PROCEEDINGS OF THE 46TH ANNUAL MEETING OF THE NORTH AMERICAN CHAPTER OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

Envisioning the Future of Mathematics Education in Uncertain Times

EDITED BY

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Proceedings of the Forty-Sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

Envisioning the Future of Mathematics Education in Uncertain Times

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PME–NA History and Goals

The International Group for the Psychology of Mathematics Education (IGPME) was founded in 1976 at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany, with the first conference held in 1977 in Utrecht in the Netherlands. The North American Chapter of PME (PME-NA) was organized not long after with the first conference held in Evanston, Illinois in 1979. This initial meeting was organized and chaired by Dr. Richard Lesh following the initial National Council of Teachers of Mathematics (NCTM) Research Presession held in 1978 (Boston, Massachusetts). The plenaries at the NCTM meeting (Heinrich Bauersfeld, Efraim Fischbein, & Hans Freudenthal) were instrumental in the founding of IGPME and encouraged Lesh and others to create a North American chapter. The initial meeting in 1979 may have only featured plenary speakers. However, with the second meeting of PME-NA being a joint meeting with IGPME, individual papers published in the conference proceedings became a mainstay of the conference. Early conferences for both IGPME and PME-NA were focused on exploration of various areas of psychology in the teaching and learning of mathematics. Since their origins, PME and PME-NA have expanded and continue to expand beyond their psychologically oriented foundations. For example, the 1981 PME-NA conference focused on "the influence of modern technology upon mathematics education and related research" (p. 11) in addition to areas of psychology.

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers; and
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

PME–NA Membership

Membership is open to people who are involved in active research consistent with PME-NA's aims or who are professionally interested in the results of such research. Membership is open on an annual basis and depends on payment of dues for the current year. Membership fees for PME-NA (but not PME International) are included in the conference fee each year. If you are unable to attend the conference but want to join or renew your membership, go to the PME-NA website at http://pmena.org. For information about membership in PME, go to http://www.igpme.org and visit the "Membership" page.

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The Local Organizing Committee is extremely appreciative of the following people for serving as Strand Leaders. They managed the reviewing process for their strand and made recommendations to the Local Organizing Committee. The conference would not have been possible without their efforts.

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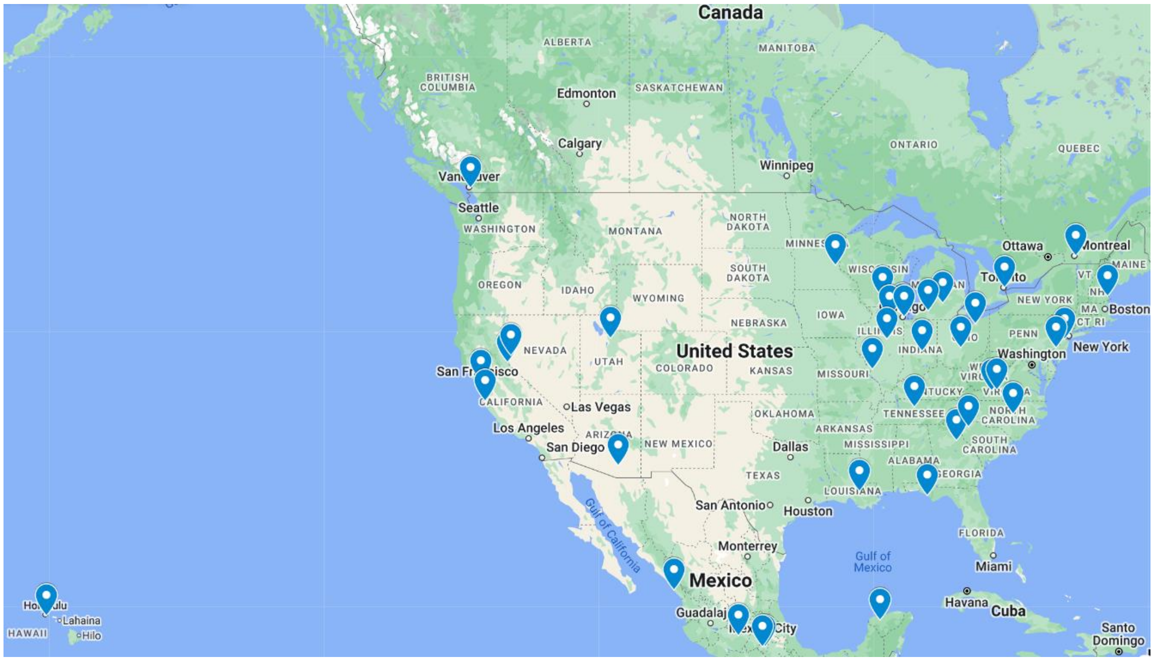

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Preface

On behalf of the 2024 PME-NA Steering Committee, the 2024 PME-NA Local Organizing Committee, and Kent State University, we welcome everyone to Cleveland, Ohio, USA, for the Forty-Sixth Annual Meeting of the International Group for the Psychology of Mathematics Education – North American Chapter, held at the Hilton Cleveland Downtown hotel and convention center.

This year's conference theme is **Envisioning the Future of Mathematics Education in Uncertain Times**. The past several years have seen significant change across North America and the world resulting from the pandemic, war, technological, political and social shifts. When we selected this theme in 2022 (two years ahead of the conference), there was no war in Ukraine or Israel, ChatGPT was virtually unknown, and each nation represented by scholars at PME-NA faced very different political landscapes. As the context of our world around us changes and evolves, mathematics education *will* also change – either solely from the pressures of the world around us, or through mathematics educators' engagement with the world. Thankfully, our field has a history of considering the profession in the midst of change across and within various contexts (society writ large, classrooms, academia, etc.). Engagement with the world for meaningful change must be informed by rigorous theory and research regarding the teaching and learning of mathematics. Additionally, because education is a caring activity, such engagement with others (students, teachers, the public, politicians, etc.) must be oriented towards not only teaching but learning from others for it to be meaningful. We encourage attendees and presenters to reflect on this as they attend various sessions at PME-NA and engage with others beyond these doors.

This year's conference will be attended by 511 researchers, faculty members, and graduate students, with presenters from around the world including Canada, México, the United States, as well as Australia, Cyprus, Denmark, the Netherlands, Norway, and Turkey. Submitted papers were reviewed in a double-blind process by multiple reviewers. After initial reviews were submitted, Strand Leaders examined feedback and provided recommendations for paper acceptance to the proceedings and conference. The local conference committee made final decisions based upon Strand Leader recommendations and reviewer comments. Overall, there were 423 submissions. The table below outlines the number of submitted proposals across each strand. Following the review and decision process, these proceedings include 81 research reports (41.8% acceptance), 170 brief research reports (73.0% acceptance), 111 posters (79.3% acceptance), and 16 Working Groups (88.9% acceptance).

Strand	Research Reports	Brief Reports	Posters
Curriculum, Assessment, and Related Topics	8	8	8
Early Algebra, Algebraic Thinking, and Function	10	2	2
Equity and Justice	18	15	9
Geometry and Measurement	9	3	5
Mathematical Knowledge for Teaching	4	5	5
Mathematical Processes and Practices	15	4	3
Number Concepts and Propoprtional Reasoning	8	1	3
Policy, Instructional Leadership, Teacher Educators	4	4	6
Pre-Calculus, Calculus, and Higher Math	13	7	1
Pre-Service Teacher Education	20	22	13
Professional Development / In-Service Teacher Education	17	13	9
Statistics, Probability, and Data Science	7	3	1
Student Learning and Related Factors	24	13	8
Teaching Practice and Classroom Activity	23	14	8
Technology and Learning Environment Design	14	6	10

We wish to extend gratitude to the many individuals who volunteered their time and effort in preparation of this conference. Thank you to the strand leaders for your efforts in assigning reviewers, examining reviews for quality and making appropriate recommendations for acceptance. We also wish to thank reviewers who engaged in this important service. We are grateful to many past conference chairs, current and past Steering Committee members, and various other colleagues for helpful feedback and support. Finally, we wish to thank Kent State University and Dean Hannon of the College of Education, Health, and Human Services (EHHS) for supporting our efforts in hosting this conference.

