Mathematical Understanding and Algorithmic Performance Related? For the Learning of Mathematics 6 (3),2-9.
Vygotsky, L. (1986). Thought and Language. Cambridge, MA: MIT Press

Discussion Group

EXPLORING THE USE OF CLINICAL INTERVIEWS IN TEACHER DEVELOPMENT

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Many consider records of practice a powerful means for not only promoting teacher growth but for assessing teacher growth as well. Clinical interviews are one such record of practice. This discussion group will provide an opportunity to consider the value of clinical interviews for teacher educators and researchers. We hope that colleagues who have used other records of practice will join us in exploring the ways in which clinical interviews differ from and augment other records of practice as research instrument and a professional development tool.

One-on-one clinical interviews with students have most typically been promoted for helping teachers understand how children think about mathematics (Buschman, 2001; Ginsburg, Jacobs, & Lopez, 1998; Long & Ben-Hur, 1991; Schorr & Lesh, 1998). We believe that interviews can also help teachers develop the expertise needed to respond to that thinking and implement the kind of instructional practices called for by reform documents. Specifically, interviews provide teachers with opportunities to practice eliciting and building on children's thinking by engaging them in discussions of problems. Because the development of this expertise takes time, videotapes or audiotapes of interviews can provide windows into teachers' developing practices thus making interviews a promising context for assessing teacher change and, in particular, the degree to which their practices align with reform recommendations. When interviews are used by researchers as a means to promote and assess teachers' growth, analytical frameworks are required to examine them.

The organizers of this discussion group have been developing analytic frameworks for evaluating teacher-conducted interviews in separate projects (Crespo & Nicol, 2003, Haydar, 2003, Jacobs & Ambrose, 2003, Moyer & Milewicz, 2002). While the teachers in the projects differ according to teaching experience (preservice vs. in-service teachers) and geographic location (U.S. vs. Lebanon), we have identified similar phenomena in teacher interviews through our informal discussions and by studying one another's writing. For example, we have all noticed that the way in which teachers ask questions and the order in which they ask such questions when they conduct clinical interviews with students can tell us a great deal about teachers' thinking and teaching practice in their classrooms. However, our characterizations differ, and we are interested in comparing and contrasting our frameworks in greater depth. We propose to apply each framework to the same interview as a way to illuminate the similarities and differences between the frameworks. In so doing we hope to better understand each framework and its advantages and constraints. This group session will help to begin to develop a common vocabulary for characterizing teacher/student interactions which should prove useful to researchers and teacher educators.

In the discussions, we will explore the following questions:

- Are interviews reasonable proxies for the kinds of interactions teachers have in their classrooms?
- How do interviews differ from other records of practices as measures of teachers' practice?

- What are the benefits and limitations of using interviews as a means of assessing teacher learning?
- What are the benefits and liabilities of each of the frameworks?
- Can the frameworks shared by the organizers be used for novice and experienced teachers?
- Can these frameworks be used across mathematics content areas and tasks?
- How can these frameworks help promote teachers' growth?
- What might teachers learn by observing and conducting interviews and how does this differ from what they learn from other analyzing other records of practice?

In our first session we will:

- Invite the audience to share their experiences with clinical interviews
- Invite audience to generate 'what they might look for' if they were asked to 'assess' a one-on-one teacher-student interview
- Share 3-4 frameworks
- Invite the audience to analyze similarities and differences among the frameworks

In our second session we will:

- Watch 3 (maybe only 2) videoclips of problem solving interviews (3-5 minutes each) with transcripts
- Invite the audience to provide an analysis of the each videoclip from each framework's perspective
- Discuss (with input from the audience) differences between the analyses
- Consider the possibility of integrating the frameworks

References

Buschman, L. (2001). Using student interviews to guide classroom instruction. *Teaching Children Mathematics,* 8 (4), 222-227.

Crespo, S. & Nicol, C. (2003). Learning to investigate students' mathematical thinking: The role of student interviews. In N. Pateman, B. Dougherty & J. Zilliox (Eds.) Proceedings of the twenty-seventh Conference of the International Group for the Psychology of Mathematics Education held jointly with the twenty-fifth Conference of PME-NA. Honululu, Hawaii, July 2003 (pp. 2-261 – 67).

Ginsburg, H. P., Jacobs, S. F., & Lopez, L. S. (1998). *The teacher's guide to flexible interviewing in the classroom: Learning what children know about math.* Boston, MA: Allyn and Bacon.

Haydar, H. (2003). Daring to ask hard questions: The effect of clinical interview training upon teachers classroom questioning. In N. Pateman, B. Dougherty & J. Zilliox (Eds.) Proceedings of the twenty-seventh Conference of the International Group for the Psychology of Mathematics Education held jointly with the twenty-fifth Conference of PME-NA. Honululu, Hawaii, July 2003 (pp. 3-33 –- 3-38).

Moyer, P.S. & Milewicz, E. (2002). Learning to question: Categories of questioning used by preservice teachers during diagnostic mathematics interviews. *Journal of Mathematics Teacher Education,* 5, 293–315.

Jacobs, V. & Ambrose, R. (2003). Individual interviews as a window into teachers' practice: A framework for understanding teacher-student interactions during mathematical problem solving. Paper presented at the Annual Meeting of the American Educational Research Association, April 2003, Chicago, IL

Long, M. J. & Ben-Hur, M. (1991, February). Informing learning through the clinical interview.

Arithmetic Teacher, 44-46.
Schorr, R. Y. & Lesh, R. (1998). Using thought-revealing activities to stimulate new instructional models for teachers. Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education (Vol. 2, 723-729), Raleigh, North Carolina.

Advanced Mathematical Thinking

APPROXIMATION AS A FOUNDATION FOR UNDERSTANDING LIMIT CONCEPTS

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Approximation metaphors emerged as productive tools for first year calculus students in this instrumentalist study of reasoning about limit concepts. Students' intuitions about approximation mirrored several important aspects of □-□ and □-N structures, were spontaneously employed to help tackle challenging problems about limits, guided exploration of the relevant formal structures of other important calculus concepts, and served as a basis for the abstraction of formal limit structures.

Introduction

In first-year calculus courses, whether limit concepts are introduced intuitively or treated with formal definitions and proofs, subsequent topics are almost always developed through an informal characterization of limits. For example, students are rarely asked to investigate the limit definition of the derivative or the fundamental theorem of calculus through ε-δ arguments. Such reasoning is notoriously difficult to the degree that they are often omitted from standard first-year calculus courses. Williams (1990, 1991) has revealed several ways in which students try to reason about limits using intuitive ideas and metaphors for the concept including boundaries, motion, and approximation with varying degrees of success. This paper presents results of a study indicating that with some direction, students' spontaneous reasoning about approximation can serve as a productive foundation for limit concepts and even for subsequent development of other major concepts in calculus and the development of their formal structures.

Most previous research on students' understanding and learning of limit concepts has focused on structural aspects of their knowledge. According to John Dewey, however, it is also necessary to examine the functional ways in which relevant tools are applied *technologically* against problematic aspects of situations (Hickman, 1990). First, this means that a cognitive tool is *active* in that it is selected and applied in a dynamic process which engages the attention of the individual. Second, the technological application of a tool is *testable* as it is used to manipulate the problem that gave rise to its selection, and reciprocally, is itself tested against the problem and evaluated for appropriateness. Finally such tool use is *productive* as fortuitous interactions between aspects of the tool and problem effect changes in both. The emergent artifacts of this dialectic are new meanings that may themselves become the objects of further inquiry.

In this study we analyze students' metaphors as technological tools. That is, we examine how a metaphorical context is leveraged against an emerging conception of limits in an active, testable, and productive fashion. Max Black's "interaction" theory of metaphorical attribution asserts that one must regard the subjects of a metaphor precisely as this type of a complex, interacting system (Black, 1962, 1977). Strong metaphors, such as those necessary for supporting creative thinking, force the concepts involved to change in response to one another. The resulting perspective created is one that would not otherwise have existed. That is, strong metaphors are ontologically creative. In such metaphorical reasoning, one cannot simply apply an antecedently formed concept of the metaphor as-is; something new and actively responsive to the situation is required of all concepts involved. If pursued, the application of such a system can support a degree of discovery that leads far beyond one's original thoughts, providing the complexity and richness of background implications necessary for generating new ways of perceiving the world.

Methodology

Students from a year-long introductory university calculus sequence participated in interviews and submitted writing samples covering their attempts to make sense of problematic situations involving limit concepts. Open ended questions such as "Let $f(x) = x^2 + 1$. Explain the meaning of $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$, and "Explain in what sense $\sin x = 1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots$"

were presented to the students to elicit application of their spontaneous concepts against challenging problems. An entire class of 120 students submitted responses to short writing assignments, 25-35 students participated in regular online writing assignments, nine students participated in clinical interviews during which interviewer prompted for detailed explanations, and follow-up interviews were conducted with an additional 11 students.

Clusters of contexts were developed during data analysis characterizing the application of particular schemas, contexts with detailed relationships and governing logic, such as approximation. Each cluster consists of a common schema, the various problems against which it was applied, the details of each application, and the conclusions drawn about both the schema and the problem context. Development and characterization of a "strong" metaphor cluster required 1) several students using a schema to respond to a given problem context, 2) sufficient depth to reveal aspects of the structures and usage of their metaphors, and 3) application against multiple problem contexts. Numerical cut-offs for each of these criteria were developed then revised based on coherence and emerging separation of clusters through subsequent rounds of coding. This paper reports on the approximation schema which was overwhelmingly applied most frequently by students and most closely resembled formal limit concepts.

Although students' spontaneous approximation metaphors were often very powerful and useful, they were also incomplete and misleading. In the final phase of this study, students were engaged in a variety of writing assignments which emphasized a version of the approximation schema identical in structure to formal limit concepts. Interviews were then conducted to reveal the influence of the intervention on students' interpretation and application of approximation metaphors. Since an attempt was made in this portion of the study to prescribe the ways in which certain language was used, these words served as markers for potentially interesting exchanges in the transcripts. Passages exclusively involving language related to a single context and those involving interaction between contexts were identified for particularly close analysis.

Approximation Metaphors

That approximation ideas emerged as the strongest and most frequently applied metaphor cluster for limits among calculus students is not surprising. Much of the subject is historically motivated by needs for numerical estimation techniques that continue to influence our classroom and textbook presentations. Although the professor involved in this study used such language and examples, the approximation schema as a construct in the data analysis was not brought to his attention until the final phase of the study when it's use was intentionally encouraged.

The main components of students' spontaneous approximation schemas are an unknown actual quantity and known approximations that are believed to be close in value to the actual quantity. For each approximation, there is an associated error,

$$\text{error} = |\ \text{actual value} - \text{approximation}\ |.$$

Consequently, a bound on the error allows one to use an approximation to restrict the range of possibilities for the actual value as in the inequality approximation – bound < actual value < approximation + bound.

An approximation is contextually judged to be accurate if the error is small, and a good approximation method allows one to improve the accuracy of the approximation so that the error is as small as desired. An approximation method is precise if there is not a significant difference among the approximations after a certain point of improving accuracy.

The structure of this schema parallels the logic of ε-*N* and ε-δdefinitions of limits. For the latter, bounding the error corresponds to the statement "then $|f(x) - L| < \varepsilon$". The need to obtain *any* predetermined degree of accuracy maps to requiring the ε-δ condition hold "for any ε>0 ." A mechanism to generate better approximations corresponds to "there exists a δsuch that whenever $0 < | x\text{-}a | < \delta$." Linking these structures gives the practical statement of being able to find a suitable approximation for any degree of accuracy on the one hand and the ε-δ definition on the other. Students' intuitive notions of precision reflect the structure of Cauchy convergence by mapping informal statements such as "There will not be a significant difference among the approximations after a certain point," to the condition "If n>N, then $|a_m - a_n| < \varepsilon$." The following table provides the four contexts in which the approximation schema was most heavily employed and the frequencies of responses involving approximation ideas.

Context Description	Total Responses	Approximation Responses	Percent
Taylor Series of sin(*x*)	35	26	74.3%
$0.\overline{9} = 1$	103	72	69.9%
Definition of the Derivative	98	34	34.7%
Volume of Revolution	31	8	25.8%

An Approximation Metaphor for Infinite Series (Infinite Decimals and Taylor Series)

The two problem contexts in which approximation metaphors were most prominant both involved infinite series. While attempting to make sense of the equality $0.\overline{9} = 1$, students were more likely to use the approximation schema (72 out of 103) than they were to even mention limits or infinite series (59 out of 103, with only 17 doing so correctly). When discussing the Taylor series of sin(*x*), a larger percentage of students used approximation metaphors than any other context in this study. In addition, they dedicated a greater portion of their responses to these ideas (an average of 15 out of 31 lines of text) than students gave to any other metaphor or problem context (an average of only 10 lines out of a 36 line response.) The application of the approximation schema was similar for problems involving infinite decimals and Taylor series. In both contexts, students described partial sums as approximations, the limit as the value being approximated, and the difference between the two as the error. Discussions of accuracy were abundant in both cases, but students only described trying to bound the error for the Taylor series. Students use of the approximation schema more closely resembled the structure of the ε-*N* definition when talking about Taylor series than when discussing infinite decimals. In the following excerpt, a student describes the Taylor series of sin *x* as an approximation to the function, with errors as the difference between the two that can be made as small as one wants:

> To think of sin *x* as a polynomial would be incorrect, because although an approximation of its value can be determined by a polynomial, the sin *x* itself is a function who will technically never equal the polynomial exactly. It can however be useful to think of sin *x* as equal to this value though, because although the power series for sin *x* and sin *x* are two different functions, their values are very close to one another. So for every day use of

values for sin x, their values will be close enough to think of as equal. In fact the power series for sin x will approximate a value infinitely close to the value of sin x and even a remainder can be calculated…. The power series of sin x continues forever depending on how close you want your value to come to the value of sin x, and since it would be impossible to have infinite time to calculate a value, the values for sin x and its power series could never be technically equal…. The remainder is designed to show how much a power series deviates from the value of a function at a particular point, so I guess they will never be equal, but since their values can come infinitely close to each other, its easy to think of a function like sin x and its power series which is a polynomial as the same thing. I guess a more accurate statement would be to say that the power series or polynomial for sin x is an approximation of its value that can be as close of value as you want it to be.

Some students explicitly described methods for being able to bound the error by a certain amount using the fact for alternating series that "the maximum error given a polynomial is the next term" or by using the Lagrange formula.

An Approximation Metaphor for the Definition of the Derivative

When students used the approximation schema to reason about the limit definition of the derivative, the slope of the tangent line was treated as the unknown quantity to be approximated and the approximations were various values of the difference quotient. In this context, there was little mention of error or the difference between the slope of the tangent line and a secant line. The following excerpt was typical of these responses.

If you want to find the slope of the tangent line at the point x of the graph *f(x)* , a good approximation would be the line between the two points $(x,f(x))$, and $(x+h, f(x+h))$, with h being a small number. The slope of that line would be $\frac{f(x+h-f(x))}{(x+h)-x}$ or $\frac{f(x+h)-f(x)}{h}$. The smaller you make your h, the better an approximation you would have, since the two points would be get closer and closer. So if you just did $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ you would have the slope of the tangent line at $(x+h, f(x+h))$, or the instantaneous rate of change of f at x.

Notice there is no explicit mention of what is meant by "the better an approximation" or why the limit yields "the instantaneous rate of change," but the student does indicate that both are related to making x and $xh+$"get closer and closer."

A mathematically equivalent, but conceptually different application of the approximation schema to the definition of the derivative is to view the tangent line as a local approximation to a differentiable function. Two students attempted to apply such a metaphor to L'Hospital's rule, explaining that taking a limit at a point makes a function "indistinguishable from" it's value at that point. They viewed differentiability as meaning that a function "is very close to its tangent line" and the difference as an "error," $E_1(x)$ that "must approach zero so fast that $\lim_{x\to a}\frac{E_1(x)}{x-a}=0$." Thus, a tangent line is an approximation to the function that can be used in computations, possibly taking appropriate care with the error.

An Approximation Metaphor for the Volume of Unbounded Solids of Revolution

Eight of the 31 students responding to a question about the finite volume of a solid of revolution described approximating the volume of the entire unbounded solid by using a very long, but bounded solid. These students then claimed either that the remaining volume was so small that it is "practically negligible" or at least "that it can be ignored compared to the large portion of volume near to $x = 1$." Notice how one student combines the use of several senses of the word "practical:"

> Even though there is a tiny hole in the end of it, at the extreme values of x for the graph of $y = 1/x$, the hole becomes practically negligible. Since there will not be a significant increase in volume after a certain point down the x-axis though values continue to increase infinitely, a volume can be estimated. This approximation is accurate enough for most practical purposes, and can be mathematically derived.

Initially, "practically negligible" indicates that the "tiny hole" *almost* vanishes. Later, "practical purposes" implies there is some *realistic use* intended. Finally, there is an implicit use of "practicality" in the final suggestion that, *in practice,* only finite volumes can be calculated.

Students' Reactions to Explicitly Presented Metaphors

The excerpts above demonstrate that students' intuitive notions about approximation lead to both impressive and erroneous reasoning about limit concepts. A noticeable weakness of their reasoning is that it did not systematically include parallels for the complete logic of ε-N and ε-δdefinitions. For example, although some students did discuss bounding errors, there was little mention of a need to obtain *any* predetermined degree of accuracy.

In an attempt to scaffold students' spontaneous application of approximation metaphors in limit problems toward more mathematically appropriate versions, students were given a series of exercises emphasizing the relevant structures in approximation. In each of several contexts such as approximating distance traveled under a varying velocity or approximating the value of πusing an alternating series, students were asked to identify what was being approximated, what the approximations were, what the errors were, how to bound the errors, and how to make the bound as small as one wanted. Limits were not explicitly mentioned. Follow-up interviews revealed that each of the students initially still applied approximation ideas in idiosyncratic ways, but also had adopted much of the prescribed structures. The following case illustrates this type of transition throughout the interview.

Moving from Idiosyncratic to Technical Understanding

One student, Bob, immediately began the interview using phrases such as "degree of accuracy" and "bounds on error." When questioned about his interpretation of these phrases, he described an idiosyncratic understanding related to being able to ignore errors. Later in the interview, however, he shifted to applying the approximation schema precisely as presented in the intervening problem set.

Bob's initial explanation of infinite decimals was that "0.9999 is equal to 1 because there's such little difference that in the approximation it really doesn't matter. If your error is wrong then you can always go out another 9 or 1000 more 9's, and you can get so small that there's practically no difference between it and 1." This explanation includes the terms of the sequence as approximations and the difference between them and 1 as the error. While the phrase "if the error is wrong," could refer to an unsuccessful attempt at bounding the error, the rest of the excerpt suggests he is simply thinking of a small, or inconsequential, error.

Throughout the next 20 minutes of the interview, Bob generated multiple examples of simply ignoring small errors, such as "whenever you get that small - maybe geneticists will get that small with DNA strands. No one else like a banker or something with money - a country doing its budget - no one is actually going to know the difference between those two numbers. They're just like the same." When asked where he would plot 0.9 on the number line, Bob responds, "If I was just drawing it, I would probably just make the line really big and I would probably just like put that practically on it - so close that you can't tell the difference no matter how close you look at it." Again, he is describing a small difference, this time in spatial terms, that can be ignored for practical purposes.

When first explaining the limit in the definition of the derivative, Bob describes what amounts to round-off error, a "margin of error" that is negligible in comparison to some reference. He compares computations for polynomials in which an h^2 term in the numerator of the difference quotient is "so small that it doesn't matter" to polynomial time algorithms that, he learned as a computer science major, have higher order terms which "dominate" lower order terms. Shortly afterwards, Bob referred to a vague description of quantum mechanics and concluded that differences on such a small scale do not matter and can be ignored.

Immediately following his example involving quantum mechanics, without prompting from the interviewer, Bob shifted to using this approximation language in a way consistent with formal ε-δ structures, saying for example that:

> Well, I was using the point nine as an example. If say your margin of error is - you know, you have point 99 then your margin of error is point 001.… If your error cannot be bigger than that, you just throw on some more nines and you can get smaller than 0.0001 when you subtract it from one. That was the example that I was using. You can always find something smaller than what they give you. Like this is pretty much our ε-δ proofs. You can always find a spot closer than where you need to be. Your margin of error is here [*holds up hands facing each other to indicate a distance*] and here's your limit [*waves one hand*] and you have to be at least in so far closer to it [*waves other hand across the space in between*]. You can always get closer to it, you know? That's the way I was looking at bounding. You can always get closer to it.

In this passage Bob uses the phrase "margin of error" to alternately mean error and a bound for the error. Nevertheless, he does put together nearly all of the components of the modified approximation schema, referring to the difference between 0.9 and 1 as error, describing in a variety of ways the need to achieve a pre-specified bound (e.g., "Your margin of error is here and here's your limit and you have to be at least in so far closer to it"), explaining how this can be achieved ("If your error cannot be bigger than that, you just throw on some more nines and you can get smaller"), and finally, noting this can be done for any bound ("You can always find something smaller than what they give you.")

After this point in the interview, Bob never returned to describing error as round off error or as being something that is "so small that it doesn't matter." As indicated in the previous excerpt, he noticed the similarity between his description and the ε-δ definition which he later accurately explained on a graph for the limit of a function. When he tried to explain the meaning of the definition of the derivative in terms of ε-δ language, however, he became extremely confused as illustrated in the first part of the following excerpt.

> B: Your ε-this - the slope of this tangent line. You want to pick a set of x's, and that's here. This x, it's barely changing such that it's equal to or less than this tangent line. That would be your δ. The slope - oh, OK. The slope of this tangent line - that's ε. The slope of this line that you're making is your δat 2. Take a δ-a slope of this line less - such that it is less than the slope of this tangent line.

I: OK. What if you were talking about it in terms of approximations?
B: Approximations? OK. [*pause*]
I: If that doesn't make sense that's fine.
B: It kind of makes sense. I'm just not sure how to do it. That's the thing.
I: What are you thinking about?
B: OK well the way I'm - you're saying how can we make this [*points at secant line*] approximately equal to this [*points at tangent line*], is that correct?
I: Yeah, describe what you were talking about with the tangent lines in terms of approximations and making errors small. Use that kind of language.
B: [*pause*] There will be - there could be a difference in the slopes of these lines. You could say that the slope of this line is approximately equal to this with a margin of error of such and such, and that margin of error can be less than that. You can choose a slope that's less than the margin of error - less than what ever you need it to be.

Although Bob thoroughly confused himself when using ε-δ language, he was able to give a correct explanation in terms of the approximation schema when asked to do so. Such differing abilities when using ε-δ and approximation language was common among students in these interviews. Those who were able to give correct interpretations using ε-δ language, however, became unable to distinguish between the two and began interchanging their terms. Jacob, for example, while explaining infinite series said, "So say we pick, you know, our allowed error to be this much, then we can find a δ that would allow us to be within there," and later explaining the definition of the derivative:

> This would be like your - the value you got. That would be - alright, that's your error, right? That would be like what we were talking about being an error. The secant line minus this actual slope. [*pause*] I guess - if you were given a - if you were given ε -sort of like I was saying, you're given a bound of error to be within, and you would get this value to be within that.

At the conclusions of the interview, Jacob expressed his view of the relationships between each of these schemas, as "I guess - now I think of them as sort of the same thing."

Conclusions

Students in this study spontaneously applied ideas about approximation, more than any other context, when facing challenging limit problems. Their intuitive notions about approximation mirrored several aspects of formal limit structures, and even led students to understand the role of limits in other major calculus concepts such as the definition of the derivative or Taylor series in ways that respect their formal definitions. With assistance, these students were able to more fully develop understandings of formal limit structures in terms of approximation and some were able to transfer this to an understanding of a wide variety of ε-δ arguments. Further research is currently being conducted to further explore the nature of students' abstraction of limit concepts using curricular materials built around these results.

References

Black, M. (1962). Metaphor. in M. Black. *Models and Metaphors*, Ithaca, NY: Cornell University Press, Chapter 3, 25-47.

Black, M. (1977). More about Metaphor. *Dialectica*, *31*, 433-57.

Hickman, L.A. (1990). *John Dewey's Pragmatic Technology.* Bloomington: Indiana Univ. Press.

Oehrtman, M. (2002). *Collapsing Dimensions, Physical Limitation, and other Student Metaphors for Limit Concepts: An Instrumentalist Investigation into Calculus Students' Spontaneous*

Reasoning, Doctoral Dissertation, The University of Texas, Austin, TX.
Oehrtman, M. (2003) Strong and Weak Metaphors for Limits. *Proceedings of the 2003 Joint Meeting of PME and PMENA, 3,* 397-404.
Williams, S. (1991). Models of Limit Held by College Calculus Students. *Journal for Research in Mathematics Education*, *22*, 219-236.
Williams, S. (2001). Predications of the Limit Concept: An Application of Repertory Grids. *Journal for Research in Mathematics Education, 32,* 341-367.

ON THE LIMIT CONCEPT IN A COOPERATIVE LEARNING ENVIRONMENT: A CASE STUDY

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The aim of this paper is to investigate teachers' reconstruction of the limit concept in a scientific debate/cooperative learning environment. Throughout their engagement with a set of designed instructional activities in a one- semester course, participants were first encouraged (in small groups) to volunteer their intuitive ideas related to potential infinity. Then, when teachers (enrolled in a graduate program) presented their results to the whole class, we tried to promote a socio cognitive conflict and to confront them about their ideas about potential and actual infinity in a scientific debate methodology. We then elicited participants' individual reflections about the same activity discussed in the small groups and in the whole class. Finally, at the end of the course, we interviewed each participant to test and document what he or she had retained. In this paper we present a general description of the entire experiment and of a specific case.

Introduction

The central concepts in calculus are all defined in terms of limits (e.g., convergence of sequences and series, continuity, derivative and integral). It is therefore clear that the concept of limit plays a major role in the understanding of concepts in calculus. The history of mathematical ideas (Cajori, 1915; Grattan-Guinness, 1970; Patras, 2001) and research in mathematics education

(Hauchart & Rouche, 1987; Cornu, 1994; Alberti et al., 2000; Hitt & Páez, 2001) show us that the concept of limit was historically and is (among students and teachers) difficult to grasp and often confused with vague and sometimes philosophical and intuitive ideas of infinity (potential versus actual infinity).

Recognizing that potential infinity expresses the idea of an ongoing process without end, philosophers and mathematicians have not definitively resolved the issue of whether a limit is ever reached — a controversy thought to have originated with Zeno's paradoxes (see Cajori, 1915). Grattan-Guinness (1970) has argued, on the contrary, that the quintessence of analysis is the limit-avoiding process, and it was with Bolzano's definition, in 1817, of the continuity of a function at a point that this essential idea began in mathematics. In that line of thinking, the modern (ε, N) definition of convergence of a sequence is relative to actual infinity, a limit-avoiding process. The historical development of the mathematical concept of actual infinity was long, spanning the fifth century B. C. through nineteen century A. C. Then, history shows us that potential infinity and the idea of reaching the limit became epistemological obstacles to comprehending the concept of actual infinity.

Several educational studies have documented that students often experience serious difficulties in understanding the concept of infinity (Hitt & Páez, 2001; Tall & Vinner, 1981; Eisenberg & Dreyfus, 1990). If students or teachers develop cognitive obstacles related to understanding the concept of limit, the question arises as to how to overcome such obstacles. This is the question we highlight in this paper. Our position is that one pedagogical road to understanding the concept of limit in a (ε, N) context (limit avoiding process), is through an intuitive idea of infinity. This road involves potential infinity and thus entails confronting the controversy of whether the limit is ever reached and then, based on our methodology to generate socio cognitive conflicts, to promote a reflection.

Goals of this research

In this study we deal with teachers as students enrolled in a graduate program in mathematics education. Our goal was to promote a process of reconstruction of concepts. Given the complexity of the concept of limit, we considered it important to promote participants' (high school teachers) active engagement with a set of activities during a semester-long course involving a co-operative learning environment. We promoted a culture of scientific debate within the classroom, with the aim that it support examining the small groups' approaches to the activities. Our aim was to uncover participants' cognitive obstacles, and, by having them confront such obstacles in the context of scientific debates, to encourage a better approach to the concepts involved. In this way, from a general point of view, as these teachers participated in a scientific debate, they learned that formulating conjectures is a useful and necessary activity and they began to see mathematics not merely as a massive and static body of knowledge, but as a natural and active way of constructing mathematical knowledge (Alibert & Thomas, 1994).

Theoretical framework

Our theoretical perspective centers on the notion of representations. In particular, we are interested in the role of semiotic representations in the construction of mathematical concepts. As Duval (1993) claims, "the integrative understanding of a concept is supported by at least a co-ordination of two registers of representations, and the spontaneity of the cognitive activity of conversion among representations" (p. 51). Following Duval's thinking, Hitt (2003) characterizes a student's knowledge as stable if he or she is capable of articulating different representations without entertaining lingering contradictions.

With regard to the construction of concepts, students' conceptions play a mayor role. Duroux (1983) interprets a conception as a student's knowledge that plays a positive role in some mathematical situations, thereby enabling them to arrive at a correct answer. In other mathematical situations, however, students' conceptions can lead them to a wrong answer. When designing the activities we took into account this point of view, and also the role of semiotic representations, within a co-operative learning and scientific debate methodology. Our approach in this research can be summarized as shown on Figure 1. Activities in the class were designed to follow this trend.

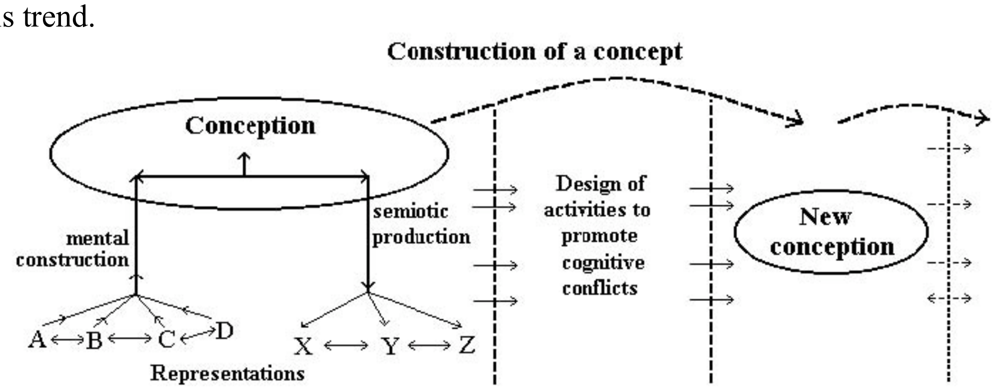

Figure 1 Construction of a concept taking into consideration a conception

Methodology

We think of this methodology of scientific debate in a co-operative learning environment as a tool for improving participants' ideas and for transforming false intuitions into consistent knowledge. Twenty-one teachers began a course in our master's degree program in mathematics education. The participants were high school teachers. Four of the 21 teachers abandoned the course at the very beginning, two of them said that the discussion was fruitless (a waste of time listening to the discussion of their classmates!). Two other participants found difficult to follow the discussion of their classmates. Twenty-two activities were designed (some taken from Hauchart & Rouche, 1987; Hitt & Páez, 2001 and Sierpinska, 1988). Our main intention in designing the activities was to create socio cognitive conflicts in the teachers' minds in order to generate a group discussion. In this way, the first activities drew out participants' intuitive ideas about potential infinity and were confronted by themselves to subsequent activities in which conceptualizing actual infinity was necessary to solve some mathematical problems related to limits of sequences and series (some convergent and some not). Our overarching aim was to We were promote a confrontation among their intuitive ideas which might then induce participants to reflect deeply about the two mathematical ideas of potential infinity and actual infinity.

Another aspect we took into account when designing activities was the conversion between representations.

In the first phase of our research, we used a diagnostic questionnaire about functions, limits and infinity. We classified the teachers as follows:

1. Those who showed no contradictory statements in their responses to the questionnaire,
2. Those who presented contradictory statements in their responses, and
3. Those who exhibited limitations in their knowledge but without contradictions in their responses.

On the basis of this classification and by considering each teacher's formal university training (in mathematics or engineering), we separated teachers into two groups.; Then, , we assigned one teacher from each category to each group. The groups worked during 14 sessions, for duration of two and a half hours per session.

Phases of the experiment

Following the methodology of co-operative learning environment (see Reynolds et al., 1995), the teachers discussed the activities in the small groups before participating in the general whole-class discussion.

First phase: The activities were designed to encourage active participation within small groups (22 activities were designed).

Second phase: Constitutions of small groups on the basis of participants' academic training and the results of the diagnostic questionnaire. When working in small groups, each member played a different role (sometimes reporting the small group work, using the calculator, or leading the discussion or presenting the results to the whole class).

Third phase: The whole-class scientific debate. When participants reached "a consensus" within the small groups, we asked for a general discussion (a sort of scientific debate in the sense of Alibert & Thomas, 1994). This phase was very important, as some teachers adhered to their initial positions, while others changed as a result of having participated in the discussion.

Fourth phase: In connection with the same activity, a reconstruction of the results was required as individual homework. At the end of the general discussion, we collected all the papers that participants used when discussing the activity. Our rationale for doing so was that without the notes they generated during the small group discussions, participants would be required to reflect more deeply on the reconstruction of the resolution that the activity entailed. In our thinking, the process of having to reconstruct what was discussed and worked out in the class could induce a better approach to the construction and/or reconstruction of concepts, changing their initial conceptions. We were promoting auto reflection, as a very important element in our methodology, in a sense consistent with Hadamard's (1975, p. 25) a notion of "incubation of ideas".

Fifth phase: In the last session teachers took a final exam and one week later we conducted 17 interviews.

In summary, we followed these important phases in the subsequent part of our methodology:

- Teachers worked on a task in small groups in a co-operative learning environment,
- Teachers participated in a plenary discussion involving the whole class in a scientific debate environment,
- Teachers re-examined the same activity (in a personal reflection as a homework) based on their previous work (small groups and scientific debate discussion),
- Teachers wrote a final exam and participated in an individual interview.

As we mentioned earlier, this research, enabled us to collect information from several angles and using different tools. We collected data through questionnaires, videos of the teachers' work carried out in the 14 sessions, homework and videos of the interviews. In this paper, we concentrate on the case of one participant that provides us some hypotheses concerning a theoretical approach related to the role of "functional representations" in the construction of mathematical concepts. This, in turn, permits us to better understand how teachers reconstruct knowledge in an environment like the one described here.

Results

As we said, we are interested in showing a case: 'Victors' process of reconstruction of the concept of limit'. It is supposed that this teacher, called Victor (pseudonym), has already an idea of the concept of limit. We would like to show the conceptions Victor had before the experiment and the changes those conceptions underwent as a result of his having participated in this learning experiment.

Diagnostic questionnaire

Our analysis of Victor's responses to particular questions uncovered some difficulties that he experienced when solving them. We can interpret Victor's answers about infinity as follows: Infinity as numerable. His idea that "There is not and end" suggests that he was thinking about potential infinity. That is, Victor's conception of limit is that of approaching, related to potential infinity.

Class sessions

In order to illustrate the environment, we show some main points of the general discussion (scientific debate) in class.

The limit figure is a hexagon or a point?

We first analyzed an activity that involved the construction of a regular hexagon, joining the half point of every side of the previous hexagon to construct another hexagon, etc. In the debate, the teachers discussed several statements, but we focus here on one: We can imagine an infinite step-by-step process without finishing it; there is not an end. The hexagons' areas are approaching zero.

The idea of approaching to the limit was confirmed by the whole class, but some participants mentioned that the limit process will never reach zero and the position of others was the opposite, for example:

Mario: The areas [of the hexagons] of course they tend to zero… well, mmm…, well it becomes one point, the hexagons, mmm…; that is, one point.

Marcos: … constructing the hexagons, the last one, so to speak, that we could construct in some way, its area would, mmm…, well be near zero…

Professor: If it is not zero, what is it? A hexagon and then with area different to zero?

Marcos: Oh! Yes, that's it! Mario: No! I mean they tend to one point. Many of the teachers said at the same time: There is a tendency to one point. Juan: I think that is really happening… the hexagon theoretically never stops being a hexagon, and then the area never becomes zero, really, there is an asymptote…

The idea of approaching the limit captivated the attention of the whole class during several sessions. In another activity similar to the hexagons, but this time with triangles, participants tried to give a meaning to the sign " = " when working with limits.

The sign " = " versus the intuitive idea of approaching

In a general discussion, teachers used the following notation $\lim_{n\to\infty}\frac{1}{2^{2n}}$ when solving a task involving an equilateral triangle divided into four equal equilateral triangles and so on. The professor demanded that participants explain what the notation means and what is the limit. Some teachers said that the limit is 0. The professor told them that Victor's group disagreed.

Víctor: "The idea is that it tends to 0, the idea of tend to 0 is, that is, very near 0, and that is what we use to denote that, it is the idea we used for infinity, then, we use that about infinity, we used that to find a number, certainly we are not finishing with the construction of, to calculate all the areas ... that number we are approaching as the value, we are expressing it as a value to which we are near, that is, that little hole that is remaining is in a way despicable..."

Professor: When you said "little hole", do you mean a little triangle?

Víctor: "about that, we are imagining it remains, always, but this is always so despicable that is not worthy to take into account".

From this point, a general discussion ensued about the role of the equal sign, in which participants said that it is preferable to exchange the equal sign with an arrow to indicate approximation. This kind of discussion recurred often in others sessions as well.

What is important to highlight here is that what has been highlighted in the history of mathematics as an obstacle, is appeared in this context as a natural way to construct the actual infinity concept. This first part provides the reader with the tone of the discussions in class. The professor's attitude was always to confront ideas rather than to provide answers (scientific debate methodology). Participants' conceptions of approaching a limit was related to potential infinity, and it led them to reject the use of the equal sign.

Individual work

After the third session, teachers were asked to write a definition of limit as part of their individual work. Victor wrote:

"For me, the expression $\lim_{x\to a} f(x) = L$ means that the value of the function f is approaching as near as you wish a value 'L' taking values nearer 'a', that is, L is a value that NEVER is reached, but we are very near it, as near as our mind can imagine. In a practical way, the value 'L' represents an ideal, something to which we want to arrive if there were 'a last' value…"

If we compare the two definitions, the initial answer in the questionnaire and this one, we can observe that Victor's intuitive idea remains; that is, the idea of approaching something without reaching it.

Work in the group

Some sessions later, in class, we were discussing the idea of convergence. Adrian explained his intuitive idea of convergence and his corresponding graphs, and Mario gave a graphical example of a convergent oscillating sequence. After this, the professor asked participants, once again, to write down a definition of convergence. Victor, explaining his ideas to Pablo, said and wrote the following:

*"We consider a sequence a_n, and its limit L, we can put some points here but not necessarily those which that we usually we draw but as the last example [given by Mario] that is going down, and some times could be a bit further, but approaching to that number [L], we must begin to say it as 'Adrian' did [about a neighborhood]. For every **r** or every distance **r** there exist a number around here [pointing to his graph] and here in this case, from this number [n] all the numbers that we obtain comparing to the distance with **L**, that distance is smaller than **r**.*

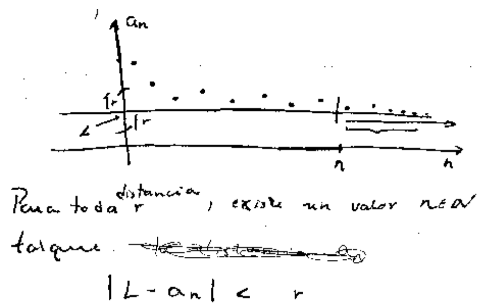

Figure 2

Comparing this definition to the others previously given by Victor, we can appreciate an important change in his intuitive ideas at the beginning and what he expressed in Figure 2. A careful analysis of Victor's oral and written explanations to Pablo suggests that he had a coherent definition in mind. That is, Victor expressed his conception of convergence using different representations (see Figure 3).

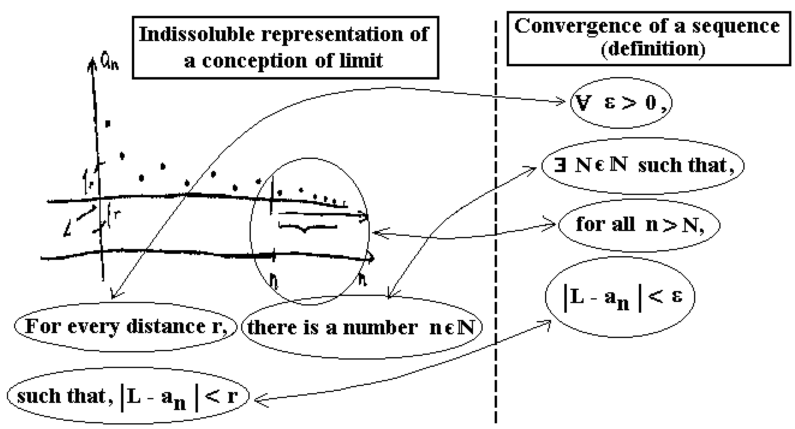

Figure 3

With this, we wish to highlight two things. First, the representations used by Victor emerged in the scientific debate (socio constructivist environment) and in a discussion in his small group. or example, Victor had used, and distinguished among, graphs of continuous and discrete functions to represent sequences. Second, for Victor functional representations when communicating his ideas played an important role in the construction of the concept of limit (see Hitt, 2003). Victor's functional representations are different from what we find in books or similar documents.

Final exam and interview

At the end of the course (after 14 sessions), participants wrote an exam. Taking into account what their overall performance during the sessions, and what they wrote in their exams, we conducted a semi-structured interview with each participant. In the case of Victor, we verified that his construction of convergence was stable. As a final comment Victor claimed that he never consulted a textbook during the 14 sessions because he realized we were doing research work with the group and he wanted to avoid contaminating our experiment. In the interview, we noted that Victor experienced difficulty in negating his definition of convergence.

Discussion

The enriched co-operative learning scientific-debate environment seems to be a powerful tool to for examining and constructing the complexity of the limit concept. The debates characterized in this analysis led participants to socio cognitive conflicts that are not easy to promote in more traditional instructional environments. For example, the discussion about changing a notation (i.e., using an arrow instead of the equal sign) was very important for the teachers. Inciting individual reflection among participants was imperative because the discussions always highlighted different ways of thinking and different arguments to test. The role of the instructor is very demanding in this context, but the results suggest it is worth investing in such contexts, as they seem to support students' learning.

From a theoretical point of view, the functional representations used by teachers when discussing the activities indicate the important role they can play in the construction and reconstruction of mathematical concepts. These functional representations have the property of expressing an idea in a coherent way by mixing language, graphics and algebraic expressions that are different from the representations we usually find in textbooks.

References

Alberti N., Andriani M., Bedulli M. et al. (2000). Sulle difficoltà di apprendimento del concetto di limite. *Riv. Mat. Univ. Parma* (6) 3, 1-21. Alibert D. & Thomas M. (1994). Research on mathematical proof. In D. Tall (Ed.) *Advanced Mathematical Thinking* (pp. 215-230). Kluwer Academic Publishers.

Cajori F. (1915). The history of Zeno's Arguments on Motion: Phases in the Development of the Theory of Limits. *American Mathematical Monthly*, Vol. XXII, pp. 1-6, 39-47,77-82, 109-115, 143-149, 179-186, 215-220, 253-258, 292-297.

Cornu B. (1994). Limits. In D. Tall (Ed.) *Advanced Mathematical Thinking* (pp. 215-230). 2nd edition. Kluwer Academic Publishers.

Duroux A. (1983). La valeur absolue. Difficultés majeurs pour une notion mineure. *Petit x*, no. 3, pp. 43-67.

Duval R. (1993). Registres de représentation sémiotique et functionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives*, 5, (1993).

Eisenberg T. & Dreyfus T. (1990). On the Reluctance to Visualize in Mathematics. In Zimmermann W. & Cunningham S. (Eds.), *Visualization in Teaching and Mathematics*, MAA Series. USA.

Grattan-Guinness I. (1970). *The Development of the Foundations of Mathematical Analysis from Euler to Riemmann.* MIT Press.

Hadamard J. (1975). Essai sur la psychologie de l'invention dans le domaine mathématiqe. Gauthier-Villars, Paris.

Hauchart C. & Rouche N. (1987). *Apprivoiser l'infini. Un enseignement des débuts de l'analyse*. GEM, CIACO Éditeur, Belgique.

Hitt F. & Páez R. (2001). The notion of limit and learning problems. *Proceedings PME-NA* XXIII, Utah, USA, 2001, Vol. 1, pp. 169-176.

Hitt F. (2003). Le caractère fonctionnel des représentations. *Annales de didactique et sciences cognitives*, vol. 8, pp. 255-271.

Patras F. (2001). La pensée mathématique contemporaine. PUF.

Reynolds B., Hagelgans N., Schwingendorf., Vidakovic D., Dubinski E., Shahin M. & Wimbish

G. (1995). *A Practical Guide to Cooperative Learning in Collegiate Mathematics.* MAA Notes Number 37.

Sierpinska A. (1988). Sur la relativité des erreurs. En Compte rendu de la 39e rencontre internationale de la CIEAEM. Sherbrooke, Canada. pp. 70-87.

Tall D. & Vinner S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. Educational Studies in Mathematics, 12, 151-169.

A FIRST-ORDER DIFFERENTIAL EQUATIONS SCHEMA

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This article presents a description of the first-order differential equations schema that developed in an individual, Hassan, through study in a modern differential equations course. His work is contrasted with the work of another student, Rich, who did not show evidence of such a schema. An important component of Hassan's differential equation schema was his ability to view a differential equation as a function. Data was collected through a series of three interviews that included open-ended prompts, a card-sorting task, and nonroutine problems. The theoretical framework to make claims about concepts and schema is based on the work of Skemp (1986). This research suggests more attention should be paid to developing the view of a differential equation as a function in modern courses on differential equations that stress multiple representations.

On the heels of calculus reform, an effort to reform differential equations curriculum and instruction is underway. This modern approach emphasizes multiple solution strategies (qualitative, analytical, and numerical) and is quite different than the recipe driven approach many of us experienced as undergraduates. These new strategies depend on multiple representations of differential equations – algebraic $[dy/dt = f(t, y)]$, graphical $[dy/dt \; vs. y]$ and direction field – and their solution functions.

Through six case studies I set out to investigate the question, what is the nature of students' understanding of first-order differential equations that result from this new approach? A theme emerged in the analysis that distinguished the participants into two general groups, those with relational understanding and those with instrumental understanding. Skemp (1978) defined relational understanding as "knowing what to do and why" (p. 9) and instrumental understanding as "rules without reasons" (p. 9). In Donovan (2002) I wrote detailed descriptions of the understanding for representatives of these groups, Hassan's relational understanding and Rich's instrumental understanding. In this paper, I extend that analysis to describe Hassan's first-order differential equation schema, providing structure to the characterization that Hassan's understanding is relational. Examples from Rich's work are used to provide a comparison.

Theoretical Perspective

The theoretical framework that guided the investigation is constructivist, and is broadly captured by Schoenfeld's (1987) assertion that, "people are interpreters of the world around them. They don't necessarily see 'what's out there' – some version of 'objective reality' – but instead they perceive what they experience in light of *interpretive frameworks* they have developed" (italics added, p. 195). This view raises the question, what interpretive frameworks develop in students that allow them to give meaning to differential equations? Skemp's (1986) notions of schema and concept provide a basis to talk about understanding as frameworks. In Skemp's seminal treatise, *The Psychology of Learning Mathematics*, he talked about schema:

> Now, when a number of suitable components are suitably connected, the resulting combination may have properties which it would have been difficult to predict from a knowledge of the properties of individual components. How many of us could have predicted from knowledge of the separate properties of transistors, condensers, resistors,

> and the like that, when these are suitably connected, the result would enable us to hear radio broadcasts? So it is with concepts and conceptual structures. The new function of the electrical structure described above is marked by a new name–transistor radio. Likewise, a conceptual structure has its own name – *schema*. (italics in original, p. 37)

The definition of schema for this study is a network of connections between concepts. This definition raises the question, what is a concept? Concepts are abstractions formed from classes of objects that share properties. Skemp (1986) states,

> *Abstracting* is an activity by which we become aware of similarities (in the everyday, not the mathematical, sense) among our experiences. *Classifying* means collecting together our experiences on the basis of these similarities. An *abstraction* is some kind of lasting mental change, the result of abstracting, which enables us to recognize new experiences as having similarities of an already formed class. Briefly, it is something learnt which enables us to classify; it is the defining property of a class. To distinguish between abstracting as an activity and an abstraction as its end-product, we shall hereafter call the later a concept (italics in original, p. 21).

He goes on to say, "The criterion for *having* a concept is not being able to say its name but behaving in a way indicative of classifying new data according to the similarities which go to form this concept" (italics in original, p. 26). Using this criterion for "having", or understanding, a concept I will make claims about the concept of first-order differential equation and the concept of solution. The data suggest that for Hassan these concepts are connected and thus form a first-order differential equation schema. Henceforth, I will use differential equation [DE] to mean first-order differential equation.

Methods

This study was conducted at a large state university in the northeastern United States. Participants came from a class titled "Introduction to Differential Equations" taught by a mathematician whose teaching was recognized with a statewide award for excellence two years prior to the study. Participants were selected on the basis of a questionnaire administered during the first week of classes. A series of three different types of tasks, administered in three different one-on-one interviews, were designed to allow participants to be repeatedly observed enacting knowledge about DEs. The data from the interviews, including audio recordings, video recordings, transcripts, and written work, were the primary data for the study. The interview sequence was timed to correspond with the schedule of the course.

The first type of task was a prompt. A DE on an otherwise blank sheet of paper was given to the participant with the directions, "tell me everything you can about it." The process was repeated for the five examples shown in Figure 1. My role as the interviewer was to explore the meaning of the things participants said and did, but not to pose questions that might provide a different stimulus. For example, I did not ask them to "solve" because I wanted to know whether the concept of solution was enacted by the participant in response to the prompt, rather than to the stimulus that might provided by the word "solve." I did not assume that the association between DE and solution would be automatic despite the fact it was fundamental to their studies.

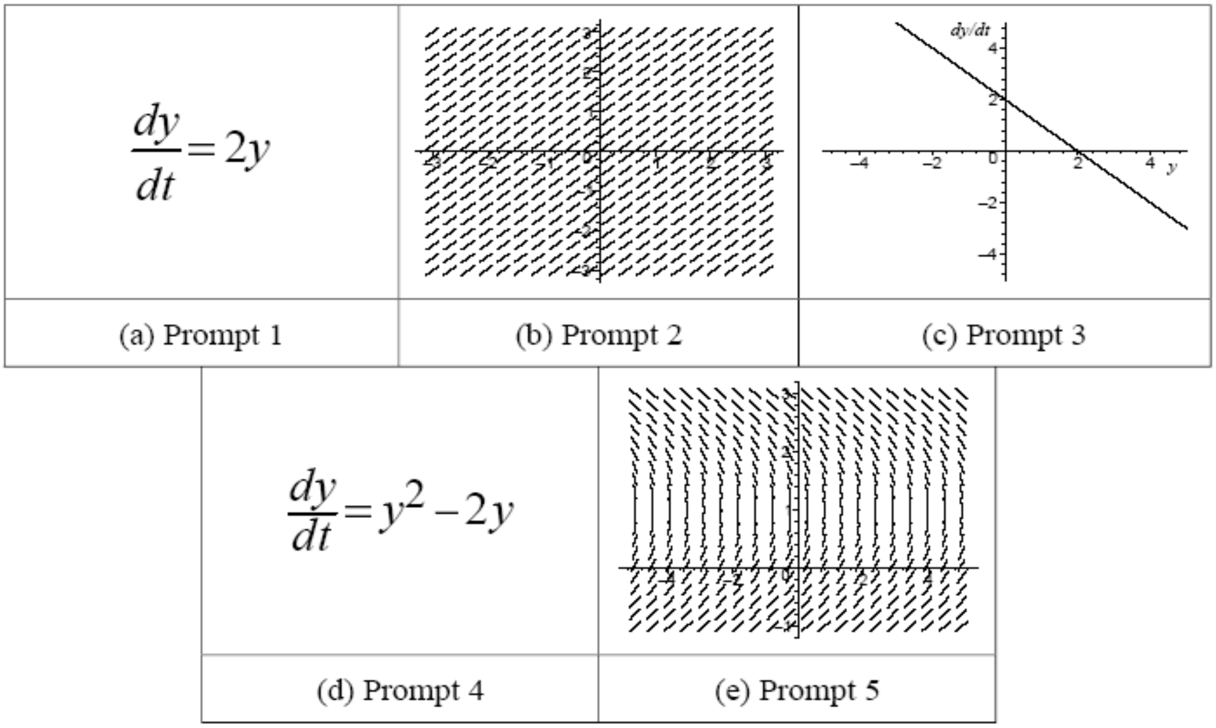
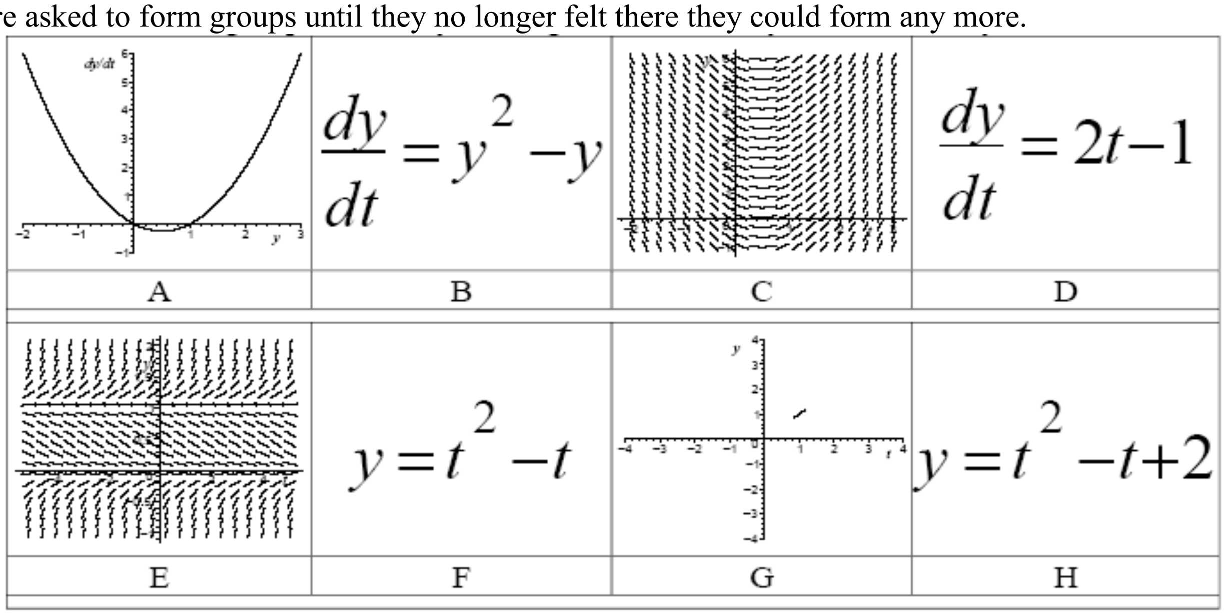


Figure 1: Prompts from interview 1

The second type of task was a card sort. Participants were asked to sort the collection of 20 equations and graphs shown in Figure 2 into groups. Each item was reproduced on a 3 inch square laminated card. This task is similar to the prompts because it is open-ended, although more structured. The groups formed were completely determined by the participant, but the sorts were limited by, and hence structured by, the examples on the cards. Connections exist among the cards, including different representations of the same DE with some of its solutions. My role in this interview was to probe the structure of the groups the participants formed, but again as with the prompts, not to introduce concepts or ideas that might serve as stimuli. Participants were asked to form groups until they no longer felt there they could form any more.



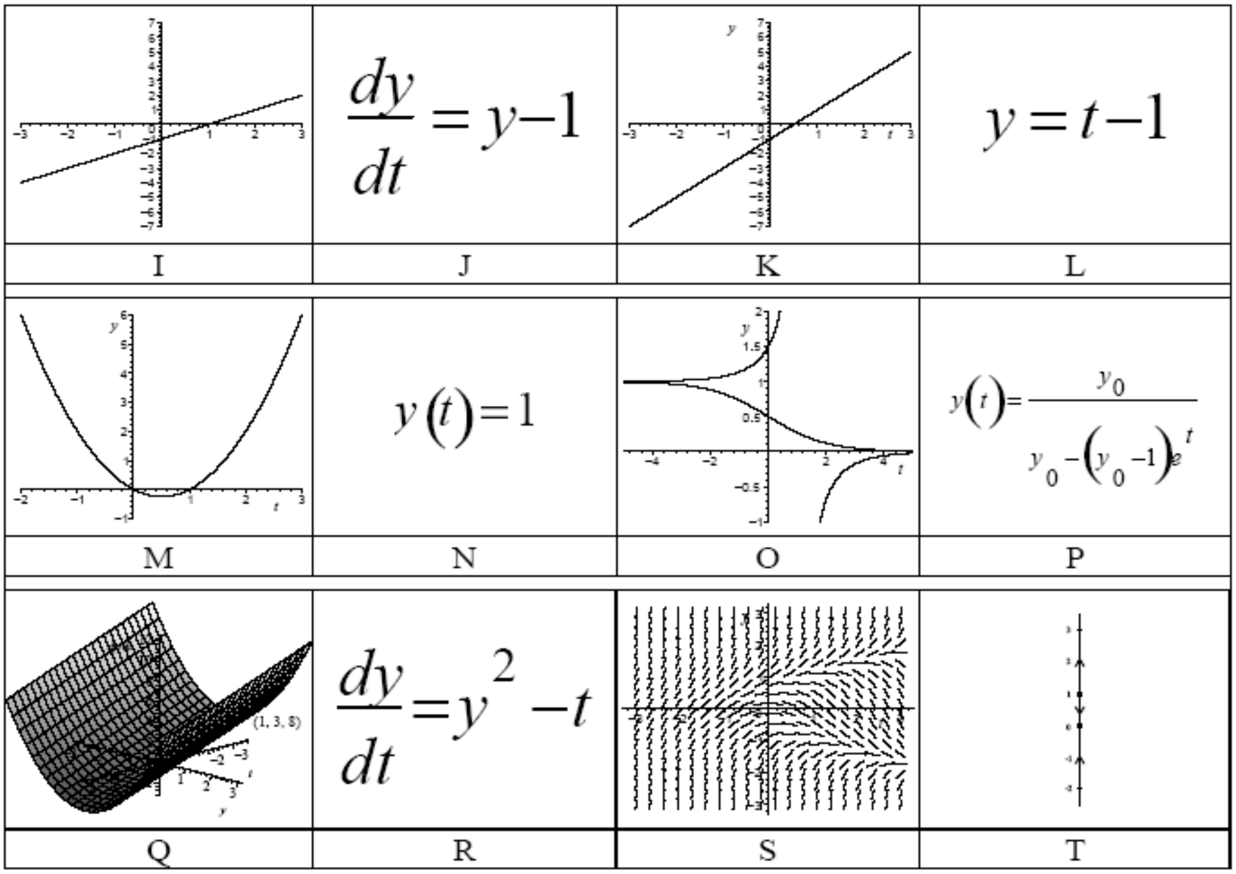


Figure 2: Representations used in card sort

The third type of task was a nonroutine problem; participants attempted four of these. Data to support the claims in this paper come from the prompts and the card sort, thus the nonroutine tasks will not be discussed (see Donovan [2002] for detailed descriptions). Taken together, the interviews offered multiple situations to observe participants' interactions with different representations of DEs and their solutions.

Participants

Bothe Hassan and Rich were in their third year of university study. Hassan's major area of study was mechanical engineering. His pseudonym was chosen to reflect his Pakistani heritage. He is a first-generation American; his parents emigrated before he was born. Neither of Hassan's parents attended college. The most influential reason Hassan was asked to participate was because he responded, "I sleep better knowing math" to a questionnaire item. Hassan enjoys mathematics and solving problems.

Rich's major area of study was mathematics, his favorite subject. He planned to become a high school mathematics teacher. Both Rich's parents were teachers; his father taught mathematics at a middle school. Growing up Rich never thought he would follow his parents into the teaching profession, but changed his mind in college. He was attracted to teaching by job security and his perception that getting a job teaching mathematics after college would be relatively easy.

Results

The protocols I designed for the prompt and card sort interviews required that I only probe things mentioned by the participant because my words might provide alternative stimuli. This was done specifically so the participants had to impose meaning based solely on given equations and graphs. The strength of this protocol is that all the ideas enacted could be directly attributed to the participant's understanding of the given object. For Hassan, the examples provided a jumping off point to explore, and allowed for a detailed account of his understanding. The same was not true for Rich. Throughout the interviews Rich struggled to make meaning of these objects. A reason for Rich's struggles is suggested by a comment he made during the card sort. One of his groupings contained cards he was confident the he would "know what to do if [he] saw that elsewhere." By themselves the graphs and equations had very little meaning for Rich, but elsewhere, i.e. in class and on homework where "what to do" was often implied if not specified directly, he was more successful. Rich was confident in his ability to follow procedures, in fact that was what he enjoys about mathematics.

During these interviews, Hassan treated the different representations of DEs as interchangeable parts, and consistently connected them to solutions. As an example, consider his work on the third prompt (the graph of $dy/dt = y + 2$ shown in Figure 1c). Visually inspecting the linear graph he wrote its equation. He then began to think about solutions. He debated whether he could solve the equation analytically saying, "And so if you want to solve that you can. No you can't. Yes you can. No you can't." Rather than pursue this debate with himself he said, "Lets go like this" and he correctly sketched the solutions using slope information from the given graph. As he considered his solution graphs, a conflict arose in his mind. Intuitively he felt the solutions should be parabolic, but his graphs were asymptotic to y =2 and thus could not be parabolic. He resolved the situation by realizing his original equation was separable which allowed him to find the algebraic solution. From the algebraic solution he concluded the solution graphs he drew were exponential. Hassan also realized why his intuition was wrong, he explained that if the original graph was labeled t instead of y on its horizontal axis, the solutions would have been parabolas. For each of the prompts, Hassan's thought patterns were similar. He pursued solutions for each, and considered different representations as he needed to. He showed a preference for analytical methods, but the above example shows he was equally capable pursuing solutions through qualitative analysis.

Further evidence of the connections Hassan saw among different representations of DEs and between DEs and their solutions can be seen in his work on the card sort task. (Each of the graphs and equations shown in Figure 2 have a corresponding letter; cards will be referenced by this letter.) Grouping L, I, and G together Hassan explained, "they all have the same derivative,… they're all in the same general solution of the differential equation." He wrote $dy/dt = 1$ on a piece of paper and put it with this group. Note, he explicitly assumed the unlabeled graph on card I was a labeled y vs. t. In this example he grouped three linear functions together that shared a common derivative; then immediately realized this meant they were in the general solution of a DE.

Another example from the card sort gives insight into the way Hassan thinks about connections between algebraic DEs and their graphs. Grouping A and B together he said, "dy/dt is just like it's own variable. … It is a function of itself but I just don't think about that, …they just match. …'Cause like say y [pointing to vertical axis on A] and x [pointing to horizontal axis on A], so [pointing to B] $y = x^2 - x$ and that's what that graph looks like." In essence Hassan was ignoring the context of these variable and viewing it as a function with the generic variables y and x. Using this view of a DE as a function he went on to combine the grouping Q and R with the grouping A and B. Hassan concluded A and B were embedded in Q along the line $y = t$.

To better appreciate Hassan's ability to smoothly ignore the context and consider a DE as a function, consider Rich's work on the third prompt (the graph of $dy/dt = y + 2$ shown in Figure1c). He began by saying, "I know that negative slope." He continued saying, "it looks like a differential equation," although he was tentative. The next things he noticed were the coordinates for the graph's intercepts. Rich was out of things to say so I prompted him to come up with an equation (because he introduced the word). He was able to generate an equation, but he struggled. His difficulty centered on his interpretation of the variables. He said, "the *mx* + *b*, …but there's no *x* in here [the graph] so I can't use that." Eventually Rich did come up with a correct equation, but he was hesitant to say it was correct because of the difficulty he had with the variables. Rich had no more to say about this task.

Discussion

The work presented above is representative of Rich's, and Hassan's, work throughout the interviews. Clearly there are vast differences in what was observed, but Hassan's work gives evidence of a DE schema, i.e. a network of connections between concepts. Hassan treated the DEs in these interviews as familiar objects. He consistently linked different representation types and treated them as if they were equivalent. This satisfies Skemp's criteria of "having similarities of an already formed class" and gives evidence that Hassan has a concept of differential equation that includes three equivalent representation types.

A feature of this concept for Hassan is his ability to view a DE as a function, the *functional view,* as decscribed above for his work linking $dy/dt = y^2 - y$ to its graph. Thinking of this DE as a function he saw dy/dt as a dependent variable and y as an independent variable. His statement, "it is a function of itself but I just don't think about that" seems like a recognition of the dual role *y* plays as dependent and independent, but one that he puts aside so he can think of dy/dt as a function. Rich's struggle to develop an equation for the graph of $dy/dt = -y + 2$ gives evidence that the transition to view a DE as a function, to adapt a function schema, is not obvious a priori.

The functional view is subtle, but important. Recognizing the DE as a part of a different class, as an example of a different concept, allows connections to schema that otherwise could be inaccessible. Skemp talks about this "integrative function of a schema" using the concept of "car" as an example:

> When we recognize something as an example of a concept we become aware of it two levels: as itself and as a member of this class. Thus, when we see some particular car, we automatically recognize it as a member of the class of private cars. But this class-concept is linked by our mental schemas with a vast number of other concepts, which are available to help us behave adaptively with respect to the many different situations in which a car can form a part. (p. 24)

In addition to viewing DEs as functions, Hassan also viewed them as objects that needed to be solved. This process of solving links the concept of differential equation to that of solution. First consider his concept of solution. During the first interview he found solutions for each of the prompts. For the third prompt he used qualitative means to obtain "all possible solutions" but still continued to work to find an algebraic solution to verify his graphs were "exponential." During the card sort he recognized cards and classified them as solutions to a DE that was not even among the cards. These examples satisfy Skemp's criteria for a concept, in this case of a DE solution.

Hassan's work on these tasks shows that he understands a concept of DE, which includes two views, and a concept of DE solution. These concepts are linked together through solutions processes and therefore satisfy the definition of schema. The structure of Hassan's schema is represented by the diagram in Figure 3.

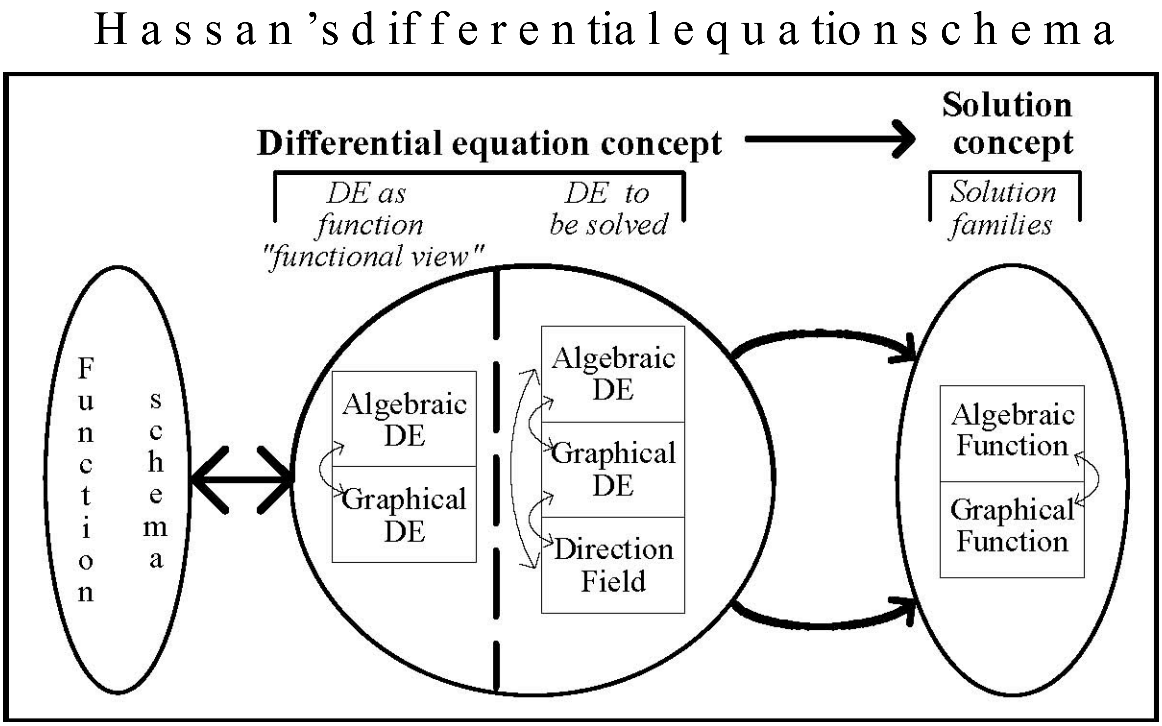


Figure 3: Hassan's DE schema

In the diagram, the large circle represents Hassan's DE concept. The circle is divided into two parts by a vertical line to represent his two views, the functional view and DE to be solved view. The bold arrows are links between concepts, i.e. his solution methods. Different representation types he used equivalently over the course of all the interviews are linked by curves with arrows at each end. The oval on the right side illustrates his concept of solution, what he referred to as "general solutions." The thin oval on the left shows his function schema, which is accessible through the functional view.

Conclusions

The formation of schema for growth and understanding is extremely important. According to Skemp (1986), "[t]o understand something means to assimilate it into an appropriate schema" (p. 29). At this level of mathematics, the development of new schema relies upon the relationships of other existing concepts and schema. For Hassan's DE schema, his function schema is a key component that allows him to move easily among different representations of DEs and their solutions. His ability to adapt a function view has implications for modern curricula. The fact that a DE is a function should be exploited to help students adapt their existing knowledge about functions.

Questions remain to be answered. What other concepts are connected to Hassan's differential equation schema? Analysis is underway to identify these concepts. How does this schema develop? With regard to concepts, in particular the concept of differential equation, what are the similarities that define a class? Does this schema model generalize?

References

Donovan, J. E., II. (2002). Students' understating of first-order differential equations. *Dissertation Abstracts International, 63*(08).

Schoenfeld, A. (1987). What's all the fuss about metacognition? In A. H. Schoenfeld (Ed.), *Cognitive Science and Mathematics Education*. Hillsdale, NJ: Erlbaum.

Skemp, R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher, 26* (3), 9-15.

Skemp, R. (1986). *The Psychology of Learning Mathematics.* New York: Penguin.

INTRODUCTION OF ‘FLEXIBILITY’ OF MATHEMATICAL CONCEPTIONS AS A LEARNING GOAL

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This Ph.D. project is rooted in the research-development project “World Class Math and Science” in upper secondary school. The main frame of the project is to study how the design of teaching with laptops can support the students’ investigative work, which in the project is linked to the notions of mathematical competencies, modeling and Realistic Math Education. A definition of ‘Flexibility of conceptions’ is introduced, based on teachers’ experiences and on literature. An empirical study of four teachers’ computer based teaching of differential equations is carried out with the aim to 1) identify signs indicating flexibility 2) interpret the influence of flexibility on the students’ further working process and 3) inquire how the elements of flexibility are provoked. Conclusions will be drawn concerning the correspondence between flexible conceptions and acting with competence. Design guidelines will be prepared for teachers.

Objective of the Project

During the first three years of the ‘World Class Math and Science’ project, the participating teachers and students have shared the experience that the use of CAS (Computer Algebraic Systems) may support the students’ learning processes by giving a better overview and a deeper understanding. (Andresen et.al. 2002 &2003). These experiences are shared with other teachers. The Ph. D. project includes a closer examination of this phenomenon as a central part. The purpose of the inquiry is double: Insight in these learning processes may contribute to the understanding of learning mathematics and ad knowledge to educational theory, and the theoretical and empirical analyses and studies of the phenomenon lead to the development of auxiliary educational concepts, useful as tools for the design of teaching that supports overview and deeper understanding. In this way, the gains in educational theory may serve as foundations for an improved teaching practice. Further, it is inquired whether some potentials of using CAS may be understood in terms of flexibility.

Therefore, there are two main objectives of the research project:

-The project aims to link teachers’ professional experiences to scientific knowledge by articulation of and conceptualising the phenomenon of students’ getting a better overview and deeper understanding when using CAS, according to themselves and their teachers. In the project, I attempt to make this phenomenon tangible and scientific legitimate by the introduction and definition of ‘flexibility’ of mathematical conceptions. In the following, the term flexibility denotes this special concept, which is claimed to capture the phenomenon.

-By setting up the concept of flexibility as a learning goal, I aim to develop a guiding tool for experienced teachers, useful in their design of teaching with and without CAS. This aim follows the claim, that rather than rigid schemes or standard prefabricated teacher-proof teaching, the teachers may benefit from a scaffold of educational theory and concepts, serving as a starting point and support for their own reflections. In this sense, the introduction of flexibility as an auxiliary educational concept aims to offer support to teachers’ professional development.

In the project, the claim is inquired that teachers may plan, analyze and reflect on teaching sequences in terms of flexibility and thereby over time support the students acting with mathematical competencies.

Perspectives and Theoretical Framework

As the definition of flexibility is rooted in teaching experiences and modified through analysis of episodes of practice and reflected in educational theory, it may be looked upon as a result of a process reminiscent of didactising in the sense Freudenthal uses the term (Freudenthal, 1991, p.45). The process is carried out by me in collaboration with teachers, although the teaching experiments were not designed, aiming explicitly to try out the concept of flexibility. Not mutually exclusive elements of three different approaches are implied in the definition: A structural point of view as seen by for example A. Sfard (Sfard, 1991), D. Tall (Gray & Tall, 1994) and E. Dubinsky (Dubinsky, 1991), a tool-or model- approach as seen by for example M. Blomhøj (Blomhøj 2003) or J. Confrey (Confrey & Costa, 1996) and the division of the expressions of mathematics into 3 representations, which is implied in the representational point of view taken by for example J. Kaput (Kaput, 1989).

The following two basic views are underlying the definition of flexibility:

-Mathematics is a human activity (Freudenthal, 1991, p.14)

-Learning is a process of constructing and modifying conceptions (Cobb, Yackel & Wood, 1992, p.2).

In consequence, the structural-, model- and representational points of view, rather than being base of static descriptions contribute to frame the elements of the students' social and mental activities. There is no built-in cognitive hierarchy in the concept of flexibility: all the processes it includes are reflexive. This does not imply that no processes should ever prerequisite others but is meant to keep the options open for reversions of thinking processes which might support the learning. The concept of flexibility aims to capture the dynamics of doing and learning mathematics, which is called 'acting with mathematical competence' in the KOM-report (KOM 2002). In the definition of flexibility, the construction of epistemic knowledge is distinguished from the construction of pragmatic knowledge in the sense described by P. Verillon and P. Rabardel (Vérillon & Rabardel, 1995). This distinction is made to prepare for an instrumental approach, in accordance with the theoretical framework presented by M. Artigue in (Artigue 2002) but till now this idea has not been further developed in the project.

There is an intended correspondence between the mathematical competencies stated as educational goals in the KOM-report (KOM 2002), and flexibility as a learning goal: The flexibility as well as the competencies is stated independently of the special topic and of the educational level in question and both relate to the potentials and abilities of the individual student. Further, the key elements of the definition of flexibility are claimed by the author to cover a mental readiness of action needed for the capture, construction and use of mathematical ideas, topics, applications and models. Such mental readiness links flexibility as a learning goal to the educational goals articulated in terms of competencies. The inquiry of the correspondence between competencies and flexibility is one main subject of the research project.

Definition

The flexibility of a mathematical conception constructed by a person is the designation of all the changes of perspective and all the changes between different representations the person can manage within this conception.

This definition is worked out in more details, with references, in M. Andresen: *'Flexibility of mathematical conceptions'* (Andresen, 2004).

Changes of Perspective

The term ‘changes of perspective’ in this definition denotes changes of view in conceptions. Use of the term is not restricted to changes of view for instance on models or changes, caused by ‘variation of the problem’ during problem solving, as described by G. Polya (Polya, 1945, p 209ff).

The changes of perspective, considered as changes within complementary pairs or dualities, should not be seen as a categorisation of all kind of changes. The group of dualities is meant to cover the field of mental operations of change. A mathematical concept may be exposed in more than one way in a given perspective (or representation), as the intention is not to establish any one-to-one correspondence between mathematical conceptions and their exposure in the different perspectives. In this comprehension, flexibility includes the following key elements:

Dualities of perspectives intrinsic to mathematics

1. Local and global position
2. General - specific
3. Analytic- constructive

Dualities of perspectives linked to construction of epistemic knowledge

4. The process - object duality.
5. Situated - decontextualised

Dualities of perspectives linked to construction of pragmatic knowledge

6. The tool - object duality
7. Model - reality
8. Model of - model for

Changes Between Different Representations

Three main representations are considered: graphic representation, analytic representation (or formal language), and natural language. A fourth, called technical representation (or computer language) is included as well, caused by the use of laptops. Each representation is split in ‘external representation’ and ‘internal representation’. This distinction may be stressed or not, depending on the context.

Methods and Modes of Inquiry

The inquiry is purely qualitative and not comparative in accordance with the type of questions it addresses. The main questions may be formulated like this:

-Does the definition of flexibility as an educational concept capture the phenomenon, which it was intended to articulate? And further, is it possible to identify visible or otherwise tangible signs of flexibility, underlying the demonstration of mathematical competence and/or linked to the use of CAS? An empirical study is carried as mentioned below.

The conclusions of the inquiry and an adapted version of the concept of ‘flexibility’ including some guiding materials will be presented to the teachers in a new established CAS – network group with the purpose of a try out.

Data Sources

The materials partly based on P. Blanchard et.al: ‘Differential Equations’ (Blanchard 2002) and Morten Blomhoej et.al: “BASE Note 1+2” (Blomhoej 2001), were prepared by three teachers, participating in the ‘World Class’-project. The use of Derive ver.5 is prerequisite. The main characteristics of the teaching materials and the teaching are described as follows by the teachers: 1) It takes a ‘dynamical systems’ point of view, 2) The qualitative, numeric and analytic approaches are presented and modelling as a tool for learning is incorporated throughout, 3) The students are gradually given more open ended and complex tasks, 4) It aims to support the students' knowledge of types of models, their ability to recognize models and their competencies in building them as well.

The materials were tried out by one of the authors of it and by three colleagues at different schools, all participants from the 'World Class'- project. At that time, the students had used the software for two years. Almost all the lessons (22) in one of the classes and sequences of lessons in the other three were observed (50 lessons in all) and data collected by me, including videotape recordings, field notes, copies of student works and interviews with the authors and groups of students. The data are or will be analysed with the purpose of identifying signs of elements of flexibility in the students' activities and to enlighten their role in the students' working process. Further, data are or will be inquired to find out how the specific changes of perspective or between different representations are provoked by the teacher, the task, the dialogue etc.

Data Episodes Showing Flexibility

EXAMPLE 1: An indication of flexibility: changes from object to tool and from formal language to graphic representation.

In the following scene A, the students' ability in combining different pieces of knowledge seems to depend on the flexibility of the concepts involved. The scene is cut from a lesson in the last year of upper secondary school. The students have been working on the subject differential equations the last couple of weeks. The lesson concerns on proving existence and uniqueness of the solution of the differential equation $\frac{dy}{dx} = b - ay, a > 0$. An unknown solution *f(x)* is considered and the auxiliary function $g(x) = \frac{b}{a} - f(x)$ is introduced.

Scene A, concerning the general solution

A student (S1) asks: How do you find the g?

Teacher (T): It happens incidentally - that is, when you try a little with this and that you will find out that this one is convenient... When making this type of a proof you know the answer before you start - (.....)

S2: And it is the same way in the statement: "The general solution is constituted by..." or are we allowed arguing that you can se it from the slope field?

In the classroom, the conversation concerns handling the analytic expressions. S1 wonder if the auxiliary function g is supposed to be taken as shared knowledge. The teacher has to explain that it is not. The explanation reveals the fact, that the proof is result of antididactical inversion. S2 associates from this situation of one knowing the auxiliary function before the proof is designed, to the situation of one knowing the solutions from a statement. The later situation is well known for students who are familiar to teaching materials that presents the topics in antididactical inversion. The two situations are linked together through the student's experience of a certain structure (antididactical inversion) of the topics. Then S2 changes his position from 'knowing the solution from a statement' to 'knowing a solution from the shape of the slope field'. This change involves a change from analytic- to graphic representation. Further, the student's question reveals his knowledge of how to find out what a solution is like. This concerns a change from tool- or maybe from process- to object perspective of solution of differential equations.

In this case, the student's ability of changing between the two representations and changing perspective from object to tool enables him to learn about "rules for arguing" by connecting two different types of arguments.

The change was not provoked intentionally, but the teacher might pick up the thread, for example by asking: What other sources for knowing that f is a solution do you know?

EXAMPLE 2, Shift from general to specific perspective on a solution of a differential equation:

The episode is cut from the analysis of a case study in the project concerning the modeling of transformation of cholesterol in the body. Parts of the data is shown in the following. The Task include several questions, concerning the differential equation:

$$\frac{dC}{dt} = 0.1(265 - C) \qquad C(0) = 180 \tag{8.2}$$

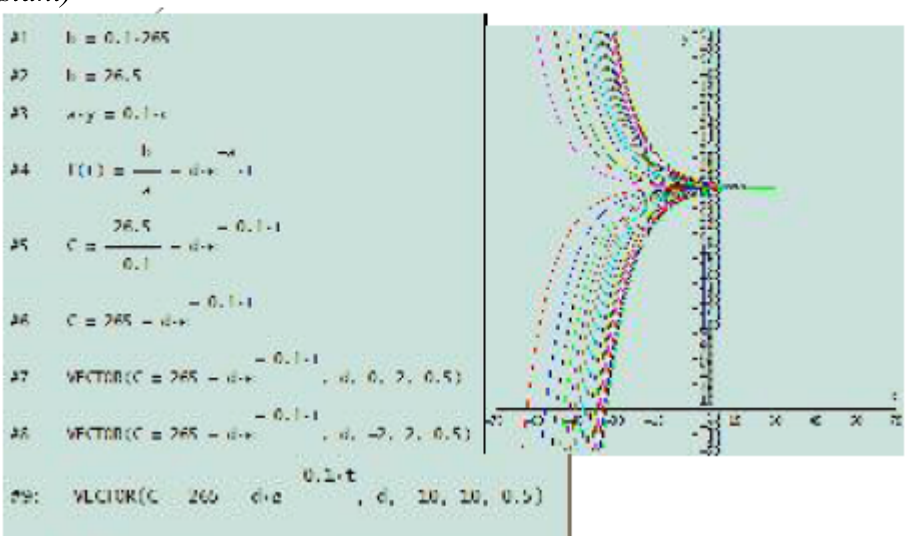
Question 2. Find the general solution to the differential equation (8.2) (Show calculations)

Question 3. Find the specific solution to the initial value problem (8.3)

In the report from group 3, question 2. is answered as follows (5), l. 50-64,one graph omitted):

General solution:

The equation for cholesterol is of the type dy/dx=b-ay and may be solved as follows: (b is a constant)



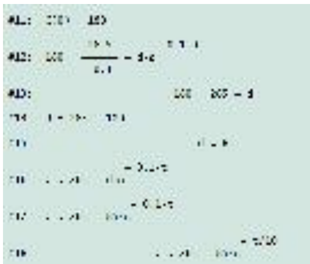
From the transcription of the video tape, recorded while the group answered the question 2 above, these remarks are cut (7)

..P3: What is it called? It isn't called a name?P1: Yes it is called a name. P2: I don't think it is called any name here. P3: (reads) ..of the type, it says. P2: Yes. ...P2: (looks in the compendium) they are not called any names – then leave it! P1-3: Yes

In the same report, question 3. is answered (5) line 65-76):

Find the specific solution to the initial value problem.

This problem is solved by substitution of the point into the equation for C and isolation of d. The result is then substituted in the original equation



Cuts from the transcription of video tape, recorded while the group answered the above question 3, (7):

..P1: We now have a point called zero point … hundred and eighty P3: You can substitute it here .. find d P1: Or.. yes, it is P2: It is the same (types in)P1: I would substitute it in an ordinary Runge Kutta P2: (points at the screen) this one P3: No, because you need some values P1: (points) You may substitute as…here and you should substitute this as t. This one P2: Yes.

Analysis of Data in Example 2:

The students' answers to both questions in the report are rather short with hardly any explanations. In the transcription, the communication between the students is strikingly poor in the use of technical terms, negotiations and considerations too. Rather than by talk and writings, their communication is mediated by the screen and the possibilities of Derive, that is, by the tool use. In the report, the graphs are not the direct result of the commands: The command VECTOR(C = 265-….) for example, will not result in graphs as shown, as far as 'C = 265…' is evaluated logically. To answer question 2 about the general solution, the students find the formula in the compendium of formulas. The transcription-cut shows, that the students are looking for anchor points or starting points for acting. The status in the students' minds of the graphs is not clearly indicated: The general solution is not underlined as a result and the graphs are not stressed as examples or specific realizations. It follows, that the students do not explicitly demonstrate understanding of the general perspective. To answer question 3, one of the students suggests the use of a numerical method, ignoring the fact that they have just found the general solution. In this suggestion there are no signs indicating understanding of the general perspective on solution too. Though, the suggestion shows a change from general to specific perspective on the conception of differential equation. Another group member runs the answering, using substitution into the general solution, without negotiating the alternative. Negotiating the two alternatives might have been fruitful by causing overall reflections changes between general and specific perspectives.

How are the changes provoked?

The questions 2 and 3 in the task intend to provoke the changes from general to specific. The questions do not reveal the connection between general and specific solutions, and no hints are given. Nevertheless, the guidance may seem obvious to experienced students.

What role do the changes play?

As the motives for finding the general solution in question 2 and the specific solution immediately afterwards in question 3 are not made explicit, a process perspective rather than a tool perspective is realized in the task. The intended change from general to specific may let the students experience the generative character of the general solution in an epistemic way.

Concluding remarks, episode 2:

The use of a general solution as a tool for generating specific solutions may be added to the students' mental readiness, through experiences with this kind of tasks. In this episode, no reflections on the possibility of generating a special differential equation from the general case, or on the correspondence between these two alternative changes from general to specific were provoked. Such reflections might have been the result of negotiation and discussion in the group, as far as one of the students suggests the alternative solution path. In that case, problem solving strategies involving the generation of specific equations from general, generating specific solutions from general and comparison of such alternative solution paths might have been added to the students' mental readiness for solving differential equation problems with the use of derive-like tools. This goal might have been reached in the actual situation by a closer guidance. The reason for not reaching it seem to be the insufficiently development of the common technical language in the group. Rather than by natural or technical language, the students' communication within the group, and in the report, is mediated by the tool use in this episode.

Remarks

One feature considered by students and by teachers as an advantage is the easy generation of series of graphs and the easy substitution of different values in expressions for the purpose of checking. The terms of flexibility are suitable for the discussion of this issue: The episode demonstrates how the former two facilitate easy changes between general and specific and changes to graphic representation. Future analyses of episodes in terms of flexibility might show how the tool offers a shortcut to the object perspective. The analyses of the data examples demonstrate that the terms of flexibility are suitable for the discussion of episodes, not involving CAS, and for discussions of obstacles and of potentials, which were not realised.

Endnote

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References

Andresen, M, Pawlik, E. & Winther, A.: 2002 and 2003: Evaluation reports (in Danish): Indsatsområde A: Højt niveau i gymnasiet på matematisk linje, Learning Lab Denmark.

Andresen, M. 2004: "Flexibility of mathematical conceptions*" ICME 10 conf. proc. TSG 15*

Artigue, Michèle 2002: "Learning Mathematics in a CAS environment: The genesis of a Reflection about instrumentation and the dialectics between technical and conceptual Work" in *International Journal of Computers for Mathematical Learning 7*

Blanchard, P. et.al. 2002: "*Differential equations*" Brooks/Cole, Thomson Learning USA

Blomhøj, M. et.al. 2001. "*BASE Note 1+2* " Roskilde (math notes)

Blomhøj, M. & Jensen, T. H. (2003): "Developing mathematical modelling competence: conceptual clarification and educational planning" in *Teaching Mathematics and its applications: An international Journal of the IMA* 22.3 pp 123-139

Cobb, P., Yackel, E. & Wood, T. (1992): " A Constructivist alternative to the representational view of mind in Mathematical Education" in *Journal of Research in Mathematics Education* vol23 no 1, pp 2-33

Confrey, Jere & Costa, Shelly 1996: "A critique of the selection of "Mathematical objects" as a central metaphor for advanced mathematical thinking" in *International Journal of Computers for Mathematical Learning* 1: pp 139-168

Dubinsky, Ed. 1991: " Reflective abstraction in advanced mathematical thinking" in D. Tall (ed):"*Advanced mathematical thinking"*, Kluwer

Freudenthal, Hans (1991): "*Revisiting mathematics Education. China Lectures*" Kluwer

Gray, E. & Tall, D. 1994: "Duality, ambiguity and flexibility: A 'proceptual' view of simple arithmetic" *Journal for Research in Mathematics Education* vol 25. N2, 116-140

Kaput, James 1989: “Linking Representations in the Symbol Systems of Algebra” in C. Kieran & S. Wagner (eds.) “*Research Issues in the Learning and Teaching of Algebra*” Erlbaum

KOM 2002, KOM-arbejdsgruppen: “*Kompetencer og matematiklæring*” IMFUFA, RUC (The CaLM-working group:”Competencies and Learning Mathematics”, Roskilde University)

Polya, G. 1945: “*How to solve it*”, Princeton

Sfard, Anna 1991: “On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin” in *Educational Studies in Mathematic* 22

Vérillon, P. & Rabardel, P. 1995: “Cognition and artefact: a contribution to the study of thought in relation to instrumented activity” in *European Journal of Psychology in Education*, volt IX n 3.

UNDERSTANDING THE MEANING AND REPRESENTATION OF STRAIGHT LINE SOLUTIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS

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This study has as its main purpose the analysis of student responses to questions related to their understanding of the meaning and representation of straight-line solutions of systems of differential equations. The theoretical framework used in the design of instruments and for the analysis of students' responses is Action-Process-Object-Schema (APOS) theory, in particular, the notions of triad and double triad. We present the analysis of students' responses to questions involving the linearity theorem in the context of systems of linear differential equation and the geometric representation of straight line solutions to these systems. Students responses provide evidence of students difficulties to relate concepts coming from different areas of mathematics even when they are able to apply solution methods without problem. Some instructional activities that seem to be successful are suggested.

Previous research results have described students' conceptions of the solution of differential equations and of systems of differential equations (Whitehead and Rasmussen; 2003; Rasmussen 2001; Trigueros, 2001; Habre, 2000; Trigueros, 2000). Results in all of these studies have shown some of the difficulties and misconceptions students have to interpret and represent solutions of differential equations; on the other hand, there are research studies on the integration of information in the solution of complex problems in calculus which suggest that the description of students' strategies when solving these kind of problems can be made in terms of the interaction between different schema (Baker, Cooley and Trigueros, 2000; Cooley, Trigueros and Baker, 2003). This ideas have been taken into account in other studies on differential equations where the interaction of two main schema: a schema for the representation of functions in parametric form, and a schema for systems of differential equations has been used to describe difficulties and strategies of students when they solve systems of differential equations. (Trigueros, 2001; Trigueros, 2000).