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A COMPARISON OF STRANDS AND COGNITIVE DEMAND LEVELS: EXAMINING UNIVERSITY ENTRANCE EXAM QUESTIONS ACROSS THREE COUNTRIES

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University Entrance Exams (UEEs) serve as the primary criteria for university admissions into universities in many countries. Given the significant impact of assessment types on teaching and learning, analyzing UEEs in different countries can reveal factors that influence students' learning. This study examined the content and the Cognitive Demand Levels (CDL) of UEE questions in Iran, Turkiye, and the United States. The results indicated that algebra was the most frequently featured content area in UEE questions. Moreover, the majority of algebra questions in all countries were categorized under procedure with connections and procedure without connections cognitive demand levels. Iran and Turkiye's UEEs included topics not covered in the NCTM strands for high school in the United States, such as set theory, graph theory, and logic. Furthermore, the SAT emphasized real-world scenarios within specific subject areas, whereas the UEEs in Iran and Turkiye integrated multiple subject areas within single questions. The paper discusses the implications of these findings.

Keywords: university entrance exams, cognitive demand levels, assessment, algebra, NCTM strands

Many high school students take the national UEEs in different countries, and their performance outcomes indicate admission to universities (Davey et al., 2007; Yildirim, 2007). Thus, it is essential that UEEs, as assessment tools, effectively measure students' knowledge and skills. Furthermore, since the assessment type greatly influences teaching and learning (Shepard, 2001), the university entrance examination plays an important role in shaping mathematics education in countries (Hong & Choi, 2011). Building on prior findings on how assessment tools, such as UEEs shape mathematics instruction, examining these assessment tools could assist educational researchers in understanding the necessities of curriculum development (Chang & Silalahi, 2017). This study is part of a larger research project that examines and compares textbooks and assessments across various countries and reports the results from comparing the mathematics sections of the UEE in Iran, Turkiye, and the United States.

Below are the research questions we aim to address in this study:

1- Which mathematics topics were included in the UEEs in Iran, Turkiye, and the United States?
2- In what ways do the cognitive demand levels of questions in UEEs differ or align across Iran, Turkiye, and the United States?

Literature Review and Theoretical Perspective

This study relies on the framework established by Smith and Stein (1998) to examine the CDL of UEE questions. Building on Doyle's (1983) work, Stein and Smith (1998) proposed the Task Analysis Guide comprised of four categories of tasks at different levels of cognitive demand: (1) memorization, (2) procedures without connections (PWOC), (3) procedures with connections (PWC), and (4) doing mathematics. According to this framework, memorization tasks involve producing previously learned facts, rules, formulae, or definitions, which are highly dependent on memorizing. PWOC tasks require using procedures specifically called for the previous instruction. These tasks do not require students to understand how the procedure works or to explain their thinking. These two levels of tasks are named low-level tasks. The next two CDL comprise high-level tasks. Although there may be some suggested pathways in the PWC tasks, these tasks require a deeper understanding of the mathematical concepts and a higher level of cognitive effort. Doing mathematics tasks requires complex and non-algorithmic thinking in which students explore and understand the nature of mathematical concepts, processes, or relationships. Studies have found that high CDL tasks provide students with more opportunities to learn (Smith & Stein, 1998).

Methods

The present study reports on the analysis of the UEEs of Iran, Turkiye and the United States. For Iran and Turkiye we selected the most recently released UEEs in 2023. However, due to restricted access to the actual SAT questions (the UEEs in the U.S.), the latest available released practice questions (Test 1) by the College Board for SAT were used for the study.

The UEE of Iran and Turkiye each comprised 40 multiple-choice questions, while the SAT practice questions comprised of 54 items, including both multiple-choice and short-answer questions. The first two authors conducted the analysis of Iran's UEEs questions, while the next two authors carried out the analysis of Turkiye's UEEs due to their proficiency in the language. For the SAT exams, all authors participated in the data analysis.

Initially, the authors employed the NCTM strands to classify each question into one of the subsequent topics: Algebra, Numbers and Operations, Data Analysis and Probability, Geometry, and Measurement. Next, the authors narrowed the focus to algebra strand questions, applying the CDL framework. To ensure consistency in the interpretation of CDL, the authors first selected a random sample from all questions (10%) for joint discussion. Prior to reconciling the analysis, they assessed the percentage of agreement as an indicator of inter-rater agreement. The results indicated high agreement rates for CDL: 87.5% for Iran, 86.67% for Turkiye', and 84.61% for the U.S.

Results

Initially, the study will explore the subject areas addressed in the exam questions using NCTM standards (Table 1). Following this, the analysis of the CDL of algebra tasks will be presented, with corresponding data available in Table 2. Lastly, a detailed exploration of the differences observed in algebra tasks among the three countries will be provided.

Table 1: ***Frequency of the UEE Questions in Each NCTM Strand***

Strands from NCTM		The U.S.		Iran		Turkiye	
		#	Percent age	#	Percent age	#	Percent age
Numbers and Operations		5	10%	3	7.5%	9	22.5%
Algebra		39	72%	16	40%	15	37.5%
Geometry		4	7.5%	8	20%	6	15%
Measurement		3	5.5%	3	7.5%	7	17.5%
Data Analysis and Probability		3	5.5%	4	10%	1	2.5%
Other topics	Set Theory	0	0%	3	7.5%	1	2.5%
	Graph theory	0	0%	2	5%	0	0%
	Logic	0	0%	1	2.5%	1	2.5%
Total		54	100%	40	100%	40	100%

In the SAT, the data indicates a heavy concentration on Algebra, 72% of the questions. In other countries, while Algebra remains the most dominant category, it represents a smaller portion of the overall exam, accounting for 40% and 37% of the questions in Iran and Turkiye, respectively (Table 1). Interestingly, the second most prominent category in Iran's UEEs was Geometry (20% of the questions), in Turkiye Numbers and Operations (22.5% of the questions), and in the U.S. Numbers and Operations (10% of the questions). A few questions in Turkey and Irans' UEEs pertained to topics not covered in the NCTM strand (Table 1)

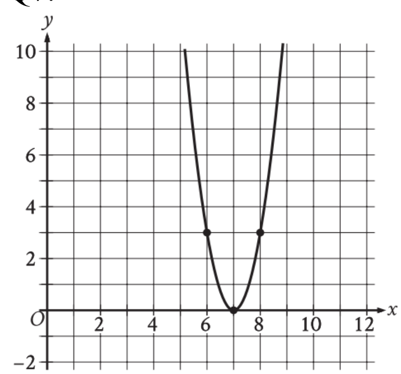
Table 2: ***CDL of the Questions for Algebra Tasks in UEEs***

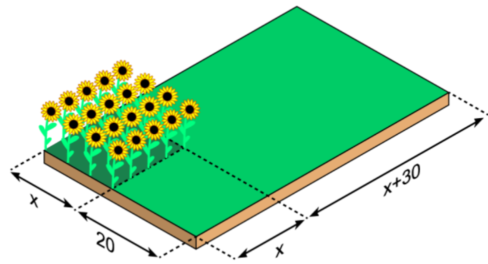
CDL Algebra Tasks		US		Iran		Turkiye	
		#	%	#	%	#	%
Low Level	Memorization	1	2.50%	0	0%	0	0%
	PWOC	23	59%	5	31%	10	67%
High Level	PWC	15	38.50%	10	63%	5	33%
	Doing Mathematics	0	0%	1	6%	0	0%
	Total	39	100%	16	100%	15	100%

We selected Algebra as our primary topic of evaluation of its CDL since it was the largest category. As Table 2 shows, in Iran's UEEs, the majority of the algebra questions were at a higher CDL (69%), while in Turkiye and the U.S., most questions were at a low level (67% in Turkiye and 61.5% in the U.S.). Furthermore, the majority of algebra questions in all countries fell within the procedural level, with or without connections (94% of the questions in Iran, 100% in Turkiye, and 97.5% in the U.S.

Notably, no questions in Iran and Turkiye, and only 2.5% of the questions in the SAT were at the Memorization level. Also, only 6% of the questions in Iran's UEEs were classified at the doing mathematics level, whereas in Turkiye and the U.S., this percentage was 0% (Table 2). Furthermore, Algebra questions were contextually framed in different ways. For instance, the SAT exam emphasized real-world scenarios within a specific subject area more than the UEEs in Iran and Turkiye. Conversely, the UEEs in Iran and Turkiye integrated multiple subject areas within a single question (see examples in Table 3).

Table 3: ***Comparison of Different Approaches to One Subject among Different Countries***

Country/Test	Content Area	CDL	Question
USA/Sample SAT Test 1 Module 1	Quadratic function	PWOC	Q4. The function g is defined by $g(x) = x^2 + 9$. For which value of x is $g(x) = 25$?
USA/Sample SAT Test 1 Module 2	Quadratic function	PWC	Q7.  The x-intercept of the graph shown is (x,0). What is the value of x?
Iran/2023 UEE	Quadratic function	PWOC	Q6. What is the sum of the root for the following equation? $\frac{1}{x^2} + \frac{1}{(1-x)^2} = \frac{160}{9}$
Iran/2023 UEE	Quadratic function	PWC	Q14. If the function is f is strictly decreasing and its domain is a set of negative values, and if $f(m^2 - m - 5) < f(-3 + 2m - m^2)$ how many integer values can m take?
Turkiye/2023 UEE	Quadratic function	PWOC	Q15. If x and y are real numbers, $x^2 + 8xy = 60$ $y^2 - 3xy = -15$

			What is the product of x.y?
Turkiye/2023 UEE	Quadratic function	PWC	Q20. Uncle Ahmet had a rectangular field with side lengths of x+20 and 2x+30. As shown in the figure in the square-shaped part of his field, he grew sunflowers with a side length of x meters.  If the area of the remaining part of the field is 1400 square meters, how many meters is the perimeter of the entire field?

Discussion and Implications

In summary, the UEE questions in Iran, Turkey, and the U.S. predominantly focused on procedural levels, both with and without connections. Additionally, many UEEs in Iran and Turkey covered topics not typically included in U.S. high school curricula. However, the SAT featured more questions related to real-world applications. This trend might mirror the nature of the mathematics curriculum in these three countries (Chang & Silalahi, 2017) which is the future studies undertaken by the authors. This way, we hope to learn more about how UEE exams align with the school curriculum and also the intricate relationship between curricular materials and expectations from students as they transition into college-level mathematics.

There were no queries at the memorization level in both countries, which provides a promising snapshot of how the focus of the examinations was not on memorizing mathematical concepts, but instead required students to apply a procedure at least. Nonetheless, it is crucial to emphasize the inclusion of doing mathematics level questions in the UEEs, which was not emphasized in any country. This emphasis aligns with the goal of preparing students to become problem solvers and preparing them to navigate situations where they may encounter unfamiliar mathematics problem-solving processes.

This study reported the content and CDLs of UEEs in three countries. The outcomes of similar studies could offer a valuable overview of mathematics education across various countries, aiding our understanding of potential factors contributing to differences in students' achievements.

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SYNTHESIZING BIAS AND FAIRNESS EVIDENCE FOR THE PSM-CAT

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Assessment is a part of teaching and learning, and recent efforts seek to improve assessment practices within K-12 education (Harris et al., 2023). There has also been a substantive shift from paper-and-pencil assessments to technology-delivered assessments (Thompson, 2017). Fairness and bias considerations are an important aspect of the assessment validation process (AERA et al., 2014; Herman & Cook, 2022). Exploring fairness and bias issues related to technology-delivered mathematics tests is necessary. This study's purpose is to synthesize bias and fairness validity evidence related to a technology-delivered, computer-adaptive mathematics problem-solving test called the PSM-CAT. The PSM-CAT is composed of standards-aligned, mathematics word problems. It is designed for students in grades 6-8 (age 11-14) and is intended to be formative in nature (see Bostic et al., 2024 for more information).

Potential respondents as well as possible test administrators and users can provide useful data regarding issues of test bias and fairness (Lane, 2014). Our team conducted 1-1 and small-group interviews with students and adults from a broad sample over two years. We purposefully and representatively interviewed (i) students and (ii) teachers, administrators, or curriculum specialists, and (iii) STEM Education faculty and professionals representing different (a) geographic regions, (b) school communities, (c) individuals with and without learning disabilities, (d) multilinguals and native English speakers, (e) genders, and (f) students representing BIPOC and White students. Interviews were audio recorded and transcribed. Data were qualitatively analyzed using a two-cycle approach to generate themes, with checks and balances throughout the analysis to promote trustworthiness (Miles et al., 2014). We report our findings from student-data (potential respondents) and adult-data (potential test users) separately.

One theme emerged from students' data: Items' contexts, language, and content were *broadly accessible to peers*; nearly all students did not perceive bias within items. We will present quotations in our poster and summarize the data as showing that 125 of 128 students (98%) reported items as fair, content-appropriate, and free from bias. A second theme came from adults: Readability, contexts, and standards-alignment was *fair and appropriate for diverse learners*. An implication from this two-year study is opening our test development scholarship on bias and fairness for scrutiny, which may be a call for other test developers to publicly share their findings. Bias and fairness are a cornerstone for a validity argument and should be explored to promote better, shorter tests and concomitantly, more time for teaching and learning (Bostic, 2023).

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COMBINED READING: A QUALITATIVE METHODOLOGY FOR RECIPROCAL ANALYSIS OF CURRICULAR DOCUMENTS

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This article introduces Combined Reading, a novel qualitative methodology for reciprocal analysis of curricular documents. We advocate that Combined Reading has the potential to highlight and unveil aspects of mathematics teaching and learning that may not be perceived in the ordinary and naturalized use of curricular orientation documents. Drawing on our previous and ongoing investigations, we underscore the richness of the process and the learning that emerges from it. We propose the Combined Reading methodology to encourage discussion about the necessary knowledge for teachers in teaching and learning mathematics.

Keywords: Research Methods, Mathematical Knowledge for Teaching, Curriculum, Elementary School Education.

Why the Combined Reading of Curricular Documents?

Significant shifts in curricular guidelines for teaching mathematics in elementary education have been observed worldwide (Li & Lappan, 2014; Shimizu & Vithal, 2018; Thompson et al., 2018). Curricular documents directly affect and shape the organization and structure of the mathematics taught in elementary school education. They highlight conceptions about mathematics and its teaching. Teachers interact with and use curriculum guidelines in their practice (Remillard, 2005; Sherin & Drake, 2009). This process allows teachers to grow their knowledge for teaching (Ball et al., 2008). Typically, a single curriculum document will guide teachers. The question that motivates us is to comprehend to what extent a combined analysis of curricular documents, one in relation to the other, offers a way of contributing to and expanding the discussion around teachers' mathematics knowledge for teaching.

> What do teachers need to know and be able to do in order to teach effectively? Or, what does effective teaching require in terms of content understanding? (…) These are centrally important questions that could be investigated in numerous ways—[for example] by examining the curriculum and standards for which teachers are responsible (…). (Ball et al., 2008, p.394)

Combined Reading aims not to compare the teaching of a certain subject in curricular documents from two or more educational systems in search of best practices (e.g. Son et al., 2017; Villalobos Torres & Trejo Sánchez, 2015). The intention is to identify similarities and differences between elements of the documents to understand the reasons that ground them and the implications of their implementation in the classroom, learning from this process. Following Remillard (2005), we believe "there is still much to learn about whether [the] use of unfamiliar curriculum materials might be viewed as a form of teacher development" (p.239). We acknowledge that social, cultural, historical, and economic aspects play different roles in the curriculum and its teaching (e.g., Bessot & Comiti, 2006; Bickmore et al., 2017). We also acknowledge that different countries have different educational particularities and demands (e.g. Acar & Serçe, 2021; Cerqueria & Silva, 2020; Pires, 2013; Wang & McDougall, 2019). However, in this study, we do not consider these aspects. We understand that, regardless, every curriculum has something to learn from and contribute to the discussion. Hence, this paper presents Combined Reading as a methodology for document reciprocal analysis that seeks to promote discussion about teachers' necessary mathematics knowledge for teaching based on the relational investigation of different curricular guidelines.