An important theorem related to the solution of linear autonomous systems of differential equations and of higher order linear equations is the linearity principle or principle of superposition which links concepts of linear algebra to those corresponding to differential equations. One interesting problem, related to the way students work when integration of different concepts, representations and tools is required is the study of how students relate the linearity principle to the methods for finding, analyzing, and representing solution curves of systems of differential equations. This study addresses this issue. In particular, this study tries to address the following research questions: How do students interpret the superposition principle when solving systems of differential equations? How students interpret straight line solutions of systems of differential equations?

Theoretical Framework

This research study uses the notions of triad and double triad for schema from APOS, in order to analyze student responses to questions related to their understanding of the meaning and representation of straight line solutions of systems of differential equations. A Schema for a concept in this theory is defined as the collection of actions, processes, objects, other schema and the set of relationships that have been established between them and that represent for the individual a coherent structure. The coherence of the structure is demonstrated by the student's possibility to determine the situations where the schema is applicable.

In this study, we consider that two schema interact in the solution of problems related to systems of differential equations. We refer to them as the parametric representation of functions schema and the solution schema.

The development of the schema parametric representation of functions involves the construction of relationship between function as a vector, parametric representation of functions, curves, derivative of functions, tangent to a curve, and can be described by means of the following stages: At the Intra-parametric level the student can interpret the parametric representation of functions in terms of the meaning of each of its components as a function of the parameter, but is not able to relate those functions with their geometric representation, the possibility of elimination of the parameter and the representation of a function in a two dimensional plane, where the parameter is not explicitly shown causes confusion. At the Interparametric level, students still show difficulties when interpreting functions as vectors and their graphic representation. At the Trans-parametric level, students are able to describe the function and its different representations in terms of the parameter involved. Coherence of the schema is demonstrated by the student's ability to describe which parametric representations are possible for a given function and how they relate to different graphical representations.

The development of the schema for the solutions of systems of differential equations involves the construction between solution as a function, solution curve, the process of finding the solution and its relationship to concepts in linear algebra, and can be described by the following stages: At the Intra-solution level the student is able to solve a system but is unable to interpret the meaning of the straight line solutions obtained with all the solutions except for being capable to write the analytical expression, and to coordinate the system and its solution with its graphical representations. At the Inter-solution level, the student is able to assign meaning to the straight line solution but difficulties to relate his or her conception of solution to the properties of a base of the vector space persist, and it is not clear to him or her when a particular representation is convenient or even possible for some systems of equations. At the Trans-solution level, students are able to solve, interpret, and describe graphically different systems of differential equations, they can also establish the relationships between the straight line solutions of a system and the properties of the system and those of a vector space. Coherence of the schema is demonstrated by the student's ability to discriminate between systems where the use of analytical methods is more appropriate from those for which graphical representations are better suited and by their ability to determine when and why the solutions of a system form a base of a vector space.

Methodology

Data were collected from an applied dynamical systems course at a small private university. The course was taught to 38 Economics students, and emphasized both the analytical and geometrical representations of solutions of linear and non-linear systems, as well as the theoretical aspects related to the solution of linear systems. A semi-structured interview on systems of differential equations was conducted to all the students of the group and an open questionnaire including some of the questions of the interview was completed by all the students before the interview to demonstrate the consistency of their knowledge. The interviews were audio-taped and transcribed, and all the work done by the students was collected. The analysis of the data from the interviews was compared to the analysis of the responses to the questionnaire for validation purposes.

The focus of analysis was on student understanding of straight-line solutions of linear systems of differential equations, including both their analytical and geometrical representations. Students responses were classified according to the characteristics that were assigned to what we describe as their schema development using the double triad described before. The questions in the interviews and questionnaire were specific examples of the following type of items: 1) Given the general solution of a linear autonomous and homogeneous system of two differential equations with constant coefficients, what is the meaning of the two parts of the solution? And how each part can be represented in the phase plane?, 2) What is the meaning of the constant in the exponent of the solutions of the system and of the constant vectors that appear in the solution?, and how are they related to the behaviour of the solutions in the phase plane?, 3) Given the representation of the straight line solutions of a system in the phase plane, draw other solutions of the system and explain how you do it.

Results

The classification of students was made on the basis of the analysis of their whole work. Here we can only show parts of the interviews that we think illustrate the use of the framework and students understanding.

When the students' various schema development levels were classified using the double triad, they were distributed throughout all levels, with a larger number at the intra-solution or intra-parametric levels. Only one student was classified at the trans-parametric, trans-solution level of the triad. This is probably a consequence of students' tendency to memorize procedures, as well as the inherent difficulties in relating concepts from calculus with concepts from linear algebra.

	Intra- Parametric	Inter- Parametric	Trans- Parametric
Intra- Solution	12	6	1
Inter- Solution	7	5	2
Trans- Solution	1	2	1

This results show that students have difficulties when relating the different concepts studied in the course. Most of the students were able to solve linear systems of differential equations. However, their understanding of the meaning of what they found was not as deep as expected.

Students at the Intra- parametric – Intra- solution level were not able to distinguish between straight line solutions and nullclines. This is evidence of a lack of coordination between their schema for functions, where function as a vector and the links between function and derivative seem to be missing, as is the relationship between the analytical representation of the function and its graphical representation. These students only show a very weak coordination between the concept of solution of a system of differential equations and its properties. They can explain certain properties, for example, what happens to the solution when the independent variable tends to infinity from the interpretation of the sign of the eigenvalues of the matrix associated to the system, but their explanations are based on memorized facts, so when the analytical representation of the solution is not presented to them, they cannot make sense of the behaviour of solution curves. The following excerpt shows a typical response of students in this group.

I. Suppose you solve a linear system of differential equations X' = AX, and you obtain the solution $X' = k_1 \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t}$, what is the meaning of the two terms of the solution? And how would you represent each part in phase plane?

S. The solution gives two functions, one is x and one is y, to give them a name…and… you mean the meaning of the terms and draw them…how would I draw them? Well… I would go back to the system and I would make x'=0 and y'=0 to find the equilibrium solution and the places where each of them is zero…but I don't have the system, I don't have an expression for x' and for y', and… well no, I cannot do it…I don't know how.

I. OK that procedure helps you to know what is the behaviour of solutions when you know the system, but here you already know the solution to the system, how can you use it?

S. I am not sure… but I think that when x'=0 and y'=0 …I am confused, maybe I did not understand the question.

I. Let me ask you something else and we will come back to this later. Can you explain to me what are the straight line solutions of the system?

S. Yes… they are the nullclines, where the derivative is zero…

I. and how are they related to the solution of the system?

S. They help you … they divide the plane in regions where the sign of the derivative is different, then you find that, and you know how to draw the solution curves

I. How do you do that?

S. Well I need the expressions for x' and y'

I. OK let´s go back to the question we were discussing. How would you graph this solution?

S. Well I know that this is six and this is minus two, different signs, so the equilibrium point is a saddle point. And… as I don't have the x' and y', I only know the components of this X, so I can draw that with the calculator and find how x behaves and how y behaves.

I. Fine, and once you have that how would you draw the solution curves in phase space?

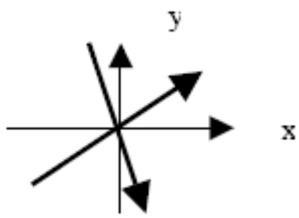
S. …(long pause) no… I really don't know how to do that, as I told you I think I need more information to do so.

At the other extreme of the evolution of the interaction between the two abovementioned schema, the student at the trans- parametric – trans- solution level in this description, show in her explanation a deeper understanding of the meaning of parametric representation of functions, of the relationship between different representations of the solution function, and of the linearity principle and the concepts that come from linear algebra. She is able to explain the behaviour of solutions for degenerate systems and shows the coherence of her schema when she can explain that the linearity principle is valid only in the case of solution of linear systems of differential equations, and that phase plane is useful only when the systems are autonomous.

The distribution of the students in the table shown above makes the complexity of the problem of understanding the meaning of the usual procedure for solution of linear systems of differential equations more apparent. Even when the students are capable to follow the solution algorithm, their understanding of function as a vector, or parametric functions, or the relationship between the derivative, also as a vector, and the properties of functions was found to be superficial. The nature of their conceptions of all of this and of the qualitative graphical tools used to understand the behaviour of solutions, and the concepts associated with properties of vector spaces, can foster or inhibit a real understanding of what is meant by the solution to a system of differential equations.

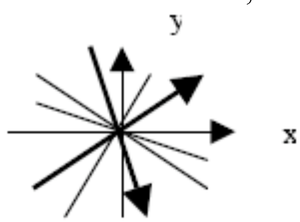
Students at the Inter- Parametric, Intra- solution level, for example, are students who can work with parametric functions, can use the analytical representation of the solution to plot solutions in *x(t)* vs *t* graphs and *y(t)* vs *t* graphs, but are not confident when they need to relate this knowledge to the behaviour of the solution in phase plane, or when they are asked to plot the family of solutions given the phase plane representation of straight line solutions, or to explain the meaning of the linearity principle in the construction of solutions, other that repeating the procedure learnt by heart.

I. The graph in the diagram shows two solutions of a linear autonomous system in the phase plane. Can you draw other solutions of the system and explain to me how you do it?



S. I know this…you told me these are two solutions, I draw these solutions, this one (pointing to one of them) for example, I know that *x* grows with time, and the same for *y*. the other decreases, both *x* and *y*, that is why the equilibrium point is a saddle point.

I. Fine, now, how would you find another solution in the same XY plane?



S. I would need to know the direction of the vectors, and I don't have that information…so only with these two… I know the system is linear, so I think they go like this. (Draws diagram on the left).

I. All of them are lines?

S. Yes, I know we drew some in class that were not lines, they were other curves but if these two are lines, then I suppose all the others are lines as well.

I. Can you tell me something about the analytical expression for these solutions?

S. A combination of exponentials. Well, linear combination of exponentials.

I. And when you draw that expression on the phase plane you say you obtain lines?

S. For this system yes…well, I believe so, for some others, no, for example if the eigenvalues are complex you have spirals, instead of lines.

In contrast, students at the Intra- parametric, Trans- solution level, are able to interpret the meaning of the linearity principle, and although they show some problems with the relationship between the analytical and graphical representation of the solution because their understanding of the parametric nature of the functions is weak, they can explain the relationship between the direction of the solution curve and the signs of the derivatives at different regions of the phase plane, even when they are not capable of relating the behaviour of straight line solutions, or other solution curves to the behaviour of each of the components of the solution, as we can see in the following excerpt:

I. What is the meaning of the constant in the exponent of the solutions of the system and of the constant vectors that appear in the solution?, and how are they related to the behaviour of the solutions in the phase plane?

S. The constant is the eigenvalue of the matrix of the system, the vectors are the eigenvectors related to those eigenvalues. Each solution is the result of adding…

of doing a linear combination of those linearly independent solutions.

I. So going back to what you drew here in your response to this other question (referring to the question where the to two straight line solutions are given in a phase plane plot), can you explain that to me again?

S. Yes…Yes, I know what you mean, I told you these curves are obtained when you have the vector field of the system and the curves are tangent to the direction vectors, now I know I can say more, these two lines are the two linear independent solutions and if you add them for different values of the k's then you get the curves.

I. so, can you now draw the behaviour of x and y with time for one of those curves?

S. It is a combination of exponentials. That I know because that is what the formula for the solution tells you, but, I still don't know how to draw that , we did that in class, but I was never able to see how to do that.

I. If I give the analytical expression to you, do you think you would be able to do it?

S. Well, I would need a calculator… what is difficult is that you don't have here in this plot the time, and I know it is possible to read it only I could not understand how, because, here you have vectors, and when you do the other plots you don't have vectors anymore….The phase plane is better because you can see everything at the same time.

I. But you just said you could not see there the time? So how do you mean by looking at everything at the same time?

S. You need another axis, that is difficult to draw, and even without that axis I know there was some way to obtain the information of time, but for me it is enough to know that the equilibrium solution is a saddle point.

The difficulties identified in the analysis of the student data were classified into three categories according to the responses given by the students to the set of questions related to the types already described: results related to the geometric representation of straight-line solutions, results related to the meaning of the characteristics of the analytical representation of the straight-line solutions, and results related to the way other solutions can be obtained from straight-line solutions. Only eight students were able to successfully plot the solutions of linear systems obtained by the usual analytical procedure and to relate the specific solutions obtained to the straight-line solutions. Six students related these solutions to the vector field or to the trajectories of solutions other than the straight-line ones. The remaining students either, were not able to determine how to represent the straight-line solutions and responded mainly by giving details on the construction of the phase plane without making reference to the solutions obtained in analytic form, or showed confusion between the nullclines and the straight-line solutions.

Students also showed a lack of understanding of the meaning of the analytical characteristics of the solution of linear systems. There were only four students who could explain clearly the meaning of the eigenvalues and eigenvectors in the expression of the solution. Seven students related the eigenvalues to the stability of the equilibrium solution and had difficulties with the meaning of the eigenvectors. Other students related the eigenvalues only to the procedure to obtain the eigenvectors and, in some cases, to the need to find linear independent solutions. However, they were not able to explain their role in the solution process.

Students exhibited fewer difficulties when asked to construct solutions if the straight-line solutions were given in an analytical form. However, even in this case there were two students who could not complete the task, and six students who had enormous difficulties in doing it. In the case when solutions were given in a geometrical context, they were asked how to construct other solutions when the straight- line solutions were presented. This question increased difficulties significantly and twelve students found the task nearly impossible. This results were surprising since these kinds of activities were stressed in the class all along the course. As in other questions, some students replied by giving a description of how to draw the phase plane starting from the direction field and were not able to relate this construction to the straight-line solutions. Other students started by drawing trajectories of solutions that were not related to the given solutions, and demonstrated difficulties in explaining their relationship. Four students drew many straight lines concurring with the given ones at the origin. When asked about the reasons for the solutions to be lines, they explained that all solutions had to be straight lines since the system was linear. Ten students were able to interpret the given solutions, four had some difficulties drawing the other solutions, and three drew several solutions but had difficulties in explaining the direction of movement on the trajectories.

Concluding remarks

The fact that it was possible to find students distributed in all the levels of the double triad defined to analyse students understanding of the meaning of solution of linear systems of differential equations provides evidence that the evolution of complex concepts where the establishment of multiple relationships is needed does not proceed in a specific direction. Each student constructs knowledge by building on what they know and making sense of different issues at different times. A good understanding of concepts pertaining to other branches of mathematics that are studied in different courses is necessary, but, what is even more important, a lot of links between those concepts have to be established. This study shows that students mathematical knowledge is somehow isolated, they can successfully solve problems they can interpret as related to specific domains in mathematics, but they have a lot of difficulties when these concepts need to be used together to solve problems or make sense of specific situations. These results, together with those from other studies, show that emphasis in relationships between concepts during lessons is not enough. More effort is needed in collaborative work between students and in the design of specific activities where students work on making those links between concepts. Some comments of students in the interviews pointed precisely to how some specific activities they worked on during the course helped them to think differently on the concepts of function and derivative that helped them in better understanding the concepts in the course. A deep understanding of the relationship between concepts involved in the solutions of systems of differential equations is particularly important since the qualitative or geometrical analysis of systems of differential equations has become very important in many different applications in today's world, and if we want students to achieve a deeper understanding of the new concepts introduced in the course and their relationship with those introduced in other courses.

References

Baker, B., L. Cooley and M. Trigueros. (2000). The Schema Triad: A Calculus Example. *Journal for Research in Mathematics Education.* Vol. 31, No. 5, 557-578.

Cooley, L. , Trigueros, M and Baker B. (2003) "Thematization of the Calculus Graphing Schema". Neil A. Pateman, Barbara J. Dougherty and Joseph Zilliox (Editors) Proceedings of the 2003 Joint Meeting of PME and PMENA. CRDG, College Education, University of Hawai'i. Honolulu, Hawai'i, USA, 13-18 julio. Vol. 2, pp. 2-57-2-64.

Habre, S. (2000) Exploring students' strategies to solve ordinary differential equations in a reformed setting. *Journal of Mathematical Behavior,* 18, 455-472

Rasmussen, C. (2001) New directions in differential equations: a framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior,* 20, 55-87

Trigueros, M. (2001) Trigueros, M. (2001) Analysis of students' strategies when solving systems of differential equations in a graphical context. Dans R. Speiser, C. A. Maher and C. N. Walters (Ed.s) *Proceedings of PME-NA XXIII,* vol. 1, 529-537. ERIC: Columbus, OH.

Trigueros, (2000) Students' conception of solution curves and equilibrium in systems of differential equations. In Fernandez, M. L.(Ed.) *Proceedings of PMENA-22* (pp. 93-97), Columbus OH ERIC.

Whitehead, K. and Rasmussen, C. (2003) Undergraduate students' mental operations in systems of differential equations. *Proceedings of the Joint Meeting of PME and PMENA.* Eds Pateman, N., Dougherty, B. and Zilliox (Eds.) V. 4 Pp. 411-418 College of Education University of Hawaii Honolulu, HI, USA.

STUDENTS' EXPLORATION OF POWERFUL MATHEMATICAL IDEAS THROUGH THE USE OF ALGEBRAIC CALCULATORS

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The students' use of technology plays an important role in their learning of mathematics. Here we report the work shown by high school students who participated in problem solving activities using an algebraic calculator TI-92. Tasks proposed by the instructors are used to illustrate approaches that appeared during the students' work. Each approach shows diverse mathematical processes and resources that helped them explore and solve the tasks at hand and the role played by the algebraic calculator as a mathematical instrument.

Introduction

The curriculum framework proposed by the NCTM (2000) emphasizes the importance of organizing students' learning activities in terms of mathematical content —for instance numbers, geometry, algebra, data analysis— and processes that appear in the practice of doing mathematics —problem solving, reasoning, communication, connections, and representations. It is also recognized that the use of calculators and computers, in particular algebraic calculators, is important to promote a mathematical way of thinking that is consistent with the practice of the discipline.

> Electronic technologies –calculators and computers- are essential tools for teaching, learning, and doing mathematics. They furnish visual images of mathematical ideas, facilitate organizing and analysing data, and compute efficiently and accurately. They can support investigation by students in every area of mathematics…when technological tools are available, students can focus on decision-making, reflection, reasoning, and problem solving (NCTM, 2000, p.24).

Along students' learning experiences, it becomes important to characterize their problem-solving approaches that emerge from the use particular tools. That is, it becomes important to investigate features of students' mathematical thinking that appear when they use technological tools in processes that involve understanding mathematical ideas, representing relationships and solving problems. In this context, we are interested in documenting methods and strategies that appear as fundamental in their problem-solving approaches. Research questions that guide this kind of study include:

What type of mathematical thinking can be enhanced via the use of technology while learning the discipline? To what extent is students' thinking compatible with approaches based on paper and pencil? What features of mathematical proof are privileged via the use of technology? What types of strategies do students exhibit in their problem solving approaches based on the use of algebraic calculators?

We concentrate on investigating students' explicit answers while working on a series of learning activities implemented throughout a ten weeks problem-solving seminar (with 12th grade students), that goes on with the work presented at PME25 (see Moreno & Santos, 2001). Here, we illustrate students use of distinct representations and problem solving strategies that emerge from dealing with a set of tasks that involves looking for patterns, analysing particular cases, and examining variation phenomena. Students used the TI-92 algebraic calculator.

Methods and General Procedures

An important component in this study was the design or selection of a set of tasks in which students had the opportunity to use algebraic calculator to represent and analyse fundamental concepts embedded in each problem. Thus, tasks used to illustrate tendencies of students' reasoning come from:

i) *Typical problems found in textbooks*. In this type of task, students are asked to use the tools to solve the problem and possible extend its initial nature. That is, students are encouraged to think of different representations of the task that lead them to consider distinct ways to approach them. In addition, students can change original conditions, add more information as a way to explore general cases or possible connections.

ii) *Students select or develop their own problems*. Here, students with the help of some dynamic software, for example, can construct a particular configuration that involves points, lines, angles, and triangles and use it as a platform to formulate questions and conjectures that need to be explored and validated. Besides, students can select problems from readings that include journals, history books or newspapers.

iii) *Those designed by the project team*. In general, students can approach these tasks by using distinct representations that include numerical, graphical, and algebraic methods. Thus, students have the opportunity to discuss advantages and limitations related to the use of the tools to solve this type of tasks.

We organized the tasks around mathematical activities that involved:

(a) Generalisation and formalisation of patterns,

(b) Representation and examination of mathematical situations through the use of algebraic symbols, and,

(c) Visual modelling to represent and analyse quantitative relationships. In particular, students had opportunities to construct dynamic representations of particular phenomenon to examine relationships between variables.

Kaput (1999) recognizes that these activities are crucial in promoting students' algebraic reasoning and should be present throughout the mathematical curriculum. A task that initially included the use of basic algebraic resources could be transformed into a platform to discuss *ideas of variation* and display the power of other mathematical representations. During the development of instruction, we followed a plan that involved several phases:

i) The instructor provides information regarding goals or aims of the task,

ii) Students work individually and then share their ideas within a small group. Later, the solutions are presented to the whole group,

iii) Students openly examine what small groups present and ask for explanations or clarification questions. The idea was to create a problem solving community in which both students and the teacher are part of an inquiry environment.

iv) Concluding remarks where students identify themes and goals that appear during their interaction with the task.

Students were encouraged to share their ideas, to listen to other students, and to communicate their approaches (Schoenfeld, 1998). Students could show their approaches to the entire class (using a view-screen) and share files with other students. In the following sections we will present examples that are representative of the collection of tasks that were implemented during the seminar. It is important to recognize that the instructors' interventions played a crucial role in orienting the students' discussions.

The discovery of patterns embedded in arithmetic expressions

Students were asked to look for expressions that approximate the value of π. The task was intended for exploring and analysing how students faced the problem of discovering patterns hidden in numerical expressions. A student brought into the class discussion the famous expression proposed by John Wallis (1616-1703).

$$\frac{\pi}{2} = \frac{2 \times 2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \times \cdots}{1 \times 3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \times 9 \times \cdots}$$

What does this expression mean? How does one figure out that it approximates the numerical value of π ? These questions and similar ones, led students to notice that the separate expressions in the numerator and in the denominator, involved large sequences of numbers that follow particular patterns of odd and even numbers. Indeed, they recognized that the expression includes an infinite numbers of factors and proposed to examine instances that involved only a finite numbers (the same for numerator and denominator) of factors.

The first goal was to introduce the corresponding formula into the calculator. But, to produce the formula, is not a simple task for students: It involves the discovery of an algebraic pattern. The tension between the processes of examining an arithmetic task (to compute partial results) and the algebraic task (finding the pattern) reflects a characteristic change introduced by the mediational role of the tool, that is, that the original paper and pencil problem activity is transformed into an algebraic one. Students noticed that numbers on the numerator were even, while those on the denominator were odd. With this information and after a period of deliberation, they proposed to use (2n)(2n) and (2n-1)(2n+1) to represent respectively the numerator and denominator and arrived at the expression

$$\prod_{1}^{\infty} \frac{2n2n}{(2n-1)(2n-1)}$$

Afterwards, they explored partial results for different values of n. Students were able to realize that when n increases the value of the expression approaches the value of π. Figure 1 shows the value of the product for some particular cases.

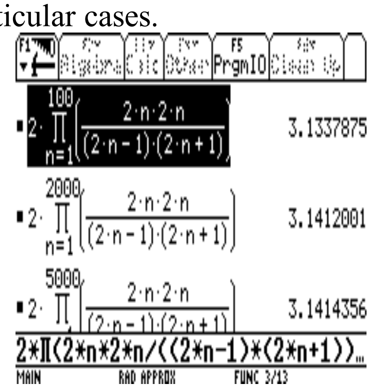

Figure 1. Partial products to approximate the value of π

It was clear that the use of the calculator helped students endow meaning to this expression and also became important to discuss ideas related to the concept of limit. Here, it is important to mention that in order to introduce Wallis' expression into the calculator they needed to transform it into a compact form. Thus, they identified patterns involved in the expression and ways to represent it.

The previous task illustrates that algebraic calculators are powerful semiotic tools for dealing with regularity in general arithmetic expressions. It is interesting to observe that the algebraic calculator functions as powerful tool to determine general formulae for the expressions and also students use it in some cases to simplify (factoring) results.

Mathematical properties

Two examples are used to illustrate students' use of the tool to represent the problem in ways that they can think of the problem in terms of mathematical properties. An example that involves an algebraic expression that led students to discuss criteria of divisibility was: show that the expression n^5-$5n^3$+4n is divisible by 120 for all positive integers n. Here, students used their calculator to factor the expression and provided arguments to support their answers. For example, they observed that the factors of the expression are five consecutive numbers: n, n-1, n-2, n+1, and n+2. What can we say about five consecutive numbers? In this example, it was evident that the use of the calculator led students to achieve an expression that needed to be interpreted in terms of mathematical properties (divisibility criteria) (figure 4). Similarly, when students dealt with the problem "how many zeros appear at the end of 100! – factorial of 100?", they realized that it was not enough to factor 100! To solve the problem, but it was necessary to reflect on the meaning attached to what they had obtained (figure 4).

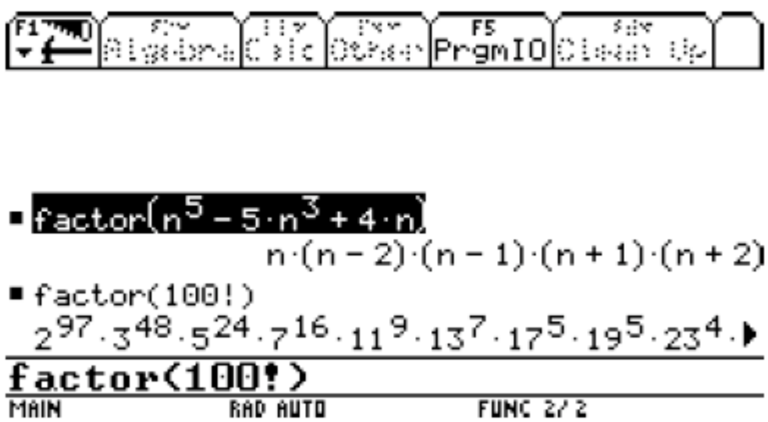

Figure 4. What properties does the factored expression involve?

To determine how many zeros appear at the end of 100!, students examined the factored expression obtained via the calculator and wondered where those zeros come from. This led them to notice that any zero comes from the product of 2 and 5. Then the goal was to count how many of these products could be identified in this expression. Since they noticed that there were more twos than fives, then the number of zeros was given by the number of fives identified in the factored expression. Again, the tool helped students generate a representation that directed their attention to those properties that were important to answer the problem.

Variation Problems: Dynamic Approach

Other group of tasks includes problems in which students were asked to examine variation phenomena. In all these problems it was important to think of a dynamic representation of the situation to model the behaviour of particular relationships. Let us introduce an example to illustrate the type of ideas and mathematical reflection that students exhibited during their work:

Find the rectangle of maximum area that can be inscribed in a given circle.

How can we represent the problem? What does it mean to reach a maximum area? How can we trace changes in the area of a family of rectangles? What conditions are needed to inscribe a rectangle in given circle as mentioned? The instructor uttered this type of questions to guide the students' discussions. The questions led the students to construct a dynamic representation of the situation using Cabri, a dynamic geometry software. In particular, students noticed that to find the rectangle of maximum area it was enough to focus their attention to only one quarter of the given circle. This is because the symmetry of the figure and, based on this consideration, students presented the following construction:

(i) They drew a line AB and constructed a circle with centre point A on the line and radius AB. Students selected point C on segment AB to construct a rectangle (figure 5). They noticed that for each point on segment AB, there was a corresponding rectangle and the goal was to quantify and examine area changes. That is, they noticed that they could draw an infinity number of rectangles and it was important to observe the behaviour of their corresponding area. With the help of the software it was easy to calculate the area of the family of rectangle generated with point C was move along segment AB.

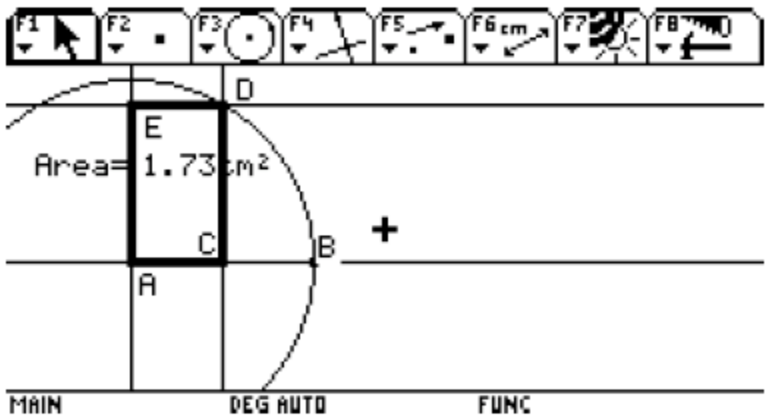

Figure 5. Dynamic representation of a rectangle inscribed in a given circle

The next stage was to analyse the area variation of that family of rectangles. Here students associate the length of side AC of each inscribed rectangle, with its corresponding area (functional relationships). By using a Cartesian System, students were able to express the relationship between the side and corresponding area of the family of rectangle graphically (see figure 6). On the Cartesian system, they transferred measures of side AC (on x-axis) and the area associated to the rectangle with that side length on y-axis to locate point P with coordinate the length of the rectangle side and its corresponding area. The complete graph is generated by point P (with the locus command, for instance) as point C moves along segment AB (figure 6). Students also registered values of sides of a set of rectangles and their corresponding areas and noted that when the length of the sides seems to appear the same then the area of such rectangle is the greatest. That is, the solution is when the rectangle becomes a square.

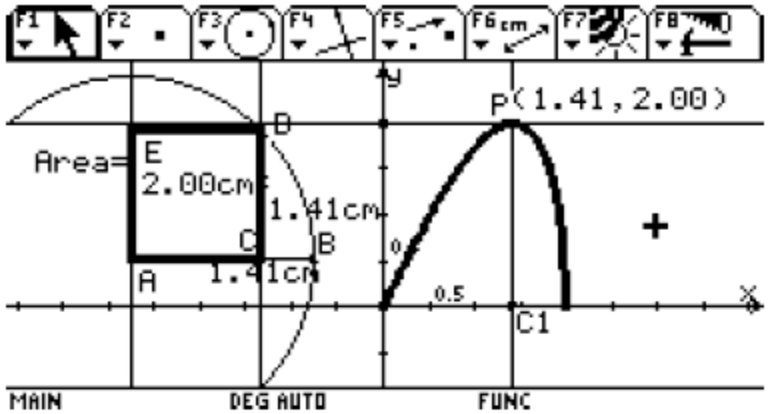

Figure 6. Locus of point P when point C moves along segment AB.

Students had the opportunity to analyse the area behaviour of family of rectangles in three related representations: The area variation of the inscribed rectangle by observing values of the area and the corresponding rectangles, the graphic representation of the function area in terms of one side, and a table (that we are not showing here) of values of the sides of the rectangles and the value of the area.

Another approach exhibited by the students involves representing the problem algebraically. On figure 7, they observed that vertex P of the rectangle has coordinates $P(x,\sqrt{2^2 - x^2})$ since P is on the circle with center the origin and radius 2. Here the area of the rectangle can be expressed as $A(x) = x(\sqrt{4 - x^2})$.

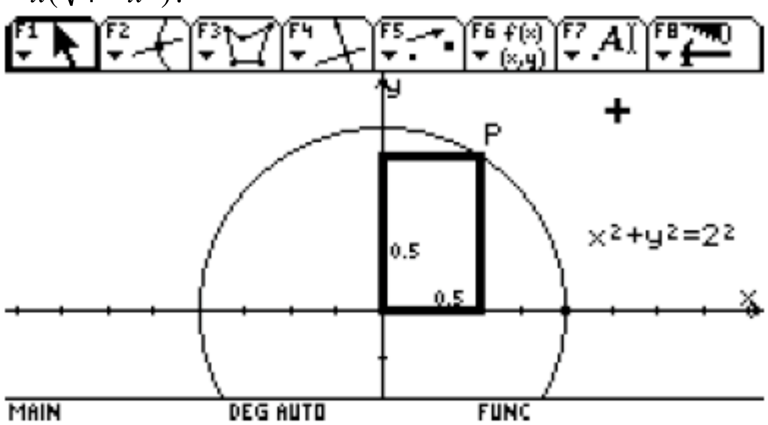

Figure 7. The circle and the rectangle

With the use of the calculator, students graphed the area expression and found directly the point where the area reaches the maximum value. That is, when the side of the rectangle becomes 1.41421, then the area value is 2, which corresponds to the maximum value (figure 8). In this problem (using the CAS), students also had opportunity to determine de derivative of the function area and calculate the point in which this function gets its maximum value. Students approached the problem using distinct representations that were important to discuss different mathematical ideas attached to this type of tasks.

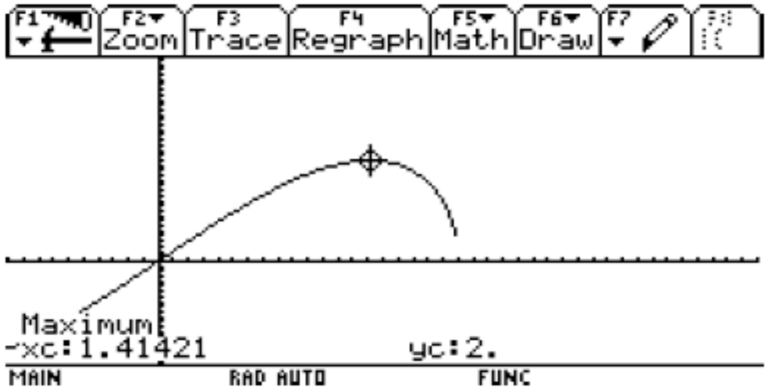

Figure 8. Finding the maximum value graphically.

An important aspect that emerged in students' problem solving instruction is that with the use of algebraic calculator they had the opportunity to engage in a story line of thinking that goes beyond reaching a particular solution or response to a particular problem.

Remarks

Students' growing familiarization with computational tools allows these tools to be transformed into mathematical *instruments* in the sense that computational resources are gradually incorporated into the student's mathematical activity. We suggest then, that exploring with computational tools eventually allows students to realize how the mediational role of these tools helps them re-organize their problem-solving strategies. Working with the virtual versions of mathematical objects provided by the algebraic calculator promotes students' constructive activities. Indeed, these virtual versions produce the feeling of material existence, because we can manipulate them where they exist: On the screen. Without these tools, it is quite difficult for students to establish conjectures, or producing a formulation associated with their explorations and to express them in the language of the computational medium wherein they are working. The computing environment is an *abstraction domain* (Noss & Hoyles, 1996), which can be understood as a scenario in which students can make it possible for their informal ideas to begin coordinating with their more formalized ideas on a subject.

Acknowledgement

The authors would like to acknowledge the support received from Conacyt (reference #42295) during the development of this work.

References

Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema & T.A. Romberg (Eds.), *Mathematics classroom that promote understanding*, 133-155. Mahwah, NJ: Lawrence Erlbaum.

Moreno, L. & Santos, M. (2001). The Students' Processes of Transforming the Use of Technology in Mathematics Problem Solving Tools. In Van den Heuvel-Panhuizen, M. (ed.) PME 25, vol. 2, Utrech, The Netherlands.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston VA: The Council.

Noss, R. &Hoyles, C. (1996). *Windows on mathematical Meanings. Learning Cultures and Computers*. Dordrecht: Kluwer Academic Publishers.

Schoenfeld, A. H. (1998). Reflections on a course in mathematical problem solving *Research in Collegiate Mathematics Education III*. 81-113.

TEACHERS' CONCEPTIONS RELATED TO DIFFERENTIAL CALCULUS' CONCEPTS

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Our aim on this paper is to analyze teachers' conceptions in understanding some concepts of differential calculus and ways teachers reconstruct knowledge in a cooperative learning environment. Participants were mostly high school teachers of mathematics who were studying a master degree program in mathematics education. In this context, we discuss here only 12 sessions related to differential calculus themes. The methodology was related to an enriched cooperative learning environment (ECLE). This methodology was as follows. First, discussion in small groups; second, a discussion with the whole class (in a sort of scientific debate methodology); and third, individual reflection was required as homework (auto-reflection in Hadamard's sense). Our theoretical relied on the role of representations in the construction of mathematical concepts and the notion of conception as important phase on the construction or reconstruction of mathematical concepts.

Introduction

In this study, we wanted to observe teachers' work when solving problems related to calculus. Our theoretical approach focuses on understanding types of conceptions held by teachers and to analyze the role they play in the construction or reconstruction of mathematical concepts, in a special design of what is called cooperative learning environment.

Following Duroux (1983), Confrey (1990), Brousseau (1997) and Balacheff (2002), a conception is knowledge that in some mathematical situations can play a positive role conducting a student to solve a problem he or she is facing, but in others, this knowledge can lead the student to an error or a difficulty; indeed, in some cases it could be an epistemological obstacle. We were interested on this approach in connection with the role of semiotic representations in the construction of concepts and problem solving.

Goals of this Research

Literature shows that learning basic calculus ideas is difficult for students (see for example Tall, 1990; delos Santos and Thomas, 2003; Giraldo, Carvalho & Tall, 2003). In our case, we wanted to show that not only theses difficulties can be found in the students' performances, but that some obstacles in understanding those calculus ideas can also be found in teachers' performances. Teachers' conceptions about certain concepts when are not coherent they need to be confronted with intuitive ideas in a mathematical situation to arrive to a new conception. From this point of view, and trying to confront their conceptions, we designed some mathematical activities in an enriched cooperative learning environment (ECLE). Because incoherent conceptions probable will become a cognitive obstacle, then, in this approach, our way to promote learning is to confront the population to several mathematical situations where their conceptions could play a positive role to solve a problem and in others situations could face a contradiction; and in this ECLE, the discussion with their classmates can provide a rich environment to construct or reconstruct their knowledge.

We agree with Thompson (2002, p. 205) that we must pay attention when "we claim that agreement has been reached" in a cooperative learning environment. In that way, in our ECLE, individual reflection was required as homework, to promote this construction or reconstruction of their knowledge.

Theoretical Framework

We are interested in the construction and reconstruction of concepts throughout manipulation of representations and problem solving. Duval (1995, 2003) stresses the fact that mathematical objects cannot be directly accessed by the senses, but only through semiotic representations.

Under this theoretical approach about the role of representations, how do we explain teachers' conceptions?

As we said before, a conception is knowledge, but what kind of knowledge? If articulation among representations is required to construct a mathematical concept, then a conception is the construction of partial connections among of the representations of a concept that are playing a positive role in certain mathematical situations when solving a problem and in others they promote errors in students or teachers' performances. These conceptions function as a unit; when a student faces a mathematical situation, in which he/she takes into account some type of representations of the situation and recalls the connections attached to those representations, as a unit knowledge, and using it to solve a problem.

From this point of view, and taking into account the role of semiotic representations, within a enriched cooperative learning environment methodology (ECLE), the population worked out the activities we designed. Our theoretical approach in this research can be summarized as showed on Figure 1. Activities in the class were designed to follow this trend.

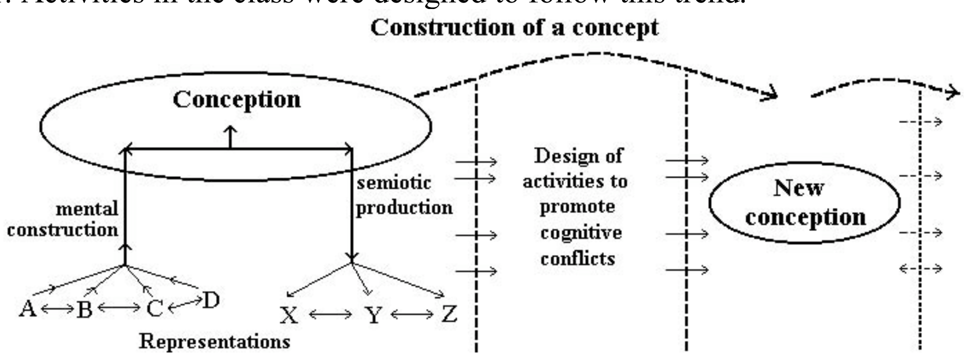


Figure 1

Methodology

We designed a one-semester course to study pre-calculus and calculus concepts. The course included two sessions of two hours and a half per week. For the discussion of this document, we are referring to nine mathematical activities, three out of nine were related to algebra and functions and the others six to differential calculus. We designed a diagnostic questionnaire and a final exam related to these activities. At the end of the sessions, we conducted semi-structured interviews with each participant.

There were ten high school teachers that followed the course. These teachers were students in a graduate program in mathematics education.

We followed the methodology of cooperative learning (Reynolds et al., 1995). Because we are interested in confrontation of conceptions we found interesting the methodology of scientific debate in Alibert and Thomas' sense (1991). We thought that to overcome an obstacle is not an easy cognitive task and because of that, we thought that an important element to add to this methodology is the auto-reflection in Hadamar's sense (1975, p. 25). Then, taking into account the results obtained from the diagnostic questionnaire and the background of the teachers (mathematician or engineer) we arranged small groups to work on each activity. Adapting the scientific debate methodology, we promoted confrontations among the small groups in a general discussion. We collected all the papers used by the teachers during each session and then we asked to reconstruct the results as individual work as homework (auto-reflection phase in our methodology).

In summary, our enriched cooperative learning environment (ECLE) methodology followed four different phases:

- Teachers worked on a task in small groups in a co-operative learning environment,
- Teachers participated in a plenary discussion with the whole class in a scientific debate environment,
- Teachers re-examined the same activity (in a personal reflection as a homework) based on their previous work (small groups and scientific debate discussion),
- Teachers wrote a final exam and participated in a personal interview.

All sessions were audio and video recorded. For each activity, one small group was chosen and video recorded, then when we discussed in a scientific debate environment, the general discussion was also video recorded. Data came also from the individual work made by every teacher and from questionnaires (diagnostic and final) and interviews.

Analysis of the Results

Particularly with derivatives and in our ECLE methodology, we would like to analyze the fifth question of the activity 5. This question involves finding the derivative of compositions of functions, but we would like to concentrate on what provoked the controversy in the ECLE.

Activity 5

Question 5. Given the function $f(x) = x^2 \sin\left(\frac{1}{x}\right)$, if $x \neq 0$, and 0 if $x = 0$; h(x) and k(x) two functions such that $h'(x) = \sin^2(\sin(x+1))$, $h(0) = 3$, $k'(x) = f(x+1)$, $k(0) = 0$. Find $(f \circ h)'(0)$, $(k \circ f)'(0)$ and $\alpha'(x^2)$ where $\alpha(x) = h(x^2)$.

A small group presented its work to the class: Wendy (pseudonym of the teacher at the blackboard) claimed that given a piecewise function, the process to find its derivative consisted in calculate the derivate for each expression separately.

She said: given the function $f(x) = x^2 \sin\left(\frac{1}{x}\right)$, if x ≠0 , and 0 if x = 0, its derivative is

$2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$, *if* $x \neq 0$, [she calculated the derivative of the first expression] *and 0 if x = 0* [because the derivative of the constant 0 is 0].

In this case the result is correct even if the process is not. As we said before, a conception can give a right answer in some mathematical situations and not in others.

When the instructor asked if everybody agreed, the rest of the teachers claimed that given a piecewise function, the process to find its derivative consisted in calculate the derivate for each expression separately and as a consequence they agree with Wendy's procedure.

The interesting thing is that one teacher (Victor) said to Wendy:

Victor: We are not sure about the result you got, because if we calculate the following limit $\lim_{x\to 0}\left(2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)\right)$ *is not giving 0*. [He was talking as a representant of his small group, even when the others agreed after Wendy's presentation].

This argument provided by Victor gave the germ to a socio-cognitive conflict. Indeed, he called the attention of the whole class and the others teachers changed their mind and agreed with Victor. At this point, only Wendy (at the blackboard) claimed that her process was right. It is interesting to say that Wendy used the function $f(x)=\sin\left(\frac{1}{x}\right)$ to make a contrast in this case and the activity. Then, she showed two graphs to support her claim and to convince their classmates that she was correct (see Figure 3), saying that in the first one there is not a derivative in $x = 0$; But the contrary in the other case(from a visual consideration).

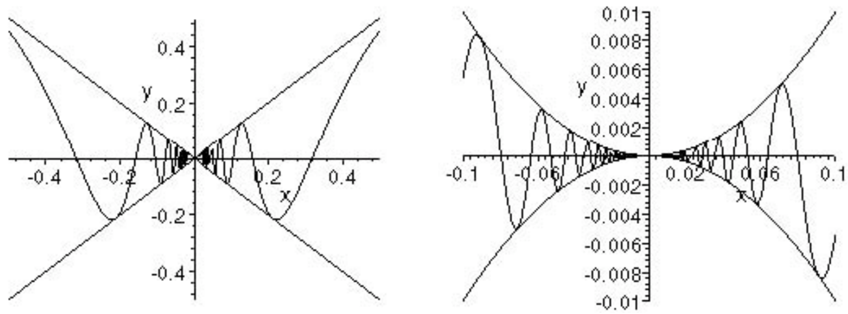


Figure 3

Wendy claimed that the initial function is 'following two parabolas' (maximums and minimums) and in zero the *'derivative must be zero'* (from a visual consideration); second, she was convinced about her result because she couldn't find an error in their algebraic process, indeed, Wendy asked to the whole class: "*If you are not convinced, where is the error in my algebraic process*". The teachers concentrated only in the second graph related to the activity and because they did not find an error, she continued trying to convince the whole class because she was loosing credibility because Victor's intervention! And we arrived to the end of the session (indeed they were discussing 30 minutes after the end of the session).

It is important to highlight here that in our methodology, the auto reflection phase was very important. That is why, when the session finished the professor said, you have a couple of days to do an individual work, reflecting about this activity and what was said in this session.

The next session, when the professor asked about what they did at home, half of teachers were in favor of one position *"there is not an error in Wendy's procedure"* and the rest with the $\lim_{x\to 0}\left(2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)\right)$ *is not giving 0"*. Then, Lidia supporting the second claim, begun the discussion saying that 'the derivative of that function is this (see Figure 4) and *0* in *x = 0*, then, there is no way to make this function continuous!

Lidia: The graph has a lot of oscillations, there, near zero; then, I had the impression that I cannot glue the zero! [She is referring to the continuity of the function in zero]…, that is, the graph is a source of information but that graph is not always reliable because this graph with a bigger scale in x, the graph is not showing the oscillations, only is showing a vertical line [they were using graphic calculators in all sessions] ...

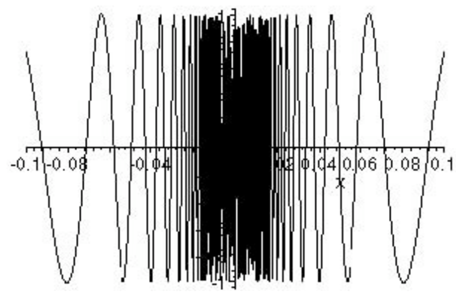


Figure 4

When Lidia presented the graph and stated that that function was the derivative but a fortiori discontinuous, the teachers had the feeling that a cognitive conflict was there (indeed, a socio-cognitive conflict). Wendy insisted that her procedure was right. Then, the instructor asked how to deal with these two positions. The instructor asked if they could conciliate the two results. He added, that we needed more time to reflect about this situation and he told them to reflect about the activity during the weekend (they had 5 days to reflect about).

Next session when the instructor asked for explanations about what they did in their individual reflection, Lidia and Victor said that Wendy's process was right, and that they learnt that the derivative of a function not necessarily is continuous. When they gave that explanation everybody agreed. The instructor could not continue the discussion about the method because from their point of view they solved their cognitive conflict.

Their initial conception was related to the fact that *"if a function has a derivative it must be continuous"*. It was a very fruitful discussion; The ECLE methodology promoted the discovery that not necessarily the derivative of a function must be continuous. One problem remained, it was about their method used to find the derivative of a piecewise function, they did not pay attention to the derivative of certain points (in this case in $x = 0$) and this approach remained unchanged. Indeed, to calculate the derivative of the function in $x = 0$ is:

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$$

The opportunity to provoke a cognitive conflict arrived at the interview. We designed some questions that they could provoke a contradiction and probably a cognitive conflict. The interviews followed a semi structured interview methodology.

We asked the definition of a derivative of a function in one point and a graph. For example, Julia gave the usual definition and the usual graph we find in books.

If the limit exist the function has a derivative en x:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

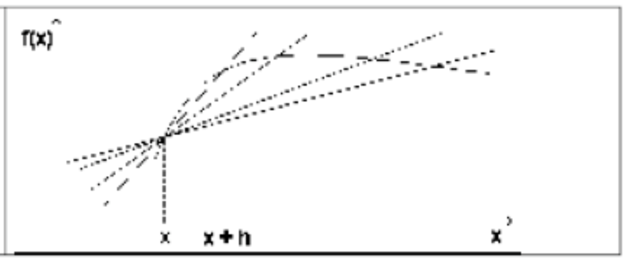


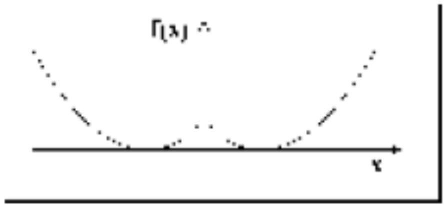
Figure 5

Then, Julia was asked to solve the following:

$$\text{Given } f(x)=\begin{cases}(x+1)^2 & \text{if} \quad x\le 0\\ (x-1)^2 & \text{if} \quad x>0\end{cases}, \text{ find its derivative in } x=0$$

She made a graph (see Figure 6, left) and claimed that the derivative of the function was 0 in $x = 0$.

Interviewer: *We can take that as a conjecture. How to know if your claim is true?*



She calculated the limit using her definition:

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{(h-1)^2-(0+1)^2}{h}$$

$$=\lim_{h\to 0}\frac{h^2-2h}{h}=\lim_{h\to 0}(h-2)=-2$$

Figure 6

She was astonished about the result and it is clear that she perceived the contradiction (a cognitive conflict took place at that moment). She couldn't find an error in her process. Then, the interviewer asked her to analyze her definition. When she was asked to make a connection between her definition and the graph, she explained that the *"h is positive"*. She was asked to explain about lateral limits and finally she arrived to the conclusion that *"h could be positive or negative"*. Then she could solve their cognitive conflict and the problem.

In general, we asked similar questions to the rest of the teachers, in the interviews; that is, piecewise functions defined with two parabolas, and their process was related to calculate derivatives of the two algebraic expressions separately. They said that the derivative in $x = 0$, do not exist (this claim from a visual consideration). When asked to use their definition they arrived to a contradiction and they perceived that contradiction. They solved their cognitive conflict stating, *"h could be positive or negative"*, changing their initial conception.

We tried to understand where this conception comes from. We found that several textbooks have a graph of secants approaching the tangent by the right. Then, a probable source to explain this kind of conception comes from those representations we find in some books and another source is coming from those representations used by their teachers when they were students.

In our study, when teachers were trying to understand what was happening (socio-cognitive conflict), we asked them to reflect about the properties of the algebraic expressions and to interpret them graphically. Indeed, we asked them to examine their processes of conversion between representations in Duval's sense. Teachers solved their cognitive conflict in that way.

Discussion

We observed that conceptions are knowledge, that knowledge is playing a positive role in some mathematical situations and conducting to errors in others. In this study, we described a conception as the construction of partial connections among representations that function as a unit of knowledge, this conception plays a positive role in the resolution of some mathematical problems and promoting errors in others situations. Some of those conceptions will become epistemological obstacles in Brousseau (1997) and Sierpinska's (1985) sense. Then, the one way we think, to promote a change in students' conceptions, is to promote a socio-cognitive conflict proposing them a mathematical situation where probably they will face a contradiction.

From our point of view, this socio-cognitive conflict can emerge when students (or in this case teachers) can perceive a contradiction. Our idea is that the role of the instructor is not to signal the contradiction. The instructor can provide (design) mathematical activities and implement them in such a way that the students' conceptions can lead them to a contradiction. Are the students in a scientific debate methodology who should perceive a contradiction and at that moment will emerge a socio-cognitive conflict. If the students deal with that mathematical contradiction, then they are in a process of modifying their conception.

Because the complexity and persistence of some conceptions, students ways to arrive to a socio-cognitive conflict is not as easy as it seems. Thinking about this complex situation, we decided not to use the classic methodology of cooperative learning. Our ECLE methodology is more related to a social construction of knowledge. We are focusing in how accelerate the teachers' understanding in a ECLE methodology taking into account their conceptions.

References

Alibert, D. & Thomas M. (1991). *Research on Mathematical Proof*. In Advanced Mathematical Thinking (Tall D., Editor), Kluwer, Netherlands.

Balacheff, N. & Gaudin N. (2002). Students conceptions: an introduction to a formal characterization. Laboratoire Leibniz-IMAG, No. 65. http://www.leibniz.imag.fr/LesCahiers/

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer Academic Publisher.

Confrey J. (1990). A review of the Research on Student Conception in Mathematics, Science, and Programming. In Courtney C. (ed.) *Review of research in Education* (pp. 3-56). American Educational Association 16.

Delos Santos A. & Thomas M. (2003). Representational ability and understanding of derivative. In (eds.) *Proceedings of PME and PME-NA* (vol. 2, pp. 325-332). Hawai.

Duroux A. (1983). La valeur absolue. Difficultés majeurs pour une notion mineure. *Petit x*, no. 3, pp. 43-67.

Duval R. (1995). *Sémiosis et pensée humaine*. Peter Lang, Berne.

Duval R. (2003). "Voir" en mathématiques. In E. Filloy, F. Hitt, C. Imaz, A. Rivera and S. Ursini (Editors), *Matemàtica Educativa, aspectos de la investigaciôn actual.* Fondo de Cultura Econômica, México, pp. 41-76.

Giraldo V., Carvalho L. & Tall D. (2003). Descriptions and definitions in the teaching of elementary calculus. In (eds.) *Proceedings of PME and PME-NA* (vol. 2, pp. 325332). Hawai.

Reynolds, B.A., Hagelgans N.L., Schwingendorf, K.E., Vidakovic, D., Dubinsky, E., Shahin, M. & Wimbish Jr., G.J. (1995). *A Practical Guide to Cooperative Learning in Collegiate Mathematics.* MAA Notes No. 37, USA.

Sierpinska A. (1985). Obstacles epistemologiques relatifs à la notion de limite. *Recherches en Didactique des Mathématiques*.

Tall D. (1990). Inconsistencies in the learning of calculus and analysis. *Focus on Learning Problems in Mathematics*, Vol 12, No 3&4, pp. 49-63.

Thompson P. (2002). Some remarks on conventions and representations. In Fernando Hitt (Editor), *Representations and Mathematics Visualization* (pp. 199-206). International Group for the Psychology of Mathematics Education North American Chapter and Cinvestav-IPN. México.

THE AESTHETIC DEVELOPMENT OF MATHEMATICIANS

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Many great mathematicians have talked about the aesthetic dimension of their mathematical work. They have claimed that for the true (read: elite) mathematician, aesthetics is essential to mathematical discovery and creation (Hadamard, 1945; Hardy, 1940; Poincaré, 1909/1956). If these mathematicians are right—and there is good reason to believe they are (see Burton, 1999; Sinclair, 2002)—then one might wonder how young mathematicians develop these essential aesthetic sensibilities. Are they developed in mathematics classrooms? Are they passed on by teachers, mentors, peers or colleagues? Are they developed through reading or working on problems? And how does this development interact with other, more well-understood cognitive and affective processes?

In previous research, using Toulmin's (1971) interdependency methodology, I identified three groups of aesthetic responses, which play three distinct roles in mathematical inquiry. These three types of aesthetic responses capture the range of ways in which successful mathematicians have described the aesthetic dimension of their practices while suggesting the roles they might play in creating mathematics. They are also useful for probing mathematicians' values and beliefs about mathematics, and thus revealing aspects of the mathematical "emotional orientation" (Drodge and Reid, 2000). That research was pragmatic in nature, and aimed at mining connections between the distant but causally-linked worlds of the professional mathematician and the classroom learner. While establishing these lateral connections—in this case, within a contemporary North American milieu—illuminated an important axis of the mathematical aesthetic, other studies are needed to delineate the socio-cultural factors determining or influencing the aesthetic responses of professional mathematician. In this research, I return to the sociocultural analysis I had previously deferred.

The most recognised and public of the three roles of the aesthetic is the *evaluative*; it concerns the aesthetic nature of mathematical entities and is involved in judgements about the "beauty," "elegance" and significance of entities such as proofs and theorems. The *generative* role of the aesthetic is a guiding one, and involves non-propositional, modes of reasoning used in the process of inquiry. I use the term generative because it is described as being responsible for generating new ideas and insights that could not be derived by logical steps alone (for example, Poincaré (1908/1956)). Lastly, the *motivational* role refers to the aesthetic responses that attract mathematicians to certain problems, and even to certain fields of mathematics.

In this paper, I aim to describe the ways in which the aesthetic sensibilities involved in the evaluative, generative and motivational roles develop in young mathematicians. This description is based on a series of interviews conducted with several distinct groups of mathematicians: undergraduate students, graduate students, post-doctoral mathematicians, pre-tenured faculty and tenured faculty. The interviews each involved a series of questions that probed the three roles. In addition, if the interview participant was involved in teaching (which many were), I also included questions on the ways in which aesthetic values or considerations emerged in their classrooms. For the smaller group of mathematicians that had graduate students, I also inquired about the specific ways in which these mathematicians attempted to support the aesthetic development of their students.

For all participants but the tenured mathematicians, the ‘code’ words ‘beauty,’ ‘elegance’ and ‘aesthetics’ were specifically avoided until—and if—the participants introduced the words themselves. This was one way for me to determine the extent to which the participants possessed the vocabulary of aesthetics common both in the mathematics literature and in discussions among many professional mathematicians. I have heard several mathematicians comment on the episodes where they learned that words such as ‘beautiful’ and ‘elegant’ were appropriate ones to use in mathematics—this often occurring in their undergraduate or graduate years. Of interest here is the extent to which young mathematicians learn how to use the aesthetic vocabulary of the mathematics community and internalize that vocabulary as they come to make sense of what is deemed beautiful and elegant in the community. Gian Carlo Rota (1997) has argued that these ‘code’ words are actually copouts: “mathematical beauty is the expression mathematicians have invented in order to obliquely admit the phenomenon of enlightenment while avoiding the fuzziness of this phenomenon (p. 132). Is it possible then, that as young mathematicians acculturate to the mathematics community, they begin using ‘code’ words to replace expressions of enlightenment? Of course, these code words aren’t the only aesthetically-relevant ones; words such as “interesting,” “important” or “surprising” are words frequently used to indicate aesthetic responses.

The first level of analysis is discourse-based, and involved tracking the choice of words used by the interview subjects in their descriptions of mathematics and their own mathematical work. For example, I looked for appeals to aesthetics in answers to the question “what is your favourite mathematical theorem?” The student interview participants tended to use non-‘code’ aesthetic words such as ‘surprising’ while the faculty participants tended to describe favourite theorems as ‘beautiful’ and ‘elegant.’ When pressed for further elaboration of these terms, some called upon the types of words used by the students, while others had more difficulty in providing any explanations.

The second level of analysis involves gauging the extent to which participants were aware of the way in which aesthetic responses played a role in their mathematical activity, using the three roles identified in my previous research. I chose to focus on participant awareness—instead of whether or not aesthetics really do “really” play a role or not—partly for methodological reasons, and partly because of the socio-cultural focus of my inquiry. Specific questions were targeted at each one of the roles. For instance, the question “Why did you choose to work on the problem you’re working on now?” was aimed at eliciting responses related to the *selective* role of the aesthetic. In case the participant had not yet ever chosen a particular problem—and this occurred frequently among the students—he/she was presented with a list of problems and asked “Given the time, which of these problems would you choose to work on?”

The third and final level of analysis is related to the aesthetic dimension of the participants’ teaching activities (including not only classroom teaching, but more informal interactions—through, say, office hours—with students). Faculty participants were asked whether and how they support the aesthetic development of their students. They were also asked to respond to a description by the mathematical physicist Frank Wilczek, working at the Institute for Advanced Study on how to teach aesthetics to young researchers (see Subotnik, 1992).

Aesthetic sensibilities are notoriously difficult to identify and tease apart from other cognitive and affective aspects of sense-making (Dewey, 1934). In fact, it is well-accepted that affective responses always accompany, and essentially alert one to, aesthetic responses (see Silver and Metzger, 1989). There also emerged in my analysis of the interviews a dual nature to the participants’ aesthetic responses. For example, one undergraduate student, in describing her first experiences with mathematics, articulated a strong aesthetic relationship to the patterns and structures of number. However, she used none of the vocabulary common in the mathematics culture, nor did she seem able to make aesthetic judgments about the significance or interest of mathematical entities, including problems and proofs. Her aesthetic relationship to mathematics provided strong initial evidence of the differences between personal aesthetics and professional ones, and revealed some similarities between mathematics and the artworld, where aesthetic responses are highly conditioned by the community of artists and critics and often in conflict with aesthetic preferences of those outside that community (consider, for example modern art, or atonal music). Thus the question of how personal, and perhaps innate aesthetic responses interact with socio-culturally acquired ones in mathematics also became a focus of inquiry in my study.

In the expanded version of this paper, I will present more specific findings about the kinds of experiences that work to acculturate young mathematicians to the mathematical aesthetic of the community. I could not identify distinct "stages" of development, as one sees in other theories of development, including Perry's (1970) ethical, Piaget's cognitive (1972) and Parson's (1987) aesthetic ones. Indeed, though Parson's theory of aesthetic development (in the arts) includes stages of growth that have some application in mathematics, it does not account for the multiple and differentiated aesthetic sensibilities play in mathematical work. I will also describe the specific ways in which the mentor mathematicians (supervisors and teachers) in my study attempt to support their students' aesthetic development, which may lead to effective recommendations in both university and K-12 contexts.

References

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37*(2), 121-143.

Dewey, J. (1934). *Art as experience*. New York: Perigree.

Drodge, E. and Reid, D. (2000). Embodied cognition and mathematical emotional orientation. *Mathematical thinking and learning, 2*(4), 249-267.

Hadamard, J. (1945). *The mathematician's mind: The psychology of invention in the mathematical field*. Princeton, NJ. Princeton University Press.

Hardy, G.H. (1940). *A mathematician's apology*. Cambridge, UK: Cambridge University Press.

Parsons, M. J. (1987). *How we understand art: A cognitive-developmental account of aesthetic experience*. New York: Cambridge University Press.

Perry, William G., Jr. (1970). *Forms of Intellectual and Ethical Development in the College Years: A Scheme.* New York: Holt.

Piaget, J. (1972). *The psychology of the child*. New York: Basic Books.

Poincaré, H. (1908/1956). Mathematical creation. In J. Newman (Ed.), *The World of mathematics* (pp. 2041-2050). New York: Simon and Schuster.

Rota, G. (1997). *Indiscrete thoughts*. Boston: Birkhauser.

Sinclair, N. (2002). *Mindful of beauty: The roles of the aesthetic in the learning and doing of mathematics*. Unpublished doctoral thesis dissertation. Queen's University, Kingston, Canada.

Silver, E., and Metzger, W. (1989). Aesthetic influences on expert mathematical problem solving. In D.B. McLeod and V.M. Adams (Eds.), *Affect and mathematical problem solving* (pp. 59-74). New York: Springer-Verlag.

Subotnik, R. (1992). "Talent Developed" Conversations with Masters of Arts and Sciences. *Journal for the Education of the Gifted, 15*(4) 370-381.

Toulmin, S. (1971). The concept of stagesí in psychological development. In T. Mischel (Ed.),

Cognitive development and epistemology. New York: Academic Press.
Tymoczko, T. (1993). Value judgements in mathematics: Can we treat mathematics as an art? In A. M. White (Ed.), *Essays in humanistic mathematics* (pp. 67-77). Washington, DC: MAA.

TRANSITION FROM SECONDARY TO POST-SECONDARY MATHEMATICS: CHANGING FEATURES OF STUDENTS' MATHEMATICAL KNOWLEDGE AND SKILLS AND THEIR INFLUENCE ON STUDENTS' SUCCESS

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This study explores the changes in knowledge of certain areas of mathematics and various skill levels of students entering first year mathematics courses, and examines their relationship to students' success in first-year university mathematics courses. Of particular interest to us are students who appear to enter university mathematics courses with low technical skills in mathematics, yet recover quickly and are successful. This exploratory study examines data collected over a three-year period from students in both the 'old' and 'new' Ontario curricula. The main focus of the current report is the difference between the incoming knowledge and skills and the performance in the first year mathematics course of these two groups of students.

Introduction

As students whose school experiences were based on a curriculum written after the NCTM *Standards* and other related reform research were published move into the postsecondary domain, research needs to examine how these students continue to do, how their curriculum experiences have effected their learning, and what changes are needed at the post secondary levels to best advantage these students. Such research is also of interest to secondary level teachers wishing to know what features of the reform based curriculum are promoting success and which areas require work.

Framework

In the Canadian province of Ontario, the new elementary curriculum was released in 1997. All years up to grade 8 were released at once, and little to no in-service support was provided. The curriculum included the five content strands as outlined in the *Standards.* The students in the latter years of elementary school may have been particularly disadvantaged by this process in that the curriculum assumed a background in the new strands, such as patterning, which they did not have. Subsequently in 1999, the secondary curriculum was released on year at a time, and simultaneously the curriculum was changed from a five year to a four year program. Thus in 2003, two groups of students graduated from high school in Ontario: those with a five year high school diploma from the 'old' curriculum [we refer to these students as OAC students, because the fifth year courses were called 'Ontario Academic Credits'], and those with a four year diploma from the 'new' curriculum [with students graduating after grade 12]. The latter group also experienced grade 7 and 8 in the new curriculum at the elementary level, learning as mentioned the new strands for the first time in grade 7. Thus in 2003 in Ontario, students entered university from these two different curricula at the same time. This group is generally referred to as the 'double cohort'.

This study forms part of a broader study which examines the transition from secondary to post-secondary mathematics courses. The focus here is on the differences between these two cohorts of students as they enter first year university mathematics courses. Some issues regarding the transition from secondary to post secondary mathematics can be expected to be common to both cohorts. For example, at the university level, more formalism and rigor are expected

(Robert and Speer, 2001, p287; Seldon and Seldon, 2001, p. 250). The ‘gap’ between the secondary and the tertiary education in mathematics is in fact a complex phenomenon covering a vast array of issues.

Although mathematics in elementary and high school has a special position in terms of time spent on it compared to other subjects, the knowledge and skills of incoming university students do not echo this fact (Artigue, 2001, p. 208). Curricular changes may also have influenced characteristics of incoming students but university teachers may largely be unaware of or unwilling to accept the magnitude of these changes. As well, the amount of research on mathematics education at the tertiary level is still modest (Seldon and Seldon, 2001, p. 245). Inquiry into both the social and cognitive backgrounds of students may provide insight into how the two factors relate to and affect one another (Ibid., p. 237). In the current study, both narratives on prior experiences in mathematics as well as the technical skills of incoming first year mathematics students were examined.

The *Standards* (NCTM, 2000) as well as many researchers such as Kamii (1994), Ball (2002) and Ma (1999) have argued persuasively that a deeper understanding of *why* mathematical ideas work rather than just *how* is crucial to retention and long term understanding. Many ‘reform based’ or ‘Standards based’ curricula were developed with this idea in mind, although the curriculum documents often contain contradictions (for example, see Davis and Whitley, 2003). Another area of confusion and controversy is the learning of technical skills (McDougall et al, 2000; Wood et al, 1981) in overall mathematics learning. The purpose of the current study, to use Schoenfeld’s categories (2001, p.222) is applied in nature; the goal is to improve mathematics instruction for better understanding, rather than to examine mathematical thinking. However, conflicts exist as to what the values should be at the post-secondary level in terms of mathematical understanding. To quote Schoenfeld, the most typical educational questions asked by mathematicians – “What works?” and “Which approach is better?” – tend to be unanswerable in principle. The reason is that what a person will think ‘works’ will depend on what a person values (Ibid., p. 223). For example, while efforts have been made to implement reform based or problems based curricula at the post secondary level (eg. Millet, 2001 and Niss, 2001), the students in the calculus course studied here received fairly traditional instruction. While this paper reports ‘success’ as success in this reasonably traditional environment as evaluated by shorter assignments, quizzes, tests and exams; success in a more problems based post secondary environment might yield quite different results. If the new curriculum is successful as to promoting the goals of reform, improved understanding of concepts by students should be noticeable. However, it is unclear if such results will be noticeable under a more traditional lens. If one gave a traditional test that leaned heavily on the ability to perform symbolic manipulations, reform students would be at a disadvantage (Schoenfeld, 2001, p223). At the moment, the best the current study can do is examine how well the grade 12 students, as the first group of students experiencing the new curriculum (for six of the 12 years spent in school) performed in a traditional university course compared to their peers who received instruction in the old curriculum for 13 years. Judgments concerning the effectiveness of one form of instruction over another will depend heavily on questions such as how much weight is put on problem solving or communication (Ibid.).

It is hoped that having the two perspectives of students from both the old and new curricula in Ontario will give us insight into changing features of students’ expectations, and their mathematics knowledge and skills, as they undergo the process of transition to university. It is hoped that these results may influence university pedagogical content knowledge, of which little exists (Seldon and Seldon, 2001, p251). The information on students' background preparation and experience in mathematics should be very valuable to university faculty, some of whom are not aware of the magnitude of changes in the incoming student population. Seldon and Seldon (Ibid.) also cite the need for a theoretical melding of the cognitive and social perspectives and we examine some of these issues as well.

Description of the Study

The following research questions are being examined in a larger study:

- With what technical skills do students begin first year mathematics courses? How much do they retain from high school?
- How do students' incoming skills effect their chances of success in university mathematics courses?
- What other factors (such as their feelings about their past mathematical experience) affect their success?
- Are there noticeable differences in students from the old and new Ontario curricula and how do their performances in the calculus course differ?

The current report focuses particularly on the last question.

A survey called the Mathematics Background Questionnaire (henceforth referred to as 'the survey') was developed to administer to students at the beginning of their first year calculus course at McMaster University. The survey itself may be viewed at http://www.math.mcmaster.ca/lovric/survey/survey.html. The survey was field tested with high school students to ensure the terminology was appropriate to students leaving high school. A second version of the survey has also been developed to examine backgrounds of education students entering Lakehead University. This latter group of students was taking a different mathematics course, and the current report focuses only on the calculus students at McMaster.

The initial survey was designed to help us better understand certain issues related to the transition from secondary to tertiary mathematics education. Given for the first time in 2001, the survey has two major parts, roughly described as 'narrative' and 'mathematical.'

Besides inquiring about basic demographic data, the 'narrative' part asks students to describe their experiences with high school mathematics and their expectations about the university mathematics courses. The larger of the two parts, the 'mathematics' part aims to identify students' strengths and weaknesses in the following areas:

- basic technical and computational skills (fractions, equations)
- basic notions for functions (range, composition)
- • familiarity with transcendental functions (exponential, logarithm)
- written communication of mathematics ideas ('explain' type of questions)
- proficiency in multi-step problems
- drawing and interpretation of graphs
- applied problems

The survey was given to incoming calculus students in September 2001, and again with minor changes in September 2002. In September 2003 the survey was slightly revised to include questions pertaining to the new curriculum, and was used with the so-called 'double cohort' students entering calculus courses. Information on students' feelings about their experiences with mathematics in high school was also collected in the survey (in a narrative form) and data from 2001 has been examined for correlations with students' success in their first-year mathematics course. The survey has been given to about 250 to 300 students per year depending on enrolment. In 2003 there were 142 OAC students, 116 grade 12 students, and 30 who did not come from Ontario or had a mixture of courses, for a total of 288 students.

Preliminary Results

Results from the administration of the survey for the first two years contain some predictable features as well as some surprises. Preliminary analysis has also been done on the fall 2003 data. As a whole, OAC students (students with five years of high school mathematics) performed slightly better overall on the initial survey than did their counterparts with only a four year high school (grade 12) background. OAC students appeared more experienced with topics such as logarithms, exponentials, sketching quadratics, and terminology; in general they were more familiar with topics pertaining directly to the university calculus course. Grade 12 students, on the other hand, seemed better at handling interpretation questions, such as analyzing features of velocity and acceleration from the pictorial information about the position function. They were also more adept at finding the roots of quadratics, as well as explaining their work.

However, these skills did not appear to directly benefit the grade 12 students with this particular (more traditional) calculus course. Some differences existed also in course performance. Specifically, the test and quiz averages for the two groups differed consistently throughout the calculus course, with final course averages being 74% for the OAC group and 69% for the grade 12 group. There were also more failures in the grade 12 group. Interestingly however, the top students in the calculus course (in terms of the final course mark) were nearly equally divided between OAC and grade 12 students.

While the course content had been modified slightly in terms of topics to suit the grade 12's, such as by teaching more trigonometry than was assumed in the past, moving more slowly, and leaving off some topics, the university course was not changed in any fundamental way in response to the new curriculum or reform ideas. Thus it is possible that the OAC courses contained topics and methodology more directly connected to the university course hence giving the OAC students the advantage.

Another interesting point is that as well as being on average one year older, the OAC students took an average of 2.4 courses in mathematics in their final year in high school while the grade 12 students averaged 1.8. Said differently, it could be hypothesized that the OAC students took an average of 1.6 more math courses before entering university (one more year of high school and an average of 0.6 more courses in the final year); a significant amount.

While other comments made by the students on the initial survey have not yet been related to their final course grades, some observations can be made. Both OAC and grade 12 groups contained comments that students are expecting to have to "think more" in university mathematics than they did in high school, rather than just "using formulas" as they did in high school. The unanimity of these comments is surprising, as one would hope that the grade 12 students would have had more opportunity to "think" and explore in the new curriculum than the OAC group. However, the grade 12 group reported here is the group of students who were taught the new elementary and secondary curricula for the first time by teachers, so it is possible that the approach did not immediately change in many classrooms and the style of teaching remained reasonably traditional in nature.

A lack of mathematical sophistication was generally noticeable in both cohorts in both the initial survey and the course itself. This had been found in previous years' administrations of the survey as well. For example, students had difficulties writing mathematical sentences that made sense, as well as justifying answers in the calculus course. Also, basic mistakes such as thinking that $1/(a+b) = 1/a + 1/b$, were in evidence with both groups on the initial survey, and, disturbingly, students often expressed the opinion that making such mistakes was "not a big deal". Apparently, there is still more work to be done in terms of students understanding basic ideas more deeply.

More detailed analysis of the narrative portions of the survey as related to individual course performance has been done with the 2001 data. We suspect to find a similar result with the subsequent years. Not surprisingly, we found that students who reported having had a positive experience in high school tended to do well in the university course. Here are a few typical comments which we interpreted as the student having had a positive high school experience, regardless of grades:

- Math was never my best subject but I gained confidence in my ability through my high school years
- I learned a lot in calculus and geometry through the [high school] lectures
- My experiences were pretty good
- The work was easy and it interested me. The math was fun to learn.
- Enjoyed high school math and found it to be interesting

Although some of the students who did report a positive high school experience did not do well on the survey, indicating perhaps a need to improve their background knowledge, the interesting result was that of the students who did not do well on the initial survey, the ones who reported a positive experience in high school were generally able to recover and obtain good marks in the calculus course.

We also found that the students who reported a negative experience in high school have, as a rule, a hard time at university. It seemed from our data that students with a negative experience in high school often do not fully recover in terms of grades from the survey to the final course grade. This group included many students who ended up with mediocre to low marks in university calculus.

Discussion

Pinpointing causes and effects in this study is hampered by the simultaneous changes from a five year to a four year program in high school at the same time as changing the curriculum. Also, measuring success with a traditional set of post secondary tests and exams as was done here may not be ideal. Several observations may be made however. Firstly, the grade 12 group did surprisingly well in calculus compared to their older peers. Similar results have been informally reported to us from other university contexts. This is especially surprising considering the 1.6 less mathematics courses the grade 12 group had previously taken. Several possibilities exist (or a combination of these). Either the new curriculum did manage to give these students a nearly equivalent background with fewer courses, or previous experience in high school is not as significant as one might expect in ensuring success in university mathematics courses. While the better interpretation skills of the grade 12 group did not appear to particularly advantage these students in the calculus course, it is as yet to be determined if in other contexts these students might be more appropriately prepared.

Since past experiences in mathematics and students' feelings about these experiences appeared to be important for success in university, more analysis is still required of the double cohort data in this area. The variable of students' feelings about past mathematical experiences may in fact warrant deeper study. What aspects of the students' past educational experiences in mathematics lead them to describe their experiences in high school as positive, regardless of course grades or background preparation? What aspects of their experiences give them an advantage? Were pedagogical differences involved? What other subsequent curricular changes might be important to support these students even more? In particular, Niss (2001, p. 165) reports that problem-oriented work at the post-secondary level serves to generate and foster students' enthusiasm. Such an approach at university might further advantage some students.

Further research is needed to see if the differing skill set of the grade 12 students is more helpful in other types of courses than the traditional calculus context. Also, more thought may be needed to examine if and how university calculus courses (and other courses, in particular, linear algebra) should in turn change in the light of changing secondary curricula.

Concluding Remarks

The data may be useful in various ways. University courses in mathematics assume certain level of technical proficiency. These assumptions will generally need to be revisited and in some cases altered. Of particular interest in the data may be initial clues as to what strengths and weaknesses students from the new Ontario curriculum bring to tertiary level mathematics courses and how these differ from those of their predecessors. This information should be useful to both secondary and tertiary level teachers of mathematics. An important outcome of this study may be increased thought on the part of both levels of teachers as to what skills are ultimately important for their students' success. However, the measurement of success may have to be rethought at the post secondary level. University teachers of mathematics may need to further examine the priorities in students' learning of mathematics and evolving values in mathematical pedagogy in light of the changing features of their incoming students' backgrounds and experiences. Further study of student success in other types of university mathematics courses such as engineering courses would also be interesting in the light of the current data.

References

Artigue, M. (2001). What can we learn from research at the university level? In D. Holton (Ed.), *The Teaching and Learning of Mathematics at the University Level: An ICMI Study* (pp. 237-254). The Netherlands: Kluwer Academic Publishers.

Ball, D. (2002). Towards a practice-based theory of mathematical knowledge for teaching. In B. Davis and E. Simmt (Eds.) *Proceedings of the Canadian Mathematics Education Study Group Conference* (pp. 3-14). Kingston, ON.

Davis, B, and Whitley, W. (2003) A mathematics curriculum manifesto. In B. Davis and E. Simmt (Eds.) *Proceedings of the 2003 Canadian Mathematics Education Study Group Conference* (pp. 79-83), Wolfville, N.S.

Kamii, C. (1994). *Young children continue to re-invent arithmetic, 3rd grade: Implications of Piaget's theory.* New York: Teachers College Press.

Ma, L. (1999). *Knowing and teaching elementary mathematics.* Mahwah, New Jersey: Lawrence Erlbaum Associates.

McDougall, D., Lawson A., Ross, J., MacLellan, J., Kajander, A. and Scane, J. (2000). *Research report: A study of the impact math implementation strategy for the Ontario mathematics curriculum, grades 7 and 8.* Toronto, Ontario: University of Toronto.

Millet, K. (2001). Making large lectures effective: An effort to increase student success. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at the University Level: An ICMI Study* (pp. 137-152). The Netherlands: Kluwer Academic Publishers.

National Council of Teachers of Mathematics. (2000). *Principles and standards of school mathematics*. Reston, Va.: NCTM.

Niss, M. (2001). University mathematics based on problem-oriented student projects: 25 years of experience with the Roskilde model. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at the University Level: An ICMI Study* (pp. 153-165). The Netherlands: Kluwer

Academic Publishers.
Robert, A. and Speer, N. (2001). Research on the teaching and learning of calculus/ elementary Analysis. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at the University Level: An ICMI Study* (pp. 283-299). The Netherlands: Kluwer Academic Publishers.
Schoenfeld, A. (2001). Purposes and methods of research in mathematics education. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at the University Level: An ICMI Study* (pp. 221-236). The Netherlands: Kluwer Academic Publishers.
Seldon, A. and Seldon, J. (2001). Tertiary mathematics education research and its Future. In D. Holton (Ed.) *The Teaching and Learning of Mathematics at the University Level: An ICMI Study* (pp. 207-220). The Netherlands: Kluwer Academic Publishers.
Wood, T., Cobb, P., and Yackel, E. (1991). Change in teaching mathematics: A case study. *American Educational Research Journal* 28(3), 587–616.

A FRAMEWORK FOR CHARACTERIZING THE TRANSITION TO ADVANCED MATHEMATICAL THINKING

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The conceptions of congruence of integers of six above-average-performing undergraduate mathematics students enrolled in a third-year introductory number theory course were examined using an exploratory case study design. A framework was developed for analyzing the degree to which these students were employing advanced mathematical thinking when solving problems related to congruence of integers. This framework represents an attempt to synthesize multiple perspectives on the nature of advanced mathematical thinking currently present in the field. The students in the study were primarily prospective secondary mathematics teachers, and since the topics studied in this type of course are closely related to those of high school mathematics, this study has implications for teacher education as well.

Currently, the precise characterization of advanced mathematical thinking (AMT) is a subject of some controversy in the mathematics education community. On both sides of the Atlantic, researchers have used the term to describe a goal we would like our undergraduates to achieve: the ability to do mathematics as mathematicians do. However, defining AMT for purposes of research has been problematic. In this paper, I present a framework that both characterizes AMT and provides a means of analyzing undergraduates' use of AMT.

Review of Literature

Elementary mathematical thinking (EMT) begins with a focus on concrete objects, such as numbers, which are studied in order to generalize to related processes and concepts. Gray, Pinto, Pitta, and Tall (1999) write that EMT is characterized by the "use of symbols as concepts and processes to calculate and manipulate" (p. 116). Tall (1992) characterized AMT as involving both precise mathematical definitions and logical deductions of theorems based upon these. He contends that the transition to AMT requires "a massive process of cognitive restructuring" (p. 508) on the part of the student. This restructuring involves the student moving from viewing a concept primarily as a process to viewing it as an object by abstracting properties of the object from a formal definition and constructing properties of the object through logical deduction. Different representations for the concept and the relationships between these are constructed by the individual.

Edwards, Dubinsky, and McDonald (2000) argue that Tall's definition of AMT is incomplete in that it does not take into account the possibility that students might reason from definitions purely formally with little or no understanding of the mathematical concepts. They define AMT as "thinking that requires rigorous, deductive reasoning about mathematical notions that are not entirely accessible to us through our five senses" (p. 3). According to this definition, a student who creates physical or written models or who reasons from examples would be demonstrating EMT. Rasmussen, Zandieh, King, and Teppa (2001) prefer to use the phrase "advancing mathematical thinking", and emphasize that AMT may be present at all grade levels, not just at the undergraduate and graduate levels of study.

These researchers appear to agree that the transition from elementary to advanced mathematical thinking requires a major shift in one's thinking processes. In advanced mathematics, instead of starting with objects, students must start with a formal definition and from it construct a mental image of the object in question. Gray et al. (1999) note that this shift in thinking constitutes a “didactic reversal ... [that] causes new kinds of cognitive difficulty for students” (p. 117). Dreyfus (1991) argues that though advanced mathematical topics can often be approached from an elementary perspective, a distinguishing feature between the elementary and advanced approaches is how one deals with the complexity of the mathematical ideas.

Theoretical Framework

The Transition to Advanced Mathematical Thinking Framework (TAMTF) was developed to place students along a continuum between elementary and advanced mathematical thinking. TAMTF represents an attempt to synthesize multiple perspectives on the nature of advanced mathematical thinking currently present in the field and draws heavily on the work of Tall, Vinner, Dreyfus, and others.

Rasmussen et al. (2001) state that a framework for characterizing AMT must account for the fact that mathematics learning occurs as an individual participates in mathematical activity. Accordingly, TAMTF was developed to characterize the study participants’ thinking as they were engaged in describing their reasoning about a problem previously solved, and in solving problems in the context of a task-based interview.

In TAMTF, it is asserted that AMT involves the following four components:

1. Coordination of concept image and concept definition (Vinner, 1991)
2. Use of definitions to construct meaning [Definitions, Objects] (Gray et al., 1999)
3. Presence of multiply-linked representations (Dreyfus, 1991)
4. Evidence of reflection on one’s learning (Dreyfus, 1991)

The terms concept image, concept definition, and representations are defined similarly as in the literature:

- Concept Image (CI): The individual’s general intuitive understanding of a concept; associated images and ideas.
- Concept Definition (CD): The individual’s definition for the concept; may be fairly disconnected from the concept image.
- Representations: Internal mental images and ways of thinking about a concept; external ways of depicting a concept.

	Elementary Mathematical Thinking	**Transitional**	**Advanced Mathematical Thinking**
Coordination of CI and CD	• Complete reliance on CI, which may be limited • Extremely limited CD, which is disconnected from the person’s CI	• Rich CI • CD is usable and includes some formal ideas • Able to apply both CI and CD in familiar situations	• Rich CI, some evidence of formal image • CD is usable and based on formal ideas about concept • Able to apply both CI and CD in familiar situations
Use definitions to construct meaning	Tend to use CI only in new situations	• Tend to use CI only in new situations • May attempt to use CD, and this may cause	Able to apply both CI and CD in new situations

		problems	
Multiply-linked representations	Few representations for the concept, few or no links	Several linked representations	Several multiply-linked representations
Reflection on problem-solving and learning	Reflective comments are focused on procedures of doing problems	Reflective comments focus on processes and strategies for solving problems in general	Reflective comments show attempts to structure understanding, fit ideas into a big picture

Table 1

Table 1 above summarizes the framework. A more detailed description of the development of TAMTF can be found in the results section of this paper, along with examples demonstrating how this framework was used to capture students' developing mathematical thinking across the semester.

Method

The results presented here are from the researcher's dissertation study (Smith, 2002). An exploratory case study design was used to investigate undergraduates' conceptions of congruence of integers in an introductory number theory course at a large state university in the southwestern US. The six participants, primarily pre-service secondary mathematics teachers, were chosen based on high scores on a first exam, in conjunction with the instructor of the course. The researcher's role was that of a participant-observer and teaching assistant in the course. The framework presented in this paper arose from the analyses of the interviews as the result of a need to delineate what would evidence elementary and advanced mathematical thinking in this context.

The six participants completed questionnaires and participated in three semi-structured phenomenological interviews over the course of the semester. The first interview took place one week prior to the exam covering congruence of integers, a second interview took place approximately two weeks later, and a third approximately three weeks after the second. In addition, class sessions during the unit covering congruence and applications of modular arithmetic were videotaped and transcribed. Data were analyzed using a grounded theory approach, which is described more thoroughly in the following section.

Results and Discussion

Coordination of Concept Image and Concept Definition

Each participant's CI and CD for congruence of integers was determined after the data were been collected, and so the questions asked in the interviews were not specifically designed to uncover these. The explanation of congruence offered by an individual was regarded as the CD, since this was given in response to the question, "What does the statement $a \equiv b \pmod{n}$ mean?" The question was asked again at the end of subsequent interviews, with the intention of giving the participant an opportunity to summarize his or her ideas about congruence that had been drawn out during the interview.

The CI was more difficult to obtain, but it was assumed that the CI would be the interpretation of congruence that the individual used in practice, while solving problems. The CIs of the participants were primarily gleaned from the explanations offered while working through problems. The determination of the CI relied heavily on the participant's use of representations and related ideas when working with congruences.

A CI was considered "rich" if the participant appeared to have many ways of thinking about congruence (representations), and seemed to connect the concept with other mathematical ideas. A CD was considered "formal" if the participant tended to use formal language and notation to describe his or her definition for congruence. After attempting to determine the nature of each participant's CI and CD, the interconnectedness of these was ascertained by asking the following question: When explaining their solutions, how much did the individual rely on the CI and how much on the CD? Use of CI and CD is classified in the framework as follows:

EMT: complete reliance on a limited CI, together with a fairly disconnected and limited CD. For example, Fran's CI of congruence was highly procedural during all three interviews. She interpreted most congruence statements in an operational way, and seemed to rely on a narrow set of procedures in order to solve simple linear congruences. Her CD for congruence consisted of interpreting the statement $a \equiv b \pmod{n}$ as $a \div n$ has remainder b. Fran applied her concept image to explain her solutions in every situation.

Transitional: possessing a rich CI, together with a usable and partially formalized CD, and a tendency to primarily rely on the CI when explaining ideas, occasionally using the CD. Dan, for example, had a fairly rich CI for congruence, which included informal notions of congruence classes of integers and of equivalence, in which congruent integers behave similarly and can be interchanged, as well as the division-remainder idea ($a \div n$ has remainder b). When explaining solutions to problems he had already solved, Dan used both his CD and his CI fairly interchangeably. For example, when explaining his solution for an item on the questionnaire,

$x = 1 \pmod{11}$, he used the standard definition: "11 divides x minus 1, so 12 fits in there, 12 plus 11 fits in there."

AMT: possessing a rich CI which seems to be well-integrated with a formal CD, with an ability to use both CI and CD when explaining ideas. Andrew's understanding of congruence provides a good example of AMT. His CI included the division-remainder interpretation ($a \div n$ has remainder b), the standard textbook definition ($n \mid a - b$), and rewriting linear congruences as linear equations in Z . In addition, he tended to work with linear congruences as if they were analogous to equations, and freely substituted equivalent integers on both sides of a congruence. His CD for congruence consisted of the standard definition ($n \mid a - b$) and the division-remainder interpretation. Andrew seemed to use his CI and CD most of the time when explaining his solutions to problems; for example, he tended to emphasize the connection between congruences and divisibility statements when discussing Fermat's "Little" Theorem. In addition, Andrew tended to use formal language and notation when discussing congruence.

Use Definitions to Construct Meaning

This part of the framework focuses more directly on the participants' use of definitions when solving problems, while the previous section focused on the participants' explanations of solutions. The focus here is on using congruence in novel situations, in which the participant was asked a question related to congruence they had not considered before. The way in which each person approached answering this question gave insight into how she or he used definitions.

The three categories for reasoning from definitions are given below with examples from the data:

EMT: tend to use only CI in novel situations. Eva, for example, twice described her solution process for solving the congruences in the first interview as looking for a number that satisfies certain conditions. She appeared to be relying on her CI in novel situations during all interviews.

Transitional: Tend to use only CI in novel situations; may occasionally attempt to use CD. In novel situations, Chris appeared to be relying both on his CI and his CD; there were no clear circumstances under which he appeared to be reasoning solely from a definition. For example, when explaining why $75 \equiv 21 \pmod{27}$ implies that $75x \equiv 21x \pmod{27}$, Chris used the fact that congruence means that two integers are considered equivalent. "It doesn't matter what x is because no matter what x is... as long as it's an integer value. Because ten 75s are still going to be equivalent to 12 (mod 27), the same as one 75 would be." Similarly, he explained why one can substitute 1 for 33 when working modulo 8: "33 and 1 are in the same set in terms of mod 8, and any number that I take to the 25th power in that set is going to be congruent to any other number in that set that's to the 25th power."

AMT: able to apply both CI and CD in novel situations. In both interviews, Andrew frequently used the standard definition of congruence when trying to answer questions posed by the interviewer. He frequently rewrote congruences as statements about divisibility; for example, $75x \equiv 39 \pmod{27}$ was rewritten as $27 \mid 75x - 39$. He also rewrote statements about divisibility as congruences: "This statement [$9 \mid 25x - 13$] could also be the same as $25x$ congruent to 13 (mod 9)."

Multiply-linked Representations for Congruence

The students' explanations and interpretations of congruence were examined in detail, and then the following types of representations for a congruence statement ($a \equiv b \pmod{n}$) emerged:

- Division-remainder (DR): $a \div n$ has a remainder of b (This was not the definition given in class, though it is applicable in certain contexts. Occasionally a participant interpreted a congruence as "a and b have the same remainder upon division by n", though this was rare.)
- Equation (EN): rewrite as $a = b + nk$ or $a - b = nk$.
- Divisibility (D): $n \mid a - b$ (This is the standard definition, given in class and in the text.)
- Multiples (M): add or subtract multiples of n to or from a or b; congruent integers a and b are a multiple of n apart; congruent integers are evenly spaced on the number line.
- Equivalence (EQ): Congruence is an equivalence relation; notion that congruent integers act the same or can be considered as the same.
- Classification (C): classifying integers into (usually informal) congruence classes. (Congruence classes were not formally defined n class.)
- Visual (V): a visual image of a clock or number line depicting congruent integers.

In line with Dreyfus (1991), it was assumed that the more closely linked the representations are in the mind of the individual, the more flexibility the individual has when solving problems. To determine the links between representations, the interview transcripts were divided into segments. Each segment represented a complete "thought" in the sense that it seemed to be a stand-alone phrase, sentence, or set of sentences that the participant used to explain an idea. If the participant used more than one representation in one segment, these representations were considered to be linked.

In the framework, the use of representations is classified as follows:

EMT: having few representations for the concept, with few or no links between them. Fran, for example, relied primarily on the division-remainder and multiples interpretations at the time of the first interview. Fran was able to connect the division-remainder interpretation to the multiples representation a few times, though she tended to use one representation to solve or explain a problem without switching representations.

Transitional: having several links between representations. Andrew primarily used division-remainder and divisibility in both interviews, though he also used multiples, equation, classifying, and equivalence. He relied heavily on the division-remainder interpretation in the first interview, and almost all links between representations involved it. He connected division-remainder with divisibility, with a visual representation, and with multiples. For example, when explaining his solution to an interview task, he said, "I was looking to see if 33 to the 5th [power] would leave a remainder of 1 [DR]. You could also look at 33 itself and see what it is in the world of mod 8 [C]." He did not appear to ever connect more than two representations at a time.

AMT: having many multiply-linked representations. Dan, for example, used many representations. He tended to interpret congruences as statements classifying integers throughout the semester, though divisibility, equation, and division-remainder were important representations as well. Dan's representations were extremely well-linked in both interviews. When he explained how he had solved a problem, he jumped from representation to representation quickly. In the first interview, he linked Classifying $\leftrightarrow$ Equation $\leftrightarrow$ Equivalence, Classifying $\leftrightarrow$ Division-remainder $\leftrightarrow$ Divisibility, and Division-remainder was linked to each of Divisibility, Multiples, and Equation. For example, when solving the linear congruence $x \equiv 3 \pmod 7$, he said, "This is the family of numbers which when divided by 7 had a remainder of 3 [DR]. It's 7 divides x minus 3 [D]. So whatever x is, you can plug in there, divide by 7. A lot of things can be reduced to 3 mod 7 [C]." Dan's links between representations had only strengthened by the last interview, and he had several instances of linking three and four representations, more than any of the other participants.

Reflection on Problem-solving and Learning

The participants made comments during the interviews that demonstrated how they were thinking about their own learning, and were approaching the concept of congruence in general. In the framework, these comments are classified into three categories:

EMT: reflective comments are focused on procedures of doing problems. Fran's comments at the time of the first interview were limited to expressing confusion and an awareness of her lack of understanding: "I don't know how to think of it another way," and "What am I thinking here? I'm not used to working in the world of mod." By the time of the last interview, Fran's comments showed that she had begun to place value on procedural understanding and that she had stopped trying to understand the concept of congruence. "I'm at the beginning stages, everything else goes away. Let me figure out how to do it, and then I'll figure out why it happens later."

Transitional: reflective comments focus on processes and strategies for solving problems in general. At the time of the first interview, Eva's comments mostly referred to her thinking processes while solving problems: "I thought about this for a long time," and "I don't see them as the same – that's interesting." At the time of the last interview, her comments were focused on the process of solving the problems and what difficulties she had. "At first I felt more comfortable with this one. I had some problems understanding this for a while."

AMT: reflective comments show attempts to structure understanding, fit ideas into a "big picture." In both interviews, Chris's comments were quite reflective and generally referred to the general thinking processes he used on problems of the type being discussed, not just that specific problem. He also tended to think in terms of the difficulties he had in coming to understand the concepts, and thought about how to explain the concepts to others in such a way that these difficulties would be addressed. "Congruence was very difficult for me in [discrete math]. I had to keep looking it up. We wanted to have examples of equivalence relations, so [congruence] was introduced for that purpose only. [...] [This class] has given me a way to think of it and to apply the equivalence relation idea to what I learned in [discrete math]."

The pseudonyms of the participants were chosen to indicate their relative exam averages in the course, as a standard of comparison throughout the study. (Andrew had the highest exam average of the six, while Fran had the lowest.) It is interesting to note that this ordering is consistent with each row of Table 2.

	EMT	Transitional	AMT
First Interview:	Fran, Eva	Dan, Chris, Barbara	Andrew
Last Interview:	Fran	Eva, Dan, Chris	Barbara, Andrew

Table 2

Growth was demonstrated by each participant, and as a result, Eva and Barbara both were classified differently after the last interview. Fran was the only student in the course who had not yet taken the department's "Introduction to Proof" course, and so perhaps it should not be surprising that she remained classified at the EMT level. The participants' success in the course is certainly correlated with these findings.

Conclusions

Congruence of integers is a concept that seems to lie in the borderlands of elementary and advanced mathematical thinking. It is a topic that can, in a limited fashion, be approached from a very elementary perspective. After all, children learn how to do "clock arithmetic" in elementary school. However, the student must make the shift to thinking about congruence from an advanced perspective in order to develop a rich understanding of congruence as an equivalence relation. It seems that advanced mathematical thinking may be necessary in order for the student to make sense of even simple statements of congruence.

The goal of the course in which the study was conducted was not to prepare students for further study in algebra or number theory. Rather, the intent of the instructor was that the students see congruence arithmetic as an interesting way of looking at numbers as well as a powerful tool for solving problems. These results suggest that even studying congruence for these "humble" purposes requires students to use advanced mathematical thinking. This study implies that an awareness of the difficulties that may be encountered by students who have not yet made the transition to advanced mathematical thinking might be helpful for instructors of number theory at this level.

References

Dreyfus, T. (1991). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 25-41). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Edwards, B., Dubinsky, E. and McDonald, M. (2000). Advanced mathematical thinking. Unpublished manuscript.

Gray, E., Pinto, M., Pitta, D., and Tall, D. (1999). Knowledge construction and divergent thinking in elementary and advanced mathematics. *Educational Studies in Mathematics*.

Rasmussen, C., Zandieh, M., King, K., and Teppa, A. (2001). Advancing mathematical thinking: Aspects of students' mathematical activity. Unpublished manuscript.

Smith, J. C. (2002). *An Investigation of Undergraduates' Understanding of Congruence of Integers*. (Doctoral dissertation, University of Arizona, 2002). Dissertation Abstracts International.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 495-511). New York: MacMillan Publishing.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall, (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht, The Netherlands: Kluwer Academic Publishers.

IRRATIONAL NUMBERS: DIMENSIONS OF KNOWLEDGE

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This report focuses on prospective secondary mathematics teachers' intuitive understanding of irrational numbers. Participants' intuitions and beliefs regarding the relations between the two sets, rational and irrational, are examined. Three issues are addressed: richness and density of numbers, the fitting of numbers, and operations. The results indicate that there are inconsistencies between participants' intuitions and their formal and algorithmic knowledge. Explanations used by vast majority of participants relied primarily on considering the infinite non-repeating decimal representations of irrationals, which provided a limited access to issues mentioned above.

This report is part of an ongoing study on prospective secondary mathematics teachers' understanding of irrationality. The purpose of the study was to provide an account of PST's understandings and misunderstandings of irrational numbers, to interpret how the understanding of irrationality is acquired, and to explain how and why difficulties occur. Another report resulting from this study, to appear in the proceedings of PME International, addresses PST's understanding of irrational numbers from the perspective of representations. This report, on the other hand, can be understood as a story of the ingenious ways in which participants strive to harmonize their intuitions with what they formally know to be true.

Background

Understanding irrationality of numbers is mentioned in literature in the research on proofs, limits and infinity (e.g. Tall, 2002). However, research focusing on the learning and understanding of irrational numbers is rather slim. Fischbein, Jehiam & Cohen (1994, 1995) are the only published reports in the literature that focus explicitly on the understanding of irrational numbers. The main objective of the two studies was to survey the knowledge that high school students and preservice teachers possess with regard to irrational numbers. This study assumed, on historical and psychological grounds, that the concept of irrational numbers faced two major intuitive obstacles, one related to the incommensurability of irrational magnitudes and the other related to the nondenummerability of the set of real numbers. Contrary to expectations, the study found that these intuitive difficulties did not manifest in the participants' reactions. Instead, it is reported that subjects at all levels were not able to define correctly the concepts of rational, irrational, and real numbers. It was concluded that the two intuitive obstacles mentioned above are not of a primitive nature – they imply a certain intellectual maturity that the subjects of this study did not possess.

Theoretical Perspective

We situate our findings using the conceptual framework suggested by Tirosh, Fischbein, Graeber, and Wilson in their study of rational numbers. The basic assumption of this framework is that learners' mathematical knowledge is embedded in a set of connections among algorithmic, intuitive and formal dimensions of knowledge. The algorithmic dimension is procedural in nature – it consists of the knowledge of rules and prescriptions with regard to a certain mathematical domain and it involves a person's capability to explain the successive steps involved in various standard operations. The formal dimension of knowledge is represented by definitions of concepts, operations, and structures as well as by theorems and their proofs. The intuitive dimension is composed of our ideas and beliefs about mathematical entities and it includes mental models we use to represent number concepts and operations. Intuitive knowledge is characterized as the type of knowledge that we tend to accept directly and confidently – it is self-evident, intrinsically necessary and psychologically resistant (Fischbein, 1987).

The three dimensions of knowledge are not discrete; they overlap considerably; however, for the purpose of analyzing the subjects' mathematical understanding of irrationality we find it useful to focus on each of them separately. Inconsistencies between a learner's algorithmic, intuitive and formal knowledge manifest as misconceptions, cognitive obstacles, and other common difficulties. The focus of this report is on the intuitive dimension of knowledge of irrational numbers and its connections to other dimensions.

Methodology and Instrument

We investigated PST's beliefs and intuitions regarding three distinct threads. First, we explored their beliefs about the relative "sizes" of the two infinite sets. To explore this we designed the following question:

1. *Suppose you pick a number at random from [0,1] interval (on the real number line). What is the probability of getting a rational number?*

Second, we looked at participants' knowledge and intuitions about how the rational and irrational numbers fit together, related to the density of both sets. The following questions were used:

2. *Is it always possible to find a rational number between any two irrational numbers? Explain your thinking.*
3. *Is it always possible to find an irrational number between any two rational numbers? Explain your thinking.*

Thirdly, we investigated how PST's respond to questions about the effects of operations between various types of numbers (for example, when is there a closure), using questions, such as:

4. *If you add two positive irrational numbers the result is always irrational. True or false? Explain your thinking.*
5. *If you multiply two different irrational numbers the result is always irrational. True or false? Explain your thinking.*

These questions were presented to a group of 46 PST's as part of a written questionnaire. Upon completion of the questionnaire, 16 volunteers from the group participated in a clinical interview, where they had the opportunity to clarify and expand upon their responses. In particular, we examined the participants' capability to produce adequate intuitive models for representing number concepts to accommodate the evidence that the existence of irrational numbers mandates. By "adequate" we mean such that will not create inconsistencies with the other two knowledge dimensions. As well, here we situate participants' difficulties in the context of the system of rational numbers and in the system in which they are embedded - the real number system.

Results and Analysis

Intuitions on richness and density

The table below shows the quantification of participants' responses regarding the order of infinity of rational numbers versus irrational numbers:

Response category	Number of participants	[%]
“Equal to 0”	2	[4.3]
“Close to 0”	9	[19.6]
“Close or equal to 50%”	10	[21.7]
“Close or equal to 100%”	8	[17.4]
“Undefined”	1	[2.2]
No answer	16	[43.8]

Table 1: Probability of picking a rational number from [0,1] interval. (n=46).

Disregarding the fact that many participants abstained from responding to this question, the results indicate a symmetric distribution centered around the majority, who thought that the two infinite sets are equally abundant, usually justified as “you can’t have one finite greater than another”. Of those that held the view that the set of irrationals is richer, three made some reference to the formal theory of cardinal sets whereas others based their responses on informal intuitive reasoning. The following excerpt from our interview with Ed exemplifies this.

I: You say that you think that irrationals are richer, meaning we have more of irrationals, how can you justify that thinking.

Ed: Because there’s always going to be more, if, if you were to *just take random digits*, but anywhere, and *pull them out of a hat*, like whatever, chances are those numbers, like say up to 1,000 digits or whatever, *chances are those numbers are going to have no pattern, there’s much bigger chance*. Like if you think about it, there’s no way you’re going to, of course you’re occasionally going to get one that has an exact pattern, but that’s less likely, it’s just in nature, in your environment, you do see more rational numbers then irrational numbers. But in the actual numbers themselves, if you, there’s probably, in my opinion how I think of it, there’d probably be way more irrationals.

Of course, this simple and intuitively sound reasoning is more likely to occur in those who see irrationals primarily as infinite non-repeating decimals. However, a heavy relying upon this representation of irrationals was found to be more of hindrance than an aid to thinking about irrational numbers.

Much more prevalent were arguments revealing misconceptions. For example, some people see rationals as *terminating* decimals and consequently claim that the probability of picking a rational is very low, close to 0. In addition, another misconception was exposed in response to this question, namely that there is a finite number of rationals and an infinite number of irrationals, and therefore the probability of hitting a rational is close to 0. Two of the participants expressed this view during the interview, and, judging from the written responses, several others may have held this belief. We propose a hypothesis that this thinking may have been induced by the exposure to cardinal infinities, in a situation where the underlying conceptions of rational, irrational and real number were underdeveloped at the time to begin with. According to common sense, “countable” means “what can be counted”, and it implies that what is countable must necessarily be finite. Although the usage of “countable” in Cantor’s theory of infinite sets is entirely different, meaning that the elements of the set can be put into one-to-one correspondence with the set of natural numbers, it is possible that some participants adopted this more colloquial meaning of the word. Further research could examine whether this is indeed a “verbal obstacle” and, if so, what its extent is. It is not clear for every participant that expressed such view whether it applied to rational numbers in general or specifically to the interval [0,1].

Another type of argument, offered by several participants in response to Question 1, employed the idea of mapping, intended to show that the set of irrationals is richer. Here is an example:

> *Irrationals are richer. If we take each element of Q and add* $\sqrt{2}$ *to each, all of those numbers are irrational. Then we could take each element of Q and add* π *to it. Already we have twice the amount of irrationals as rationals. We could do this forever, so the set of irrationals is much richer.*

Although these intuitions lead to a correct conclusion, they are not formally correct, unless it is already known that the set of irrationals in nondenumerable. Instead, these arguments seem to imply that $\aleph_0 \times \aleph_0 \neq \aleph_0$ which is not the case according to the proof of denumerability of the set of rational numbers. These responses reflect the application of finite experience to infinite sets, in particular that part is smaller than the whole or that infinity plus infinity is twice as large as the original infinity.

Intuitions regarding the fitting of numbers

Almost all of the PSTs' explanations in response to questions on how numbers from the two sets fit amongst each other were intuitively based, with the majority relying almost entirely on the decimal representation of numbers. Following are some examples of common responses, categorized according to whether or not they are consistent with the formal dimension of knowledge.

Justifications inconsistent with the formal dimension of knowledge

- Irrational numbers are so dense, you can find two that do not have a rational in between.
- There will be two irrational numbers that are closest to one another; that is, spaces between irrational numbers can be infinitely small so that nothing could fit between.
- I believe numbers alternate: rational, irrational, rational, irrational, … So there will be some closest rational numbers where only an irrational will be found. Similarly, between any two closest irrationals, you'd find a rational, not an irrational.
- Two non-patterned decimals can exist without a number that has a pattern existing between them. The two irrational numbers can be very close, but not the same.

Justifications consistent with the formal dimension of knowledge

- Let a, b $\in$ Irrational and a□b. There must exist (a+b)/2 which could be rounded to some nearby rational number so that this number would fall between a and b.
- You can find a rational number between any two irrationals by terminating the decimal expansion of the larger number such that you create a number bigger than one and smaller than the other.
- There should always be a terminating decimal between any two infinite non-repeating decimals.
- It is always possible to find an irrational number between any two rationals: just expand the decimal expansion so that it neither terminates or repeats *and* it is bigger than one and smaller than the other.

It seems that the reasons for many of the ill intuitions regarding the fitting of numbers lie in the non-intuitive character of the infinite. In addition, rational numbers, in spite of being everywhere dense, are in fact very sparse in comparison to irrationals and moving between these two conflicting ideas may cause inconsistencies to erupt. All this is non-obvious, and often not convincing. Cantor's proofs are both simple yet very sophisticated, leaving many who have contemplated them still in doubts. Sometimes these doubts come not from what they show us, but from what new questions they open up for us, and leave unanswered. Therefore, there are epistemological obstacles that may account for the difficulties preventing learners from concluding, for example, that there is a rational number between any two irrationals. Furthermore, the formal knowledge that the irrationals by far outnumber the rationals, encourages the thinking that there must be some closest, neighbouring irrationals between which no rational number can be found. An excerpt from interview with Kyra responding to whether a rational number can always be found between two irrationals exemplify this.

Kyra: No, just because it's so, it's so dense, **the amount of irrational numbers is so dense**, I don't think, I don't think in every case you would find, because if you could find a rational number between any two irrational numbers, that would mean that the richness, that wouldn't hold, it would have to be equal richness, in order to find one, so to be consistent, I would have to say no. . .

What we see here, is the mind's desperate effort to accommodate new evidence brought about by the exposure to new formal knowledge, such as that of cardinal infinities. Sometimes consistent connections fail to be created.

Intuitions on operations

The table below shows the quantification of responses to Questions 4 and 5. For easier reference, the correct response is shaded.

Item	**False [%]**	**True [%]**	**No answer [%]**
(4) irrational + irrational = irrational (always?)	19 [41.3]	23 [50]	4 [8.7]
(5) irrational × irrational = irrational (always?)	16 [34.8]	21 [45.6]	9 [19.6]

Table 9: Quantification of responses to Questions 4 and 5 – The sum of two positive irrationals is always irrational? The product of two different irrationals is always irrational. (n=46).

Looking at these results, what stands out is that the majority of responses to questions 4 and 5 were incorrect. We suggest that one of the main reasons for difficulties is the disposition towards closure of operations within a number set, in this case, the set of irrational numbers. Further, the results indicate that the majority of participants' have difficulties conceiving of an irrational number as an object. Instead, there is a perception that a number with infinite decimals as being constructed in time by the process of endless summation of its decimals. The following responses exemplify this view.

Justifications inconsistent with the formal dimension of knowledge

- When you multiply two numbers each with an infinite number of digits together, the result will still be a number with an infinite number of digits.
- I use decimal representations. Because the decimal representations of irrational numbers cannot be terminated, the sum of such numbers will be a decimal that cannot be terminated.
- You cannot add $\sqrt{2}+\pi$, but you can add their decimal representations. The sum cannot be a terminating decimal.
- The sum of two irrational numbers is irrational because $2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$.
- No pattern × no pattern = no pattern. You can't create a pattern through multiplication

from something that has not pattern to begin with.

- You cannot add two irrational numbers because they both continue forever so you would be adding infinitely.

Justifications consistent with the formal dimension of knowledge

In contrast, the following responses utilize the formal knowledge of operations with irrational numbers and also exhibit a higher level of concept development.

- $(2+\sqrt{3})+(2-\sqrt{3})=4$
- $\sqrt{3}\times 2\sqrt{3}=2\times 3=6$ (c)
- Proof by counterexample. You can find two irrational numbers that create a repeating decimal expansion. For example,

$$\begin{array}{r} 0.12122122212222\ldots \\ +\,0.21211211121111\ldots \\ \hline 0.33333333333333\ldots \end{array}$$

As a general observation, we found that there was a great reliance on decimal representation, even when a symbolic representation would be more appropriate and revealing, and a general lack of competency in evaluating the adequacy of statements related to operations with irrationals. An ability to flexibly move between representations in considering the truth of these statements was exhibited by as few as 4 participants. Vast majority of participants incorrectly argued that adding two positive irrational numbers will always produce an irrational number, and likewise, that multiplying two different irrational numbers must result in an irrational number. Interestingly, there was not a single case of drawing upon a standard procedure, such as the commonly used "rationalizing the denominator". This reveals that algorithmic knowledge can become highly procedural and rote for the learner to the extent where the very purpose of using such procedures may be completely lost. It indicates there is a problem in the integration of algorithmic, formal and intuitive knowledge. We interpret the strikingly poor performance on these items as an indication that the notion of irrational number, such as $5+\sqrt{2}$ for example, is commonly conceived operationally (as a process) rather than structurally (as an object) (Sfard, 1991).

Conclusion

In this report, we centered our attention on the complex notion of intuition as manifested in the participants' responses regarding the relations between the two infinite sets (rationals and irrationals) that comprise the set of real numbers. Our findings indicate that underdeveloped intuitions are often related to weaknesses in formal knowledge and to the lack of algorithmic experience. Constructing consistent connections among algorithms, intuitions and concepts is essential for having a vital (as opposed to rote) knowledge of any mathematical domain, and therefore also for understanding irrationality. It is clear that intuitions cannot develop in a vacuum. What is often missing, particularly in this domain, is the attention to algorithmic dimension that may serve as a basis for securing the formal knowledge, by means other than solely based on the definition. In cases where formal knowledge had been secured, learners were capable of seeking acceptable explanations, such that would not violate their formal knowledge. As anticipated, intuitions and beliefs that individuals held revealed a great deal about their understanding of number in general, and also about their formal knowledge of irrational number in particular. This is not surprising given that most of the questions posed can only be considered after the concepts of irrational and real numbers have been solidified into objects and seen as new members in the category of number. Only in such state can one meaningfully investigate general properties of various sets of numbers and relations between their representatives, or solve problems involving finding all the instances of the category which fulfill a given condition. The evidence suggests that only about 10% of participants of this study achieved this stage of the concept development.

References

Fischbein, E., (1987). Intuition in Science and Mathematics: An educational Approach. D. Reidel, Dordrecht.

Fischbein, E.: 1994, 'The irrational numbers and the corresponding epistemological obstacles', Proceedings of PME-XVIII, vol. 2, pp. 352-359, Lisboa

Fischbein, E., Jehiam, R., & Cohen, C. (1995). The concept of irrational number in high school student and prospective teachers. Educational Studies in Mathematics, 29. 29-44.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in mathematics.* v22, p 1-36.

Stevenson, W.S. (2000). *Exploring the Real Numbers*. New Jersey: Prentice Hall.

Tall, D.O. (2001). Natural and Formal Infinities. *Educational Studies in Mathematics*. v48, n2-3, p.199-238.

Tirosh, D., Fischbein, E., Graeber, A.O., & Wilson, J. Prospective elementary teachers' conceptions of rational numbers. Taken from World Wide Web on June 5, 2003. http://jwilson.coe.uga.edu/Texts.Folder/Tirosh/Pros.El.Tchrs.html

HOW ARE STUDENTS' UNDERSTANDINGS OF FUNCTION AFFECTED BY ENGAGING IN A CURRICULUM MODULE IN KNOT THEORY?

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We report on a study of how undergraduate mathematics students' conceptions of function develop in the context of a curriculum module that focused on classification of knots. Knot invariants, functions from the set of knots into various other sets (including numbers and polynomials), provided tools to engage in the study of advanced mathematics but also provided challenges for students as they tried to reconcile these invariants with their concept images of function.

Introduction

There is wide agreement among mathematicians and mathematics educators about the importance of the function concept in mathematics. The recent report, *The Mathematical Education of Teachers* (CBMS, 2001) states, "Prospective high school mathematics teachers need to acquire deep understanding of the concept of a function." Refined and developed understandings of functions are essential for teachers to teach mathematics with understanding. An example is Lloyd and Wilson's (1998) case study, which suggested, "teachers' comprehensive and well-organized conceptions contribute to instruction characterized by emphases on conceptual connections" (p. 270). However, research indicates that students do not tend to develop an advanced notion of function through their school experiences (Breidenbach, Dubinsky, Hawks & Nichols, 1992; Dreyfus, 1990).

As a way to address these points, one of us taught a two-week curriculum module in knot theory as part of a college geometry course to 21 mathematics majors (of which 12 were preservice secondary teachers). The module focused on the classification of knots, a major question in the field, using various knot invariants including the Jones polynomial. A knot invariant is a function from the set of all knots to some other set (e.g., integers or polynomials). The curriculum made explicit connections between the term "invariant" and the mathematical construct of function.

In this report, we investigate how students' understandings of function were affected by their study of knot theory. This work forms part of a larger study that addresses questions related to various aspects of students' engagement in the curriculum module.

Literature Review

There is a considerable body of literature dealing with students' understandings of the concept of function (e.g. Markovits, Eylon, & Bruckheimer, 1988; Sfard, 1991). However, there are few studies that address students' understandings of function in contexts more advanced than algebra and single- or multi-variable calculus (for an example, see Asiala, Brown, Kleiman, & Mathews, 1998). There is wide agreement about the importance of the function concept: function is a unifying theme in mathematics, its historical evolution can reflect the development of the individual's understanding, and it is a primary concern of mathematics educators (Kleiner, 1989; MacLane, 1986; Sfard, 1991).

Theories of Lakoff & Nunéz; Piaget; Sfard; Tall, Thomas, Davis, Gray, & Simpson helped frame the study. Tall and colleagues (2002) discuss various theories which describe students' transitions from viewing mathematical ideas as processes to viewing them as objects. These transitions are sometimes called “reification” (Sfard, 1991). In this study we discuss the operational and structural aspects of students’ conceptions of function in the non-traditional context of knot invariants.

The interplay of empirical and pseudo-empirical abstraction (Piaget, 1972) with the use of conceptual metaphors (Lakoff & Nunéz, 2000) is useful in understanding how students attempt to make sense of functions in a novel context. In the curriculum, functions are knot invariants which have as domains the set of all knots (a knot being an equivalence classes of 2-dimensional representations called knot projections) and whose co-domains include both sets of numbers and sets of polynomials.

In this study we address the following questions:

- How do students understand the function concept in the context of our knot theory curriculum?
- Do students view knot invariants as functions?
- What issues arise as students engage in mathematics that uses an expanded notion of the function concept?

Data Sources

Inquiry into the subtleties of the development of students’ understandings of function is well served by clinical interviews, as well as collection of student work and observations of classroom activity. These were the data sources employed in this study.

Participants volunteered for the study. Of the 21 students, 18 offered their work and 12 offered to participate in interviews. Eight students were chosen for interviews based on criteria that would provide interviewees with a range of abilities, difference in gender, and variation in proposed career. The semi-structured clinical interviews were administered and audio taped by the researcher who was not instructing the class.

The first interview focused on questions ranging from “What do mathematicians do? ” to “Tell me what you know about functions.” The exit interviews probed into students’ experiences with the curriculum with questions such as “What can you tell me about functions in regard to knot theory?”

Analysis

Observations of all classroom meetings (5), student work from 18 participants, and transcripts of preliminary and exit interviews with eight participants were analyzed using open and selective coding techniques (Strauss & Corbin, 1998). Initial coding of interview transcripts was based on areas of understanding identified in the literature, for example, indicators of Sfard's categories of interiorization, condensation, and reification. The data was rich in metaphors for function as input/output machines and as mappings or directed links. After initial analysis, additional issues were identified and transcripts were coded further, noting students’ understandings concerning various attributes of functions, such as domain, range, univalence and one-to-one.

Mathematics

A knot invariant is a function from the set of knots to some co-domain. The numeric invariants, crossing number, unknotting number, and linking number, each map knots to integer values. The Jones polynomial has as co-domain integer-coefficient polynomials in the indeterminate x. Knot theorists hope to find a complete, or one-to-one, invariant as that would amount to a classification of knots. However, to date, no useful complete invariant has been found. A knot can be thought of as an equivalence class of very different looking (2-dimensional) projections which all represent the same embedding of a circle in three space. In addition to knot invariants, the curriculum dealt with functions from the set of projections including writhe (co-domain: integers) and the Lake and Island (L&I) polynomial (co-domain: polynomials).

Results

This study indicates a wide variation among students in the depth of understanding of functions and in their flexibility to move from their experience with real functions to a context in which function provides a framework for the study of advanced mathematics. A substantial portion of the population did not have the ability to see a knot invariant as a function or to differentiate between a relation and a function in the context of knot theory, whereas a small number of students were more successful in the transition.

Among the eight interviewees, three levels of understanding are worthy of delineation:

- Low extreme: two students barely recalled function as an aspect of the knot theory curriculum.
- Mid-range: three students developed an expanded understanding of functions that include function as input/output machine with new domains and co-domains and vague notions of functions as mapping or directed links.
- Advanced: three students demonstrated understandings of function that included univalence, arbitrariness of sets, using both input/output machine and directed links metaphors. They demonstrated a rich understanding of knot invariants as functions.

A particular focus of our report is a comparison of students in the mid- and advanced ranges. The contrast allows us to highlight the transition through interiorization and condensation of the mid-range students vis-à-vis advanced colleagues who have reified the concept of function. We identify initial tendencies of mid-range students to view knot invariants operationally rather than structurally. These students began to exhibit a shift in understanding of function from operational toward structural conceptions. We examine the extent to which the advanced and mid-range students have assimilated function using conceptual metaphors.

Advanced students

Ken and Rick have no difficulty recognizing knot invariants as functions. Ken: "we put the knot into a function … and … output some sort of result." Rick: "the function takes the knot and yields a result, often numeric." They are so comfortable with the function concept as to appear nonchalant. Ken says functions are "just another tool that we're using." Rick has assimilated the idea of knot invariant as function to the point where he assumes the interviewer must be probing for something else when asking about the uses of function in knot theory: "except to find various properties of the knot, none that I could think of." These students have no difficulty adding knot invariants to their robust understanding of functions. Indeed Ken points out how functions are useful precisely because they are "so general."

Mid-range students

First we describe each of the three mid-range students, Mark, Dave, and Jesse, in detail and then we compare understandings of these students. Mark seems comfortable thinking of function as a "mapping of one set of numbers onto another set of numbers." He is very aware that the knot invariants are not complete invariants, i.e., not one-to-one functions. This awareness seems to make him reluctant to call the knot invariants functions. In fact, he even asserts that to be a function a mapping must be onto saying "if it's not onto, it's not one-to-one, it's not a function…in the normal algebraic sense."

In spite of this confusion, Mark is capable of bringing a linking metaphor for function to mind on prompting ("What's the mapping?") saying, "The knot is the dependent variable… and the actual result [crossing number, unknotting number, linking number] is the dependent variable." And he reminds himself that the Jones polynomial is also a function in the mapping sense: "you're taking a discrete object such as a knot and mapping that into something else, in this case a polynomial."

Polynomial invariants are also a source of confusion; polynomials "are clearly functions of something," but he's not sure what. In addition he wonders, "What does x represent as far as the knot?" A crossing number of three says something clear about the knot, but what does x-cubed say about the knot?

Finally, he resolves his dissonant cognitions about knot invariants as functions by creating two senses of function – the algebraic and "the knot sense of function."

Dave thinks of functions operationally, with inputs and outputs and is comfortable with the range being a set of numbers. At first he is fairly confident that crossing number and linking number are functions. He contends that you put in a knot (or a projection of a knot; he seems uncertain, perhaps due to a homework problem) and you get out a number.

Dave contends he thought of the "polynomials as functions just because I have always associated polynomials with functions" (and wonders why we never looked at graphs of the polynomials). But then he makes sense of the Jones polynomial as a function (operationally) noting that to construct it, "you have to put the writhe in, you put the L&I in, and out pops your Jones polynomial." For Dave, the Jones polynomial is "like" a function because of the input/output process.

Dave connects the ability, or inability, to distinguish knots to the one-to-one property of function. He notes that the crossing number is a function but it can't distinguish a knot. He said, "It wasn't like a one-to-one function per se. You can't just say the crossing number had 9, so it's this sort of knot. Where the polynomials, you could do that." He is aware that crossing number is not one-to-one but apparently believes that "the polynomials" are.

Jesse was asked if functions came up during the study of knot theory and replied, "Um, you know what? He did talk about functions. The Jones polynomial takes all knots and maps it to individual Jones polynomials or whatever." So, in his disinterested manner, Jesse expresses his understanding that the Jones polynomial is a mapping from knots to Jones polynomials, and might think it's a one-to-one mapping. When asked about other mappings in knot theory he recalls writing a lot of things using function notation, like "W of K, which was like the writhe." As he thinks further he expresses a belief that writhe is not a function because "there's lots of different knots with writhe zero." Here he mistakes multiple-to-one as multivalent, something he seems to know a function can't be. He concludes that although function notation was used for things like writhe, they weren't necessarily functions.

After the interviewer asks Jesse if these were functions he hesitates, saying "it just seems like they, like I was thinking of a function as only, well, no." Upon his conclusion they were not functions the interviewer asks, "What is a function?" Jesse expresses his understanding of function using a directed link metaphor, it "takes a collection or set of numbers and puts it on or connects it to another set," and input/output machine metaphor, it "takes the numbers in the set, turns them into these numbers over here." He sees sets of numbers as the domain and range of functions. This is when he realizes it's okay to have "two different numbers in the first group go to the same number in the second group." Further, he articulates the univalent requirement, saying "the thing that makes it not a function is if one number in the first set goes to two different numbers in the second set."

Jesse concludes, "it would be a function because I was thinking if different knots go to the same writhe or same crossing number that that wouldn't make it a function, but that's still okay." Upon questioning he identifies, for the writhe, crossing number, and unknotting number, the domain as the set of knots and the range as "just numbers."

Comparisons of mid-range students

There are some interesting common features of the mid-range students' understandings of the function concept. Jesse and Mark are both familiar enough with the concept of functions to be able to correctly recall the idea of mapping a set to another. When they make this connection, they are able to see how the knot invariants are functions. They are functions in this mapping sense. However, this mapping sense is not a primary sense of function for them. It requires some prompting and a conscious effort to bring the mapping picture of function into their minds. And even when they recall this for a minute, they are likely to quickly forget about it soon after. The mapping idea of function is something that they are familiar with and can talk about, however it is not yet part of their immediate associations with function.

Rather, they immediately associate functions with mappings of numbers. Polynomial invariants are especially confusing. As polynomials they are immediately identified as a type of function in the number mapping sense. However, this leads to questions like Mark's, "What does it mean to substitute x=2?" and Dave's query about the graph of a Jones polynomial.

All three confuse function with complete invariant or one-to-one function. This is perhaps a reflection of the way they were taught. The instructor emphasized that none of the invariants discussed in the module were complete invariants. Unfortunately, this seems to have left these students with the impression (perhaps due to a confusion between univalence and one-tooneness) that the invariants are also not functions.

Jesse and Dave bring out further possible sources of confusion. In a homework assignment, students were asked to find how many times a projection crossed over itself. This use of the idea of crossing number for both projections and knots seems to have left them confused about the domain of the crossing number function. Also, functional notation was used in the same way for functions with domain in the set of knots and functions with domain in the set of projections. Failing to make the distinction in the notation may be another source of confusion.

There is evidence that the mid-range students are expanding their notion of function as a result of our curriculum. Jesse and Mark have begun to see knot invariants as mappings. The issues with univalence and one-to-one characteristics of functions present challenges and require students to re-visit their notions of function. This not only clarifies the meaning of these characteristics, but also expands the students' concept images of function to include those with polynomial co-domains and those with domains consisting of equivalence classes.

Conclusions

- How do students understand the function concept in the context of our knot theory curriculum?

Although a focus on students' understandings of function was a major factor in the design of the curriculum, some students remained unaware of any connection between functions and knot theory. Other students were very comfortable with the idea of function as a useful tool in understanding knots. In the mid-range, several students were clearly challenged by the curriculum to expand their notions of function. Although these students have not yet reified the function concept, the curriculum seems to be pushing them toward a deeper understanding of function.

- Do students view knot invariants as functions?

Aside from a few exceptional students, this seems to be a very difficult connection for students to make. This difficulty manifests in various ways. While some students saw invariants as completely divorced from function, others could be made to make the connection with prodding. On the other hand, these mid-range students also came up with strategies like differentiating between a "knot-theory" sense of function and an "algebraic" sense of function. The association of knot invariant with function was problematic because of confusions between univalence and the one-to-one property, because of uncertainty about domains (projections versus knots), and because polynomials as co-domain invited unwanted associations of polynomials with functions.

- What issues arise as students engage in mathematics that uses an expanded notion of the function concept?

Such mathematics brings to the forefront weaknesses in students' understanding of the function concept. This is valuable as it provides concrete evidence that earlier school experiences had not been sufficient to induce reification of the function concept in the majority of our students. Although our curriculum also did not bring students to the reification stage, it did force them to grapple with their understanding of functions. It is only through such efforts to stretch themselves that students can hope to attain mastery of the function concept.

Recommendations

The results suggest that incorporation of contexts for function, such as in the study of knots, that include domains and co-domains other than that of sets of numbers might serve to expand student's understandings of function in a way useful for their learning of advanced mathematics. Our experience points out the challenges that students face when presented with extending their notions of function to include unfamiliar domains and co-domains. Instruction that makes explicit these extensions, and the connections between students' prior knowledge and new ideas, might serve to deepen students' understandings of mathematical constructs.

References

Asiala, M., Brown A., Kleiman, J., and Mathews, D. (1998). The development of students' understanding of permutations and symmetries. *International Journal for Computers in Mathematical Learning*, 3, 13-43.

Breidenbach, D.E.; Dubinsky, E.; Hawks, J. & Nichols, D. (1992), Development of the process conception of function. *Educational Studies in Mathematics*, 23, 247-285.

Conference Board of the Mathematical Sciences (2001). The mathematical education of teachers. *Issues in Mathematics Education: Vol. 11*. Providence: The American Mathematical Society.

Dreyfus, T. (1990). Advanced mathematical thinking. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition: A research synthesis by the International Group for the Psychology of Mathematics Education*. Cambridge: Cambridge University Press.

Kleiner, I. (1989) Evolution of the Function Concept: A Brief Survey, *College Mathematics Journal,* 20, 282-300.

Lakoff, G. and Núñez, R.E. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In Lyn E. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images*, 21-89. Mahwah, NJ: Lawrence Erlbaum Associates.

Lloyd, G. M. and Wilson, M. R. (1998). Supporting innovation: The impact of a teacher's conceptions of functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education*, 29 (3), 248-274.

Markovits, Z. Eylon, B. S., & Bruckheimer, M. (1988). Difficulties students have with the function concept. In A. F. Coxford (Ed.), *The ideas of algebra* (33-42). Reston, VA: The National Council of Teachers of Mathematics.

Piaget, J. (1972). *The principles of genetic epistemology*. London: Routledge & Kegan Paul.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Strauss, A. & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks: Sage Publications.

Tall, D., Thomas, M., Davis, G., Gray, E., Simpson, A. (2000) What is the Object of the Encapsulation of a Process? *Journal of Mathematical Behavior*, 18(2), 223 – 241.

RESEARCH AND PRACTICE AT UNIVERSITY LEVEL. THE IMPROPER INTEGRAL

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Introduction and Theoretical Framework

In this work the general outlines of the design of a didactic engineering for First Year University students about the concept of improper integral are shown. The class where our design was developed had about 23 students.

Our design has been built taking into account, primarily, the variations in the didactic contract and the importance of the *medium*[1] proposed by Brousseau (Brousseau, 1998). The elements that characterise our design are the recourse to the graphic register, the use of examples and counter-examples, the production of debates and work in small groups and the explicit use of concepts previously learnt, such as definite integrals and series. As for the cognitive dimension, an analysis of the student's learning of these concepts was done. In order to achieve this, Duval's theory of semiotic representation registers (Duval, 1993) has been used, as some later contributions on the role of errors and problem solving in the theory of representation registers (Hitt, 2000).

Objectives

Our design arises as a response to a group of difficulties, obstacles and errors detected in our students after a traditional teaching about improper integration (González-Martín & Camacho, 2004). In our study about the cognitive dimension of this concept (developed in 2002 with a group of 31 studens), we found that students, after a traditional teaching (lectures at University), do not learn it adequately and that this produces important difficulties, bringing forward a merely algorithmic use of it. On the other hand, it is learnt detached from other concepts and without achieving any coordination between the graphic and algebraic registers.Therefore, we had three main objectives: 1) to analyse the processes involved in the development of the concept of improper integral on the part of the students; 2) to investigate the most common obstacles, difficulties and errors that might arise; 3) to develop a teaching sequence that might help the students overcoming those difficulties.

In the design of our sequence, we must point out that the general aims are: 1) to generate a teaching sequence which includes explicitly both the graphic and algebraic registers; 2) to analyse to what extent the changes in the *didactic contract* modify the students' attitude towards improper integrals; 3) to analyse whether the active use of examples and counter-examples in teaching, as the use of the graphic register, might improve students' learning; 4) to compare the level of understanding of the students who follow our proposal with that of other students who did not experienced our learning sequence, in particular with the group of 31 students previously cited; 5) to identify the strongest difficulties, obstacles and errors linked to this concept.

Data Collection

We have used a number of data collection sources. Firstly, the classroom sessions with the students were videotaped. Moreover, sheets for the work in groups of several activities proposed and individual work sheets have been collected. Lastly, a questionnaire of contents where questions of non-routine type (González-Martín & Camacho, 2004) and reasoning questions addressing the most important aspects of our design, both from the algebraic and graphic point of view, has been developed and answered by the students, as well as an opinion test about the methodology used.

Results and Discussion

The analysis of the questionnaire of contents and the individual and group work sheets shows a greater acceptance of the students towards the graphic register and even its use to justify certain questions (particularly, the divergence of many integrals). Some clearer relations with contents about series also appear. On the other hand, non-routine type questions have a lower no-answer rate than in our cognitive study.

The opinion tests also throw some interesting data. The students state in general that using the graphic register helps them to understand theoretical issues; moreover, several students affirm that the sequence helped them to review contents about definite integration and series. Furthermore, the changes in the didactic contract and the use of small working groups to tackle some questions turned out to be helpful to reason these questions before being institutionalised.

From what has been said, the importance of the *feed-back* between research in the cognitive dimension of students' learning and research in the elaboration of new didactic proposals which try to minimize the obstacles, difficulties and errors detected is evident in our work. The embodiment in teaching of the difficulties and obstacles detected as explicit elements made the students more aware of their learning.

On the other hand, the explicit use of previous knowledge and the graphic register, the construction of examples and counter-examples and the variations in the didactic contract are elements that have enriched our design and have produced a good attitudinal response in our students. Moreover, the results in the questionnaire of contents are encouraging.

It remains as a task the design of new didactic sequences for other concepts at the undergraduate level. In fact, one of the biggest difficulties of our implementation has been the difficulties students show in concepts such as limit, sequence or definite integral. The challenge of constructing new sequences from theoretical results obtained at the undergraduate level appears to be crucial.

Endnotes

1. This is how the French term *'milieu'* is usually translated.
2. This work is partially supported by the contract nº 1802010402 and by grant AP2000-2106 of Spanish MCT.

References

Brousseau, G. (1988). Le contrat didactique: Le milieu. *Recherches en Didactique des Mathématiques.* 9(3), 309-336.

Duval, R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives,* 5, 37-65. IREM de Strasbourg.

González-Martín, A. S. & Camacho, M. (2004). What is First-Year Mathematics students' actual knowledge about improper integrals?, *International Journal of Mathematical Education in Science and Technology,* 35 (1), 73-89.

Hitt, F. (2000). *Construcción de conceptos matemáticos y de Estructuras Cognitivas*, second version of the paper presented in the Working Group: *Representations and mathematics visualization* of PME-NA, Tucson, Arizona 2000, 131-147.

OVERCOMING THE OF TALL VINNER PROBLEMATICS IN THE TEACHING OF THE CONCEPT OF FUNCTION WITH THE ASSISTANCE OF MATHEMATICA

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The purpose of this project was, in the first place, to verify if freshmen students, at the Universidad del Valle, presented similar comprehension problems as those reported by Tall & Vinner (1981) and Vinner (1991) in relation to the concept of function and what was the evolution of these problems at the end of the first academic semester. Secondly, if Mathematica, could contribute positively to the elaboration of a teaching proposal to teach the concept of function in the high school – university transition, which would aid students in overcoming these problems. The project was carried out with physics majors at the Universidad del Valle who were taking a basic mathematics course. This course could be considered a transitional level course from high school to university mathematics. The project obtained positive answers to both questions.

Introduction

The development of this project has to do with the interest in understanding and helping to solve the learning problems in mathematics, presented by high school students who begin technical and scientific studies in Colombian Universities, particularly at Universidad del Valle. In this specific context the difficulties in the understanding of the concept of function are particularly noticeable and, given the mathematical importance of the concept, they are quite influential in the high rate of student failure observed in the first university courses of mathematics.

The notions of c*oncept image, evoked concept image, personal definition* and *institutional definition* of a concept, introduced by Tall & Vinner (1981) and Vinner (1991) were used to address comprehension problems regarding the concept of function. While reviewing Tall and Vinner's propositions, it seemed appropriate to introduce (Alvarez et al.,2001) the following concepts: s*table personal definition, mathematically well-adapted personal definition*, and *local and global coherence* of a stable personal definition .

The term "Tall-Vinner Problematics" was coined (Alvarez & Delgado, 2002) to refer to the following problems:

A. The concept-image constructed by a subject for a given mathematical concept may not be consistent each time the subject tackles a different problem situation, and may present also a number of mathematical-adaptation problems with the institutional definition of said concept (e.g.: The concept-image may be significantly restricted in terms of the more general mathematical definition of the concept). In other instances, the meaning associated with the concept-image, which is put into action in some situations, may conflict with the socially–accepted meaning (mathematical meaning) of the concept.

B. A subject may possess a stable definition that is mathematically well-adapted or mathematically maladapted, and that remains isolated with respect to the concept-image as a whole. There may be instances in which a stable definition is mathematically maladapted and also coherent. However, it is common among university students not to have a stable personal definition with regards to a previously studied concept that is both mathematically well-adapted and coherent with the concept-image evoked when the subject takes on different problem situations.

From this perspective, the understanding of a concept is associated with overcoming the Tall Vinner Problematics. When this is the case it is said that the subject has constructed or appropriated; regarding a given concept, *a well-constituted or mathematically well-adapted personal concept.*

Methodology Used

It was composed of an experimental group (GE) (10 students), a control group (GC) (11 students), and a group called open (GA) (40 students). GE and GC were made up of students in a basic mathematics course offered to physics and science students. The GA group was composed of engineering students who were taking at the same time, a beginning course in calculus without any relationship to the two previous groups. *The Process*

In the context of the mentioned course, the GE students were taught a chapter on functions, elaborated by the project group. Mathematica was used to teach it. The GC students, who formed part of a larger group which was taking the same course of basic mathematics, was also taught a chapter on functions using the same materials and homework of the experimental group, with adaptations for not using Mathematica. Both groups had different professors but the courses maintained academic coordination to ensure coverage of the same contents and emphasis. The GA group was not targeted in our intervention.

In order to investigate the state of the personal concept of function of students before and after the intervention, the three groups were given the same exam at the beginning (moment A) and at the end of the semester (moment B).

Results

The table below shows the percentages of students, that according to the vector valued indicator IBCF, have levels 1 and 2 of basic comprehension of function, at moment A as well at moment B.

Exam	Level	GE Group	GC Group	GA Group
A	1	0%	0%	17.1%
	2	0%	0%	0%
B	1	63.6%	30.0%	27.3%
	2	45.5%	10.0%	9.1%

It is interesting to know the students´ answers to the Vinner question (the .function defined by parts). At moment A, practically 90% of the students of the GE and GC groups did not identify, as such, the proposed function. At moment B, after the teaching of the concept of function, these percentages decreased to 0% in the GE group, but remained at 50% in the GC group. In the case of the GA group, at moment A, 58.5% responded that there was no function and at moment B, 18.2%, maintained the negative response.

The better results obtained with the GE group lead to the question if the differences with other groups are statistically significant. The non parametric Mann-Whitney test, for small unequal samples was applied. When the GE group is compared with the GC and GA groups, the p values, associated with the contrast statistics, are low for some indicators and relatively high for others. This does not represent any contradiction.

Conclusions

A review of the results obtained demonstrate that, among Colombian students, similar problems to those reported by Tall and Vinner, regarding the comprehension of the concept of function, are presented. The results obtained with the Vinner example is a proof of this.

The great majority of students who enter the Universidad del Valle have a very precarious personal concept of function, inappropriate for successful study of the first course in calculus and much less in linear algebra. This result validates hypothesis 1i. Comparing the results of the GA group in the first moment, with the results of the other groups, it can be concluded that the situation is more critical among science students than among engineering students.

At the end of the semester there are still appreciable percentages of students, in all groups who have not achieved acceptable levels of comprehension of the concept of function, being particularly high in the GC and GA groups. The low evolution which is observed in this respect demonstrates that the Tall-Vinner problematics is not overcome spontaneously, and that a student, even after a first calculus course,might still have serious comprehension problems of the concept of function, with its negative implications in his/her learning. This result validates hypothesis 1ii.

The comparison of the results of the GE, GC and GA groups, which clearly favors the GE group, indicates that the teaching proposal that was elaborated with the help of Mathematica favours the overcoming of the Tall – Vinner problematics, in the case of function, and suggests the validity of hypothesis 2i.

The results indicates significant statistical differences in some indicators but it can not be concluded decisively that said differences can be attributed exclusively to the presence of Mathematica in the elaboration of the proposal.

There are reasons to believe, however, that the advantages that Mathematica offers to carry out conversions among representations that were considered in the teaching proposal, and the way it can be used to transform teacher –students interactions, are factors which play an important role in the positive results of the proposal. It is worthwhile to mention, however, that the utilization of Mathematica guided by another kind of approach, could strengthen among the students the conception of function as a mathematical expression, restricting the general meaning of the concept.

References

Álvarez, J., & Delgado, C. (2002). The Tall-Vinner problem. An operative reformulation In A. Cockburn, & E. Nardi (Ed.), *Proceedings of the 26th Annual: Vol. 1.* (pp. 261). Norwicch: University of East Anglia.

Álvarez, J., Delgado, C., Espinosa, A., Pinzón, M., Hoyos, D., & Mora, H. (2001). El problema de la imagen conceptual en el aprendizaje de función. (Tech. Rep. No 3). Cali, Colombia. Universidad del Valle, Grupo Educación Matemática. Final Report presented to COLCIENCIAS.

Tall, D., & Vinner, S. (1981). Concept images and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12,* 151-169.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. advanced mathematical thinking. In D. Tall (Ed.), *Advance mathematical thinking* (pp. 65 – 81). Kluwer Academic Publishers.

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THE ROLE OF THE REVERSE THINKING IN THE CONTEXTS OF LIMIT

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The concept of limits has been regarded as one of the most foundational concepts in mathematics. However, educational studies indicate that students have experienced difficulties in understaning the concept of limits (Szydlik, 2000; Tall, 1992; Tall & Vinner, 1981; Williams, 1991). Students' difficulties in learning limits tend to make it hard for them to learn other related topics such as continuity or infinite series (Cornu, 1991; Merenluoto & Lehtinen, 2000; Tall & Schwarzenberger, 1978).

This study explores the role of the reverse thinking, appeared in the formal definition, in intuitively understanding the concept of limits. Intuition is regarded as the immediate cognition that is accepted directly without the feeling that any kind of rigorous justification is required (Fischbein, 1994). In line with this viewpoint, we use "intuitively understanding" in this study as immediately cognizing some concepts without any kind of rigorous justification process. On the other hand, the reverse thinking has been regarded as the ability to get the solution or the result of a problem as well as to understand the process by working backward from the answer to the starting point (Adi 1978; Bryant et al., 1999; Driscoll & Moyer, 2001; Ferrandez-Reinisch, 1985; Inhelder & Piaget, 1958; Krutetskii, 1969; Wagner et al., 1984). In the case of limits, when we read the symbol $\lim_{n\to\infty} = A$, we say that as n goes to infinity, a approaches A . However, in the case of the formal definition of limits, we have to reverse the thinking process in the following sense: A sequence a is convergent to a real number A if for any positive number ε, there is a natural number N such that $|\ a_n - A\ | <$ε whenever n N. (Apostol, 1974, p. 183.) As above, we first choose any positive number ε, and next take a proper index number N of the sequence. Hence, we call such a process the reverse thinking in the formal definition of limits. The reverse thinking plays a crucial role in representing the formal definition of limits; nevertheless, there is dearth of empirical studies investigating the role of the reverse thinking in understanding the concept of limits. The research question guiding this study is addressed as follows: *How does the reverse thinking play roles in intuitively determining the convergence and the limit of a sequence?*

Methodology for this study is in line with the qualitative research. We conducted this study in a midwestern university with a fairly diverse department of mathematics. A subset of students enrolled in second quarter calculus courses for biology majors had voluntarily taken the pretest. Four students were selected to complete 1-hour semi-structured interviews once a week for 5 weeks. Interviewees were asked to determine the convergence and the limit of a sequence given in the numerical context, the graphical context, and the visual context of the reverse thinking. In particular, conducting the visual context of the reverse thinking, strips with various widths, corresponding to ε, were provided to students and students were asked to put each strip, centered at the anticipated limit, on the graph of a sequence and to describe the distribution of points inside and outside each strip. All interview sessions were videotaped and transcribed. We also analyzed class activities observed, homework, and exams in order to build grounded theory around the main themes emerged.

Through analying the study, we result in that students who considered the reverse thinking in intuitively determining the convergence and the limit of a sequence clearly described their responses; on the other hand, those who did not considered the reverse thinking were confused or inappropriately described their responses. Hence, the reverse thinking plays a crucial role in leading students to proper response in intuitively determining the convergence and the limit. Therefore, the reverse thinking should be considered as an important factor even in intuitively learning the concept of limits as well as rigorously.

References

Adi, H. (1978). Intellectual development and reversibility of thought in equation solving. *Journal for Research in Mathematics Education, 9*, 204-213.

Apostol, T. (1974). *Mathematical Analysis*. Philippines: Addison-Welsely.

Bryant, P., Christie, C., & Rendu, A. (1999). Children's understanding of the relation between addition and subtraction: Inversion, identity, and decomposition. *Journal of Experimental Child Psychology, 74*, 194-212.

Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 153-166). Dordrecht, The Netherlands: Kluwer Academic.

Driscoll, M., & Moyer, J. (2001). Using students' work as a lens on algebraic thinking. *Mathematics Teaching in the Middle School*, *6*, 282-287.

Ferrandez-Reinisch, A.(1985). The acquisition of inverse proportionality: A training experiment. *Journal for Research in Mathematics Education*, 16, 132-140.

Fischbein, E. (1994). The interaction between the formal, the algorithmic, and the intuitive components in a mathematical activity. In R. Biehler, R. W. Scholz, R. Sträßer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp.231-245). Dordrecht, The Netherlands: Kluwer Academic.

Inhelder, B., & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence*. (A. Parsons & S. Milgram, Trans.). New York: Basic Books.

Krutetskii, V. A. (1969). An analysis of the individual structure of mathematical abilities in school children. In J. Kilpatrick & I. Wirszup (Eds.), *Soviet studies in the psychology of learning and teaching mathematics* (Vol. 2). Standford: School Mathematics Study Group.

Merenluoto, K., & Lehtinen, E. (2000, April 24-28). *Do theories of conceptual change explain the difficulties of enlarging the number concept in mathematics learning?* Poster session presented at the AREA annual meeting, New Orleans, LA.

Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of function. *Journal for Research in Mathematics Education*, *31*, 256-276.

Tall, D. (1992). The transition to advanced mathematical thinking: functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495-511). New York: Macmillan.

Tall, D. O., & Schwarzenberger, R. L. E. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, *8*(*2*), 44-49.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151-169.

Wagner, S., Rachlin, S. L., & Jensen, R. J. (1984). Algebra learning project (NIE No. 400-81-0028).

Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, *22*, 219-236.

LEARNERS' CONCEPTIONS OF THE LIMIT IN CALCULUS

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The limit concept in calculus remains troublesome for learners to grasp. Two factors stand out: the limit concept is complex, and it poses particular cognitive challenges. This study investigated from a qualitative, cognitive theoretical perspective, how postsecondary students who have successfully completed an introductory calculus course conceptualised the limit. The findings reveal that participants' conceptualisations of limit vary by mathematical context and in relation to the participants' academic backgrounds. The study suggests follow-up investigation of what it means to learn a difficult mathematical concept in general, questions the inclusion of limit in introductory calculus courses, and suggests further investigation of the relationship between instruction and students' conceptions as a basis for improving learning of limit and other difficult mathematics concepts.

Introduction

The intuitive idea of a limit involves getting as close as one wishes to some goal. This was known to the classical Greek mathematicians over two thousand years ago, and enabled scientists and engineers through the centuries to solve problems by calculating quantities that were not otherwise accessible. This made the limit concept important in calculus and in learning calculus. However, the limit concept has been, and still remains, a troublesome concept for learners to grasp. Two factors stand out: the limit concept is complex, and it poses particular cognitive challenges. Calculus reform has resulted in new curriculum approaches, but is still controversial. Descriptive constructs such as concept image and concept definition (Vinner, 1991), and the role of symbols to link these, help to account for observations of learners. However, the learning problems persist (Przenioslo, 2004). Clearly, more needs yet to be known about learner thinking about limits.

Most studies involving the limit concept, in addition to not providing conclusive answers to remedy the challenges students face in learning limits, have been confined to a relatively narrow mathematical context, primarily analytical, and to a relatively homogeneous group of learners, primarily undergraduate mathematics majors not long out of high school. So, the outcomes of previous studies were constrained by these factors. Learners also have difficulty transferring their mathematical knowledge to new contexts. Accordingly, the present study tries to help to understand learner conceptions of the limit concept by relating them to different mathematical contexts and learners' experiences. Specific focus of this study, which is continuing, was on how postsecondary students who have successfully completed an introductory calculus course conceptualise the limit concept, and possible implications for mathematics education.

Theoretical Perspective

This study draws from a humanistic and an empirical perspective of qualitative research, collecting qualitative data by observation, document review and in-depth interviewing (Best and Kahn, 2003). Analysis of the data involved data reduction, matrix display and examination, and conclusion drawing and verification (Sowden and Keeves, 1988). The two key constructs of the study are limits and conceptions. Conceptions and learning are defined from a cognitive perspective, which serves as the underlying theoretical perspective of the study. The limit concept is defined in terms of its history and characteristics of mathematical concepts.

Data Collection and Analysis

In seeking to uncover conceptualisations, not comparisons, the only requirements were that the participants had been exposed to the concepts of function and limit, and that they were willing to participate in the study. Fourteen volunteers participated. They had diverse backgrounds in mathematics, careers, maturity, life circumstances, and the elapsed time since learning the limit in calculus. Nine were current students or recent graduates, two of whom were in a university mathematics program, the others being engineering related. Of the remaining five, one is employed as an economist, one in telecommunications, one supervises an electronics teaching laboratory, and two are teachers in engineering. Volunteers were difficult to find and more than half were invited personally. It was possible to retain some diversity of academic and personal background, but engineering and business academic backgrounds predominated. Several of the volunteers had not revisited the concepts in the five or more years since they graduated.

Data collection began with task-based interviews (Goldin, 1998) incorporating concept maps and other mathematical activities. The script was first tested with one participant, then modified before being given to three others, and again with the first participant several months later. However, this provided data with insufficient depth, resulting in the decision to extend the number of participants and thus a shift in the nature of the research design to include a qualitative empirical component through the use of a questionnaire. The questionnaire was tested on five volunteers, but required only minor formatting modifications. The original four interviewees completed the questionnaire offsite, but the others completed it in one sitting at a specific location. There was a gap of several months between interviews and interviewees doing the questionnaire, during which time their thinking might have evolved, and which may have resulted in inconsistencies in the data.

Findings

The findings reveal that the participants' conceptualisations of limit varies by mathematical context and in relation to the participants' academic backgrounds. The study clearly established that limit is a difficult concept to learn and suggests follow-up investigation of what it means to learn a difficult mathematical concept in general. Since there was no evidence of understanding the limit concept, yet many participants had passed introductory calculus course, and one an advanced course, the study raises the question of the necessity to include limit in introductory calculus courses and suggests follow-up studies to investigate the impact on learning calculus with and without limit. Finally, further investigation is suggested of the relationship between instruction and students' conceptions as a basis for improving learning of limit and other difficult mathematics concepts.

References

Best, J. W. and Kahn, J. V., 2003, *Research in Education* (9th ed.), Boston: Allyn and Bacon.

Goldin, G. A., 1998, Observing mathematical problem solving through task-based interviews. *JRME Monograph Number 9: Qualitative research methods in mathematics education*, pp. 40-62

Przenioslo, M., 2004, Images of the limit of function formed in the course of mathematical studies at the university. *Educational Studies in Mathematics* v. 55, pp. 103–132.

Vinner, S., 1991, The role of definitions in the teaching and learning of mathematics. *Advanced Mathematical Thinking*, D. Tall (ed.), Dordrecht: Kluwer Academic.

ELEMENTARY AND SECONDARY SCHOOL STUDENTS' MISCONCEPTIONS OF MATHEMATICAL SET CONCEPT

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In this research paper, it is intended to determine what misconceptions of set concept seventh- and 10th-grade students have and if these misconceptions disappear as an effect of time and more formal education of sets. Several misconceptions have been determined among both elementary and secondary school students.

The concept of set is abstract, and the beginning of abstract thinking has not been determined precisely. During the process of abstraction, the concept turns into a hidden model in the brain (Ayas, et al., 2001). People often apply to this model in developing new concepts. Likewise, students frequently apply to this model in their mind during the formation of mathematical set concept. This model called *collection model*. We have found in the literature we reviewed that many of the mathematical misconceptions of set concepts are due to this model (Linchevski & Vinner, 1988). In fact in many texts, the concept of set is defined as "a collection of objects." A major difficulty for students is the fact that students accept *collection model* intuitively in contrast to the formal definition including a number of formal properties and aspects contradicting the initial "collection model" (Fischbein & Baltsan, 1999).

In this research paper, our purpose was to determine what misconceptions of the set concept seventh and 10th-grade students have and if these misconceptions disappear as an effect of time and more formal education of sets.

Data Collection

Mathematical set concept is taught beginning from the first grade to the sixth-grade elementary school and also in ninth-grade high school mathematics curricula in Turkey. Participants in our research were six seventh-grade elementary school students and six 10thgrade secondary school students in Turkey. Students were identified by their mathematics teachers based on the following criteria: low, middle, high in their mathematical abilities. The data collected through written assessment and interviews. The written assessment included open-ended questions (i.e., can the collection of two whole numbers less than 10 form a set and explain why?). All students' answers were recorded in terms of Correct/Incorrect decision and its correctness according to mathematical conventions. Semi-structured interviews were conducted with 12 participants after completion of the written assessment. The interviews were summarized by the interviewer immediately after the interview. During the interviews, students were prompted for understanding that may not have been apparent from their initial responses.

Results

Our analysis focused on the misconceptions of seventh- and 10th-grade students. Our analysis revealed that from Grade 7 to Grade 10 the number of misconceptions students have tended to decrease. This finding contradicts with Fischbein and Baltsan's (1999) study. However, the misconceptions students have, especially the ones about empty set, supported the findings of other studies (Fischbein & Baltsan, 1999; Linchevski &Vinner, 1988). The following misconceptions were identified in Grade Seven students' responses to open-ended questions:

a. Only objects such as numbers or geometric shapes can compose a set. Thus, living organisms cannot form a set. (Five out of six students)
b. The idea of empty set is rejected. In other words, there cannot be such set without any element. (Three out of six students)
c. Empty set cannot be an element of another set. (Three out of six students)
d. A set with an element of zero defines an empty set. (Three out of six students) The same misconceptions in students' assessment occurred with only one or two students in Grade 10. The interviews with students also revealed some other misconceptions in both grade groups. Although these were not observed in the majority of the students, we think that it is important to report them here so that they can be investigated further by researchers.
e. To form a set, it is enough to have a "collection" of elements.
f. Elements without any boundaries such as parentheses or a Venn diagram cannot form a set.
g. The elements of a set must possess a certain explicit common property.

The misconceptions in "a.", and "f." might be related to the presentation of set examples both in Turkish mathematics textbooks and by the classroom teachers. Turkish elementary and secondary school texts we reviewed (i.e., see Demiralp, et al., 1999; Ekmekci, et al., 1999) had very few examples of sets including living organisms. One reason behind the misconceptions in "b.", "c.", "e.", and "g." might be the use of the term "collection" in describing mathematical sets. For instance, in daily life when we talk about stamp collections or toy car collections, we are talking about things that have common characteristics. Stamp collection would not include a small toy and vice versa. Likewise, one cannot discuss a toy car "collection" without having any toy car. Students are usually not aware of the possibility of an inadequate interpretation of an abstract concept because of "hidden" model acting behind the scenes. Finally, our interpretation of misconception in "d." is that in daily life, "absence" of something is represented with "zero". Thus, by transferring the meanings between these two words, students interpret the set with an element of zero as equal to an empty set. "Language is essentially metaphorical, that is, we are profoundly used to automatic transfers of meaning, very often to unconscious transfers of meaning." (Fischbein & Baltsan, 1999, p.4)

References

Ayas, A., Karamustafaoglu, S., Cerrah, L., & Karamustafaoglu, O. (2001, June). *Fen bilimlerinde ogrencilerdeki kavram anlama seviyelerini ve yanilgilarini belirleme yontemleri uzerine bir inceleme* [A study on determining students' conception levels and misconceptions in Science]. Paper presented at the meeting of the 10. Ulusal Egitim Bilimleri Kongresi [10th National Educational Sciences Congress], Bolu, Turkey.

Demiralp, A., Gurkan, M., & Pelit, T. (1999). *Liseler icin matematik-1 ders kitabi* [Mathematics-I for high school textbook]. Ankara: Basari.

Ekmekci, S., Yildirim, H., Ayhan, K., & Yildirim, U. (1999). *Ilkogretim matematik 6. sinif ders kitabi* [Mathematics 6 elementary textbook]. Ankara: Yildirim.

Fischbein, E., & Baltsan, M. (1999). The mathematical concept of set and the 'collection' model, *Educational Studies in Mathematics, 37*, 1-22.

Linchevski, L., & Vinner, S. (1988). The naive concept of sets in elementary teachers. *Proceedings of the 12th International Conference, Psychology of Mathematics Education, Hungary, 11,* 471-478.

CONCEPTIONS OF CONTINUITY:
ONE ADVANCED PLACEMENT CALCULUS STUDENT'S THINKING

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The results of this paper are part of a larger study of high school Advanced Placement calculus students' conceptions of continuity. Data collected for this study consisted of a survey and interview. The student we focus on had difficulty distinguishing continuity from the existence of the function. Results elaborated the contextual dependence of reasoning about continuity that suggests a link between the functional representation used and students' determination of whether a function is continuous.

Over the past two decades, researchers have focused on the ways that students think about functions (Dubinsky & Harel, 1992; Leinhardt, Zaslavsky, & Stein, 1990). This research has shown that many students conceptualize functions as mathematical objects that (a) can be represented by a formula, (b) are differentiable or smooth, and (c) are continuous. These conceptions can cause difficulty as students enter calculus and are asked to begin considering functions that may not fit any of the above three criteria.

Tall and Vinner (1981) found that while students may have a strong mathematical concept definition for a mathematical idea such as continuity, their concept image might not include all of the same complexities. However, students are more likely to rely on their images to help them reason through a problem. Thus, many students have not made a connection between their definitions and images of continuity. This is similar to the disconnect in knowledge that Lauten, Graham and Ferrini-Mundy (1994) describe. They found that the student in their study approached problems quite differently based on whether the problem was presented graphically or symbolically. Students also have a tendency to conflate the concepts of (1) functional existence, (2) existence of limits of functions, (3) continuity of functions, and (4) differentiability of functions (Baker, Cooley, & Trigueros, 2000; Bezuidenhout, 2001). One might interpret these studies from a models and modeling perspective (Lesh & Doerr, 2003) suggesting that students have competing models of continuity and based on the context, they choose the most appropriate reasoning to apply.

This study was conducted in two phases. We began by collecting data on a larger group of students through a questionnaire that would provide us with a picture of the range of students' conceptions. The second part of the study investigated the understandings of a small group of students by engaging them with problems to work on and then, through probing questions to attempt to understand the students' logic and how they are conceptualizing continuous functions. Our goal in these activities was to be able to understand the students' ways of thinking by seeing how these ways of thinking affect the way that the students interact with problems and how the context affects students' interpretations of continuity-related problems. In this paper, we report on our interview with Mandy. Mandy was a senior at Washington High School at the time of the interview.

The results of the analysis of Mandy's questionnaire and interview yielded two findings. We describe each of these findings as part of Mandy's conceptual model and illuminate how she shifted between elements of her model and appeared to hold conflicting models at the same time.

One key element of Mandy's model of continuity was her understanding of the role that the domain of the function plays in determining if a function is continuous. She described her strategy for determining if a function was continuous as:

> I kind of looked to see… to make sure there weren't jumps or breaks… and every now and then, I would remember to look at the domain and see where it was… what the domain was.

Later in the interview she said, "Whatever part they specify in the domain is the only part you have to worry about". She sees the domain as being more important in her decision process than looking for whether the function had jumps or breaks, or was smooth.

Mandy seems to have already incorporated some ideas into her model that helped her to move beyond an intuitive idea of continuity. She seems most able to use this sort of thinking when she can look at a graph of the function. In fact, she said, "here, there was no graph, so I was like… I didn't know what to do 'cause… I don't know, it kind of threw me off a little bit". So, when she is not able to work with her most comfortable representation, a graph, she has trouble figuring out what it would mean for a function to be continuous.

When dealing only with symbolic representations, Mandy favored a functional existence model of continuity. In other words, when she doesn't have a graph to rely on, she starts to equate existence of the function at a point with continuity. This is clear when she considers the Dirichlet function:

> I said yes because I think a number has to be rational or irrational, so… it would have to be continuous because there wouldn't be any x's that weren't defined.

Thus, while she has a fairly sophisticated continuity model when dealing with graphical representations, this is not incorporated with her model for dealing with symbolic representations. When prompted, she begins to tie these two parts of her model together, but it is a very tenuous link. She is not at all confident that her thinking in the graphical examples will apply to the symbolic situations.

Our findings indicated that some students at the high school level are able to incorporate information about the domain of the function into their models of continuity. Results elaborated on the contextual dependence of reasoning about continuity that suggests a link between the functional representation used and students' determination of whether a function is continuous.

References

Baker, B., Cooley, L. A., & Trigueros, M. (2000). A calculus graphing schema. *Journal for Research in Mathematics Education, 31*(5), 557-578.

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology, 32*(4), 487500.

Ferrini-Mundy, J., & Graham, K. (1994). Research in calculus learning: Understanding limits, derivatives, and integrals. In E. Dubinsky & J. Kaput (Eds.), *Research issues in undergraduate mathematics* (pp. 31-45). Washington, DC: Mathematical Association of America.

Lauten, A. D., Graham, K., & Ferrini-Mundy, J. (1994). Student understanding of basic calculus concepts: Interaction with the graphics calculator. *Journal of Mathematical Behavior, 13*(2), 225-237.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151-169.

MATHEMATIZING INTUITIVE NOTIONS OF SYMMETRY AND TRANSFORMATIONS FOR USE IN MORE FORMAL REASONING

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The purpose of this paper is to describe the complexities involved in moving from intuitive knowledge of symmetry and congruence to using mathematized versions of these ideas in more formal arguments. We will investigate students' mathematical activity (Rassmussen, Zandieh, King, & Teppo, in press; Freudenthal, 1973) as they attempt to incorporate intuitive (Fishchbein, 1982, 1999) notions of transformations and symmetry into their reasoning. Freudenthal (1973) describes mathematizing as "the process by which reality is trimmed to the mathematician's needs and preferences." Activities such as experimenting, conjecturing, organizing, generalizing and formalizing, are all examples of the kinds of mathematizing that might contribute to the development of mathematical proof. Raman (2003) defines a *key idea* as a "heuristic idea that can be mapped to a formal proof with the appropriate sense of rigor."

Our research indicates that notions of symmetry and transformation can act as powerful key ideas for students as they attempt to prove geometric statements. On the other hand, these ideas often seem intuitive in the sense that they are immediately self evident and hence are seen as being *obviously true*. This self evidence sometimes seems to serve as a barrier, inhibiting the mathematization of these ideas into a form that can be utilized in rigorous arguments.

Methods

The data analyzed for this report comes from a semester-long classroom teaching experiment conducted in a geometry course of 28 students at a large university. Data consisted of videotape recordings of each class session, copies of students' written work, researcher field notes, and videotaped debriefing sessions involving all members of the research team. The text for the course was Henderson's (2001) geometry text. Typical class sessions consisted of a brief introduction of the problem by the instructor (the first author), followed by small group work on the problem and whole class discussion of students' reasoning, interpretations and solutions.

Results and Discussion

In our paper we will draw upon an example from students' work with the Isosceles Triangle Theorem (ITT) to illustrate the complex role that key ideas and intuition play in students' mathematical activity as they attempt to prove geometric statements using transformations and symmetry. Henderson (2001) states the ITT problem as, "Given a triangle with two of its sides congruent, then are the two angles opposite those sides congruent?"

The small group discussion excerpt below illustrates the positive role that symmetry can play as a key idea. In this case the idea of symmetry ignites an explosion of mathematizing resulting in the construction of the outline of a proof that the students in this group find convincing.

Alice: The book says to use symmetries
Emily: Symmetries?
Valerie: That angle…
Alice: [Interrupting Valerie's thoughts] Okay! Yeah! Yeah…
Valerie: …equals that angle [mumbling and thinking out loud to herself].
Alice: If you have, yeah! If you have, like, a bisected angle…
Emily: You do the angle bisector…

Alice: Yeah! And then this [side of the triangle] matches this [other side of the triangle] because it can lay right on top of it! Because then you like rotate it.

Emily: You do a reflection over the perpendicular bisector of the angle.

Alice: Yes! And then it proves it!

Emily: Or, not the perpendicular but the bisector angle.

Alice: The…the bisector angle. But you do make it [angle bisector], I make it…but it is [also] perpendicular. That's why you can do it. It's 'cause it's like this line [angle bisector], this one's perpendicular here [at base of triangle].

Emily: It [angle bisector] becomes perpendicular to the…

Alice: Yeah, so these [angles] are the same, and this [side] folds right onto that [side].

Valerie: So, I agree that they're perpendicular, but you know she's going to ask, "Well, how do you know that the angle bisector is also a perpendicular bisector?"

Valerie's challenge leads to an interesting discussion during which Alice struggles to move beyond the apparent obviousness of her assertion that the angle bisector will be a perpendicular bisector. She attempts to convince the other members of the group by demonstrating how to construct both the perpendicular bisector and the angle bisector, and arguing that these constructions give the same line. The other students continue to press for a more rigorous argument. As this goes on, Alice gets more and more frustrated with them for asking her to prove something that is obvious to her, while they become more and more frustrated with her apparent inability to see that there is something to prove.

Unfortunately this controversy was never resolved. The students were asked to write a proof of ITT for their homework. Alice wrote up the proof she described in the small group discussion, complete with the unproven assertion that the angle bisector was the same as the perpendicular bisector in this case. The other students in the group abandoned this proof in favor of a different approach presented by another group in the whole class discussion.

While this is only one example, our data shows many other examples where the key idea (often related to symmetry or transformations) leads to an outburst of productive but informal mathematical activity. Students then struggle to mathematize this informal activity in order to construct a rigorous definition or argument. Often this process is impeded, at least temporarily, by the apparent obviousness of some intuitive notion. This research explores how students are able to succeed in the process of mathematizing their intuitive notions of symmetry and transformations as they define, conjecture and prove.

References

Fischbein, E. (1999). Intuitions and schemata in mathematical reasoning. *Educational Studies in Mathematics*, *38*, 11-50. Fischbein, E. (1982). Intuition and Proof. *For the Learning of Mathematics, 3(2)*, 9-18, 24.

Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht: Reidel. Henderson, D. (2001). *Experiencing geometry on plane and sphere*. Upper Saddle River, NJ:

Prentice Hall. Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics, 52*, 319-325.

Rasmussen, C., Zandieh, M., King, K., & Teppo, A. (in press). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking.

STUDENTS' COLLOQUIAL DISCOURSE ON INFINITY AND LIMIT AND MATHEMATICAL DISCOURSE: THE CASE OF AMERICAN AND KOREAN STUDENTS

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Limits are tightly related to the notions of infinity in mathematical contexts. Infinity and limit are the terms used loosely in everyday language. Students' difficulties in their understanding of limit can come from not only inside of mathematics classrooms but also within non-school contexts. Thus, it is important to know how students use the notions of infinity and limit in colloquial discourse to focus on how students learn infinity and limit in mathematical discourse.

Theoretical Background

Most students have extreme difficulties in acquiring the limit concept because of its abstract, formal definition and its precision (Mamona-Downs, 2001). Although Mamona-Downs (2001) points out that the concept of infinity is never directly experienced by one's senses in the physical world, we believe that students enter the classroom with preexisting conceptions of infinity from everyday language. Therefore, the ways that students take up classroom or disciplinary discourses may be shaped by the everyday discourses they bring to classroom (Moje et al., 2001). The focus of this study is on the relationships between students' uses and understanding of the notions of infinity and limit in mathematical and everyday discourse in the view of differences between American and Korean students. Mathematical activity can be seen as a form of communication. Thinking can be regarded as a special case of the activity of communication (Ben-Yehuda et al., 2003).

Methodology

Each linguistically different group includes one elementary student, one middle school student, one high school student, and one university undergraduate. Data are analyzed based on three distinctive features of mathematical discourses (Ben-Yehuda, 2003): mathematical uses of words, discursive routines, and endorsed narratives.

References

Ben-Yehuda, M., Lavy, I., Linchevski, L., Sfard, A. (in press). Doing wrong with words or what bars students' access to Arithmetical discourses. *Journal for Research in Mathematics Education*.

Mamona-Downs, J. (2001). Letting the intuitive bear on the formal; A didactical approaching the understanding of the limit of a sequence. *Educational Studies in Mathematics,* 48, 259-288

Moje, E. B., Collazo, T., Carrillo, R., & Marx, R. W. (2000). "Maestro, what is 'quality'?": Language, literacy, and discourse in project-based science. *Journal of Research in Science Teaching, 38 (4)*, 469-498.

PROCESS OR OBJECT? – STUDENTS' THINKING ABOUT FUNCTIONS AND GRAPHS

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The concept of function can be seen throughout the curriculum in high school. The definition of function is easy to state, but the concept is hard to understand thoroughly. Because of its complex nature, students often glean a superficial understanding and fragile internal representation. Graphs are used extensively in the function curriculum; they are seen as transparent external representations useful for helping students understand the concept of a function. However, the graph might not the best introduction for many students.

Sfard proposes that there is a dual nature of mathematical representation of concepts. (Sfard, 1991) In her proposal, mathematical concepts can either be seen either as processes or objects. However, it is only in the ability to see a concept as an object that a person is capable of deeper understanding. She also claims that people must be able to conceptualize both representations to fully comprehend the concept. For this poster session, I present data showing students' thinking about these dual representations in the domain of functions and graphs, with a specific emphasis towards trigonometric functions. Students' dominant internal representation is examined, and how a student holding only a process representation might be restricted in their ability to use appropriate tools for problem solving about functions and graphs.

In part of my analysis, I gave a problem to 58 students in an Algebra II / Trigonometry classroom. It asked the students to estimate the value of f(1), given a functional equation and a graph of the function. The equation was a particularly complex one, but the graph displayed that f(1) was approximately zero. Everyone who chose to focus on the graph (20) got the problem correct. However, almost half of the students (24) tried to evaluate the function at x=1, in spite of the fact that the graph made the problem easier to answer. Only 8 of those who focused on the equation got the right answer. (This included three that chose to focus on both.) Further interviewing implied that those students who didn't use the graph also had a more limited view of the concept of a function.

I propose that at least part of the difficulty of mathematics learning lies in the curricular presentation of material. This is because curricula often present material requiring both process and object representations simultaneously, and do not do so in an appropriate order or time span that allows for student understanding. In addition, curricular shifts from one representation type to another pose problems in students' progress towards understanding if they aren't capable of concurrently seeing the duality. These conclusions will be shown with the use of classroom assessments, individual interviews, and field notes taken during class and tutoring sessions.

In addition to the analysis done, there will be an introduction to an alternate representation for trigonometry, the use of sound to convey mathematical concepts of function. This representation invites an object interpretation, and might be helpful in planning new curricula that take advantage of knowledge of this duality.

References

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics,* 22(1), 1-36.

THE RELATIONSHIPS AMONG INFORMAL STRATEGIES STUDENTS USE IN SOLVING PROBLEMS IN PROPORTIONAL SITUATIONS

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The existing literature shows that students use correct and incorrect informal strategies in proportional situations (e.g., Lamon, 1994; Lamon, 1995; Kaput & West, 1994; Lo & Watanabe, 1997). One of those strategies is known as "incorrect addition strategy" (e.g., Noelting, 1980). The incorrect addition strategy is based on the use of addition where the quantities are supposed to be compared multiplicatively (Noelting, 1980). In addition, different researchers point to different hierarchical levels of student's thinking when they use informal strategies. According to Kaput & West (1994), the hierarchy is as follows: the first level is build-up strategy; the second level is abbreviated build-up strategy and the third level is unit factor approaches. Lamon (1993) claims that the first level of students' informal ways of thinking in solving proportion problems is single-unit strategy and the second one is build-up strategy. Another informal strategy provided in the literature is the ratio-unit/ build-up method (Lo & Watanabe, 1997).

The aim of the study is to articulate similarities and differences among those informal strategies discussed in the existing literature. In a one- hour interview, a fourth grade student completed several proportion problems. This student's informal strategies are analyzed to shed light on the nuances among the informal strategies reported in the literature.

References

Lamon, S. J. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J.Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 89-123). Albany, NY: SUNY Press.

Lamon, S.J. (1995). Ratio and proportion: Elementary didactical phenomenology. In B. Schappelle (Ed.), *Providing a foundation for teaching mathematics in the middle school grades* (pp. 167-198). Albany: State University of New York.

Kaput. J. & West, M.M(1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J.Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). Albany, NY: SUNY Press.

Noelting, G. (1980). The development of proportional reasoning and the ratio concept. Part II--Problem-structure at successive stages; Problem-solving strategies and the mechanism of adaptive restructuring. *Educational Studies in Mathematics, 11*(3), 331-363.

Lo, J. & Watanabe, T. (1997) Developing ratio and proportion schemes: A story of a fifth grader." *Journal for Research in Mathematics Education*, 28(2), 216-236, March, 1997.

MATHEMATICAL MATURITY AND MATHEMATICAL SKILL AT THE UNDERGRADUATE LEVEL (COMPONENTS OF A CONTINUUM)

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This 'Poster Presentation' compares the components of mathematical maturity and mathematical skill. Furthermore, the research that supports this poster presentation has situated mathematical maturity and mathematical skill, as they pertain to pure and applied mathematics from a cognitive perspective, into components of a 'continuum'. This continuum and comparison has been developed from information gathered from interviews with an applied mathematician and a pure mathematician during a dissertational research. To configure a continuum based on a comparison of mathematical maturity and mathematical from a cognitive perspective, the author also utilized publications from the Mathematical Association of America. These publications included the 'Discussion Papers about Mathematics and Mathematical Sciences in 2010: What Should Students Know?', and the Undergraduate Programs and Courses in the Mathematical Sciences: Committee on the Undergraduate Program in Mathematics (CUPM) Curriculum Guide 2004.

STUDENT USE (AND MISUSE) OF GRAPHS AND RATES OF CHANGE IN ECONOMICS

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This study examines the use of mathematics by successful students in economics. Our preliminary findings are the students have constructed a set of economics representations that *parallel* rather than *build upon* the typical mathematical representations of algebraic, graphical, numerical and verbal. Students did not show particular skill moving between the economics representations and mathematical representations.

The economics education literature holds several studies that find a positive correlation between students' mathematics preparation and success in undergraduate economics courses However, the role of mathematics in the economics curriculum is an unresolved issue (Von Allmen, et al., 1998). Because of this positive correlation, it becomes an important issue to examine case studies of how economic students build on their math knowledge to understand economics.

The case studies are based on videotaped interviews of four high-achieving economics students as they worked on two undergraduate level microeconomics problems. The students all had received A's in microeconomics and each had completed at least one year of calculus. The students worked two problems on Indifference Curve Analysis and Price, Output and Profit of the Purely Competitive Firm. They were given several minutes to think about a problem and then asked to reason aloud and illustrate their thinking at a whiteboard. The data was coded using open-coding and analyzed for emergent themes, grounded theory style.

For this set of case studies, we focused on the use of graphical representations in economics analysis and in particular, in situations requiring covariational reasoning (Carlson, et al. 2003). Our preliminary results show that students in many cases did not use their formal mathematics knowledge about Cartesian graphs nor needed to use it to solve the two graphically based problems. The students' economics knowledge was well equipped with its own conventions, rules, examples, and graphical images which they used with varying success.

However, when pressed to justify their use of graphical representations and rules of thumb, students had great difficulty. Often students could not justify their conclusions using mathematics at all. Sometimes they were able to call upon pointwise and linear mathematical knowledge, such as when dealing with straight lines, plotting and analyzing points and solving and evaluating linear equations. In general, when analyzing non-linear curves and continuously changing slopes, they resorted to economics reasoning, such as reference to economic properties and quoted examples and/or special cases.

References

Carlson, M., Jacobs, S., Coe, T., Larsen, S., & Hsu, E. (2002). Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study. *Journal of Research on Mathematics Education, 33*(5), 352--378.

Von Allmen, P. & Brower, G. (1998). Calculus and the Teaching of Intermediate Microeconomics: Results from a Survey. *Journal of Economics Education*, (Summer), 277-284.

Algebraic Thinking

STRATEGIES USED BY FIRST GRADE OF SECONDARY SCHOOL STUDENTS WHEN SOLVING WORD PROBLEMS OF UNEQUAL SHARING

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This paper reports strategies used by first grade of secondary school students when solving algebraic word problem of unequal sharing (Bednarz & Janvier, 1994). The particular way students solve this kind of problems, helps us to understand how certain —and many times efficient—strategies arise even before students have taken a formal course of Algebra. These solving strategies used, helped us to confirm an intrinsic difficulty in word problems, because such difficulty was conceived a priory with aid of a theoretical frame. Students' solving processes helped us to understand certain characteristics of algebraic thinking.

Introduction

Learn Mathematics is more than a set of rules and procedures to solve problems. It implies that students develop their own strategies and attitudes according to mathematical work (Santos, 2004, p. 314). Santos states that in a solving problem environment, students can solve and express conjectures; discuss several meanings of unknown; observe and explore diverse patterns –when exist- and; finally, to communicate their findings, as result of their observations, to their classmates or teacher.

Frequently, secondary school graders consider Mathematics like something complex, governed by rules, whose learning mostly depend on memorizing such rules. Thus, when solving Algebra problems (when we talk about Algebra, we are referring to school Algebra), students use characteristic rules and procedures of such discipline, although a reflexion about its use has not taken place. This way of working leads many students to a wrong belief that such manners of facing problems in this discipline represent Algebra essence.

In present work, we stress in the analysis of solving methods pupils used in the experimental stage of this study. Students' solving strategies are in close relation to their domain of previous arithmetic knowledge, acquired in elementary school. Taking this domain as a referent to carry out our analysis, we try to explain the origin of their initial algebraic thinking, according to characteristics of their own way of working when using arithmetic, as a basis to construct their strategies for solving posed problems. Results produced by this study helped us to understand, to a certain point, the existing complexity in transition from Arithmetic to Algebra; however, we think that an adequate selection of problems may favor students' transition to algebraic thinking.

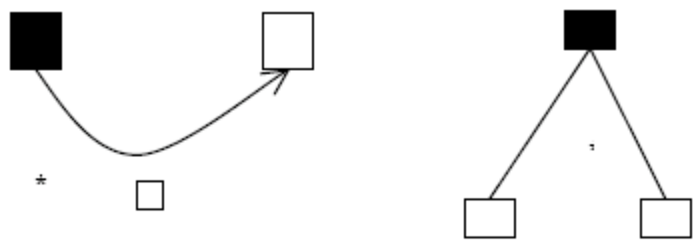
Theoretical frame

Bednarz & Janvier (1994, 1996) analyzed diverse algebraic word problems in order to propose its classification. They identified nature of relation among data, unknowns and the connection amongst them. So, they succeeded to show the complexity a priori of those problems students face in regular Algebra courses.

These authors pointed out the relevance to identify general structure of a word problem starting from involved (known and unknown) quantities, relationship between them (connection between quantities), and type of involved relation (additive or multiplicative comparison). Bednarz & Janvier (*Ibid*, 1996) proposed a "theoretical tool" (*grille d'analyse*) to classify word problem used in teaching, which remain as *arithmetic and algebraic*.

One of the characteristics of the so called arithmetic problems is that they allow constructing bridges between known and unknown states, starting from one of their data and taking in account present relation in order to find the unknown. On the contrary, algebraic problems do not allow a settlement of direct bridges between data and unknowns. Unknown should be considered as a known *object* and operate it through those given relations in problem in order to solve it. Algebraic problems force students to carry out a symbolic treatment of the unknown and of present relations. Another characteristic of algebraic problems is that there are no states and relations known in a consecutive way and so one must simultaneously operate with two states.

Bednarz & Janvier (*Ibid,* 1996) created a symbolism to support their theory: known quantities are represented by black square; unknown quantities are represented by white squares. Such quantities have been denominated states. They classified relation between quantities into two types: comparison and operation (Figure 1).