We believe that considerations emerging from Combined Reading are of the same nature as those carried out by a teacher when deciding, based on curricular guidance documents, how to teach, what students should learn, or what pedagogical resources they should use. The premise of Combined Reading is the recognition of teachers' leading role in the curriculum implementation process (Valverde et al., 2002). It is the teacher's work that ultimately "brings life" to the curriculum in the classroom. Therefore, the Combined Reading of curriculum documents encourages insightful reflections that can support translating the intended curriculum into the implemented curriculum (United Nations Educational, Scientific and Cultural Organization [UNESCO], 2016). Founded on an interactive process that promotes thinking through comparison, Combined Reading allows us to investigate alternating perspectives from different curricular documents. This integrated and comprehensive analysis reveals general aspects of mathematics teaching that go beyond the particularities of a specific curriculum.

What is Combined Reading?

As we designed, Combined Reading (Corrêa & Rangel, in press) is a methodology based on the reciprocal analysis of two or more curricular documents (Figure 1). The process foresees different sequential emphases with their own characteristics. Initially, the curricular documents under analysis must be read in parallel, that is, concomitantly, to provide a panoramic view of the documents. From this overview, correlated elements are identified in the documents, that is, elements with similar functions that allow for a correspondence to be established. In the absence of correlated elements, Combined Reading becomes unfeasible since correlations cannot be determined for document analysis. Correlated elements can have different natures, and their identification is tied to the research question. They can, for example, be structural, such as learning outcomes to be achieved, or they can have a foundational nature, such as principles, values, beliefs, and concepts that underpin the curriculum.

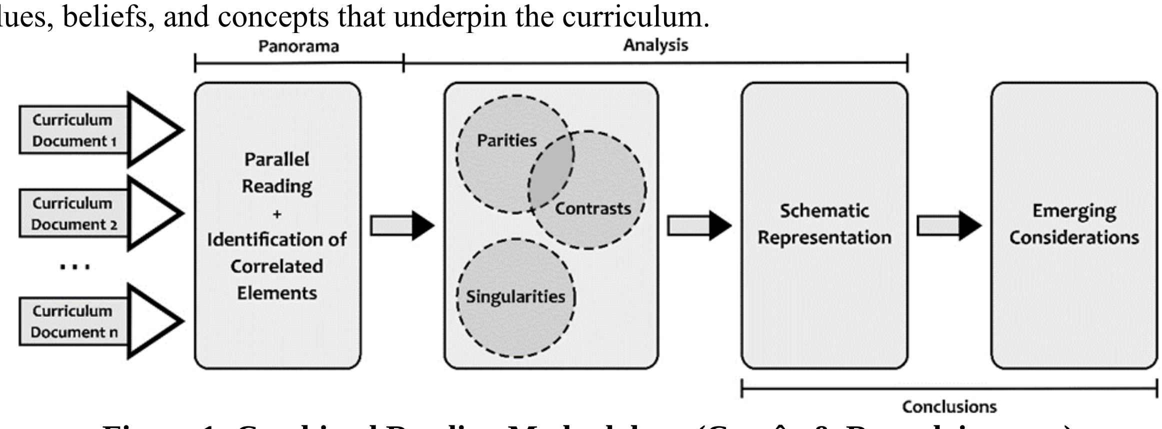

Figure 1: Combined Reading Methodology (Corrêa & Rangel, in press)

Once correlated elements are identified and a correlation can be established between the documents, the analysis begins, seeking parities, contrasts, and singularities in the documents. Parities are identified when correlated elements present high similarity, equivalence, or convergence. This is the case, for example, of identifying that, in the analysis of learning outcomes, two or more curricula propose teaching fractions starting with unit fractions. Contrasts are identified when correlated elements present significant differences, divergences, or contradictions. For example, when the teaching of integers is covered in grade 6 in one curriculum and not mentioned in another one at the same grade level. Finally, singularities are identified when a document presents particularities not included in other documents under analysis. For example, a unique curriculum that proposes Financial Mathematics as one of its strands. Following the identification and categorization of parities, contrasts, and singularities, the subsequent emphasis involves the development of schematic representations that organize and portray the established relationships. These representations are visual and multidimensional, varying according to the objectives and nature of the investigation. This emphasis also supports the analysis, as it highlights the characteristics of one curricular document in relation to another one. Conclusions and considerations emerge from the schematic representations, revealing observations, conjectures, and learning relevant to mathematics teaching.

Example of Combined Reading

We have applied Combined Reading in investigations aimed at discussing the teaching of mathematics in elementary education based on different themes: the teaching of numbers (Corrêa & Rangel, 2021a), teaching approaches to fractions (Corrêa & Rangel, 2021b), the teaching of probability and statistics (Rangel et al., 2024), and curricular terms and expressions (in progress). To illustrate the application of the Combined Reading framework and highlight its potential for shedding light on mathematics knowledge for teaching (Ball et al., 2008), we describe the study focused on Statistics and Probability (Rangel et al., 2024), given that this topic has been a recommendation and a concern in elementary education (Zieffler et al., 2018).

In Rangel et al. (2024), Combined Reading supported an investigation into the teaching of probability and statistics in elementary education. The analysis associated the probability and statistics strand from the Brazilian National Common Curricular Framework (BNCC) (Brasil, 2018) with the Guidelines for Assessment and Instruction in Statistics Education (GAISE) (Franklin et al., 2005; Bargagliotti et al., 2020). GAISE, a United States reference, is not precisely a curricular document; however, it proposes a two-dimensional model to observe the development of statistical literacy (Gal, 2021) that relates the four steps of the Process for Solving a Statistical Research Problem (PRPIE) – formulation of an investigative statistical question, data collection, data analysis, and interpretation of results – with skills that distinguish three levels of statistical literacy – Level A, beginner, Level B, intermediate, and Level C, advanced. The levels that mark the GAISE model are not intended to be directly related to the stages of school education; still, they can coherently guide such an organization.

The authors identified as correlated elements the BNCC skills (Brasil, 2018) for probability and statistics in the first six years of elementary school and the GAISE Level A skills (Bargagliotti et al., 2020). Thus, the study focuses on the initial stage of learning statistics. The qualitative analysis investigated to what extent the BNCC's probability and statistics skills can potentially develop statistical literacy following the GAISE model. The analysis was based on identifying three categories, which reflected the intensity of the relationship between the skills compared: (i) parity – indicating a strong relationship between what was intended by the BNCC and what was indicated in GAISE. In this case, when having a BNCC skill as a learning objective, it is quite possible to achieve the corresponding skill of the GAISE model; (ii) conditioned parity – indicating that the relationship between what is intended by the BNCC and what is indicated in GAISE is not immediate, and is possibly conditioned by external factors, such as a didactic-pedagogical action that requires intentional teacher intervention. Therefore, when having a BNCC skill as a learning objective, it is not natural to also achieve the corresponding skill from the GAISE model; (iii) contrast – indicating that there is no relationship between what the BNCC intends and what is proposed in GAISE. Hence, the skills seem to have no relationship; when one of them is sought, it is unlikely that the other will be achieved. The schematic representation of this study portrays a table in which colours represent the different categories: green for parity, yellow for conditioned parity, and red for contrast (Figure 2).

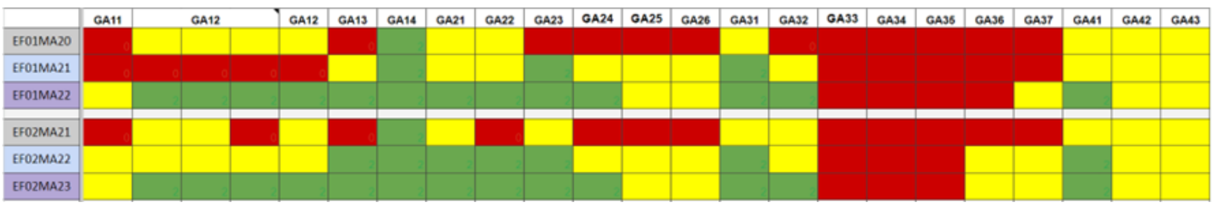

Figure 2: Extract of the Schematic Representation – Study on Probability and Statistics (Rangel et al., 2024)

Emerging considerations from this investigation include the understanding that the BNCC has the potential to develop statistical literacy in the early years of elementary school through the proposition of problems of statistical nature in contexts of interest to students. However, the study highlights that the order in which BNCC skills are listed can compromise the approach to problems, for example, by leaving statistical investigations until last. Furthermore, it was found that the analyzed BNCC skills give little or no emphasis to summary measures, which usually emerge in the data analysis stage of investigations or problems of a statistical nature. Finally, it is noteworthy that the study points out that the BNCC skills regard probability as a separate discipline of statistics; it does not establish, as recommended, a clear connection with statistical investigation. These research considerations contribute to teachers' knowledge for teaching and offer opportunities for improvement in probability and statistics teaching practices.

Final Considerations

Developing and implementing the Combined Reading methodology led to fruitful learning (Corrêa & Rangel, in press). We learned, for example, that the Combined Reading between curriculum guidelines can expand our understanding and perspective on mathematics teaching, addressing curricular requirements and teaching possibilities and opportunities at the elementary level. We learned that Combined Reading has a unique role in observing demands specific to mathematics teaching that are not evident in the individual reading of a curriculum, pointing out convergences, advances, and refinements in curricular documents. We learned that Combined Reading can contribute to converting intended curricula into implemented curricula. We understand that the emerging learnings offered by Combined Reading promote discussion and reflection, contributing to teachers' mathematics knowledge for teaching (Ball et al., 2008).

As an exploratory methodology, Combined Reading also has its limitations. Perhaps the main one is linked to a feature that grounds and supports it: the ease of access to different curricula widely available in digital media. The digital dimension of curricular documents gives them life and dynamism. It is not uncommon for adjustments and changes to be detected in curricular documents amid a Combined Reading. We understand that this limitation does not compromise the value of the reflections emerging from the various studies. Our experience shows that it can even promote new pertinent and enriching reflections. We present a new methodological proposal that we believe may be promising. Publicization, discussion, and peer evaluation are essential for its improvement. In the spirit of collaboration and recognizing the nature and relevance of this conference, we present this methodology. We have learned a lot from it.

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"I HAD THIS ROUTINE IN MY HEAD": THE INFLUENCE OF INSTRUCTIONAL ORIENTATIONS AND CURRICULAR CONCEPTIONS

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This study investigates how a mathematics teacher's curricular conceptions and instructional orientations relate to their planned lessons as they interact with curriculum materials that reflect different design principles. An analysis of five interviews with six high school teachers from the Northeast revealed that all six teachers planned lessons with consistent orientations regardless of the alignment of the curriculum materials. However, the teachers' interactions with the materials differed based on their instructional orientations and curricular conceptions.

Keywords: Curriculum, Problem-Based Learning, High School Education

Despite significant investment in improving mathematical learning by designing and testing new curriculum materials, little has changed in mathematics instruction at the high school level. Even with research-based curriculum materials and professional learning, the goal of engaging students in mathematical thinking is often unrealized (e.g., McCaffrey et al., 2001). One reason could be that when research-based curriculum materials are not aligned with the teacher's assumptions about how mathematics can and should be taught (a teacher's *instructional orientation*, Lloyd & Behm, 2005), the materials are largely abandoned (e.g., left on the shelf). However, it could also be that even when teachers plan lessons with curriculum materials that are not aligned with their instructional orientation, the resulting lessons remain oriented toward their existing assumptions for how mathematics lessons should play out (e.g., "I do, we do, you do").

This study investigates whether and how a teacher's instructional orientation and their assumptions and perspectives about mathematics curriculum materials, what we refer to as their *curricular conceptions*, may influence the ways the teacher interacts with curriculum materials and the lessons they plan. We wonder how a teacher's curricular conceptions might influence their interpretations of curriculum materials during the planning process, and whether these interpretations might explain how teachers use curriculum materials in different ways than they are intended. For example, a teacher who is oriented towards direct instruction and who views textbooks as a mechanism to deliver information might be inclined to interpret all problems in textbooks as opportunities to practice known procedures, even when they plan with a textbook that is primarily designed for students to learn through problem solving. In such a case, they might even adjust tasks so that they fit their instructional vision (e.g., shifting a task designed for students to become a demonstration by the teacher). On the other hand, a teacher who is more oriented towards fostering student exploration in their classes and who views curriculum materials as tools to promote student thinking might interpret exercises in curriculum materials centered on explicit statements of facts and demonstrations of procedures for their potential to inspire student thinking, and thus design a lesson to exploit this potential.

Specifically, we explore the research question: *How are a teacher's curricular conceptions and instructional orientations related to their interactions with textbook materials and their planned lessons?* Understanding whether and how these factors influence teachers' interactions with curriculum materials and the lessons they plan can enable future research to identify ways to support curricular change.

Theoretical Framework

In this study, *curriculum materials* (or *textbooks*) refer to intentionally-designed mathematical content for the purpose of learning. Although curriculum materials come in a variety of forms, we focus on commercially-available paper-bound textbooks since those are often adopted by school districts as mechanisms to influence instruction. Typically, curriculum materials for secondary mathematics courses contain teacher support as well as portions of text designed for student use. The student portion of curriculum materials often contain a variety of elements, such as *tasks* (all prompts designed to elicit student actions, including problems, activities, and exercises), *exposition* (expository text that provides information), and *worked examples* (problems with detailed solutions). Even when textbooks claim to address the same content goals, there is wide variance in how they are designed: some engage students in thinking and reasoning (*thinking devices* or "TD"), others expect students learn through explicit statements of facts and procedures (*delivery mechanisms* or "DM") (Choppin et al., 2015), while others contain a mixture of both approaches.

Dietiker and Richman (2021) demonstrated that even when two textbook lessons appear similar in style (e.g., exploratory) and content (e.g., same topic), they can have significant differences in how the mathematical content is designed to emerge and change as the lesson unfolds, what we call the *mathematical story* (Dietiker, 2015). The sequence of parts of a lesson through which it advances what is known mathematically (what we refer to as the story's *acts*) impacts the potential for a lesson to inspire curiosity and support inquiry, and changing the sequence changes the mathematical story and its potential impact on students (Dietiker, 2016).

Teachers who base their instruction on curriculum materials interact in ways that can influence their decisions, and thus, their planned lessons. Dingman et al. (2021) identified five types of curricular reasoning teachers employ while interacting with curriculum materials: *viewing mathematics from the learner's perspective*, *mapping learning trajectories, analyzing curriculum materials, considering mathematical meanings, and revising curriculum materials.*

As teachers draw from curriculum materials, they mediate the way content unfolds in their lesson plans. Teachers hold conceptions of curriculum, that is, their perspectives of the role of textbooks and assumptions about what constitutes a mathematical lesson to their curricular work (Behm & Lloyd, 2009; Brown, 2009). For example, some teachers may see curriculum materials as a script to follow (e.g., Lloyd & Behm, 2005), while others might make modifications while using the curriculum materials such as overly scaffolding tasks that are intended to be more exploratory in nature (Lloyd, 1999). In addition, teachers bring their *instructional orientations* to their curricular decisions (Lloyd & Behm, 2005); some teachers assume that to enable learning, students must be given information, often by the teacher or textbook (*teacher- or text-centered instruction*). In contrast, other teachers assume that students can develop understanding through opportunities to explore new ideas, solve problems, and negotiate meaning with other students (*student-centered instruction*). Still, other teachers assume instruction should include a mixture of both approaches.

Methods

This is an exploratory qualitative study comparing the lessons a teacher plans with different types of instructional materials to learn whether and how their lesson plans are related to the design principles of the textbooks. To see if the patterns are related to the instructional orientations and conceptions of curriculum of the teacher, we compared patterns across multiple teachers with contrasting instructional and curricular perspectives.

Data Collection

To learn about the relationship of the design of curriculum materials on teachers' interactions with the curriculum as well as their planned lessons, we selected six mathematics teachers who, as a group, had experience planning with a mixture of curriculum materials (i.e., DM, TD). The selected teachers had between 5-29 years of experience. Inviting teachers with extensive teaching experience increased the likelihood that they had established lesson planning practices.

To explore teachers' assumptions about the shape and sequence of content within a lesson, each teacher was interviewed five times for about 1 hour on each occasion. All interviews were conducted over Zoom and recorded. Interview 1 was conducted in a semi-structured fashion (Weiss, 1994) to learn about each teacher's professional and curricular history. This interview prompted teachers to describe their "typical lesson" with questions such as, "*If I picked a random lesson to observe, what types of activities would you expect I would see?"* Interviews 2 through 5 used the Staged Lesson Planning Interview Protocol, as described by McDuffie, Choppin, Drake, and Davis (2018), which involves asking teachers to plan a lesson with a given set of curriculum materials. No lesson plan templates or expectations of what a lesson plan would include were provided. Prior to each interview, participants were provided a PDF of a lesson to read before the interview. Then, in each interview, the participant was asked to make assumptions about what content preceded and followed the lesson and to plan a lesson for a group of students of their choice (such as one of their current algebra classes). As the teachers planned, they shared their screen with the PDF of the curriculum materials so the researcher could view which parts of the materials to which the teachers were attending. The researcher also asked follow-up questions to gain more information about the participant's visions, intentions, and rationales.