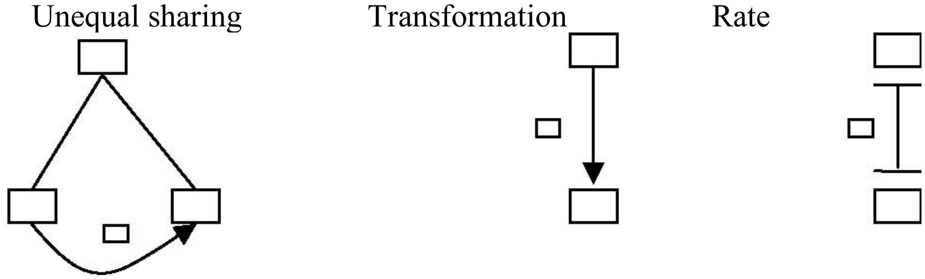
(a) Comparison (b) Operation

Asterisk indicates which operation will take place: addition, sustraction, multiplication or division.

Figure 1. Symbolism of relations: comparison and operation

From identification of nature of data and the structure of involved relations (mathematic-relational structure of problem), these researchers classified problems into three types: *Unequal sharing, transformation and rate*. In Figure 2, we show schemes of this type of problems, according to symbology used by Bednarz and Janvier (1994).

Unequal sharing Transformation Rate



(a) (b) (c)

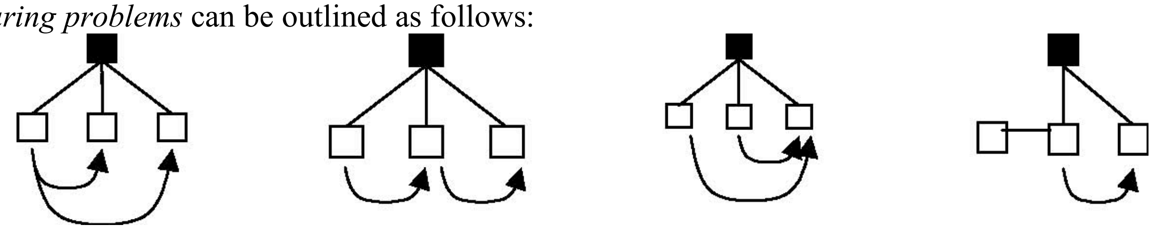
Figure 2. Schemes of the three principal types of problems

Rate problems are characterized by the existence of comparisons between no-homogeneous quantities. For instance, kilometers per hour, salary per day, legs per animal, etc. A question mark is written in white squares, which represent magnitudes to be found in a posed problem.

Transformation problems are characterized because of relations between data and unknowns are dynamics. Relation existing between two states is symbolized by an arrow that might point to one or another direction and a small square that represents the operation (addition, multiplication, etc.). For example, a boy had a certain quantity of marbles. He wins 23 marbles and now he has 34, ¿how many marbles did he has at first?

Unequal sharing problems are characterized by the existence of a total quantity related to its parts (*states* that can be known or unknown). In Figure 3, scheme (a), upper square usually represents the total quantity given in a problem enunciation (known quantity, although it can be unknown, as we show it in this study: problems 8, 9, 10, 11 and 12). Lower squares of scheme represent those quantities into which upper quantity is shared (unknown quantities, although if we talk of an arithmetic problem can be known quantities). Along this research, we only used *Unequal sharing* algebraic problems.

In these problems, relations of states can be of diverse types: *source relation*, when relation starts in a same state and goes to one or more states; *well relation*, when two or more relations arrive to a same state; *composition relation*, when a relation arrives to state and emerges another relation; and, *non-usual structure*, when data only involved two relations (Vargas, 2001). The mathematic-relational structure of these problems is represented in a *classical* way; through *simple tree schemes* (lines that go from upper sates to lower ones are called *branches*). *Unequal sharing problems* can be outlined as follows:



(a) Source (b) Composition (c) Well (d) Non-usual structure

Figure 3. Schemes of four types of relations in Unequal Sharing problems

Theoretical frame in this research is also based on relational calculus (Vergnaud, 1991), which involves problems' representation and solution (nature of relations between data, and connection of relations, etc).

Methodology

Grasping data

Our data arose from work in groups with nine students (11-12 year olds) in first grade of secondary school. All posed problems (see Annex 1) correspond to *Unequal sharing*. We selected one arithmetic and seven algebraic problems, which varied relations between data and unknown and type of involved relation. Students had 30 minutes to solve, in group, for each of these problems. Researcher participated only posing questions and giving some suggestions to students to encourage the solving problem process.

The following section includes text of posed problems as well as an example of how problems were solved by some of the participants in the study. In each one of figures 4, 5, 6, 7, 8 and 9 we indicate the name of the strategy used.

In this paper we do not provide evidences of work with three students, who individually solved five *Unequal sharing* problems, where quantity to be allotted was unknown (more complex problems that the ones presented before) However, obtained results at this point of the study indicates us the complexity of solution of this kind of problems, when entering upon by students who have not studied Algebra in formal courses. Our research project is more extensive, that is, in this paper we report only obtained results in part of what constitutes the pilot stage.

Results and Discussion

Strategies emerged during the process of working with pupils (according to Bednarz and Guzman's classification, 2003) and systematically used by them, were as follow:

E1. *Identify structure*. Students detect the *structure of problem*; that is, they identify relation between involved quantities and type of involved relation. For example,

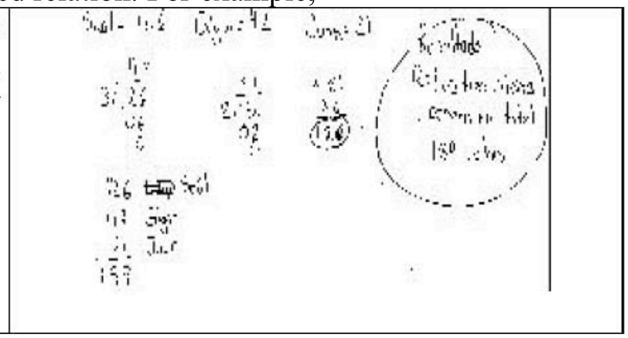
Problem 1. Saul has three times more candies than Edgar, and six times more than Juan. Saul has 126 candies, how many candies do the three boys have?	

Figure 4

E2. *Transform structure*. Students do not restrain connection of present relations in problems. For example,

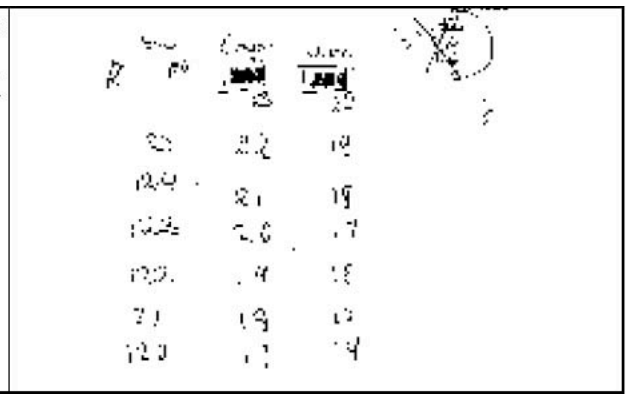
Problem 1. Saul has three times more candies than Edgar, and six times more than Juan. Saul has 126 candies, how many candies do the three boys have?	

Figure 5

E3. *Proceed by means of numerical attempt (trial)*. Students start allotting in two or three equal parts, according to initial instructions of problem. Obtained result is used as a producer to find other quantities. Along the solution process, students make their own *pertinent adjustments*. For example,

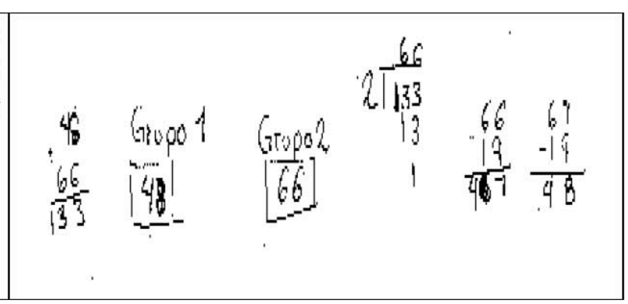
Problem 2. 133 chocolates are allotted in two groups; the second group has 19 more chocolates than the first one. How many chocolates does each group receive?	

Figure 6

E4. *Use a numeric game*. Students break quantity to be shared without considering present relations in problem. For example,

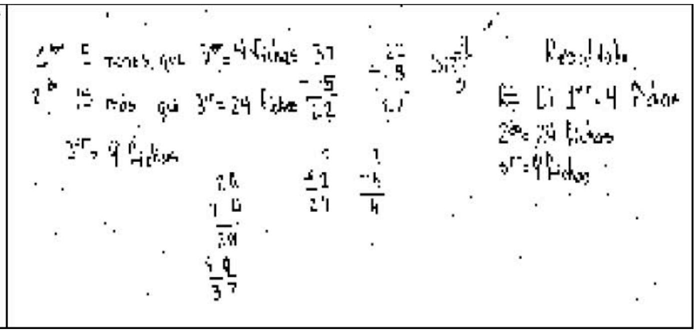
Problem 3. We have three piles of pieces. In the first pile there are 5 pieces less than in third pile, and in second pile there are15 more pieces than in the third one. There are 37 pieces in total. How many pieces are there in each pile?	

Figure 7

E5. *Numerical break of given quantities*. Students resign themselves to find numbers that give the total quantity without consider present relations. For instance,

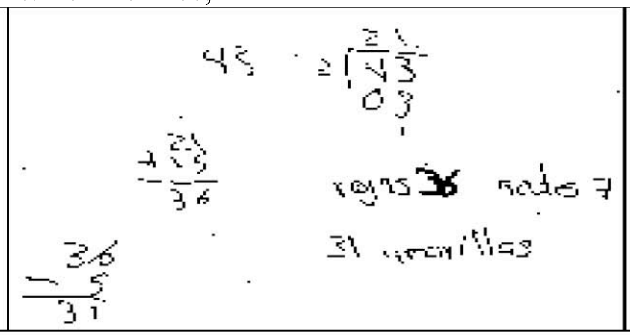
Problem 3. We have three piles of pieces. The first one is formed by yellow pieces; the second one by red pieces, and the third one by blue pieces. Yellow one are 5 less than the blue pieces, and red pieces are 15 more than blue ones. Blue a red pieces sum 43. How many pieces are there in each pile?	

Figure 8

E6. *Use of numeric relations*. Students make an estimation of the quantity they are looking for, without paying attention to relations. For example,

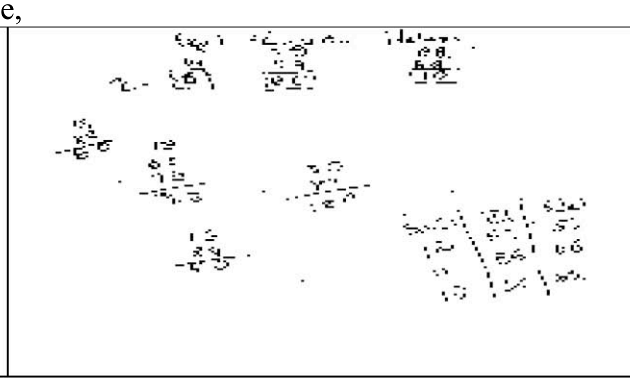
Problem 4. We want to allot 181 chocolates among Saul, Ricardo and Nelson in such a way that Ricardo receives 4 times the number of chocolates than Saul, and Nelson receives the same than Ricardo plus 10 chocolates. How many chocolates correspond to each one?	

Figure 9

Some students used strategy *E1* to solve problems. In students' records fulfilled during the solution process, we detected some characteristics pertaining to algebraic thinking (to work from general to particular, to pay attention to the unknown, among others). There are evidences that students understood the problem structure (they identified relation between involved quantities and type of involved relation) other students were not able to perceive the problem structure and used numeric attempts as a recurrent strategy to solve problems.

The solutions given by three students to five problems where total quantity to allot is unknown (see Annex 2) although they identified relations between data and unknowns, are quite interesting. Only one of the three students could correctly solve the five problems by means of the use of strategy *E2*. The two other students, who used strategies *E4*, *E5* and *E6* had difficulties to solve problems and had no success in problem solving.

The mentioned five problems, where total quantity is unknown, were very difficult to solve for the rest of the group. As we supposed at the beginning of the selection of such problems, students were not able to use their own arithmetic strategies to solve them. This indicates us that this type of problem can, in a certain way, to encourage students in the use of symbology related to Algebra.

Preliminary conclusions

Although strategies *E2, E3,·E4, E5* and *E6* seem to be linked to arithmetic thinking, its use may favor an evolution in students' thinking, when working with a certain kind of problems, which forced them to go beyond their arithmetic knowledge. Solving processes carried out by students indicate that it is possible to favor the emergence and development of algebraic thinking in a solving problem context. Students faced these seven problems in such a way that they started using —in naive way— symbols to express the unknown and the present relations in those problems.

Preliminary results show that by means of a certain type of word problem —*Unequal sharing*, in this case— and under didactic conditions students may eventually produce strategies that gradually lead them toward algebraic thinking.

References

Bednarz, N. y B. Janvier. (1994). *The emergence and development of algebra in a problem solving context: An analysis of problems.* En: J. P. da Ponte y J. F. Matos (Eds.), Proceedings of the 18th *International Conference for the Psychology of Mathematics Education,* 2, 64-71. Lisbon, Portugal: PME Program Committee.

Bednarz, N. y B. Janvier. (1996). *Emergence and development of algebra as a problem-solving tool. Continuities and discontinuities with arithmetic.* En: Bednarz, Nadine; Kieran, Carolyn, and Lee, Lesley (Eds.), *Approaches to Algebra,* 115-136. Dordretch, Holanda: Kluwer Academic Publishers.

Bednarz, N. y Guzmán J. (2003) *¿Cómo abordan los estudiantes de secundaria la resolución de problemas antes de ser introducidos al álgebra? Un estudio exploratorio. México-Quebec.* Fondo de Cultura Económica. México, pp. 11-40

Santos, L. (2003) *Hacia una instrucción que promueva los procesos de pensamiento matemático,* en Matemática Educativa. Aspectos de la investigación actual. Cinvestav-IPN, Fondo de Cultura Económica, México, pp. 11-40.

Vargas, V. (2001) *Problemas de enunciado verbal en los libros de texto de educación primaria.* Tesis de Maestría, Universidad Autónoma del Estado de Morelos, México, pp. 14-20.

Vergnaud, G. (1991). *El niño, las matemáticas y la realidad. Problemas de la enseñanza de las matemáticas en la escuela primaria.* (Primera Edición). Ed. Trillas. México

Annex 1

1. Saul has three times more candies than Edgar, and six times more than Juan. Saul has 126 candies, how many candies do the three boys have?
2. 133 chocolates are allotted in two groups; the second group has 19 more chocolates than the first one. How many chocolates does each group receive?
3. We have three piles of pieces. In the first pile there are 5 pieces less than in third pile, and in second pile there are15 more pieces than in the third one. There are 37 pieces in total. How many pieces are there in each pile?

4. We want to allot 181 chocolates among Saul, Ricardo and Nelson in such a way that Ricardo receives 4 times the number of chocolates than Saul, and Nelson receives the same than Ricardo plus 10 chocolates. How many chocolates correspond to each one?

5. There are 224 animals (ducks and hens) in a courtyard. Hens are six times more than ducks. How many ducks and how many hens are there in the courtyard?

6. In a school cooperative shop were sold 324 soft drinks of different sizes (small, medium and big). Medium size was sold two more times than big one and three times more small soft drinks than big ones. How many soft drinks of each size were sold?

7. A bookcase has three divisions. A certain amount of books is located in first division there; in the second division there are 13 more books than in the first one and in third one there are 19 more books than in the second one. The whole amount of books is 96, how many books are there in each division?

Annex 2

8. A box of chocolates was allotted among Alma, Maria and Silvia. Silvia received 5 more chocolates than Alma, and Maria 5 more chocolates than Silvia. Both, Silvia and Maria had 37 chocolates. How many chocolates were allotted?

9. Mr. Dominguez shared his stamp collection with his three sons: Javier, Raul and Luis. Luis received 5 more times the number of stamps than Javier did, and 4 less stamps than those received by Raul. The whole quantity received by Javier and Raul is 22 stamps. How many stamps did Mr. Dominguez allot?

10. An ice-cream parlor offers three savors: strawberry, lemon and pineapple. Pineapple ice-cream was sold three more times than strawberry ice-cream; and lemon ice-cream was sold twice more than pineapple ice-cream. 63 strawberry and lemon ice-creams were sold. Which is the whole amount of ice-cream sold?

11. Mrs. Mercedes sells *tamales*. She sold four more sweet *tamales* than *tamales* of mole and three times more sweet *tamales* than *tamales* of chili. She sold 21 tamales of mole and chili. How many tamales did she sell?

12. A gym offers three activities in a summer course: soccer, swimming and basketball. The number of children that preferred basketball was a half of those registered in soccer, and twelve more children in swimming than in basketball. There were 63 students registered in soccer and swimming, how many boys were registered in the summer course?

DESIGN OF ACTIVITIES TO OBSERVE THE COGNITIVE STRUCTURE OF STUDENTS EXPOSED TO TASKS WHICH INVOLVE COVARIATION OF QUANTITIES

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What thinking process do engineering students show when asked to represent graphically relation among quantities which are changing simultaneously? What type of activities should we design to promote covariational thinking which would be the base for the students to understand the fundamental calculus concepts? This study document the behaviors of engineering students who were exposed to task involving covariation of quantities. The outstanding results of this investigation showed that students do not conceive that a quantity can always increase but its rate of change decrease at the same time.

Introduction

Some of the recurring difficulties shown by students in calculus courses include the inability to recognize or identify changes in a variable with respect to changes in another variable and to identify and represent relationships within a situation which involves variation phenomena (Carlson, et al, 2002, 2003). Thus, it is relevant to study the students' difficulties related with tasks that demand to interpret situations which involve covariation of quantities. To develop this ability it is essential to comprehend the fundamental concepts of calculus (Thompson, 1994).

In order to promote students' understanding of fundamental ideas associated with covariational reasoning phenomena, it is important to know their ideas or the cognitive reasoning they show when working with learning activities embedded in variation phenomena.

The aim of this study is to document the cognitive characteristics of first semester engineering students from the University of México at the beginning of a calculus course. The students were asked to identify, interpret and represent graphically the changes in one variable with respect to changes in another variable in different settings. To carry out this activity the tasks demanded visualization and coordination of rate of change (average or instantaneous) of one variable with respect to another constantly changing variable. This study forms part of a larger project, which includes the design of tasks that involve covariation of quantities in a computational environment. For example, to simulate on a computer the inflow and outflow of water into a tank. On the screen we can see how the level of the water increases or decreases in the tank. The control of the inflow and outflow is with the *mouse.* While manipulating the mouse two graphs appear simultaneously. One graph represents the variations of the volume in the tank. The other graph represents the handling of the mouse (first derivative). Some of the tasks of students are as follows: The students would see a graph on the screen representing volume of water in the tank, based on this the students should describe verbally and graphically what happened with the "tap" controlling the inflow and outflow (first derivative) of the water in the tank or given a graph which represents the handling of the *mouse* (first derivative), now based on this graph the students should to describe verbally and graphically what happened with the volume of water in the tank (volume vs. time)

The questions used as a guide to carry out the study and also to observe the cognitive aspects which interested us are as follows:

1. Which outstanding covariational characteristics and reasoning were shown by the students when interacting with the activity?
2. At what point in their thinking process did students establish a connection between the changes in one quantity with respect to changes in another quantity?
3. At what point did the students connect the change in one quantity (increase o decrease) and its rate of change? Did they realize that one variable can be increasing but its rate of change (average or instantaneous) can be decreasing (e.g., the increase in the level of water in a conical shaped container which is being filled at a constant rate, the rate of change in the level of water decreases)
4. What type difficulties did students experience when solving the task?

Conceptual Framework

The conceptual framework used to observe the covariational thought of students was that of Carlson, et al., (2003). A concept in this framework is *covariational reasoning*. This is the ability to visualize, coordinate and represent the changing nature of two variables which vary at the same time, e.g., to visualize the movement of a ladder with one end leaning against the wall and the other end being pulled away at a constant speed. This mental action demands coordinating the rate of change of the end against the wall with the rate of change of the other end on the ground as it is being pulled away. Carlson points out five mental actions:

Mental Action 1: Changes in one variable with change in the other variable, that is, the recognition that the value of the **y-**coordinate changes with the changing **x-**coordinate

Mental Action 2: The direction of change in one variable with changes in the other variable, e.g., to represent graphically an imaginary line as it is going up while the other variable is increasing

Mental Action 3: The amount of change of one variable with changes in the other variable, e.g., focuses on the amount of change of the ouput –height-while considering changes in the input

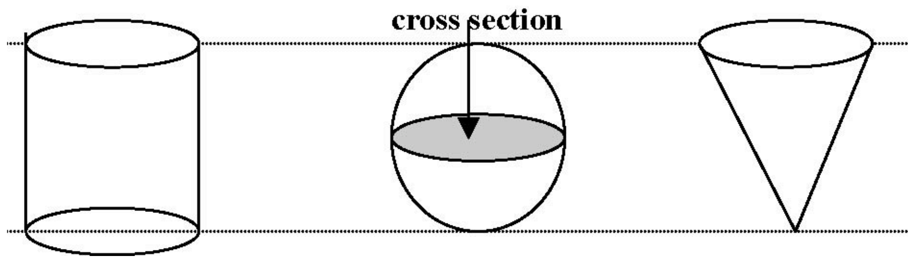
Mental Action 4: The average rate-of-change of a function with changes of the independent variable, e.g., focuses on the *rate of change* of the ouput with respect to the to the input for uniform increments of the input

Mental Action 5: The instantaneous rate-of-change of the function with continuous changes in the input variable, e.g., constructing a smooth curve with clear indications of concavity changes

Methodology and Procedures

Forty six first semester calculus engineering students from of the University of México participated in this study. The students worked individually on the following task and handed in written answers. Two students were chosen to work together and they were audio taped. We selected a pair because in written individual work the students leave gaps which create difficulties in interpreting the mental process used by the students, the interaction of the pair permitted us to observe in more detail the thinking processes of the students. These two types of data allowed us to draw up a "picture" of the characteristics of the cognitive development of the students. The activity designed to study the behavior of the students was the following:

The figures below show three containers of water each having a different shape: cylinder, sphere and cone. The three containers have the same capacity (10 liters) and the same height. Water enters at a constant rate, 1 liter/min.



Based on the previous information, the students were asked to answer the following questions:

1. Do you think that the radii are different or the same?
2. Do the three containers fill up at the same time?
3. Does the level of water in the three containers rise at the same speed?
4. What happens with the radii of the cross section in each of the containers?
5. Draw a graph which represents the height of the water with respect to the volume of the water flowing into each of the containers.
6. Draw a graph showing the radii of the cross section with respect to the volume of the water flowing into each of the three containers.
7. What differences do you see in the graphs drawn for 5 and 6?

Some of the characteristics of the activity are: the level of the water in the three containers always increases but the rate of change is different, e.g., at the beginning the rise in the sphere is fast and then as the water reaches the middle it begins to slow down and afterwards it begins to speed up, however, in the cylinder the rate of level of water is constant. The last question was asked in order to call the attention of the students to similarities or differences of the shapes on the graphs, that is, from concave to convex. This task had been already piloted.

Discussion of Results

The answers of the students were classified and analyzed based on the conceptual framework. Below we cite the most outstanding features shown by the students in their interactions with the task.

The tendencies shown by the students in question 1 were: "they are the same because they have the same capacity and height". Another explanation: "I believe they are different because the containers have different shapes, although the same height and capacity". The first tendency showed no mental operations with the objets. These were seen as "static". The second attended the geometric shapes. Only one student gave a more interesting reasoning which indicated mental process with the figures, that is, imagining or "seeing" dynamically the narrowing in the capacity of the cone: "different, if the cone would have the same radius as the cylinder, its capacity would decrease because it lacks volume in the pointed part, the sphere is similar except it would lack volume in the upper and lower part". It is worthwhile to note that the majority of the students did not use an algebraic approach except for one pupil and the pair. However, the pair only wrote the formulae to find the volume of each container but they did not know how to use these formulae to solve the question and decided to base on comparison or visualization among figures: "to answer this question I think we would have to know how to calculate the volume of each container" (pupil **A**) (they wrote the formula for each of the three containers), pupil **B** suggested "I think that they are different because this and this have the same height and capacity but this is higher than this".

Eighteen students (39 %) of a total of forty six answered the question incorrectly.

Question 2 was answered correctly by the majority of the students. The tendency was "because the water enters at a constant rate and the three containers have the same capacity, one would not fill before the other, but all three at the same time". The pair thought likewise, but student **A** was more cautions "yes, I think so, but it seems too easy… I have to think". These students noticed the rate of change which could be considered as mental action 4, on the other hand, some students who answered incorrectly based their answer on the shape of the containers "no, because the cone would fill more quickly than the other two because of its shape"

Question three was answered correctly by the majority of the students, that is the level of the water would not increase at the same speed. The answers were based on different considerations:

i. **The shape of the containers**: "no, because the shape of the figures are different and there may be a part where the level rises faster or slower in relation with the others".

ii. **Comparison among the levels of the water in the containers by assigning numbers** : "no, even though the rate is the same, the shape of each container is different, e.g., if the cylinder has 2 cm., of water, the cone won't have the same amount, they would fill up at the same time but the levels would not be the same for the three containers, therefore the levels would rise at different speeds".

iii. **Differences in the rate of change in the level**. "no, only in the cylinder would it be constant, but in the sphere first it is fast, then slow and then again fast. In the cone first it is fast and then it begins to slow down".

Answers i) y ii) are examples of mental action 3 , answer iii) is a example of mental 4. The pair showed a similarity with iii): "I think that, e.g., at the beginning , in the cone it is fast and in the cylinder it would go slower" (pupil **B**).

The difficulties of students emerged when trying to convert this verbal description to graphic representation, in other words the students' difficulties arose when trying to convert their ideas graphically e.g., in the case of the sphere only eighteen (39%) of forty six drew the graph correctly showing the change in the concavity of the curve (Mental Action 5).

The pair also showed difficulties to represent graphically the change in the level of the water of the sphere. However, the interaction between the two students helped them change their view point. At first, the pair drew a concave curve (parabola). Pupil **A** stated: "if we talk about differences, maybe we could talk about the concavity of the curve… the sphere is concave down (figure1) and the cone is concave up (figure3)… what else… the sphere reaches a maximum, I don't see any other differences".

figure 1

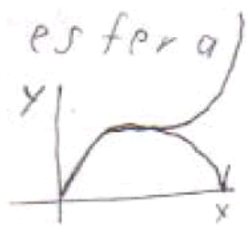


figure 2

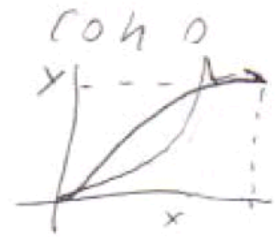


figure 3

The shift in outlook came in question 6. Here they drew a graph which represented the radius of the cross section of the sphere, but they realized that this is the same as the one drawn for question 3 (the height of the level of the water for the sphere). Here pupil **A** said "wait a minute, we may have a confusion here… something is weird". To overcome this confusion, the pupil **B** presented a new idea of two cones joined at the base "imagine if this cone had another cone on top… here the water would full faster and then slow down, then again very fast". This idea led them to correct the graphs of the level of the water of the sphere and the cone which had already been drawn (figures 2 and 3): "in the sphere it first bends down then up".

As we noted previously the pair incorrectly drew the change in the level of the water of the cone "in the cone it would be like the curve that started in the cylinder but would increase more each time" (pupil **B**). They changed their point of view when they noticed the rate of change: "may be this is a curve… because the level would rise faster at the beginning , but after it would became higher and then rise slower" (pupil **A**).

Only eleven of the above mentioned students, represented correctly the graph of the water level of the cone.

Finally in question #6 in the graph representation of the change in the radius of the cross section of the sphere and cone, the following patterns were shown: "in the sphere it rises slowly" or "in the sphere the water passes this section". Notice that descriptions are very poor. In contrast the pupils who answered correctly gave clear explanations, that is to say, they noted the outstanding attribute in the situation: "in the sphere the water rose to its maximum radius and after it started to slow down, until it was filled". Sixty percent of the pupils answered incorrectly.

Conclusions

Based on the above, we can state that one of the major conflicts of the pupils is to coordinate the increase of a variable (height of the water in the sphere) with the decrease or increase of its instantaneous rate of change. In other words, the students cannot establish the fact that a variable can always increase and its "rate of change" goes from decreasing to increasing or increasing to decreasing. This coordination is essential to understand what happened with the instantaneous rate of change at the inflection point of the curve, that is to say, at said inflexion point its instantaneous rate of change goes from increasing to decreasing or vice versa, therefore at these inflection points the instantaneous rate of change reaches its maximum or minimum value. This weakness may explain why the students showed difficulty in converting a written or verbal description into graphic representation as seen in the variation of the level of water in the sphere.

As we can see from the above, the covariational reasoning of students is very weak, therefore, it is necessary to design such activities which would develop and strengthen this type of thought

References

Carlson, M., Larsen, S., & Jacobs, S. (2001). An investigation of covariational reasoning and its role in learning the concepts of limit and accumulation. *Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH: Eric Clearinghouse, 2, 145-153.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational Reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352-378.

Carlson, M., Person, J., Smith, N. (2003). Developing and connecting calculus students' notions of rate-of-change and accumulation: The fundamental theorem of calculus. *Proceedings of the 2003 Joint Meeting of PME and PMENA*. Honolulu, HI: CRDG, College of Education, University of Hawaii, *2*, 173-179.

Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics, 26*, 229-274.

VARIATION VARIABLES AND SEMIOTIC MEDIATION IN A DYNAMICAL ENVIRONMENT

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We study variation and variables in a dynamic geometry environment. The dragging implemented in this environment is a tool well adapted to explore these ideas and becomes instrumental to develop a symbolic level that opens the door to a genuine visualization process involved in the understanding of these concepts.

Introduction

Understanding variables has been extensively discussed in Mathematics Education research (Moreno, L. and Santillán, M. A., 2002; N. Bednarz, C. Kieran and L. Lee (eds), 1996; R. Sutherland, T. Rojano, A. Bell and R. Lins (eds) 2001). Operating the unknown and operating the variable are quite different activities, mainly for a beginner student. In just one problem, it is feasible that variables exhibit all their complexity becoming an obstacle beyond students' present abilities. In this paper, we approach the study of variables and variation supported by Dynamic Geometry (Cabri-Geometre), using *dragging* as the main exploration tool. In this way, we will be able to deal with perceptual and cognitive difficulties as this tool allows to identify, all together, the elements that change in a Cabri-figure as well as those that remain invariant during a deformation while dragging the figure. In the context of geometry instruction, this facility has been used as a tool to distinguish between a drawing and the corresponding geometrical object ––the deep geometric structure corresponding to what you see on the screen.

Perception of variation and related problems

The immediacy of perception makes one to believe that this is a universal and homogeneous ability; that is is ready to help us process information whatever be the environment. Yet, this is not so: One can recognize a familiar face among others, but decoding a functional graphic, for instance, poses a higher cognitive demand on students (Duval, 1995; Leinhardt, Stein, Zaslavsky, 1990). Recognizing variation is an even harder problem. A research reporting students earlier difficulties with problems involving variation in a dynamic environment, is introduced in Moreno-Santillan (2002). In the following pages, we will be reviewing our new findings on this research questions. Excerpts from two interviews will exhibit some of the cognitive difficulties students face when dealing with these problems.

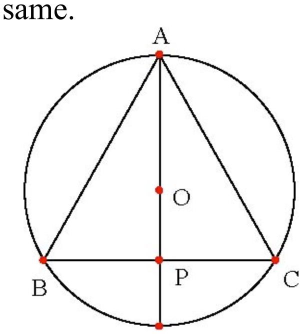
First interview

Some interesting cognitive behavior was made apparent in the interview with Estella, a student 15 years old. She is a bright student with a good working knowledge in mathematics. The instructor poses the following question: An isosceles triangle is inscribed in a circle; one of the triangle's vertices A, coincides with an endpoint of a diameter and the remaining two vertices B, C, are the endpoints of a chord BC that is perpendicular to the mentioned diameter. The height of the triangle from A, intersect BC in P.

The instructor makes clear that the point P can be displaced along the diameter (the only one to be considered here is the one mentioned above). When the point P moves along this diameter, the chord BC also is displaced and remains perpendicular to the diameter. In the following translation we will try to keep close to the student's vocabulary during the interview.

Instructor: When I displace point P (under the above constraints) what happens to the triangle? Does its area change? Does it remain the same?

Estela : The shape changes…it seems that the triangle remains as isosceles…but its area is the same.

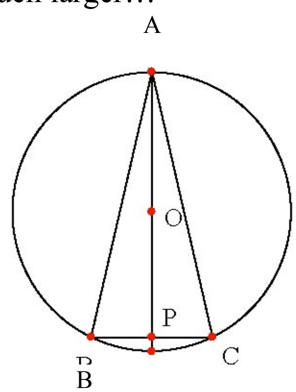


Instructor: Can you explain?

Estela : Because here (indicating the sides AB and AC) we are making the triangle smaller and here (indicating BC) we are making it larger.

The instructor continues, asking if there is a position of point P wherein the area changes, " where it is smaller or larger", and the student's answer is:

Estela: No, the area remains the same…(Then she draws, on the slate, a triangle with BC very short)…for instance in this triangle, BC is smaller but AB and AC are much larger…



The instructor asks Estella to construct the figure on the screen (students were using a TI-92 calculator). Afterwards, she begins displacing point P along the diameter without reaching the center O of the circle.

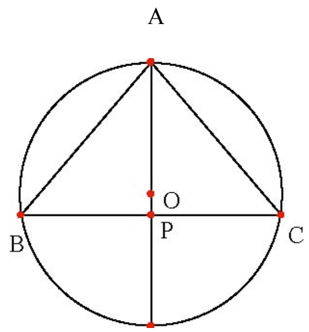
Instructor: Does the area change?

Estela: No…it does not.

Instructor: Why?

Before answering, Estela displaces P, very slowly, without reaching O.

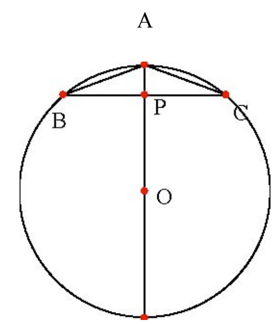
Estela: If the side BC grows, the sides AC and AB are diminishing…



Instructor: The area of the triangle…does it ever change?

Estela: No, because…

Then, Estela moves P towards O and, for the first time during the interview, P goes beyond O until almost touching A, and then drags P back beyond O. At that very moment she discovers something new, and says:



Estela: When I drag P beyond O, AB and AC gets shorter and BC also gets shorter…

Estela seems concerned and she drags P again. It is clear that something is disturbing her, attracting her attention.

Instructor: What is going on?

Estela: I was looking at the area…it…it diminishes, the area diminishes!

Instructor: When does it diminish?

Looking at the slate she says:

Estela: If I drags P beyond O, the area is smaller.

After the interview, Estela remarked:

"I discovered that the area would diminish because, if the base BC and the sides AB and AC are all smaller (than previously), then the area has to be smaller. At the same time, if I drag P from A, without reaching O, the BC gets larger, AB and AC also gets larger, so the area grows".

Estela sees the simultaneous variation of three magnitudes when she displaces P along the diameter. Yet what she observes on the screen is not the variation of the area itself but an ***index*** (in the sense given to this word by Peirce; see Deacon, T. 1997, p. 71) of this variation, a signal that engenders an inference: *the area becomes smaller if the sides become shorter*.

Let us reflect on the concept of symbol to establish a framework to interpret these results. What is an index? Let us suggest an answer inspired by Deacon (1997, pp.59-68): When a bird sings, an adult, by the side of a child, will indicate the bird and will utter: "bird". By pointing the bird out, the adult creates the first level of reference —an *indexical* level of reference. Next time the child hears the song of the bird, he will utter 'bird" even *without seeing the bird*. He has established a connection between the song and the word "bird". Later, the child will be able to enlarge her bird-experiences and "graft" them into the word "bird". This hierarchic and dialectic process will engender, gradually, what we refer to as the **Reference Field** of the word (conceived of as a symbol). The word (symbol) and the reference field appear as one and the same. But, in fact, *the signifier, the signified, and the interpreter are never dissociated*. There is an old Chinese saying: The fish is the last one to realize that it lives in the water. The fish is inseparable from the water. The complexity of the relations among signifier, signified and interpreter is more resilient than any attempt to fracture it. The symbolic experience results from the "blending" of interpreter and symbol. If you are using an elevator and you find a button with the word "close" written on it, you interpret that word as meaning something different if you find the same word written on the door of a bookstore. These are two different symbolic experiences. There has been a long and fierce debate on the nature and definition of symbols; on the relationships between the signifier and the signified; on the role of the interpreter and the reference field lodged within a symbol. There is a social agreement as to what the words mean. It seems that the connections between "bird" and all that comes to mind when one hears the word, is arbitrary but *a posteriori* those conventions are not arbitrary. Thus, *the symbol refers to something that although arbitrary, is shared and agreed by a community*. Nevertheless, there is always room for personal interpretations: The experienced world inside each of us plays a considerable role in our quest for meaning but, for communication to work, we need to share a substantial part of the reference fields of our symbolic systems. This is worth taking into account when analyzing the learning process of students.

The reference field lodged within a symbol can be greatly enhanced when that symbol is part of a network of symbols. Emergent meanings come to light because of the new links among symbols. This phenomenon can be termed the *semiotic capacity* of a symbol system. An obvious example is provided by natural languages wherein the meaning of a word can be found inside the net of relations that are established with other words in utterances or texts. Reading in a foreign language illustrates this situation very well. With a high frequency you can realize the meaning of an unknown word thanks to the context (sentences, paragraphs) wherein that word appear**.**

The symbolic system drives the production of diverse *levels of crystallization* for our own thinking. Writing is indeed one of the best instruments to reach these levels in our thinking. As an example, consider the symbolic system called Arithmetic. As your level of proficiency grows, the symbols 1, 2, 3, …become *crystallized beings* in themselves, for you, the interpreter. An important trait of symbols consists of the capacity to crystallize an experience and explore it (that experience) by means of the objective model thus constructed. In that dynamical process, signifier and signified spiral one around the other incessantly...but do not forget: that spiraling takes place in the cognition of the interpreter. Or the student.

Previously we said that Estella observes on the screen an *index of variation*. In other words, she observes a signal that engenders an inference: *the area becomes smaller if the sides become shorter*. This inference is a first step to understanding the variation of area. Here is where a perceptual phenomenon becomes an *act of visualization*, that is, perception controlled by an interpretation. A reference field in then associated with the phenomena observed on the screen. Cabri-Geometre is an adequate instrument to develop geometric visualization –a crucial cognitive problem–, and a sense of variation. This is instrumental –along these lines of development– for an understanding of symbolization of variables within a functional relation.

Language and instrumental mediation

In a general sense, we can say that a tool constraints the activity that is feasible with it. There is a complex of intentional activities embodied within a tool. Thus a user only can proceed in agreement with the legitimacy that is imposed from outside. The tool crystallizes a complex of activities, in a sense, equivalent to the tool itself. In the hands of the student, the calculator induces the use of certain vocabulary associated with the words in the menus. For instance students say: “use compass” or “use perpendicular bisector”. All these expressions, adopted from the menus, become part of a “language” that refers to practical activities within the dynamic environment. In this sense, these words are used as markers, or indexes, for reflecting on the instrumental activity.

Each tool has a vocabulary, a small set of symbolic expressions, that the user must know in order to use the tool creatively and share this use with other users. This is part of the process of *socialization* of the tool. Needless to say, this is crucial in the classroom. A symbolic tool is a mediatory tool for communication and thinking. As Chassapis (1999, p. 276) has remarked, “mediatory means, thinking processes, and human practical activities become functionally intertwined in their development, shaping each other in a dialectical interdependence” and hence, their use has an impact on the process of concept development.

Second interview

Let us describe, briefly, another activity developed with 13-14 olds.

Consider a segment AB and a moving point P on this segment.

After measuring the lengths of segments AP, PB and AB, students must establish a valid relation among these lengths. To do so, they are asked to produce a table with these values (AB, AP, PB) as P moves along AB. The first answers were: “The length AB is the sum of the lengths AP and PB”, “AP plus PB equals AB”. Later, we were able to hear expressions like: “AP and PB are variables but AB is constant”. Or, “when we drag P, AP and PB change but not AB”. Afterwards, we explicitly asked to write a symbolic expression corresponding to these previous natural language expressions. Finally they wrote: AP+PB=AB. Then the instructor proposed to write x=AP and y=PB. It was very difficult for the students to arrive at x+y=5 (the original length of the segment AB). It follows that dragging, naming symbolically the variable magnitudes contribute to the building of a bridge between natural language and algebraic language. Cabri- figures are *iconic referents* for variation (see Deacon, T. op. cit. p. 71, for a discussion on icons). The names AP, and PB are used as indexes that indicate the variability of those segments.

Remarks

We hope this little study provides some elements to establish that within a digital tool as the one selected here, Cabri-Geometre, there are elements that can be used to explore variation and variability with a specific intentionality to help pave the way towards their symbolic representation. In our study, the variable is an abstraction that generalizes a visual dynamic pattern and that finds its objective existence in the symbol.

References

N. Bednarz, C. Kieran, and L. Lee (eds) (1996). *Approaches to Algebra: Perspectives for Research and Teaching*, Kluwer.

D. Chassapis (1999).The mediation of tools in the development of formal mathematical concepts:the compass and the circle as an example. *Educational Studies in Mathematics,* 37: 275-293.

Deacon, T. (1997). *The Symbolic Species.* Norton, New York.

Moreno, L. & Santillán, M. A. (2002). Visualizing and understanding variation. In D. S. Mewborn, et al. (eds), *PME-NA XXIV,* Vol. 2, pp. 907- 914.

Duval, R. (1995). *Semiosis et Pensée Humaine. Registres Semiotiques et Apprentissages Intellectuels.* Peter Lang S.A. Edition Scientifiques Europeennes.

Leinhardt, G., Zaslavsky, O. & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks. Learning, and Teaching. *Review of Educational Research,* Vol. 60, No. 1, pp. 1-64.

FROM ARITHMETIC TO ALGEBRA: A STUDY OF THE SPECIAL CASE OF GEOMETRIC FORMULAS

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This paper presents the results of a research study which had as its initial objective to investigate students' difficulties working with geometric formulas before any formal training in algebraic ideas. Although we feel that this is not the proper order, it is the current practice in elementary schools in Mexico. We describe and compare the responses of two groups of students, one at the elementary level (11-13 year olds) and the other at the secondary level (14-16 year olds) to a questionnaire designed for this purpose and to the interviews that followed. The analysis showed that the students lack the basic algebraic concepts required to face the complexity of geometric formulas. Thus, there is a need to reformulate this practice and to put forward approaches for the classroom in which algebraic concepts are learnt through geometric contexts to smooth out the transition from arithmetic thinking to algebraic-geometry thinking.

Introduction and Theoretical Framework

The difficulties of students in the transition from arithmetic to algebraic ideas have been studied and documented thoroughly (Boulton-Lewis et al. 1998; Reggianni, 1994). A book edited by Bernardz, Kieran and Lee (1996) compiles the different perspectives of algebra (functional, generalization, problem solving and modeling) and the efforts made, trying to smooth out this transition.

However, there is a similar situation within the subject of geometry that has been mostly ignored. In fact, we observe in the classroom practices that geometric formulas of perimeter, area and volume are introduced before any serious study of algebra.

Not only do the analytic expressions in geometry contain the variable as a functional relationship but in fact, geometric formulas relate several variables in very complex ways. To show this point, it is enough to give the following two examples:

$$A = \frac{(B+b)h}{2} \qquad V = \pi\, r^2 h$$

Both of these formulas are introduced at the elementary level, to 10-12 year old students, before any of the elements of algebra have been taught.

Several studies on the conceptualization of the variable have shown that students at all levels have difficulties understanding the different meanings of the variable (Bills, 2001; Ursini and Trigeros, 2001; Stacey and McGregor, 1997; Warren, 1999).

The variable has three main uses (Usiskin (1988); Ursini et al. 2002): as an unknown, as a generalized number and within a functional relationship. According to these three uses, within the context of geometric formulas, students would need the following basic ideas:

- To differentiate between variables and constants like π.
- To interpret the variables in a formula as an entity that can take any value.
- To determine the values of the dependent variable given the others.
- To determine the value of one of the independent variable given the others.

- To recognize the joint variation of the variables in a formula.
- To think of a symbolic variable as an entity that can be operated on.
- To express as a formula, a functional relationship between variables.

What we describe in this article is the first stage of a research project which has as it main objective to integrate the learning of algebraic and geometric ideas. In the final stage of this project we are planning to develop a learning sequence with geometric contexts which has the learning of geometric ideas as its main purpose, but at the same time, advance the students from arithmetic to algebraic thinking. In this way, geometric ideas can be used to introduce algebraic concepts and to deepen students' understanding of algebra.

Consequently, we advanced a research project which had as its first objective to find out what specific difficulties students have in working with geometric formulas before any basic training in algebra (as it is currently done in the practice of elementary schools in Mexico.)

Methodology

Before we can address the more significant question of how to use a geometrical context to help students advance from arithmetic thinking to algebraic thinking, we would like to find out first the kinds of algebraic thinking that are done when students work in geometrical contexts.

At this stage, the research questions are limited to: (a) what meaning do elementary and secondary students assign to geometric formulas and (b) how their ways of understanding variables and equations (which is different in both groups) affect their ability to work with geometric formulas.

For this purpose, a questionnaire (see appendix) with 10 problems was designed and applied to two groups of students. The first consisted of 28 students finishing elementary school (11-13 year olds). The second had 25 students finishing secondary level (14-16 year olds). Both groups were chosen at random from the same zone in the south of Mexico City.

In some of the questions we compared numerical tasks with similar algebraic tasks. In others, we studied the different uses of the variable within a geometric context. Yet, in others, we observed the abilities and difficulties of understanding the syntax of formulas and substituting values on them.

After the answers of the questionnaires were analyzed and compared, we selected six students from each group for interviews to probe their ideas further.

We centered the analysis on what students are capable of doing, the meaning they attach to geometric variables and formulas and their difficulties in each of the tasks.

Results

First we will give the results of the analysis to some of the tasks in the questionnaire, comparing the two groups of students.

The first exercise of the questionnaire contains three tasks about a square: a) calculate the area from numerical data, b) express the formula for the area and c) express the formula for the perimeter. The second exercise contains two tasks about a parallelogram: a) calculate the perimeter from numerical data and b) express the formula for the perimeter.

The number of correct answers for the two groups of students in these two questions are summarized in the following table:

	1 a)	1 b)	1 c)	2 a)	2 b)
6th graders	18 + 9*	17	14	24	3
9th graders	24	24	19	19	16

The 9 marked with an *, calculated the perimeter instead of the area.

Two comments on these results are worth mentioning. There is a drop from numerical exercises (1a and 2a) to the algebraic ones (1b, 1c and 2b) especially in the 6th graders. But what really stands out are the 3 correct responses of the 6th graders in the question 2b (some of the others wrote known formulas for the area of a rectangle or a triangle.) This shows that they have difficulty in expressing the perimeter formula of a parallelogram (the other two questions requiring formulas didn't show this drastic fall probably because the elementary school children knew by heart the formulas of a square.)

The first part of the fourth exercise requested a calculation of the area of a rhombus. 57% of the 6th graders did this calculation correctly as opposed to 100% of the 9th graders. However, the third part of this question was the most interesting. None of the 6th graders accepted as valid the second form of the formula. Some stated: "it has to have the × because without it, you don't know if to multiply, add or subtract" or "without the ×, it doesn't mean anything". On the other hand, half of the 9th graders stated that the formula still has meaning and was correctly written.

The first part of the fifth exercise asks for the calculation of the area of a trapezium given its formula (with the sign × explicitly shown in it). Here, 75% of the 6th graders and 92% of the 9th graders did this calculation correctly. This shows that this task of plugging numbers into a formula is not difficult for these students. The second part, asked why letters are used in formulas. Twelve 6th graders gave the interpretation of "initials" for each of the sides, and fourteen of them gave an operational meaning like: "to know what you have to do" or "because I added 3 and 2, multiplied it by 4 and divided it by 2". On the other hand, 21 of the 25 9th graders gave the "labels" interpretation: "each letter is the label of one of its sides".

In question 6 (part c), students were asked to calculate the circumference of a circle given the value of its radius and the corresponding formula. Only 6 students of the 6th graders and 12 students of the 9th graders performed the correct procedure, even though, 20 and 23 students respectively knew the value of pi (part a) of the same question). This shows that even for this relatively "simple" formula, the majority of students had trouble using it. When asked, if one of the three forms given was wrongly written, 26 of the 28 6th graders said yes. 65% of those choosing a specific form, selected the third one because "you don't understand it, if it doesn't have signs" or "you cannot get the result like that". Only half of the 9th graders, said yes to this question (the rest said that all of them are valid) and again, the third form was the most frequently picked. This shows the difficulty these students had to obtain a result from a formula that doesn't have meaning to them.

In question 7, only 6 students (6th graders) related the "a" to the side of the cube. The rest gave different answers like "apothem" or "area". Eleven of them associated the 3 in the formula to a cube power. The others mentioned different meanings like "times 3", "to the square". Only half the 6th grade students calculated correctly the volume as opposed to 100% of the 9th graders. Once again, most students of both grades chose the last form as the one wrongly written because "it doesn't have signs" or "it doesn't say anything".

In exercise 9, only 5 of the 6th graders could calculate the perimeter using the formula. Most of them obtained other values like: (3, 4 and 5), (6, 8 and 10) or (3L, 4L and 5L). This shows that they have difficulties even in the task of plugging numbers into a simple formula (80% of the 9th graders did this correctly).

In part b) of exercise 9, surprisingly enough, 60% of 6th graders could give the length of the side given the perimeter. This doesn't mean that they can calculate this value "from the formula" but most likely their strategy was to guess the answer based on the fact that an octagon has 8 sides. By comparison, 85% of the 9th graders got this correctly.

In question 10, only 3 of the 6th graders calculated the area correctly (as opposed to 22 of 9th graders). Other frequent answers were: "3.14×10", "3.14", "3.14×10/2" or "3.14×20". In part b) of this exercise, 26 students of 6th grade chose the third form as the one wrongly written because "you don't know what it means" or "it cannot have two r's" or "it doesn't have how many times to multiply".

In the interviews we observed many of the difficulties students have with algebra but within the geometric context. We will concentrate on describing the answers given by one 6th grade student (L – Luisa; I – interviewer):

In the following two sections of the interview, we observe a confusion between area and perimeter not only conceptually, but generated by the formulas used:

I: How did you calculate the area of the square?

L: First, I multiply the side, 6 times 6 equals 36, and then I multiply by 2 to get 72.

I: Why times two?

L: Because 6 times 6 gives me 36 and the other two sides missing, so I don't have to add all of them. I could do it also by 6 times 4 or by 6 times 6 equal 36 times two equal 72.

Later in the same interview:

I: In the formula for the area of a square, why did you write L × L?

L: Because it is side times side times side times side.

Later on, in the same interview, the student shows that she cannot operate on variables like b+ L:

I: For the formula of the perimeter of a parallelogram of base b and side L you wrote b + L. Do you think it is enough to calculate the perimeter?

L: No.

I: What is missing?

L: Yes it is base plus side, and later if I would have a number, what I get I would multiply it by two since there are four sides.

When asked how many values can the L in the formula P = 3 L take, she interprets the right side of this formula as "3 sides" and concludes it is the perimeter of a triangle. She then expresses that the L should take only the value 3 because L is the number of sides.

Many students interpret the variables as simple labels. For example, Luisa states that the "a" in the formula $V = a^3$ means "area". When she calculated the volume "according" to this formula, she stated: "I first obtain the area" and then she multiplies 5 by 5. In the next part of the same question, she chooses the formula V = a a a as wrongly written since "it doesn't have any signs and therefore we wouldn't know if to multiply or add". This was a common feature of many younger students which do not accept that the multiplication signs in a formula can be omitted.

In exercises 7 and 10 where powers appear, some students ignored them. For instance, in question 10b, a student said that the third formula is not the same as the other two because "it has an extra r".

We observe in general that the younger students still have conceptual problems with the differences between area and perimeter that crept into their reasoning. Although they can perform calculations with known formulas, they had a lot of trouble with the unknown of complicated formulas because they still do not understand the algebraic language of an equation. Although, they can calculate something numerically, they cannot transfer this process to the algebraic world. Thus, they have difficulties to express a relationship with a formula since they cannot operate on variables.

They have major difficulties in understanding the syntax of algebraic formulas. They still need an operation sign between the variables of a formula to tell them what to do with the values they substitute in it (as in exercise 4). Variables with powers also present difficulties for them (they use the power as a factor).

The older students have developed a better understanding of algebraic ideas that allows them to better perform in all these tasks. However, even at this age, the same difficulties appear to a smaller extent.

In the following paragraphs we describe some of the answers given by one 9th grade student (IM – Imelda; I – interviewer):

In the following section of the interview, we observed that she doesn't have problems operating on variables or transferring to the algebraic domain (her answer to problem 2b was b×2 + L×2):

I: If instead of numbers, you have letters, do you do the same?

IM: Yes, here the same, there is a side measuring L and another one measuring b. You multiply b times two because it has two equal sides and L times two because it has another two sides equal and then you add the results.

I: How is it simpler, with letters or with numbers?

IM: With letters because you can give them any value and thus, it is useful for many.

Imelda still prefers formulas with signs because "you need a sign to specify the operation".

Conclusions

Algebra and geometry are treated separately in the math curriculum. This assumes that there is little connection between these subjects and that the understanding and handling of formulas in geometry does not required algebraic thinking.

In this study, we can observe within geometry, the same conceptual difficulties encountered in the algebraic content. We saw for example that the students cannot transfer from the numeric world to the algebraic-geometric world. The analysis also showed that the students lack the basic algebraic concepts required to face the complexity of geometric formulas.

Thus, there is a need to put forward approaches for the classroom in which algebraic concepts are learn through geometric contexts to smooth out the transition from arithmetic thinking to algebraic-geometry thinking.

Students' understanding of geometric ideas and formulas, involve inevitably algebraic concepts like notion of variables, functions, joint variation, etc. This study hopes to promote research that will help making connections across these two domains.

This is an example of a research area (algebraic ideas) which had had an extensive influence on its own practice, but has not impacted on the practice of the same ideas within a related topic (geometric ideas.)

References

Bernardz, N., Kieran, C. and Lee, L. (eds, 1996) *Perspectives for Research and Teaching*, Départment de Mathématiques, Université du Quebec á Montreal. Kluwer Academic Publishers, Netherlands.

Bills, L. (2001) Shifts in the meanings of literal symbol, *Proceedings PME 25*, Vol. 2, 161-168.

Boulton-Lewis G. M., Atweh B., Pillay H. and Wilss L. (1998) Pre-algebra: A Cognitive Perspective, *Proceedings PME 22*, Vol. 2, 144-151.

Stacey K. and McGregor M. (1997) Multiple Referents and Shifting Meaning of Unknowns in Students' Use of Algebra, *Proceedings PME 21*, Vol. 4, 190-197.

Reggiani, M. (1994) Generalisation as a Basis for Algebraic Thinking: Observations with 11-12

Year Old Pupils, *Proceedings PME 18*, Vol. 4, 97-104.
Ursini S. and Trigueros M. (2001) A model of the uses of variable in elementary algebra, *Proceedings PME 25*, Vol. 4, 327-334.
Ursini S., Trigueros M., Montes, D. and Escareño, F. (2002) Teaching Algebra using the 3UV model, *Proceedings PME-NA 24*, Vol. 1, 495-507.
Usiskin Z. (1988) Conceptions of School Algebra and Uses of Variables, in Coxford A. F. And Shulte A. P. (Eds.), *The Ideas of Algebra K-12,* 8-19.
Warren E. (1999) The concept of variable; gauging students' understanding, *Proceedings PME 23*, Vol. 4, 313-320.

Appendix

In this appendix, we reproduce the statements of the ten problems used in the questionnaire (we will only describe the figures shown in it.)

1. a) Calculate the area of the following square (a square with measures 6 cm. and 6 cm. is shown).
   b) Write the formula for the area of the following square of side L (a square with measures L and L is shown).
   c) Write a formula expressing that the perimeter of a square is the sum of the measures of all its sides.
2. a) Calculate the perimeter of the following parallelogram (a parallelogram with measures 8 m., 5m., 8 m. and 5 m. is shown).
   b) Write the formula for the perimeter of a parallelogram of base b and side L (a parallelogram with measures b and L is shown).
3. a) How many values can the P in the formula P = 3 L take?
   b) How many values can the L in the same formula take?
   c) Circle the pairs of values that correspond to the previous formula:
   L = 6 and P = 18 , L = 4 and P = 10 , P = 15 and L = 5 , P = 1 and L = 3
4. The area of a rhombus is given by: $A = \frac{D \times d}{2}$, where D and d represent the length of the diagonals (a rhombus with diagonals D and d is shown).
   a) Calculate the area if D = 8 cm. and d = 4 cm.
   b) What is the meaning of the × in the formula?
   c) Somebody writes the formula as $a = \frac{Dd}{2}$. Is it well written? Explain.
5. The area of a trapezoid is given by: $A = \frac{(B + b) \times h}{2}$ (a corresponding trapezoid is shown).
   a) For B = 3 m., b = 2 m. and h = 4 m., calculate the area of the trapezoid.
   b) Explain why letters are used in a formula like this one.

6. The length of the circumference of a circle is given by C = 2 π r (a circle of radius r is shown).
   a) In this formula, what value or values should be given to π("pi")?
   b) What value or values should be given to r?
   c) Calculate the length of the circumference (a circle of radius 8 cm. is given).
   d) The previous formula is written in the following three forms:

$C = 2\ \pi \times r$ $\qquad$ $C = 2 \times \pi \times r$ $\qquad$ $C = 2\ \pi r$

Is any of them written incorrectly? Why?

7. The volume of a cube is given by the formula: $V = a^3$ (a cube with sides "a" is shown).
   a) What does the "a" in this formula represent?
   b) What is the meaning of the 3 in the formula?
   c) Calculate the volume of the cube: (a cube of side 5 is shown).
   d) Compare the formulas for the volume of a cube:

   $V = a^3$ $\qquad$ $V = a \times a \times a$ $\qquad$ $V = a\ a\ a$

   Which is the best written? Why?

   Is any of them written incorrectly? Why?

8. (a triangle of base b and height h is shown).

   From the formulas $A = \frac{L \times L}{2}$ $\quad$ $A = \frac{b \times h}{2}$ $\quad$ $A = \frac{b \times a}{2}$ that allow you to calculate the area of a triangle, which seems the best written to you? Why?

9. (a regular octagon is shown).

To obtain the perimeter of a regular octagon the following formula is used: P = 8 L. For L =

3, what is the value of P? For L = 4, what is the value of P? For L = 5, what is the value of P?

a) Which other value, in addition to 3, 4 and 5, you can give to L to calculate P?

b) If the perimeter of a regular octagon is 56, can you find out the length of its sides? What is the length?

10. The area of a circle is given by $A = \pi r^2$ (a circle with radius r is shown).

a) Calculate the area of the following circle (take the value of πas 3.14) (a circle of radius 10

m. is shown).

b) The previous formula is written in the following three forms:

$A = \pi r^2$ $\qquad$ $C = r^2\ \pi$ $\qquad$ $C = \pi r\ r$

Which is the best written? Why?

Is any of them written incorrectly? Why?

A TEACHER'S MODEL OF HIS STUDENTS' ALGEBRAIC THINKING: "WAYS OF THINKING" SHEETS

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This research report describes a subset of the findings of a study on mathematics teachers' models or interpretations of students' responses to middle school algebraic tasks involving equivalent expressions. A cohort of five teachers implemented the reform-based Connected Mathematics Project (Lappan et al, 1998) for the first time. The theoretical framework and related research design drew upon a models and modeling perspective of teacher development (Doerr & Lesh, 2003). The teachers in this study created "Ways of Thinking" sheets to help them recognize the multiple ways in which their students interpreted the given tasks. Results of the analysis of the practice of one of the teachers showed that the teacher became aware of the multiple ways his students solved these algebraic tasks, including the use of visual representations and conjoining expressions.

Introduction

Since the nature of algebraic instruction is changing (NCTM, 2000), it is important to study how the changes have impacted classrooms and teaching. Reform-based curriculums often use open-ended questions. The use of open-ended questions offers teachers the possibility of detailed information from which to examine students reasoning processes (Chamberlain, 2003; Moskal & Magone, 2000; Van den Heuvel-Panhuizen, 1994). These studies collectively reveal that not all teachers acquire the same information nor interpret it in consistent ways. The primary goal of this study was to focus on the nature of the teachers' developing ideas about their students' algebraic thinking as the students completed a series of open-ended tasks on equivalent expressions. The teachers engaged in a model-eliciting (or thought-revealing) activity designed to reveal the multiple ways their students interpreted mathematical problems by creating "Ways of Thinking" (WOT) sheets (explained fully below). Specifically, the core research question addressed in the study was: How do teachers interpret their practice when they focus on their students' algebraic thinking about equivalent expressions by creating WOT sheets?

Kieran (1992) documented that the leap from arithmetic to algebra is often related to instructional strategies. Research shows implementing reform-based instructional programs can successfully advance students' conceptual thinking and skills (Hiebert, 1999). Students often want a numerical answer for algebraic problems (Booth, 1988; Stacey & MacGregor, 2000). Under a reformed vision of algebraic instruction, students will make connections between conceptual (or structural) features and procedural (or operational) features. In addition, the research base about teachers' knowledge for algebraic instruction is still quite limited (Doerr, in press). Research on teachers' knowledge related to equivalent expressions showed that neither novice nor expert teachers used spatial arrangements to help students see that two expressions might be equivalent (Even, Tirosh, & Robinson, 1993). Tirosh, Even, and Robinson (1998) showed that experienced teachers were aware of students' tendencies to conjoin open expressions, and novice teachers were not aware.

Theoretical Framework

A models and modeling perspective of teacher development guided the development of this study and framed my examination of the way teachers think in the context of their work. To make sense of complex situations, teachers need to develop systems of interpretation or models that account for their experiences (Lesh, Doerr, Carmona, & Hjalmarson, 2003). A modeling perspective of teacher development focuses upon the ways teachers think about and interpret their practice (Doerr & Lesh, 2003). Through the development of model-eliciting (or thought-revealing) activities, teachers recognize the multiple ways their students interpret mathematical problems. This perspective asserts that teachers' models serve as interpretive and explanatory frameworks to make sense of their students' mathematical thinking. What teachers do is inherently complex. This perspective draws upon the mathematical knowledge teachers currently possess and uses that base to engage teachers in expressing, revising, and refining their knowledge. The intent is to extend that knowledge into increasingly powerful models of classroom teaching. Model-eliciting or (thought-revealing) activities for teachers help account for the evolving nature of the teachers' learning, and attempt to help them become reflective of their teaching efforts.

Methodology and Data Sources

Multi-tiered Teaching Experiment

The goal of the study was to develop a description of the teachers' developing models of their students' algebraic thinking. Five teachers from two urban middle school settings participated in a multi-tiered teaching experiment designed to perturb and promote the development of teachers' knowledge (Lesh & Kelly, 1999). In the multi-tiered research design each of the participants, students, teachers, and researchers, play a unique role as follows (Lesh & Kelly, adapted from p. 198):

Tier 3: Researcher Level	Researchers develop models to make sense of teachers' and/or students' model-eliciting activities. Researchers reveal their interpretations as they create learning situations for teachers and/or students, and as they describe, explain, and predict teachers' and/or students' behaviors.
Tier 2: Teacher Level	Teachers develop shared tools (such as WOT sheets or libraries of student work). As teachers describe, explain, and predict students' behaviors, they construct and refine models to make sense of students' activities.
Tier 1: Student Level	Students work on a series of activities, in which the goals include constructing and refining models (descriptions, explanations, justifications) that reveal how they are interpreting a mathematical situation.

Figure 1. The design of multi-tiered teaching experiments.

Data Sources

A thought-revealing activity should result in a shared and reusable artifact (Schorr & Lesh, 2003). The thought-revealing activity used in this study consisted of asking teachers to create a WOT sheet based upon anticipated and then actual students' responses to the selected assessment tasks (Doerr & Lesh, 2003). WOT sheets are fully described as "a task that engages teachers in the task of anticipating and evaluating how their students' ideas might develop" (Doerr, in press). Teachers were asked to complete the WOT sheets before the students began the tasks, immediately after implementing the tasks, and about a month later. For each iteration of the WOT sheets, teachers were directed to think of what might be useful to a pre-service teacher and write down such things as:

Ways of Thinking Sheet
Hints about the students' mathematical thinking
Mistakes students might make in their mathematical thinking

Figure 2. Sample Ways of Thinking Sheet

The teachers were asked to create the WOT sheets over time so as to have the opportunity to test, revise and refine their thinking about the students' work.

The primary data source used in this study were the WOT sheets. Other data sources included fieldnotes from classroom observations, student work samples collected immediately after implementation of the problem, and transcripts from teacher interviews conducted before the tasks were taught, immediately after the tasks were completed, and again about a month later.

Data Analysis

Data analysis drew upon a grounded theory approach (Strauss & Corbin, 1998) where the theory was derived from the data, and systematically gathered and analyzed throughout the research process. Analysis began as each initial interview was transcribed. Analysis within this study was a continual process of simultaneously coding and analyzing data. The analytic process involved reading and rereading the data, and coding and refining the codes, throughout the duration of the study in order to gain insight to the research question. Consistent with a models and modeling perspective, the records the teachers produced in the form of WOT sheets and samples of student work helped produce a continuous trail of documentation, and these artifacts were used to reflect on the nature of the teachers' developing models (Lesh & Kelly, 1999). These reiterative cycles promoted the teachers and researcher to test, revise, and extend their knowledge development. From this process, I developed a profile of each teacher. The teacher being reported on in this research report was implementing CMP for the first time. Bruce (a pseudonym) had 18 years of experience teaching middle school mathematics in an urban setting. He was one of five eighth-grade teachers in this study.

Student Mathematical Problem

Students were asked to solve a series of tasks involving equivalent expressions. The tasks were drawn from the Connected Mathematics Project (CMP) book *Say It With Symbols* (Lappan et al., 1998). Students were first asked to find the number of 1-foot square tiles surrounding different sized square pools and then to find an equation for the number of tiles, N, needed to form a border for a square pool with sides of length s feet. Later, the students drew representations for the following problem, and finally they applied the distributive property.

Given a square pool as shown, draw a picture to illustrate the border of a square pool in four different ways:
a. $4(s + 1)$
b. $s + s + s + s + 4$
c. $2s + 2(s + 2)$
d. $4(s + 2) - 4$
e. Explain why each expression in parts a-d is equivalent to $4s + 4$.

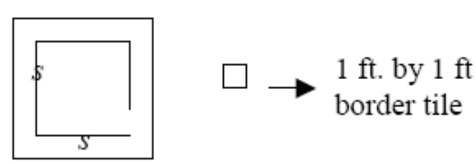

Figure 3. Student problem (Lappan et al., adapted from p. 22).

Results

This research report focuses upon one teacher's (code named Bruce) interpretations of his practice. Two results emanated from the study. First, Bruce gave a significant amount of time to the implementation of the unit. Second, Bruce was beginning to develop significant insight into the students' ways of thinking. These results are intimately connected with one another. Bruce's support of the implementation led to his acceptance of the new ideas about how to teach, which in turn led to his new insights on helping his students learn algebra in meaningful ways. I continue with excerpts of Bruce's responses to interview questions that show support for the implementation of the curriculum, and then excerpts from his WOT sheets that demonstrate his insight into his students' algebraic thinking on equivalent expressions. Throughout the study, Bruce became increasingly aware of the multiple ways his students solved the given tasks. This was Bruce's first time engaged in creating WOT sheets, and it is likely that his perceptions would grow over time.

First, Bruce was committed to teaching the task sequence in its entirety at the pace dictated by the students' progress. Bruce spent more time than any other teacher in this study that amounted to about one and one-half times as long to teach the tasks than recommended by the curriculum developers. This appeared to be due to his willingness to give his students ample time to investigate the problems. Bruce acknowledged that "*Time is always one of the biggest factors, you know. I am always fighting with that...*" However, he concluded at the end of the study, *"I know that I will be doing this again. I will never do it the way I did in the past. I think this is better."* Bruce came to embrace the curriculum during the course of the study.

Second, I will chronologically describe Bruce's responses on his WOT sheets. Before implementing the unit, Bruce offered these potential mistakes:

Ways of Thinking Sheet	
Hints about the students' mathematical thinking	• None.
Mistakes students might make in their mathematical thinking	• Students won't use inverse operations to correctly solve equations for variables. • Students won't correctly define a variable and write a corresponding equation.

Figure 4. Bruce's First WOT

Both of these mistakes are related to solving equations, and not necessarily linked to the material on equivalent expressions. Solving equations does appear at the end of the *Say It With Symbols* unit, but is not a part of these tasks. Bruce was perhaps thinking more globally, and this may allow him to be open to variations in student work. Alternatively, he may not have examined the material on equivalent expressions in detail.

In Bruce's second iteration of the WOT sheets, he began to develop insight into his students' algebraic thinking about equivalent expressions. Bruce reported as follows:

Ways of Thinking Sheet	
Hints about the students' mathematical thinking	• Using a visual or geometric representation helps the students.
Mistakes students made in their mathematical thinking	• It was difficult for students to find algebraic equations for the number of 1foot square tiles surrounding a square pool. • Students incorrectly labeled the corners in their representations of the different expressions for the number of tiles.

Figure 5. Bruce's Second WOT

He now perceived that visual representations helped his students. When describing visual representations, he said:

> *For example, having a student come to the board to explain how they are dividing up the border of tiles around that pool to present that as breaking up that border into pieces… And showing the pieces. And they are relating the pieces to the parts of the expression.*

Bruce also described:

> *I did have him show me his expression and he did for the s + 2… on both ends with two tiles and then again down here, the two sides…I was surprised [because of] the student that it was. He is one of my students who is generally not that "with it."*

Bruce now observed that many of his students connected the visual representation with a given algebraic expression, although some students at times incorrectly labeled the corners. Bruce also noted that his students could readily find the exact number of tiles surrounding a given sized pool, but that they had trouble writing an algebraic generalization. Consistent with prior research on student knowledge, Bruce found that his students preferred a numerical answer.

In the third iteration of his WOT sheets, he continued to develop insight:

Ways of Thinking Sheet	
Hints about the students' mathematical thinking	• Using a visual or geometric representation helps the students. • Students might add $16x + 2$ together.
Mistakes students made in their mathematical thinking	• Students can set up equations properly for such problems as write an equation for the money raised in a walk-a-thon if Alena earns a \$5/donation and \$.50/mile for each sponsor, and if Gilberto asks his sponsors to pledge \$2/mile. However, the students could not apply the distributive property and order of operations consistently.

Figure 5. Bruce's Third WOT

At the conclusion to the study, Bruce emphatically stated (about his hints), "*using a visual of geometric representation helps the students*." The more his students used these representations, the more Bruce felt that they understood the problems. Also, now that Bruce had taught the entire CMP unit, his students solved equations. Bruce now noted that some of his students had a tendency to conjoin expressions. Bruce previously did not mention this possibility. Bruce became increasingly focused on his students' thinking and his responses became more specific to the algebraic material related to equivalent expressions as his WOT sheets progressed through the three cycles of iterative refinement. In his last interview, Bruce remarked that his students' understanding of variable grew during the instructional sequence. The curricular activities guided Bruce and his students to using variables in contexts different from those that Bruce used in the past. In prior years, Bruce lectured about properties like the distributive property. At the end of the study, Bruce commented, *"They are using pictures and diagrams and they are labeling the different parts of the diagrams with variable terms, and then expressing those areas in different ways."* The visual representations helped his students make sense of the algebraic expressions. Bruce also concluded, *"The transition… from arithmetic to algebra is something that has got to be taken I think a little more seriously."* He began to think more deeply about students' ways of thinking and algebraic instruction during the course of the study.

Discussions and Conclusions

Since teachers' knowledge is complex, teachers need multiple models illuminating the myriad of possible interpretations of situations for teaching and learning (Doerr & Lesh, 2003). The emphasis under the models and modeling perspective is on the way in which teachers interpret situations. These interpretations are assumed to be in a constant state of development. Bruce's practice was shaped by the implementation of a reform-based curriculum and the related model-eliciting WOT activity. During the course of the study, Bruce perceived that his teaching practice changed because he would never teach it again the way he had in the past. Bruce developed new insight into the usefulness of visual strategies as well as a recognition of conjoining expressions. Contrary to Even, Tirosh and Robinson (1993), Bruce identified visual arrangements as a highly useful instructional tool. Tirosh, Even, and Robinson (1998) found that experienced teachers were aware of students' tendencies to conjoin expressions. Bruce did identify this as significant later in the study where conjoining became both an important hint and a topic ripe for mistakes. Bruce's students behaved consistent with research on wanting a numerical answer, but Bruce's comments demonstrate that he was able to move in a direction away from emphasizing procedural operations (that he identified as salient in the beginning of the study) to a more structural understanding of teaching equivalent expressions incorporating visual representations.

Ultimately it is the teacher who interprets the curriculum and carries out the instructional process. Teachers must be aware of the students' ways of thinking to help students' self correct and improve. The more a teacher can anticipate, the more potential exits for dynamic instruction. The literature is replete with situations describing students' difficulties and the area is a difficult one to teach. A collection of the teachers' WOT sheets may provide other groups of pre-service or in-service teachers with examples that are useful for algebraic instruction.

Continued research on the development of teachers' knowledge of algebraic instruction is needed to improve and inform instruction. Instructional strategies are important to students' success. There are many opportunities for further research. First, research is warranted even on the same series of lessons in this study. This study was conducted with teachers implementing the lessons for the first time. Further research is needed to see how teachers' models may evolve in subsequent years. A second area concerns the use of different model-eliciting activities that might extend or refine the models presented in this study. A third possibility is to extend the research framework to other algebraic tasks.

References

Booth, L. (1988). Children's difficulties in beginning algebra. In A. Coxford (Ed.), *The ideas of algebra, k-12* (pp. 20-32). Reston, VA: National Council of Teachers of Mathematics.

Chamberlin, M. T. (2003). Teacher investigations of students' work: Meeting the challenge of attending to students' thinking. In N. A. Pateman, B J. Dougherty, and J. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of the International Group for the Psychology of Mathematics Education and the North-American Chapter of the International Group for the Psychology of Mathematics Education: Vol. 2* (pp. 189-196). Honolulu: University of Hawaii Center for Research and Development Group.

Doerr, H. M. (in press). Teachers' knowledge and practice and the teaching of algebra. Paper to be published in book by K. Stacey, M. Kendal & H. Chick (Eds.), the Twelfth ICMI Study on *The Future of the Teaching and Learning of Algebra*.

Doerr, H., & Lesh, R. (2003). A modeling perspective on teacher development. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective* (pp. 125-140).

Mahwah, NJ: Lawrence Erlbaum Associates.
Even, R., Tirosh, D., & Robinson, N. (1993). Connectedness in teaching equivalent algebraic expressions: Novice versus expert teachers. *Mathematics Education Research Journal, 5*(1), 50-59.
Hiebert, J. (1999). Relationships between research and the NCTM standards. *Journal for Research in Mathematics Education, 30*, 3-19.
Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan Publishing Company.
Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (1998). *Connected mathematics*. White Plains, NY: Dale Seymour Publications.
Lesh, R., Doerr, H., Carmona, G., & Hjalmarson, M. (2003). Mathematical Thinking and Learning. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective* (pp. 125-140). Mahwah, NJ: Lawrence Erlbaum Associates.
Lesh, R., & Kelly, A. (1999). Multitiered teaching experiments. In A. Kelly & R. Lesh (Eds.), *Handbook of research in mathematics and science education* (pp. 197-230). Mahwah, NJ: Lawrence Erlbaum Associates.
Moskal, B. M., & Magone, M. E. (2000). Making sense of what students know: Examining the referents, relationships and modes students displayed in response to a decimal task. *Educational Studies in Mathematics, 43*(3), 313-335.
National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.
National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
Schorr, R., & Lesh, R. (2003). A modeling approach for providing teacher development. In R. Lesh & H. Doerr (Eds.), *Beyond constructivism: A models and modeling perspective* (pp. 141-158). Mahwah, NJ: Lawrence Erlbaum Associates.
Stacey, K., & MacGregor, M. (2000). Learning the algebraic method of solving problems. *Journal of Mathematical Behavior, 18*(2), 149-167.
Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks, California: SAGE Publications, Inc.
Tirosh, D., Even, R., & Robinson, N. (1998). Simplifying algebraic expressions: Teacher awareness and teaching approaches. *Educational Studies in Mathematics, 35*(1), 51-64.
Van den Heuvel-Panhuizen, M. (1994). Improvement of (didactical) assessment by improvement of problems: An attempt with respect to percentage. *Educational Studies in Mathematics, 27*, 341-372.

TRACKING PRIMARY STUDENTS' UNDERSTANDING OF PATTERNS

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Pattern generalizing is considered an important element in the development of children's algebraic thinking—with particular importance for the development of the concepts of variable and function. The NCTM Standards 2000 recommend work with patterns within the algebra strand starting in kindergarten. We are engaged in constructing a research instrument for tracking algebraic thinking across the primary grades and have been conducting exploratory classroom research on the many components of algebraic thinking since early 2003. In this paper, we report on our exploration of pattern generalizing with K-6 students. An analysis of questionnaires administered to a kindergarten, a grade three and a grade six class in a Montreal area school allows us to raise a number of questions about pattern work in the early grades and to suggest some key questions for tracking students' thinking and for comparing them across curricula.

Introduction

Over a decade of research into students' understanding of introductory algebra (Hart, 1981; Booth, 1984, 1988; Lee & Wheeler, 1987; Kieran, 1992; etc.) confirmed what was already popularly recognized: algebra constitutes a major obstacle for a significant number of middle and high school learners. There have been a number of responses to this problem that have mainly centered on reforming the teaching of high school algebra. One response has recently found expression in a large number of curricular reforms in North America. The American *Standards 2000* has enshrined the response of the early introduction of algebra by creating an "algebra strand" running in parallel to the traditional arithmetic and geometry from kindergarten onwards and many primary curriculum reforms have followed suit. The task of creating appropriate didactical tools for the teaching of algebra in the early grades—textbooks, in-service and pre-service teacher training programs and evaluation tools—has barely begun. NCTM has suggested a few classroom activities in its *Navigating through algebra* (2001) series. Carpenter, Franke and Levi (2003) have recently produced a textbook for use in teacher training, *Thinking mathematically: Integrating arithmetic and algebra in elementary school.*

Our research falls within the "evaluation tools" category; it aims at tracking students' algebraic thinking throughout primary school. This tool, intended chiefly for researchers, will allow us to compare various primary school programs—those that aim at the introduction of algebraic thinking to varying degrees, those that build algebra on arithmetic, those that develop algebra independently or in relation to the entire curriculum and those that begin school with algebra and build arithmetic on a basis of algebra—and to trace the development of algebraic thought in individual students as they move through the grades within these programs. In this paper, we look at one widely recognized element of algebraic thinking, pattern generalization, both for the lessons and concerns that have come out of our research to date and as an example of the construction and use of this instrument.

Studies of Pattern Work

Most research work in the area of patterns has involved high school students—for example, Lee & Wheeler (1987) or more recently, Radford (2000). At the elementary level, glimpses into children's thinking about patterns have concentrated on specific pattern work with children at the beginning and end of primary schooling. Threlfall (1999) looked at younger primary students' work on repeating patterns while Stacey (1989) studied 9 to 13 year old students doing linear patterns. Our work confirmed a number of results of that small body of research, such as: the more positive observation, that children engage easily and enthusiastically in pattern work having little or no difficulty in "seeing a pattern" and the more negative observation, that the final stage of pattern work, checking the generalization, appears to be absent. There was evidence of the strategy of proportional reasoning—underlined in the Stacey work—where some children, in their search for the 10th form of the pattern, found the 5th and doubled it.

Pattern Work and Algebraic Thinking

Three issues arise whenever pattern generalization becomes the object of research focus. The first is one of scope—many researchers are of the opinion that pattern work permeates all of mathematics. The second is the question of what pattern work belongs to mathematics; the third, and most important here, is question of the relationship of pattern work to the development of algebraic thinking. While we cannot deal with these issues within the context of this paper, we will situate our own work relative to them. While we agree that pattern work is omnipresent in mathematics, we believe that children's attention needs to be focused on the nature of that activity and that skills need to be explicitly worked on. We recognize the need to think more about what pattern work is specifically mathematical. One could make the case that all our learning involves pattern work and that recognizing, expressing and extending patterns is the basis of all organized knowledge fields. In our work, we are focusing on pattern work that is widely recognized as being mathematical in nature: work with geometric and numeric patterns mainly. We have not, for instance, done any work with sound or color patterns. Although we did do a bit of work with repeating patterns with kindergarten children, our main focus has been with growing patterns. The final issue might be reformulated as a question: What kind of early pattern work leads to the development of algebraic thinking? Pattern work was undertaken in primary schools long before algebra was a strand and pattern work placed within it. It was usually presented as an important problem solving tool. In both primary and middle school, pattern work often deteriorated into a disconnected "topic" leading to neither problem solving skills nor algebraic thinking. We think the mathematics education community still needs to do some serious thinking about how exactly pattern work leads to functions and algebra. An examination of some of the textbooks we are familiar with would indicate it does not, in spite of being presented in a "pre-algebra" format. Orton and Orton (1999) note that most children find the step of formulating an expression for the nth term in a number pattern beyond their capabilities and therefore question the "pattern route" to algebra: "Thus, as an approach to algebra, it is reasonable to ask to what extent children are able to perceive, understand and generalize in a wide variety of patterns of numbers. Just how successful is this new route?"(p. 104). While wrestling with these questions, we have nonetheless spent a number of classroom research days exploring children's understandings of some geometric patterns and analyzing their work in the light of Lee's (1997) model of algebra and algebraic understanding.

Our Research

Context

In the spring of 2003, we began exploring a number of "early algebra" themes in kindergarten (35 students, ages 5-6), two grade three classes (20 and 11 students, ages 8-9) and a grade six class (23 students, ages 11-12) in a French Montreal area private primary school with a math enrichment program for all students. Our study of children's understanding of pattern work involved from two to three hours of videotaped classroom time with each group. We began each class with a brief "warm up" whole class discussion, children at all three grade levels were asked to work on some geometric patterns on paper with instructions appropriate to their age level: continuing the pattern in kindergarten; continuing and finding the 10th form in grade 3; and finding the next, the 10th and the general forms in grade 6.

Results

Kindergarten children were remarkably successful in continuing geometric patterns of the repeating (Table 1) and growing (Table 2) types; their main difficulty lay in recognizing when patterns had switched from the first to the second type.

Kindergarten, repeating patterns	Pattern 1 OO OO	Pattern 2 O OOOOO	Pattern 3 OOOOO
correct	29	30	22
Not correct	1	0	7
Not done	0	0	1

Table 1. Kindergarten children and repeating patterns

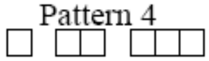
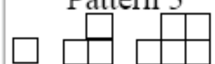
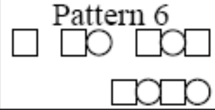
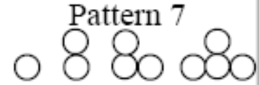
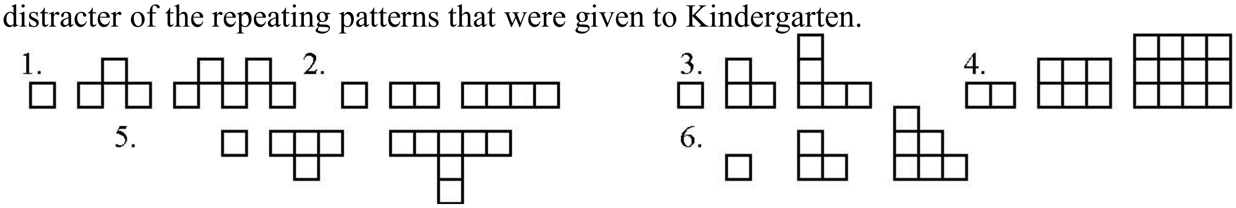
Kindergarten, growing patterns	Pattern 4 	Pattern 5 	Pattern 6 	Pattern 7 
correct	20	7	11	16
Not correct	9	20	15	12
Not done	1	3	4	2

Table 2. Kindergarten children and growing patterns

As we see from Table 2, Pattern 5 was the most difficult. Many children drew the correct number of shapes but did not respect the geometric arrangement. 9 children continued it as repeating pattern taking the given figures as the repeating unit; 5 others continued it as a + 1 pattern (drawing 5, 6, 7, 8 squares), 2 children 'corrected' the given pattern adding 'missing' terms 2 and 4 and then continued adding 1 square for each new figure. In Pattern 6, 12 children reverted to a repeating pattern perception (the alternation of squares and circles as in the first three patterns might have contributed to this phenomenon). Interestingly, one kindergarten child gave only number sequences for the patterns without drawing any shapes.

Grade 3 students were given a slightly more complex set of growing patterns but without the distracter of the repeating patterns that were given to Kindergarten.



They were quite successful in continuing their patterns (finding the next term, Table 3)

Grade 3	Pattern 1	Pattern 2	Pattern 3	Pattern 4	Pattern 5	Pattern 6
Correct	26	13	26	20	24	18
Not correct	1	12	1	7	2	8
Not done	0	0	0	0	1	1

Table 3. Grade 3 (2 groups, 27 children) finding the next figure in growing patterns

Pattern 2 was the least successful although a number of configurations were accepted for the fourth figure. Many children continued by adding either one or two boxes each time (perhaps ignoring the third or the first figure).

Table 4 focuses on the work of one class of Grade 3 students (n = 19) and their responses to the request to draw the 10^{th} figure.

Grade 3	Pattern 1	Pattern 2	Pattern 3	Pattern 4	Pattern 5	Pattern 6
Correct	12	5	14	3	2	3
Not correct	2	7	1	4	3	2
Not given	5	7	4	12	13	13
Not done	0	0	0	0	1	1

Table 4. Grade 3 children finding the 10^{th} figure in growing patterns

Interestingly, these students were more successful at expressing the rule governing the pattern. A number of children who did not give the 10^{th} figure responded to the request to give their pattern rule. For example, while only 2 children correctly drew the 10^{th} figure in Pattern 5, the growing T, 8 expressed the rule correctly. Table 5 is perhaps more illustrative of Grade 3 students' abilities with these patterns because it shows that a significant number are able to express their patterns.

Grade 3	Pattern 1	Pattern 2	Pattern 3	Pattern 4	Pattern 5	Pattern 6
Rule given	14	11	14	6	8	7
Not given	5	7	5	13	10	11
Not done	0	0	0	0	1	1

Table 5. Grade 3 children expressing the pattern rule

It is difficult to compare K and Gr. 3 students on the criteria of continuing their growing patterns. Although a couple of patterns seem to be of comparable complexity, the Kindergarten children had the added obstacle of having to make the abrupt leap from repeating to growing patterns. If we had accepted the solutions of children who continued to treat all patterns as repeating ones, they would have performed as well or better than the Grade 3 group. Certainly we cannot affirm that the Grade 3 students outperformed the Kindergarten children on their respective tasks.

Grade 6 students were given many dot patterns and asked to draw the next figure (Table 6), to find the 10th figure (Table 7a) and to express the pattern rule (Table 7b). The first three dot patterns are presented here since the majority of students attempted them.

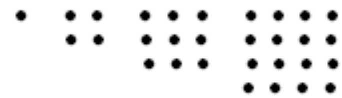
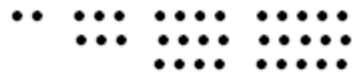
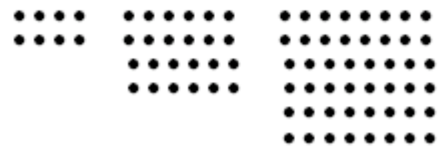
Grade 6 (n = 20)	Pattern 1 	Pattern 2 	Pattern 3 
Correct	15	10	5
Not correct	1	3	2
Not drawn	3	3	0
Not done	1	4	13

Table 6. Grade 6 finding the next figure in growing patterns

Pattern	#1	#2	#3
Correct	10	10	1
Not correct	1	1	1
Not given	8	5	5
Not done	1	4	13

Table 7a. Grade 6 finding the 10th term

Pattern	#1	#2	#3
Correct rule	9	8	0
Wrong rule	0	0	5
Not given	10	8	2
Not done	1	4	13

Table 7b. Grade 6 finding the rule

Unlike the Grade 3 students, the grade six students who did not draw the next figure did not show they nevertheless saw the pattern on the tasks of finding the 10^{th} figure or the rule governing the pattern. Out of 20 students, 9, 8, and 0 expressed the correct pattern rule for patterns 1 to 3 respectively. In fact the Grade 3 students appear to outperform the Grade 6 students even on the task of finding the general rule. Patterns 4 and 2 in grades 3 and 6 respectively are identical except that the first is made up of boxes and the second of dots. 74% of Grade 3 students and 50% of Grade 6 students drew the next figure while equivalent proportions of students were able to give the pattern rule.

In general, within the constraints of what was requested in each grade, we did not find significant improvement in performance with pattern work across the grades. Yet these same children—contrary to those studied by Falkner, Levi, & Carpenter (1999)—had shown significant improvement in their understanding of another recognized element of algebraic development, the equal sign, across the grades (Freiman & Lee, in press).

Future Directions

Questions & concerns

While this study was exploratory, it did lead us to a number of questions and concerns about the teaching of patterns in elementary school. The study confirmed that even very young children are able to see patterns. What surprised us was how strong that urge to see a pattern is – strong enough to compel the student to impose a pattern by modifying or ignoring some elements in a given configuration. If nearly all students "see" a pattern, then we need to look more carefully at what they actually see if we are to judge their ability to move to the next step of "saying" or expressing the pattern rule. Future questionnaires need to allow us to determine the student's perception of pattern – possibly by requesting the next two or three elements in the pattern. This would require limiting the size of geometric configurations so that students will not skip the drawing bit as did a number of the Grade 3 and Grade 6 students.

The difficulty of comparing students across the grades was evident in this study. A few identical patterns need to be included in each test in order to allow for this. In the future we will avoid the use of repeating patterns for two reasons. The first comes out of the Kindergarten experience where the repeating patterns interfered with their handling of the subsequent growing patterns. The second reason for avoiding repeating patterns is that we question their mathematical richness and more particularly, their link to algebraic thinking. Why, in the context of mathematics, do we use repeating patterns with young children? What algebraic thinking is involved?

We used geometric patterns in this initial work because we felt they were more likely than numeric patterns to elicit the expression of rich and varied pattern perceptions. The variety of student responses to these problems—which cannot be reflected in a paper of this length—confirms our initial conjecture in choosing these problems. However, most pattern research has involved number patterns and if only for comparison purposes, we will consider integrating a few such problems in our future work. What are the other forms of pattern work that elicit algebraic thinking? Finding the pattern in a single example or figure may be one.

We have also been forced to look more deeply into the question of where the algebraic thinking comes into play in pattern problems—and the very real issue that children may become successful at "doing patterns" without developing any algebraic insights at all. When patterns were introduced in middle school, the tendency was to introduce them in the same order as their related algebraic formulas were introduced—from linear to quadratic to exponential. We are now starting to question that ordering of topics even for algebra. Are "linear" patterns somehow "easier" than "quadratic" problems? What makes a pattern complex? What patterns lead to richer algebraic thinking? While there is a large quantity of literature now that promotes pattern work in order to develop algebraic thinking, the research literature that analyzes the nature of that thinking and the particular pattern problems that give rise to the many aspects of algebraic thought is rare. What patterns evoke which elements of algebraic thinking? Is it possible that a student can become very competent at pattern work without developing any algebraic habits of mind?

Towards a research instrument

Some of the questions we used with students have proven fruitful and will be integrated into our research instrument for tracking algebraic thinking across the grades. We will continue to use a number of geometric patterns with some running across the grades—for example, the growing T or growing L patterns used in grade 3. As mentioned, we will eliminate the repeating patterns and introduce some number patterns in future questionnaires. However, this exploratory work has led us to focus more on the issue of algebraic thinking in pattern problems and to build and test new questions and formats for getting at this. We are also building in some student interviews to enrich the results of the questionnaires and to explore other avenues of pattern work and representations—such as working with manipulatives.

A team of teachers at the school where this study was done have indicated their interest in pursuing the study there. As well as providing a site for further testing, this will allow us to track many of the kindergarten and grade three students as they move across the grades. Other research sites in very varied contexts have opened up due to both the researchers' geographical displacements since the time of this study. Lee is now working in Hawaii and the Pacific Islands; Freiman now works in Moncton, New Brunswick. With the new variety of research sites, we will be able to compare pattern thinking—as well as all the other elements of algebraic thinking—across a wide spectrum of primary curricula and contexts producing an instrument for a much wider research community interested in the development of algebraic thinking in the primary grades. Perhaps an even more important result of the discussion around the construction of this instrument will be the furthering of our thinking on what we mean by algebraic thinking and how it can be fostered in the primary school setting.

References

Booth, L. (1984). *Algebra: Children's strategies and errors*. Windsor, UK: NFER-NELSON Publishing Co.

Booth, L. (1988). Children's difficulties in beginning algebra. In A.F. Coxford and A.P. Schulte (Eds.), *The ideas of algebra, K-12. Yearbook*. Reston, VA: NCTM.

Falkner, K. P., Levi, L. & Carpenter, T. P. (1999). Children's Understanding of Equality: A Foundation for Algebra. *Teaching Children Mathematics, 6*(4), 232-236.

Freiman, V., & Lee, L. (in press). Tracking Primary Students' Understanding of the Equality Sign. *Proceedings of the 28th International Conference for the Psychology of Mathematics Education*. Bergen, Norway.

Hart, K. M. (1981). *Children's understanding of mathematics: 11-16*. London: John Murray.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Lee, L. (1997). *Algebraic Understanding: The search for a model in the mathematics education community.* Doctoral dissertation. Université du Québec à Montréal.

Lee, L., & Wheeler, D. (1987). *Algebraic thinking in high school students: Their conceptions of generalisation and justification* (Research Report). Montréal, Canada: Concordia University, Mathematics Department.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

Orton, A., & Orton, J. (1999). Pattern and the approach to algebra. In A. Orton (Ed.), *Patterns in the Teaching and Learning of Mathematics* (pp.104-120). London: Cassell

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: a semiotic analysis. *Educational Studies in Mathematics, 42*, 237-268.

Stacey, K. (1989). Finding and using patterns in linear generalizing problems. *Educational Studies in Mathematics, 20*, 147-164.

Threlfall, J. (1999). Repeating patterns in the primary years. In A. Orton (Ed.), *Patterns in the Teaching and Learning of Mathematics* (pp. 18-30). London: Cassell.

COGNITIVE ABILITIES AND ERRORS OF STUDENTS IN SECONDARY SCHOOL IN ALGEBRAIC LANGUAGE PROCESSES

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In this work, we give a research report carried out on three specific algebraic processes: formal substitution, generalization and modeling. We studied the abilities of a group of secondary school students by means of a questionnaire designed for this purpose and analyzed whether a hierarchy might be established between the aforementioned processes in order to aid the development of students' algebraic thinking. Our work also highlights the errors made by students when carrying out these processes in order to establish procedures to help students correct their errors. We also make a comparative study of students' abilities at two different school levels: Compulsory Secondary Schooling ("ESO") and Post-Compulsory Higher Certificate Secondary Education ("Bachillerato").

Introduction and Objectives

Algebra and algebraic processes have always been a matter of concern for teachers and researchers, perhaps because of the major difficulties that learning these areas causes students. At present, the main ways for introducing algebra during school years are: generalization, problem solving, modeling and the functional approach (Bernard, Kieran and Lee, 1996). Thus our interest in specifically studying formal substitution, generalization and modeling, these being processes that form part of the skills needed to understand and use algebra, and which are implicit in the school mathematics curriculum.

In this study we have not set out to determine which of these approaches is the best, but rather set out the advantages and disadvantages of using one approach or the other. In particular, our aims are to provide information on Secondary School students' abilities with regard to these processes, as well as analyze and classify errors made by students when carrying out related tasks. A joint study of these two questions will provide us with information on the didactic implications for the teaching/learning of algebra using one or others of these three approaches: formal substitution, generalization or modeling.

Theoretical Framework

It should be pointed out that the act of attributing an unknown letter or number to a mathematical object is necessarily a formal substitution and also at the same time a generalization. The act of establishing a relationship between a letter and number is, obviously, a form of modeling in which formal substitution and generalization are involved. However, in spite of this necessary and implicit relationship between the three processes in mathematical culture, we believe that we can differentiate them epistemologically.

Formal substitution is an important algebraic instrument that is involved in many processes: generalization, simplification, elimination, structural complication, particularization, etc. (Freudenthal, 1983). Formal substitution is of such importance because its validity goes beyond substitution in given expressions. Within formal substitution, we are going to deal with four types of actions: undertaking transformations in a single register, recognition of a substitution, substitutions and changes of register.

We understand generalization in two dimensions: seeing a generality through the particular and seeing the particular in the general (Reggiani, 1994). Within generalization we differentiate the following actions: particularization, iteration, recursion, and generalization.

Finally, we differentiate three phases in the modeling process, one more than Janvier (1996): (a) Explication and recognition of the rule, whereby recognition and familiarization with the problem situation are achieved. (b) Formulation and solution in terms of the rule. (c) Validation and interpretation of the results in the problem.

Study of errors made in the teaching-learning of algebra can be based on cognitive psychology theories. Thus, students' minds are not blank pages, but rather students possess prior knowledge, which appears sufficient and thereby establishes certain equilibrium in their minds. Errors especially appear in students' work when they are confronted with new knowledge that makes them revise or restructure what they already knew (Matz, 1980). Thus, we understand that error is the result of different causes, but it should always be considered as an inadequate cognitive scheme and not only as the result of a lack of knowledge or a lapse.

We undertook our study of error by taking the theoretical framework described in Socas (1997) as our reference; here three connected axes are contemplated, allowing us to analyze the sources of errors. Thus, we can situate errors made by students in relation to three different sources: affective attitudes, lack of meaning and obstacle.

By obstacle we refer to acquired knowledge, and not a lack of knowledge, which has been shown to be effective in other contexts. When the student uses this knowledge outside these contexts, inadequate answers are produced (Brousseau, 1983). Those obstacles arising in the didactic system can be organized (Socas, 2001) in accordance with the whether the errors are epistemological, didactic or cognitive in nature.

Errors caused by a lack of meaning have their source in the various stages of development (semiotic, structural and autonomous) evident in systems of representation, allowing us to differentiate them according to the different stages: (a) Algebra errors originating in arithmetic. (b) Procedural errors. Students use formulas or procedural rules inadequately. (c) Algebra errors due to the properties themselves of algebraic language.

Errors caused by students' affective attitudes are of different types, such as lack of concentration (over-confidence), blockages, forgetfulness, etc.

Methodology

This research is essentially descriptive as we attempt to show the situation a group of students find themselves in when undertaking formal substitution, generalization and modeling. The main instrument used to collect information is a questionnaire which is applied both quantitatively (abilities analysis) as well as qualitatively (error analysis).

A questionnaire was designed containing 15 questions and 43 items and covering the study processes. In order to set up this questionnaire several questionnaires about algebra were taken into account: Hart (1981), Palarea (1998), Rojano (1985) and Socas, Camacho, Palarea and Hernández (1989). Finally, to complete our questionnaire, the researchers set two questions of their own.

The questionnaire was set out in two parts, C1 and C2. Part C1 contains 9 questions, with a total of 28 items. As for the study processes covered, the first 4 questions are devoted to formal substitution (13 items) and the rest to generalization. Part C2 comprises 6 questions (20 items) of which the first three are about generalization, and the rest modeling.

The student sample was 60 students from the San Matías High School (Santa Cruz de Tenerife): two groups from the 4th Year of Compulsory Education, "ESO", (Option B), with 21 and 25 students, respectively, and one group from 1st Year Technology Post-Compulsory Education, “Bachillerato” (14 students). The students had received no prior instruction before doing the questionnaire and were given no type of help. After correcting the questionnaire, the sample was reduced to 43 students (13, 19 and 11 students, respectively, in each group).

For this research two categories of analysis were used: (1) Study of cognitive abilities related to the processes formal substitution, generalization and modeling. (2) Study of errors related to the same processes. In order to analyze the information related to the first category we relied fundamentally on the use of tables. Also, we established four levels of difficulty ranging from a maximum [0,25] to a minimum [75,100], where the three processes under study were located in accordance with the results obtained from the questionnaire. To organize the information related to the second category, we used several analytical procedures. Classification of errors was made based on analytical schemata. In some questions with more open answers, we adapted the systematic networks for information collection set out in Bliss, Monk and Ogborn (1983), and included some modifications to adapt them to our purpose. Concretely, we included a square on the right side of the network with the number of the student who had followed the particular route.

Abilities Analysis

As mentioned above, the study was undertaken with three groups of students, two from 4th Year of Secondary Education (“ESO”) and one group from 1st Year Post-compulsory Secondary Education (“Bachillerato”).

Comparing the results for the two groups of 4th Year “ESO” and the group of 1st Year “Bachillerato”, we can see that, generally speaking, the “Bachillerato” students possess greater cognitive abilities relating to the three processes under study.

	4th Year “ESO”		1st Year “Bachillerato”	
[0,25]	Modeling	7,81%		
			Modeling	21,21%
(25,50]	Generalization	43,05%		
			Generalization	47,14%
(50,75]	Formal substitution	53,61%		
			Formal substitution	53,85%
(75,100]				

Table 1

In this table we can see that although the three processes are at the same level of difficulty for both “ESO” as well as “Bachillerato” students, there are notable differences between the two types of students. A hierarchy can be established between the levels of difficulty for each process, each being situated at a different level. Differences between the two types of students regarding formal substitution process are insignificant. On the other hand, differences regarding the process of modeling are indeed significant. In spite of the fact that both can be found at the same level of difficulty, students from 4th Year “ESO” are near the lower limit of the interval, while “Bachillerato” students are near the upper limit. The percentages for generalization are similar for both groups, though slightly higher for the “Bachillerato” one.

If we compare the results for both sets of students according to the tasks undertaken for the various processes, we get the following table:

Process	Task	% ESO	NºESO	%Bach	NºBach
Formal substitution	Undertaking transformations	76,04%	1	87,88%	1
	Recognition of a substitution	54,69%	4	59,09%	2
	Substitutions	4,69%	10	9,09%	10
	Change of register	60,16%	2	45,45%	4
Generalization	Particularization	58,86%	3	49,40%	3
	Iteration	48,89%	5	39,29%	5
	Recursion	10,42%	7	7,58%	11
	Inductive method	10,32%	8	14,85%	8
Modeling	Recognition of the model	16,39%	6	36,37%	6
	Formulation of the model	6,25%	9	21,21%	7
	Validation of the model	3,13%	11	13,64%	9

Table 2

We can see that of the 11 types of task set, the students from 4th Year "ESO" achieve higher percentages of correct answers than "Bachillerato" students in four of these tasks: one relating to formal substitution and the other three related to the process of generalization.

If we analyze each process separately, we can see that the percentages for tasks related to formal substitution are quite similar for both sets of students. The greatest difference in favor of the 4th year ESO students is in the task of changing register. With the exception of the task "substitutions", for the other tasks both sets of students have fairly high percentages of correct answers, thus allowing us to infer that, generally speaking, students do not find formal substitution excessively difficult. It can also be pointed out that the "Bachillerato" students have a greater tendency to develop expressions and simplify them than students from 4th Year "ESO". With regard to the process of generalization, we can see that the differences between the two sets of students are not too great when carrying out the various tasks, in no case reaching 10%. What is most noteworthy in this process is that 4th Year "ESO" students achieve a higher percentage of correct answers than "Bachillerato" students in all tasks other than the use of the inductive method. As regards modeling, the differences between the two sets of students are indeed great, being more than 10% in all of the tasks, and reaching 20% in recognizing the model. In this process 4th Year "ESO" students never achieve higher percentages of correct answers in the tasks into which the modeling process has been divided.

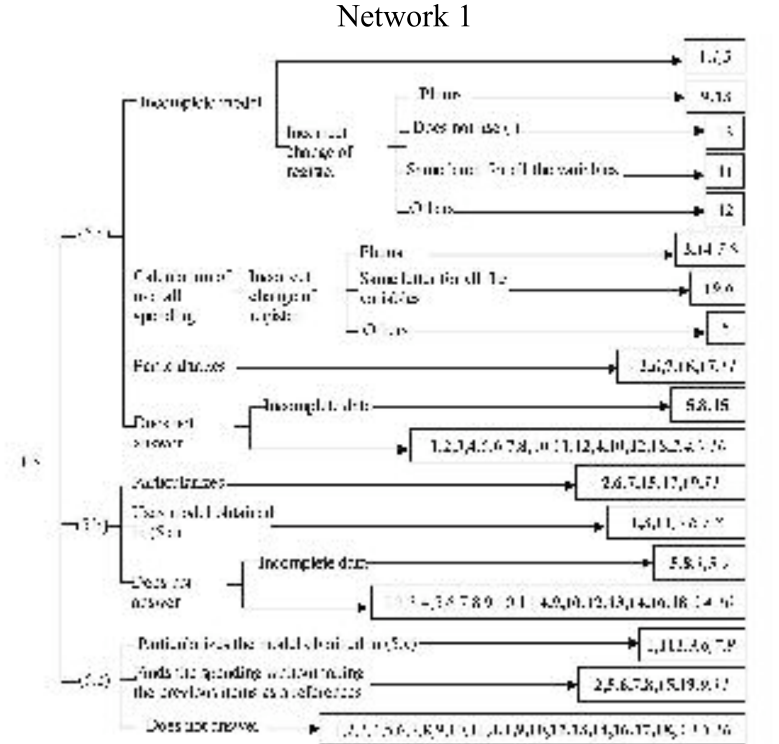
Error Analysis

In order to analyze and classify errors, we chose one of the 15 questions from the questionnaire, Nº. 5 from Part C2, which covers modeling, as shown as follows:

5. At the supermarket, a kilo of pears costs 125 pesetas, a kilo of bananas 60 pesetas, a kilo of plums 325 pesetas, a kilo of oranges 210 pesetas and each kiwi fruit costs b pesetas. Ana's family makes the following purchases of fruit: they buy every week 2 kg of pears, p kilos of bananas, 3 kilos more of plums than of bananas, and 6 kiwi fruit.
   a) Can you say how much Ana's family spends on fruit every week?
   b) Can you say how much Ana's family spends on fruit every month, supposing that they consume the same amount of fruit every week and that one month is 4 weeks?
   c) If the price of kiwi fruit is 20 pesetas and they buy 1 kilo of bananas per week, can you say how much Ana's family spends on fruit every week?

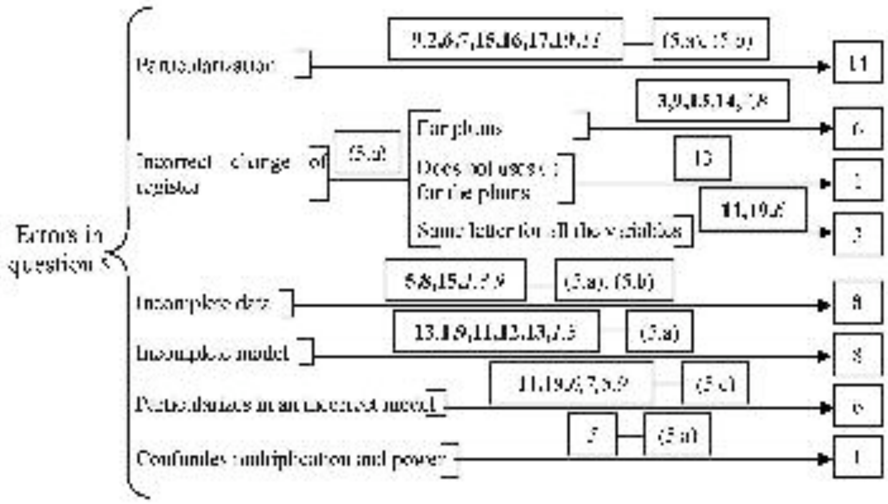
In order to schematize the errors made in this question we have organized the answers in a systematic network given below:

Network 1



This network shows the different routes taken by the students in order to solve each of the sections of the question set. For our error analysis, we only take into account the erroneous reasoning. Thus, with the help of this network, we can organize the errors in the following way: the numbers appearing in the first box above the arrow represent the numbers assigned to the student that have made the errors with different types of letters – normal, in bold, or in italics – according to whether they belong to Groups 1, 2 or 3, respectively. Next to this rectangle there is another one in which the item where the error that has been made is indicated, and, finally, in the box after the arrow can be seen the total number of errors of every type made.

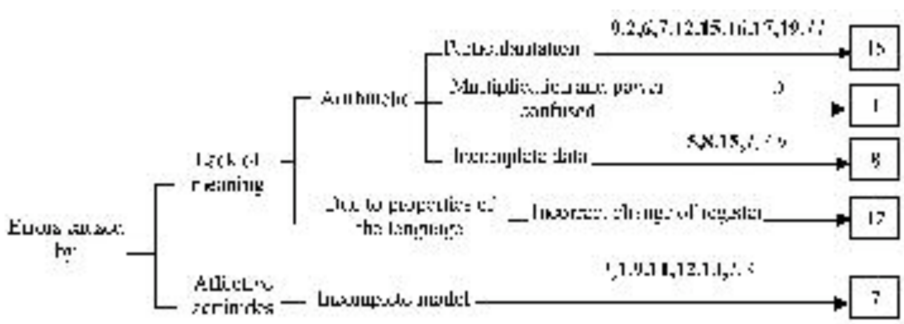
Schema 1



The error most often repeated is the need for particularization. This is usually due to a lack of meaning. Although the students recognize the model, they cannot find any meaning when using the letters and necessarily go back to using numbers to solve the problem. Another fairly frequent error is the incorrect change of register. Errors made when carrying out formal substitution tasks, such as the change of register, are usually caused by a lack of meaning due to the properties themselves of algebraic language. Other students believed that that the problem could not be solved because there was not enough data. This idea of not needing to use letters is due to lack of meaning and might show that students continue to think numerically. Within the group of errors, we have called "incomplete model" are those students who have correctly carried out the changes of register but who have only partially solved the problem. This situation whereby students interpret the various part of a whole correctly but then fail to juxtapose them might be due to carelessness. The confusion between multiplication and power might, we believe, be an error that is due to lack of meaning.

This analysis leads us to schematize the errors in function of the causes as follows:

Schema 2



Conclusions and Didactic Implications

Finally, we put forward here some of the conclusions we have drawn from our research. In addition, we find it interesting to underline some didactic implications derived from this study and make suggestions that might lead to improvements in the teaching/learning of algebra.

Methodology:

We would like to make it clear that we are aware of limitations in our study. Firstly, there is the small sample size (43) for a quantative study that needs greater guarantees. Secondly, there is the absence of clinical interviews which would have allowed us to identify more faithfully the various causes of the errors and which would have supplied us with details about the process followed and the students' abilities regarding the processes of formal substitution, generalization and modeling.

These limitations do, however, have a positive dimension in that they suggest the way in which our research work might naturally continue. In any case, this research work constitutes a pilot test for work that will go into greater detail regarding the role these processes play in the transition from arithmetic to algebra.

Students' abilities:

Our sample, though small, shows evident differences between the three processes which can be placed at different levels of difficulty (Table 1). There are also differences within each of the processes for the different tasks set. A hierarchy, then, can be established for the tasks involved in each process (Table 2).

Students have a greater control over questions involving formal substitution and, in this type of question, those concerning transformations, which get the highest percentages of correct answers. As for generalization, there are considerable differences depending on the tasks undertaken. It seem that students do not find particularization and iteration especially difficult, some such tasks getting even higher percentages of correct answers than formal substitution. It does seem, however, that anything to do with models, their interpretation and use, gives rise to considerable problems for students. Within the modeling process, students have fewest problems recognizing the model, and this type of task achieves higher percentages of correct answers than some related to generalization and formal substitution.

The fact that there are various levels of difficulty associated with the processes of formal substitution, generalization, and modeling suggest that we need to be cautious when dealing with these areas in the classroom and, once the signs that allow us to separate these processes have been found, put forward more coherent proposals for teaching-learning.

Comparison of abilities:

We can see that there is a cognitive difference between Compulsory Secondary Education "ESO" and Post-Compulsory Secondary "Bachillerato" students, although not as great as might be expected. We can note that although the three processes are at the same level of difficulty in both Compulsory Education "ESO" and "Bachillerato", there are considerable differences between the two (Table 1), especially when it comes to the process of modeling.

If we compare the results for both sets of students according to the tasks performed in the various processes (Table 2), it can be noted that of the 11 types of task set, the 4th Year "ESO" students achieve higher percentages of correct answers than the "Bachillerato" students in 4 of them: one relating to formal substitution and the other three related to the generalization.

If we analyze each process separately, we can see that the percentages for formal substitution are similar for both sets of students; the greatest difference is for tasks of change of register. With regard to the process of generalization, what is most noteworthy is that the 4th Year "ESO" students achieve greater percentages of corrects answers than the "Bachillerato" students in all tasks except for the use of the inductive method. As for modeling, the differences between the two sets of students are indeed great: the 4th Year "ESO" students do not achieve higher percentages of correct answers than the "Bachillerato" students in any of the modeling tasks.

Errors

In general, errors depend on the contents of the tasks set and the process. However, some errors repeatedly occur independently of the process carried out: the need for closure, particularization of expressions, incorrect use of parentheses and the confusion between multiplication and power. We would strongly recommend paying special attention in order to prevent and rectify these errors when introducing Algebraic Language, and pay particular heed to the causes of these errors.

Regarding the causes of errors, we can note that, generally speaking, there are three different sources: affective attitudes, lack of meaning and obstacles, although it should be pointed out that no error caused by obstacle was found in the process of modeling. The most common source of error in all three processes was lack of meaning, and on most occasions, this is the result of aspects of arithmetic and geometry which have yet to be resolved. It is important, then, to identify them in order to try to correct them in both the learning of arithmetic and geometry so that they do not form an added problem when introducing the students to algebra. In the process of generalization, lack of meaning was mainly the result of the properties themselves of algebraic language. In order to correct this type of error we suggest that use should be made of situations that create schemes which are easily assimilable and which are based on various systems of representation and not only on formal scripts. Also, we must fully take into account those errors that are caused by affective attitudes, as well as identify the reasons why students fail to answer some questions. Often blockages caused by students' attitudes are the main reason for error.

References

Bednarz, N., Kieran, C. y Lee, L. (1996). *Approaches to Algebra. Perspectives for Research and Teaching.* Dordrecht: Kluwer.

Bliss, J.; Monk, M.; Ogborn, J. (1987). *Qualitative Data Analysis For Educational Research.*USA: Croom Helm.

Brousseau, G. (1983). Les obstacles épistémologiques et les problèmes en mathématiques. *Recherches en Didactique des Mathématiques 4 (2),* 165-198.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Dordrecht: Reidel.

Hart, K.M. (1981). *Children's Understanding of Mathematics*, pp 11-16. London: John Murray.

Janvier, C. (1996). "Modeling and the initiation into Algebra." En Bednarz, Kieran y Lee. *Approaches to Algebra*, pp 225-236. Dordrecht: Kluwer.

Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Children's Mathematical Behavior, 3, 1,* 93-166.

Palarea, M.M (1998). *La adquisición del lenguaje algebraico y la detección de errores comunes cometidos por alumnos de 12 a 14 años.* Tesis doctoral. Universidad de La Laguna.

Reggiani (1994). "Generalization as a basis for Algebraic thinking: observations with 11-12 year old pupils." PME XVIII. Vol. IV, pp 97-104. Portugal: University of Lisbou.

Rojano, T (1985). *De la aritmética al álgebra (estudio clínico con niños de 12 y 13 años de edad).* Tesis de Doctorado. México: Cinvestv del IPN.

Socas, M. M., Camacho, M., Palarea, M. y Hernández, J. (1989). *Iniciación al Álgebra.* Matemáticas cultura y aprendizaje núm. 23. Madrid: Síntesis.

Socas, M. M. (1997). Dificultades, obstáculos y errores en el aprendizaje de las Matemáticas en la Educación Secundaria. (Cap.V, pp. 125-154). En Rico, L. y otros: *La Educación Matemática en la Enseñanza Secundaria.* Horsori: Barcelona.

Socas, M. M. (2001). *Investigación en Didáctica de la Matemática vía Modelos de competencia.*

Un estudio en relación con el Lenguaje Algebraico. Departamento de Análisis Matemático. Universidad de La Laguna.

SPATIAL ABILITY, ACHIEVEMENT, AND USE OF MULTIPLE REPRESENTATIONS IN MATHEMATICS

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This study focused on how mathematics students at different spatial ability and achievement levels use multiple representations. Sixteen interviews were conducted with four 8th grade students: high achieving-high spatial ability, high achieving-low spatial ability, low achieving-high spatial ability, and low achieving-low spatial ability. The students were asked linear function problems in the interviews. Additionally, their mathematics class was observed for 7 hours. The Wheatley Spatial Ability test was applied to determine the spatial ability levels of the students. Achievement levels were determined from students' State Comprehensive Assessment Test scores, and consultation with the teacher. The findings suggest that students at different levels of achievement and spatial ability used multiple representations differently. High spatial ability students of this study had more access to multiple representations than low spatial ability students. Similarly, high achieving students had more access to multiple representations than low achieving students. However, the use of different representations seemed to help all four students.

Recently, there has been a growing interest in research on students' use of multiple representations in mathematics education. Research suggests that using multiple representations in mathematics education enhances understanding of concepts, improves problem solving ability, and strengthens the learning process by providing mutual sources of information (Borba & Confrey, 1993; Brenner, Mayer, Moseley, Brar, Durán, Reed, & Webb, 1997; Porzio, 1999). However, little is known about possible factors and characteristics that may affect students' use of multiple representations. This study investigated how spatial ability and achievement levels may be related to students' use of multiple representations. Case studies were conducted to understand how students at high and low spatial ability levels and high and low achievement levels use multiple representations when problem solving and reasoning mathematically.

Theoretical Background and Rationale

A representation is "something that stands for something else" (Palmer, 1978, p.262). According to Palmer, a representation system involves two entities: the represented world and the representing world. The represented world is composed of the mathematical concepts that individuals represent, and the representing world consists of the tools, multiple representations, used to represent the mathematical concepts. Pape and Tchoshanov (2001) contended that the cognitive capacity of the human brain is aligned with multiple representational patterns. Additionally, there exists a mutual influence between internal and external representations. If students learn a concept with more than one representation, they can make connections between them and improve mental schemes that results in better learning.

Pavio (1971) proposed one stimulus composes both a linguistic and an imagery system wherein a person can translate visual information into verbal and verbal information into visual. McKim (1972) suggested that using representations to develop students' thinking skills is related to students' ability to operate with mental images (e.g., visualizing external representations). It is reasonable to argue that there is a relation between the use of representations and spatial ability or imagery. In this study, spatial ability was defined as the ability for mentally rotating and comparing two-dimensional objects as measured in The Wheatley Spatial Ability Test (Wheatley, 1978/1996).

In a quantitative study involving arithmetic with signed numbers, Moreno and Mayer (1999) found that in a multiple representation environment, high achieving students produced a more significant pretest to posttest gain than low achieving students and, high spatial ability students produced a more significant pretest to posttest gain than low spatial ability students. These findings suggest a need to explore in depth how students at high and low spatial ability levels and high and low achievement levels use multiple representations when thinking and reasoning mathematically. To this end, a qualitative study involving interviews and classroom observations was designed.

The present study focused on verbal/contextual, graphical, numerical, and symbolic representations of linear functions. Verbal/contextual representation will refer to any narrative descriptions that students use for interpreting the linear functions. Numerical representations will be referred to primarily as tables. Graphical representations will refer to lines or points graphed on the coordinate axes. Symbolic representations will be referred to primarily as linear equations.

The framework for investigating students' understanding and use of multiple representations was based on the work of Leinhardt, Zaslavsky, and Stein, (1990), and O'Callaghan (1998), as well as, an interest in students' tendencies to use different representations. Data was collected and analyzed within four areas of linear functions: translation, classification, interpretation, and preference and tendency. Translation will refer to constructing a representation of a linear function when it is given in another representation. Classification will refer to identifying if a linear function is increasing, decreasing, or constant. Interpretation will refer to how students make sense of the linear relationships such as how they explain what y=5 means. Preference will refer to the choice of representation of a student when he/she is given more than one representation. Tendency will refer to what representation a student uses more often in solving problems dealing with linear functions.

Methods and Data Sources

A case study methodology was used. To select the participants, the Wheatley Spatial Ability Test (WSAT) was applied to a regular mathematics class of 8^{th} graders. This test is valid and reliable for determining spatial ability levels with a high internal consistency (K-R=.92) (Wheatley, 1978, 1996). The students' achievement levels were determined from students' state comprehensive assessment test (SCAT) scores, and consultation with the classroom teacher whose suggestions were based on students' previous exam scores in that class as well as their participation in the lessons. Based on these data, four students were selected: one high achieving-high spatial ability (HA-HS), one high achieving-low spatial ability (HA-LS), one low achieving-high spatial ability (LA-HS), and one low achieving-low spatial ability (LA-LS). The reader can examine Table 1 for a summary of the achievement and spatial ability scores for the four students in this study and their class means and ranges. Table 1 includes WSAT scores, achievement levels for the criterion-referenced mathematics test section of the SCAT, the National Percentile Rank [NPR] scores for the norm-referenced mathematics test section of the SCAT.

Table 1: Achievement and Spatial Ability Scores

	WSAT	SCAT Level 1-low to 5-high	SCAT NPR
David (LA-LS)	54.4	2	52
John (LA-HS)	95.5	2	62
Lara (HA-LS)	26	3	81
Rose (HA-HS)	97.5	4	97
Class mean	69.8	2.8	78.7
Class range	25-97.5	1-4	14-97

Four 45 min. video-taped interviews were conducted with each student. Questions involved linear function problems requiring the use of different representations. The purpose was to understand how and to what extent achievement and spatial ability influences students' problem solving strategies with respect to use of multiple representations, connections between different representations, and depth of understanding of multiple representations. One of the interview questions is given in Figure 1 as an example. Additionally, the students' mathematics class was observed for 7 hours and field notes were gathered.

Data from the 16 interviews and seven classroom observations were analyzed in terms of each student's translations between multiple representations, classification of linear functions, interpretation of linear functions, and preference and tendency toward representations. Data collection and analysis was ongoing. Tables summarizing students' responses in the four areas of interest were formed. Summary tables, interview transcripts, students' written work, and field notes from observations were used to develop individual cases, seeking evidence confirming and disconfirming developing themes. Individual cases were used to conduct a cross-case analysis seeking similarities and differences among the cases.

Results

In this study, we sought to understand how students at high and low spatial ability levels and high and low achievement levels use multiple representations when problem solving and reasoning mathematically. Analysis of data revealed differences in students' use of multiple representations among the four students in this study.

Rose, the high achieving-high spatial ability student, had the strongest connections between multiple representations. An example for her strong connections between representations is from one of the class observations, where students did a worksheet about rule finding from tables. Rose was the only student among the four students of this study who realized that the coefficient of the x value (slope) in the symbolic linear equation is the same as the difference in x values in the table in which y values increased by 1. Although Rose was flexible in using any kind of representation, her tendency was to use graphs. Among the many examples confirming this conclusion, one was as follows. When Rose was asked to compare the temperatures in two cities given by two equations, after translating the equations to tables, she graphed the equations and then compared the temperatures of the two cities by using the graphs.

Lara, the high achieving-low spatial ability student, had a good understanding of the concept of linear functions. She demonstrated connections between multiple representations but they were not very strong. For a question requiring translation from a table to equation, where the table contained y values for x=0, 1… 6, Lara created the equation correctly. However, when she was asked to find the y value for x is 11, she formed the table up to x=11, and then found the y value correctly instead of plugging 11 for x in the equation. Lara had a tendency to use tables to interpret, classify, and compare the other representations. For instance, in order to compare temperatures in two cities given by two equations, she translated equations to tables and then compared the temperatures of two cities by using the tables.

Figure 1. An example of an interview question

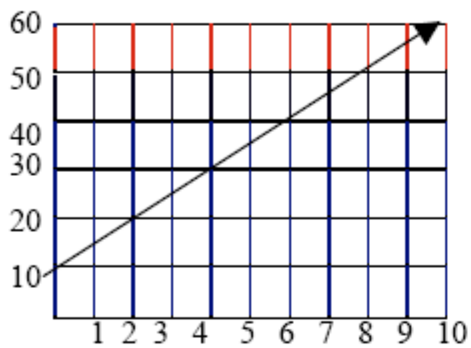
The savings of Dina, Yonni, Moshon, and Danny changed during the last year, as described below. The numbers indicate amounts of money (in dollars) at the end of each week.

Dina: The table shows how much money Dina had saved at the end of each week. The table continues in the same way for the rest of the year.

Week	1	2	3	4	5	6	7	8	9
Amount	7	14	21	28	35	42	49	56	63

Yonni: Yonni kept his savings at $300 throughout the year.

Moshon: The graph describes Moshon's savings at the end of the first 10 weeks. The graph continues in the same way for the rest of the year.



Danny: Danny's savings can be described by the expression y=200-5x, where x stands for the number of the weeks and y is his money for that week.

a. What is the saving of each child at the end of the 8th week? 40th week?

b. Describe in words how the savings of each child changes throughout the year.

c. Compare the savings of two out of the four children. Use words like "the saving increase (or decrease)", "who has a larger (or smaller) amount at the beginning (or end) of the year". Use tables, graphs, expressions, and explanations.

Modified from Friedlander, & Tabach, 2001.

John, the low achieving-high spatial ability student, lacked knowledge of the concept. For instance, he did not know how to find the slope of a linear function. He also had difficulty in classification of graphs and equations. For an increasing graph, he said: "I was thinking since like the line is in the middle of the graph, it is going both down and up so this one is neutral." However, he was good at the procedure of translating multiple representations to each other and was able to use different solution strategies. For example, for Moshon's savings (see Figure 1) he used proportional reasoning as follows to find his savings for the 40^{th} week: "… there are 3 more 10 weeks. If you add 3 more sets of 10 weeks that would be 40…I just did 60 times 4, which gives you 240 dollars what you get in 40 weeks." John tended to use tables extensively during the problem solving process and when interpreting multiple representations of linear relationships. For instance, when he was asked to explain what y=5 means, he made a table and explained the relationship as "stays the same."

David, the low achieving-low spatial ability student, had the weakest understanding of the concept. Similar to John, he often classified equations and graphs incorrectly. He classified y=2x-1 as increasing, because "the positive is higher than negative". David also did not show an understanding of the concept of slope. Typically, David did not tend to use multiple representations during the problem solving process. The majority of the time, if he could not solve a problem with the given representation, he stopped trying to solve that problem while the other three students tended to use other representations to reach a solution. Nevertheless, David was successful at interpreting contextual problems and created contexts to make sense of problems without a context. For example, for Danny's savings context (see Figure 1), even though, 200 is greater than 5 (an incorrect strategy he applied previously), he classified 200-5x as a decreasing relationship by referring to the context. David preferred verbal representations.

David and John were both low achieving but at different spatial ability levels. Both students were able to interpret tables and verbal/contextual representations. However, they both showed a lack of understanding of graphs and equations. The main difference was that John often made a table to classify an equation, to translate to equations or to interpret multiple representations whereas David did not tend to use tables or any other representations in his problem solving process. Lara and Rose were both high achieving but at different spatial ability levels. Both Lara and Rose were able to classify tables, verbal statements, and graphs and had difficulties in classifying equations. When we asked for an explanation for the classification of an equation, Rose translated equations to graphs and Lara translated equations to tables to give an explanation. Another difference was that Rose always related slope with the classification of a relationship whereas Lara did not. This may be related to Rose's preference of graphs. Since she was good with graphs and the class learned slope in relation to graphs, Rose's understanding of slope was also good and she connected it with the classification of relationships.

In this study, the high achieving students showed greater knowledge of linear functions and each representation than the low achieving students. However, given the same achievement levels, the high spatial ability students were able to access more representations than the low spatial ability students. While Rose was flexible in using any kind of representation and in particular graphs, Lara mostly used tables, equations, and verbal representations. Although John demonstrated lack of understanding of the concept, he used multiple representations, particularly tables, during the problem solving process whereas the majority of the time David did not attempt to use different representations.

It was found that the high spatial ability students, Rose and John, were better at pattern finding and producing different solution methods than the low spatial ability students. In one of the class periods, students played a game called "guess my rule". The students were supposed to determine the rule in the teacher's mind by giving her input numbers and getting output numbers from her. In that class period, Rose and John were actively involved in the game whereas David and Lara were rather quiet. Also in their problem solving process during the interviews, we observed that when they were translating tables to equations, Rose and John were more successful than David and Lara. Mathematics is a science of patterns; therefore, it is important for students to recognize patterns in mathematical relationships. These findings suggest that developing spatial ability through meaningful experiences may help develop mathematical ability. Moreover, similar to the study of Hines, Klanderman, and Khoury (2001), we observed that tables helped students reflect upon and reorganize their thinking of linear functions, and recognize patterns.

Another finding was that while solving contextual problems the high achieving students sometimes ignored the context and focused only on the numbers. Classroom observations revealed that students rarely used contexts in their mathematics class. The interviews revealed examples where Rose and Lara ignored the problem context and could not reach a solution. For example, given a graph of a child's weekly savings for 10 weeks, one question asked for the amount of savings on the 40th week (see Figure 1). Both Rose and Lara created tables to transfer the graph into an equation. However, they could not reach an equation when they focused only on the numbers. After the interviewer reminded them to consider the context, they were able to make the translation and reach a solution. Drawing on problem contexts seemed to help all four students to better understand problems and arrive at solutions. This seemed particularly helpful for David, the low achieving-low spatial ability student. He even made up contexts for non-contextual questions. For instance, in order to classify the equation y=-5x, David related it to a savings: "3 times negative 5 will be negative 15 because she takes 5 out each day. She takes 5 out one day, two days 10, and 3 days 15. So she will be decreasing".

Some of our findings are consistent with the work of Nathan, Stephens, Masarik, Alibali, and Koedinger (2002) who found that students showed fluency with instance-based representations (tables and point-wise graphs) before holistic representations (symbolic equations and verbal expressions). All four students in our study utilized tables when making translations among multiple representations, and classifying and interpreting linear relationships. Tables seem to have potential for bridging arithmetic thinking to algebraic thinking. The easiest type of translation for the students in this study was translating other representations to tables. All four could classify tables and verbal contexts as increasing, decreasing or constant relationships and all had difficulty classifying equations. Mathematics instruction should seek ways of connecting understanding of tables and verbal contexts with equations, in order to enhance student understanding and use of symbolic representations. However, the use of tables must be considered carefully. Tables may cause students to think point-wise instead of considering a general interpretation over a function as in the case of John, who tended to use tables extensively. In order to graph the equation y=-4x+1, John plotted only points (he translated the equation to a table to graph it); he did not connect them to make a line while the other three students connected them. Regarding the growth of two babies that were represented by a graph, John explained that baby-A was growing faster than baby-B by referring to specific points: "Because, for the 6th month, when baby-A was 8 pounds and baby-B was 7 pounds." Instead of observing the general growth, as the other three students did, John compared the babies' weight at a specific time. David, the low achieving-low spatial ability student, interpreted the same graph by making a global comparison of the growth of the two babies saying baby A was growing faster "because baby-A increases by 1 pound and baby-B increases a half pound in each month." This example also points out that student communication can be beneficial for mathematics learning since each student may show a different perspective about a problem.

Conclusion

This study revealed that four students at different levels of spatial ability and achievement had different tendencies toward and fluency with uses of representations. Although the low achieving-low spatial ability student tended to use different representations less than the other three students, when the interviewer suggested he use other representations he could figure out solutions. The use of multiple representations was found to help students at different spatial ability and achievement levels think about mathematical ideas and solve mathematics problems. This study adds to the existing literature that suggests the use of multiple representations will improve students' mathematics understandings (Borba & Confrey, 1993; Brenner, Mayer, Moseley, Brar, Durán, Reed, & Webb, 1997; Porzio, 1999). Additionally, the use of contextual problems has potential for improving the mathematical understanding of diverse students given the benefits produced for all four students, particularly, the low achieving-low spatial ability student. A shortcoming of the study by Moreno and Mayer (1999) was the lack of contextual problems possibly influencing why the multiple representation environment had less affect on gain scores of the low achieving and low spatial ability students in comparison to the other students.

References

Borba, M., & Confrey, J. (1993). The role of teaching experiment: Students' construction of transformations in a multiple representational environment. Paper presented at the Annual Meeting of the American Educational Research Association, Atlanta, GA.

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Durán, R., Reed, B. S., & Webb, D. (1997). Learning by understanding: The role of multiple representations in learning algebra. *American Educational Research Journal,* 37(4), 663-689.

Friedlander, A., & Tabach, M. (2001). Promoting multiple representations in algebra. In A. A. Cuoco, & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 17385). Reston, Virginia: The National Council of Teachers of Mathematics, INC.

Hines, E., Klanderman, D. B., & Khoury, H. (2001). The tabular mode: Not just another way to represent a function. *School Science and Mathematics, 101*(7), 362-371.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, graphing: Tasks, learning, and teaching. *Review of Educational Research, 60*, 1-64.

McKim, R. (1972). *Experiences in visual thinking.* Monterey, CA: Brooks/Cole.

Moreno, R., & Mayer, R. E. (1999). Multimedia-supported metaphors for meaning making in mathematics. *Cognition and Instruction, 17*(3), 215-248.

Nathan, M. J., Stephens, A. C., Masarik, K., Alibali, M. W., and Koedinger K. R. (2002). Representational fluency in middle school: a classroom study. Proceedings of the 24th Annual Meeting of PMENA.

O'Callaghan, R. B. (1998). Computer-Intensive algebra and students' conceptual knowledge of functions. *Journal for Research in Mathematics Education, 29*(1), 21-40.

Paivio, A. (1971). *Imagery and verbal processes*. New York: Holt, Rinehart, and Winston.

Palmer S. E. (1978). Fundamental aspects of cognitive representation. In Rosch, E. & Lloyd, B. B. (Eds.), *Cognition and categorization (pp.259-303).* Hillsdale, NJ: Lawrence Erlbaum Associates.

Pape, S. J., & Tchoshanov, M. A. (2001). The role of representation(s) in developing mathematical understanding. *Theory into Practice, 40*(2), 118-27.

Porzio, D. (1999). Effects of differing emphases in the use of multiple representations and technology on students' understanding of calculus concepts. *Focus on Learning Problems in Mathematics, 21*(3), 1-29.

Wheatley (1978, 1996). *The Wheatley spatial ability test form B*. West Lafayette, IN: Purdue University.

MIDDLE-SCHOOL STUDENTS' EXPERIENCE WITH THE EQUAL SIGN: SAXON MATH DOES NOT EQUAL CONNECTED MATHEMATICS

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We examined how two middle-school textbook series, Saxon Math *and* Connected Mathematics*, present the equal sign. We compared differences across grade level and textbook series. Following Seo and Ginsburg (2003), we examined the proportion of equal sign instances presented in an "operations-equals-answer" context. We found that the proportion of these instances decreases from sixth to eighth grade and that* Saxon Math *has a greater proportion of these instances than does* Connected Mathematics. *Following McNeil and Alibali (in press), we examined the proportion of equal sign instances in equations with operations on both sides of the equal sign. We found that there were very few of these instances in either series, but that the proportion of these instances increases in eighth grade.*

Purpose

The call for algebra as a K-12 strand has received much attention in the mathematics education research community. Central to the inclusion of algebra in the earlier grades, researchers argue, is a focus on students' understandings of equality and the equal sign (e.g., Carpenter, Franke, & Levi, 2003). Although the equal sign is rarely an explicit focus of instruction, students' conceptions of this symbol are undoubtedly shaped by the nature of their experiences with it. The present study investigates a component of middle-school students' experience with the equal sign by examining how the equal sign is presented in textbooks (grades 6-8). Although students' experiences with the equal sign certainly extend beyond textbooks, we chose to focus on textbooks because they have a large impact on the instructional practices of American mathematics teachers (Kulm, 1999; Nathan, Long, & Alibali, 2002; Roseman, Kulm, & Shuttleworth, 2001).

We examine differences in equal sign presentation across the grade levels (6-8). We also compare how the equal sign is presented in two very different middle-school textbook series: a skills-based textbook series, *Saxon Math*, and an NSF-funded, standards-based textbook series, *Connected Mathematics*. Although both textbook series aim to provide students with a solid foundation for success in algebra, the two differ in their philosophies regarding the best way to prepare students for algebra. *Saxon Math* focuses on incremental learning of basic skills by repeated practice (Saxon Publishers, 2004), whereas *Connected Mathematics* emphasizes the learning of patterns and relationships by representing ideas in multiple forms (Connected Mathematics Project, 2004). Our goal was to examine the instances of the equal sign throughout the textbooks to determine if they promote relational understanding, which is essential for understanding algebraic equations.

Theoretical Framework

Students must interpret the equal sign as a relational symbol of equivalence if they are to understand certain areas of higher-level mathematics (e.g., complex equation solving).

Unfortunately, most elementary school students interpret the equal sign as an operator meaning "the total" or "the answer," rather than as a relational symbol of equivalence (Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; Carpenter, Franke, & Levi, 2003; Carpenter & Levi, 2000; Kieran, 1981; McNeil & Alibali, 2000, 2002, in press; Rittle-Johnson & Alibali, 1999; Seo & Ginsburg, 2003). When asked to rate the "smartness" of various definitions, most rate definitions such as "the total" or "the answer" as smarter than definitions such as "two amounts are the same" or "equal to" (McNeil & Alibali, 2000, 2002, in press; Rittle-Johnson & Alibali, 1999).

Much less is known about students' equal sign understandings past elementary school. One might assume that students develop a relational understanding of the equal sign as they progress in mathematics. However, some researchers suggest that middle school students, high school students, and even undergraduates maintain the operational interpretation in some contexts (Kieran, 1981). For example, McNeil and Alibali (in press) showed that seventh-grade students interpret the equal sign as a relational symbol of equivalence only when it is presented in the context of an equation with operations on both sides of the equal sign (e.g., 3 + 4 + 5 = 3 + 9). When the equal sign is presented alone or in an operation-equals-answer context (e.g., 3 + 4 + 5 + 3 = 15), seventh-grade students maintain the operational interpretation.

Some researchers (e.g., Baroody & Ginsburg, 1983; McNeil & Alibali, 2002; Seo & Ginsburg, 2003) have suggested that students' operational interpretations of the equal sign are a byproduct of their experiences with it. However, only Seo and Ginsburg have systematically examined how the equal sign is presented to students. In their case study of one second-grade classroom, they analyzed two mathematics textbooks used in the classroom and found that the equal sign was nearly always presented in the operations-equals-answer context (e.g., 3 + 4 = 7). This finding supports the argument that students' interpretations of the equal sign are correlated with their experiences with it.

In the present study, we expand upon the work of Seo and Ginsburg (2003) by analyzing how the equal sign is presented in two middle-school textbook series: *Saxon Math* and *Connected Mathematics*. Examining middle-school students' experiences with the equal sign is important given that one goal of middle school mathematics is to prepare students for algebra, and a relational understanding of the equal sign is necessary for understanding algebraic equations. Similar to Seo and Ginsburg, we examined the proportion of equal sign instances presented in the operations-equals-answer context. We were additionally interested in the extent to which equal signs are presented in the context of equations with operations on both sides of the equal sign (e.g., 3 + 4 + 5 = 3 + __) because those equations have been shown to activate the relational view of the equal sign in middle-school students (McNeil & Alibali, in press).

Method

Materials

We examined two middle-school textbook series (grades 6-8): (1) *Saxon Math* (2004 edition), a skills-based textbook series, and (2) *Connected Mathematics* (2002 edition), an NSF-funded, standards-based textbook series.

Procedure

We coded instances of the equal sign on a randomly-selected 50% sample of the pages in every book. Each instance of the equal sign was transcribed and identified as presented in an operations-equals-answer context (e.g., 3 + 4 = __) or a non-operations-equals-answer context (e.g., __ = 3 + 4). We define the *operations-equals-answer* context as any equation containing operations on the left-hand side of the equal sign, and either one number (e.g., 3 + 4 = 7) or an unknown quantity (e.g., 3 + 4 = __, 3 + 4 = x) on the right-hand side of the equal sign. For the purposes of this analysis, fractions (e.g., 1/2) were treated as numbers, not operations. We define the *non-operations-equals-answer* context as any equation that does not follow the operations-equals-answer form (e.g., 1 foot = 12 inches, 7 = 7). For non-operations-equals-answer instances, we further coded each instance as having operations on both sides of the equal sign (e.g., 5 + 2 = 3 + 4) or not having operations on both sides of the equal sign (e.g., 7 = 3 + 4.)

Results

We examined the number of equal sign instances found in each textbook. Table 1 presents the total number of equal sign instances found in each textbook, along with the number of pages in the 50% sample for each textbook and the average number of equal sign instances per page. As can be seen in the table, the number of equal sign instances increases from sixth to eighth grade. This is true whether one considers the total number of equal sign instances, or the average number of equal sign instances per page. Across the three grade levels, *Saxon Math* presents far more equal sign instances than does *Connected Mathematics*. In terms of the average number of equal sign instances per page, *Saxon Math* presents more equal sign instances per page than *Connected Mathematics* in sixth and seventh grade; however, the two textbook series even out in eighth grade.

Table 1: Number of equal sign instances, number of pages sampled, and average number of equal sign instances per page in each grade level and textbook series.

		Sixth	Seventh	Eighth
Saxon	Instances	280	558	999
Math	Pages	317	327 1.71	424
	Instances/Page	0.88		2.36
Connected	Instances	46	315	707
Mathematics	Pages	314	327 0.96	285
	Instances/Page	0.15		2.48

We used logistic regression to examine the odds that an equal sign would be presented in an operations-equals-answer context. Predictor variables were grade level (6, 7, or 8) and textbook series (*Saxon Math* or *Connected Mathematics)*. The effect of grade level was significant when controlling for textbook series, Wald (2, N = 2905) = 76.647, $p < .001$. The effect of textbook series was also significant when controlling for grade level, Wald (1, N = 2905) = 387.759, $p < .001$. Table 2 presents the proportion of equal sign instances in each grade level and textbook series that are presented in the operations-equals-answer context.

Table 2: Proportion of equal sign instances in each grade level and textbook series presented in the operations-equals-answer context.

	Sixth	Seventh	Eighth
Saxon Math	.70	.69	.45
Connected Mathematics	.24	.10	.16

The significant effect of grade level is due to a negative linear relationship between the operations-equals-answer context and grade level, $\hat{\beta} = -0.662$, z = -6.968. The model estimates that the odds of finding an equal sign in the operations-equals-answer context are cut almost in half for each increase in grade level.

The equal sign is more likely to be presented in an operations-equals-answer context in *Saxon Math* than in *Connected Mathematics*, $\hat{\beta} = 1.988$, z = 19.683. The model estimates that the odds of finding an equal sign in the operations-equals-answer context are more than seven times greater in *Saxon Math* than in *Connected Mathematics*.

Equations with operations on both sides of the equal sign (e.g., 3 + 4 = 5 + 2) accounted for only a very small proportion (0.06) of the equal sign instances overall. We examined the odds that an equal sign would be presented in this context. The effect of grade level was significant when controlling for textbook series, Wald (2, N = 2905) = 45.016, p < .001. The effect of textbook series was not significant when controlling for grade level, Wald (1, N = 2905) = 0.131, p = .72. Table 3 presents the proportion of equal sign instances in each grade level and textbook series that are presented in an equation with operations on both sides of the equal sign.

Table 3: Proportion of equal sign instances in each grade level and textbook series presented in an equation with operations on both sides of the equal sign.

	Sixth	Seventh	Eighth
Saxon Math	.03	.02	.08
Connected Mathematics	.00	.003	.09

The significant effect of grade level is due to an increased proportion of equations with operations on both sides of the equal sign in eighth-grade textbooks. The odds of finding an equal sign in the context of an equation with operations on both sides of the equal sign are greater in the eighth-grade textbooks than they are in the sixth- and seventh-grade textbooks, $\hat{\beta} = 1.448$, z = 6.16. The model estimates that the odds of finding an equal sign in an equation with operations on both sides of the equal sign are over 4 times higher in the eighth-grade textbooks than they are in sixth- and seventh-grade textbooks combined.

As can be seen in Table 3, both *Connected Mathematics* and *Saxon Math* have a very low proportion of equations with operations on both sides of the equal sign. This is especially true of the sixth and seventh grade textbooks in the *Connected Mathematics* series.

Conclusions

The present study examined how the equal sign is presented in two middle-school mathematics textbook series, which are designed, in part, to prepare students for algebra. The odds of finding an equal sign in an operation-equals-answer context declines from sixth to eighth grade, and the odds of finding an equation with operations on both sides of the equal sign increase in eighth grade. Averaging across grade level, *Saxon Math*, a skills-based textbook series, presents a greater proportion of equal sign instances in an operation-equals-answer context than does *Connected Mathematics*, a standards-based textbook series. Few equations with operations on both sides of the equal sign were found in either textbook series (only 6% of equal sign instances overall). This may be problematic, given that this context has been shown to activate students' knowledge of the equal sign as a relational symbol of equivalence (McNeil & Alibali, in press).

In the future, we will explore whether or not there are strong advantages to exposing students to equations with operations on both sides of the equal sign. We also intend to extend the textbook analysis to a wider range of textbook series and grade levels. The goal is to contribute to our understanding of students' experience with the equal sign and how it changes over time. The equal sign is arguably the most fundamental symbol in all of mathematics and science. If we are to understand students' misconceptions fully, we need to characterize the nature of students' experiences.

References

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the "equals" sign. *Elementary School Journal, 84*, 199-212.

Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equal sign. *Mathematics Teaching, 92*, 13-15.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann.

Carpenter, T. P., & Levi, L. (2000). *Developing conceptions of algebraic reasoning in the primary grades*. Madison, WI: National Center for Improving Student Learning and Achievement in Mathematics and Science.

Connected Mathematics Project (2004). http://www.mth.msu.edu/cmp.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317-326.

Kulm, G. (1999). Evaluating mathematics textbooks. *Basic Education, 43*.

McNeil, N. M., & Alibali, M. W. (2000). Learning mathematics from procedural instruction: Externally imposed goals influence what is learned. *Journal of Educational Psychology, 92*, 734-744.

McNeil, N. M., & Alibali, M. W. (2002). A strong schema can interfere with learning: The case of children's typical addition schema. In C. D. Schunn & W. Gray (Eds.), *Proceedings of the Twenty-Fourth Annual Conference of the Cognitive Science Society*. Mahwah, NJ: Lawrence Erlbaum Associates.

McNeil, N. M., & Alibali, M. W. (in press). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development.*

Nathan, M. J., Long, S. D., & Alibali, M. W. (2002). The symbol precedence view of mathematical development: A corpus analysis of the rhetorical structure of textbooks. *Discourse Processes, 33*, 1-21.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural knowledge of mathematics: Does one lead to the other? *Journal of Educational Psychology, 91*, 175-189.

Roseman, J. E., Kulm, G., & Shuttleworth, S. (2001). Putting textbooks to the test. *ENC Focus, 8*, 56-59. Saxon Publishers (2004). http://www.saxonpublishers.com/school/math/index.jsp.

Seo, K.-H., & Ginsburg, H. P. (2003). "You've got to carefully read the math sentence...": Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills* (pp. 161-187). Mahwah, NJ: Lawrence Erlbaum Associates.

BUILDING UP THE NOTION OF DEPENDENCE RELATIONSHIP BETWEEN VARIABLES: A CASE STUDY WITH 10 TO 12-YEAR OLD STUDENTS WORKING WITH SIMCALC

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This paper reports the results from a study with 10 to 12-year old students working on activities involving various functional representations (graphs, tables, and numerical relationships) in a motion phenomena simulation environment such Math Worlds. Results from the study suggest that pupils that have not been received formal instruction in algebra symbolism able to evolve towards a better understanding of functional relationships, when working with a variety of representation systems. Duval's registers theory was used for activity design and data analysis.

Background

There has been recently a growing interest on children's potential to learn algebra at early stages of their development. In some of these *early algebra* studies the possibility of teaching algebra to young students (7 to 8 year old) has been explored through problem solving activities that elicit the algebraic nature of arithmetic competency (Carraher et al, 1999 and 2000). Other approaches emphasize the role of young children drawings and representations in word problem solving processes as a basis to develop algebraic ideas (Dougherty and Zolliox, 2003). Smith and Davis (2001) say that the history of algebra may be used as a source of information about the possible difficulties faced by young students when they are introduced to algebraic thinking. L. Lee stresses out the idea of considering the relevant aspects of different methods used to teach algebra (as a language, a way of thinking, a tool, or generalized arithmetic) to encourage the learning of algebra (Lee, 2001). Every study, however, has reported the feasibility of introducing young students to the algebra domain either by using algebra symbolism or through other representations. The purpose of this study is to investigate the possibility of introducing fundamental algebraic ideas to students from elementary schools through the use of representation systems such as Cartesian graphics and numerical tables generated by a learning tool that includes a motion phenomena simulation environment.

Theoretical Framework

Theoretical references on representation devices are based on R. Duval (1999), who describes how semiotic registers provide an effective way to materialize knowledge and deal with mathematical objects. In this regard it is necessary to promote a kind of learning, where several representation devices are integrated and coordinated in such a way that the student does not mix up the mathematical object and its semiotic representation, and relates the mathematical object to several representations. R. Duval claimed that it is necessary to encourage three cognitive activities: 1) *formation* (create a representation to describe an object); 2) *treatment* (transform the representation into the device); and 3) *conversion* (transform the representation of a device into another).

Our research takes these elements as a basis for the development of a didactic strategy to design learning activities, which enable students to approach up algebraic concepts such as functional variation through the cognitive activities of *treatment*, *conversion*, and *formation*.

We must highlight the fact, however, that in certain situations students may or may not be aware of such cognitive activities; for example, when they create a functional table from position graphics (corresponding to a phenomenon of constant and positive speed). This was possible because a software application (a simulator) was used as a mediation tool to create links between the students' algebraic knowledge and cognitive processes. The idea was to introduce 10 to 12-year old students, who had never received formal education on algebra, into the notion of functional relationships through the use of the SimCalc Math Worlds computing environment.

Math Worlds provides animated worlds, where animations move according to changes in graphics. Graphics are represented through rectangles meaning speed: The height of a rectangle means "how fast", and the width means "how long". Position, speed, and acceleration graphs are dynamically linked. If there is a change in speed, the corresponding changes in the position or acceleration graphs are instantly displayed.

As for environmental usage, theoretical references are taken from J. Roschelle (1998) and J. Kaput (1998). Ideas such as functions, equations, and variables are used to promote the development of skills in the initiation of algebra.

In this paper we discuss the results from the students' analysis of position and speed graphics that help them gain a better understanding of dependence relationship between two variables, as well as to provide concepts such as "it goes faster" or "this is quicker" with a mathematical meaning.

As other studies based on a functional approach to algebra (Kieran et al, 1996; Nemirovsky, 1966: Heid, 1966), our study used the computing environment to design modeling activities that allowed students to explore quantitative changes in variables, and analyze how these changes modify functional representations.

Methodology

-12 activities in two different versions (Math Worlds and spreadsheets) were designed to encourage the three cognitive operations of *formation*, *treatment* and *conversion* among registers (Duval, R. 1998, 1999).

-Such activities are used to promote i) the use of more than one representation register: a) interpreting position graphs, b) building up tables and position graphs, and c) interpreting speed graphs; ii) the notion of constant speed (functional relationship); and iii) problem solving in a motion phenomena simulation environment. This paper is focused on aspects i) and ii).

-A database consisting of: a) a diagnosis questionnaire on basic notions of physics (speed), pre-algebraic operations, arithmetic, and reading and gathering data from tables and graphs; b) structured guided interviews (A. Brown et al, 1998); c) students production from learning activities with Math Worlds (student records); d) videos from interviews; e) learning activities aimed to encourage the *formation*, *treatment* and *conversion* of registers to gain evidence on the students understanding of notions such as variable, functional relationship, speed, and others.

-The participants were 10 to 12-year old children from elementary schools, who had never received formal education on algebraic symbolic language, and were selected from their answers to the diagnostic questionnaire. They worked in two groups of six pupils each, one with spreadsheets an the other math MathWorlds This paper reports on the results from the six-student group working with Math Worlds.

Diagnosis Questionnaire Results

Based on a diagnosis questionnaire, subjects were grouped in three levels, according to their arithmetic competency, register-handling abilities (data gathering from tables and graphs), and notion of constant speed. Levels I, II, and III vary from a substantial proficiency on arithmetic (Level I) to a limited competency (Level III).

Based on this questionnaire, six students (3 from 5^{th} grade and 3 from 6^{th} grade) were chosen to participate in the study. They took part in 12 sessions, working with the computing environment on:

a) the use of more than one representation system to interpret position graphs; build up position graphs and charts; interpret speed graphs;
b) the notion of constant speed as a functional relationship;
c) relating problems to motion phenomena.

In the middle and at the end of each session sequence individual interviews were carried out with all six students to analyze the strategies they use to interpret different representations as they went on solving the proposed activities. This paper discusses the results related to a) and b).

10-year old Students working with SimCalc.

All three students required the use of simulations along the learning activities to verify their answers, complete tables or build up graphs. Only the Level I student (Erick) built up continuous graphs from the first activity, compared to the other two students, who did it in a discrete way, by running the simulation step-by-step or making operations to identify the next section on the graph. In addition, Level III student (Rodrigo) found out in the last two activities that he could build up the position graph continuously by knowing the total distance and duration of a path, as shown in the following table:

The two students from levels II and III seemed to focus their attention on the use of graphs to calculate speed, while Erick showed at first some reluctance to use tables to identify speed when requested to do so, but once convinced, he worked on the table to identify the required information and calculate the answer.

Toward the notion of speed (10-year old children) Level I (Erick).

SimCalc incorporates motion of various characters, such as frogs or lifts (according to the World chosen). From the first sessions he shows to deal with a notion of speed involving both variables, and providing explanations such as "the frog moves forward four meters every second, and the clown two meters." After becoming familiar with the simulator, his explanations were "the frog moves forward twice as fast as the clown", or "it moves three times faster than the green lift." As his identification of the variables improves, his answers to the notion of speed include elements such as "frog 2 moves forward three meters per second, and frog 1 one and a half meter per second."

In the middle of the activity sequence, Erick is requested to give a definition of speed. His answer, "speed is what it runs in one second," could be considered as focusing his attention on one of the variables. At the end of the sequence he was requested again to provide a new explanation of speed. This time he uses a specific example, but after a few questions he comes out with a more general notion pointing out both variables:

Er: It is... it moves forward one point five meters (the character's speed in the simulation).

E: And speed involves... what?

Er: Seconds and distance.

Level II (Ana Karen)

At the beginning of the session she provided explanations such as "The frog's steps are four meters long," or "The clown's steps are two meters long." After comparing the speed of some characters she expressed the following: “The truck goes faster, and the car slower.” After moving on through the activities, Ana managed to include a little more information: “The slower clown moves five meters, and the faster clown moves eight meters.”

In the middle of the sequence her definition of speed is “the number of kilometers a car or anything else moves forward.” At this point she also perceived the possibility of using distance and time to calculate the speed from the information contained in the position graph. By the end of the sequence she employed a particular situation to explain the notion of speed, taking into consideration the time and distance variables, but on a particular example:

E: How would you explain speed?

Ak: Speed is the distance and time a car travels.

E: How do you read speed?

Ak: If we take meters and seconds, then it will be 81 meters per second.

Level III (Rodrigo)

His first explanations about motion took into account the physical features of characters, with definitions such as “The clown is small, and that is why his legs go slower,” or “The tires of the truck are bigger, and that is why it goes faster.” After completing the first activities he included in his explanations elements related to the characters movement, such as “the red one moves slow, and the green one moves fast,” until he finally takes into consideration both variables: “It advances one third every second,” or “the red one goes up two floors every second.”

When first asked about his notion of speed, his answer was “every lift goes up a number of floors per second,” making use of an example to generate an explanation. Once the activities sequence was concluded, his notion of speed evolved to include both variables, distance and time, making it easier for him to calculate speeds from a position graph, but avoiding any oral explanation:

Ro: Then we looked at the graph.

E: And what did we notice there?

Ro: The hours *(he writes his answer: “We looked at the graphic, and paid attention to the hours and kms.”)*

E: OK. Now, how would you explain speed to your schoolmates?

Ro: I don’t remember *(he could not provide an explanation).*

11 to 12-year old Students working with SimCalc.

On both ends of the table, Level I student (Eduardo), and Level III student (Rafael) draw graphs step by step, and it was not until the last activities that they started on building up continuous graphs, using as reference the total distance and period. With this information they didn’t have to make any further calculations to identify every portion of the graph. On the other hand, Level II student (Clara) drew continuous graphs from the beginning, when she was asked to build up a register.

As for the verification process, all three levels repeatedly used simulation to get data and match their results. In those situations where the environment didn’t provide directly the required information, they made calculations using pencil and paper, an electronic calculator or mentally. For the final activities the use of the simulator in all three cases was restricted to the analysis of situations rather than to getting information to give an answer.

As for the preference on the kind of representations, only Clara (Level II) showed a clear disposition to use graphs rather than tables to get information. Eduardo and Rafael used indistinctively data from graphs and tables to make calculations, solve problems, and answer questions.

Toward the notion of speed (11 and 12-year old children) Level I
(Eduardo)

From the beginning he took into consideration the two variables to build up the notion of speed, with answers such as "Because the frog went over more meters in a second than the clown," or "Clown 1 goes six meters in one second, and clown 2 goes eight meters." These answers were consistent along the sequence. When he was asked to give a definition of speed, he said:

E: How would you explain to your schoolmates what speed is?
Ed: Through distance and time. You can say that it represents how fast two people or two objects go.

Together with the comments above, he recognizes the need to divide distance by time to calculate speed either from a table or a graph: "We can see how far did he go, and the time, and all that."

Level II (Clara)

Her first explanations about the movement of characters indicate her focus on the physical conditions of the phenomena: "It runs more and gets first", or "It walks slower". After the third activity, she includes other elements such as distance and time: "The second frog, because it runs more meters per second."

For her first explanation of speed, she uses a specific example: "The number of floors per second it climbs, the distance it travels per hour." Once the sequence is complete, she takes into consideration the variables involved in her notion of speed, even if she requires a specific example:

E: How would you explain to your schoolmates the concept of speed?
Cl: The distance traveled every second.

We may consider her definition of speed after completing the sequence as an evolution from a very intuitive and inconsistent notion of speed at the beginning of the study.

Level III (Rafael)

At the beginning of the sessions, his explanations were: "it goes farther than the frog", or "it goes two by two and is slower." His answers indicate that he takes into consideration one variable. From the third activity, however, he includes in his explanations both variables as follows: "it goes only a few meters in many seconds," and "the red one, because every second it climbs three floors."

His first definition of speed indicates the relationship between the distance and time variables: "the distance traveled in a period of time." This may suggest that he is adjusting his notion of speed, since he also recognizes the need to identify distance and time in order to calculate speed:

E: What elements should be taken into consideration to calculate speed?
Ra: Floors andseconds.

In spite of recognizing both variables in the notion of speed, his defective use of the division produces false explanations, as the following:

E: And now what did you do to calculate speed?
Ra: A division
E: What did you divide?
Ra: Hours and meters.

This was a continuous obstacle throughout Rafael's work, and it could not be overcome.

Results

After working with the activities designed for *SimCalc*, the participants showed an improvement on their understanding of motion phenomena and on the dependence relationship between variables. There was, however, a differentiated use of the representational means available in the computing environment, according to children's level of arithmetic competency (identified in the pre-questionnaire). In addition, it was observed that whereas some children during all the sessions required the simulation to analyze the dependence between variables, others were able to focus their analyses on mathematical representations (Cartesian graphs and function tables). These children in turn succeeded in building up a quantitative notion of constant speed. For example, in the final session Erick (11 years old) described his notion of speed as "the frog moves forward four meters every second, and the clown two meters" (referring to the characters in the simulation environment). When he was asked about his conception of "speed", he said: "Speed is what it runs in one second."

Final Comment

Results from this study suggest that students at a pre-symbolic stage (pupils that have not received formal instruction in algebra symbolism) are able to evolve towards a better understanding of functional relationships, making use of a variety of representation systems (including simulations) to analyze different aspects of motion phenomena. Duval´s *registers* theory was not only used at the activity design stage, but also helped us to define an analysis framework that enables feasible explanations of the way children make use of representational systems when solving problems in a computing environment.

References

Carraher, D.; Schliemann, A. and Brizuela, B. (1999). Bringing out the Algebraic Character of Arithmetic. In *Proceedings of Annual Meeting of the American Educational Research Association*, Montreal, Canada.

Carraher, D.; Schliemann, A. and Brizuela, B. (2000). Early Algebra, Early Arithmetic. In *Proceedings of 22th Annual Meeting of the American Educational Research Association*, Arizona, USA.

Duval, R. (1998). Registros de Representación Semiótica y Funcionamiento Cognitivo del Pensamiento. Hitt, F. (ed.). In *Investigaciones en Matemática Educativa II*. México: Grupo Editorial Iberoamérica.

Duval, R. (1999) *Semiosis y Pensamiento Humano. Registros semióticos y aprendizajes intelectuales*.

Colombia: Universidad del Valle. Dougherty, B. and Zolliox, J. (2003). Voyaging from Theory to Practice in Teaching and Learning: A View From Hawai'i. In *Proceedings of the International Group for the Psychology of Mathematics Education*., Hawai'i, USA, 1,17-23.

Heid, M. (1996). A Technology-Intensive Functional Approach to the Emergence of Algebraic Thinking. In Bednarz, N. Kieran, C. and Lee, L. (1996). *Approaches to Algebra. Perspectives for Research and Teaching*. The Netherlands: Kluwer Academic Publishers.

Janvier, C. (1996). Modeling and the Initiation into Algebra. In Bednarz, N. Kieran, C. and Lee, L. (eds.). *Approaches to Algebra. Perspectives for Research and Teaching*. The Netherlands: Kluwer Academic Publishers.

Kaput, J. (1998a). Teaching and Learning A New Algebra with Understanding.

Kaput, J. (1998b). Technology As A Transformative Force In Education: What Else Is Needed To Make It Work?

Kieran, C., Boileau, A and Garançon, M. (1996). Introducing Algebra by Means of a Technology

Supported, Functional Approach. In Benders, N. Kieran, C. and Lee, L. (1996). *Approaches to Algebra. Perspectives for Research and Teaching*. The Netherlands: Kluwer Academic Publishers.
Lee, L. (2001). Early Algebra – but Which Algebra? 392-399.
Nemirovsky, R. (1996). Mathematical Narratives, Modeling, and Algebra. In Bednarz, N. Kieran, C. and Lee, L. (1996). *Approaches to Algebra. Perspectives for Research and Teaching*. The Netherlands Kluwer Academic Publishers.
Perrusquia, E. and Rojano, T. (2002). The Role of Representation Devices in the Algebraic Conceptualization of Change: A study with 10 to 11 year-olds. In Proceedings of the XXVI International Group for the Psychology of Mathematics Education., Norwich, UK. 1, 311.
Roschelle, J.; Kaput, J. and Stroup, W. (1998). SimCalc: Accelerating Student's Engagement with the Mathematics of Change [On-line] Available: http://www.simcalc.umassd.edu
Smith, N. and Davis, G. (2001). Implications of the History of Algebra for Teaching Early Algebra, 548555.

A SEMIOTIC FRAMEWORK FOR VARIABLES

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Starting from a semiotic perspective, the TAAR project at the University of Wisconsin-Madison is developing a framework to analyze middle-school students' conceptions of variables. The framework describes the ways in which we use and represent variables, noting the importance of discourse and experience in shaping students' interpretations and incorporates a process-object psychological model. Student written work and interviews from a three-year longitudinal study complement the framework; we present and briefly analyze two excerpts here.

Algebra has been called the study of the 24th letter of the alphabet. Although this characterization is somewhat facetious, it underscores the importance of developing a meaningful conception of variable quantities if one wishes to successfully solve algebraic problems. In investigating students' developing conceptions of algebra, the TAAR project at the University of Wisconsin-Madison has developed a framework to describe and understand students' conceptions of variable quantities. This framework is being used to analyze—and is supported by—initial data in a three-year longitudinal study of middle-school students' developing conceptions of algebra. The framework is based on a semiotic model and a process-object psychological model.

Theoretical Background

Our goal in teaching any topic in mathematics is to enable students to participate meaningfully in mathematical discourse. In the case of variables, this would mean that a student is able to understand the idea of a varying quantity (along with its domain), recognize and interpret mathematical situations and tasks involving variables, manipulate variables using both internal and external representations, "create" variables to mathematize situations, and communicate these ideas to other people.

Along with the goals of teaching mathematics, our choice of learning paradigm will have implications for the framework. The paradigm of social interactionism combined with some ideas from social constructivism will be the basis for the arguments in this paper. Social interactionism asserts that learning involves the negotiation of meaning within the context of situated activities (Forman, 1996). In this mediation between individualism and collectivism, objects of the classroom discourse are initially ambiguous. The process of negotiation is used to produce meanings that are taken-as-shared between two people or within a group of people (Voigt, 1995). Combining interactionism and constructivism, we use the idea that individuals develop their personal interpretations while they participate in the negotiation of social and sociomathematical classroom norms (Yackel & Cobb, 1996). These interpretations, combined with previous knowledge and experience, allow individuals to construct and reconstruct personal knowledge, although cultural and social practices are still integral to mathematical activity.

Semiotic Model

Greeno and Hall (1997) among others have suggested attending to the distinction between representation and referent; many others (e.g. Ernest, 2002) have called for using semiotic analysis to study mathematics education. In particular, Radford (2001) provides a compelling example of using semiotics to understand different techniques of mathematical proof and justification. Presmeg (2001) describes a Peircean model that provides a basis for the semiotic model in this framework.

The terminology used by Peirce is different from that of Saussure's dyadic model. A *sign* is the indissoluble union of three components: an *object*, a *representamen* and an *interpretant*. The representamen is experientially accessible and represents the object; it may be spoken, drawn, written or encoded in some other medium. The object is the "thing" being represented by the representamen. An interpretant (interpretation) of the object-representamen pair completes the semiotic object. Signs are located within semiotic systems (Ernest, 2002), which consist of a set of signs, a set of relationships between the signs and a set of rules of sign production.

Object

Usiskin (1988) presents four conceptions of algebra, each of which affords a different conception of what a variable is depending on how it is used.

The first conception takes algebra as generalized arithmetic. For example, the statement $a + b = b + a$ is a statement true for any real numbers a and b. Variables are treated as pattern generalizers and are used to translate and generalize patterns. The second conception is as a study of procedures for solving certain kinds of problems. For example, a word problem may be translated into mathematical notation as $x^2 + 6x + 5 = 0$. Variables are treated as unknowns or constants and used to simplify and solve equations. The third conception is as the study of relationships among quantities. It is closely related to working with functions. Variables are used as arguments (an element in the domain or substitution set) or parameters (upon which other numbers depend). It is in this conception that context-dependent uses of literal symbols most often arise. The fourth conception of algebra is as the study of structures. At higher levels, the study of algebra involves structures such as groups, rings and modules. Variables are treated as abstract elements of these structures, as arbitrary marks on paper, without a referent.

While it is important to keep in mind all potential conceptions of the symbols used to represent varying quantities, we will restrict our attention to the main "types" of varying quantities encountered by middle-school students, based on Usiskin's descriptions. The four categories into which our objects may fall are

- variables (quantities that may take on any value from a specified or implicit domain),
- unknowns (quantities that may take on any value from the domain but have a subset of the domain that makes a given statement true),
- parameters (quantities that are usually fixed within a problem but may take on different values in other situations) and
- indeterminates (generalized numbers).

For example, in $ax^2 + bx + c$ we usually view x as able to take on any value from the real numbers (making it a variable) and a, b and c as values that will be fixed (making them parameters). If we change this symbol string into $ax^2 + bx + c = 0$ then x becomes an unknown, able to take on zero, one or two values. It is important to emphasize that variables are usually used in a functional relationship, as implied by Usiskin's conception of algebra as the study of relationships among quantities.

Representamen

Whatever the nature of these signified variables, we are faced with "the impossibility of any direct access to mathematical objects and the ensuing need to render them sensible" (Radford, 2003 p. 43). We can turn variables into objects of discourse by using the Radford's idea of semiotic means of objectification. These are the "objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities" (Ibid. p. 41).

For example, if someone is thinking about a variable and wants to communicate this idea to other people, they must somehow make the variable an object that is accessible to everyone participating in the discourse. This could be accomplished by, for example, writing symbols on a piece of paper, drawing a picture or a graph, or making verbal utterances or gestures.

Although students may spontaneously generate many different representations of variables, four basic types are used in math texts. The first is a class of symbolic representations consisting of letters (literal symbols), letters in other alphabets (such as α and β) and iconic symbols such as blanks, boxes and question marks. The second class consists of graphical representations. Although graphs on Euclidean planes typically convey functional relationships, they can also convey parametric information (by showing several functions drawn on the same set of axes) or the idea of an unknown (an intersection of two functions). Variables can be represented graphically in a less-explicit functional relationship showing movement along a number line (instead of a plane). The third class consists of tabular representations of variation. Although it could be argued that tables can't fully represent variation over an infinite domain (and are generally only used to represent functional relationships), tables are used to convey the *idea* of variation and as such are a semiotic means of objectification. The fourth class consists of verbal or written descriptions of variation. For example, in the statement "My friend gives me some money and then I tripled the amount of money he gave me," we can interpret "some money" as a varying or unknown quantity.

Interpretant

How might someone interpret the symbol $2c$? If you're in an algebra class, it could represent the product of a constant and a variable quantity, the number resulting from taking the product, or just a symbol string that you could manipulate according to a set of rules. If you haven't been introduced to algebraic notation, it could represent a number between 20 and 29, or perhaps a number that starts with the digit 2 (such as 237). If you're cooking, it might represent "two cups." If you live in an apartment building, it might represent your neighbor's address.

How do individuals decide how to interpret a symbol and give it meaning? Dörfler (2000) claims that an interpretation is the product of participation in discursive activity. Cobb (2000) observes that "The ways that symbols are used and the meanings they come to have are mutually constitutive and emerge together" (p. 18). Sfard (2000) argues that not only is the act of symbolizing integral to the process of developing individual as well as collective interpretations, but mathematical discourse is an autopoietic system. That is, the social situations in which individuals interact are created through discourse, and these social situations both constrain and create what can be said and done in them. The ideas of social constructivism align with this idea: "Mathematical truths arise from the definitional truths of natural language, acquired by social interaction" (Ernest, 1999). Clearly discourse plays an important role in shaping students' interpretations of symbols.

MacGregor and Stacey (1997) emphasize the role experience plays in understanding algebraic notation. Specifically, they link students' difficulties to (among other things) "intuitive assumptions and sensible, pragmatic reasoning about an unfamiliar notation system [and] analogies with symbol systems used in everyday life, in other parts of mathematics or in other school subjects" (p. 1). They found that students' facility at interpreting variables greatly improved after receiving additional experience through instruction. In addition, the errors that the students made repeatedly could be linked to improper use of learned procedures and improper invocation of schemas.

Our experiences with the various representations of variables—the ways in which we have seen them used, the settings in which we use them and the social situations when we use them—create possibilities of their creation and existence. That is, an individual will interpret a symbol as a variable if he or she has had prior experience interpreting a similar symbol as a variable or is participating in discourse that facilitates such an interpretation.

Psychological Model

Reflection plays an important role in mathematical development. It is intrinsically related to the dialogicity of a text, in that reflection occurs when one generates meaning from the ideas and utterances of either other people or oneself. McClain & Cobb (2001) assert that the act of mathematizing involves making mathematical ideas the object of reflection. Piaget claimed that a defining feature of mathematics is that the actions on mathematical objects become objects themselves (Cobb & Boufi, 1997).

Incorporating this idea of constructing mental mathematical objects, our psychological viewpoint is based on the process-object theories of RUMEC (Weller et. al., in review) and Sfard (1991). Although we will use the "Action-Process-Object-Schema" (APOS) language of the RUMEC framework, Sfard's "Interiorization-Condensation-Reification" uses essentially the same ideas. Variables are intimately linked with ideas of functions, and we will base our descriptions of variables on Breidenbach, Dubinsky, Hawks and Nichols' (1992) description of students' conceptions of function.

An *action* is "any transformation of objects to obtain other objects" (Weller et al., p. 5). It is viewed as an explicit sequence of actions that are external to the observer. If an object is conceived of as an action, it is viewed as a repeatable set of operations performed on a supplied input. We will say that a student has an action conception of a variable if they can replace the variable with a value from its domain (or multiple values from the domain one at a time).

A *process* is an action that has been interiorized and no longer requires an explicit sequence of steps. It is "characterized by an individual's ability to describe, to reflect upon, or to reverse the steps of a transformation without actually having to perform the steps explicitly" (Weller et al., p. 5). Breidenbach et al. (1992) consider the specification of the distinction between an action and a process to be "an important open question" (p. 250). We will say that a student has a process conception of a variable if they view the variable as able to take on multiple values from its domain simultaneously. The student is no longer concerned with the act of substituting a value for the variable and can recognize patterns that emerge in basic functional relationships.

An *object* is derived from a process or a schema through encapsulation (Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas & Vidakovic, 1996 p. 172). It represents the process as a whole, and as something that can be acted upon. We will say that a student has an object conception of variable if they view the variable as the *result* of substituting multiple values for the variable.

A *schema* is a coherent collection of objects and processes linked by the context of their use; it is invoked to deal with a given problem situation. Although schemas are an integral part of the APOS framework, we will focus only on the ideas of action, process and object; schemas describe the relationship between conceptions of mathematical entities, while we are more interested in describing conceptions of entities themselves.

This idea of processes being turned into objects can be extended by incorporating objects as pieces of new processes (e.g. Sfard, 1991). For example, a variable as an object can be incorporated into an algebraic expression that can be in turn thought of as a new process or object. Symbolically this could be represented as x, representing an arbitrary number, being incorporated into $3x + 2$ which could then be thought of as (for example) a number itself. This idea is a parallel to Presmeg's (2001) presentation of semiotic chaining, where representations are turned into discursive objects.

One objection to the process-object models is that their existence can be interpreted as conflicting with the social nature of our learning paradigm. That is, "the object... is reified and abstracted from its historical and cultural context and is conceived as standing in front of the subject as if the object was a mere item in the ecology of the subject" (Radford, 2000 p. 240). Emphasizing the discursive nature of these process-object ideas along with employing them as cognitive descriptions serves to reaffirm their social nature and incorporate them as a complementary component of the framework. We will say that a student has a particular conception of variable if they tend to use variables as actions, processes, or objects in mathematical discourse.

Employing the Framework

We will use our framework to analyze two short examples from work done by the TAAR project at the University of Wisconsin-Madison.

In the first example, in one-on-one interviews, 32 sixth-graders were asked to represent mathematically the situation "I have some number of pencils and then get three more." Many of the students had previously encountered problems such as $?+ 3 = 10$ asking them to find the missing number. Although the students used many different symbols to represent "some number" of pencils, almost all of the students wrote an equation of the form $M + 3 = N$. The situation does not explicitly ask students to represent the total number of pencils. The students' insertion of an equals sign could be explained in part by the students' failure to encapsulate the addition into an object that could be an answer. In addition, the students transformed what was intended to be a representation of a variable into a literal-symbol representation of an unknown. This can be explained in more detail by our framework in two ways. First, the students could be transplanting representations from their prior experience trying to solve unknown quantities to deal with this new situation. Second, their symbolic representation of variables may not have developed to a point where they can use it fluently as a replacement for a textural variable.

The second example further supports the theory that students' understanding of a verbally presented variable may differ from their understanding of a variable represented by a literal symbol. In a written assessment administered to sixth, seventh and eighth-grade students (n=373) students were asked one of the following two questions:

i. Can you tell which is larger, $3n$ or $n + 6$? Please explain your answer.
ii. A fried gives you some money. Can you tell which is larger, the amount of money your friend gives you plus six more dollars, OR three times the amount of money your friend gives you? Please explain your answer.

These questions were designed to present the same variable in two different representations; a student was said to have given a correct response by indicating that you couldn't tell which amount was larger. In the sixth grade, students who were given the second version of the question were significantly more likely to give a correct response than those who were given the first version (40% vs. 10%). In 7th and 8th grade the representation had no significant effect on the percentage of correct responses, although students were increasingly likely to give a correct response (50% of 7th graders responded correctly and 62% of 8th graders responded correctly).

This data suggests that students can develop a good conception of a variable before having much experience with literal symbols, indicating that the concept of variation (the variable object) is in part independent of its representations.

Conclusion

Students' conceptions of varying quantities are diverse and often difficult to understand. The framework developed by the TAAR project has synthesized several psychological and experiential ideas in the context of semiotics. It allows us to paint a broad yet relatively detailed picture of students' understanding of variable quantities while emphasizing important features we can attend to in an instructional setting.

References

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992) Development of the process conception of function. *Educational Studies in Mathematics*, 23, 247 – 285.

Cobb, P. (2000). From representations to symbolizing: Introductory comments on semiotics and mathematical learning. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating: perspectives on Mathematical Discourse, Tools, and Instructional Design* (pp. 17-36). Mahwah, NJ: Erlbaum.

Cobb, P. & Boufi, A. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education*, 28 (3), 258-278.

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K. & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process schema. *Journal of Mathematical Behavior*, 15, 167–192.

Dörfler, W (2000). Means for Meaning. In P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and communicating: perspectives on Mathematical Discourse, Tools, and Instructional Design* (pp. 99-132). Mahwah, NJ: Erlbaum.

Ernest, P. (1999). *Social Constructivism as a Philosophy of Mathematics: Radical Constructivism Rehabilitated?* Retrieved June 15, 2004, from http://www.ex.ac.uk/~PErnest/soccon.htm

Ernest, P. (2002) A semiotic perspective of mathematical activity. In A. Cockburn & E. Nardi (Eds.) *Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education* (26th, Norwich, England, July 21-26, 2002).

Forman, E. (1996). Learning mathematics as participation in classroom practice: Implications of sociocultural theory for educational reform. In P. Nesher, L. P. Steffe, P. Cobb, C. Goldin & B. Creer (Eds.), *Theories of Mathematical Learning*, (pp. 115-130). Hillsdale, NJ: Lawrence Erlbaum Associates.

Greeno, J. & Hall, R. (1997). Practicing representation: Learning with and about representational forms. *Phi Delta Kappan*, 78, 361-67.

MacGregor, M. & Stacey, K. (1997) Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics*, 33, 1-19.

McClain, K. & Cobb, P. (2001). An analysis of development of sociomathematical norms in one first-grade classroom. *Journal for Research in Mathematics Education*, 32 (3), 236-266.

Presmeg, N. (2001). Progressive mathematizing using semiotic chaining. Paper presented to the Discussion Group on Semiotics and Mathematics Education at the 25th PME International Conference, The Netherlands, University of Utrecht, July 12-17, 2001.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics*, 42 (3), 237-268.

Radford, L. (2001). On the relevance of Semiotics in Mathematics Education. Paper presented to the Discussion Group on Semiotics and Mathematics Education at the 25th PME International Conference, The Netherlands, University of Utrecht, July 12-17, 2001.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning* 5(1), 37-70.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. (2000). Symbolizing mathematical reality into being: How mathematical discourse and mathematical objects create each other. In P. Cobb, K. E. Yackel, & K. McClain (Eds.), *Symbolizing and communicating: perspectives on Mathematical Discourse, Tools, and Instructional Design* (pp. 37-98). Mahwah, NJ: Erlbaum.

Usiskin, Z. (1988) Conceptions of school algebra and uses of variables. In A. F. Coxford, (Ed). *The Ideas of Algebra, K-12* (pp. 8-19). Reston, VA: NCTM.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In P. Cobb, & H. Bauersfeld (Eds.), *The Emergence of Mathematical Meaning* (pp. 163-201). Hillsdale, NJ: Lawrence Erlbaum Associates.

Weller, K., Clark, J., Dubinsky, E., Loch, S., McDonald, M., & Merkovsky, R., In review. An examination of student performance data in recent RUMEC studies.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education*, 27 (4), 458-477.

DIFFERENT FORMS OF MATHEMATICAL THINKING IN HIGH SCHOOL ACTIVITIES

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Introduction and Objectives

The attempt to understand the nature of mathematical thinking processes has been an object of research in the fields of philosophy and psychology as well as in that of mathematics education for many years. Because of this diversion in the fields of research, a variety of theoretical and methodological perspectives (see for example Sternberg & Ben-Zeev, 1996) have resulted. Although based on different aspects of human activity, the studies coincide in signaling that mathematic thinking processes can be developed through experience and instruction. In other words, teaching, besides helping with the learning of concepts and procedures, offers opportunities for the students to develop their thinking.

The vision of learning mathematics represented in the Principles and Standards for Student Mathematics (NCTM, 2000) proposed that learning be developed upon the base of previous knowledge and experience (ibid, p. 20), with an emphasis on learning through understanding, wherein reflection and communication are the most important parts of the process (Hiebert et al., 1997). This representation of the process of teaching-learning includes the development of mathematical thinking as one of its fundamental components.