To identify relationships between the curriculum materials and the lessons that teachers plan, we selected lessons from four textbooks located at different points on the DM-TD continuum (see Figure 1). **Lesson A,** which introduces new content through exposition and worked examples, with student exercises, was selected from Glencoe's *Algebra 1* (Lesson 7-1, 2003). **Lesson B**, selected from *Big Ideas Math Algebra 1: A Common Core Curriculum* (Lesson 5.4, 2015), introduces new content through a mixture of worked examples, exposition, student explorations. **Lesson C,** which develops new content through multiple scaffolded tasks, was selected from CPM's *Core Connections Algebra* (Lesson 4.2.3, 2013). Finally, **Lesson D**, which introduces new content through an open-ended exploration, was selected from *IMP Year 1* (Lesson "Get the Point," 2015). Since Lessons A and B introduce new content through worked examples and exposition, we refer to both as DM materials (even though Lesson B also contains explorations). Similarly, since Lessons C and D introduce new content through problem solving, we classify both as TD materials (even though Lesson C is more incrementally structured).

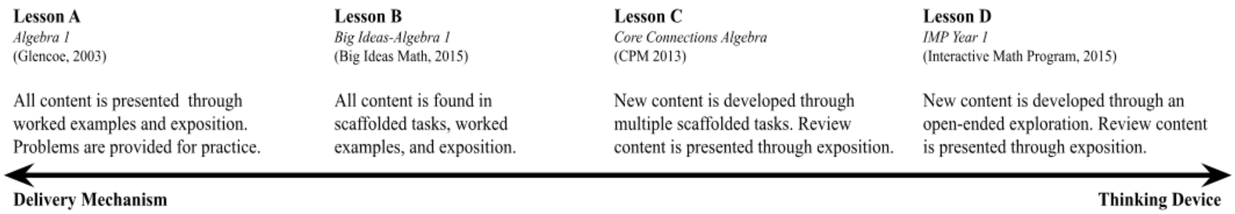

Figure 1: Categorization of Lessons A, B, C, and D along a continuum of DM to TD.

From the teacher version of each textbook (which also includes student-facing materials), we selected a lesson focused on solving systems of equations. This topic was chosen because it is a central topic in Algebra 1 and any teacher with algebra teaching experience would have likely taught a lesson in this topic, minimizing the impact of teacher content knowledge on our results. However, although all lessons addressed systems of equations, each lesson had a different focus: solving by graphing (Lesson A), solving special systems with no solution or infinite solutions (Lesson B), solving using the Elimination Method (Lesson C), and solving using the Substitution Method (Lesson D). Interviews were paced with three or more weeks between each, and the textbook lessons were assigned to each teacher in a different, randomly-generated order.

Data Analysis

We started our analysis by segmenting the transcript of each teacher's lesson plan into sequential acts. To be considered an act, the portion of the lesson must mathematically advance (e.g., moving between the different mathematical tasks or shifting to using elimination after substitution for the same system).

Furthermore, for each act, we recorded its connection to the textbook materials, how the act was intended to be carried out (e.g., were the students expected to work in groups?), and the teacher's rationale for their plan. In addition, we took notes about the teachers' views of the textbook materials, key moments of consideration and decision-making, and their goals and lesson objectives. For Interviews 2 through 5, we listened to each recording at least twice and made additions and modifications to our spreadsheet. While conducting this analysis, we used the information from Interview 1 to see if the participants made significant changes to the typical lessons they usually have in their classrooms. Moreover, we used the Interview 1 data to resolve ambiguities we encountered while looking at the data from Interviews 2 through 5. For instance, when one of the participants told us that students worked on a particular problem in partners and didn't give sufficient logistics about the role of the teacher during the partner work, we returned to their Interview 1 to learn what partner work means to them in their teaching.

To reveal the patterns in the planned lessons, we coded each act based on two dimensions: *who was expected to generate the math ideas* (i.e., teacher/text or student) and *whether the content in focus was new or review of previously-learned content* (see Figure 2). After color-coding the acts, we compared the overall patterns for all participants to group teachers based on whether their lessons contained more student-centered instruction, more teacher-focused, or a mixture of approaches (i.e., some of both). We then compared each group of teachers for the patterns of lesson designs and their relationship with the design of the textbook (e.g., usually having a teacher demonstration before independent practice with TD materials) and any patterns of interactions with the curriculum materials (e.g., looking for opportunities for student exploration or solving problems and tasks as they read DM materials).

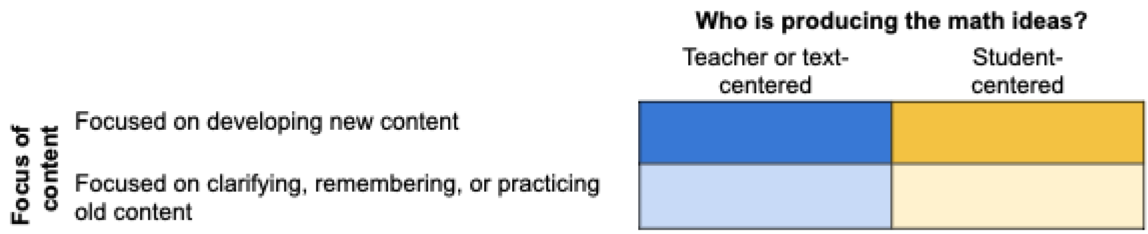

Figure 2: Coding framework to compare the planned lessons.

Findings

Overall, all teachers planned lessons with consistent orientations regardless of the alignment of curriculum materials. Despite this similarity, those teachers oriented to using some exploration and problem solving interacted with DM and TD curriculum materials differently than those teachers who were more inclined to lecture. To explain these findings in more detail, we share patterns in the lesson plans and highlight relationships between the lesson plans and curriculum materials. We also describe how the teachers interacted with the curriculum materials to make sense of the content and to plan their lessons.

Emerging Patterns of Lesson Designs and their Relationship to the Curriculum Materials

Through analyzing patterns, we categorized teachers along a continuum from *teacher-centered* to *student-centered* (see Figure 3). Teachers 1 and 2 intended new content to emerge through lecture and teacher-led discussion (dark blue). When they intended students to engage mathematically, it was to practice known procedures (light yellow). In contrast, Teachers 5 and 6 designed lessons so that new content would primarily emerge through student-centered activity such as student small group problem solving on unfamiliar problems or student-led presentations (gold), along with some teacher summary of content (light blue) and student practice (light yellow). Finally, Teachers 3 and 4 planned new content to emerge through a mix of teacher-centered instruction and student-centered learning (dark blue, gold), along with student practice of existing strategies (light yellow).

What follows is an analysis for each subgroup of teachers.

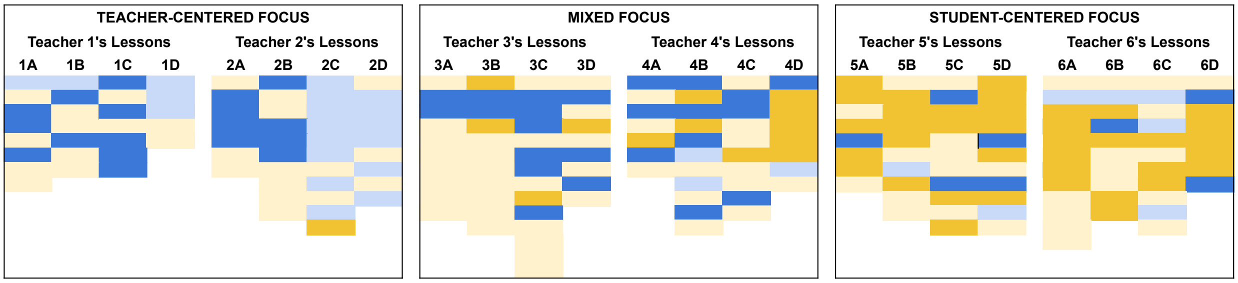

Figure 3: Planned lessons by Teachers 1, 2, 3, 4, 5, and 6 with textbook lessons A, B, C, and D. Acts are color coded as shown in Figure 2.