In this work we present a categorization of mathematical thinking in terms of basic processes in the task-work of mathematics. This proposal comes from the necessity of defining characteristics that can be translated with ease into promotional activities for the high school students. If this categorization is practical and can be based in different concepts of mathematical thinking, it can also deepen different aspects of the fundamental importance of mathematical task-work. It could be applied, for example, to analyze the potential that the study offers for teaching methods, an activity that can be seen as stressed today in the emphasis on curriculum proposals and guides presented in the document from NCTM mentioned above.

Theoretical Perspective

The investigation in mathematics education has been aided by psychological research and that of other disciplines in the attempt to understand the nature of mathematical thinking. In particular, the constructivist perspective has been helpful in the understanding some processes and difficulties that students encounter during their development. This perspective supposes that learning consists over all in the active construction of signifiers (Romberg & Carpenter, 1986), in that interrelated aspects of psychological character intervene, epistemological and socio-cultural factors significantly influence individual thinking processes. These aspects have generated, over time, different concepts about what mathematical thinking is. For example, Polya (1996) argued that it deals with all those processes that allow understanding of phenomena, such as exploration, experimentation, posing and testing hypotheses, compiling and analysis of data. For Polya, this is the product of the search for patterns and principles (induction), and the construction of links of inference aided by logical argument (deduction).

More recently, the work of Schoenfeld (1998) suggests that in the classroom, activities such as those practiced by mathematicians must be promoted so that the students can develop a *mathematical point of view* (Schoenfeld, 1998). In other words, students must learn which and what mathematical tools to use in observation, analysis and solution of problems. For Schoenfeld, to learn to think mathematically means “much more than simply ‘to know’ a particular body of mathematics or the skills of solving problems. It means having a predilection for seeing the world through a type of mathematic lenses (a perspective specifically mathematical), and to have the tools to act mathematically when it is appropriate” (ibid, p. 93).

Tall (1991) mentions that previous experience and human cultural activities intervene in such a way that they develop thought, generating a variety of types of mathematical minds. Tall and his colleagues (2001) also speak of different types of cognitive development (perception, action and reflection) that can produce different mathematics: “Our vision is that perception, action and reflection occur in various combinations at any given time and the emphasis in one more than another can bring about very different types of mathematics” (ibid, p. 82).

From our point of view, perception, action and reflection are basic mathematical thinking processes that do not precisely bring about different types of mathematics but rather different forms of mathematical thinking. Therefore, the analysis of the resources that aids a mathematician during his daily practice permits the differentiation of basic categories, those which we call forms of mathematical thought. These are: inductive thinking (perception), deductive thinking (action) and logic thinking (reflection) that we also call mathematical reasoning.

On occasion, mathematical thinking and reasoning are spoken of as if they were synonymous, but we consider that mathematical reasoning is only one of many forms of thought, because mathematical thinking thought includes other processes of coming to conclusions and formal logic is a characteristic of mathematical reasoning. Many times what is relevant in mathematical activities is not formal logic but all those resources that permit a mathematician to sketch out a possible reasoning. For example, individual experience, the sensibility to perceive certain things over others, speculations, presentments, feelings and inspired ideas among other aspects are involved. These resources can lead to conjectures and bring to them a certain grade of credibility. In this sense they all form part of the whole of evidence that can be the base of mathematical reasoning.

Deductive thought includes those situations where in a fact is applied that, if its origin was helped by mathematical reasoning, in the moment of its application it might not be explicit. While mathematical reasoning requires conscious reflection for justification, to recuperate the significance of the objects or to give meaning to the results of a process, deductive thought is of a more practical order: it permits work with mathematical objects leaving to one side their origin. This, for example, is one of the potentialities of symbolic manipulation.

Deductive thinking differs to induction in the type of resources that sustain the activity. While deduction is aided by mathematical reasoning, in induction resources intervene that are not necessarily of the logical type, such as intuition or perception. Inductive thinking is frequently a resource of mathematicians in the formation of conjectures which are based on evidence found that, to a lesser or greater degree, make for plausible affirmation. This form of think involves mental processes such as examination, intuition, testing, perception, making analogies. All of these are associated with the activity of experimental mathematics.

When we design activities for students, in this case at the high school level, it is important to be clear about the aspects of mathematical thought we wish to promote. This proposed categorization can be of much utility to the nature of the practice. If all the different theories about mathematical thinking show great complexity that attempt to define it in terms of a combination of characteristics, in mathematics instruction we must be clear about the aspects of mathematical thinking that we wish to promote, and these must be demonstrated in the activities we propose.

References

Burton, L. (1999) Mathematics and their Epistemologies – and the Learning of Mathematics. In: Inge Schwank (Ed.), *Proceedings of the First Conference of the European Society for Research in Mathematics Education* (pp. 87-102). Volume I. Forschungsinstitut fuer Mathematikdidaktik, Osnabrueck. Internet version. Retrieved March 18, 2003, from: http://www.fmd.uni-osnabrueck.de/ebooks/erme/cerme1-proceedings/cerme1-proceedings-1-v1-0-2.pdf.

Hiebert, J., Carpenter, T., Fennema, E., Fuson, K., Wearne, D., Murray, H., Oliver, A., Human, P. (1997). *Making Sense: Teaching and Learning with Understanding*. Portsmounth, NH: Heinemann.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

Polya, G. (1966). *Matemáticas y Razonamiento Plausible*. Madrid: Editorial Tecnos.

Romberg, T., Carpenter, T. (1986). Research on Teaching and Learning Mathematics. Two Disciplines of Scientific Inquiry. In: MC Witrock (Ed.), *The Third Handbook of Research on Teaching* (pp. 850873). NY: Macmillan Publishing.

Schoenfeld, A. (1998). Reflections on a Course in Mathematical Problem Solving. In: Alan Schoenfeld, James Kaput, and Ed Dubinsky (Eds.), *Research in Collegiate Mathematics Education III*. (pp.81-113). Issues in Mathematics Education, Volume 7. Washington D.C.: American Mathematical Society.

Sternberg, R. (1996) What is Mathematical Thinking? In: Robert Sternberg, and Talia Ben-Zeev (Eds.), *The Nature of Mathematical Thinking*, (pp. 303-318). NJ: Lawrence Erlbaum.

Tall, D. (1991). The Psychology of Advanced Mathematical Thinking. In: David Tall (Ed.) *Advanced Mathematical Thinking* (pp. 3-21). Netherlands: Kluwer Academic Publishers.

Tall, D., Gray, E., Ali, M., Crowley, L., DeMarois, P., McGowen, M., Pitta, D., Pinto, M., Thomas, M., Yusof,

Y. (2001). Symbols and the Bifurcation between Procedural and Conceptual Thinking. In: *Canadian Journal of Mathematics, Science and Technology Education* (pp. 81-104), volume 1.

UNDERSTANDING EARLY ALGEBRA STUDENTS' NOTIONS OF FORMAL REPRESENTATIONS USING THINK-ALOUD PROTOCOL ANALYSES

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In order to better understand students' ideas about formal representations early in their algebra instruction, a pilot study was conducted with five middle school students using think-aloud methodology. The goal of the pilot study was to gain experience and insight into how students reacted to the study questions and to use this information to design assessment questions for a larger study of the process of middle school students' development of algebraic thinking.

In response to the developing interest in constructivist teaching and its implementation in school systems, new approaches to teacher-student communication and classroom assessment are important to the success and effectiveness of new teaching styles. Constructivist curricula are designed to value creative student thinking in both individual and collaborative activities (Simon, 1995). Traditional mathematics classes are often taught by demonstration, lecture and individual practice, sometimes referred to as skill and drill. Teachers accustomed to these methods rely on written tests for assessment of student accomplishment, and test items are graded as right or wrong. As teachers begin to use constructivist lessons they will need to communicate with students in new ways, and use new assessment tools that measure thinking development, learning readiness, and flexible solution methods.

As a step toward designing a research project to study the development of students' representational fluency (Nathan et al., 2002), five students in 6th through 8th grade were given algebraic items to work on while thinking aloud, and the interviews were recorded on audio tape for later analysis. Items were designed by the research team to measure what types of representational input were most easily interpreted by middle school students and how students could use one mathematical representation to think abut or generate another, equivalent one. The items were represented in verbal, graphical, and equation form. Students shared their thinking process out loud, and the recordings, along with observer's notes, were transcribed and studied using protocol analysis methods (Ericsson & Simon, 1993).

Our analyses show strong evidence that middle school students often have complex thoughts about early algebra structures and operations, but are also confused about the formal properties of the representations. To a teacher or researcher surveying student responses it might at first appear that student answers are wrong, but the level of development toward a correct operation would be missed by this traditional evaluation. When the interviews are analyzed by studying written work, verbalized ideas, and gestures, the developing processes of algebraic thinking can be seen even when the student cannot produce a correct final answer.

Analysis of the interviews showed that students struggle when trying to implement formal approaches to problem solving. Middle school students have fragments of mathematical knowledge, but often have little understanding of the unifying rules about how to work with numbers. They use a combination of guessing, estimating, and occasionally implementing fragments of knowledge that they think might relate to the given problem. Sometimes this eclectic method produces correct answers, but often it results in erroneous answers that still reveal valuable mathematical reasoning.

Two of the interviews with sixth graders working on graphical representations showed how students implement their developing knowledge and reasoning. One student revealed a common struggle when reading a graph without grid lines, and although he was familiar with the use of graphs he could not read the values from this representation. He viewed the paper from different angles, appearing to try to sight along the line shown on the graph. He finally decided to draw in the grid lines in an effort to figure out what the graph meant, but after twenty minutes of work he gave up in frustration. A second student with weak graph reading skills showed an understanding of how values can be composed proportionally (Kaput & West, 1994). This allowed him to reason about quantities beyond the scale of the graph, and to provide a generalized rule for computing values using the graphed information. Standards for student performance, such as the Colorado Model Content Standards for Mathematics, state that students will have certain mathematical skills by each grade level and yet this preliminary investigation suggests that many students may lack the expected level of mastery. A larger study will be designed to explore the ways that students' developing ideas can be identified, supported, measured, and strengthened by mathematics reform curricula and teaching strategies (Nathan & Koedinger, 2000).

The traditional teaching model of lecture presentations, homework assignments, and written testing is being replaced with a more interactive model where students participate by using their existing knowledge to build new knowledge (Bransford, Brown & Cocking, 2000). The new model depends on communication between teachers and students, where student thinking is understood and where teachers have learned how to listen to students (Stigler & Hiebert, 1999). The findings from this pilot study suggest that further research on how students think about mathematics would provide important information for teachers. Teachers could learn how to listen to student ideas as part of the assessment process and to understand the thinking process of novice learners (Stein, Smith, Hennigsen, & Silver, 2000). This information would help teachers discover effective ways to implement reform mathematics programs and become successful constructivist teachers.

References

Bransford, J. D., Brown, A. L., & Cocking, R. R. (Eds.). (2000). *How people learn: Brain, mind, experience, and school.* Washington, D.C.: National Academy Press.

Ericsson, K. A., & Simon, H. A., (1993). *Protocol analysis.* Cambridge, MA: The MIT Press.

Kaput, J., & West, M. (1994). Missing value proportional reasoning problems: Factors affecting informal reasoning pattersn. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp 235-287). Albany, NY: State University of New York Press.

Nathan, M. J. & Koedinger, K. R. (2000). Moving beyond teachers' intuitive beliefs about algebra learning. *Mathematics Teacher, 93,* 218-223.

Nathan, M. J., Stephens, A. C., Masarik, D. K., Alibali, M. W., & Koedinger, K. R. (October, 2002). Representational fluency in middle school: A classroom based study. In Mewborn , D., Sztajn, P., White, D., Wiegel, H., Bryant, R., & Nooney, K. (Eds.), *Proceedings of the twenty-fourth annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education*. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26,* 114-145.

Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction.* New York: Teachers College Press.

Stigler, J. W. & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: The Free Press.

THE COGNITIVE DEVELOPMENT OF STUDENTS FROM 9TH GRADE TO COLLEGE IN THE LEARNING OF LINEAR AND QUADRATIC FUNCTIONS

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The function concept plays a central and fundamental role in modern mathematics, (Dreyfus & Eisenberg, 1983; Eisenberg, 1991; Leinhardt, Zaslavsky & Stein, 1990) permeating virtually all the areas of the subject (Selden & Selden, 1992). Understanding the concept of function is not trivial for many students, whether they are in high school or are college undergraduates. They start learning about linear and quadratic functions from middle grades through college. However, despite all those years studying functions, the difficulties solving linear and quadratic functions persist. (Dubinsky, 1991; Even, 1990; Kieran, 1993; Moschokovich, Schoenfeld, & Arcavi, 1993; Tall, Thomas, Davis, Gray & Simpson, 2000). Research is needed in an effort to understand how the function concept develops through the learning sequence from school mathematics to college mathematics.

In this study, I will examine the cognitive development of students from 9th grade to college in the learning of linear and quadratic functions in terms of an operational-structural model of learning. Toward this aim, I will use a three-phase model of conceptual development. The three-phase model was developed by Sfard (1991) in an effort to understand the students' mathematical concept formation. The three phases in the evolution of the operational-structural continuum are: interiorization, condensation, and reification. This three-phase model of concept formation is hierarchical: one stage cannot be reached before all the former steps are taken. That is, what appears to be a process at one level is transformed into an abstract object at a higher level to become a building block of more advanced mathematical constructs (Sfard, 1993).

I will translate Sfard's Three-Phase Model of concept formation to my inquiry topic of how students change their conception from a mathematical process to a mathematical object while learning the concept of function. I will consider the three stages as follows:

Interiorization occurs when the learner is capable of dealing with operational processes on symbols, for example, when the student uses the idea of variables in order to manipulate a formula and find values. In this stage, processes can be executed without necessarily running through all of the specific steps. *Condensation* occurs when the student has developed the ability to use mapping as a whole, without looking into its specific values. Eventually, he or she can investigate functions, draw their graphs, and combine couples of functions. *Reification* occurs when the student is able to handle functions as objects. For example, when the "unknowns" or "variables" are functions and the student has the ability to talk about general properties of different processes performed on functions as integration processes.

This study hypothesizes that students of upper academic levels, who have been working longer with the function concept, move more in the evolution of the process-object continuum toward the reification stage than students at lower academic levels.

The participants were students in the early tenth grade who were planning to enroll in the school of engineering, students in the early first semester at the school of engineering who have just finished high school, students in the fourth semester at school of engineering, and students in the ninth and last semester of the school of engineering. All the 150 participants were attending a university in a northeastern state of Mexico. A research instrument was designed to measure the students' development in the understanding of the function concept for a wide range of academic levels. All the problems I used involved different kinds of representation and their translations as a tool to look at the students' understanding of the linear and quadratic function in light of Sfard's Model. A rubric for each of the exam items according to Sfard's Three-Phase Model and a problem solution guide for each question were constructed. In addition to the quantitative component involving the participants, this study has also a qualitative component involving follow-up oral interviews with a total of 24 students, 6 students from each of the 4 academic levels.

The results obtained for the linear and quadratic function were similar. For both, the linear and quadratic function, there were significant differences regarding the student`s stages of conception among the high school students in their tenth grade and the undergraduate students in their first, fourth, and last semester at the school of Engineering. No significant differences were found among the undergraduate students in their first, fourth, and last semester at the School of Engineering. Changes observed in the students` understanding of the notion of linear and quadratic functions were observed as they progress through the grades, but these changes go more on the direction of the operational conception.

References

Dreyfus, T., & Eisenberg, T. (1983). The function concept in college students: Linearity, smoothness and periodicity. *Focus on Learning Problems in Mathematics, 5.* (Summer-Fall Edition), 119-132.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Netherlands: Kluwer Academic Publishers.

Eisenberg, T. (1991). Functions and associated learning difficulties. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 140-152). Netherlands: Kluwer Academic Publishers.

Even, R. (1990). Subject matter knowledge for teaching and the case of functions. *Educational Studies in Mathematics*, 21, 521-544.

Kieran, C. (1993). Functions, graphing, and technology: Integrating research on learning and instruction. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions*. (pp. 189-237). Hillsdale, NJ: Erlbalum.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of education research, 60*, (1), 1-64.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 69-100). Hillsdale, NJ: Erlbaum.

Selden, A., & Selden, J. (1992). Research perspectives on conceptions of function. In G. Harel, & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology MAA Notes, 25* (pp. 1-16). Washington, DC: Mathematical Association of America.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics, 22* (1), 1-36.

Tall, D., Thomas, M., Davis, G., Gray, E., & Simpson, A. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior, 18,* (2) 223-241.

AN UNDERGRADUATE STUDENT'S UNDERSTANDING OF ALGEBRA: A NUMERICAL APPROACH

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Introduction

Some researchers (Booth, 1984; Linchevski & Livneh, 1999) believe that students' difficulties in algebra stem from their lack of understanding of arithmetic. Several methods have been suggested to help students construct equations as well as several processes that lead students to make mistakes.

Background

It is suggested that teachers start with students' knowledge of arithmetic (Van Amerom, 2003), and one purpose of using a numerical approach is to help students make the transition from arithmetic to algebra. Rubio (1990) used numerical explorations in order to analyze a problem and reach a solution. In this study, a numerical approach, which is developed as a result of reasoning about the problem prior to attempting to construct an equation, was initially used in order to construct a meaningful equation.

Methods

The participant in the study, Eric, was a senior undergraduate student in physics. The highest level of mathematics he had taken was engineering mathematics. In this study, three interviews were conducted in 9 months. The second interview was conducted three months after the first interview and the third interview was conducted 6 months after the second interview. During the time between the first and third interviews, Eric had not received any algebra instructions and did not take any mathematics class. He started taking some engineering classes, when he changed his major to civil engineering. The purpose of solving the problems after the first interview and asking the same questions in subsequent interviews was to investigate whether or not the participant would later use the same strategies that led him to make mistakes in the first interview. In addition, by conducting interviews with long intervals, the researcher wanted to determine whether Eric would retain his understanding of numerical approach and would be able to solve the problems in the third interview.

Results

The results of this study revealed that Eric had little understanding of constructing equations for problems, although he took some advanced mathematics classes. Eric employed the syntactic word order matching process (Clement, 1982) to construct the equation for the first question in the first interview. He did not make the same mistake in subsequent interviews and constructed the equations for questions 1 and 3 correctly. However, he had difficulties in interpreting the equation in the second question and considered the coefficients as the values of the variables. In addition, although he successfully solved the equation for B and R in the first and second interviews, he could not determine which number was bigger. It was observed that the numerical approach helped him understand the relationship between the coefficients and the variables. He realized that his way of using the coefficients as the values of the variables would lead to errors. In addition, the numerical approach helped him understand that the bigger the coefficient the smaller the value of the variable. When he was asked the same questions 6 months later, he used this numerical approach to verify and interpret the equations, and with this approach, Eric was able to create meaning for the equations given in the questions. Although the relationship between students' knowledge of numbers and letters is too complicated to be described by any simple models (Wong, 1997), the results of this study indicate that numerical approach could be helpful for students to construct equations as well as to make inferences from the equations.

Although further study needs to de conducted, the results of this study have shown two things. Eric, who had difficulties in constructing equations and making inferences from equations, was not able to retain his understanding of his former, and generally more typical, strategies for solving problems three months after the first interview. Using this numerical approach, employing the least common multiple as a problem solving strategy, Eric was able to retain his understanding and solve problems after six months.

Interview Questions

Question 1: Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors." (Rosnick & Clement, 1980)

Question 2: At Vallapart Motors the equation 5B = 4R describes the relationship, which exists between B, the number of blue cars produced and R, the number of red cars produced. Next to each of the following statements place a T if the statement follows from the equation, an F if the statement contradicts the equation, and a U if there is no certain connection.

a) There are 5 blue cars produced for every 4 red cars

b) The ratio of red to blue cars is five to four.

c) More blue cars are produced than red cars. (Kaput & Sims-Knight, 1983)

Question 3: Write an equation using the variables C and S to represent the following statement: "At Mindy's restaurant, for every four people who order cheesecake, there are five people who order strudel." Let C represent the number of cheesecakes and S represent the number of strudels ordered (Lochhead, J., & Mestre, J.P., 1988).

References

Booth, L. R.(1984). *Algebra: Children's Strategies and Errors*. Windsor, UK: NFER-Nelson.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education*, 13, 16-30.

Kaput, J.J., & Sims-Knight, J.E. (1983). Errors in translations to algebraic equations: Roots and implications. *Focus on Learning Problems in Mathematics*, 5, 63-78.

Lochhead, J., & Mestre, J.P. (1988). From words to algebra: Mending misconceptions. In A.F. Coxford, & A.P. Shulte (Ed.), *The ideas of algebra, K-12* (pp. 127-135). Reston, Virginia: The National Council of Teachers of Mathematics.

Linchevski, L. & Livneh, D. (1999). Structure sense: The relationship between algebraic and numerical contexts. *Educational Studies in Mathematics*, 40, 173-196.

Rosnick, P., & Clement, R. (1980). Learning without understanding: The effect of tutoring strategies on algebra misconceptions. *Journal of Mathematical Behaviour*, 3, 3-27.

Rubio, G. (1990). Algebra word problems: A numerical approach for its resolution: A teaching experiment in the classroom. *Proceedings of the Annual Conference of the International Group for the Psychology of Mathematics Education with the North American Chapter*, Mexico, 2, 138-145.

Van Amerom, B.A. (2003). Focusing on informal strategies when linking arithmetic to early algebra. *Educational Studies in Mathematics*, 54, 63-75.

Wong, M.P. (1997). Numbers versus letters in algebraic manipulation: Which is more difficult? *Proceedings of the 21st Conference for the Psychology of Mathematics Education*, Finland, 4, 285-290.

QUANTITATIVE OPERATIONS AS A BASIS FOR ALGEBRAIC REASONING AND TEACHING PRACTICES

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Abstract: *We present initial analysis of data from a long-term constructivist teaching experiment whose purpose is to understand how middle grade students can build algebraic reasoning out of their evolving quantitative operations and quantitative reasoning. We taught 4 pairs of 6th graders at a rural middle school in Georgia from October 2003 to May 2004 and will work with them through AY 2004-2005. Teaching episodes occurred biweekly, lasted thirty minutes, were videotaped, and included the use of computer software—TIMA: Sticks and Javabars. For this presentation we focus on 2 pairs of students, describing how they built a quantitative scheme that we believe underlies the construction and solution of basic linear equations (i.e., of the form* ax = b.*) In this process we discuss the connection between quantitative schemes and basic linear equations, a hallmark of algebra.*

In a teaching experiment we seek to understand and explain how students operate mathematically and how their ways of operating change in the context of teaching. Teaching practices involve presenting students with problem situations, assessing students' responses as indications of students' current operations, and determining problem situations that might allow students to build other operations. For us, *quantitative operations* include actions performed mentally in building and analyzing relationships between two known quantities or between a known and an unknown quantity; a *quantitative scheme* is an assemblage of quantitative operations; and *quantitative reasoning* is the purposeful functioning of quantitative schemes.

We believe quantitative reasoning becomes *algebraic* as unknowns are operated on explicitly and reasoning is notated. That is, we do not take as given the use of basic linear equations such as $(3/5)x = 7$ to solve, e.g., the problem of finding the length of a candy bar if a 7inch bar is 3/5 of that bar (CB problem). Instead, we are interested in how students construct a

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quantitative scheme to solve such a problem, become aware of operations and relationships implicit in the functioning of such a scheme, and symbolize these operations and relationships. For us, symbolization is written notation that comes to stand for, and is used prior to, enacting quantitative schemes; operations on *notation* stand for, and curtail, carrying out these schemes.

Our conceptual analysis indicated that *reversibility of basic fraction schemes* and the *sharing with distribution scheme* were foundational in students' eventual construction of a quantitative scheme sufficiently powerful to solve the CB problem. *Reversibility of basic fraction schemes* entails creating a whole given a fractional part (e.g., creating a whole candy bar given 3/5 of the bar.) In the CB problem, our students used *reversibility of their fraction scheme* on the 7-inch bar to create the unknown bar. In order to find the number of inches of the unknown bar, the students then *shared with distribution,* which involves being able to "take" any fractional part of any length by taking fractional parts of subdivisions of the length (e.g., 1/3 of 7 inches is 1/3 of each inch 7 times.) Both schemes were key in students' construction and use of the *reversible quantitative scheme* that enabled them to solve the CB problem.

Our students made significant progress in explaining how they operated using natural language; they seemed to be aware of how they operated a posteriori but not a priori. When students began to develop written algebraic notation, we found that their quantitative operations were suppressed rather than symbolized. A key element in this gap between the students' reversible quantitative scheme and their use of notation was that they did not conceive of fractions as operations (Steffe, 2002). Our hypothesis was that using natural language to explain their use of the reversible quantitative scheme a posteriori would engender abstracting a fraction as an ensemble of operations, but the hypothesis was not corroborated.

References

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 20*, 267-307.

USING THE SOLO TAXONOMY TO EVALUATE STUDENT LEARNING OF FUNCTION CONCEPTS IN DEVELOPMENTAL ALGEBRA

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The purpose of this study was to investigate the potential benefits of a multi-representational curriculum on students' understanding of and connections among graphical, tabular, and symbolic representations of algebraic concepts. Qualitative interviews were conducted with sixteen participants from two schools to examine the connections that students were making and their ability to move flexibly among the representations. The interviews were analyzed using Biggs and Collis's SOLO Taxonomy. This research showed that a multi-representational curriculum could be effective in expanding students' web of connected knowledge of algebraic and functional concepts. The SOLO Taxonomy and rubric gives researchers an effective way of measuring connections students make between representational forms.

Many teachers of developmental college mathematics have observed that employing a traditional teaching approach, one that emphasizes symbolic manipulation, has had limited success in student achievement in both two and four-year college situations (Laughbaum, 1992). One teaching approach that has been advocated is to teach developmental mathematics from a functional perspective, utilizing technology to rapidly produce multiple representations of functions (Conference Board of the Mathematical Sciences, 2001). Teachers who have employed these methods believe students gain flexibility in their problem solving and make more connections between representations, though there have been mixed empirical research to support these beliefs (Even 1998; Knuth, 2000).

This research examined connections students made between graphical, tabular, and symbolic representations of functions, their flexibility to solve problems using alternative representations other than symbolic, and the extent to which their development of a well-connected web of knowledge of these representations of basic function concepts was an indicator of conceptual understanding. This research employed the SOLO Taxonomy (Biggs & Collis, 1982) to examine connections that students made between representations and to evaluate their learning outcomes on basic function concepts tested.

Conceptual Framework

The conceptual framework for studying students' understanding was grounded in the theory of constructivism. The majority of researchers in the domain of representations share this theory of how knowledge is developed (Janvier, 1987). Conceptual knowledge is assumed to be constructed by assimilation of new relationships and is stored as a linked network of concepts. Procedural knowledge is gained by practice involving performing a routine in response to a certain stimulus (Galbraith & Haines, 2000). This study specifically addressed students' conceptual understanding of functions by evaluating the strength of their connections between algebraic, graphical, and tabular representations. These connections are fundamental to functional understanding and developing function sense (Eisenburg, 1992). At the end of the study, if it was shown that students could successfully solve problems using alternative representations and make connections between representations, it could be concluded that they have made progress toward concept formation.

A key aspect to this research was determining if students have increased their conceptual knowledge of functions. This could be indicated if students could make connections between representations, implying relational understanding (Skemp, 1987). If students could solve a problem with more than one representation but viewed those representations as separate entities, this would indicate they have expanded their instrumental understanding in that they now had more than one procedure to solve a problem, but not an increased conceptual knowledge because they did not link representations.

In order to assess a students' web of connected knowledge this study employed the SOLO Taxonomy (Biggs & Collis, 1982). SOLO is an acronym for Structure of Observed Learning Outcomes. The SOLO Taxonomy separates answers into five qualitative categories, evaluated according to the stage of concept formation: prestructural, unistructural, multistructural, relational, and extended abstract. Students having no knowledge of the element being assessed manifest prestructural concept formation. A student at this stage might have pieces of isolated information but which make no sense in the problem situation. At the unistructural level, students focus on a single portion of the task, for example, they can solve a problem symbolically but make no association to graphical or tabular representations, nor can they solve a problem with another representation. They may make associations between representations that are obvious but not understand their significance. The multistructural level describes concept development in which students focus on several aspects of the task, but connections between them are not apparent. A student as this level may be able to solve a problem with two or more representations, but will not make any connections between them. The student would not be concerned if they arrived at different solutions for three representations of the same problem as they see them as three separate problems. At the relational level, students see the parts as they relate to the whole that is not only can they solve problems using more than one representation but also they make connections between representations and can use an answer to one representation to check the others. A student at this level would be bothered by answers that did not match. The extended abstract level is reached when students can make connections within the task can also transfer ideas beyond the problem situation.

Methodology

Sixteen semi-structured task based interviews were conducted. Eight students were from a traditionally taught development algebra curriculum that emphasized symbolic manipulation. Although students in this curriculum were taught graphical and tabular representations, they were taught in traditional ways, with graphs primarily being viewed as a answer and a table as a means of producing a graph. Eight students were from a reform-based curriculum where representations were introduced simultaneously throughout the course, with no one representation being preferred over another. These sixteen students were given a test instrument with five basic problems dealing with concepts common to most developmental algebra courses: slope and y-intercept of a linear function, solving a linear system, factoring a polynomial function, finding zeros of a quadratic functions, and function identification. Each problem was presented in symbolic, graphical, and tabular form and appeared on a single page. Students were asked to solve a representation of their choice. They were then asked to solve the problem using the two remaining representations and asked a series of questions from a pre-determined protocol to establish their ability to make connections between representations.

The researcher developed a classification rubric based on the SOLO Taxonomy to measure student learning and concept development and interviews were analyzed based on this rubric. Each student was rated on each problem using the rubric and a taxonomy mapping sheet was employed which synthesized number of representations used, correctness of answers, and number of connections made between representations. The researcher did not know the curriculum of the student until analysis was complete and ratings assigned.

Data Sources/Results

Analysis was summarized by student and problem for: order of representation, type of representation, number of connections made, number of correct answers, and an explanation of misconceptions. This allowed for numerical rating of each student on the SOLO Taxonomy by problem. The levels were number from zero to four, with zero corresponding to a prestructural understanding and four representing extended abstract. It is important to note that the taxonomy assigns levels to students learning outcomes for a particular question, not as a classification for a student in general (Watson, Chick, & Collis, 1988).

Students in the multi-representational curriculum were found to have higher SOLO levels that were mostly above the multistructural level. In contrast, the eight students from the traditional curriculum were mainly in the prestructural through multistructural level across all questions. The students in the traditional curriculum had a median rating of one or less on four of five questions on the test instrument. This corresponds to a median below a unistructural level on four of five basic function problems. This indicates that students had not mastered even the symbolic representation. The students from the multi-representational curriculum ranged from a low of a median of 2.125 which would correspond to slightly above a multistructual level to a high on two questions of a median of 2.875 which corresponds to a low relational level of understanding. This indicates that traditionally taught students were still were at an instrumental understanding, whereas students taught using reform methods were at a relational level of understanding.

Conclusions

Traditionally, algebra has been taught in an equation-to-graph direction with tabular representations being an intermediate step. The reverse path of graph-to-equation gets little focus in a traditional curriculum (Eisenberg, 1992). The path which begins with tabular representations is even less emphasized by teachers and therefore by students. These alternate paths have long been thought of as implicit skills that students will develop though not explicitly taught. For many mathematicians and scientists this is true, however for developmental college algebra students who have typically had two pre-college algebra classes and have not mastered skills of flexibility and reversibility in dealing with multiple representations it is not. This research showed that a teaching approach that utilizes a functional perspective with multiple representations of functions and highlighting connections between those representations could be effective in helping students increase their conceptual knowledge of functional concepts in developmental mathematics.

References

Biggs, J. & Collis, K. (1982). *Evaluating the quality of learning: The SOLO taxonomy.* New York: Academic Press. Conference Board of Mathematical Sciences (2001). *Mathematical education of teachers.* Washington, DC: Author.

Eisenburg, T. (1992). On the development of a sense for functions. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (Vol. 25, pp. 153-174): Mathematical Association of America.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior, 17*(1), 105-121.

Galbraith, P., & Haines, C. (2000). Conceptual mis(understandings) of beginning

undergraduates. *International Journal of Mathematical Education in Science and Technology, 31*(5), 651-678.
Janvier, C. (1987). Translation process in mathematics education. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 27-32). Hillsdale, NJ: Lawrence Erlbaum Associates.
Knuth, E. J. (2000). Student understanding of the cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500-508.
Laughbaum, E. (1992). A time for change in remedial mathematics. *The AMATYC review, 13(2*), 7-10.
Skemp, R. R. (1987). *The psychology of Learning Mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associations.
Watson, J. M., Chick, H. L., & Collis, K. F. (1988). Applying the SOLO taxonomy to errors on area problems. In J. Pegg (Ed.), Mathematical Interfaces, *Proceedings of the 12th Biennial Conference of the Australian Association of Mathematics Teachers*, *Armidale, NSW*, 260281.

LARGE NUMBERS AND GENERALIZATION IN THE ABSENCE OF ALGEBRAIC NOTATION

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This report will detail a case study in which three students attempt to construct a general procedure for answering a certain class of numerical questions. We will trace the development of their thinking through several stages, culminating in a verbal description of such a procedure. We find that their interpretation of the general question changes over time as they ask and answer intermediate questions, and that this contributes to their appreciation of what it would take to answer this general question.

Research context and data

This presentation reports on an undergraduate research project forming part of a larger study based in an after-school program. This program involved 30 sixth-graders in an economically depressed, urban school district with 98% African American and Latino students. The 90-minute sessions followed the format used in previous studies (e.g. Maher & Martino 2000; Steencken & Maher 2003). That is, a researcher posed a question, students thought about this question in small groups, researchers used strategic questioning to elicit justification for possible answers and students presented their findings at an overhead projector. Throughout, four cameras record the proceedings.

Initial sessions focused on fractions. Each student was supplied with a set of Cuisenaire rods in which each length of rod had a different color. They worked with individual rods and with "trains" of rods. A typical example of the problems posed was:

> "Suppose we give the orange rod the number name '1'. What number name do we give the yellow rod?" (where an orange rod is two times as long as a yellow rod)

During the two sessions considered here, such questions were extended by asking students to make a train of one red rod and one orange rod (which is five times as long as the red rod), and to call it a *rorange* rod. Students were then asked the question above with rorange in place of yellow. Next, the number of rorange rods in the train was then progressively increased. Finally, students were asked questions such as:

> "Suppose a friend is given a train consisting of an unknown number of orange rods, and that train is given the number name '1'. How does your friend find the number name for red?"

The aim was to elicit from the students a general procedure for answering this type of question.

Analysis

This presentation will report on analysis of one group in which three students worked with a researcher. During the session, the researcher posed the question regarding a general train of rorange rods several times. Each time the question was restated, the students responded to it differently, moving through stages of interpretation that led to two of the students making good attempts at an answer. The focus of the analysis was on the students' capacity to progressively develop a deeper understanding of what would constitute a satisfactory response. In particular, it focused on:

- The role of progressively larger and eventually unspecified numbers in promoting generalization (following Zazkis, 2001).
- Students' of deictic terms such as "this" and "it" (*cf.* Radford, 2002) to accurately describe the application of a procedure to a general object prior to experience with algebraic notation.

Report

The presentation will describe this analysis as it pertains to the students' progress through the following stages in responding to re-statements of the general question:

1. Students ignore the general question and answer as though about a specific train.
2. Students comprehend the generality of the question and argue about whether or not it can be answered.
3. Students suggest cheating and/or thinking about the "longest" train.
4. Students suggest posing specific-train problems with number name '1' given to progressively longer trains of rorange rods.
5. Students state procedure answering the general question, but often revert to illustrations using specific numbers of rorange rods.

The presentation will focus particularly on the fact that on a number of occasions when the same problem was restated, students treated the problem as if it were new. Only at the end of the session did one student note that the question just asked was identical to that asked before. It seems that, even without the assistance of algebraic notation, experience in thinking about larger numbers of rods helped these students to move toward a statement of a general procedure and an appreciation of the power of such a procedure.

References

Maher, C.A. & Martino, A.M., (2000), "Teacher questioning to promote justification and generalization in mathematics: what research has taught us", *Journal of Mathematical Behavior,* 18(1), 53-78.

Radford, L., (2002), "The seen, the spoken and the written: a semiotic approach to the problem of objectification of mathematical knowledge", *For the Learning of Mathematics,* 22(2), 14-23.

Steencken, E.P. & Maher, C.A., (2003), "Tracing fourth graders' learning of fractions: early episodes from a year-long teaching experiment", *The Journal of Mathematical Behavior,* 22(2), 113-132.

Zazkis, R., (2001), "From Arithmetic to Algebra via Big Numbers", *Proceedings of the 12th ICMI Study Conference "The Future of the Teaching and Learning of Algebra",* Melbourne, Australia, 676-681.

A MICROANALYSIS OF PLANNING AND IMPLEMENTING AN INTRODUCTORY LESSON ON LINEAR FUNCTIONS

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This study examined a seventh grade mathematics lesson in which students were developing understandings for linear functions. The purpose was to analyze what and how the teacher's considerations, decisions, and actions contributed to and limited students' conceptual learning in planning and implementing a lesson. The study reported here was part of a larger study in which I (a university mathematics educator) collaborated with a middle school mathematics teacher, Ms. Lerenz (a pseudonym), who had been teaching for nine years. We conducted a teaching experiment throughout a school year. I documented the nature of our work and Lerenz' practice while supporting Lerenz in examining her practice. Our collaboration revolved around the teaching cycle of planning, teaching, and reflecting. This manuscript reports on a retrospective analysis I conducted of a lesson from the end of the year. Data sources consisted of fieldnotes for all activities, audio tapes for meetings, video tapes of classroom observations, archived email correspondences, lesson plans and student work. Initially, I analyzed data by analytic induction (Bogdan & Biklen, 1992) using open coding to search for patterns of similarities and difference in lesson planning and implementation. After preliminary analysis, I selected a lesson for microanalysis using text indexing (Miles & Huberman, 1994) that was representative of Lerenz' practice at the end of the year. To analyze lesson planning and implementation while focusing on each student's individual mathematical activity, I used Simon & Tzur (2004) steps for designing and analyzing lessons. These steps include: (1) understanding students' current conceptions, (2) stating an instructional goal, (3) identifying an activity sequence that is part of the students' current understandings (Simon et al. consider an activity to mean a set of actions a student uses – not synonymous with a task), and (4) designing or selecting a task. To analyze social interactions between and among students and teacher, I used Brendefeur and Frykolm's (2000) four hierarchical categories of classroom communication. These categories range from unidirectional (teacher talking at students) to instructive (interactions that lead to modifications in students' understandings and instructional sequences).

The study focused on a 3-day lesson in May from one of Lerenz' mathematics classes. The students were predominantly Hispanic, with Spanish as their first language. Class size fluctuated, ranging from 26 to 20 students, due to students from migrant families leaving and returning to school. During planning meetings, Lerenz stated that she wanted a task that introduced linear relationships. She perceived that the textbooks's chapter on this topic focused on input and output of values without providing a context for students' to make meaning for change processes. Lerenz aimed to approach linear functions as a family of functions (Step 2 in Simon et al.'s [2004] framework). After we reviewed several resources, Lerenz selected the "Stacking Cups" task (Friel, Rachlin, & Doyle, 2001, pp. 41 – 43) (Step 4). In this task students develop and explain a prediction for the height of a stack of 50 cups, given a set of 4 cups. In examining the task, Lerenz believed that the hands-on approach of stacking and measuring cups (Step 3) would help her students experience the change process and provide an opportunity to move students from what Sfard (1994) referred to as a process-oriented conception of linear functions to a structural conception (Step 1 and Step 2). During the lesson while interacting with a group of students, Lerenz realized that using four cups to stack was not sufficient for students to discover a rule. Students needed to repeat a process of conjecture, stack, measure, and check the conjecture. At this point, she adjusted the activity to include up to 20 additional cups.

Although Lerenz identified stacking cups as an activity to help students gain a sense of linear relationships, prior to teaching she had not anticipated the need for iterating the planned activity of stacking four cups to achieve this understanding (Simon & Tzur's [2004] Step 3). In recognizing and making this adjustment, the lesson turned from students not discovering the relationship to students "seeing" a rule emerge. While Simon and Tzur's steps did not occur in a distinct order (Simon and Tzur indeed argued that some steps may occur reflexively), they were each important in the success of the lesson. In regard to social interactions, Lerenz' questioning served as important scaffolding for her students as they made sense of the relationship between cup height and stack height and the need of a constant in the function. Moreover, the students' communication of their understandings to Lerenz shaped her subsequent decision to change the activity of the task. In doing so, she achieved Brendefeur and Frykolm's (2000) highest level of communication with her students. Thus, communication both shaped her actions and contributed to her students' learning.

To the reader it may seem that Lerenz' adjustment to extend/iterate the activity was an obvious choice for fostering learning. However, this adjustment was an intuitive decision in the moment, and even during initial reflections, Lerenz attributed the success of the lesson to surface features of the task (e.g., involving "hands-on" work and a real world connection). These reflections did not serve to parse out the specific actions and interactions that drove the lesson to higher level learning. Simon and Tzur's (2004) framework helped to identify, as a critical instructional move, the moment of the lesson that Lerenz recognized the need for iterating the students' activity of stacking and measuring cups. This move was an essential element for learning for these students. Indeed, if Lerenz had focused on Step 3 more deliberately in planning the lesson, she might have considered that four cups were not sufficient for students to identify the change pattern. Additionally, Brendefeur & Frykolm's (2000) framework revealed the nature and role of questioning in facilitating students' sense making. The use of both frameworks provided a more complete understanding of successful and limiting elements of the lesson to better inform future practice. This kind of microanalysis is critical both for mathematics educators to clarify and understand teaching and learning issues for specific topics and for teachers to have an instrument for designing and analyzing lessons.

References

Bogdan, R., & Biklen, S. (1992). *Qualitative research for education: An introduction to theory and methods.* Boston, MA: Allyn and Bacon.

Brendefur, J. & Frykholm, J. (2000). Promoting mathematical communication in the classroom: Two preservice teachers' conceptions and practices. *Journal of Mathematics Teacher Education,* 3 (2), 125-153.

Friel, S., Rachlin, S., & Doyle, D. (2001). *Navigating through algebra in grades 6 – 8.* Reston, VA: National Council of Teachers of Mathematics.

Miles, M., & Huberman, A. (1994). *Qualitative data analysis.* Thousand Oaks, CA: Sage.

Sfard, A., & Linchevski, L. (1994). The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics,* 26, 191-228.

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaborating of the Hypothetical Learning Trajectory. *Mathematical Thinking and Learning,* 6 (2), 91-104.

CONFRONTING TEACHERS' BELIEFS ABOUT STUDENTS' ALGEBRA DEVELOPMENT: AN APPROACH FOR PROFESSIONAL DEVELOPMENT

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Teachers' knowledge and beliefs are powerful mediators of decision-making and action (e.g., Sherin, 2002). For example, teachers generally report that their perceptions of students are the most important factors in instructional planning, and teachers consider student ability to be the characteristic that has greatest influence on their planning decisions (e.g., Ball, 1988; Borko & Shavelson, 1990; Carpenter et al., 1989; Peterson et al., 1989; Thompson, 1984). Yet teachers' beliefs and expectations of students' behaviors are not always accurate. This paper focuses on a method for influencing the beliefs of algebra teachers so that they are more closely aligned with actual student performance data. We report on the initial and changing views exhibited by teachers about the nature of algebraic development and instruction, and discuss why the method has promise for affecting teachers' knowledge of students more generally.

Few studies (e.g., Carpenter et al., 1989; Peterson et al., 1989) have looked at the relation between teachers' beliefs about student reasoning, and students' actual problem-solving performance. Of these, most have focused on elementary level mathematics. Lacking is a similar emphasis at the secondary level. In one such study, Nathan and Koedinger (2000a) found that high school algebra teachers (76%; N=67) inaccurately predicted that symbolic equations would be easiest for algebra students to solve. Student data showed the opposite (Koedinger & Nathan, 2004): Algebraic equations were most difficult for students, even though they were carefully matched to context-based story problems and context-lacking word-equation problems (e.g., If I start with some number, multiply it by 6 and then add 66, I get 81.9. What number did I start with?). Teachers justified their response by arguing that symbolic reasoning was a necessary precursor to leaning to solve story problems, and that symbolic representations were more "familiar," "straightforward," and mathematically "pure." This symbol precedence view (SPV) has since been found to be widely held among practicing and pre-service mathematics teachers, while a comparable view more generally favoring formalisms over informal presentations of material has been seen in a number of other subject areas (Nathan & Petrosino, 2003).

Beliefs about student development of algebraic reasoning that are at odds with student performance may lead to sub-optimal instruction. However, any attempts to change teachers' views need to explicitly address teachers' existing beliefs in the context of student reasoning. This tenet guided the current intervention. Five high school mathematics teachers from an urban, ethnically and socio-economically diverse school district participated in a professional development session aimed at influencing deep-seated and widespread misconceptions about students' development of algebraic reasoning. As with prior teacher cohorts, participants of this study initially exhibited a SPV of algebra development. The composite group ranking was strongly correlated with an idealized SPV ranking that predicted symbolic problem solving was easier than verbal problem solving, Pearson's $r = 0.9$. To corroborate this, the ranking of each participant was correlated with the idealized SPV rank. The correlations range from 0.49 to 0.9 with a mean of 0.7. This distribution of correlation measures yields a 95% confidence interval that ranges from 0.55 to 0.84 (SD = .17). The constant comparative method was employed to establish a grounded coding system for teachers' justifications. The resulting categories revealed that symbolic representations were favored because they were considered more basic and familiar to students. Three categories covered 80% of teacher justifications: Students have greater familiarity and skill with symbols than words; Verbal problems are solved via translation to symbolic representations; and Arithmetic word problems tell you exactly what to do.

Following this belief-elicitation activity, the professional development team gave a 30-min presentation showing the ranking data for other mathematics teachers and researchers typically followed the SPV with similar rationale given favoring the development and use of symbolic reasoning before verbal applications. This helped to establish for participants in the current session that they had views similar to the mathematics education community at large. Student work was then presented showing symbol equation use, common conceptual errors in symbolic representation and manipulation (along with frequency data; slip type errors were ignored). Participants also saw student uses of alternative solution strategies that led to the verbal advantage (Koedinger & Nathan, 2004), along with frequency, error patterns, and data on likelihood of success when applied to symbolic and verbal algebra problems. Teachers then were asked to apply a general rubric for evaluating students' written work to four example solutions that captured the major features of the student performance data seen in previous studies. This was intended to enhance the development of teachers' "algebraic eyes and ears" (Blanton & Kaput, 2002) by focusing teachers on the problem-solving processes and representations that could be inferred from students' written work.

Teachers participated in a new ranking task mailed to them one month later. The group level analyses showed a relatively low correlation with the idealized SPV ranking, Pearson's $r = 0.15$. The individual level analyses showed correlations between 0.6 and -0.09 with a mean correlation of 0.13 ($SD = .27$). The 95% confidence interval includes 0. Both analyses led to similar conclusions—teachers' post-intervention views show little resemblance with SPV. There was a clear change in their thinking about students' algebraic reasoning. Additionally, teachers' justifications showed greater awareness of the difficulties students have with formal algebraic notation as well as the facilitating effects of verbal representations for solving algebra-level problems. Four categories covered 86% of teachers' justifications: Symbol manipulation is difficult or error prone; Context helps in problem solving; Arithmetic word problems tell you exactly what to do; and Arithmetic skill strictly precedes algebraic reasoning.

Teachers' views of student development must be open to examination. And in a manner consistent with constructivist pedagogy at the student level, teacher learning must be rooted in teachers' prior knowledge and beliefs. The ranking task assesses one aspect of pedagogical content knowledge for teaching algebra; namely, teachers' expectations about the relative difficulty students experience for problems presented in more or less formal representations, while controlling for the underlying quantitative structure. To enhance the validity of this study, teachers evaluated the relative difficulty of specific mathematics problems, rather than making statements about student algebraic reasoning in the abstract. If teachers misperceive the relative difficulties of equations, they may introduce them prematurely, or inappropriately withhold story problems from a student. Improving teachers' expectations of problem difficulty is important because these beliefs affect instructional planning and assessment design. This seems especially important given national and international data (e.g., NAEP, TIMSS) showing poor student performance in secondary mathematics topics, and as US schools explore how to teach algebra in the middle and elementary grades. The effectiveness of a method of professional development as measured by changes in the expectations of a small sample of urban high school teachers can now be explored as part of a larger professional development program aimed at enhancing algebra teachers' pedagogical content knowledge.

REFERENCES

Ball, D. L. (1988). Unlearning to teach mathematics. For the Learning of Mathematics, 8, 40-48.

Blanton, M. L., & Kaput, J. J. (2003). Developing elementary teachers' "algebra eyes and ears". Teaching Children Mathematics, 10(2), 70-77.

Borko, H., & Shavelson, R. (1990). Teacher decision making. In B. F. Jones & L. Idol (Eds.), Dimensions of Thinking and Cognitive Instruction. 311-346.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P. & Loef, M. (1989). Using knowledge of children's mathematical thinking in classroom teaching: An experimental study. American Educational Research Journal, 26, 499-531.

Koedinger, K. R., & Nathan, M. J. (2004). The real story behind story problems: Effects of representation on quantitative reasoning. Journal of the Learning Sciences, 13(2), 129-164.

Nathan, M. J., & Koedinger, K. R. (2000a). Teachers' and researchers' beliefs about the development of algebraic reasoning. Journal for Research in Mathematics Education, 31, 168-190.

Nathan, M. J., & Koedinger, K. R. (2000b). An investigation of teachers' beliefs of students' algebra development. Cognition and Instruction, 18(2), 207-235.

Nathan, M. J. & Petrosino, A. J. (2004). Expert blind spot among preservice teachers. American Educational Research Journal, 40(4), 905-928.

Peterson, P. L., Carpenter, T. C., and Fennema, E. (1989). Teachers' knowledge of students' knowledge of mathematics problem solving: Correlational and case study analyses. Journal of Educational Psychology, 81, 558-569.

Sherin, M. G. (2002). When Teaching Becomes Learning, Cognition and Instruction, 20(2), 119–150.

Thompson, A. (1984). The relationship of teachers' conceptions of mathematics teaching to instructional practice. Educational Studies in Mathematics, 15, 105-127.

FIFTEEN YEARS LATER: MULTIPLE REPRESENTATIONS IN UPPER LEVEL HIGH SCHOOL MATHEMATICS

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Fifteen years have elapsed since the National Council of Teachers of Mathematics (1989) advocated the use of multiple representations of functions. The release of NCTM's principles and Standards for School Mathematics (2000) reinforced the notion of multiple representations of functions as an essential element of teaching and assessing mathematics. Research, however, suggests that students continue to rely on algebraic representations as a primary method for solving a particular task, often at the expense of a more efficient approach utilizing a graph (Knuth, 2000) or a table.

The purpose of my investigation addressed the question 'If there is a direct solution between representations, do upper level mathematics students (above Algebra II) find it necessary to use an intermediate algebraic equation to create a table or a graph?' My inquiry focused on perpendicular lines. Each task involved a direct solution between two representations chosen from algebraic, graphical, and tabular to emphasize the bidirectional link between representations (i.e. graphical to tabular vs. tabular to graphical). Students were required to write an equation, graph, or complete a table for a perpendicular line based on a given line and point. The format of the given information addressed algebraic, graphical, and tabular representations, as did the required responses.

Using written teacher responses and surveys coupled with videotaped student interviews, I found that not only did students demonstrate a strong dependence on algebraic representations, but also teachers showed a strong initial inclination toward algebraic solution methods. Subsequent interventions with students showed some progress in using the strengths of each representation (graphical or tabular), and follow-up interviews documented less reliance on algebraic equations or formulas. My findings support the fact that, despite clear and consistent standards for instruction and assessment, much work still needs to be done in the area of multiple representations in high school mathematics. Neither the areas of individual teachers' practices nor its impact on student performance were addressed in my investigation, but this may serve as a later topic of investigation.

The format of my poster includes written documentation of teacher and student responses, video documentation of student interviews, and a power point presentation highlighting optimal solutions to tasks, as well as analysis of student solutions.

References

Knuth, E. (2000). Understanding connections between equations and graphs. *The Mathematics Teacher*, 93(1), 48-53.

National Council of Teachers of Mathematics (NCTM). (1989) *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM). (2000) *Principles and standards for school mathematics.* Reston, VA: Author.

LEARNING DISABILITY CHILDREN MAKE SENSE OF THE EQUAL SIGN: MOVING FROM CONCRETE TO ABSTRACTIONS AND GENERALIZATIONS

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Past research has indicated that normally developing children misconceive the equals sign as a signal to perform a computation and record the answer, rather than as a symbol that denotes a relationship between two mathematical expressions that hold the same value. For the majority of elementary children this misconception about the function of the equals sign becomes firmly entrenched early on (Falkner, Levi & Carpenter, 1999). However, research has also shown that children as young as 4 seem to have no problem determining whether groups of concrete objects are equivalent (Baroody, 2000).

Children's misconceptions may be based on repeated experiences with only one mathematical grammatical form of equation (Falkner, Levi & Carpenter, 1999). This mathematical syntactical constraint directly relates to a semantic constraint, i.e. children are never explicitly taught the symbolic meaning of the equals sign which leads them to formulate their own interpretations. The fact that this is such a universal interpretation for primary children lends credence to the hypothesis that this "misconception" is in fact a very sophisticated interpretation springing from children's active construction of knowledge both of mathematical syntax (the order in which the symbols make sense) and English language semantics (that equals means "makes" or "results in").

The purpose of this project was two-fold: to determine whether children with learning disabilities also have an innate sense of equality in terms of groups of concrete objects, and to investigate the effects of an intervention designed by the author to explicitly teach the meaning of the equals sign by using specific language and linking concrete manipulations to symbolic representations.

Five grade three students with learning disabilities participated in this study. Prior to the intervention, baseline data for students' understanding of the equals sign and their ability to solve algebraic equations was collected. Next the students engaged in activities designed to access conceptual knowledge of equivalence in terms of concrete objects, and these activities were then linked to symbolic representations of equations. Students were also introduced to and encouraged to generate different forms of equations, and solve for unknown quantities. Immediately following instruction the students completed a brief assessment to discover whether they had progressed in their understanding of the equals sign.

Preliminary analyses reveal that these learning disabled students conceived a more accurate understanding of the equals sign, and demonstrated their ability to abstract and generalize that understanding by generating and solving different forms of equations, and generating and solving equations with unknown quantities. A repeat post-test is planned for the end of the school year to ascertain whether these changes in the children's understanding as a result of the intervention are sustained.

This poster session will describe the teaching intervention in detail, and preliminary measures and post-intervention results of children's understanding of the equals sign. Insights gained from the analyses of the children's developing understanding of the symmetrical properties of equations and implications for instruction with respect to the teaching and learning of the equals sign and equations for students with learning disabilities will also be presented.

References

Baroody, A.J. (2000). Does mathematics instruction for three- to five-year-olds really make sense? *Young Children,* (2000).

Carpenter, T.P. & Levi, L. (2000). Developing Conceptions of Algebraic Reasoning in the Primary Grades. Wisconsin University, National Center for Improving Student Learning and Achievement in Mathematics and Science.

Falkner, K.P., Levi, L., & Carpenter, T.P. (1999). Children's understanding of equality: a foundation for algebra. *Teaching Children Mathematics,* 6(4).

CHILDREN'S EMERGING INFORMAL UNDERSTANDING OF MULTIPLICATION IN THE CONTEXT OF GEOMETRIC PATTERNS AND FUNCTION MACHINES

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This study investigates the evolution of primary students' understanding of multiplicative structures (Vergnaud, 1988). Earlier this year, three grade two classes and one control group participated in a teaching intervention for patterning and early algebra. The instructional sequence, designed to foster an integration of students' understanding of geometric and numeric patterns, followed a specific sequence. To begin, students explored geometric growing patterns using pattern blocks. The children's natural way of predicting what would come next was based on counting-up strategy (a recursive strategy; please see Rubenstein, 2002). In order to help the students focus on the mathematical function that held across the numbers, we introduced T-charts and modeled how students' patterns could be recorded. Next we provided a colourful function machine (i.e. Willoughby, 1997) and involved children in activities in which they created a rule. They then used T-charts to record unordered series of inputs and outputs (Carraher & Earnest, 2003) that they used as challenges for their classmates. In the final instructional phase, the students worked with activities in which they used a rule as a basis for building geometric patterns.

It is important to note that neither prior to the instruction nor at any time during the experimental intervention were the children taught any formal lessons in multiplication. However, the rules that they uncovered for geometric patterns and the rules that they invented for functions often corresponded to multiplicative functions. Data sources for this study include transcripts of audio and videotaped lessons, images of the students' work, the results of pre and post-tests, and data from the Knowledge Forum® databases in which students pose questions, contribute theories and debate ideas.

References

Anghileri, J., & Johnson, D. C. (1988). Arithmetic operations on whole numbers: Multiplication and Division. In T. R. Post (Ed), *Teaching mathematics in grades K-8* (pp. 146-189). Boston, MA: Allyn and Bacon.

Carraher, D. & Earnest, D. (2003). Guess my rule revisited. In N. Pateman, B.Dougherty, & J. Zilliox (Eds.), *Proceedings of the Twenty-seventh International Conference for the Psychology of Mathematics Education* (vol. 2, pp. 173-180). Honolulu, Hawaii: University of Hawaii.

Fosnot, C.T., & Dolk, M. (2001). Constructing multiplication and division. Young mathematicians at work. Heinemann, Portsmouth NH

Rubenstein, R. (2002). Building Explicit and Recursive Forms of Patterns with the Function Game. *Teaching Mathematics in the Middle School* (April 2002):426-452

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert, & M. Behr (ED.), *Number concepts and operations in the middle grades* (pp. 141-161). Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Lawrence Erlbaum.

Willoughby, S. (1997). "Functions from Kindergarten through Sixth Grade." *Teaching Children Mathematics* 3 (February): 314–18.

REVERSIBILITY OF CHILDREN'S THINKING ABOUT THE RELATIONSHIP BETWEEN NUMERIC AND GEOMETRIC PATTERNS

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In the past, teachers of early elementary mathematics programs have placed emphasis on repeating geometric patterns with limited efforts to help children understand relationships between geometric and numeric patterns. Recently the NCTM (2000) has placed emphasis on the need for early elementary students to recognize and extend both numeric and geometric patterns and to translate from one representation to another. In addition, the 1997 Ontario Curriculum documents have specified the ability to identify relationships between and among patterns as an overall expectation for the early elementary grades (1-4).

In a current project involving three experimental second grade classrooms (n=56) and one control second grade classroom, (n = 22), students have been exploring both geometric and numerical patterns and the mathematical functions that underlie them. As a part of this project students have participated in a number of learning activities that require them to coordinate their understanding of the relationship between geometric and numeric patterns, an ability that appears to be slow to develop as indicated by recent NAEP materials (Kenney, 1997). Observations of the children's participation so far have revealed an interesting phenomenon in children's developing understanding of numeric rules and how they apply to patterns. The majority of the children have demonstrated an ability to build geometric growing patterns and identify potential rules that could help them to anticipate extension of the pattern to the next position and the n^{th} position. For example, given the first-(3 tiles), second-(6 tiles) and third- (9tiles) positions in a grow by three pattern; children were able to build the fourth and fifth positions and predict what would occur in the tenth and the n^{th} positions. In addition children noticed that, " the number of tiles in each potion is equal to three times the position number". The students have also shown a great interest and capacity to select their own rules and to use them appropriately for function machines (see Willoughby, 1997). For example, students might make up a rule (i.e. the output equals the input number multiplied by 2 + 1) and then apply their rule to unordered series of input numbers and correctly find the corresponding output. In contrast however, the students appear to have difficulty working in the reverse direction: that is to build geometric patterns based on a given functional rule.

In this poster we will present details of the lesson sequence that we employed as well as analyses of students' developing ability to apply their knowledge of functional relationships and general rules to geometric patterns. Specifically, we will focus on the reversibility of these concepts as demonstrated by the children. We are interested in the development of the children's ability to articulate a mathematical rule before building a pattern as well as their ability to see complex numeric patterns in geometric representations. The data for this poster will include examples of student work, analysis of student participation in class discussions, and analysis of student contributions to a collaborative data base, Knowledge Forum®, that allows students to post questions, contribute theories and debate ideas (e.g., Bereiter & Scardamalia, in press; Scardamalia &Bereiter, 2003). Individual student progress will also be compared to their performance on a pre-test interview.

References

Kenney, P. & Silver, E.A. (1997, February). Probing the Foundations of Algebra: Grade-4 Pattern Items in NAEP. *Teaching Children Mathematics.*

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics

Ontario Ministry of Education and Training (1997). *The Ontario Curriculum, Grades 1-8: Mathematics*. Toronto: Queen's Printer for Ontario.

Scardamalia, M., & Bereiter, C. (2003, Fall). Beyond brainstorming: Sustained creative work with ideas. *Education Canada, 43*(4), 4-7 & 44.

Willoughby, S.(1997) "Functions from Kindergarten through Sixth Grade." *Teaching Children Mathematics* 3 (February): 314–18.

MATHEMATICAL KNOWLEDGE BUILDING: STUDENTS COLLABORATE TO IMPROVE IDEAS OF GROWING PATTERNS

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Within a knowledge building approach to understanding, ideas are at the centre of classroom life. Activities emerge from those ideas, which will help students deepen their understanding of the concepts being explored. Within mathematics, the potential exists for the most concrete activities to become rote if students are not given the opportunity to "reflect on their actions." (Clements, p.271) The current emphasis on math journals and the mandate from NCTM (2000) on describing mathematical thinking is consistent with this view.

It is the purpose of this poster to describe the learning environment that evolved in three second grade classrooms where an attempt was made to implement a knowledge building approach to mathematics. It was the goal of the project to provide opportunities for children to actively play with patterns, ask questions and develop deep understandings of early algebra concepts. Through participation in a knowledge building environment, children worked together to think, talk and

write about math, collaboratively creating knowledge. Knowledge Forum®, a computer database, was used to enhance the children's theory building by allowing students to post questions, contribute theories and discuss ideas in the form of notes (Scardamalia & Bereiter 2003). To our knowledge, no guidelines were available and Knowledge Forum® had not been used to explore mathematical concepts in primary classrooms.

Traditionally, teachers focus on repeating geometric patterns in early elementary years. Although numeric patterning might be taught, there is often little relationship drawn between the two types of patterns. Within this study we presented children with patterns that are familiar in their lives as a bridge to understanding more abstract numerical patterns and generalizations. Students were involved in a variety of activities using concrete materials, T-charts and a function machine. Students were challenged to generalize rules within and between concrete and numerical growing patterns. During each of the lessons we took digital photographs of the patterning work that the students did. Our plan was to enter the digital photos into the database; this way student designed patterns became objects of inquiry for the children. Student reflections, which were entered into Knowledge Forum, enhanced the learning process, as well as teacher understanding of student beliefs.

The project was a collaborative initiative between The Institute of Child Study (OISE/UofT) and The School at Columbia University. There are three experimental classrooms (n=57) and one control classroom (n=22). Our emphasis on technology not only allowed for ongoing communication between the researchers through video conferencing but will also enabled the exchange of ideas between the students in the two cities. All classes have some familiarity with Knowledge Forum software.

In this poster, we will present data collected from pre and post test interviews as well as transcripts of videotaped lessons. Particular emphasis will be placed on data in the form of student ideas from the Knowledge Forum database. Growth in students' understanding will be examined within their collaborative contributions to the database as well as in portfolio notes that were entered before and after the study.

References

Clements, D. H., & McMillen, S. (1996). Rethinking “concrete manipulatives.” *Teaching Children Mathematics*, 2, 270-279.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Scardamalia, M., & Bereiter, C. (2003, Fall). Beyond brainstorming: Sustained creative work with ideas. *Education Canada, 43*(4), 4-7 & 44.

MAKING SENSE (OR NOT) OUT OF LINEAR EQUATION SOLVING

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This research describes the development of two students' strategies for solving linear equations. Of particular interest is how students do or do not discover efficient solution methods. Students' problem solving on linear equations has been previously studied (e.g., Matz, 1980; Sleeman, 1984), but typically with a focus on the types of procedural errors that students make. Less emphasis has been placed on students' strategy development, as is the focus of the present work.

The two students, Jack and Brenda, were among 160 students who participated in a weeklong summer "camp" for students entering seventh grade. The students came for one hour each day. Twenty-four of the students (including Jack and Brenda) were randomly selected to be interviewed. On the first day, the students completed a pretest and were then introduced to four different steps that could be used to solve algebraic equations (adding to both sides, multiplying on both sides, distributing, and combining like terms). Students then spent three days working through a series of linear equations. The interviewed students worked individually with a tutor/interviewer who checked their work. On the last day of the project, students completed a posttest.

At the beginning of the project, both Jack and Brenda struggled to solve linear equations. However, by the end, Brenda could consistently solve linear equations using logical, efficient problem solving strategies while Jack consistently failed to solve linear equations and exhibited inefficient, illogical, and inappropriately-executed strategies. Careful examination of Jack and Brenda's work shows very different problem solving behaviors, including differences in how they treated variables, used problem solving steps, and worked with equations. While Brenda exhibited conventional behaviors in each of these categories, Jack was usually unconventional. For example, Jack frequently ignored variables, leaving them out of equations that he was rewriting. Jack was also very different from Brenda in his use of problem solving steps, particularly his preference for dividing both sides of an equation by a constant. Finally, Jack used the equals sign as a way to divide the problem into parts. He would solve each side separately and then add the two sides together. For example, for $5x + 3 = 2x + 6$, Jack wrote $8x = 8x$, then $8x + 8x = 16x$.

When Jack's atypical actions are contrasted with Brenda's traditional strategies, they seem random and irrational. However, close examination reveals patterns in Jack's behaviors, indicating that he was making purposeful, not arbitrary, choices. The regularity of his actions hints that Jack had constructed his own rules for working with linear equations. In fact, Jack's rules may be considered as one alternative way students might approach linear equations.

References

Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Mathematical Behavior, 3*(1), 93-166.

Sleeman, D. H. (1984). An attempt to understand students' understanding of basic algebra. *Cognitive Science, 8*(4), 387-412.

MATHEMATICAL MODELLING THROUGH KNOWLEDGE BUILDING: EXPLORATIONS OF ALGEBRA, NUMERIC FUNCTIONS AND GEOMETRIC PATTERNS IN GRADE 2 CLASSROOMS

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Mathematical modelling has been described as the engagement with authentic real world problems (Palm, 2004) to which there are multiple possible solutions that are expressed mathematically through such representations as equations, diagrammes, charts, graphs, or computer programmes (Lesh & Doerr, 2000). While it has been clearly established that secondary and university level mathematics students can successfully engage in mathematical modelling, the potential for this approach has not yet been fully explored with elementary school children.

Key mathematics understandings embedded in mathematical modelling include algebraic reasoning and a grasp of functions, among others. The notion of what constitutes a mathematical model has recently been debated (ICMI, 2004). Within the framework of Realistic Mathematics Education (RME), Gravmeijer (2004) differentiates mathematical modelling (models of formal mathematical understandings within the discipline) from emergent modelling (local representations of informal mathematical understandings). While the usefulness of separating emergent from mathematical models has been challenged (e.g. Lesh, 2004), even within a theoretical structure that accepts emergent modelling as distinct, it may be argued that degree of sophistication of the model itself is not a criterion for whether or not a representation properly constitutes a mathematical model. Critical features of a mathematical model, then, involve: the abstracted representation of a real world situation expressed in some mathematical form; generalisability of a "rule" or function at play; and transferability to other applications. Clearly within complex authentic situations that call for increasingly sophisticated responses, the models generated will require several cycles of critique and refinement (Blomhøj & Højgaard Jensen, 2004). However, in the early grades, an immediately successful match without many cycles of revision serves to support the mathematics learning of very young children (van den Heuvel-Panhuizen, 2004).

This poster will present results of a study in three Grade 2 classrooms (n = 56) in which students have engaged in mathematical modelling involving algebraic reasoning and use of functions. Students explored both geometric and numeric patterns and the mathematical functions that underlie them. They were asked to create both geometric "growing" patterns using coloured tiles and pattern blocks, and numeric functions using T charts and a "function machine". The students then created algebraic expressions for the rules that would predict what would occur in the next position and in the nth position, for both ordered geometric patterns that have numbered positions in a sequence, and unordered series of input numbers in numeric functions.

This poster will focus on key pedagogical strengths of this mathematical modelling project, which include a relocation of authority in mathematics learning from the teacher and textbook to the learner (London McNab, Moss, Woodruff & Nason, 2004), as well as the opportunity to engage in student led collaborative knowledge building (Scardemalia & Bereiter, 1998). These investigations by the Grade 2 students involved classroom use of Knowledge Forum software through which the children engaged in complete repeated cycles of mathematical modelling, including theorizing, questioning, critiquing, revising and justifying one another's models.

References

Blomhøj, M. and Højgaard Jensen, T. (2004). *What's all the fuss about competencies? Experiences from using a competence perspective on mathematics education to develop the teaching of mathematical modelling.* Plenary presentation: ICMI Study 14: Applications and modelling in mathematics education.

Gravemeijer, K. (2004) *Emergent modelling as a precursor to mathematical modelling.* Conference proceedings: ICMI Study 14: Applications and modelling in mathematics education.

Lesh, R. & Doerr, H.M. (2000). *Symbolizing, communicating and mathematizing: Key components of models and modelling.* In P. Cobb & E. Yackel (Eds.), Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructional design (pp. 361-383). Mahway, NJ: Erlbaum.

London McNab, S., Moss, J., Woodruff, E. & Nason, R. (2004). *Investigating fairness in ranking commonwealth games performance: Collaborative mathematical modelling in a Grade 5/6 classroom.* Conference proceedings: ICMI Study 14: Applications and modelling in mathematics education.

Palm, T. (2004). *Features and impact of the authenticity of applied mathematical school tasks.* Conference proceedings: ICMI Study 14: Applications and modelling in mathematics education.

Scardamalia, M. & Bereiter, C. (1998). *Web Knowledge Forum. User guide.* Santa Cruz, CA: Learning in Motion.

Van den Heuvel-Panhuizen, M. (2004) *Shifts in understanding—the didactical use of models in mathematics education.* Conference proceedings: ICMI Study 14: Applications and modelling in mathematics education.

A COLLABORATIVE TEACHING APPROACH TO ALGEBRAIC REASONING WITH GRADE FOUR STUDENTS: UNDERSTANDING FUNCTIONS THROUGH MATHEMATICAL DISCOURSE

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In recent years there has been a gradual shift in thinking about the teaching and learning of algebra. Rather than delaying algebra until secondary school it has been recommended that algebraic reasoning should be introduced in the elementary years and be integrated throughout the entire mathematics curriculum (Kaput & Blanton, 2003). Literature within the field of early algebraic thinking indicates that although students are competent at continuing patterns, they often encounter difficulty in making far generalizations using rules (Kenney & Silver, 1997). Moreover, it has been noted that students have difficulty explaining their thinking in formal mathematical language. Strategies to overcome these difficulties have become a topic of significant interest since both the National Council of Teachers of Mathematics (2000) and the Ontario Mathematics Curriculum (1998) mandate that children explain their thinking in mathematical language.

The purpose of this study was to investigate the progress of students in two grade 4 classrooms in their ability to make far generalizations and their ability to express those understandings through oral language, using diagrams, and in more conventional mathematical notation. Our research involved creating a unit of instruction that put rule-making at the center and used it as a means of helping students describe and understand numeric and geometric growing patterns.

We also investigated the use of technology to support the students' ability to communicate their mathematical thinking. This was done by electronically posting algebraic word problems using *Knowledge Forum*®computer software. *Knowledge Forum*®is a database where student can enter their own theories and read and challenge ideas written by their peers. This software was designed with the goal of building knowledge as a community and has successfully achieved this goal in the realm of science (e.g. Scardamelia & Bereiter, 1996; 1999).

The participants included 22 grade four students from a small independent school and 18 students from one classroom in a large urban public school. The classes were selected because they already used *Knowledge Forum*® as a component of their regular classroom program. Our goal in connecting the classrooms electronically was to provide an authentic framework that necessitated the occurrence of mathematical discourse.

In this poster we will present statistical data comparing the pre- and post- test measures, which included items from the National Assessment of Educational Progress. We will also display anecdotal notes and artifacts from the teaching lessons that will document the students' growth in their thinking over the course of the unit. The artifacts will include work samples, digital photographs of growing patterns, and particular strings of notes from the joint *Knowledge Forum* ® database to highlight the potential of deep, sustained mathematics discussions and investigations.

References

Kaput, J and Blanton, M, (2003). Developing elementary teachers' "algebra eyes and ears" *Teaching Children Mathematics, 10*(2), 70-77.

Kenny, P.A., & Silver, E.A. (1997). Probing the foundations of algebra: Grade 4 pattern items in NAEP. *Teaching Children Mathematics* 3 (February 1997): 268-74. National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Ontario Ministry of Education and Training (1997). *The Ontario Curriculum, Grades 1-8: Mathematics*. Toronto: Queen's Printer for Ontario.

Scardamalia, M., & Bereiter, C. (1996). Adaptation and understanding: A case for new cultures of schooling. In S. Vosniadou, E. De Corte, R. Glaser, & H. Mandel (Eds.), *International perspectives on the psychological foundations of technology-based learning environments* (pp. 149-165). Mahwah, NJ: Lawrence Erlbaum Associates

Scardamalia, M., & Bereiter, C. (1999). School as knowledge-building organizations. In D. P. Keating & C. Hertzman, ed., *Developmental health and the wealth of nations*. New York: The Guildford Press.

NUMBERS CAN'T BE PATTERNS BECAUSE THEY GO ON FOREVER: CHILDREN TRY TO RECONCILE THEIR BELIEFS THAT PATTERNS CAN ONLY REPEAT WITH THEIR DEVELOPING KNOWLEDGE OF GROWTH PATTERNS

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The NCTM has prioritized pattern learning as part of its push for early algebra development. While goals and potential for pattern learning for the development of early algebraic reasoning is increasingly well articulated e.g. Copley 2000; Greenes et al 2001, Van de Walle and Folk, 2003; there is still a paucity of research that examines students' abilities to make sense of patterns, particularly of growing patterns. One of the research findings that will be presented in this poster, along with corresponding findings from informal cross-sectional assessments carried out in the Lab School at OISE/UT, is that the great majority of K-grade 3 students hold an unbending, rigid belief that a series of numbers or geometric objects can only be called patterns if they repeat, even after spending substantial time working with numeric patterns.

In this poster I will present analyses of grade 2 students' developing understanding of growth patterns. Data will come from two separate sources. First, students' responses taken from a pair of daily activities which are designed to give students increasingly sophisticated ways of working with growth patterns: tracking the number of days in school on a cumulative number line and generating number facts whose answer is the date. Second, data generated from larger study for the development of early algebraic reasoning from four grade two classes. The central thrust of this intervention is to foster an integration of students' knowledge of numerical and geometric patterns through work with pattern blocks, T-charts and a function machine where they are challenged to extend the given patterns and to find generalizations and functional relationships that will tell them what the pattern will be at any point along the way.

Data sources include pre and post-test interviews, protocol taken from field notes and transcripts of classroom lessons, children's portfolio notes from the beginning and end of the study and samples of student notes about theories about pattern from our Knowledge Forum database.

References

Copley, J V. (2000). The Young Child and Mathematics. Reston, VA: NCTM.

Greenes, C., Cavanagh, M., Dacey, L., Findell, C. & Small, M. (2001). Navigating Through Algebra in Prekindergarten-Grade 2. NCTM: Reston VA.

Van de Walle, John and Folk, Sandra (2004). Elementary and Middle School Mathematics: Teaching Developmentally. Pearson: Boston.

THREE GRADE 2 CLASSROOMS PARTICIPATE IN A KNOWLEDGE BUILDING APPROACH TO PATTERN EXPLORATION

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In this poster we report the methods and preliminary results of an ongoing collaborative research project for the teaching and learning of patterning and early algebra. This project shares many of the goals of other research projects in early algebra (e.g., Carpenter et al 2003; Carraher et al, 2000; Kaput & Blanton, 2001). However, there are also differences: 1) the focus in our study is primarily on patterning and functions; 2) the lesson sequence, based on previous Neo-Piagetian approaches to mathematics learning (i.e. Griffin & Case, 1997; Kalchman, Moss, & Case 2000; Moss & Case 1999), is designed to help students to integrate their separate visual (geometric) and numeric (i.e. skip counting) patterning abilities; 3) this intervention involves a knowledge-building software program, Knowledge Forum®, in which students post questions and contribute theories in the form of notes into a collaborative database (e.g., Scardamalia & Bereiter 2003). Digital photos of children's patterns are downloaded, allowing student-designed patterns to become objects of inquiry.

Subjects were students in three intact Grade 2 classes (n = 78), two in a downtown school in New York City, and a third in a private school in Toronto. A fourth class serves as a control group.

Quantitative analyses were conducted based on a pre- and post-interview measure of patterning items. Qualitative analyses were conducted using coded transcriptions of videotaped sessions and notes that students had contributed to the Knowledge Forum® database.

The lessons begin with geometric growing patterns. Students initially work with square tiles to built growing patterns and use special number cards (called position cards) which they place under each element of the pattern. The number cards enable students to relate the number of tiles in each element of a pattern with the position of the element. For example, in one activity students build the first 3 elements of a pattern and then challenge others to predict the number of tiles required to build the 5^{th} position, the 10^{th} position and eventually the *nth* position. The children also use T-charts on which they record the position number in the left-hand column and the corresponding number of blocks in the right-hand column. These kinds of activities along with lessons using functions machines (Willoughby,1997), support students' reasoning about the functional relations that underlie the patterns. In the final phase the students work with activities in which they design and build geometric patterns based on given rules. At various points student-designed patterns are digitally photographed and downloaded to the Knowledge Forum® database for discussion. In the poster we present details of the instructional intervention as well as preliminary analyses of results.

References

Bereiter, C. (2002). *Education and mind in the knowledge age*. Mahwah, NJ.

Carpenter, T.P., Franke, M., & Levi, L. (2003). *Thinking Mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann

Carraher, D., Brizuela, B. & Schliemann, A., (2000). Bringing out the algebraic character of arithmetic. Instantiating variables in addition and subtraction. In T. Nakahara & M. Koyama (Eds.) Proceedings of the XXIV Conference of the International group for the Psychology of Mathematics Education. Hiroshima, Japan, Vol. 2, 145-52.

Griffin, S., & Case, R. (1997). Re-thinking the primary school math curriculum an approach based on cognitive science. *Issues in Education*, *3* (1), 1-49.

Kalchman, M., Moss, J., and Case, R., (2001). Psychological models for the development of mathematical understanding: Rational numbers and functions. In S. Carver and D. Klahr (Eds.), *Cognition and Instruction: 25 years of progress*.

Kaput, J and Blanton, M, (2001, October). Student achievement in algebraic thinking: A comparison of 3rd graders' performance on a state 4th grade assessment. In R. Speiser, C. Maher, & C. Walter (Eds.), *The Proceedings of the 23rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Vol. 1, 99–107. Columbus, OH.

Moss, J., & Case, R. (1999). Developing Children's Understanding of Rational Numbers: A New Model and Experimental Curriculum. *Journal for Research in MathematicsEducation,* 30(2), 122–147, 119.

Scardamalia, M., & Bereiter, C. (2003, Fall). Beyond brainstorming: Sustained creative work with ideas. *Education Canada, 43*(4), 4-7 & 44.

Willoughby, S.(1997) "Functions from Kindergarten through Sixth Grade." *Teaching Children Mathematics* 3 (February): 314–18.

Assessment

INDIVIDUAL GAIN AND ENGAGEMENT WITH TEACHING GOALS

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Our aim in this paper is to relate pre-service elementary teachers' mathematics gain in test scores, initial-test to final-test, to psychological profiles, attitudes, and dispositions to learning mathematics. Our approach is "shamelessly eclectic" in the sense of Rossman & Wilson (1994), integrating both quantitative and qualitative methods to tell our story, because our story is one that tries to link numerical tests scores with psychological dispositions of students. Thus, we extend Hake's (1998) findings on average gain to individual students. The gain statistic assesses the amount individual students increase their test scores from initial-test to final-test, as a proportion of the possible increase each student. We examine the written work of students with very high gain and those with very low gain and show that these groups exhibit distinct psychological attitudes and dispositions to learning mathematics. In an appendix we examine a common belief that students with low initial-test scores have higher gains, and students with high initial-test scores have lower gains, and show this is not correct for a cohort of pre-service elementary teachers.

Introduction

As teachers of mathematics we want to know that our students have learned something from their class experiences, and we want to be able to assess what form that learning takes. Commonly, teachers will use a test at the beginning of an instructional sequence and an identical or similar test at the end. Increased scores from initial to final tests tell us little about the nature of student learning, its qualitative character, or how students exercised flexible thinking or a different and productive focus of attention in their learning. Initial to final test score comparisons simply tell us that a student's test score increased by such and such an amount. Yet in 1998, Richard Hake made an interesting discovery related to initial and final test scores. Hake (1998), in defining the average gain statistic as the amount students increase their test score on average from initial-test to final-test as a proportion of the possible average increase, found that traditional lecture courses in undergraduate physics are associated with a relatively low value for the average gain while courses with less emphasis on lectures and more on participation, are associated with relatively high average gain. Thus, Hake showed that a statistic obtained from pre/post test scores could distinguish lecturing style in undergraduate physics.

In Fall 2000 we began to teach mathematics to pre-service elementary teachers differently. Starting from a model for helping students make explicit their implicit memories of mathematics (Davis, Hill & Smith, 2000) we designed an introductory series of isomorphic combinatorial problems designed to assist students to see connections between apparently disparate problem situations, without their being told explicitly about those connections. At the same time we were looking for measures, or numerical indicators, of student growth and understanding of mathematics that were based on tests we gave at the beginning and end of a semester. We came up with the difference between final-test and initial-test scores, scaled to take into account the total marks a student might obtain from the initial to the final test – essentially Hake's gain, but calculated for each individual student rather than as an average over a class. The term *gain* will have a technical meaning throughout this paper, as defined in the section below on the gain statistic.

The Gain Statistic

Hake (1998) introduced the mean gain—denoted <gain>—for a class of students who were given an initial-test and a final-test in undergraduate physics:

<gain> = (mean final-test – mean initial-test) / (1 – mean initial-test)

This is a measure of what fraction, on average, students achieved of the possible marks they could achieve from initial-test to final-test. Hake studied the mean gain for classes consisting of over 6,000 undergraduate physics students in total, and concluded that generally high mean gains (0.48) were associated with classes with a focus on participation and interaction, while low mean gains (0.23) were associated with traditional lecture classes. In earlier studies involving pre-service elementary teachers as well as developmental algebra students we calculated, independent of Hake, an individual gain for each student as

gain = (final-test – initial-test) / (1 – initial-test)

(Davis & McGowen, 2001; McGowen & Davis, 2001a, 2001b, 2002). Note that gain is undefined if the initial-test score is 1, a situation we have not encountered in the data reported here, nor in similar data over 7 years.

In this paper we discuss this statistic in some detail, placing it in perspective with relative change functions. The individual gain statistic is related to a class of relative change functions (Tornqvist, Vartia & Vartia, 1985; Bonate, 2000, pp. 75-90). Two examples of commonly used relative change functions are (i) the proportional change score *(final-test score – initial-test score)/initial-test score* and (ii) the logarithmic difference, *log(final-test score/initial-test score)*. Change functions are characterized by a list of relatively straightforward properties, but the individual gain fails to be a relative change function in the sense of Tornqvist, Vartia & Vartia (1985) due to the fairly trivial fact that the denominator in the definition of the gain statistic rules out a requisite scaling property of a relative change function (see Appendix B).

Hake (1999) points out that the average gain in his physics studies correlates poorly with initial-test scores, a finding that is in accord with our studies. This is in marked contrast to the logarithmic difference, log(final-test score/initial-test score), which correlates linearly for our data with initial-test scores ($r^2 = 0.83$). The percentage change, (final-test score - initial-test score)/initial-test score, correlates quadratically with initial-test scores ($r^2 = 0.88$). In our context, therefore, the gain function provides *significant* extra statistical information beyond initial-test scores.

Initial-Tests and Final Tests

Pre-tests and post-tests are commonly thought of as part of quasi-experimental design and as such are subject to numerous confounding variables that affect internal validity. (Campbell & Stanley, 1966, pp. 7-12; see Bonate, 2000, for a detailed discussion of pre-test/post-test design and analysis). A common objection to using similar, but not identical, initial and final tests is that in comparing student scores from one to the other we are trying to "compare apples and oranges." This would be a valid objection if we were asserting that an intervention was associated with a change in test score, initial to final test. But that is not our purpose in this article: our aim is to understand how we might assess growth in student mathematical development and understanding across a semester, given the usual assessment tools and practices available to a classroom teacher. Essentially we use a statistic based on initial and final test scores to disaggregate student data in meaningful ways. Note that the nature of the test items is not relevant to the present discussion and analysis, even though the test items *are* integral to understanding the nature of the course. Our analysis is based around the different attitudes to learning and engagement exhibited by students with differing gain statistic, as obtained from a final and initial test that we, the instructors, deemed relevant to the course itself. In this paper we assume test scores have been normalized so as to lie between 0 and 1. In the application of the gain statistic, we are interested in student attitudes and dispositions to learning mathematics, rather than comparing mean test scores before and after an instructional treatment. We use different, yet strongly related, tests in a sequence—one near the beginning of a course, one nearer to the end. We use the terms "initial" and "final" test to alleviate confusion that might result from use of pre-test and post-test in these circumstances.

Data Sources/Evidence

We consider pooled data from 4 classes of a 16 week pre-service elementary mathematics course. The cohort for whom complete data was available consisted of 65 students. Students were given a written mathematics competency test (referred to as "initial-test") the first week of the semester. The students sat a final written examination at the end of the course. This written final examination contained problems that required students to recognize the mathematics and skills in contextual situations along with problems similar to those included on the competency test that tested skills. We refer to the final examination as "final-test". Test scores have been scaled so as to represent numbers in the range 0 through 1.

We examine the written work over a semester of those students who had gain more than one standard deviation above or below the mean cohort gain, for evidence of attitudes and dispositions to learning mathematics. Note that the students with very high gain necessarily had high final test scores because

final-test score = gain + [initial-test score × (1– gain)] and the second term on the right side of the equation is non-negative. Importantly, however, not all students with high final-test scores had high gain.

Results

Very High Gain

This group of students had gain statistic more than one standard deviation above the class average of 0.57. There were 8 students in this group (12.3% of the cohort), with a mean initial-test score 0.49, mean final-test score 0.94, and mean gain 0.87. Students in this group, like most of the cohort, characterized their prior mathematics learning as instrumental (Skemp,1976).

> "I have never been taught a math course by relational understanding. All of my classes were learning rules and applying them." LT
>
> "I think most of my learning in math was done instrumentally. We were taught the rules and how to use them." SM

However, they stated explicitly that they focused in this course on re-learning basic mathematics:

> "I had to re-learn basic math in order to eventually teach it to children." JH
>
> "We have essentially (to my mind) be re-learning mathematics." JK
>
> "I felt like I am re-learning everything." SM

They consistently looked for relationships and connections, and stated that how they approach a mathematics problem had changed:

> "My self-confidence in my ability to do mathematics has increased. Mathematics is making a lot of sense to me now. A lot of the mathematics we learned has connections to

something else we learned. I definitely approach math differently than I used to in high school. I now know why I use a particular method or formula." JW

"… I found that I was making connections I had not before. These connections made it easier to understand what and why we were doing things in class. This influenced my attitude to change for the better. I'm more willing to learn new concepts and apply them to mathematics." JV

They wrote that their organizational skills had increased:

"I believe my organization skills have improved, … Do I know what to do, and why I should do it? This is what I ask myself with each assignment. Organization, effort, and willingness to learn from your mistakes are the way to truly learn math." JH

"The connections have also helped my organization. I couldn't organize my thoughts. It was like I knew what I meant, but I couldn't explain it. … My thoughts have become clearer ever since I've made better connections." SM

"Three habits of mind which make math a lot easier to complete are: think, estimate answers, and learn to use patterns. In using these three habits, I have grown in my organizational skills as well." HH

Principally, these students had become more reflective problem solvers–a change from their prior mathematical experiences. They were able to elaborate what they did and did not know in very specific detail. They focused on truly understanding a problem and being able to solve it in an efficient and elegant way, and they utilized and understood appropriate mathematical terminology. Students in this group were able to see a problem and think of different ways to solve it: they focused on what the problem was asking. This group tended not to over-generalize, and were aware of what is appropriate to use in a given situation. They emphasized the importance of being systematic in approaching mathematical problems, and focused explicitly on organizational skills. They stressed organization, effort, and willingness to learn from mistakes. They had a focus on looking for relationships—not only looking for isomorphic problem situations.

Very Low Gain

This group of students had gain statistic more than one standard deviation below the class average. There were 11 students in this group (16.9% of the cohort), with a mean initial-test score 0.48, mean final-test score 0.64, and mean gain 0.28. This group of students split naturally into three subgroups—Group A: 5 students; Group B: 2 students; Group C: 4 students.

Group A

This group had mean initial-test score 0.24, mean final-test score 0.50, and mean gain 0.33. All students in this group expressed confidence at the end of the course in their ability to do mathematics. There was, however, a marked disconnect between what these students thought their understanding was, and what we thought it was. For example, in the final examination students in this group rated themselves as "Exemplary (all the time) 5/5" in creating a general rule or formula, despite their writing consistently throughout the semester that they had trouble coming up with an equation. All these students characterized themselves as hands-on and visual learners and claimed to have problems with oral or written explanations. However their expressed view of being a visual learner meant seeing a problem worked on the board, not thinking in visual images. This group of students claimed to learn better from examples. They all expressed a belief that learning mathematics is about the teacher showing how to do a problem. Then, and only then, they said, could they understand what was done. The following comments are typical:

"I am more of a hands-on or visual type of learner when it comes to any subject. When a teacher verbally explains how to do some sort of math problem, I have a harder time grasping the concept. If a teacher shows and explains a problem on the board, I can actually understand. I can also learn better from examples. I also can teach myself many things, just by looking at an example." CL

"I am a visual person. In order to understand a math problem, I need examples of the same type of problem. Usually, I can figure out how a teacher came to an answer just by looking at his/her example, and then I do really well on assignments. If I don't know how to do a problem, and we go over problems in class, I raise my hand and explain what I don't know. By the time the teacher finishes the problem, I feel better understanding how he/she got to it. However, if a teacher does not teach, I get lost." BK

Despite claims to the contrary, these students persisted in the belief that teaching means the teacher "shows me how to do it" and then "I can understand what was done".

What it means to learn mathematics and to teach mathematics remained instrumental for these students. Their focus of attention was on learning how to do a procedure:

"To really learn math, a person has got to have a feel for it. This can be accomplished by having specific examples and a concrete way of learning it. Algorithms give you a sure plan on how to do the problem, it helps you understanding how you want to work it, and it gives you full directions on how it is supposed to be done. " NA

"The way I like to learn is when a teacher goes up to the blackboard or overhead projector, and demonstrates the mathematics by showing the process." BK

They persisted with inappropriate word usage. The examples below, in which the students use the inappropriate word "equation" instead of "expression", are typical:

"Finding an equation to match a pattern is a different story. CL

"So 2^n is not the correct equation." CL

Despite claims that their goal was to learn different algorithms to do a problem, these students did not use multiple representations to solve problems and stated that being shown more than one way to do a problem is confusing. They held to working one way— the way they were most comfortable. For example, on the final test, a student in this group was unable to demonstrate more than one way to compute subtraction problems using whole numbers and mixed numbers, and was unable to divide mixed numbers correctly at all. The problem asked students to use (a) missing factor; (b) "you don't have to multiply", and (c) standard algorithm. Given a shaded array, this student was unable to identify the fraction multiplication problem indicated by the drawing. This type of response was typical for this group.

Their reflections were frequently written as instructions to a third person—a teacher:

"If a teacher can give you a rule and then explain and show the children how and why it works, they believe you more." CL

"Teachers need to understand the algorithm and make sense of it. Students need rules with reasons. They need to be taught or shown different methods of finding an answer." LL

"You, the teacher, need to not only be able to solve the problem but to learn different process of how to arrive at an answer." BK

Group B

There were two students in this group, with mean initial-test score 0.82, mean final-test score 0.86, and mean gain 0.22. These two students were computationally competent, and saw no need to re-think basic mathematics. They viewed teaching as direct instruction and focused on the importance of knowing how children think:

"I have learned how I thought about problems. By knowing how I think about problems and the different ways to think about them, I am able to see how students work their problems so I can either help them or I can learn from them. A very important aspect in teaching I believe I have learned is that you need to know how a student thinks about a problem before you tell him how to work it." AS

"This experience has taught me just how imperative it is for a teacher to be sensitive to each individual child's learning. The biggest aspect of this class that I will take with me is the idea that each child has a different learning process regarding mathematics and it is the job of the teacher to recognize these different methods in other to help the child understand." NM

Group C

This group had mean initial-test score 0.6, mean final-test score 0.7, and mean gain 0.25. Students in this group, like the majority of the cohort, began the course with a very procedural approach to mathematics. Unlike the very high gain group they did not break out of this procedural approach to mathematics. They were different from students in group A, however, in that they did know a correct procedure to use, and when they were asked to use a procedure they knew, they could work the problem correctly. In their reflections and self-evaluations, they described what they should learn and their limitations:

"I should be able to think flexibly once I look at math problems. I shouldn't be stuck with one rule or certain ways to solve it." KN

"I need to develop skills of having more flexible thinking. I get frustrated easily when I can't figure out a problem." ME

"Different algorithms should be used to work out different problems. You need to know when to use which algorithm in different problems." GY

Conclusions

Students in this study with very high and very low individual gains had markedly different psychological profiles in relation to attitudes to study and the course material.

Students with gain more than one standard deviation above the cohort mean worked hard and smart, particularly in relation to learning to be more organized. They focused on looking for relationships and re-learning basic mathematics so as to teach it better.

Students with gain more than one standard deviation below the cohort mean were not homogeneous in their attitudes and dispositions toward learning mathematics. The largest group began and remained very instrumental, highly dependent on being shown how to do worked examples. A second group comprised two students with high initial-test score and very low gain. These two students were competent in terms of mathematical computation and viewed teaching only as an instructional process. A third group comprised students with moderate initial-test scores and very low gain. These students were competent in using algorithms, but showed no flexibility in their approach to problems.

In the context of this study, very high gain meant engagement with the explicit aims and goals of the course and a willingness to take risks in learning mathematics. Although there were three identifiably different groups of students having very low gains, all these students showed a lack of engagement with the aims of the course—albeit in different ways—and also showed no evidence of risk-taking in relation to learning mathematics.

On the basis of the test scores and written work of these 65 pre-service teachers we are now inclined to see high individual gain as an indication of engagement with the aims and goals of an instructional sequence, and low individual gain as a lack of such engagement for varying reasons.

We believe that single classroom data from initial and final tests can be very useful provided one disaggregates the data and relates a statistic such as gain to attitudes to learning of individual students. The salient point for us is our hypothesis that irrespective of the initial-test and final-test questions—so long as they relate to the instructor's aims and goals for the course—we will see similar psychological profiles among the students with very high and very low gain. This hypothesis is eminently testable, and is not dependent on the precise nature of the test questions. Thus, our focus is not on the exact nature of test questions, nor on the details of an instructor's aims and goals. Rather, our focus is on whether a single change statistic—the individual gain—does consistently disaggregate students in the ways we have indicated in this study.

References

Bonate, P.L. (2000) *Analysis of Pretest-Posttest Designs*. Chapman & Hall.

Davis, G., Hill, D. & Smith, N. (2000) A memory-based model for aspects of mathematics teaching. In T. Nakahara & M. Koyama (Eds.) *Proceedings of the 24th conference of the International Group for the Psychology of Mathematics Education*, vol 2, p.225-232. Hiroshima, Japan: Hiroshima University.

Davis, G.E. & McGowen, M.A. (2001) Jennifer's journey: Seeing and remembering mathematical connections in a pre-service elementary teachers' course. In M. van den Heuvel-Panhuizen (Ed.), *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education*, vol 2, pp. 305-312. Utrecht: Freudenthal Institute, Utrecht University, The Netherlands.

Davis, G.E. & McGowen, M.A. (2003) Characterizing the gain function. Retrieved on December 18, 2003, from www.wsu.edu/~gedavis/gain.htm

Hake, R.R. (1998) Interactive-engagement vs. traditional methods: A six-thousand-student survey of mechanics test data for introductory physics courses. *American Journal of Physics* 66, 64–74. Online at http://www.physics.indiana.edu/~sdi

Hake, R.R. (1999) Analyzing change/gain scores. http://www.physics.indiana.edu/~sdi McGowen, M.A. and Davis, G.E. (2001a) Changing elementary teachers' attitudes to algebra. In Chick, H., Stacey, K., Vincent, J. & Vincent, J (Eds.) *Proceedings of the 12th ICMI Study on The Future of the Teaching and Learning of Algebra*, Vol. 2, 438-335. Melbourne: University of Melbourne.

McGowen, Mercedes A. and Davis, Gary E. (2001b). What mathematical knowledge do pre-service elementary teachers value and remember? In R. Speiser, C.A. Maher, & C. N. Walter (Eds.) *Proceedings of the XXIII Annual Meeting, North American Chapter of International Group for the Psychology of Mathematics Education.* Snowbird, Utah, Vol. 2, 875-884.

McGowen, Mercedes A. and Davis, Gary E. (2002). Growth and Developent of Pre-Service Elementary Teachers' Mathematical Knowledge. In Denise S. Mewborn (Ed.) *Proceedings of the XIX Annual Meeting, North American Chapter of International Group for the Psychology of Mathematics Education*. Athens, GA. Vol. 3, 1135-1144.

Rossman, G. and Wilson, B. (1994) Number and words revisited: Being shamelessly eclectic. *Quality and Quantity*, 28, 315-327.

Skemp, R.R. (1976) Relational understanding and instrumental understanding. *Mathematics Teaching*, 77, 153-166.

Tornqvist, L., Vartia, P. & Vartia, Y.O. (1985) How should relative change be measured? *American Statistician*, 39, 43-46.

Appendix A: Gain & Initial-Test Score

In this appendix we address the question of whether low initial-test score implies high, and high initial-test score implies low gain. At presentations of work on the gain statistic we often hear assertions like: "of course, students with low initial-test score will have high gain." The argument is that a student with a low initial-test score has a lot more room for improvement, and so a potentially higher gain, than a student with a high initial-test score. A natural corollary of this line of reasoning is that a student with a high initial-test score will have a generally lower gain than other students, "because it's harder to achieve a higher final-test score starting from a high initial-test score." The mean initial-test score for the cohort of 65 pre-service teachers in the present study was 0.39, the mean final test score was 0.74, and the mean gain was 0.57. The group of students (n = 36) with below average initial-test scores had a statistically significantly lower average final-test score than those students (n = 29) with above average initial-test score (mean of 0.70 as compared to a mean of 0.79; $p < 0.002$). However, the average gain for the group with below average initial-test score was not statistically significantly different from that of the group with above average initial-test scores (gain of 0.60 compared with a gain of 0.53; $p > 0.1$) A similar situation pertains if we take the group with initial-test z-score < -0.5 on the one hand, and the group with initial-test z-score > 0.5 on the other. For this cohort, therefore, there seems to be little basis for the claim that low initial-test score entails high gain on average. Indeed, there are significant numbers of students who had below average initial-test scores and below average gains, as well as significant numbers with above average initial-test scores and above average gains: 13/65 = 20.0% of the cohort had below average initial-test score and below average gain (95% confidence interval = [12.1%, 31.3%]), and 14/65 = 21.5% of the cohort had above average initial-test score and above average gain (95% confidence interval = [13.3%, 33.0%]).

Of the 36 students with below average initial-test score, 13/36 = 36.1% had below average gain (95% confidence interval = [22.5%, 52.4%]), and of the 29 students with above average initial-test score, 14/29 = 48.3% had above average gain (95% confidence interval = [31.4%, 65.6%]). Thus, for this cohort of 65 students, given a student had a less than average initial-test score there was more than a 1 in 3 chance that the student had a less than average gain. Equally, given a student had a greater than average initial-test score there was about a 1 in 2 chance that the student had a greater than average gain. These proportions are not inconsiderable, and while it is more likely that a student with a below (*resp*. above) average initial-test score will have an above (*resp*. below) average gain, it is by no means a foregone conclusion.

Appendix B: Relative Change Functions

A change function in the sense of Tornqvist, Vartia & Vartia (1985), is a function C of two non-negative real variables x (initial-test score) and y (final-test score) with the following properties:

1. 1. $C(x, y) = 0$ when $y = x$
2. 2. $C(x, y) > 0$ when $y > x$
3. 3. $C(x, y) < 0$ when $y < x$
4. 4. For all $\lambda > 0$, $C(\lambda x, \lambda y) = C(x, y)$
5. For each x, the function $y \rightarrow (x, y)$ is continuous and increasing

For example, the commonly used proportional change score $C(x, y) = (y-x)/x$ (Bonate, 2000) clearly has properties (1) – (5) above. In contrast the individual gain $g(x, y) = (y-x)/(1-x)$, where x and y are normalized so as to lie between 0 and 1, satisfies (1) – (4), but trivially fails to satisfy

(5). The proportional change function can be written as $C(x, y)= y/x–1$ and so, in common with other change functions, can be expressed as a function of y/x. The gain function, in contrast, cannot be so expressed, due of course to the normalization of the test scores in calculating the gain.

Further, the gain function is characterized by its preservation of the binary operation $x*y := x+y–x\times y$, namely $g(x, y)$ is the unique function C:[0,1)×[0,1] → (-∞, 1] satisfying:

A. $C(x, x) = 0$ for all $0 \leq x < 1$

B. $C (0, y) = y$ for all $0 \leq y \leq 1$

C. $C(x, z) = C(x, y) + C(y, z) – C(x, y) \times C(y, z)$ for all $0 \leq x, y < 1, 0 \leq z \leq 1$

One checks easily that $g(x, y)$ has these properties and, conversely, if C is such a function then from $0 = C(x, x) = C(x, 0)*C(0, x) = C(x, 0) + C(0,x) - C(x, 0)C(0, x) = C(x,0) + x - xC(x,0)$ and $C(x, y) = C(x,0)*C(0, y) = C(x, 0) + C(0, y) - C(x, 0)C(0, y),$ we see easily that $C(x, y) = (y-x)/(1-x) = g(x, y)$ for all *0 □x<1and0□y□1.*

This feature of the gain function places it more clearly in perspective with the logarithmic difference function □(x, y) = *log(y/x)* which is the unique relative change function satisfying the additivity property □(x, z) = □(x, y) + □(y, z) (Torqvist, Vartia & Vartia, 1985). Because of formula C —reminiscent of a measure in the sense of measure theory —we interpret individual gain as a numerical indicator of the “size” of change from one test to a succeeding test. The gain function, therefore, is of theoretical interest as the unique measure of relative change satisfying A – C above, and a statistic that has low correlation with initial-test scores.

QUANTITATIVE LITERACY: THE CREATION OF A FRAMEWORK TO ANALYZE INSTRUCTIONAL MATERIALS FOR THE PROMOTION OF A LITERATE SOCIETY

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This study focuses on the creation of a framework to analyze the extent to which Standards*-based instructional materials promote quantitative literacy among learners. The framework, created to aid in future curriculum development and textbook adoption processes and also to add to our overall understanding of quantitative literacy, was developed using relevant theory and research related to the definition and measure of quantitative literacy, as well as several major textbook analysis frameworks. One middle school curriculum program is evaluated using the proposed framework.*

To function in society, one must be literate. Although the traditional definition of literacy is the ability to read and write, it is imperative as we move into the 21st century to expand that definition for a changing, growing society. No longer will the ability to read and write alone help citizens make sense of critical issues such as health care, politics, finance, and technology. As quantitative literacy becomes an increasingly important idea in our lives, many researchers and educators have attempted to define what we really mean by this term.

Although not all encompassing, most would agree that part of being quantitatively literate means developing an understanding of a certain level of mathematics content. Dossey (1997) indicates, for example, the importance of data representation and interpretation, number and operation sense, measurement, variables and relationships, and spatial visualization when defining quantitative literacy. Some stress the importance of everyday functional mathematics such as statistics and probability, the ability to estimate, interpret graphs and draw inferences (Devlin, 2000; Kolata, 1997; Steen, 1999; Wilkins, 2000). Others argue for the inclusion of reasoning processes when defining quantitative literacy, stating the importance of such things as informal reasoning, the use of data and display, and the ability to identify problems (Atkins & Helms, 1993; National Council of Teachers of Mathematics [NCTM], 1989, Organisation for Economic Cooperation and Development [OECD], 2000). Still others define a quantitatively literate individual as possessing certain habits of mind, beliefs, dispositions and attitudes towards mathematics, science and technology (American Association for the Advancement of Science [AAAS], 1990; Atkins & Helms, 1993; Devlin, 2000).

NCTM's (2000) *Principles and Standards for School Mathematics* document highlights the importance of curricula that "emphasize the mathematics processes and skills that support the quantitative literacy of students" (p. 16). It seems critical then to evaluate the instructional materials that arose out of an effort to interpret and implement the new vision of school mathematics in regard to their promotion of this literacy. While there have been numerous studies on the impact of these curricula on student learning of mathematics (Senk & Thompson, 2003), there is rarely mention in these reports of levels of quantitative literacy among students using the curricula. There has also been little attempt to identify specific features within curriculum materials that might promote quantitative literacy among learners. The framework proposed in this paper intends to provide a tool that might help future researchers and curriculum scholars find ways to fill this gap in the literature.

Theoretical Framework

Although much has been written in an attempt to define quantitative literacy, much less has been written about the attempt to measure it. There have, however, been a few theoretical and empirical studies that move us towards an understanding of quantifying quantitative literacy (Orpwood & Garden, 1998; OECD, 2000; Wilkins 2000, 2003). While these studies provide tools with which to evaluate levels of quantitative literacy among students, it is still unclear exactly what types of instructional activities and curriculum designs within school-based subjects might help create more quantitatively literate students. With this in mind, it seems imperative to create a framework designed to evaluate certain texts currently supported by national mathematics, science, and technology standards in relation to their level of promotion of quantitative literacy among learners.

The main components of quantitative literacy as proposed by Wilkins (2003) -- (1) the cognitive domain (including content and reasoning in mathematics), (2) mathematical beliefs, and (3) dispositions -- provide the structure for the overall textbook analysis framework. It is important to note here the existence of a benchmark-based textbook analysis procedure created by AAAS (2000). This evaluation procedure focuses on two main components - content analysis based upon national science and mathematics standards and instructional analysis based upon a set of criteria considered to represent good instructional design. Although AAAS provided invaluable information to educators, it is the opinion of the authors, based upon a review of the literature on quantitative literacy, that the evaluation components are missing key elements in determining whether or not the curriculum materials evaluated promote quantitative literacy. The framework proposed here will modify the content evaluated by AAAS (2000) and add to the framework by including an evaluation of the types of mathematical reasoning, beliefs, and dispositions promoted through the use of the curricula evaluated.

Proposed Analytic Framework for the Promotion of Quantitative Literacy

A review of the literature in several areas of quantitative literacy contributed to the creation of the proposed framework (see Tables 1 and 2). Due to the straightforward nature of evaluating the existence of the cognitive component of quantitative literacy among texts, criteria under both the content and reasoning components came directly from the literature (AAAS, 1999; Kolata, 1997; NCTM, 1989, 2000; OECD, 2000; Steen, 1999; Wilkins, 2000). Different from the AAAS content knowledge component, the content analysis in this framework focuses in particular on the mathematics needed to function in everyday life. The following content areas are included in the framework for evaluation: (1) percents, ratios, and proportions, (2) relationships and/or functions, (3) measurement, (4) graphing and/or tabling data, (5) estimation, (6) probability, and (7) sampling. Mathematical reasoning is the ability to make conjectures, interpret data, make inferences, judge the validity of arguments, and build sound arguments. These informal reasoning skills will enable students to "interpret current issues, synthesize available information, ask informed questions and make informed decisions" (Wilkins, 2000, p. 407). The reasoning component analyzes the extent to which curriculum materials provide students with the opportunity to (1) communicate in written and oral form, (2) interpret mathematical models in terms of reality, (3) reflect on, analyze and critique mathematical models, (4) choose and switch between different forms of mathematical representations, (5) make conjectures and inferences, and (6) interpret current issues.

The beliefs and disposition components of quantitative literacy, given their interpretive nature, are much harder to measure. Because this analytic framework is attempting to measure the extent to which certain beliefs and dispositions in regard to mathematics are promoted by curriculum materials use, it was necessary to first make sense of literature on these affective aspects in mathematics education. Although most studies have focused on how predetermined attitudes and beliefs among students affected their achievement and performance in mathematics, more recently studies have attempted instead to measure how instructional practices and curriculum materials affect students' beliefs about and dispositions towards mathematics. This body of literature (e.g. Fauvel, 1991; Franke & Carey, 1997; Higgins, 1997; McLeod, 1992; Stander, 1989) helped to determine what types of activities and curriculum promote certain beliefs, attitudes, and dispositions in students and was included as such in the analysis framework. The beliefs component, focusing on beliefs about the nature and social importance of mathematics, analyzes the extent to which curriculum materials encourage students to (1) discover multiple ways of solving a problem, (2) check if answers are reasonable, (3) solve problems without the use of algorithms, (4) explain and justify their answers, (5) create their own solution strategies, (6) learn about the history of mathematics, (7) study multicultural aspects of mathematics, and (8) recognize the social importance and impact of mathematics. The dispositions component analyzes the extent to which curriculum materials include activities that promote (1) problem solving and (2) cooperative learning.

Implementation of the Proposed Analytic Framework

The framework proposed here was first implemented through the analysis of MathScape, a National Science Foundation - funded middle school curriculum developed by the Seeing and Thinking Mathematically project (STM). MathScape is composed of twenty-one units intended for use in grades 6, 7 and 8. There are 12 lessons within each unit and therefore a total of 252 lessons throughout the entire MathScape series. Given the large number of lessons, analysis was based upon three randomly selected lessons within each of 10 units, totaling 30 lessons analyzed in all. MathScape was chosen for analysis due to the accessibility of the entire program to the authors. Although not included here due to space limitations, each lesson was analyzed individually using the framework outlined in the table. A summary report of this analysis is included in Tables 1 and 2.

Results

Mathematical Content

Of the MathScape lessons sampled, relationships and/or functions comprised the most highly represented quantitative literacy mathematical content component (77%). The content components least covered throughout the sampled lessons were estimation, probability and sampling, although all three components were apparent in at least 20% of the lessons sampled. Given the broad nature of the content components included within this framework, it is interesting to note that all 30 lessons sampled included at least one of the seven content components proposed in this framework as promoting quantitative literacy among students. Even more interesting is the inclusion of four or more of the content components in close to one-third of the lessons sampled. See Table 1 for a complete summary.

Reasoning

Within the MathScape series, students are given the most opportunity to communicate in written and oral form (23 of the 30 lessons sampled). Five of the six components included in the reasoning portion of the framework were present in at least 40% of the lessons sampled. All lessons included at least one reasoning component and seven of the 30 lessons sampled included four or more of the reasoning components included in the framework. An examination of each individual reasoning component reveals less of an opportunity for students to interpret current issues. Only five of the 30 lessons sampled included this component. See Table 1 for a summary.

Table 1. Frequency and Percentage of Sampled Lessons from MathScape that included Mathematical Content and Mathematical Reasoning QLT Components (N = 30)

	Frequency	%
Mathematical Content – *Lessons promoting an understanding of or activities needing the use of…*		
…Percents, ratios, proportions	14	46.7
…Relationships and/or Functions	23	76.7
…Measurement	13	43.3
…Graphing and/or tabling data	13	43.3
…Estimation	7	23.3
…Probability	6	20.0
…Sampling	7	23.3
Lessons including …		
…1 QLT mathematical content component	30	100
…2 to 3 QLT mathematical content components	27	90.0
…4 or more QLT mathematical content components	9	30.0
Reasoning – *Provides students opportunities to…*		
…communicate in written and oral form	23	76.7
…interpret mathematical models in terms of reality	14	46.7
…reflect on, analyze and critique mathematical models	12	40.0
…choose and switch between different forms of mathematical representations	10	33.3
…make conjectures and inferences	16	53.3
…interpret current issues	5	16.7
Lessons including …		
…1 QLT reasoning component	30	100
…2 to 3 QLT reasoning components	27	90.0
…4 or more QLT reasoning components	7	23.3

Beliefs

In order to promote positive beliefs about the nature and utility of mathematics, MathScape lessons most often encourage students to solve problems without the use of algorithms (25 out of 30 lessons sampled). Approximately half of the lessons encourage students to create their own solution strategies and explain and justify their answers. Impressively, almost all lessons sampled (29 out of 30) included at least one of the beliefs components included in the framework. There are however three belief components given significantly less emphasis in the curriculum. Only two of the 30 lessons sampled focus on the history of mathematics, three of the 30 focus on the social importance and impact of mathematics, and just four of the 30 lessons focus on multicultural aspects of mathematics. See Table 2 for a complete summary.

Dispositions

The most prominent instructional activity included within the lessons sampled was problem solving, with 25 of the 30 lessons sampled including the component somewhere throughout the lesson. Approximately one-fourth of the lessons included both problem-solving activities and cooperative learning opportunities. See Table 2 for a complete summary.

Table 2. Frequency and Percentage of Sampled Lessons from MathScape that included Beliefs and Dispositions QLT Components (N = 30)

	Frequency	%
Beliefs – *Encourages students to…*		
…discover multiple ways of solving a problem	8	26.7
…check if answers are reasonable	12	40.0
…solve problems without the use of algorithms	25	83.3
…explain and justify their answers	15	50.0
…create their own solution strategies	14	46.7
…learn about the history of mathematics	2	6.7
…study multicultural aspects of mathematics	4	13.3
…recognize the social importance and impact of mathematics	3	10.0
Lessons including …		
…1 QLT beliefs component	29	96.7
…2 to 3 QLT beliefs components	23	76.7
…4 or more QLT beliefs components	7	23.3
Dispositions – *Includes activities that promote…*		
…problem-solving	25	83.3
…cooperative learning	12	40.0
Lessons including …		
…1 of the QLT disposition components	21	70.0
…Both of the QLT disposition components	8	26.7

Discussion and Implications

The analytic framework proposed here was utilized to analyze one NSF-funded middle school curriculum. Results of the analysis highlight important ideas about both the curriculum materials evaluated and the overall usefulness of the framework in determining the existence of components thought to promote quantitative literacy. To focus first in particular on the MathScape series, the results of this analysis bring to our attention three key elements inherent in the curriculum

First, we identified at least four separate content components in close to one-third of the lessons sampled. This finding suggests a focus on integration of content throughout the series, an important idea inherent in the current reform movement in mathematics education; and although not explicit in the framework, an important aspect for promoting beliefs about the dynamic nature of mathematics consistent with being quantitatively literate.

While the analysis of the MathScape series also indicated frequent inclusion of components shown to promote reasoning throughout the lessons sampled, one reasoning element – the opportunity for students to interpret current issues using mathematics – was emphasized less.

While one recommendation based on this result is the inclusion of more such opportunities in future versions of the curricula, the solution might not be so simple. This finding might instead suggest to teachers using these materials – or any materials – the importance of incorporating current events into mathematics lessons in conjunction with the curricula being used. The very nature of current events – that relevant issues change over time and place – makes this finding in a sense very understandable. Yet, the finding does highlight the potential need to revise future curricula that would allow the inclusion of current events to be more easily accomplished by teachers.

While the reasoning component of the analytic framework highlighted one area in particular less emphasized in the MathScape curricula, the beliefs section highlighted three areas – the history of mathematics, multicultural aspects of mathematics, and the social impact of mathematics – of lesser emphasis. The lack of opportunity for students to understand the humanistic nature of mathematical knowledge and realize past obstacles in the development of mathematics – similar at times to their own struggles and obstacles to understanding mathematics – is unfortunate. Additionally, little opportunity for students to recognize the social importance and impact of mathematics seems critical. Only when students and adults alike recognize the critical role that mathematics plays in our lives will it become an important area to study and understand. It would be relatively easy to include more history, multiculturalism, and instances highlighting the social importance of mathematics in mathematics curricula.

While the framework outlined here was developed with the explicit purpose of analyzing curriculum materials – and in fact has been the focus of much of this paper – the process of developing the framework and its existence, as preliminary as it may be, can be seen as another tool or resource that helps us get closer to understanding what it means to be quantitatively literate in today's society. More importantly, it highlights the essential inclusion of beliefs and dispositions in that definition. Given that quantitative literacy is focused on the mathematical skills needed to function in *society*, it seems critical to reflect upon two findings in particular – the lesser emphasis on current issues and the opportunity for students to recognize the social importance and impact of mathematics in the curricula evaluated. These components in particular seem critical to the development of quantitative literacy, yet are limited in their inclusion in the curriculum evaluated.

If utilized extensively, this framework and others like it have the potential to guide future curriculum development plans, structure local and national mathematics, science and technology standards and provide valuable insight into textbook and curriculum selection within our classrooms. Utilized at a local level, this framework can be used by teachers and administrators to first determine what components are of greatest importance to their students and their school and to then determine the best curricula to help in developing or enhancing those components of quantitative literacy among their students. Used more extensively, this framework can help in the redesign of reform materials – a process currently underway for most NSF-funded series.

References

American Association for the Advancement of Science. (2000). *Middle Grades Mathematics Textbooks: A Benchmarks-Based Evaluation.* Washington, DC: American Association for the Advancement of Science.

American Association for the Advancement of Science. (1990). Habits of Mind. In *Science for all Americans,* Chapter 12. http://www.project2061.org/tools/sfaaol/sfaatoc.htm

Atkins, J. M. & Helms, J. (1993). Getting serious about priorities in science education. *Studies in Science Education, 21,* 1-20.

Devlin, K. (2000). The four faces of mathematics. In M. J Burke and F. R. Curcio (Eds.), *Learning Mathematics for a New Century* (pp. 112-126). Reston, Va.: National Council of Teachers of Mathematics.

Dossey, J. A. (1997). Defining and measuring quantitative literacy. In L. A. Steen, (Ed.) *Why numbers count: Quantitative literacy for tomorrow's America.* New York: College Entrance Examination Board.

Fauvel, J. (1991). Using history in mathematics education. *For the Learning of Mathematics, 11,* 3-6.

Franke, M. L. & Carey, D. A. (1997). Young children's perceptions of mathematics in problem-solving environments. *Journal for Research in Mathematics Education, 28,* 8-25.

Higgins, K. M. (1997). The effect of year-long instruction in mathematical problem solving on middle-school students' attitudes, beliefs and abilities. *The Journal of Experimental Education, 66*(1), 5-28.

Kolata, G. (1997). Understanding the news. In L. A. Steen, (Ed.) *Why numbers count: Quantitative literacy for tomorrow's America.* New York: College Entrance Examination Board.

McLeod, D. B. (1992). Research on affect in mathematics education: A reconceptualization. In D. A. Grouws, (Ed.) *Handbook of Research on Mathematics Teaching and Learning: A project of the National Council of Teachers of Mathematics.* New York: MacMillan Publishing Company.

National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston, Va.: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Organisation for Economic Cooperation and Development. (2000). *Knowledge and skills for life: First results from the OECD programme for international student assessment (PISA) 2000.* Paris, France: OECD Publications.

Orpwood, G & Garden, R. A. (1998). *Assessing mathematics and science literacy*. Vancouver, BC: Pacific Educational Press.

Senk, S. L., & Thompson, D. R. (Eds.) (2003). *Standards-based school mathematics curricula*. Mahwah, NJ: Lawrence Erlbaum Associates.

Stander, D. (1989). The use of the history of mathematics in teaching. In P. Ernest (Ed.), *Mathematics Teaching: The State of the Art*. New York: The Falmer Press.

Steen, L. A. (1999). Numeracy: The new literacy for a data-drenched society. *Educational Leadership, 57*(2), 8-13.

Wilkins, J. L. M. (2000). Preparing for the 21st century: The status of quantitative literacy in the United States. *School Science and Mathematics, 100*(8), 405-418.

Wilkins, J. L. M. (2003). *Validating a model of quantitative literacy.* Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.

UNDERSTANDING MATHEMATICAL CREATIVITY: A FRAMEWORK FOR ASSESSMENT IN THE HIGH SCHOOL CLASSROOM

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Creativity in mathematics is often looked at as the exclusive domain of professional mathematicians. There have been few studies that have examined creativity in the high school classroom, particularly in its manifestation in students with exceptional mathematical abilities. One of the obstacles to the study of mathematical creativity is the "fuzziness" of the construct of creativity. In this paper the notion of mathematical creativity is developed from literature in psychology. In particular the evolving systems model (Gruber & Wallace, 2000) is used to construct a framework for studying and assessing creativity at the high school level. The key elements from a case study involving the discovery of a mathematical structure are used to illustrate this approach to the study of creativity.

Introduction and Background

Research on creativity has been on the fringes of psychology, educational psychology, and mathematics education for almost a century. It is only in the last ten years that there has been a renewed interest in the phenomenon of creativity. It is in the best interest of the field of education that we identify and nurture creative talent in the mathematics classroom. Creativity as a feature of mathematical thinking is not a patent of the mathematician (Krutetskii, 1976). Most studies on creativity have focussed on eminent individuals (Gruber, 1981). However studying non-eminent samples such as high school students would reveal more to the mathematics education research community about creativity at the secondary level. In particular the questions for exploration are: (1) Does mathematical creativity manifest in the high school classroom? (2) How can the teacher identify and assess creative work?

Literature

The definitions of creativity found in the mathematics and mathematics education literature is vague. For instance Poincaré (1948) defined creativity as the ability of making "mathematical" choices that leads to fruition as opposed to a dead end. Gruber & Wallace (2000) propose a model that treats each individual as a unique evolving system of creativity and ideas, and claim that each individual's creative work must be studied on its own. This model calls for "detailed analytic and sometimes narrative descriptions of each case and efforts to understand each case as a unique functioning system" (p.93). It is important to note that the emphasis of this model is not to explain the origins of creativity, nor is it the personality of the creative individual, but on how creative work evolves. In this model creative work is defined as one that is novel and has value. This definition is consistent with that used by current researchers in creativity (Sternberg & Lubart, 2000). The case study as an evolving system has the following components to it. First, it views creative work as multi-faceted. In constructing a case study of a creative work, relevant facets such as (1) Uniqueness of the work; (2) a narrative of what the creator achieved; (3) systems of belief; (4) multiple time-scales (persistence); (5) problem solving; and (6) contextual frames, are distilled and used to re-create the creative work. The question then is what are the types of tasks that allow for creativity to manifest and how do we go about mediating the conditions under which can study the evolution of the creative work of high school students?

Methodology

In order to answer these questions, the author, in the capacity of a full time teacher designed several teaching experiments with ninth grade students in a rural American public school according to the following criteria. Since discovering an underlying principle or structure is considered as a valid working hypothesis for studying creativity (Gruber, 1981; Gruber & Wallace, 2000), eight contextually different problems were assigned, three of which had an underlying combinatorial structure, namely Steiner Triple Systems. For instance problem 1 was as follows: A woman plans to invite 15 friends to dinner. For 35 days she wants to have dinner with exactly three friends a day, and she wants to arrange the triplets in such a way that each pair of friends will come only once. Is this arrangement possible? (Gardner, 1997). It should be noted that the problems with the structural similarities were not assigned in sequence but were spread out over a 3-month period. The sequence in which these problems were assigned was 1-3-6. Students were especially asked to keep a trail of recorded attempts including trial and error work and asked to provide a written summary of things tried in addition to being asked to reflect on possible similarities of the given problem to problems previously assigned (Sriraman, 2004). These cues were provided with the explicit hope that they would allow for creative traits to manifest in the solutions. Creative individuals are prone to reformulating the problem or finding analogous problems (Frensch & Sternberg, 1992) and are different from their peers in that they tend to reflect a great deal (Policastro & Gardner, 2000). The author's hypothesis was that some of the students in this course would be able to discern the underlying structure of a class of problems by discovering the structural similarities that characterized problems 1-3-6.

Results & Implications

A Steiner Triple System (STS) consists of a set X of n points, and a collection B of subsets of X called *blocks* or *triples*, such that each block contains exactly 3 points, and any two points lie together in exactly one block. In other words, a STS is an arrangement of 'n' objects in triplets such that every pair of objects appears in a triplet exactly once. One of the oldest results in block design theory is Kirkman's existence theorem for Steiner Triple systems, which states that a STS of order n exists if and only if n is congruent to 1 or 3 mod 6 (Hall 1967). The sequence of possible values for n are 3, 7, 9, 13, etc. It follows that there exist ½ n (n-1) such pairs and that the number of required triplets is one-third the number of pairs. Therefore it is possible to construct a STS only when each object is in ½ (n-1) triples, with the restriction that these numbers are integers. STS are fairly sophisticated mathematical objects encountered in undergraduate combinatorics courses. It is therefore quite an accomplishment for any ninth grader to construct these systems and get an insight into how they work. As stated earlier problems 1-3-6 were contextually different problem -solving situations that required the construction of Steiner triplets. Problem 8 was a general/structure problem posed to students who were successful in solving problems 1-3-6 and who reported an isomorphism emerging in the structure of their solutions. Farram was one of the ninth grade students in this classroom who was able to get a deep insight into the structure of STS. The question then is whether or not his insight was a result of creative behavior? Viewing his case study as an evolving system of ideas (Gruber & Wallace, 2000) enables us to assess whether or not his work was creative or not. As mentioned earlier there are several facets that need to be considered when constructing a case study as an evolving system of ideas. Farram's work met the criteria of uniqueness because he was among three students out of the sixteen students in the class who not only conceptually linked the problems that were similar but also gained an insight into the underlying mathematical structure of the problems. It met the criteria of persistence because it took Farram five weeks to construct a solution that had the beginnings of the modular relation that applied to the related problems. His work over the 3-month period showed both intrinsic motivation and persistence. In terms of problem-solving behavior Farram spent a considerable amount of time on global

planning and devising a "general" counting strategy that could easily be adapted to problems which he perceived required an efficient triplet-sorting algorithm. In generating the values of "n" that were the required to induce the modular relation, Farram made the correct and crucial choice (Poincaré, 1948) of reworking relevant cases of previously related problems to generate the correct data to detect the underlying pattern. This was the most intense and frustrating stage where conceptual (creative) activity occurred in his discovery of structure. Reconstructing his case study as a system of evolving ideas revealed that it met most of the facets outlined by Gruber & Wallace (2000). Such case studies have several ramifications for classrooms with students of high abilities. It shows that mathematical creativity does manifest in the high school classroom. Given the opportunity to tackle non-routine problems with complexity, students like Farram rise up to the challenge, and are intrinsically motivated to solve such problems. This implies that teachers should recognize the value of allowing students to reflect on previously solved problems and draw comparisons between various problems. Encouraging students to look for similarities in a class of problems also fosters abstractive behavior, leading some students to discover fairly sophisticated mathematical structures such as Steiner Triple systems (Sriraman, 2004).

Acknowledgement

This work was partly supported by the University of Montana, Small Grants Program.

References

Frensch, P. & Sternberg, R. (1992). *Complex problem solving: principles and mechanisms.* Lawrence Erlbaum and Associates, New Jersey.

Gardner, M. (1997). *The Last Recreations.* Springer-Verlag: New York.

Gruber, H.E. (1981). *Darwin on man.* Chicago: University of Chicago Press.

Gruber, H.E., & Wallace, D.B. (2000). The case study method and evolving systems approach for understanding unique creative people at work. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 93-115). Cambridge University Press.

Hall, M. (1967). *Combinatorial Theory.* Blaisdell Waltham, Massachusetts.

Krutetskii, V. A.(1976). *The psychology of mathematical abilities in school children.* (J. Teller, trans. And J. Kilpatrick & I. Wirszup, Eds.). Chicago: University of Chicago Press.

Poincaré, H. (1948). *Science and method.* Dover: New York.

Policastro, E., & Gardner, H. (2000). From case studies to robust generalizations: An approach to the study of creativity. In R. J. Sternberg (Ed.), *Handbook of creativity* (pp. 213-225). Cambridge University Press.

Sriraman, B. (2004). Discovering Steiner Triple Systems through problem solving. *The Mathematics Teacher,* vol. 97, no.5, pp. 320-326.

Sternberg. R. J., & Lubart, T.I. (2000). The concept of creativity: prospects and paradigms. In R. J. Sternberg (Ed.)., *Handbook of creativity* (pp. 93-115). Cambridge University Press.

REFORMING CALCULUS TEACHING: ORAL ASSESSMENT BEFORE TESTS

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Purpose and Rationale

Hundreds of thousands of students take and fail calculus every year in the United States. Since Calculus I is the gatekeeper to many high status, high paying jobs in science, engineering and technology, these excessive failure rates are limiting the futures of many American students. This research report details an examination of one two-semester Calculus I class where reform mathematics was implemented in an effort to improve conceptual understanding, grades and retention rates for at-risk students.

Robin Wilson (1997) asserted that approximately 40% of all college students take Calculus I. Peterson (1987) reported that in any given semester about 12,000 Calculus instructors were teaching more than 750,000 students in 7,500 universities, colleges and high schools. Each year two hundred-fifty million dollars in tuition and fees, and millions in textbook costs are being spent to teach calculus (Peterson, 1987) at a time when money for higher education has been severely limited by state tax shortfalls.

In a meeting in Washington in 1987, mathematicians reported that "as many as 40% of undergraduates were failing introductory calculus, and even those who passed did not appreciate the subject's relevance" (Wilson, 1997, p. A12). This research was undertaken to address failure rates in Calculus I especially for at-risk students.

Theoretical Framework

Reform-based mathematics is an embodiment of constructivist views of learning that suggest the development of conceptual understanding is enhanced when students are given meaningful opportunities to share solution strategies with peers, to explain their mathematical thinking and to ground new concepts on practical examples that provide connections for student learning. In addition to promoting learning, these strategies provide important feedback for teachers and students alike (Black and Wiliam, 1998)

Critical components of these kinds of classroom activities include the discourse patterns that support student learning. The critical nature of discourse in conceptual understanding is supported on many fronts (Ball, 1991; Cobb, Boufi, McLain and Whitenack, 1997). Other essential elements of the treatment relate to formative assessment (Black and Wiliam, 1998), relatedness (Light, 2001) and identification of misconceptions (Chi & Roscoe, 2002).

Research Question and Methods

Building upon these theoretical ideas, the research question of this paper revolves around the question, "Do oral assessments promote greater conceptual understanding and better grades?" Before every test, students were given the opportunity to take a small group oral assessment. The orals were optional. About half the class took the first oral, but the number of students "volunteering" to takes these orals exceeded 90% by the last exam. Students attended in groups of 3-5 and were asked to explain the major concepts that would be covered on the up-coming exam. I tried to establish whether each member of the group understood the main ideas, was comfortable with the procedures necessary to solve problems, and saw the relationship between the procedures and the concepts. All orals were audiotaped and were an important source of feedback about what students knew. They helped me to scaffold student learning.

In order to be sure students who participate were not being exposed to something that the other students miss, I always made certain that in the last class before the test, I covered all of the questions that were used in orals. I asked parallel questions so that those participating in orals were not faced with exactly the same questions.

The majority of students who took orals, reported that orals were the key to mastering calculus. Students offered that orals helped to see how other students think about the problems. Students claimed that orals motivated them to think more deeply about "what was happening" and provided a way to self-assess.

Findings and Conclusions

An ANCOVA was carried out comparing the grades of students attending orals before the first test to those who did not participate. Those participating did significantly better ($p = .05$). Placement scores were used as a covariate. Though students self-selected whether to participate in orals or not, there is a strong correlation between test success and participation in oral exams. Motivation is a definite limitation. To address motivation in some small way, I examined the first two course tests. Some students chose to participate in orals for only one of the first two tests. Of those who took orals for only one of the first two tests, the average gain when they participated in orals was 8.2 percent, almost a letter grade.

Implications and Future Research

It appears that giving students the opportunity to explain their thinking and talk through the concepts central to calculus helps them to gain conceptual understanding and makes problem solving easier for them. When funding is available, two university instructors, two community college instructors and one high school teacher will each teach both a treatment and control group of calculus I students. The paired courses will be taught in the same manner, but the treatment groups will be required to take part in oral exams before each test. We will then analyze differences in grades, retention rates, and conceptual understanding.

References

Ball, D. (1991). Implementing the Professional standards for teaching mathematics: What's all this talk about discourse? *Arithmetic Teacher*, 39, 44-48.

Black, P. & Wiliam, D. (1998). Assessment and classroom learning. *Assessment in Education*, 5(1), 7-73.

Chi, M.T.H. & Roscoe, R.D. (2002). The processes and challenges of conceptual change. In M. Limon and L Mason (Eds). *Reconsidering Conceptual Change: Issues in Theory and Practice*. Kluwer Academic Publishers, The Netherlands, 3-27.

Cobb, P., Boufi, A. McClain, K. & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education*, 17, 21-30.

Light, R. (2001). *Making the Most of College: Students Speak Their Minds*. Cambridge, Massachusetts: Harvard University Press. Peterson,

Ivars (1987). Calculus reform: Catching the wave? *Science News*. November 14, 1987, 132(20), 317.

Wilson, R. (1997). A decade of teaching reform Calculus' has been a disaster, critics charge. *Chronicle of Higher Education*, 43(22), A12-A13.

DOMAIN-SPECIFIC MATHEMATICS ACHIEVEMENT AND SOCIO-ECONOMIC GRADIENTS: A COMPARISON OF CANADIAN AND UNITED STATES EDUCATION SYSTEMS

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The paper employs the 1999 Trends in Mathematics and Science Study (TIMSS-99) to examine differences in mathematics learning outcomes of grade 8 students in Canadian and the United States education systems. The substantive interest in this study is to explore the extent to which differences in the mathematics achievement levels of the two education systems is attributable to school influences on their students from disadvantaged socioeconomic backgrounds. The analysis indicates that, in all mathematics domains, the two education systems attain equal levels of success for their students from high socio-economic families but the socio-economic disadvantaged students in Canada are more successful learning mathematics than their counterparts in the United States. In the United States, the poor performance of these disadvantaged students is attributable to tracking where students from disadvantaged socio-economic backgrounds are often placed in classrooms with limited opportunity to learn mathematics.

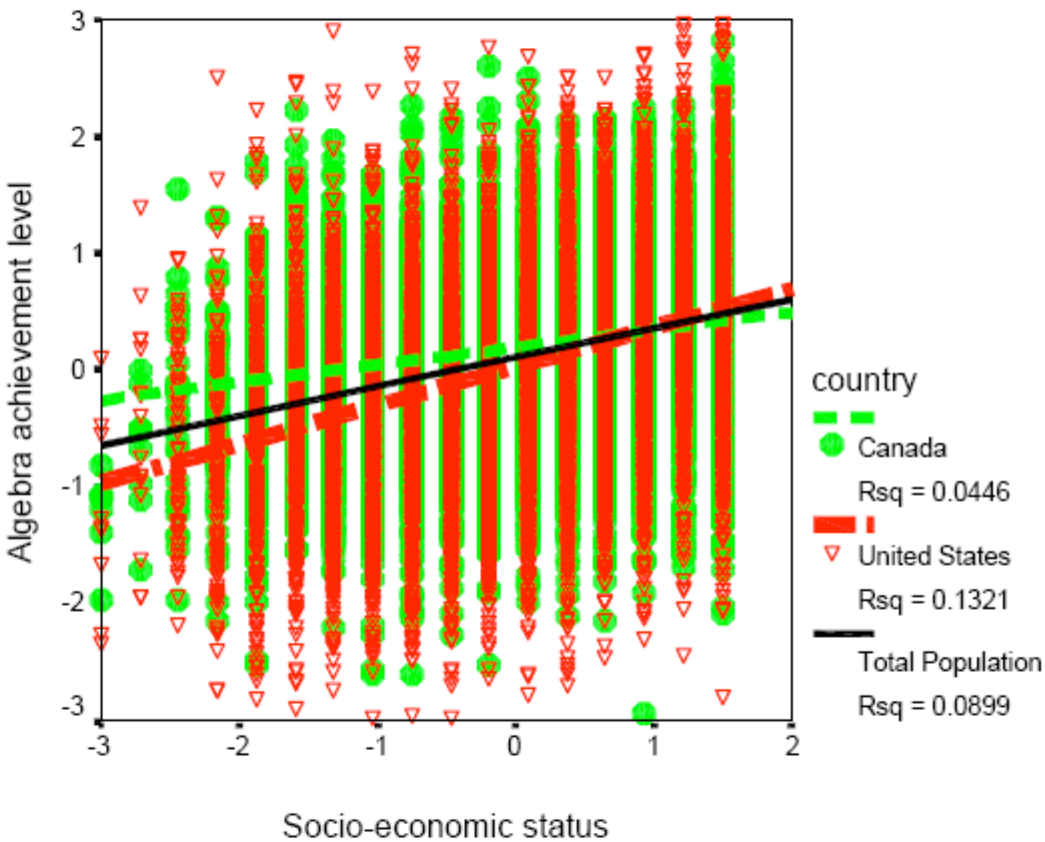
Introduction

This study employs the concept of 'socio-economic gradient' to compare the mathematics outcomes of students in the United States and Canadian grade 8 classrooms. Socio-economic gradient refers to the relationship between an individuals=educational outcomes and their socio-economic status (SES). SES describes a person=s access to, and control over wealth, prestige and power. Measures of SES usually include factors such as income, the prestige of a person=s occupation, and their level of education (White, 1982). Gradients can also refer to gaps in educational outcomes between minority and majority groups, or between males and females, or between immigrants and nonimmigrants. A number of studies have demonstrated consistently the poor school outcomes of students from disadvantaged backgrounds. Thus, socio-economic gradient is a reliable indicator to highlight the gap in school achievement between advantaged and disadvantaged groups. Shallow gradient indicates schooling outcomes are distributed equitable among students of varying SES, while steep gradient demonstrates less equitable distribution.

The concept is particularly useful in assessing the effectiveness of education systems in terms of their success with students from disadvantaged backgrounds. Analyses of the 1995 Trends in Mathematics and Science Study (TIMSS) data for Canada, and the International Adult Literacy Study (IALS) data for the U.S., Canada, and several other OECD countries, reveal that children and youth from advantaged backgrounds tend to fare well in any type of setting; what differentiates countries or states (and provinces) is their performance of students from less advantaged backgrounds (Willms, 1999). This finding suggests that variation amongst these schooling systems in their students' performance is determined mainly by their success with disadvantaged groups and therefore the need to understand how students of differing status perform across different domains of achievement, and whether their performance is related to particular schooling processes.

Statistical Analyses and Findings

The statistical analyses ranged from a simple scatter plot of school outcomes versus SES that provided the initial estimates of gradients, to complex models with multilevel statistical procedures in an attempt to explain differences in socioeconomic gradients among the two education systems. The analyses were done using data describing the six domains of mathematics achievement: fractions, geometry, algebra, statistics, measurement, and proportionality. One of the major finding from the analysis is that, in all mathematics domains, the socio-economic gradient is steeper for United States than Canada. However, the line describing the socio-economic mathematics achievement trend intersect at the high SES levels (see the figure below) suggesting that the Canadian advantage in mathematics is largely due to the successful learning outcomes of her students from disadvantaged socio-economic backgrounds. The poor performance of disadvantaged students in the United State is attributed to tracking that places these students in learning environments with limited opportunities to understand mathematics (Oakes, 1990). This finding suggests that, efforts in the United States to raise their achievement levels in international mathematics assessments should emphasise opportunities to enhance the mathematics learning of students from disadvantaged socio-economic backgrounds.



References

Oakes, J. (1990). Multiplying inequalities: The effect of race, social class, and tracking on opportunities to learn mathematics and science. Santa Monica, CA, RAND.

White, K. R. (1982). The relation between socio-economic status and academic achievement. Psychological Bulletin 91, 461- 481.

Willms, J. D. (1999). Inequalities in literacy skills among youth in Canada and the United States. (International Adult Literacy Survey No. 6). Ottawa, ON: Human Resources Development Canada and National Literacy Secretariat.

ORAL EXAMS AS A TOOL FOR TEACHING AND ASSESSMENT

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Even if we agree that teaching is an interactive process, still the question is how we can invite our math students to actively participate. While learning new material, how can we develop their ability to listen and respond; how can we teach them to think? How can we evaluate their progress in this direction? In this paper I wish to share one possible approach to the problem and my own experience with it. That is, to supplement a written exam by an oral one. I will provide details and results of the experiment I conduct with my students in various mathematics courses for several years, and discuss possible benefits for the students and the teacher.

Motivation. Nowadays North American undergraduate math students are assessed primarily through written exams given during and at the end of each term. This is justified by their more objective character compared to oral exams. On the other hand, in many cases a short conversation with a student can significantly clarity his/her level of understanding, as well as to help the student in further studies. Communicative activities in general enhance understanding in students via articulation of their ideas. But just any discussion will not automatically lead to better understanding: students must receive a professional feedback to direct, if needed, their ways of problem solving, and to assist their reasoning and concept formation. Also, students are often reluctant to participate in group discussions about mathematics mostly due to lack of expertise and mathematical culture. The right time to start such a discussion could be after the midterm test and the subject would be the material covered by the test.

Action. Oral exam is an interview with a student on a given topic or a set of problems. The student gets points later converted to a grade. The teacher conducting the interview has an opportunity to reformulate the question if necessary and to observe a way the student approaches the problem. In my case, oral exam was given as a continuation of a midterm written exam. By that time, the written exam had been graded and returned to the students for review. The students willing to improve their current grade (up to 10%) were invited to talk about test problems, to explain their mistakes, and to solve similar problems.

Benefits. 1. Discover student's background. What is hard and unclear. It helps to plan further lectures and learning activities. 2. Help a student to locate his/her own problems of understanding and to suggest ways to overcome them. 3. Focus on student's positive experience, on moments of capturing the right idea. It gives the student self-confidence, establishes a habit of critical thinking, develops a potential for future growth. 4. Provide a better assessment. Students willing to work receive better grade. 5. Enhance interpersonal relations with the class.

I believe that this approach also accommodates the idea that effective pedagogy must be based on what students currently know, must affirm active cognitive construction and make the student consciously self-aware of its implicit epistemology.

References

Kondratieva, M.F. (2001). The conversational component: a ten minute exam. *Instructional Development. University of Minnesota-Duluth*, XVIII (1), 1-3. Kudryavtsev , L.D.(1985). *Modern mathematics and its teaching.* Moscow: Nauka. (in Russian).

DECONSTRUCTING 'ALTERNATIVE' ASSESSMENT: MOVING BEYOND/WITHIN A DISCURSIVE 'OTHER' TO LINK CURRICULUM, INSTRUCTION AND ASSESSMENT IN MATHEMATICS

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The word 'alternative' is commonly used to refer to something that is 'other', a choice between several possibilities. When used to describe assessment in mathematics, alternative often denotes something other than the traditional tests and quizzes used to assess students' learning in mathematics. Unfortunately, as long as the word alternative is used it is likely that tests and quizzes will remain mainstream and all other forms of assessment in mathematics will be seen as diversions or detours from *real* tests of knowledge.

This poster presents research on the deconstruction of the notion of 'other' that is tied to alternative (or non-traditional) forms of instruction and assessment in mathematics. This is achieved through theoretical and practical discussions of what it means to know (in) mathematics and how elementary pre-service teachers' experiences of learning mathematics have shaped their images of knowing (in) mathematics (Nolan, 2001). The discussions are intimately connected to the need for acknowledging diverse experiences in the expression of mathematical knowledge through a careful examination of instruction and assessment practices.

> Assessments are not just a set of tools that we can add to teachers' curriculum and instruction. They are intrinsically tied to what teachers value, what they teach, the amount of control over learning they share with students, and what they think they are responsible for measuring. (Martin-Kniep, 1998, p. 11)

In deconstructing the use of the word 'alternative', this research draws attention to the need for forms of assessment in mathematics that critically highlight the links between curriculum, instruction and assessment. These links are discussed in this poster presentation in terms of issues such as authentic practice, meaningful and integrative learning, and the nature(s) of mathematics.

The research seeks to inform (and be informed by) the teaching of middle years mathematics curriculum and instruction courses, thus shaping a reciprocal relationship between research and teaching in mathematics teacher education. The poster presents data on recent experiences of incorporating non-traditional forms of assessment into mathematics methods courses. The context for the discussion and presentation of the research is based in experiences with two particular 'alternative' instruction and assessment strategies in mathematics: portfolios and problem-based learning. While still considered 'alternative', mathematics portfolios and problem-based learning approaches represent instructional and assessment techniques that acknowledge the importance of meaningful learning and diverse ways of knowing (in) mathematics.

References

Martin-Kniep, G. (1998). Why am I doing this? *Purposeful teaching through portfolio assessment.* Portsmouth, NH: Heinemann.

Nolan, K.T. (2001). Shadowed by light, knowing by heart: Preservice teachers' images of knowing (in) math and science. Unpublished doctoral dissertation, University of Regina, Regina, SK.

Geometry

THE DIFFICULTY OF UNDERSTANDING 'LENGTH X WIDTH': DOES IT HELP TO GIVE SQUARES TO MAKE IT UNDERSTANDABLE?

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Four tasks were presented to students in grades 4-9 to test the null hypothesis that a square is the unit of an area for middle-school students. This hypothesis was rejected. A developmental trend was found showing that a square begins to become the unit of an area for the majority at sometime between grades six and eight. The findings were interpreted and educational implications drawn in light of Piaget's theory.

"Length x width" remains difficult even in middle school in spite of the repeated instruction given in grades 4-8. For more than 20 years, NAEP has reported that this formula is used by only half of the seventh graders and 65 % of the eighth graders when a rectangle is presented with its dimensions (Lindquist, Carpenter, Silver, & Matthews, 1983; Lindquist & Kouba, 1989; Kenney & Kouba, 1997). The solution to this problem usually recommended by educators is to ask students to cover a rectangle with small squares and to explain the multiplicative relationship with columns and rows (Martin & Stretchens, 2000).

Many data have suggested, however, that a square may not be the unit of an area for students. NAEP has presented rectangles with grids drawn in them, and all that students had to do to give the correct answer was to count the squares. However, only 70%, 64%, and 56%, respectively, of the middle-school students gave the correct answer in the Second, Third, and Fourth NAEP.

In a Michigan State Assessment (Coburn, Beardsley, & Payne, 1975), part of a grid was shaded to show a shape consisting of 2 squares and 4 triangles made by cutting 2 squares diagonally into halves. The students were asked, "What is the area of this shape?" and only half of the seventh graders chose the correct answer of "4 square units." Nineteen percent chose the answer of "6 square units."

Heraud (1987) presented third graders with a large square (40 cm side), a large circle (40 cm diameter), a large equilateral triangle (40 cm side), and a large rectangle (48 x 32 cm), as well as a pile of each of the following small shapes: squares (8 cm side), equilateral triangles (8 cm side), circles (8 cm diameter), rectangles (made with two squares), rhombuses (made with two triangles), and trapezoids (made with three triangles). Each child was asked to choose a big shape and to find out how many small shapes were needed to cover up each one of the big shapes. The children were free to choose the big shapes in any order, and the small shapes were returned to their original positions after each response so that the same piles would always be available to choose from. Heraud found that the third graders most often used small rectangles to cover up the big rectangle, small squares to cover up the big square, small triangles to cover up the big triangle, and small circles to cover up the big circle. He concluded that a square is not the "natural" unit of area for children.

Piaget, Inhelder, and Szeminska (1948/1960) pointed out that it is very easy for children to cover an area with squares and use multiplication because squares are *discrete* objects. However, lengths and areas are *continuous* quantities. Furthermore, they said, length is a *unidimensional* quantity, and the difficulty for children is to understand how a *unidimensional* quantity multiplied by another *unidimensional* quantity can become one *two-dimensional* quantity. When children are presented with two lengths perpendicular to one another starting from the same point, they become able to conceive of a two-dimensional, continuous entity only when they become able to think about the area as a *matrix* consisting of an infinite set of parallel lines "infinitesimally close to one another (p. 350)."

The child does not have to be *conscious* of a matrix consisting of infinitely close lines to be able of thinking about an area. Piaget and Inhelder (1948/1956) showed a straight line (or a square, circle, triangle, etc.) to children and asked them to draw another line half its length (or size), then half of the half, half of the half of the half, and so on. They found in this way that it is at about age 11 or 12 that children become able to think of a line (or a square, circle, triangle, etc.) as consisting of infinitely small points. No one can empirically see infinitely big or small points because infinity is a theoretical construct children become able to create in their minds as they become capable of formal operations (Piaget, 1974/1980). Twelve-year-olds are not *consciously* aware that a line consists of infinitely small points, but when they are asked in a task like the Piagetian task just described, they reveal their ability to think about infinitely small points.

Based on the preceding theory and research, I devised four tasks (among others) to test the null hypothesis that a square is the unit of an area for middle-school students. If this hypothesis is supported, we would be able to conclude that it makes sense to give small squares to children to explain "length x width." If, on the other hand, the null hypothesis is not supported, we would have to conclude that the use of small squares is off the mark, as a square is *not* the unit of an area for students.

Method

In Task 1, a total of 210 students in grades 4-9 were individually interviewed in three schools serving a surburban, middle-to-upper-middle-class neighborhood--38 students in grade 4, 65 in grade 6, 60 in grade 8, and 57 in grade 9. The fourth graders had not been taught "length x width," but they had been asked to cover rectangles with square blocks and to use multiplication to calculate the number necessary to cover each rectangle. In grades 5-9, the formula was reviewed and explained repeatedly every year.

In Task 1, each student was presented with two geoboards, each showing a rectangle (3 x 3 and 2 x 4). The question posed was "If these were chocolate bars, and I asked you to choose the bigger one that has more to eat, which one would you choose? I want you to choose one of these and then count whatever you need to count to prove to me that it really has more to eat." The word "area" was not used in this task because amounts of chocolate clearly implied the size of the surface area. Squares and pegs were equally visible on the geoboards, and the children were free to count either of them depending on what seemed to them to be the appropriate unit of quantification.

Tasks 2, 3, and 4 were presented to 72 eighth graders in one of the three suburban schools mentioned above. Only those in Regular eighth grade math were interviewed with Tasks 2, 3, and 4 because Task 1 had shown those in Advanced sections to be much more advanced in their development than those in Regular sections. Eighth grade was chosen because (a) Task 1 had shown that a square was beginning to become a unit of area for Regular eighth graders, and (b) eighth graders had repeatedly been taught "length x width."

In Task 2, the students were individually shown the L shape illustrated in Table 2 and were asked, "What is the area of this shape?" This task used a small shape with marks along the periphery that made it easy to imagine squares in it. But the marks could also be used to quantify length.

Task 3 involved the use of 9 Color Tiles in two ways. One way was to make a 3 x 3 square and ask, "What is the area of this shape?" The second way was to make an irregular shape with the same 9 tiles and ask, "What is the area of this shape?" The 3 x 3 arrangement was presented first to every other student, and the irregular arrangement was presented first to all the other students. Task 3 was given to find out whether or not students thought about the *space* covered by the 9 Tiles. If a student said "9 units," "9 squares," or "9 square inches" both times, we would conclude that he or she was thinking about the *space* covered by the tiles. If, on the other hand, the student gave different numbers for the two arrangements, we would conclude that he or she was thinking about something else.

In Task 4, each student was shown a card on which a 3 x 6 rectangle was drawn with a grid in it showing 18 squares. Each student was presented with a strip which also had a grid drawn on it. The strip was 4 squares wide, and the student was asked to draw a straight line to show where a straight cut would have to be made to make the strip have exactly the same amount of space as the 3 x 6 rectangle. This task was taken from Hirstein, Lamb, and Osborne (1978), who used it with children in grades 3-6. They reported that many of these children responded that the only way to get 18 squares in the strip was to make a zigzag cut because 4 x 4 would be too small, and 5 x 4 would be too big. Because the children insisted on not cutting the squares into halves, Hirstein et al. concluded that, for these children, squares were rigidly discrete and inviolable objects without any space-filling characteristic.

Results

Table 1 shows the percentage of children at each grade level who counted various parts of the geoboards in Task 1. Squares and pegs were equally observable on the geoboards, but it can be seen in Table 1 that only in the Advanced sections of eighth and ninth grades did the great majority (83% and 93%, respectively) count the squares. Most of the other students in grades 4-8 counted pegs, which are discrete objects. The percentage counting pegs generally decreased from 68 in fourth grade to 47 in the Regular sections of ninth grade. Task 1 thus showed that a square is *not* the unit of area for most students even in eighth grade.

Table 2 shows what eighth graders in Regular math sections counted (or calculated) when asked in Task 2, "What is the area of this shape?" It can be seen that less than half (43%) counted (or calculated) the squares that could easily be imagined. The majority counted the marks (24%), the marks and some or all the corners (23%), or units of length (10%).

In Task 3, when the students were shown 9 Color Tiles arranged in two different ways and asked "What is the area of this shape?" 67% replied that the area was 9 both times. The other eighth graders gave a surprising variety of responses. Some counted the number of sides that were touching or not touching, and others added or multiplied these numbers.

The results from Task 4 are presemted in Table 3. The squares were completely obvious in this task, and all the eighth graders understood that the strip had to have 18 squares. However, only 6% thought about cutting the strip between the fourth and fifth lines (because 4.5 x 4 = 18). The others either drew a zigzag line (7%, because 4 x 4 + 2 = 18) or said the task was impossible (because the only possible solution was a zigzag line (87%)).

Discussion

The null hypothesis of this study stated that a square is the unit of an area for middle-school students. The findings from Task 1 lead to a rejection of this hypothesis. In this task with two geoboards, the squares and pegs were clearly visible, but the unit of area was found to be pegs for the majority in grades 4-8. A strong developmental trend was found, however, with 16% already counting squares in fourth grade, and 41% and 83%, respectively, counting squares in

Regular and Advanced eithth-grade math classes. The unit of an area is not a square for middle-school students in general, but it seems to begin to become a square sometime between grades six and eight.

Task 2 confirmed the findings from Task 1. Squares were strongly suggested by the marks around the L-shaped figure, but when direclty asked "What is the area of this shape?" only 43% of the Regular eighth graders thought that the unit of quantification was squares. All the others counted other variables such as marks, corners, and lengths.

Task 3 involving 9 Color Tiles arranged in two different ways revealed that, for 33% of the Regular eighth graders, the 9 tiles were one thing and the space covered by them was something else. These students gave widely different answers when asked each time, "What is the area of this arrangement?" Empirically *covering* an area was clearly one thing, and quantifying the *space covered by the tiles* was something else for these students.

Squares were explicitly and inescapably visible in Task 4, and this task revealed that eighth graders did not think about them as units of space. Only 6% of those in Regular eighth grade math thought that the way to make a *straight* cut to get 18 squares on a strip 4 squares wide was to cut between the fourth and fifth lines. For all the others, the squares were rigidly inviolable objects having no space-filling function as Hirstein et al. stated in 1978. Hirstein et al.'s students in grades 3-6 objected when it was suggested at the end of the interview that squares could be cut in the middle. The eighth graders in the present study readily accepted this solution, and this acceptance, too, shows a developmental trend. Eighth graders are beginning to accept the space-filling function of squares.

Piaget (Bringuier, 1977/1980) said that human beings do not see reality as it is "out there" and that we see reality with our minds rather than with our eyes. When our logic is more advanced, he said, we see things we could not see before. When eighth graders' logic became more advanced, they began to "see" the squares on the geoboards and in the L-shaped figure. They could likewise begin to "see" the squares in Task 4 as having a space-filling function.

It is often said that "length x width" is difficult for students because they confuse area with perimeter. It is true that this confusion occurs frequently, but the confuson does not explain why students counted pegs in Task 1 or counted marks and/or corners in Task 2. Neither does it explain why the eighth graders did not think about cutting a grid between two lines in Task 4. Why is it that a square is not the unit of an area for students before eighth grade? I have not found a more convincing explanation than Piaget's. As stated earlier, he said that covering an area empirically with squares and using multiplication is easy even for fourth graders. However, elementary-school children cannot understand how two unidimensional, continuous lengths can become an area. To understand this relationship, children have to become able to conceive of a *matrix* consisting of infinitesimally close parallel lines. Since infinitely close lines cannot be observed empirically, they have to be *constructed* by the mind. When a matrix of infinitely close lines can be constructed, it becomes possible for children to think about a square as the unit of an area. This is why, in Task 1, 83% and 93%, respectively, of the eighth and ninth graders in Advanced math sections became able to think of squares as units of areas.

The educational implication of this study is that, at the present time, "length x width" does not appears to become understandable by the majority before six or eighth grade. Further research is necessary to clarify this point. However, it may well become understandable earlier if children were encouraged to *think* more since the preschool years.

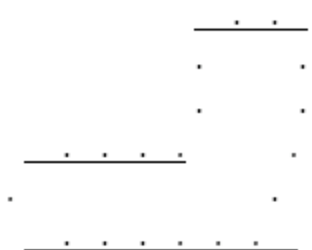
The construction of a matrix consisting of infinitely close parallel lines grows out of a complex network of mental relationships that children have built over many years and is not teachable through "this specific activity" or "that specific activity." An example of encouraging children to think in elementary school is to encourage them to invent their own procedures for multidigit addition, subtraction, multiplication, and division rather than teaching them conventional algorithms to mimic. In the measurement of length, children can be asked to solve problems like figuring out how much paper to bring from the office to cover up a bulletin board. Empirical procedures such as aligning paper clips along a pencil and counting them are mindless and useless as explained in Kamii and Clark (1997). In dealing with areas, too, children should be given problems to solve rather than formulas to memorize. A problem like the one involving chocolate bars is a good possibility. Another example might be to choose between two plots that Daddy says a child can use to raise vegetables to sell during the summer months--one plot being 10 yards long and 10 meters wide, and another plot being 12 yards long and 8 meters wide.

In the sociomoral realm, too, children are now educated mindlessly with ready-made rules and reward and punishment, but they should be encouraged to think more and to construct moral rules and values from within. For example, on the first day of school, many children are now greeted with a set of rules such as "Raise your hand and wait to be called on (rather than speaking all at once)." It would be much better for children's sociomoral development as well as their intellectual development if teachers waited for problems to arise and asked the class, "What can we do about this problem?" Many other examples of encouraging children to think all day long can be found in Kamii (1994, 2000, 2003, 2004).

Table 1. Students in Grades 4-9 Who Counted Squares, Pegs, and Other Units to Decide Which Chocolate Bars Had More to Eat (in Percent) Grade level

	Grade level					
	4	6	8		9	
			Reg.	Adv.	Reg.	Adv.
	n=38	n=65	n=27	n=23	n=30	n=27
Counting squares (9 vs. 8)	16	56	41	83	53	93
Counting pegs (16 vs. 15)	68	38	59	17	47	7
Others (e.g., counting units of length)	16	6	0	0	0	0

Table 2. What Eighth Graders Counted When Asked, "What is the area of this shape?"



Counting or calculating the number of squares (23) 43%
Counting marks (18) 24%
Counting marks and some or all the corners 23%
Counting units of length (e.g., perimeter) 10%

Table 3. Eighth Graders' Ways of Cutting a Strip Four Inches Wide to Make a Rectangle Having an Area of 18 Squares

Cutting at 4.5	6%
Making a zigzag cut	7%
Others (e.g. saying that the task was impossible)	87%

References

Bringuier, J.-C. (1980). *Conversations with Jean Piaget*. Chicago: University of Chicago Press. (Original work published 1977)

Coburn, T. G., Beardsley, L. J. & Payne, J. N. (1975) *Michigan Educational Assessment Program, Mathematics Interpretive Report, 1973, Grade 4 and 7 Tests. Guidelines for Quality Mathematics Teaching Monograph Series, No. 7*. Birmingham, MI: Michigan Council of Teachers of Mathematics.

Hirstein, J. J., Lamb, C. E., & Osborne, A. (1978). Student misconceptions about area measure. *Arithmetic Teacher,* 25(6), 10-16.

Kamii, C. (1994). *Young children continue to reinvent arithmetic, 3rd grade*. New York: Teachers College Press.

Kamii, C. (2000). *Young children reinvent arithmetic,* 2nd ed. New York: Teachers College Press.

Kamii, C. (2003). Modifying a board game to foster kindergartners' logico-mathematical thinking. *Young Children*, 58(5), 20-26.

Kamii, C. (2004). *Young children continue to reinvent arithmetic, 2nd grade*, 2nd ed. New York: Teachers College Press.

Kamii, C., & Clark, F. (1997). Measurement of length: The need for a better approach to teaching. *School Science and Mathematics*, 97(3), 116-121.

Kenney, P. A., & Kouba, V. L. (1997). What do students know about measurement? In Kenney, P. A., & Silver, E. A. (Eds.) *Results from the Sixth Mathematics Assessment of the National Assessment of Educational Progress*, 141-163. Reston, VA: National Council of Teachers of Mathematics.

Lindquist, M. M.; Carpenter, T. P.; Silver, E. A.; & Matthews, W. (1983) The Third National Mathematics Assessment: Results and implications for elementary and middle schools. *Arithmetic Teacher*, 31(4), 14-19.

Lindquist, M. M., & Kouba, V. L. (1989). Measurement. In Lindquist, M. M. (Ed.), *Results from the Fourth Mathematics Assessment of the National Assessment of Educational Progress,* 35-43. Reston, VA: National Council of Teachers of Mathematics.

Martin, W. G., & Stretchens, M. E. (2000). Geometry and Measurement. In Silver, E. A, & P. A. Kenney (Eds.), *Results from the seventh mathematics assessment of the National Assessment of Educational Progress*, 193-234. Reston, VA: National Council of Teachers of Mathematics.

Piaget, J. (1980). *Experiments in Contradiction*. Chicago: University of Chicago Press. (Original work published 1974).

Piaget, J., & Inhelder, B. (1956). *The child's conception of space*. London: Routledge & Kegan Paul. (Original work published 1948).

Piaget, J. Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry*. London: Routledge & Kegan Paul. (Original work published 1948).

CHILDREN'S EVOLVING UNDERSTANDING OF POLYHEDRA IN THE CLASSROOM

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Most previous research on children's geometric thinking has taken place in clinical settings. We add to this research base by providing data about changes in children's thinking over time as a result of classroom instruction. Children's written work and class discussions were used to consider the attributes to which children attended as they described polyhedra that they had built. We found that recent theories reformulating the van Hiele levels adequate to characterize children's thinking. In particular, the syncretic level, proposed by Clements et al. (1999) described the children who remained at this level for the whole year. Within that level children changed by attending more to properties and less to visual impressions although they had trouble attending to more than two properties at a time. Specific examples from a classroom episode are provided to illustrate how analytic property-based thinking was fostered in the classroom.

This project represents an effort to bring research on children's geometric thinking into a classroom to test theory and to develop tasks that instantiate theory. We are particularly interested in this domain because although the NCTM Standards include geometry as a critical component of mathematics instruction, geometry tends to be overlooked in the U.S. (Battista, 1999). We chose a focus on children's conception and creation of three-dimensional forms because of children's interest in building and because manipulation of shapes is theorized to be critical to children's learning of geometry (Clements & Battista, 1992).

We take as our starting point the five van Hiele levels (Clements & Battista, 1992), particularly the first two, which are most relevant to early elementary school. They are as follows:

Level 1 (Visual): Students attend to the overall visual appearance of geometric figures. When naming figures, they use familiar prototypes, saying, for example, "That looks like a door so it must be a rectangle." Students do not yet associate particular properties with classes of figures.

Level 2 (Descriptive/Analytic): Students attend to component parts of figures; they can discover properties of figures and classify into one class all shapes that have the same properties, noting, for example, "That has short sides and long sides, so it must be a rectangle." Students do not yet order classes of figures or see relationships between classes of figures.

Several researchers have suggested alterations to the levels. Burger and Shaughnessy (1986) found that many students in their study were at transition points, showing that students' thinking could be at more than one level during a single task. Lehrer and his colleagues (Lehrer, Jenkins, & Osana, 1998) found that children's thinking was better characterized by variability than by consistency. Clements, Swaminathan, Hannibal, & Sarama (1999), proposed a reconceptualization of van Hiele level 1, proposing the term *syncretic level* to account for the fact that children use both visual thinking and declarative knowledge of properties. A child at this level might categorize a non-square rhombus as a square because he is applying his idea that "squares have four sides the same and four points". Because his verbal description does not include information about right angles, he is led to make a mis-categorization that he probably would not have made had he only been relying on his visual prototype of a square. Taking in to account this *syncretic level,* Clements et al. recommended that instruction promote imagery that

is flexible, dynamic, abstract, and manipulable, along with "development of robust, explicit knowledge of components and properties of geometric shapes" (p. 208). They proposed doing this by having children describe why figures belong (or do not belong) to shape categories as they work with a wide array of exemplars and non-exemplars. By discussing their categorization of shapes, children begin to develop declarative knowledge to describe and categorize those images.

The goal of this research was to extend previous research, which had taken place in clinical settings, by taking it into the classroom to test out Clements et al.'s (1999) instructional recommendations and also to provide data about changes in children's thinking over time as a result of instruction. We were curious to see if activities like those suggested would advance student thinking. We wanted to see how the distinction between visual thinking and property oriented thinking would inform instruction. We also wanted to explore if and how the amended van Hiele levels applied to children's thinking about three-dimensional figures.

Method

This study was conducted in a third-grade classroom as a *conjecture-driven* teaching experiment (Confrey & Lachance, 2000). The conjecture driving the project was that students, through their use of building and describing structures, would become increasingly sophisticated in their examination of three-dimensional forms progressing along the continuum from visual thinking based on holistic impressions toward more analytic thinking based on attention to specific properties of figures. All classroom activities included some building component using Polydrons to provide the tactical/kinesthetic experience that stimulates learning in this domain. The analysis used for this paper represents preliminary analysis performed as part of the ongoing work of the project.

Our instructional design followed the hypothesis that children's thinking would progress as they articulated the visual images, which formed the basis of their thinking. We were following the van Hiele theory that advancement in thinking is achieved as the implicit becomes explicit. We identified those features that children noticed and then designed activities that drew upon these salient features and required the students to elaborate on the specifics of the features to which they were attending. For example, we noticed that some children were describing some polyhedra as having "pointy tops". In making these observations they seemed to be drawing on their prototype of a triangle demonstrating visual thinking. We developed an activity involving pyramids in which children would have opportunities to specify what the features of a "pointy top" were. We were hoping that questions like, "what do you mean by pointy top?" would evoke definitions like, "it's the highest point and it's where triangles meet." We hypothesized that students would be able to come to some agreement about the specifics so that they could defend their characterizations of their constructions.

Data

Nineteen children from a low-performing school in an urban area participated in the study (2 moved away from the school during the year and one joined the class). Eleven of the children spoke English as their second language. We conducted 16 lessons in the classroom on a biweekly basis from late October until early May. The data corpus consisted of pre-and post-interviews and written assessments, videotapes of classroom segments, photographs of student constructions and students' written classwork. We videotaped individuals who were talking about their constructions, and each class discussion was logged to allow for future reference. For the purposes of this paper we focus on the 12 students who completed both the pre-and post-written assessment.

Students' work and discussions were coded according to how they described the polyhedra that they built. Our coding scheme was emergent, changing to accommodate new features that children included in their work. We adopted the two superordinate categories used in the Clements et al. study (1999) that correspond to the kinds of thinking indicative of the *syncretic* level; visual and property. Responses could have more than one code depending on the variety of attributes a child noted in their response. Some interpretation was required to do the coding because the third graders writing skills were limited due to language issues and limited attention to writing in daily instruction. For the purposes of clarity in reporting, children's spelling has been corrected.

In order to determine changes in children's thinking, we considered their written descriptions of polyhedra written at 5 different points in time – 10/24/03 (the pre-assessment), 11/13/03 (near the beginning of instruction), 3/25/04 (in the middle of instruction), 5/06/04 (near the end of instruction), and 6/2/04 (the post-assessment). Overall counts demonstrate the variety of things students included in their descriptions. Percentages indicate the degree to which students employed visual thinking vs. more analytic thinking involving properties. One of the limitations of this analysis is that during the class sessions the children were describing different polyhedra depending on what each individual happened to build on that particular day. In order to better understand the relationship between changes in children's thinking and classroom activity, we analyzed the discussions and interactions that were recorded during class on 12/11/04. This analysis helped us to understand the degree to which we were successful in building on children's thinking.

Results

Pre/Post- Assessments

Three tasks given on the pre-and post-written assessment were used for analysis because of their suitability for consideration of the children's use of properties. In two of the tasks, the children were asked to compare a pair of polyhedra – (1) a cube vs. a square pyramid and (2) a hexagonal prism vs. a hexagonal antiprism. In the other task, the children were asked to describe a compound polyhedron with a cube on the bottom and a square pyramid on top (elongated square pyramid). The children included visual and property responses on both the pre- and post-assessment. Table 1 shows how children's descriptions of polyhedra changed over time. At the beginning of the study, their descriptions were brief. Of the 75 codes assigned to the 60 responses, 32% were property codes, and 43% were visual codes. At the end of the study, the students' responses were longer as is evidenced by the 92 codes assigned to the 60 responses on the post-assessment, 64% of those were property codes and 30% were visual. These results show that at the end of the study a larger percentage of the children's responses included observations about the properties of polyhedra and that children continued to discuss visual features up until the end of the study.

During instruction children's written responses tended to focus on property features. Over 70% of the codes for the responses for each of the instruction days considered here were property codes. Children went from writing about the types of 2D shapes in polyhedra to writing about the number of 2D shapes. It is notable that on 11/13/03, some students counted edges and vertices but no students discussed this on the post-assessment. We hypothesize that this is because in

Figure 1: Coding of Children's Descriptions	Whole Class (12 students)					Janet		Fang		Yessenia		Peter	
	Pre-test (5 items)	11/13/03 (1 item)	3/25/04 (1 item)	5/6/04 (1 item)	Post-test (5 items)	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Property													
Type of 2D shape	18	5	1	1	32	1	1	1	2		2	5	4
Num. of 2D shape		7	9	8	13		1		1		1		1
Num. of faces	3	4	2	1	2	1							1
Num. of edges		3		1									
Num. of vertices		1	1	1									
Decompose shape	3				4						1		
Refer to net			1										
Refer to apex				1	4		1						1
Refer to base					2								1
Refer to symmetry				4	2								1
Total Property	24	20	14	17	59	2	3	1	3	0	4	5	9
% of total codes	32%	71%	82%	77%	64%								
Visual													
Top, bottom, sides	8			4	16	1	2		1		2	1	1
Refer to "pointy"	1									1			
Notice orientation					1		1						
Refer to colors	1	4			1						1		
Familiar association	10	2	2		5	1	2	1	1	1		1	
Name a 2D shape	3												
"Same"-ness	6				4	1		1					
Mention size	3		1		1			1		1	1		
Total Visual	32	6	3	4	28	3	5	3	2	3	4	2	1
% of total codes	43%	21%	18%	18%	30%								
Vocabulary													
Unspecific vocab.	15	1			1			1		3	1		
Correct association	4	1	1	1	4						1	1	
Blank / Unknown	4		1	1	1								

introducing polyhedra terminology to the students we defined the words vertices and edges and some of them were intrigued with using these new words even though they may not have otherwise attended to those features of the polyhedra. Students tended not to appropriate this way of analyzing polyhedra perhaps because these attributes were not salient to them.

Individual children varied in the extent to which their responses changed. The median change of number of property codes from pre- to post- was 3 and the median change in the number of visual codes was 0. Table 1 contains pre/post- data for four children who represent the diversity of change observed in individuals. The child who changed the least on the written assessment, Janet, continued to write about the appearance of polyhedra. For example in responding to the prompt to describe the elongated square pyramid on the pre-assessment, she wrote, "It looks like a house, shark and very weird." Her response to the post-assessment was similar. She wrote, "It looks like a house, bird's beak and pour spout." Both of these responses were coded as FAMILIAR ASSOCIATION, a visual code, because the child associated the figure with an everyday item (a house). While Janet demonstrated the ability to be more analytic in her descriptions, as will be demonstrated below, a visual approach dominated her thinking. The child whose change was identical to the median change for the class, Fang, initially wrote about the difference between a cube and an antiprism as, "they are different because they don't look the same," which was coded as SAME, a visual code, because he based his comparison on a visual inspection without mentioning specific attributes of the polyhedra. In the post-assessment Fang wrote, "they are different because it don't have a lot of squares," which was coded as TYPE OF 2D SHAPE, because he noticed that the square pyramid had fewer squares than the cube. Fang, like many of his classmates, tended to focus on the 2D shapes that polyhedra were made of and typically wrote about that in his descriptions.

The child, whose change was among the most dramatic in the class, Yessenia, initially described the elongated square pyramid as "it looks like a house." At the end of the study she wrote, "It has a pyramid on the top and a cube on the bottom and it top of the cube is open up (sic)." She went from a FAMILIAR ASSOCIATION to a more analytic response where she considered DECOMPOSE(ing) the shape into its component parts. She also noted that when the two parts were put together the top of the cube had to be taken off. Peter, the student with the most codes on the post-assessment, began the study describing the elongated pyramid as "it is shaped like a house, lots of squares and lots of triangles, like a cube." This was coded as FAMILIAR ASSOCIATION (shaped like a house), TYPE OF 2D SHAPE (lots of squares and lots of triangles), and CORRECT ASSOCIATION (like a cube). Even at the beginning of the study Peter was attending to several features in his description. At the end of the study he wrote, "It has 4 triangles, 5 squares, a base, an apex, can spin, 9 faces." This was coded as NUMBER OF SHAPES (4 triangles, 5 squares), BASE, APEX, SYMMETRY (can spin), and NUMBER OF FACES showing that Peter considered a variety of attributes as he looked at polyhedra and all were related to properties rather than visual images. The children's results showed that they used some properties in their analysis, most of which were related to type of shapes, while they continued to refer to visual properties. Some children adopted a more analytic approach than others.

Classwork

After four class sessions, which included building and describing polyhedra, the class engaged in a sorting and building session to promote understanding of pyramids. To begin the activity the children were shown a large triangular pyramid and a large square pyramid and asked what they had in common. After the group established that these pyramids had "mostly" triangles, and pointy tops (we provided them with the term apex), each child was given a polyhedron and asked to determine if it was a pyramid or not. In progressing through the sorting activity, two ideas came up – all of the triangles around the sides of the pyramid had to come all the way down to the base and a pyramid was all triangles except possibly for the base which could be some other shape. While the wording for these properties may not have been succinct, the ideas were mathematically correct and supported the children in sorting through 20 different polyhedra.

After the discussion each child responded in writing to the question, "What is a pyramid?". Two children provided visual responses. one wrote, "a pyramid is a triangle that is so big that you could sneak in it." Four children included visual responses along with one property. One wrote, "a pyramid is like a triangle. A pyramid is made with triangles. A pyramid is a tall thing." Five children wrote about two properties, one of which was related to the base. Peter wrote, "It's something built long ago. They all have to touch the base. There are different kinds of them. All triangles join together at the apex." One student wrote about the three properties that had been brought up in the discussion. He wrote, "A pyramid is a shape with triangles, with an apex. The triangles have to touch the base." While this exercise might be construed as simply asking the children to parrot back what had just been discussed, the responses make it clear that only one child was able to keep track of all of the properties and many continued to discuss visual properties.

After defining pyramids the children were asked to build one. Janet who had defined pyramids as, "a shape with one apex, base and however many sides the base has," went on to build a polyhedron which had the overall triangular appearance of a pyramid with a hexagonal base and three layers of lateral faces which included squares and triangles. She claimed it was a pyramid because, "It has a bottom and an apex" however it was not a true pyramid. She focused on two properties of pyramids and correctly used them in building but could not attend to all the features that define one. We hypothesize that the visual orientation which became apparent on the post-assessment led her to consider her figure a pyramid because it had the overall appearance of one. Yessenia who defined pyramid as, "a shape that has triangles, and apex and bases", built a complex shape made exclusively of triangles. In describing it, she noted, "it only has triangles. It has green, yellow, red and blue. It has 12 triangles." Yessenia's shape did not have the overall appearance of a triangle leading Janet to remark, "it doesn't look like a pyramid." Both children built complicated shapes that differed from several classmates who duplicated something from the set that had been discussed earlier. Janet depended in part on the overall visual appearance to drive her thinking while Yessenia depended on one property to the exclusion of the visual appearance. We argue that both were at the *syncretic* level in their thinking in trying to apply some properties that define a pyramid but not being able to account for all of the properties.

The whole class assembled to discuss the polyhedra that children had built. In defending why her figure was a pyramid Yessenia pointed to one of the vertices and identified it as an apex. She also mentioned that her figure had triangles. When asked to give his opinion about whether Yessenia's figure was a pyramid, Peter provided a *syncretic* response combining a property and visual response in noting that "it is made out of triangles (TYPE OF 2D SHAPE) but it doesn't have like a triangle shape (FAMILIAR ASSOCIATION)." Another child remarked that he thought it Yessenia's figure had two bases (BASE) and another noted that it was not a pyramid because it was "tilty" (FAMILIAR ASSOCIATION). In discussing her figure, Janet noted that her triangles did not go all the way down to the base. She classified her figure with the two others in the class that had been called “almost pyramids.” Here again we see children proposing visual judgments and property judgments.

Discussion

We found that recent theories reformulating the van Hiele levels adequate to characterize children’s thinking. The children in this study remained at what Clements et al. (1999) referred to as the *syncretic* level for the entire year. The children continued to evaluate polyhedra based on their overall appearance as they began to use some properties in their descriptions. Analysis of the written data show that the balance between visual and property descriptions changed in favor of property descriptions but children used both approaches in their work even after a year of activity. In describing figures children typically noted one or two features but successfully classifying shapes required that they attend to more than two. In the domain of 3D geometry, the property that children initially attended to was the component parts of the polyhedron, specifically the shapes of which it was composed. This may have been due to the fact that these countable features were the most concrete. Other properties were not only less obvious but their definitions tended to elude children. For example, many children readily adopted the term apex, and used it when they would have said “pointy top,” making the implicit become explicit as suggested by van Hiele. The word “apex” seemed to be linked to a visual image rather than a specific property. Some children began to refer to any vertex as an apex and others applied it more conservatively relating it to a high point in the middle of a polyhedron. Even though we had discussed that an apex was where all of the triangles on a pyramid met, none of the children recalled those specific details of the definition.

We were convinced that Clements et al’s (1999) instructional recommendations were sound but found them challenging to implement. For us the challenge in building on children’s thinking was that their ability to distinguish shapes based on visual cues served them quite well in most cases (they usually could distinguish a pyramid from a not-pyramid). It was difficult to develop tasks that would help them to see the importance of specific definitions that go beyond visual descriptions but it is just these tasks that are necessary to promote more analytic thinking.

References

Battista, M.T. (1999). Geometry results from the Third International Mathematics and Science Study. *Teaching Children Mathematics, 5* (6), 367-373.

Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education, 17,* 31–48.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 420–464). New York: Macmillan.

Clements, D. Swaminathan, S., Hannibal, M.A.Z. & Sarama, J. (1999). Young children’s concepts of shape. *Journal for research in Mathematics Education, 30,* 192–212.

Confrey, J., & Lachance, A. (2000). Transformative teaching experiments through conjecture-driven research design. In A. Kelly & R. Lesh (Eds.), *Research design in mathematics and science education* (pp. 231-266). Mahwah, NJ: Erlbaum.

Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children’s reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137–167). Mahwah, NJ: Erlbaum.

LEARNING TO REMEMBER: MATHEMATICS TEACHING PRACTICE AND STUDENTS' MEMORY

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The goal of the paper is to examine teaching practices with respect to students' memory in the hopes of identifying and describing those practices which are used in helping students learn mathematics. The questions of how teaching practice influences what students remember, how teaching practice aids in the recall of mathematical knowledge, and how those practices influence the mathematics being learned in the classroom are the focus of this study.

Consider three events. First, a non-scientific survey of friends and co-workers shows that most remember a few key words from their high school mathematics experience. Those asked could certainly recall FOIL and SOH-CAH-TOA as more than a kitchen wrap or tones of the scale; indeed, many could recall "factoring" and "trigonometry" even if they could not recall the details of what those terms meant or how those mnemonics might be useful. Second, a teacher and mathematician chuckle over the fact that whenever someone hears of their career path, they are suddenly bombarded with statements of triangle congruency theorems or the quadratic formula, even though with some follow-up those memories are little more than names and formulas known by name but not by function. And third, a student questions the teacher as to why it is necessary to memorize so many theorems in geometry class. After all, he argues, most of the theorems about parallelograms follow simply from those learned about parallel lines and transversals. His classmates struggle to catch on to his comment until the segments of a parallelogram are extended to lines and the relationships are pointed out by the teacher. The use of theorems unique to parallelograms continues and transversals are forgotten again. These snippets are part of a broader sense that funny things happen to students' memory inside the mathematics classroom. Despite the fact that the memories were created around mathematical content, the memories that remain often times seem to have nothing to do with mathematics.

Theoretical Perspective

The teacher sits in a precarious place within the institution of school, caught between at least three, often varying, sets of demands: the school, the student, and the knowledge at stake. Because the teacher and students function within the school environment, they are bound to what Brousseau (1997) calls the *didactical contract*, an implicit understanding that teachers are responsible for teaching students the necessary concepts, definitions, or theorems that they need to know and students are responsible for learning. The measurement of the success of this contract comes through the assessment of students' ability to recall and use the information that both parties are held responsible for (whether by chapter tests, assigned grades, or state testing standards). Thus, it is within the best interest of the teacher to not only teach students mathematics, but to make those concepts accessible for students to use in situations of assessment. Such assessment tools exhibit publicly that the contract between student and teacher has been fulfilled: the desired mathematics has been successfully displayed, thus students have learned and teachers must have taught. It is assumed that the negotiations involved in the learning of the subject matter have upheld the mathematics of the situation. But as we see in the examples above, while the contract is often times appears fulfilled, to what extent is the memory of the mathematical knowledge at play in the classroom accounted for? What mathematics is taken away from students' classroom experiences?