Teacher-centered teachers. In general, although there were differences, there were strong similarities between the lessons planned by Teachers 1 and 2. For both, portions of the lesson that were teacher-led (including both new and review content) were often followed by some student-led engagement on review (i.e., practice), and when new content emerged, it was almost always introduced through teacher-led instruction. However, although both teachers interpreted the DM materials as having new content, this only occurred for one of the four TD lessons planned by these teachers (namely, Lesson 1C). When Teachers 1 and 2 planned for students to engage in the tasks in each of the TD materials, the tasks were used as practice of prior knowledge. In Lesson 1C, the teacher introduced new content through teacher-led instruction, reframing the tasks in the textbook as practice. For example, when Teacher 1 encountered a task designed to have the Elimination Method emerge through student sense-making in Lesson C, she decided to teach the concept directly first: “Once I introduce the Elimination Method, that’s when I would move on to that hands-on stuff.” For Lessons 1D and 2D, however, these teachers instead interpreted the lessons as entirely reviewing prior content (i.e., no dark blue or gold).

Mixed-focus teachers. In contrast with Teachers 1 and 2, Teachers 3 and 4 planned a majority of their lessons so that new content was introduced by *both* the teacher and through student-led solving of tasks. For Teacher 4, each lesson had at least one portion focused on having new ideas emerge through student-centered problem solving while for Teacher 3, this was the case for the three of the four lessons she planned. For Teacher 3, the only lesson that didn’t result in new content being generated through student sense-making was Lesson A, which was on the far-left end of the spectrum for DM materials.

Intriguingly, these mixed-instruction teachers had different reactions to the TD materials than Teachers 1 and 2. Instead of interpreting all content as review, both Teachers 3 and 4 sought ways to keep the opportunities for content to emerge through problem solving. However, there were differences in their approach. Teacher 4 planned lessons that intended new content to emerge through student problem solving without much teacher intervention, particularly for Lesson D. In contrast, when planning with Lessons C and D, Teacher 3 planned to lead most of the efforts of solving the problems so that the new content would emerge. She expressed concern that students may not be able to generate the desired conclusions without teacher intervention. For example, while working with Lesson C, Teacher 3 commented “maybe I'm selling them short... There's not a lot of precedent for multiplying both sides by a –1" when expressing her disbelief about students generating a strategy to solve an unfamiliar system of equations.

Student-centered teachers. Differing from the lessons of teacher-centered teachers and mixed-focus teachers, Teachers 5's and 6's lessons mostly had student-centered elements. Both teachers, whose pedagogical approach is centered around student sense-making and problem solving, afforded students with multiple opportunities to generate new content without explicit teacher guidance. Regardless of the textbook, both participants emphasized their preference for student-directed activities, such as Teacher 6 remarking, "But that's for [the students] to figure out, not for me to tell them what to do" while planning with Lesson D and Teacher 5 stating, "I feel instead of sharing this table with the students, … it's easier for students to remember or understand if they derived or did something with their team members to discover" while planning with Lesson A. All the lessons they planned included such narratives. In fact, instances of new content being introduced were rare or non-existent for both teachers' lessons, regardless of the textbook materials involved.

Interactions with Textbook Materials

Although all participants read the curriculum materials (both student facing and teacher guidance) to make sense of the resource and to identify potential instructional opportunities, when we compared how the groups of teachers interacted with the textbook materials, interesting themes emerged that distinguished the ways the *teacher-centered* participants interacted with curriculum materials from those in the *mixed* and *student-centered* groups.

Considering student perspectives. When interacting with the curriculum materials, both the *mixed* and *student-centered* teachers considered student perspectives at different points of the planning process. For example, these teachers often based curricular decisions, such as whether or not to select or adjust a task, based on how they predicted students would view particular tasks. For example, when planning a lesson with Lesson A, Teacher 4 avoided tasks that he felt would be boring for students and sought problems that he felt would be relevant to students. In addition, some teachers predicted the ways students would approach solving problems, identified challenges they might confront, and described potential ways students would interpret problems. For example, when planning with Lesson B, Teacher 6 reasoned that students might experience difficulty understanding what the variables represent in the word problems about systems of equations and thus decided to rephrase the problems in a way that asks students to define and explain their variables.

Notably, both *mixed and student-centered* teachers mathematically solved the tasks when confronted with unfamiliar approaches within the textbooks. For instance, Teacher 3 analyzed the systems in Lesson D to determine if they encourage students to set y values to equal each other: "You're going to get a lot of fractions in there, which is ok, but it's unnecessary. It's tedious." From this, she decided the task needed adjustment and weighed whether to change the system's numbers to allow for students to work with integers or to prompt them to substitute for y. In another case, when trying to figure out how students would use algebra tiles to solve a system with the Elimination Method in Lesson C, Teacher 4 solved the problem using this approach. As a result, he reconsidered his approach to this topic: rather than teaching students to "just add the other equation," as he had in the past, he recognized how the textbook approach centered the additive property of equality. Still later, he reflected on the fact that although he never teaches with algebra tiles, this lesson made him reconsider this stance: "I realize [using algebra tiles] takes some time, but I can see how that would be beneficial."

In contrast, *teacher-centered* participants did not solve tasks or otherwise analyze the mathematical affordances of tasks as they planned their lessons. These teachers tended to make rapid assumptions about the complexity or characteristics of tasks. For example, when planning with Lesson D, Teacher 2 selected systems of equations in the textbook indiscriminately. She explained, "maybe, the [task (a)] would be you had to solve it through Elimination, (b) would be Substitution, (c) might be Elimination, …." By the speed at which she considered them, it was evident that the particular systems did not matter to her plan; this teacher did not pause to consider the ramifications of the selection, such as whether the systems contained any special cases (e.g., those without solutions) or other unexpected difficulties (such as those for which the Substitution Method might be particularly challenging).

Reading for content development across the lesson. Another pattern that distinguished the groups of teachers involved how the teachers interpreted the textbooks. All teachers read the DM-oriented curriculum materials as potential components of lessons to draw from. However, with regard to TD materials, the *mixed* and *student-centered* teachers at times read the curriculum materials for how the mathematical ideas emerge and change across a lesson. For example, at the start of planning with Lesson C, Teacher 4 was not sure about its approach, but reading for the nuanced way the content developed enabled this teacher to ultimately make sense of and appreciate it. He noticed that the first task, a problem prompting students to solve a system using the Substitution Method, was designed to set up to motivate the Elimination Method. Then, as he read through remaining tasks, he acknowledged the sequence ("and then they do this…") and described how the sequence might influence the ways in which the students might approach the problems. Another teacher (Teacher 6) interpreted Lesson D as a sequence when she compared it to curriculum materials she had used in the past, noting, "This is very different from a regular textbook. It doesn't have like a couple of examples and a bunch of stuff…Like this is, to me, a planned-out lesson."

In contrast, Teachers 1 and 2 did not analyze how the content developed over the course of any of the textbook lessons. Even when planning with Lessons C and D, in which content was designed to emerge as students solved a sequence of tasks, Teachers 1 and 2 interpreted subparts (a), (b), etc. as separate independent tasks (i.e., practice problems) and did not note their interrelationships. For example, when Teacher 1 encountered sequential subparts that contain clear links, such as a sequence of subparts of a task in Lesson C that sequentially develops the Elimination Method, she construed them as independent exercises. In another case, Teacher 2 suggested that a sequence of systems of equations were interchangeable without regard to any mathematical features that subtly distinguished them (e.g., no solution, or infinite solutions).

Discussion

The results of this study help explain why reforming curriculum materials does not necessarily influence the intended ways mathematics is taught. That is, our data suggests that high school mathematics teachers often plan consistent forms of lessons no matter the design principles of materials with which they plan. Each group of teachers (teacher-centered, mixed, and student-centered) appeared to approach the task of planning with a preconceived image of how their planned lesson would flow. As Teacher 4 said, when explaining why he added a new lesson component to what was offered in the textbook materials, “I had a routine in my head.” Since all the teachers’ planned lessons with consistent orientations (e.g., teacher-centered or student centered), we suspect that a teacher’s preconceived lesson image is closely related to their instructional orientations. Furthermore, since the interactions with the DM and TD materials differed for some teachers (the mixed and student-centered) but not others (the teacher-centered), we propose that a teacher’s curricular conceptions may develop to support the teacher’s instructional orientation (e.g., learning to recognize tasks in TD materials as learning opportunities). Thus, changing instruction may require developing new curricular conceptions, which in turn can support shifts in instruction.