The goal of the paper is to examine teaching practices with respect to students' memory in the hopes of identifying and describing those practices which are used in helping students learn mathematics; such issues are important to consider as the study of memory is so closely tied to the study of learning (Tulving, 1968). The questions of how teaching practice influences what students remember, how teaching practice aids in the recall of mathematical knowledge, and how those practices influence the mathematics being learned in the classroom are the focus of this study.

In identifying teaching practices used in helping students learn mathematics, it is beneficial to consider a model of learning; here, I rely upon the information processing model. In review of several popular texts, this model is what teachers are likely to be familiar with from their studies in educational psychology. Moreover, the information processing model, with its emphasis on memory and recall via connections between knowledge, also parallels much of the educational literature including the National Council of Teachers of Mathematics (2000) document *Principles and Standards for the Teaching of Mathematics*, which places emphasis on the importance of building on and making connections to students' previous mathematical knowledge (see also, Hiebert & Carpenter, 1992).

In brief, the information processing model views memory as a three-part process. These three processes - acquisition, retention, and recall - are important in guiding the examination of data for instances of teaching practice related to students' remembering of mathematics. The teacher has (at least) three jobs in the teaching of mathematics: getting students to acquire new mathematical knowledge, helping students retain that knowledge over time, and initiating practice in the recall and use of that knowledge as it is needed. In the information processing model, the acquisition of new knowledge comes through encounters with new concepts or ideas and the subsequent entry of those concepts as nodes in the network of knowledge. However, concepts are not merely entered into memory in isolation, but rather, are part of a "web" of knowledge.

Keeping our knowledge from being lost is the outcome of retention and is believed to be a consequence of the value assigned to something as it is acquired and as a consequence of the strength of the connections between that piece of knowledge and other things one knows. Knowledge may be lost if it cannot be accessed readily because of the way it was acquired initially or if it has not been used and the connections are relatively weak. In the mathematics classroom, retention is the reason we provide our students with numerous practice problems, chapter reviews, or study guides for the final exam.

Finally, what good is our knowledge if we are unable to use it? Recall is often expedited via the concepts in memory that are linked to the desired concept. Thus, when I see a face I recognize, I may attempt to recall where I have seen it before, the other people I was with, or the first letter of the name. That is, I access the name through calling upon my memories of the characteristics that are linked to that knowledge. Similarly, in the classroom, teachers make attempts to link concepts to particular problem types, drawings, or heuristics.

Methods and Data Sources

The data for this study come from a year of observation in two high school geometry classrooms, taught by teachers that I call Cecilia and Megan. Both classrooms are from the same high school, in a midsize city in the Midwest. The classrooms were studied as part of a larger project that seeks to uncover ways in which teachers of geometry use and think about using proof in their classrooms. The students involved in these lessons are part of the accelerated track within the school's mathematics department and are almost all freshmen in high school.

Because the questions at hand call for an examination of teaching, the primary source of data for this project comes from the library of tapes collected as samples of teaching from each teacher. Geometry makes an excellent candidate for a study of the students' remembering. High school geometry is typically the first place where students are exposed to the structure and methods of the mathematician. Learning in geometry requires understanding postulates and how theorems build off of those accepted truths to create further knowledge in contrast to traditional views of mathematics as the teaching of procedures. Geometry requires students to create and discover "new" pieces of knowledge. Furthermore, in geometry concepts are repeatedly recalled and connected through the process of proof. Students must not only remember what they have already learned, but they must also be able to call upon that knowledge in multiple situations (including proofs) which utilize that knowledge. These situations often appear very dissimilar to the eye of the student than those situations where the knowledge was acquired (say for example using triangle congruency proofs to prove properties of circles), testing students' ability to recall prior knowledge.

Videos from each of the lessons taught by Cecilia and Megan were reviewed according to the processes described by Powell, Francisco, & Maher (2003). Videotaped lessons were viewed and coding categories developed across lessons from both teachers. Episodes throughout the lessons were selected for coding if they addressed the teacher's role in helping students to acquire new knowledge or in helping students recall their knowledge. Having these criteria in mind, the majority of episodes coded occur at times when new concepts or theorems are being introduced or when the teacher is reviewing homework or other assigned problems with the class as a whole.

Results

Through the processes of coding the video data, eleven categories were identified as types of teaching practice that influence both what students understand to be memorable ideas, concepts, and events in the classroom as well as ways in which teachers help strengthen and organize students' knowledge for later access and use. In this paper, I report a brief description of each of these categories; for further description and examples of episodes which characterize each category the reader may refer to a larger report of this same study (Brach, 2004). The various codes generated reflect the dimensions of the work of the teacher in terms of acquisition, retention, and recall of knowledge.

Teaching practices and the acquisition of knowledge

Auditory & visual cues. Oliver Wendell Holmes writes (1858), "Memory is a net; one finds it full of fish when he takes it from the brook; but a dozen miles of water have run through it without sticking". These first two codes, auditory and visual cues, are likely in place to help the teacher identify what it is that should be "caught" by the mind during class discussions. Of the two teachers studied here, one relied heavily on auditory cues to identify important concepts that students are responsible for remembering. Changes in tempo and tone of voice were particularly common in Cecilia's class and are identified by the slowing of tempo and the intensity of her voice when new vocabulary is introduced. In contrast, the monotone quality of her voice when introducing "unnecessary" concepts (theorems that are simple corollaries of others or vocabulary of secondary importance) is noticeable by the observer.

Megan relies primarily on visual cues. In Megan's class, important pieces of knowledge are emphasized by including them in her notes at the board and highlighting important terms by underlining. Student interviews provide evidence for the importance of this practice. When students in Megan's class participated in an alternative lesson characterized by its emphasis on students generating their own definitions and theorems, the most common complaint from students was that they weren't sure what they were supposed to be learning from the lessons because of the lack of notes with important terms underlined.

The auditory and visual cues presented to students are one mechanism for helping students to identify and acquire the relevant ideas introduced in the classroom. Research in psychology points out that because we are constantly inundated with experiences, facts, and knowledge our brains require mechanisms for the selection of stimuli to focus on (Langer, 1978). Such mechanisms can be described as scripts or schemas and are activated in particular situations in order to help identify what is important for later recall (Fiske, 1993). Thus, students reliance on auditory or visual cues for identifying what they are responsible for learning in the classroom can be explained as a product of their experiences in classrooms and their learning of the relevant mechanisms a teacher uses in helping students to "catch" the ideas they are responsible for learning while other things happening in the classroom are ignored. Indeed, educational psychology texts suggest such things as varying voice intensity, pitch, and tone (Borich & Tombari, 1997) and establishing in one's notes ways that students can learn to pay attention to what features of a lesson are most important to attend to (Hohn, 1995).

Accountability cues. A teacher states clearly, "You need to know this, it's going to be on the test." Phrases like this which assign value to the statement of a concept or problem-type have been categorized as accountability cues. While a type of verbal cue, these episodes more blatantly grab the attention of the student and can be viewed as a part of the didactical contract for the same reasons as the verbal and visual cues – teaching practice focused on accountability identifies for students what they are responsible for knowing.

Repetition. If we continue with the metaphor of fishing, I suppose that even the smallest type of fish has greater chance of being caught in the net the more times it passes through the net. Repetition, the action of repeating or saying the same word or phrase over again, was perhaps the most common teaching practice utilized by these two teachers in the introduction of a new concept.

Teaching practices and the retention of knowledge

In many cases, the codes which describe certain teaching practices geared towards helping students retain knowledge have some overlap either with the acquisition of new knowledge or the recall of old knowledge. In categorizing certain codes as cases of retention, what I looked for were indications that connections were the focus of instruction. In the case of acquisition, teaching practices where new knowledge was introduced had the knowledge as the focus of instruction; and in recall, it is the knowledge of when and how to use particular pieces of knowledge that is the focus of the paper. Regarding retention, when the teaching practice was geared towards the making or strengthening of connections between pieces of knowledge, retention codes were used.

Making connections to prior knowledge. The teaching of mathematics often relies upon the relationships between the various concepts introduced. Episodes where new or existing knowledge was linked to other mathematical concepts were classified as instances of "Making Connections". Such practices, of pointing out connections between mathematics concepts or experiences, are touted as leading to a more profound, flexible, and lasting understanding of the subject (NCTM, 2000). With this in mind, the practice of pointing out connections in mathematics helps to fulfill the didactical contract – students are given the opportunity to create stronger connections which in turn should lead to better memory and increased possibility that a student will be able to use their knowledge.

Motivation for new knowledge. Episodes of teaching utilizing this code can be characterized by the practice of providing a reason for why a concept or theorem is necessary. Most often, these episodes occurred in the context of an exploratory or inquiry-based lesson which presented students with a question or problem such that the new concept emerges in order to answer an important question. One instance, occurring in Megan's class as students are presented with questions about the side lengths of non-right triangles. The students recognize and want to use the trigonometric functions, but realize the limitations of these with non-right triangles. This interaction lays the seeds for the necessity and usefulness of the Law of Sines.

Appealing to logic. In the geometry classroom where one of the primary goals is the learning of deductive reasoning, the appearance of the practice of using deduction to generate a new idea or theorem seems appropriate. In episodes which were coded as "appealing to logic" previously learned concepts were built upon to generate something new. For instance, both Cecilia and Megan introduced the equation of a circle in similar ways by presenting students with a sketch of a circle and asking students how they might write an equation with x's and y's for such a line. Many students were able to use their knowledge of the distance formula to conjecture and justify the claim that a circle has the equation $x^2 + y^2 = r^2$

Review through practice and Review through reasoning. "Practice makes perfect" is the old adage that is the bane of the high school student's existence when it comes to the mathematics classroom. The belief that the more students use their knowledge the better and more efficient they will be in using that knowledge is exactly the reason why students are assigned homework problems, often dozens each night.

But there appeared to be another practice, apparent in Megan's teaching, where review of concepts had more depth than just repetitive practice. Particularly apparent in discussions of proofs, students were asked not only to justify the steps of their proofs through providing reasons but were also required to explain why those reasons were applicable by reviewing the hypotheses of the theorem. For instance, a student might justify a step in a proof incorrectly and be asked to explain what one "needs" in order to use the theorem given as a rationale. Megan would ask others in the class to help by suggesting an alternative reason and explaining how the terms of the theorem were met and why one was able to use it. The questions asked in this type of interaction require the student to review (and thus strengthen) the hypotheses of that theorem.

Teaching practice and the recall of knowledge

Certain teaching practices appear to be used in helping students identify when and how to use the knowledge they have acquired in the classroom. I contend that these practices play a key role in the appearance of successful completion of the didactical contract. The teacher, who may be aware of the design of assessment tools, benefits from equipping her students with knowledge of when and how to use mathematics. To be less cynical, many researchers contend that no piece of knowledge is accessible (or even learned) without knowledge of when and how to use it and argue that knowledge is situated and useful only in situations where the individual has experienced the knowledge before (Lave, 1988). The codes identified serve the function of helping students situate their knowledge so that they have an identifiable group of experiences where knowledge ought to be used.

Cues. Cues include words or phrases, diagrams, and scenarios that are typically associated with a particular concept and can be used by students to gain access to knowledge. In information processing terms, cues are visual items that connect the situation at hand to knowledge that might be useful to it. One example has Megan telling her students that when a problem involves ratios they should think of similar triangles because that is the only chapter where they have encountered ratios. Student interviews provide evidence that students look at the diagrams involved in a proof to reference the concepts involved in proofs with similar diagrams in the past. Cues function by helping students connect elements of the situation they are faced with to relevant knowledge for handling that situation.

Heuristics. Like cues, heuristics involve the association of words and phrases, diagrams, and scenarios with a particular concept, but here these elements of the situation become associated with a particular set of actions to be taken. Episodes identified as illustrating heuristics often occur at the end of the introduction of new material while the teacher talks about the "types" of problems on the homework; they can be easily recognized by statements that look like "when you see x, you do y". Other ways that teachers teach heuristics is by modeling their own thinking for students as they solve problems. Heuristics function similarly to cues by providing students with identifiable elements in a situation that can lead them to processes for solving problems.

Conclusions

Student memory is a significant issue in the psychology of mathematics education as well as for teachers. While a great deal of study has documented the inability of students to take their knowledge outside of classroom settings (Nunes et al., 1993; Lave, 1988), relatively little work has been done in understanding how instructional practice relates to how students structure and access the memories created in the classroom. Because how students view mathematics is a result of how they have learned the subject, gaining an understanding of the nature of the mathematics being taught and how it is changed (or not) by the way in which memories created in the classroom is an important undertaking.

The type of data collected for this study and the methods of analysis involved make it impossible to know what kind of learning is occurring in students' heads; however, one can take an objective stance as an observer and think critically about how teaching practices may change the mathematical content of a course. As Ball (1993) points to, current reform efforts in mathematics call for students to do more than just acquire mathematics terms and algorithms; instead, students should be involved in the "doing [of] mathematics" (NCTM, 1989, p. 7), which includes exploration, conjecturing, and reasoning (NCTM, 1989). In this study of geometry teaching, in a place where students are purported to have an opportunity to learn to reason and conjecture like the mathematician, we instead see evidence to the contrary with teaching practices focused predominantly on the acquisition and recall of terminology and heuristics for using knowledge over meaningful connections.

In the anecdote presented at the beginning of this paper, why is it so surprising that the student of geometry would suggest that parallelogram theorems are redundant? Why shouldn't this student want to minimize the number of things he must remember? On the other hand, why isn't it surprising that the teacher merely acknowledges his suggestion and continues formally introducing the theorems of this chapter? Why not reformulate the mathematics being learned to accommodate this student's suggestion? Now that would be surprising! After all, this teacher is responsible for claiming that her students have acquired the knowledge they are supposed to about parallelograms, and if she acknowledges the suggestion and runs with the idea, that knowledge may not be known by name or apparent to other teachers or assessors.

Thus, it is not without surprise that those codes which dominated the landscapes of these two classrooms were those focused on helping students acquire new concepts and recall those concepts when appropriate. Moreover, many of the practices found in this study are intuitive and clearly serve a justifiable purpose for teachers and learners. What is far rarer in these classrooms were episodes depicting motivating, connecting, and deducing new ideas. What are the implications of this? What geometry are they learning – or not learning – by again experiencing the learning of mathematics as the acquisition of new concepts even as they are taught that geometry is somehow a reflection of what the mathematician does? The need for further study examining students' knowledge of mathematics as a result of acquiring knowledge in different ways is apparent and has implications for shaping the nature of the classrooms we teach in and the knowledge that we teach.

Endnote

The present paper reports on data collected by project GRIP (Geometry, Reasoning, and Instructional Practices), directed by Patricio G. Herbst. Support for the research has been provided by the National Science Foundation, through grant ROLE 0133619; all opinions expressed are the sole responsibility of the author and do not represent the views of the Foundation. The author acknowledges valuable comments from Patricio Herbst and other members of the GRIP team.

References

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal, 93*(4), 373-397.

Biehler, Robert F., Snowman, Jack (1986). *Psychology applied to teaching* (5th ed.). Boston: Houghton Mifflin.

Borich, G. D., & Tombari, M. L. (1997). *Educational psychology: a contemporary approach.* New York: Longman.

Brach, C. A. (2004). Learning to remember: Mathematics teaching practice and students' memory. Unpublished manuscript, The University of Michigan.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des Mathematiques 1970-1990* (N. Balacheff, M. Cooper, R. Sutherland, and V. Warfield, Eds. And Trans.). Dordrecht, The Netherlands: Kluwer.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.

Holmes, Oliver Wendell (1858). *Autocrat of the breakfast table.* Boston: Phillips, Sampson, and Company.

Lave, J. (1988). *Cognition in Practice*. Cambridge: Cambridge University Press.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Nunes, T., Schliemann, A., & Carraher, D. (1993). *Street mathematics and school mathematics.* Cambridge, MA: Cambridge University Press.

Powell, A. B., Francisco, J. M., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior, 22*, 405-435.

Tulving, E. (1968). Theoretical issues in free recall. In T. R. Dixon & D. L. Horton (Eds.), *Verbal behavior and general behavior theory.* Englewood Cliffs, NJ: Prentice Hall.

Woolfolk, A. (1998). *Educational Psychology* 7th ed.). Boston: Allyn and Bacon.

STUDENTS' STRATEGIES FOR MEASURING THE LENGTH OF DIAGONAL LINE SEGMENTS ON A GRID

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This study investigated seventh graders strategies to measure the length of a diagonal line segment situated on a grid before they were introduced to the Pythagorean Theorem. Two novel strategies that students used were the taxi-cab metric, $|x_1 - x_2|+|y_1-y_2|$*, the other strategy was equivalent to the max norm, max* $[|x_1 - x_2|, |y_1-y_2|]$*. Students saw no conflict between assigning the same length to segments that were actually different lengths because of the way they were situated on the grid. Implications for instruction on measurement and the Pythagorean Theorem are given.*

When seventh graders have the task of finding the length of a diagonal line segment situated on a grid, what do they do? Do the students see a need for a mathematical relationship between the length of the diagonal line and the grid spaces, the Pythagorean Theorem? What strategies do students come up with to measure the length of a diagonal line segment before they have been taught the Pythagorean Theorem or the related distance formula? This brief paper reports the findings of a study that gathered information pertaining to these questions.

Background

Measurement is an important topic in K-12 mathematics. The Principles and Standards for School Mathematics (NCTM, 2000) and the National Assessment of Educational Progress (National Assessment Governing Board, 2000) count measurement as one of the five basic content strands of school mathematics. Most studies investigating length have been done with elementary or preschool aged children (NCTM, 2003). Less research has been done with students past elementary school. This study used two classes of seventh graders to investigate how students anticipate lengths of diagonal lines on a grid. The task of finding the length of a diagonal line, or the distance between two points on a grid, is one of the first situations where length or distance can not be derived directly by counting units. Another example of indirect measurement in the school curriculum is finding the circumference of a circle given the radius or diameter. This study explored ways that was done two weeks before the seventh graders were introduced to the Pythagorean Theorem, which is the most notable tool to calculate distances or lengths on a grid.

Methods

The study involved forty seventh grade students in two different classes but taught by the same instructor. Students worked in pairs on a single computer. Students worked on three different tasks set up on a dynamic geometry environment, the Geometer's Sketchpad. The choice to use computers minimized students measuring strategies relying on physical objects, like rulers, and made them more dependent on the grid. As the students worked in pairs interviewers asked the students questions about their choices of strategies and understanding of the task. Sometimes the interviewer posed different tasks to further clarify student thinking about the tasks and about their strategies for calculating length.

Results

Some students recognized that the length of the diagonal line segment was more than either on of the "legs," imagining the diagonal line as the hypotenuse of a right triangle. Some of these students still tried to measure the length of the diagonal line by one of the two novel strategies described below. In spite of using the computer some students still tried to use a ruler, placing it on the computer screen, to measure and compare lengths.

Students tried two novel measuring strategies. The first strategy is equivalent to the taxi-cab geometry method of assigning lengths to diagonally situated lines. In taxi-cab geometry the length of a diagonal line is measured by constructing a right triangle where the diagonal is the hypotenuse and adding up the lengths of the legs of the right triangle. Mathematically this norm is equivalent to calculating the distance between the points (x_1, y_1) and (x_2, y_2) to be $|x_1 - x_2|+|y_1-y_2|$.

The second strategy is called the max norm. Some students counted the number of tiles a diagonal line spanned in the vertical and horizontal directions and then assigned the largest one of these to be the length of the diagonal line. I call this the max norm because it is equivalent to defining the distance between points (x_1, y_1) and (x_2, y_2) to be max $[|x_1 - x_2|, |y_1-y_2|]$. In one instance the instructor constructed a right triangle with both legs ten units in length using the geometry software and asked a pair of students to find the length of the hypotenuse. The students counted the length of both legs and when they saw that they were both ten units long they told the instructor that the hypotenuse was also ten units long. When the instructor changed the triangle so one leg was twelve units and the other leg was ten units the students again counted the lengths of the legs and concluded that the hypotenuse had length twelve.

Conclusions

Some students appropriately recognized that they did not have the tools to find the length of a diagonal line. Some thought that the length of the diagonal line was equivalent to the length of one of the legs, while others thought that it was the length of the sum of the length of the legs. The interviewers posed problems to the students aimed to create conflict between the apparent length of line segments and the student-assigned length of the line segments. Students saw no conflict between assigning the same length to segments that were actually different lengths because of the way they were situated on the grid. For example, the length of a leg and the length of a hypotenuse looked the same because they both started and ended at the same vertical lines. In conclusion the findings of this study suggests that: middle school students may need instruction or experiences that can help them correctly compare lengths of lines situated on a grid correctly, some seventh graders don't recognize that the length of the hypotenuse of a right triangle must be more than the length of either of its legs, and posing tasks that require students to measure the length of lines situated diagonally on a grid and discussing strategies as a class could help students see an intellectual need for the Pythagorean Theorem.

References

National Assessment Governing Board (2000). *Mathematics Framework for the 1996 and 2000 National Assessment of Educational Progress*. Washington, D.C.: NAGB.

National Council of Teachers of Mathematics (2003). *Learning and teaching measurement: 2003 yearbook.* Reston, Va.: Author.

National Council of Teachers of Mathematics (2000). *Principles and standards of school mathematics.* Reston, Va.: Author.

INTERDISCIPLINARY RESEARCH ON SPATIAL SENSE LEARNING WITH THREE COMMUNITIES OF SECONDARY STUDENTS

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The source of this project lies in the interaction between my two different teaching practices: mathematics and figure skating. I observed some differences in the learning of spatial sense between pupils from regular classrooms and athletes. This study explores the learning and the teaching of spatial sense in a new approach.

Objectives

In Quebec, geometry is an important part of the mathematic curriculum of elementary and secondary schools. Spatial sense plays an important role in the learning of mathematics (Wheatley, 1990). Researchers have shown that pupils have difficulties mastering that type of knowledge (Parzysz, 1991) and our study on the educational programs and handbooks points out some weaknesses and gaps in the transition from elementary to secondary levels. This qualitative and explorational study involved pupils of three different learning communities: advanced math class, sport and music vocational program. The project aims to bring some answers to two specific questions : how to characterize a teaching approach on spatial sense aiming on pupils action (abstract and concrete) of the pupils in comparison with a more conventional approach? And, how do athletes, musicians and advanced students react to these two approaches?

Theoretical framework

This research relies on the work of Piaget and Inhelder (1948) for whom action is the starting point of all learning, including abstract knowledge. It also relies on sports research which shows that abstract and concrete actions work together to enhance the athlete's performance. This insertion in sports psychology gave us some markers on the effects of the interaction of the two types of actions and the teaching approach to use (Orlick, 1990). Our framework also looked at the meaning and the models that has been given to spatial sense in comparison to geometrics knowledges (Clements & Battista, 1992; Berthelot & Salin, 1992; and Piaget & Inhelder, 1948) and at the type of questioning we can use in our mathematics classroom (Nesbitt Vacc, 1993). All aspects of this framework were put together to create a learning approach based on the actions of pupils and also to develop a grid that has been used to analyze our research results.

Methods

To answer these questions, we have developed one teaching approach favoring students' action (concrete and abstract) and compared it with a more conventional approach. The two lessons were experimented with smalls groups of volunteer students from the three learning communities (totalling sixty students). Each lesson was videotaped and analyzed with a grid based on our theoretical framework, characterizing the learning in terms of interventions, resolutions, performance and difficulties, and the teaching in terms of interventions, tasks, action, content and questioning (Gauthier, 1997). Each group was interviewed afterwards to discuss their reasoning, difficulties and to comment on the activities. This enabled us to characterize the two approaches by highlighting the similar and different aspects of each and the learning of each community of pupils in comparison to one another (advanced, music or sports).

Results and Conclusions

The most significant results are those related to the differences between the two teaching approaches rather than to the differences between communities. Three major differences were found between the two approaches:

1. The action approach aims more at abstract actions in the teaching interventions than the conventional approach (53% of the time compared to 5%). This is characterized by the fact that the teacher using the action approach put the emphasis on asking pupils what they saw mentally and anticipate the amount of material needed for their task.
2. The type of activities asked of students differs from one approach to the other. For example, with the action approach, students had to visualize objects, anticipate and describe their spatial transformation during the activities. On the other hand, students in a more conventional approach were asked to describe and define the objects in geometric terms.
3. The nature of questions asked differs from one approach to the other. For instance, in the action approach, questions were mostly about spatial sense (52%) and the reasoning of pupils (54%): "Can you turn the object in your head? What can you see?" In opposition, the conventional approach asked mostly questions on geometric knowledge (53%) and results (81%): "What is the name of this solid? What do these straws represent mathematically?"

In general, the action approach centers on spatial sense, abstract actions and tasks like visualization, anticipation and description. The conventional approach tends to stay on concrete actions and drift to more usual geometric tasks like vocabulary and definitions as anticipated by Berthelot and Salin (1992). Several recommendations for the learning and teaching of spatial sense came out of this research, among them:

-To develop concrete actions even in secondary classes and also focusing on the abstract action that follows. The intervention of the teacher has to center on abstract actions.

-To increase teachers' aweness of drifting from the spatial sense to more used geometric knowledge and the effects that this could have on students.

-To explore the pratice of visualization used in the sports psychology field, to switch from concrete to abstract actions.

This study is considered a starting point in a potential new field of research. Thus, the next step is to elaborate a collaborative study with teachers and construct a concrete and more complete approach for the learning of spatial sense.

References

Berthelot, R. & Salin M.-H. (1992). L'enseignement de l'espace et de la géométrie dans la scolarité obligatoire. Thèse de doctorat, Université de Bordeaux I.

Clements, D. h. & Battista M. T. (1992). Geometry and spatial reasoning. Dans *Handbook of research on mathematics teaching and learning*. Douglas A. et Grouws, Macmillan publishing compagny, New York. Chap. 18, p.420-464.

Gauthier, C. (1997). *Pour une théorie de la pédagogie. Recherches contemporaines sur le savoir des enseignants*. Les presses de l'Université Laval. 352 pages.

Nesbitt Vacc, N. (1993). *Questioning in the mathematics classroom*. Arithmetic teacher, octobre, 1993. vol. 41, no.2, pp.88-91.

Orlick, T. (1990). Chapter 8 : Mental Imagery. *In Pursuit of Excellence : How to Win in Sport and Life Through Mental Training*. Champaign, IL : Leisure Press, . pp.65-77.

Parzysz, Bernard (1991). Espace, Géométrie et Dessin. Une ingénierie didactique pour l'apprentissage, l'enseignement et l'utilisation de la perspective parallèle au lycée.

Recherches en didactiques des mathématiques. Vol. 11, no. 23, pp.211-240.

Piaget, J. et Inhelder, B. (1948). *La représentation de l'espace chez l'enfant*. Presses Universitaires de France, Paris, (ed. 1977).

Wheatley, Grayson H. (1990). Spatial sense and mathematics learning. *Arithmetic teacher*. Février 1990, pp.10-11.

DYNAMIC AREA CONCEPTS INSTRUCTION DESIGN BASED ON REPRESENTATION PERSPECTIVES

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This study firstly explores what kind of error patterns students may have, and then exploits the features of computer software application to design a situation by applying EXCEL & GSP according to the types of incorrect area concept elementary school students usually have for learning area. The windows environment will change visually in sync with mouse control or numerical input to provide learners with more immediate and precise feedback as compared with traditional methods. Hence, learners will acquire the concept of area and the formula of calculating area.

Introduction

Most of the computer software currently sold in the market is the goal-oriented tool software, however. It implies that between the input which the user enters in formatted way and the output which is the possible result of software feedback is like just a input and output linked bond, there seems to be a neglect of the learning process of calculation rules which is required in the instruction principles of mathematics (Hsieh, 2000). Besides, the rare Chinese version, high price and poor popularity of the existing tool software make general teachers step back from their idea of application.

Excel is one of the tools that can solve this problem. With Chinese interface and popularized price, it is an essential numerical calculation tool for general families or schools. Excel itself is an environment cut into 65536*230 cells. Each cell can be an input cell as well as an output cell. Any cell can be entered with words and numbers. By means of the function internally constructed, an output of the result is given. And the input and output cell is a dynamic link. It refers that merely a change of input cell can make the linked output cell to reflect the relevant result passively (Hsieh, 2000). Besides the vivid visual learning environment of computer, dynamic instruction under windows environment does not demand an investment of a lot of labor and time because of the convenient acquisition and use of software and the low development cost. It also does not neglect the merits of the interaction with students in the traditional instruction methods as well as the students' active operation and construction. It increases the students' creativity in the learning process. Meanwhile, teachers can observe the formation and change of the students' concepts from their dynamic learning process.

Geometer's Sketchpad, (GSP) is another solution. It has the characteristics of drawing graphic by ruler, dynamic continuous changes, structure keeping, recording of graphical drawing process, etc. It not only provides precise geometric graphic, but also gives instructional interaction between teachers and students and feedback opportunities at the right timing (Lin, 1996). This research paper attempts to use Excel and GSP to construct the instruction situation of area measurement concepts. In the process of construction, this research adopts multiple representations to create text mode – a presentation of problem situation; numerical mode – an investigation of various possible numerical changes; and graphical mode – a graphical change corresponding to numerical change (Hsieh, 2000). There are dynamic linkages among various modes. Through the numerical change of the text mode (exploration of different situations so as to make sure of understanding the problem) and the fast change of the relevant numerical or graphical mode, students can construct the area concepts. This learning process meets the spirit of constructivism in the new curricula. To the students, it is a meaningful learning.

Review of Literature

- Representation Inference Process
- Multiple representations and Mathematics Learning

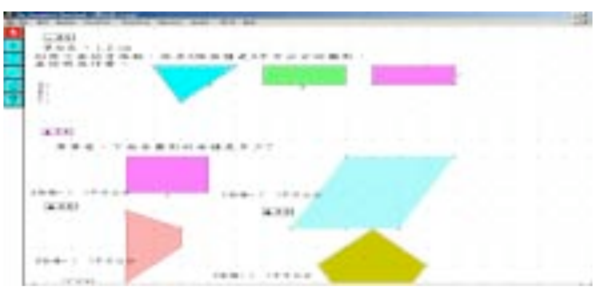
Method

Four major procedures are designed as follows:

- Collect the students' types of mistakes.
- Refer to the editing principles and the concepts handling procedures of the existing teaching materials, and find out the possible reasons for the unfavorable learning of students.
- Edit teaching materials after referring to learning psychology following the procedures of concept formation.
- Make teaching materials lively, add in the situations that students are interested in, and design prompt feedback system.

For example：The use of GSP software to design the teaching material of area concepts has the following characteristics:

Characteristic of numerical presentation of integer by grid dots: It can be used to design on the nail board the number of dots in the area that students feel difficult. The numerical presentation of the whole square can reduce beginners' cognitive burden towards the concepts of area. e.g. Title of Unit – How to Use Nail Board .Using the grid dot function of GSP, students can pull out various kinds of graphics, and at the same time observe the numerical change of their areas. Here students can easily cut and eorganize the area easily, as shown in Figure 1 below:



Conclusions

Most of the students think it easy to operate the soft-ware design of this system instruction software, and learn from it easily.

Using on instruction the dynamic system designed according to representation theory can arouse the students' learning interests and stimulate their thinking.

It can raise the students' problem-solving challenges to themselves.

As instruction goes on, teacher still play an indispensable role.

Most of the students like to use computer to learn mathematics.

This instruction system has satisfactory assistance effects to the learning motives and thinking inspiration of students.

Every student has different demand of extrinsic representation.

References

Hsieh, Che-jen (2000). "Research on Workability of Using Spreadsheet on Mathematics of Senior High School." *Academic Researches of Mei Ho Institute of Technology*, 18, 118-128.

Lin, Pao-ping (1996). "Application of Geometer's Sketchpad (GSP) on Instruction." Collection of Research Papers on Guidance for Teachers on Local Education at Guided Regions for Discussion Activity Held in the 14th Academic Year.

Palmer, S. E. (1977). Foundational Aspects of Cognitive Representation. In E. Rosch & B. B. Loyd (Eds). Cognition and Categorization. Hillsdale, NJ: Lawrence Erlbaum Association.

Tsai, Chia-tseng (1997). "Research on Relationship between Working Memories and Problem Solving Process of Mathematics Essay Questions of Primary School Students." Master Thesis, Graduate School of National Education, National Tainan Teachers College.

THE ROLE OF PEIRCE'S "INTERPRETANT" IN COMMUNITIES OF GEOMETRIC INQUIRY – PAST, PRESENT, AND FUTURE

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The geometric nature of this topic makes it suitable for visual presentation in a poster. In Peirce's triadic model of semiosis, the interpretant is the third component in which a cognizing being makes sense of the relationship between the other two components, the object, and the representamen that stands for it in some way. This model includes the need for expression or communication: "Expression is a kind of representation or signification. A sign is a third mediating between the mind addressed and the object represented" (Peirce, 1992, p. 281). In an act of communication there are also three kinds of interpretant, as follows:

- the "Intensional Interpretant, which is a determination of the mind of the utterer";
- the "Effectual Interpretant, which is a determination of the mind of the interpreter"; and
- the "Communicational Interpretant, or say the Cominterpretant, which is a determination of that mind into which the minds of utterer and interpreter have to be fused in order that any communication should take place" (Peirce, 1998, p. 478, his emphasis).

It is the latter fused mind that Peirce designated the commens. For the continuity of mathematical ideas and their evolution in the history of mathematics, a central requirement is that there be a community of thinkers who share a "fused mind" sufficiently to communicate effectively with one another – and with posterity through their artifacts – through this commens.

Peirce (1992, pp 54-55) considered that "reality depends on the ultimate decision of the community;" and that "the existence of thought now, depends on what is to be hereafter; so that it has only a potential existence, dependent on the future thought of the community." In light of hindsight, and the future's characterization of the geometrical though of the age of classical geometry as extremely powerful, some specific questions are addressed.

- More than 2000 years ago, Archimedes employed a method of exhaustion (also known to Eudoxus and used by him and by others in that community) to calculate the area enclosed by a parabola and a line segment. Why was it only in the 17th century that the development of such methods became widespread with the advent of integral calculus?
- Hipparchus of Crete generated some excitement when he figured out that the area of the "lune" was the same as that of a right triangle whose hypotenuse was the diameter of the lune. Why was this discovery important in the geometry of the time?
- Why did it take two millennia for the consequences of challenging Euclid's parallel postulate to be brought to fruition in systems of hyperbolic and spherical geometries (or hyperbolic, parabolic, and elliptic geometries, as Klein called the three forms of geometry in 1871)?

In a visual display, these questions are considered in the light of Peirce's commens, and hence it is argued that the Peircean idea of the role of the interpretant in forging communities of inquirers through communication, are quite current, and have bearing on the classroom teaching and learning of mathematics today. Examples from current mathematics education literature that addresses the role of discourse and its reflexive relationship with the creation of mathematical objects by individual learners in a discourse community are presented.

References

Peirce, C. S. (1992). *The essential Peirce*. Volume 1, edited by N. Houser & C. Kloesel. Bloomington: Indiana University Press.

Peirce, C. S. (1998). *The essential Peirce*. Volume 2, edited by the Peirce Edition Project. Bloomington: Indiana University Press.

Probability and Statistics

MOTIVATING STATISTICAL REASONING: COMPARING EQUAL AND UNEQUAL-SIZE GROUPS

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ABSTRACT

Statistical reasoning begins with a process of data analysis – an iterative process that includes asking a question, collecting data, analyzing the data, and forming and communicating conclusions. In this paper, we argue that this kind of reasoning is facilitated with tasks that are designed to engage students in comparing distributions of two or more data sets (e.g., Russell, Schifter & Bastable, 2002; Friel, in press, 2002; McClain, in press). Using Brown's (1992) characterization of design research, we have attempted to "engineer innovative educational environments and simultaneously conduct experimental studies of those innovations" (p. 141). Our analysis therefore highlights the close relation between 1) our design efforts related to engaging students in comparing data sets and 2) students' ways of reasoning statistically.

INTRODUCTION

Recent mathematics standards (NCTM, 2000) place central importance on the development of mathematical reasoning. Reasoning involves more than dealing with individual concepts and focusing on conventions, skills, and techniques that produce answers to "type problems." Reasoning involves creating a rich cognitive map of content along with habits of mind that help connect and make sense of mathematics. Many areas of mathematics have recently been framed by describing the nature of applicable mathematical reasoning; for example, geometric reasoning, proportional reasoning, algebraic reasoning, and statistical reasoning. While there are similarities across all types of reasoning (e.g., deductive, inductive, and intuitive reasoning), there are also content-specific reasoning processes.

As an example, statistical reasoning begins with a process of data analysis – an iterative process that includes asking a question, collecting data, analyzing the data, and forming and communicating conclusions. Learning statistics therefore includes two central components: the concepts and conventions of the discipline itself (Konold, 2002) and the ability to apply this knowledge in a broader, more flexible, and skeptical manner when analyzing, interpreting and/or communicating results obtained by statistical analysis. Garfield (2002) concurs when she defines statistical reasoning as "the way people reason with statistical ideas and make sense of statistical information. This involves making interpretations based on sets of data, graphical representations, and statistical summaries" (p. 1).

For the past six years, we have both been involved in research focused on supporting students' and teachers' understandings of statistical data analysis. This has included the development and testing of instructional materials and computer-based tools for analysis. In this process, we have advocated a shift in what it means to "know statistics" at the middle grades from computations of measures of center and conventions for graphs to involving students in the process of *genuine data analysis* (cf. Cobb, G., 1990; 1992; Cobb, G. & Moore, D., 1997; Moore, D., 1993). In our work, we have focused on the design of data analysis tasks and computer-based tools that place students' diverse ways of reasoning at the forefront of instructional decision-making by treating them as resources for teaching. In this approach, the process of data analysis becomes the mechanism for learning conventions. As a result, we see our work located at the intersection of students' diverse ways of reasoning, the proactive role of the teacher, and tasks and tools designed to highlight significant mathematical issues.

In this paper, we build on the research literature and our current work to argue that competence in statistical reasoning is facilitated by tasks that are designed to engage students in comparing distributions of two or more data sets (e.g., Russell, Schifter & Bastable, 2002; Friel, in press, 2002; Friel & O'Conner, submitted; McClain, in press). In doing so, we build from our previous work (cf. McClain, 2003; McClain & Cobb, 2001) to document shifts is students' reasoning that result from design decisions concerning task and tool development formulated around comparing data sets. Although we have worked at independent sites, our research reveals strong connections between the design of tasks and tools and the support they provide for teaching and learning across the two lines of research. In particular, we claim that situations in which students are asked to conduct an analysis in which they have to make a decision based on a comparative analysis create settings that parallel the work of professionals. (In addition, it provides a strong contrast to traditional instruction in statistics.) We argue that this type of comparative reasoning is inherent in all statistics, even when single measures are reported. As an example, knowing that the unemployment rate is 4% is not useful unless one can compare that to previous data or trends in the data over time. In this example, although the comparison is implied, it is still central to making sense of the "data." In the analyses reported in this paper, we therefore focus on tasks and tools designed to require students to make comparisons across data sets. In doing so, we will document the significant mathematical issues that emerged from students in the course of analyzing comparison tasks and how those tasks have provided means of supporting student learning in the context of our work with students and teachers. This analysis will therefore highlight the close relation between 1) our design efforts, 2) the students' strategies that emerge, and 3) how those strategies provide resources for supporting learning.

THEORETICAL PERSPECTIVE/MODE OF INQUIRY

Our work involves the interaction of two categories of research: *developmental research* (Gravemeijer, 1994) and *design research* (Brown, 1992; Cobb, et al, 2003). Using Brown's characterization of design research, we have attempted to "engineer innovative educational environments and simultaneously conduct experimental studies of those innovations" (p. 141). This involves iterative cycles of design that include theory development (Gravemeijer, 1994) and research where conjectures about the learning of students and the means of support are continually tested and revised in the course of ongoing interactions. Our analysis therefore highlights the close relation between 1) our design efforts related to engaging students in comparing data sets and 2) students' ways of reasoning statistically.

DATA

Data for the analyses were taken from two different contexts. McClain's work involved the use of design experiments conducted with middle-grades students and teachers.[1] The data from these experiments consist of video recordings of each session accompanied by field notes detailing events. Copies of all student and teacher work are also part of the data corpus. Friel, as researcher/curriculum developer, was involved in the development and field-testing of a middle-grades data analysis unit at multiple pilot sites.[2] Data from each site were provided in the form of after-the-fact teacher reports and samples of student work. In addition, at some of the sites, the researcher/curriculum developer was also an observer and support instructor during parts of the instructional unit.

RESULTS

In our analysis, we have separated comparing equal-size groups and comparing unequal-size groups since the latter supports the need for multiplicative reasoning strategies as part of the reasoning process. This was a major consideration in designing data sets with unequal N's. In particular, at both sites the problematic nature of direct additive comparisons across data sets with unequal N's gave rise to the need for a different form of analysis. In this way, multiplicative reasoning and multiplicative ways of structuring data then emerged from the students problem-solving efforts as a solution to a problem. It is in this manner that we speak of designing by building on students' current understandings to promote learning.

A central question focused our analysis:

> How do learners naturally compare and contrast two or more data sets (with equal *N*'s or unequal *N*'s), making use of a knowledge base that involves reasoning by describing data, drawing on a repertoire of statistical measures, and appreciating and making use of features of distributions?

It is also important to note that both our research contexts involved the use of computer-based tools for analysis either as special-purpose applets or as an exploratory software program (e.g. *Tinkerplots*). Both tools permitted students to separate data into groups and/or make separate graphs, one for each group being compared. This was an important component in students making comparisons across data sets, some involving large numbers of data items.

COMPARING EQUAL-SIZE GROUPS

The data reported in this section were taken from tasks developed as part of a classroom design experiment conducted in the fall of 1997 with a group of 29 students and a follow-up design experiment conducted with a group of middle-school teachers. One goal of the seventh-grade design experiment was to develop, test and refine a coherent instructional sequence that would tie together the separate, loosely related topics that typically characterize American middle-school statistics curricula. The statistical notion that emerged as central to instruction from our synthesis of the literature was that of distribution. Moore (1990) points to distribution as an "important part of learning to look at data" (p. 106). In the case of univariate data sets, for example, the focus on distribution enabled the research team[3] to treat measures of center, spreadout-ness, skewness, and relative frequency as characteristics of the way the data are distributed. In addition, it allowed the research team to view various conventional graphs such as histograms and box-and-whiskers plots as different ways of structuring distributions. The instructional goal was therefore to support the development of a single, multi-faceted notion, that of distribution, rather than a collection of topics to be taught as separate components of a curriculum unit.

The follow-up work with middle-grades teachers involved using the refined instructional sequence as a basis of collaboration around what it means to reason statistically. Teachers initially engaged in the data analysis tasks as learners and subsequently took the tasks back to their students as teachers.

In initial activities from the instructional sequence, students typically viewed data as sets of numbers that needed to be "reduced" by identifying the mean, median, mode, and/or range. Although measures of center and spread are valid statistical measures for making comparisons of data in many situations, the goal was to provide students with alternative resources for analyzing data and to help them develop conceptual understandings of these measures as descriptors of distributions instead of ways to eliminate variability. As an example, data on the number of hours of television watched by a group of seventh-grade students was averaged in order to "make it easier to see." When the teacher tried to focus the students' attention on features of the distribution, she was unable to generate a need for such specificity in the description of the data. Students viewed their task as finding a way to reduce the complexity of the data set by eliminating the variability instead of finding ways to effectively *capture* the variability. This occurred repeatedly throughout the design experiment when students were provided a single data set. It was only when students were asked to tease out differences *across* data sets that they began to direct their attention to features of the variability and thus, the distributions.

This shift can be seen in the activity of students who were given data on the resting heart rate of 84 non-smokers, twenty-five years of age and 84 smokers of the same age. The data is shown in Figure 1 as a screen capture of the computer-based tool used for analysis where the data sets have been separated.

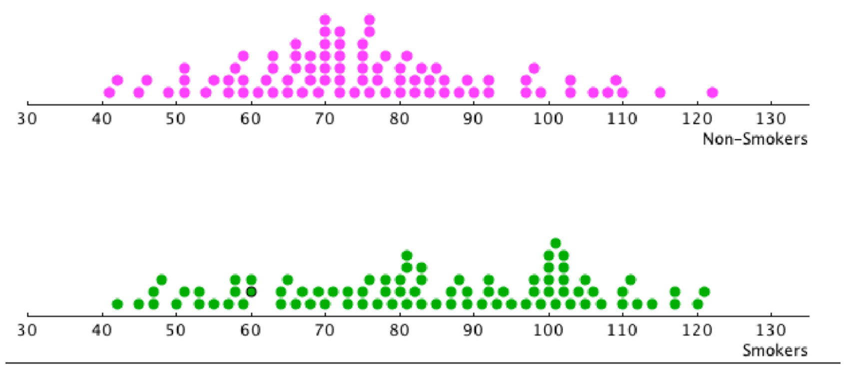


Figure 1. Data on resting heart rate of smokers and non-smokers.

The intent of the task was to problematize a focus only on measures of center or spread. In particular, it would be insufficient to note that the two data sets had similar ranges or to specify the difference in means or medians. Thus, the process of reduction of the data to a single number would now be inadequate. An adequate analysis would require that the students tease out the salient aspects of each of the two distributions. This would be necessary in order to analyze the data in two ways — first, by characterizing the distribution *within* the data sets so that they might then be able to compare differences in the distributions *across* the two data sets.

As an initial strategy, students used the *two-equal-groups* feature on the computer tool to identify the median and extremes as shown in Figure 2. In their analysis, the students focused on the shift in what they called the "cluster" or "clump" of the data and argued that although the two data sets had "almost the same range" and similar medians (e.g.72 versus 84), the significant features of the data were captured in how the data on the non-smokers were "clumped up" around the median while the data on the smokers was "more spread out in the top half." As a result of a shift in focus to characterizing the distributions, these analyses provided students with a variety of strategies (e.g. partitioning, focus on perceptual patterns in the data) that later emerged into more sophisticated ways of reasoning statistically about data sets.

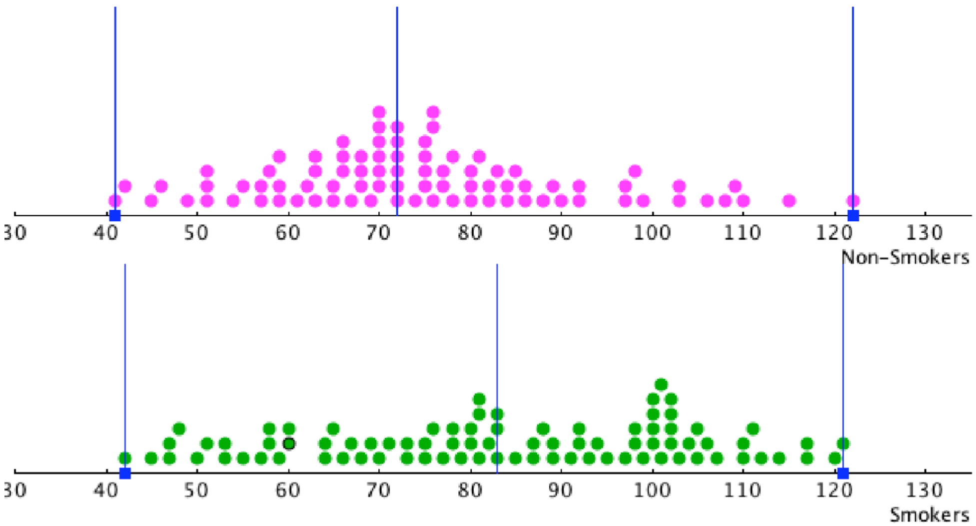
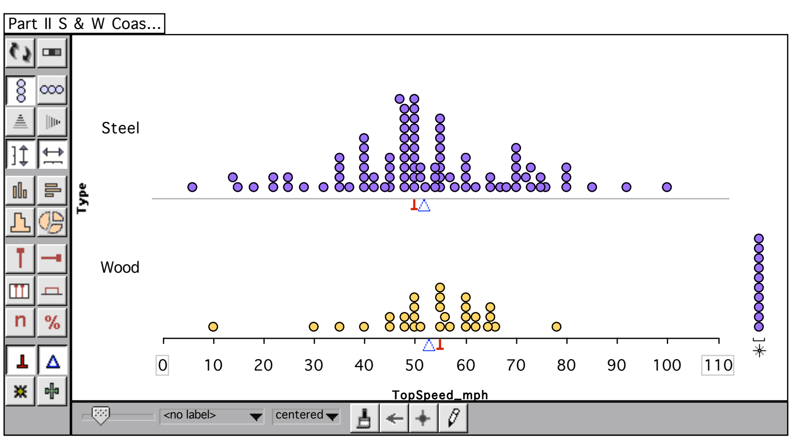


Figure 2. Resting heart rate data with median and extremes marked.

COMPARING UNEQUAL-SIZE GROUPS

Data on the speeds of wooden (30 coasters) and steel roller coasters (100 coasters) were used to answer the question, which is faster, wooden roller coasters or steel roller coasters? Traditional measures of center and spread provided one source of information. Both distributions appeared to be somewhat bell-shaped and the measures of center were similarly located. Speeds of steel coasters were more variable, ranging from 5 – 100 mph, while wood coaster speeds ranged from 10 – 78 mph.



However, students developed a strategy for comparing the distributions that involved "partitioning the data" or comparing subsections of the distributions. For example, students used the software to determine that there are 9 steel coasters and 2 wooden coasters with speeds ≤ 30 mph.[1] Students then partitioned the data in order to examine what it means to be faster. They decided that a highway speed of 55 mph or 65 mph would be a good metric. Using the software, they compared speeds of wooden and steel coasters at or above these benchmark speeds.

Because this task involved the issue of comparing data sets with unequal N's, relative frequencies were needed. However, students struggled with comparing counts versus comparing relative frequencies. As a result, students using partitioning strategies with benchmark speeds (e.g., 65 mph), found answering this question problematic. Although some students were willing to compare parts of the two distributions (e.g., coasters with speeds ≥ 65 mph), other students struggled with making sense of these qualified comparisons and preferred simple comparisons such as "steel coasters are faster than wood coasters."

In general, students were challenged by comparisons using unequal-size data sets, particularly when examining subsections such as speeds in the interval ≥ 65 mph. In addition, the data were displayed as individual case icons so students' perceptions related to actual counts may have conflicted with information provided by relative frequencies reported as percents.

CONCLUSION

We want to clarify that we view the design of tasks and tools as critical *means of support*. In doing so, we do not give agency to the tasks, but view them as a resource to support learning. In particular, we posit that the design of the tasks that require *comparison* puts the importance of quantifying difference in the foreground. This requires students to find measures and representations that capture the features and variability within a distribution instead of ways to reduce this complexity. In this process, students reconceptualize their understanding of what it means to know and do statistics as they compare and contrast different analyses. As a result, the question typically changes from *which is the faster roller coaster?* to *how can I best make my argument based on the data?* This process highlights the importance of tasks designed to provide opportunities for students to engage in statistical reasoning that support the development of central mathematical concepts (cf. Cobb, G., 1992; Cobb, G. & Moore, 1997; Moore, 1990). Here we have argued that *distribution* is one such concept and that comparing data sets is a means of supporting students' understanding of that concept.

Notes

1. The work reported in this paper was supported by the National Science Foundation under grants Nos. REC-0135062 and REC-0230137 and the Office of Educational Research and Improvement under grant number R305A60007.
2. The work reported in this paper was supported by the National Science Foundation through grants in support of *The Connected Math Project* and Tinkerplots.
3. Members of the research team for the classroom design experiment were McClain, Paul Cobb, Koeno Gravemeijer, Maggie McGatha, Jose Cortina, and Lynn Hodge.
4. Some students wondered about the coasters with speeds ≤ 30 mph. In examining the data, most of these coasters were junior coasters or mine trains – categories of coasters we would expect would have slower speeds.

REFERENCES

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classrooms. *Journal of the Learning Sciences*, *2*, 141-178.

Cobb. G. (1992). Teaching statistics. In L. Steen (Ed.) *Heeding the call for change: Suggestions for curricular action*. MAA Notes 22. Washington, D.C: Mathematical Association of America.

Cobb, G. W., & Moore, D. S. (1997). Mathematics, statistics, and teaching. *American Mathematical Monthly*, *104* 801-823.

Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9-13.

Friel, S.N. (2002). Wooden or steel roller coasters: What's the choice? *New England Mathematics Journal, 34,* 45-54.

Friel, S. N. (in press). The research frontier: Where technology interacts with the teaching and learning of data analysis and statistics. In M. K. Heid & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Syntheses and perspectives, Volume 1*. Greenwich, CT: Information Age Publishing, Inc.

Friel, S. N., & O'Connor, W. (submitted)). Statistics and data analysis in grades 6–8: What middle grades students need to know and be able to do. In P. Elliot (Ed.). Thinking and reasoning with data and chance (2006 Yearbook). Reston, VA: NCTM.

Garfield, J. (2002). The challenge of developing statistical reasoning. *Journal of Statistics Education, 10*, (3) (www.amstat.org/publications/jse/v10n3/garfield.html, Accessed July, 2003).

Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education, 25*, 443-471.

Konold, C. (2002). Teaching concepts rather than conventions. *New England Mathematics Journal, 34*(2), 69-81.

McClain, K. (in press). The Mathematics behind the graph: Discussions of data. In R. Nemirovsky, A. Rosebery, B. Warren, and J. Solomon (Eds.), *Everyday matters in science and mathematics,* Mahwah, NJ: Lawrence Erlbaum Associates.

McClain, K. (2003, July). *Supporting teacher change: A case from statistics*. Paper presented at the annual meeting of the Twenty-seventh International Conference for the Psychology of Mathematics Education, Oahu, Hawaii.

McClain, K., & Cobb, P. (2001 a). Supporting students' ability to reason about data. *Educational Studies in Mathematics, 45*(1-3), 103-129

Moore, D. (1990). Uncertainty. In L. Steen (Ed.), *On the shoulders of giants*, (pp. 95–138). Washington, D.C.: National Academy Press.

National Council of Teachers of Mathematics (2000). *Principles and standards of school mathematics.* Reston, VA: NCTM.

Russell, S. J., Schifter, D., & Bastable, V. (2002). *Working with data.* Parsippany, NJ: Dale Seymour Publications.

A STUDY OF STUDENTS' UNDERSTANDING OF PROBLEMS INVOLVING CONDITIONAL PROBABILITIES THROUGH A SEMIOTIC ANALYSIS OF THEIR REPRESENTATION OF THE PROBLEMS

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A study of undergraduate students' understanding of problems involving conditional probabilities by a semiotic analysis of their representations of the problems. The study was done through the application of six problems involving conditional probabilities to 49 computer science majors. The analysis of students' representation of the problem showed that they often were not able to model the problem correctly, not being able to determine to which population or subset of the population the data of the problem referred to. This was the source of confusion between the concepts of conditional probabilities and probability of an intersection of events.

The objective of this study was to analyze students' understanding of problems involving conditional probabilities by a semiotic analysis of their representations of the problems.

The participants in the study were 49 computer science majors enrolled in the course "Probability and Stochastical Processes" in a private university in Brazil.

The study was done through the application of six problems involving conditional probabilities. The administration of the problems was done after instruction on conditional probabilities, and to encourage involvement with the problems, it was considered part of the formal assessment in the course, therefore counting towards students' grades.

Studies on the learning of conditional probabilities have suggested that the use of different representations of probability affects students' performance in problems (Gigerenzer, 1996; Pollatsek et al., 1987; Shaughnessy, Michael, 1992). In this study, we wanted to investigate not only if the use of different representations influenced students' scores on a set of problems, but also how students used the information represented on problems to model a situation involving conditional probabilities. A justification for this approach can be found in Bar-Hillel and Falk's (1982) argument that textbooks and problems used in instruction often use some "non-epistemic' kind of language, usually with keywords such as "given" for conditional probabilities, that "hand out in silver platters" which is the conditioning event, and that to truly understand and construct an appropriate notion of the concepts of conditional probability and probability of an intersection of events, it is important that students be able to model problems and to identify the appropriate probability space in which the data was obtained. Garutti, Dapueto and Boero (2003) also provided a theoretical support for this study, since they highlight the role of semiotic tools in probabilistic thinking, and investigate students' representations in the development of probabilistic models. We use the same idea to analyze students' mathematization of a word problem – which can be considered an important part of the modeling process.

In this study we varied the structure of the sentences that present the data relative to the conditional probabilities and probabilities of intersection of events, since a confusion between these two concepts has been suggested by Pollatsek et al. (1987) as a source of error. Three general structures were used for the problems:

Two of them (problems 6 and 3) could be solved if the students were able to establish a relation between conditional and probabilities of intersections of events:

$P(A\backslash B) = P(A\cap B)/P(B)$

Two problems (problems 2 and 4) could be solved if the students were able to establish a relation between P(A\B) and P(B\A), that is, probability of an event A given that B occurred and probability of event B given that A occurred (Bayes' rule);

Two problems (problems 1 and 5) involved the partition of the conditioning event:

$P(A)=P(A\cap B)+P(A\cap\sim B)$

Also, for each type of structure above, one problem had its data presented as percentages, and one had its data presented as decimals.

The use of key words that usually prompt the interpretation of data as either conditional or probabilities intersections, such as "and" and "given that" was avoided, and some phrasings had, for example, the word "and" (usually expected to denote the probability of two events happening simultaneously) used when the data referred to the conditional probability. The problems were written in students' native language, Portuguese.

The representations used by students in their solutions to the problems – equations, tables, tree diagrams – were analyzed to see how they understood the structure of the problems.

Results and Conclusions

By analyzing students' representations of the problem, it was found that students fell into four different categories:

a) students who adequately modeled the situation in the problem and characterized also properly those probabilities or percentages as characterizing either conditional or probabilities of intersection of events;
b) students who inadequately modeled the situation in the problem, but then coherently characterizing the data as either conditional or probabilities of intersections;
c) students who adequately modeled the situation in the problem, but then considered the probabilities that used the subset of the population as its sample space a probability of an intersection of events, and vice-versa;
d) students who inadequately modeled the situation in the problem, but then, since they also took the concept of P(A\B) for $P(A\cap B)$ and vice-versa, they gave a correct answer to the problem by using a formula.

The data suggests that the source of confusion between the concepts of conditional and probability of intersection of events is the inability to correctly interpret to which population or subset of the population the data of the problem referred to.

References

Bar-Hillel, M., and Falk, R. (1982). Some teasers concerning conditional probabilities. *Cognition*, 11, 109-122.

Garuti R., Dapueto P. Boero P. (2003) Evolution of forms of representation in a modelling activity: a case study. In: Pateman, Neil A., Dougherty Barbara J., Zilliox Joseph T. (ed.) *Proceedings of the 27th conference of the International group for the Psychology of Mathematics Education,* 2, 413-420.

Gigerenzer, G. (1996). The psychology of good judgment: Frequency formats and simple algorithms. *Journal of Medical Decision Making, 16*, 273-280.

Pollatsek, A., Well, A., Konold, C., Hardiman, P. e Cobb, G. (1987). Understanding conditional probabilities. *Organizational Behavior and Human Decision Processes*, 40, 255-269.

Shaughnessy, M. (1992). Research in probability and statistics: Reflections and directions. In: D. A. Grouws (Ed.) *Handbook of Research in Mathematics Teaching and Learning*, 465494. New York: Macmillan.

AN INVESTIGATION OF TEACHERS' PERSONAL AND PEDAGOGICAL UNDERSTANDING OF PROBABILITY SAMPLING AND STATISTICAL INFERENCE

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The purpose of this study is to develop an insight into teachers' personal and pedagogical understandings of probability and statistical inference. To this end, we undertook a teaching experiment with eight high school mathematics teachers. The teaching experiment was designed with the purpose of provoking the teachers to express and to reflect upon their instructional goals, objectives, and practices in teaching probability and statistics. Our results indicated that teachers had a complicated, inconsistent mix of meanings with regard to the ideas of probability, sampling distribution, hypothesis testing, and margin of error.

Statement of Problem

Teachers' understanding of significant mathematical ideas has profound influence on their capacity to teach mathematics effectively (Ball & McDiarmid, 1990; Ball, 1990; Ball & Bass, 2000; Borko et al., 1992; Eisenhart et al., 1993; Simon, 1994; Sowder, Philipp, Armstrong, & Schapelle, 1998; Thompson, 1984; Thompson & Thompson, 1996), and, in turn, on what students end up learning and how well they learn (Begle, 1972; 1979). This has important implications for how teacher educators think about ways of supporting teachers' professional development—that supporting the transformation of teaching practices takes careful analysis of teachers' personal and pedagogical understandings of what they teach.

Probability and statistical inference are among the most important and challenging ideas that we expect students to understand in high school. They have had an enormous impact on scientific and cultural development since its origin in the mid-seventeen century. The range of their applications spread from gambling problems to jurisprudence, data analysis, inductive inference, and insurance in eighteen century, to sociology, physics, biology and psychology in nineteenth, and on to agronomy, polling, medical testing, baseball and innumerable other practical matters in twentieth (Gigerenzer et al., 1989). Along with this expansion of applications as well as the concurrent modification of the theories themselves, probability and statistical inference have shaped modern science and transformed our ideas of nature, mind, and society. Given the extraordinary range and significance of these transformations and their influence on the structure of knowledge and power, and on issues of opportunity and equity in our society, the question of how to support the development of coherent understandings of probability and statistical inference takes on increased importance.

There have been many investigations of ways people understand probability and statistical inference. Psychological and instructional studies consistently documented poor understanding or misconceptions of these ideas among different populations across different settings (Fischbein & Schnarch, 1997; Kahneman & Tversky, 1973; Konold, 1989; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993a). The challenges for students are not associated merely with acquiring new skills but with overcoming ways of thinking that are unpropitious for reasoning probabilistically (Kahneman, Slovic, & Tversky, 1982; Konold, 1989, 1991) or for understanding probability as a mathematical model (Shaughnessy, 1992; Piaget & Inhelder, 1975), and dealing with the pervasive traditional teaching that works against efforts to make sense of stochastic concepts and ideas (Fischbein, 1975). Contrary to the overwhelming evidence of individuals' difficulties in learning probability and statistical inference, there is a general lack of insight into mechanisms by which transmission of this knowledge in classroom happens. Particularly, research on statistics education has not attended to teachers' understanding of probability and statistics (Garfield & Ben-Zvi, 2003).

The goal of this study is to explore teachers' personal and pedagogical understanding of probability and statistical inference. To this end, we undertook a teaching experiment with eight high school mathematics teachers. This teaching experiment is an early, highly exploratory step of a larger research program which aims to understand ways of supporting teachers learning and their transformations of teaching practices into one that is propitious for students learning in the context of probability and statistics instruction. We designed the teaching experiment with the purpose of provoking the teachers to express and to reflect upon their instructional goals, objectives, and practices in teaching probability and statistics. Our primary goal was to gain an insight into the issues, both conceptual and pedagogical, that teachers grapple with in order to teach probability and statistics effectively in the classroom.

Research Design

This study is the last teaching experiment in a research project that entailed a total of 5 teaching experiments conducted over a 40-month period and involved three different groups of participants. The prior four teaching experiments investigated high school students' thinking as they participated in classroom instruction designed to support their learning of sampling, probability, and statistical inference as a scheme of interrelated ideas. The aim was to develop epistemological analyses of these ideas (Thompson & Saldanha, 2000)—ways of thinking about them that are schematic, imagistic, and dynamic—and hypotheses about their development in relation to students' engagement in classroom instruction. Using the products and insights we obtained from these previous teaching experiments (Saldanha, 2003; Saldanha & Thompson, 2002; Thompson & Liu, 2002), in this study we engaged a group of high school teachers in rethinking what they hope students learn from statistics instruction and in reflecting on ways of affecting students' learning.

Methodology

We conducted the study using a combination of design experiment and constructivist teaching experiment methodologies. Design experiment methodology (Gravemeijer, 1994) provides an emphasis on cycles of design, implementation, evaluation, and redesign. Constructivist teaching experiment methodology (Steffe & Thompson, 2000) provides an emphasis on conceptual analysis of mathematical ideas and analyzing subjects' participation in classroom discussions and interviews for cues about their ways of knowing mathematical ideas.

Design and Implementation

Eight high school mathematics teachers—six female and two male teachers— participated in the workshop. Among these eight teachers, two had 2-3 years of teaching experience, three had 7-9 years of teaching experience, and three had 21-28 years. Three teachers taught AP statistics. Five taught a probability and statistics chapter in high school mathematics course. The research team consisted of the PI (the second author), a collaborating teacher, and three graduate students. The team designed the workshop activities and artifacts during 9 months prior to the workshop. The collaborating teacher hosted most of the workshop activities and conversations. The PI served as an observer of the workshop and occasionally hosted the workshop or participated in the conversation. One graduate student took field notes and managed miscellaneous logistic work. The first author and another graduate student recorded the workshop sessions with front and back cameras, and made observations and notes during the workshop.

The workshop discussion progressed over eight sessions in two weeks. The workshop began at 9am each day and concluded at 3pm, with a 30- minute lunch break. At the end of each day, the research team met briefly to discuss our observations and suggestions on modification of next days' activities. We also made photocopies of teachers' notes at the end of each day. Each teacher was interviewed 3 times for about 45 to 60 minutes each time. Interviews were conducted once before the workshop and at the end of each week. We video recorded all interviews and kept record of teachers' work during the interviews.

We engineered the discussions so that teachers first worked on and discussed the problems as first-order participants. We used these occasions to construct models of teachers' personal understanding of the ideas of probability and statistical inference. We then initiated pedagogical conversations about these ideas—given these ways of understanding these ideas, what are the implications for teaching them? We intended to elicit reflective conversations in the sense that what was previously discussed became objects of thoughts and conversation (Cobb, Boufi, McClain, & Whitenack, 1997). The interviews were designed to include general questions concerning teachers' stochastic reasoning, as well as specific questions that were tailored to each teacher according to our observation and conjectures about his or her knowledge and beliefs.

Data Analysis

The data for analysis include video recordings of all workshop sessions made with two cameras (36.5 hours) and individual interviews (approximately 24 hours), the teachers' written work, field notes, and documents made during the planning of the workshop. The analytical approach we employed in generating descriptions and explanations was consistent with Cobb and Whitenack's (1996) method for conducting longitudinal analyses of qualitative data and Glaser and Strauss' (1967) grounded theory, both of which highlights an iterative process of generating and modifying hypotheses in light of the data. Analyses generated by iterating this process were aimed to develop increasingly stable and viable hypotheses and models of teachers' understanding.

We began by first reviewing the entire collection of videotaped workshop sessions and interviews. Our primary goal in this process was to develop an overall sense of what had transpired in the workshop, and to identify video segments that seemed potentially useful for gaining insight into one or more teachers' personal and pedagogical understandings of probability and statistical inference. Segments containing direct evidence of teachers' thinking, miscommunication in discussions of problems or ideas, or controversy about mathematical meanings or pedagogical practices were especially significant. Later, these video segments were transcribed. We then annotated the transcripts with the purpose of developing hypotheses of teacher's understanding. The guiding question for making sense of a teacher' utterance is: *What might he have been thinking (or seeing the situation, or interpreting the previous conversation) so that what he said made sense to himself?*

Results

Our analysis indicated that, collectively and individually, teachers had a complicated, inconsistent mix of meanings with regard to ideas of probability and statistical inference.

Probability

Teachers' understanding of probability covered a broad spectrum:

• thinking that probability is a (subjective) judgment based on personal experiences;

- thinking that probability is about predicting the state of a specific completed (or to-be-completed) event about which one does not know the actual result;
- thinking that probability is about selecting one outcome from a set of possible outcomes;
- thinking that probability is about imagining a collective of results all generated by a single process that yields results that are more dense in some regions of possible values than in other regions, i.e., a stochastic conception of probability (Thompson & Liu, 2002).

We found that the representation of probability questions or statements has a direct association with teachers' interpretation of probability. When a probability question or statement explicitly states a collection, e.g., "A study of over 88,000 women found that total folate intake was not associated with the overall risk of breast cancer," or "drivers with three or more speeding tickets are twice as likely to be in a fatal accident as are drivers with fewer than three tickets," teachers were more likely to interpret probability as a group characteristic. On the other hand, when a probability question or statement is expressed as about a single event, e.g., "your risk of being killed on an amusement park ride? One in 250 million." or "what is the probability that my car is red?" teachers were less likely to have a background image of a collection of similar events against which the probability is evaluated.

Even when some teachers later did conceive situations stochastically, they did not understand, or were reluctant to accept, that probability is determined by the stochastic processes one imagines/constructs. Teachers argued whether a probabilistic situation could have multiple interpretations and thus have different values depending on how one interprets it. The group that argued against it seemed to believe that a probabilistic situation should not subject to multiple interpretations. They did not see, or refused to accept, a distinction between a situation as it is stated/written and its (possibly multiple and conflicting) interpretations. However, both groups (one who accepts multiple interpretations and one who don't) shared a commitment to "finding one correct answer". They believed that computer simulation could decide a correct answer, without realizing that simulations are designed according to one's interpretation of the underlying stochastic process, and thus only confirms one's interpretation of a probabilistic situation.

Sampling Distribution

Our previous teaching experiments revealed students' profound difficulties in conceiving a distribution of sample statistics (Saldanha, 2003). Unlike students, teachers seemed to have a good grip on the hierarchical process that generates a distribution of sample statistics. We observed only one instance in which a teacher referred to a proportion of samples as a sample proportion.

Teachers also understood the relationship between sample size and variability, i.e., as sample size increases, the variability of the distribution of sample statistics decreases. However, it was not a general idea for them in the sense that their understanding was very contextualized (Thompson, Liu, & Saldanha, 2004). For example, teachers examined the distribution of sample proportions of samples of size 100 from a population a proportion of which have a certain characteristic. They repeated the simulation several times taking 150 samples to see how stable the distribution was from trial to trial, then repeated it several times taking 500 samples, then repeated it several times taking 1000 samples. Only one teacher recognized that "taking *x* samples of size 100" was a stochastic process, and therefore that the relationship between sample size and variability that they had just stated also applied to their investigation.

Teacher also claimed that if the population size increases, then the sample size has to increase proportionally in order to maintain the same variability of the distribution.

Hypothesis Testing

Teachers were unfamiliar with the logic of hypothesis testing. For example, we gave the teachers this scenario: “Assume that sampling procedures are acceptable and that a sample is collected having 60% favoring Pepsi. Argue for or against this conclusion: This sample suggests that there are more people in the sampled population who prefer Pepsi than prefer Coca Cola.” (With a list of 135 simulated samples of size 100 from a population split 50-50 in preference). Teachers initially took the position that there was no basis for arguing this conclusion because the data confirmed that the population percent was 50% (the null hypothesis). Two Teachers eventually concurred with the workshop leader that the data suggested that samples of 60% or more were sufficiently rare so as to reject the null hypothesis. Some teachers were reluctant to accept this logic. One teacher argued that one could not make any judgment based on the fact that a rare sample occurred, because the sample *could* occur however rare its chance of occurrence might be. This argument revealed the teacher’s commitment to having a *single* valid judgment, which is inconsistent with the idea that a decision rule does not decide a single valid judgment, but only ensure that the error rate is small over the long run.

In one scenario, when asking to create a decision rule for testing a hypothesis, some teachers proposed one hypothetical simulation result that would confirm alternative hypothesis instead of a general principle for rejecting a null hypothesis. As a result, they could not reconcile the difference between the hypothetical simulation result and the actual simulation result. Only one teacher realized that hypothesis testing was like proof by contradiction.

Margin of Error

We also found a common misunderstanding of margin of error—all the teachers initially believed that a poll result of "76% with margin of error +4%" (confidence level 95%) meant that 95% of the sample proportions would fall within the interval [76%-4%, 76%+4%]. Most teachers came to see the error in this thinking. During the second interview, 7 out of 8 teachers correctly interpreted margin of error, i.e., 95% of the sample statistics would be expected to fall within 4% point of the true parameter.

After teachers understood the concept of margin of error, they raised an interesting question: “Why bother taking a poll if we don’t know *for sure* if the poll result will be among those 95% of the poll results that are within a certain interval of the true population parameter?” The conversation around this question revealed that 1) teachers were uncomfortable with the uncertainty that is the very reason for resorting to statistical information, and 2) teachers had a tendency to fixate on individual cases (in this case, the poll result) as opposed to statistical patterns in the long run.

Pedagogical Understanding

Teachers exhibited a strong commitment to “finding the right answer”. Many of the tasks we designed were conceptually challenging—the answers were not obvious, and sometimes, they may have different answers depending on different interpretations. We found many instances in which teachers could not move on to pedagogical conversation if an agreement on one correct answer was not achieved.

Teachers believed that students learn an idea by “hearing it said correctly”. When trying to learn ideas from each other during the workshop, teachers seemed to place low confidence in their own understanding, but readily accept ideas uttered by those whom they perceived as experts. For example, when one teacher questioned the validity of “equal likelihood”, the other defended it by saying "the formula of probability is based on equal likelihood, therefore it must be valid." This conception of learning projected potential ways of teaching that are insensitive to students' understanding.

This way of teaching was also projected by the ways in which teachers negotiate meanings. We found that teachers often dismissed the opposing point of view as "mistakes", convincing others that they are wrong by telling them they are wrong, instead of trying to understand the other person's intention.

Significance

This study makes three contributions to our understanding of the teaching and learning of probability and statistics. It provides greater insight into stochastic reasoning as a foundation for understanding probability and statistical inference. It suggests limitations to which we should be alert in high school teachers' understanding. And last, it suggests areas in which teachers' development of personal and pedagogical understanding in probability and statistics may need support.

References

Ball, D., & McDiarmid, G. W. (1990). The subject matter preparation of teachers. In W. R. Houston (Ed.), *Handbook of research on teacher education* (pp. 437-449). New York: Macmillam.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *Elementary School Journal, 90*, 449–466.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on mathematics teaching and learning* (pp. 83-106). Stamford, CT: Ablex.

Begle, E. G. (1972). *Teacher knowledge and pupil achievement in algebra* (NLSMA Technical Report No. Number 9). Palo Alto, CA: Stanford University, School Mathematics Study Group.

Begle, E. G. (1979). *Critical variables in mathematics education: Findings from a survey of the empirical literature*. Reston, VA: National Council of Teachers of Mathematics.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*(3), 194–222.

Cobb, P., Boufi, A., McClain, K., & Whitenack, J. (1997). Reflective discourse and collective reflection. *Journal for Research in Mathematics Education, 28*(3), 258-277.

Cobb, P., & Whitenack, J. (1996). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Educational Studies in Mathematics, 30*(3), 213-228.

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. C. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics. *Journal for Research in Mathematics Education, 24*(1), 8–40.

Fischbein, E., & Schnarch, D. (1997). The evolution with age of probabilistic, intuitively based misconceptions. *Journal for Research in Mathematics Education, 28*(1), 96-105.

Garfield, J., & Ben-Zvi, D. (2003). Research on statistical literacy, reasoning, and thinking: issues, challenges, and implications. In D. Ben-Zvi & J. Garfield (Eds.), *The challenges of developing statistical literacy, reasoning, and thinking*. Netherlands: Kluwer.

Gigerenzer, G., Swijtink, Z., Porter, T., Daston, L., Beatty, J., & Kruger, L. (1989). *The empire of chance: How probability changed science and everyday life*. Cambridge: Cambridge University Press.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.

Gravemeijer, K. P. E. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education, 25*(5), 443–471.

Kahneman, D., & Tversky, A. (1973). On the psychology of prediction. *Psychological Review, 80*, 237–251.

Konold, C. (1989). Informal conceptions of probability. *Cognition and Instruction, 6*(1), 59-98.

Konold, C., Pollatsek, A., Well, A., Lohmeier, J., & Lipson, A. (1993a). Inconsistencies in students' reasoning about probability. *Journal for Research in Mathematics Education, 24*(5), 393-414.

Saldanha, L. A. (2003). *"Is this sample unusual?": An investigation of students exploring connections between sampling distributions and statistical inference.* Unpublished doctoral dissertation, Vanderbilt University, Nashville, TN.

Saldanha, L. A., & Thompson, P. W. (2002). Conceptions of sample and their relationship to statistical inference. *Educational Studies in Mathematics, 51*(3), 257-270.

Simon, M. A. (1994). Learning mathematics and learning to teach: Learning cycles in mathematics teacher education. *Educational Studies in Mathematics, 26*(1), 71–94.

Sowder, J. T., Philipp, R. A., Armstrong, B. E., & Schapelle, B. P. (1998). *Middle-Grade Teachers ' Mathematical Knowledge and Its Relationship to Instruction: A Research Monograph*. Albany: State University of New York Press.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. Kelly (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15*, 105–127.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, Part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*(1), 2-24.

Thompson, P., Liu, Y., & Saldanha, L. A. (2004). *Intricacies of statistical inference and teachers' understanding of them.* Paper presented at the 33rd Carnegie Symposium on Cognition, "Thinking with data", Pittsburgh, PA.

Thompson, P. W., & Liu, Y. (2002). *What is the probability that my car is red: tensions in the development of probabilistic reasoning.* Paper presented at the Annual conference of American Educational Research Association, New Orleans.

Thompson, P. W., & Saldanha, L. A. (2000). *Epistemological analyses of mathematical ideas: A research methodology.* Paper presented at the Twenty Second Annual Meeting of the International Group for the Psychology of Mathematics Education - North American Chapter, Tucson, AZ.

HIGH SCHOOL STUDENTS' LEVELS OF THINKING IN REGARD TO ANALYZING UNIVARIATE DATA SETS

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This study investigated levels of thinking about the analysis of univariate data sets. The two primary research questions addressed were: (1) What are some of the levels of thinking among high school students in regard to comparing univariate data sets?; (2) What are some of the levels of thinking among high school students in regard to analyzing the impact of a linear transformation upon the center and spread of a univariate data set? Three levels of thinking were identified for each research question.

Theoretical Perspective on Levels of Thinking

The study used the Structure of the Observed Learning Outcome (SOLO) Taxonomy of Biggs and Collis (1991) in order to differentiate among levels of thinking. The unistructural, multistructural, and relational levels of SOLO were of the greatest relevance to the present study. The same theoretical perspective has previously been employed in describing levels of statistical thinking for elementary and middle school students (Jones et al, 2000; Mooney, 2002).

Methodology

Participants

Purposeful sampling (Patton, 1990) was used in the selection of study participants. Fifteen students representing a range of experiences with high school mathematics were recruited and volunteered to be interviewed for the study. Since participants were selected in this manner, this study did not attempt to statistically generalize findings across a large population. Instead, it described the levels of thinking exhibited by students within the diverse sample chosen.

Procedure and instruments

An interview protocol (Groth, 2003) designed to elicit statistical thinking was administered to each of the study participants. This paper reports upon the levels of thinking that were evident in the responses to three questions on the interview protocol. The first two questions of interest asked students to compare two sets of univariate data. They were given the opportunity to compare data sets presented in both graphical and tabular form. The last question of interest asked students to determine what would happen to the center and spread of a specific univariate data set if each of the values in the data set was increased by a constant.

Data gathering and analysis

As students were interviewed, their responses were audiotaped, and the interviewer took observational notes. The audiotapes were later transcribed for analysis. Written work completed by the students during the interviews was also kept for analysis. Coding of students' responses was influenced by the SOLO Taxonomy, since interview responses were grouped into categories on the basis of the number of relevant attributes included in each response and whether or not connections were made among the relevant attributes incorporated.

Results

Levels of Thinking in Regard to Comparing Univariate Data Sets

The patterns of response for comparing univariate data sets represented one complete unistructural-multistructural-relational (UMR) cycle. The cycle progressively built to the point that it became evident that students perceived each of the data sets being compared as aggregates. The least sophisticated pattern of response was unistructural because only one strategy was used for comparing the data sets: point-by-point comparisons. In the second pattern, comparisons included the use of one of the characteristics of aggregate data sets such as shape, center, or spread. Hence, these responses showed a progression toward the view of the data sets as aggregates. However, the second pattern of response could only be considered multistructural, since the only relevant accompanying strategies were point-by-point comparisons, which did not provide further evidence that the data sets being compared were viewed as aggregates. The third pattern of response was characteristic of the relational level, since multiple relevant aggregate characteristics were used in concert for the purpose of comparing two data sets.

Levels of Thinking in Regard to the Impact of Data Transformation upon Center and Spread

There were three patterns of thinking for this topic reflecting varying degrees of sophistication. The first pattern was that at which students correctly stated the end of result data transformation for just one of the measures in question: either the center or the spread. The mastery of just one relevant aspect indicated unistructural level thinking. In the next pattern, students recognized the direction of the impact on a measure of center and realized that a measure of spread would be unchanged. However, while the direction of impact on a measure of center was given, the impact was not quantified. Since one more relevant aspect was incorporated, but the key theme of quantification was not, the second pattern of response was considered multistructural. Responses reflecting the most sophisticated pattern did bring the key aspect of quantification into play, so they were considered relational.

Conclusion

The present study has implications for teaching and also points out directions for further research. Some of the high school students interviewed exhibited thinking similar to that of elementary and middle school students in previous studies (Jones et al., 2000; Mooney, 2002) in that they reasoned point-by-point rather than globally when comparing data sets. This means that curriculum developers and teachers need to be aware that some high school students do not yet view data sets as aggregates, and that instructional sequences need to be adjusted accordingly. Teachers also need to be aware that understanding the impact of adding a constant to each value of a univariate data set is a non-trivial matter. Directions for further research include follow-up studies where the influence of problem context upon levels of response is investigated, and also studies in which the impact of different instructional techniques upon levels of response is described.

References

Biggs, J.B. & Collis, K.F. (1991) Multimodal learning and quality of intelligent behavior. In H.A.H. Rowe (Ed.), *Intelligence: Reconceptualization and measurement* (pp. 57-66), Hillsdale, NJ: Erlbaum.

Groth, R.E. (2003). *Development of a high school statistical thinking framework*. Unpublished doctoral dissertation: Illinois State University.

Jones, G.A., Thornton, C.A., Langrall, C.W., Mooney, E.S., Perry, B., & Putt, I.J. (2000). A framework for characterizing children's statistical thinking. *Mathematical Thinking and Learning, 2*, 269-307.

Mooney, E.S. (2002). A framework for characterizing middle school students' statistical thinking. *Mathematical Thinking and Learning, 4*, 23-64. Patton, M. (1990). *Qualitative evaluation and research methods* (2nd ed.). Newbury Park, CA, Sage.

UNDERGRADUTE STUDENTS' MEANINGS ABOUT EFFECT OF SAMPLE SIZE IN THE SHAPE AND VARIABILITY OF SAMPLING DISTRIBUTIONS

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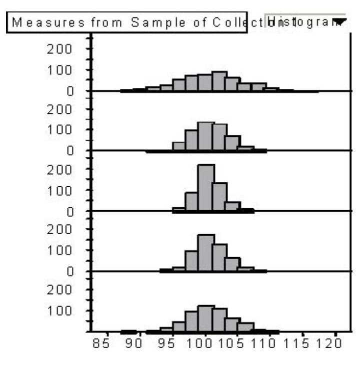
The sample size is an element closely related to the behavior (shape and variability) of the sampling distribution. The students' understanding of this relation constitutes an important factor in the development of a scheme suitable for the learning of the sampling distribution and its application in the reasoning of statistical inference (Lipson, 2002).

The purpose of this study was to explore the meanings that undergraduate students concluding a first course of statistics have of the effect of the sample size in the behavior of the sampling distributions. We interpreted these meanings by means of Godino and Batanero's (1998) framework. The meaning of the mathematical objects is conceived as the system of practices linked to specific fields of problems from which the objects emerge. These systems can be shared and accepted by a group of professionals who are involved in the study of the object (institutional meaning) or can be attributed to a person (personal meaning), as it would be the case of a student solving problems in which the object is comprised. In order to describe the mathematical activity, the model considers diverse entities that constitute the meaning elements. These entities are *language*, *situations, actions*, *concepts*, *properties* and *argumentations.* The understanding of a mathematical object by a subject is interpreted as the appropriation of the different elements that compose its institutional meaning. In order to study the students' meanings, a questionnaire was applied to 73 students.

1. Of a population with normal distribution, 500 random samples of each size were extracted (5, 10, 15, 20, 25). The average of each sample was calculated and the results were drawn in the histograms showed in the following figure. Students were asked to place next to each histogram the sample size that corresponded and to explain or make the calculations that they considered pertinent.



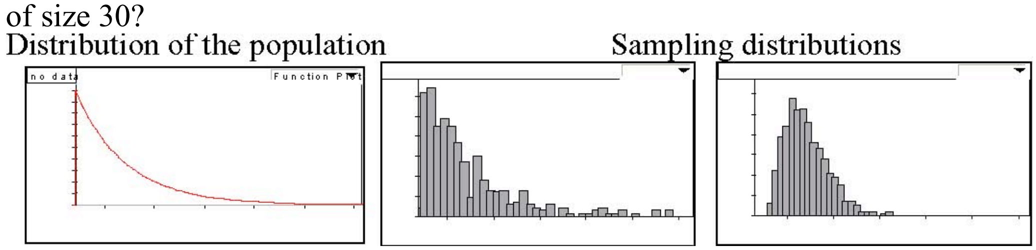
It was expected that the main concept that the students would apply was the standard deviation of the sampling distribution, in which the sample size ($\sigma_{\bar{x}} = \sigma\sigma_x = \sigma / \sqrt{n}$ is involved. It was expected that the students were conscious of the property "to greater sample size there is less variability".

Nevertheless, in their argumentations the use of this property was not explicit. In the actions that followed to assign the sample sizes, we observed two different strategies in most of the students' answers:

Strategy 1: This group used the strategy of the height of the histograms, which lead them to a correct allocation of the sample sizes. The allocation pattern was that to a greater size of sample a smaller variability. Some typical argumentations of this group were: *"I assigned the size of the sample considering the height of the histogram"*. *Strategy 2:* This group used the strategy of the number of bars of the histogram, which lead them to an incorrect allocation of the sample sizes. The allocation pattern resulted in to greater size of sample, a greater variability. Some representative argumentations of this group were: *"Because the size increased, the intervals similarly increased"*.

Nevertheless the students who followed strategy 1 assigned the sizes of sample correctly, their personal meaning does not contemplate the properties or the language established in the institutional meaning and showed misconceptions of the variability of a distribution.

2. The figure shows the distribution of a population and four possible sampling distributions of means for extracted random samples of the population. The sample sizes were 1, 5, 15 and 30 and 500 samples of each size were extracted. In your opinion, which graph represents a sampling distribution for samples of size 5? and which graph represents a sampling distribution for samples of size 30?



Sample size (n)	Sampling distribution is resembled to distribution of the population	Sampling distribution is resembled to normal distribution
Small (n=5)	35%	36%
Large (n=30)	37%	7%

It was expected that the main concept that the students would use to answer this item was the central limit theorem. Nevertheless, their argumentations did not give account of a suitable handling of this property or of the corresponding language, reason why their personal meaning did not agree with the meaning of reference.

As a conclusion we can say that the students' meaning about the effect of sample size in the behavior of the sampling distributions did not contain the meaning elements established in the reference meaning, which demonstrates their difficulties of learning and the complexity of the object, because although during the course a frequency approach was not emphasized, the effect of sample size extends to all the inference course.

References

Godino, J. & Batanero, C. (1998). Clarifying the Meaning of Mathematical Objects as a Priority Area of Research in Mathematics Education. In A. Sierpinska & J. Kilpatrick (Eds.) *Mathematics Education as a Research Domain: A Search for Identity,* 177-195. Kluwer A. P.

Lipson, K. (2002). The Role of Computer Based Technology in Developing Understanding of the Concept of Sampling Distribution. In B. Phillips (Ed.). *Proceedings of the Sixth International Conference on Teaching Statistics,* South Africa.

THE ROLE OF INTUITION IN PRE-SERVICE TEACHERS' PROBABILISTIC PROBLEM-SOLVING

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*Abstract The research question in this study was "What are the probability solving capacity of pre service teachers and their primary intuitions?" The research instrument was a 3item questionnaire concerning tossing dice and tossing a coin. When confronting a one dimensional event - tossing one die, 58 of the 99 (58.6%) gave the correct answer. 29 (29.3%) gave the answer 1/3 - the result of primary intuition. In a two-dimensional event, when tossing two dice - only 36 (36.7%) gave the right answer. Wrong answers included 1/12 in which the denominator comes from the intuition to add (6+6) instead of multiplying (6*6)-11(11/1%) gave this answer. In a compound event -getting the sum of 11, only 18 (18.3%) gave correct answers. 23 (23.2%) gave the answer 1/36, the primary-intuition result of not viewing 5+6 as a different outcome from 6+5.*

THE MEANING OF 'MEAN': TEACHER PERCEPTION OF STUDENT UNDERSTANDING WITHIN A COLLEGE STATISTICS COURSE

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The role of understanding, and the multi-contextual nature of the term itself, is of great interest and debate within the mathematics and statistics education communities. Every researcher within these domains must be wary of their own treatment of understanding; yet, care must also be taken with the implicit assumptions of teachers who help shape the ways in which students come to their own understanding(s). The main purpose of this pilot study is to examine the relationship between students' understanding of the concept of arithmetic mean, and that of their current statistics instructor. Borrowing heavily from Sfard's recursive model of process, object, and reification, I choose to treat understanding as a fluid, cyclic pattern of growth. By comparing the results of a student survey with the instructor's conceptions of both student understanding and beliefs about mathematical/statistical understanding in general, light is shed on the complexities of understanding-oriented research. Students were asked to explain the purpose of statistics and also explain the mean concept, and their instructor was later interviewed regarding the responses which she examined. Results indicate that a question about student beliefs turned into an assessment of statistical understanding by the instructor. The terminology and discourse in a classroom changes, yet whether this change is made explicit by instructor or student is another matter that can not be ignored in research on understanding. Also, differences in time between data collection and direct instruction on the topic were central in the analysis of student understanding. The instructor suggests that students see the mean as a process solely because of the extensively process-oriented nature by which it is treated throughout the course since its introduction – therefore, reification can be delayed or even prevented by curricular and instructional decisions. To be sure, the ubiquity of highly specific examples and calculations in student responses seems to suggest exactly that. This study points to the ways in which the discourse surrounding statistical terminology is shaped, refined during the learning process, and possibly ignored over the span of a course, and how that might influence both the community within the classroom and students' access to the larger discourse regarding statistics. The above revelations would not have emerged had a standard student-based assessment been the sole source of data. The instructor in this study helped mediate the nature of student responses, the growth of student understanding as it supports a recursive, process-object model, and the specific contexts in which the students and the topics on the survey were situated. By ignoring the symbiotic nature of student understanding, the discourse of statistics, and teacher perception of both, researchers will not achieve a robust and thorough perspective on the ways in which students come to learn statistics. Looking at students' responses in conjunction with their teachers, finding out what teachers think about the responses within their own class contexts, is critical for gaining a more complete picture of the growth of "understanding."

Reference

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.