However, this was a small exploratory study. Further research is needed to learn whether this phenomenon is found more generally, or whether it is limited to a particular grade band, geographic region, or other characteristic. If this pattern holds more broadly, however, then this would suggest that shifting instruction at a local level requires teacher education that aims to support mathematics teachers in developing expanded visions on what is possible in mathematics classrooms. Our data suggests three possible ways to expand this curricular vision. First, we note that both *teacher-centered* participants did not interpret problems in the IMP lesson as opportunities for new ideas to be learned. Given this, mathematics teachers need support in their curricular noticing (Dietiker et al., 2018), particularly in seeing familiar parts of textbook materials in new ways and recognizing instructional opportunities in unfamiliar materials that weren’t previously visible to them. Secondly, of Dingman et al.’s (2021) aspects of curricular interactions, two stand out in our study as ways to support instructional change: (a) our findings suggest that solving the mathematical tasks during the planning process can enable teachers to shift their ways of interacting with textbook materials by *adopting a student’s perspective*, and (b) since those teachers who designed lessons with student-centered approaches were the teachers who interpreted the curriculum materials for how the content emerged and changed across a lesson (i.e., *mapping learning trajectories*), we wonder if reading the curriculum materials for this quality supports instruction that is exploratory and student-centered. Future studies could learn whether having teachers learn to interpret curriculum materials as a mathematical story enables them to recognize and incorporate opportunities for student reasoning in their lesson plans.

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EXAMINING VALIDATION PRACTICES FOR MEASURES OF MATHEMATICS TEACHER AFFECT AND BEHAVIOR

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This research is part of a larger project cataloging measures used in mathematics education research, and evidence of validity related to those instruments. This study examines the degree to which validity considerations for measures of mathematics teacher affect and/or behavior are consistent with the guidelines set forth in the Standards for Educational and Psychological Testing. Findings suggest validity evaluations largely rely on (a) evidence of test content and (b) evidence of internal structure. However, explicit claims of how such evidence supports the interpretation and use of test scores are typically absent from validity considerations.

Keywords: Assessment; Teacher Beliefs; Affect, Emotion, Beliefs, and Attitudes

Testing has significant implications for educational practices, research, and policy. Testing programs used in K-12 education influence both the content and pedagogy of teaching and learning (e.g., Kane, 2013), and inferences drawn from quantitative data are often used to inform decisions on education policy (e.g., Hill & Shih, 2009). Validity is a fundamental concern of testing. Validity is "the degree to which evidence and theory support the interpretations of test scores for proposed uses of tests" (AERA et al., 2014, p. 11), and validation refers to the process through which validity is evaluated. The term *test* is broadly defined such that rating scales, inventories, and observation protocols are all examples of instruments categorized as a test (AERA et al., 2014). For this study, we focus on tests of (a) affective characteristics of teachers (e.g., attitudes), and (b) constructs related to teacher behaviors (e.g., instructional practice).

A vision for the future of mathematics education research includes promoting access to, and encouraging the use of, robust measures (i.e., tests) to engage in research that implicates mathematics teaching, policy, and teacher education (Bostic, 2023; Zelkowski et al., 2024). Currently, conceptualizing validity and validation is ambiguous in the mathematics education community (e.g., Bostic et al., 2021), despite guidelines for evaluating validity put forth in the *Standards for Educational and Psychological Testing* (*Standards*; AERA et al., 1999, 2014). The purpose of this study was to examine the degree to which validity considerations related to tests measuring mathematics teacher affect and behaviors are consistent with the *Standards*, and thereby gain an understanding of the aspects of instrument development for which we, as a field, need to continue to grow. Two research questions guided this study: (1) What source(s) of validity evidence were most commonly and least commonly used in evaluating validity related to measures of teacher behavior and affect? (2) To what degree have researchers reported evidence of validity in relation to claims underlying the intended interpretation and use of test scores?

Theoretical Framework

The *Standards* represents consensus in conceptualizing validity among three leading U.S. social-science research organizations related to the field of measurement (Sireci, 2016). Validity is an attribute of a test-score interpretation and use, validity is not an attribute of a test (AERA et al., 1999, 2014; Messick, 1995; Shepard, 2016). For this study, the terms *test, measure,* and *instrument* are used synonymously. Validity is a unitary concept. Describing distinct types of validity, such as *criterion validity*, is an outdated practice (AERA et al., 1999, 2014; Folger et al., 2023). However, distinct types of validity maintain merit as they are reflected in the *Standards*' five sources of validity evidence: test content, response processes, internal structure, relations to other variables, and consequences of testing (AERA et al., 2014).

Current best practices for validation include (a) clearly describing the intended test-score interpretation and use, (b) identifying claims underlying the test-score interpretation and use, and (c) gathering evidence in relation to those claims (AERA et al., 2014; Kane, 2013; Sireci & Benitez, 2023). Test-score interpretation reflects the meaning of test scores. Use statements may refer to actions or decisions arising from test-score interpretation, or use statements may denote a given purpose for testing (Folger et al., 2023). Consider, for example, a test designed to measure educators' self-efficacy for teaching mathematics. Validation would begin with defining "self-efficacy for teaching mathematics" and describing how to draw meaning from test scores (AERA et al., 2014). Next, several claims or assumptions about the test could be raised, such as assuming the items represent the construct (Folger et al., 2023). Evidence is then gathered to ideally warrant these claims (Kane, 2013). Data from subject matter experts, for example, can support the claim that items align to the construct (Sireci & Benitez, 2023). Subsequently, this evidence would be categorized as validity evidence based on test content (AERA et al., 2014).

Not all five sources of validity evidence are needed to establish some degree of validity, nor does every possible claim underlying the test-score interpretation and use need to be examined to establish some degree of validity (AERA et al., 2014; Sireci & Benitez, 2023). It is generally accepted, however, that validation involves collecting evidence from multiple sources (e.g., Bostic, 2023; Kane, 2013; Sireci & Benitez, 2023). In particular, complex test-score interpretations produce more complex claims, which require greater validity evidence (Folger et al., 2023). Put simply, as the complexity of test-score interpretation and use increases, the need for more evidence from multiple sources also increases (AERA et al., 2014). The purpose of this study was to examine the consistency of validation practices related to measures of mathematics teacher behavior and affect with the guidelines set forth in the *Standards*.

Method

We are part of a larger team cataloging tests used in mathematics education. Our team built upon Thunder and Berry's (2016) steps for a qualitative review (see Table 1) to systematically capture measures of teachers' affect and behavior used in research published from 2000-2020.

Data Collection and Analysis

Approximately 2300 articles representing 24 mathematics education journals were reviewed to some degree (i.e., abstract, methods, full article) in an initial search for quantitative measures (i.e., tests) related to teacher behavior and affect. The qualitative review process yielded 255 published unique tests. Six researchers were assigned specific tests for which they (a) searched Google Scholar for studies potentially using the test or reporting on the test, (b) selected studies which potentially contained evidence of validity and/or reliability related to the test, and (c) documented validity and reliability evidence for the respective test. The search for evidence included reviewing peer-reviewed journal articles, conference proceedings, dissertations, and white papers. The goal of this stage was to capture any validity evidence for a respective test. Each test was assigned to two coders, and coders met periodically to discuss identified validity and reliability evidence and reconcile any differences.

Our research team used descriptive statistics to evaluate the research questions for this study. We used frequency counts to determine the source(s) of validity evidence that are most/least commonly reported (i.e., RQ1). To examine the degree to which researchers present evidence of validity in relation to claims underlying the intended interpretation and use of test scores (i.e., RQ2), data were disaggregated by test to assess the presence of interpretation and/or use statements, claims, and the sources of validity evidence represented.

Table 1: Comparison of Our Process with that of Thunder and Berry (2016)

Step	Thunder & Berry (2016)	Our Process
1	Determine a research question	Same
2	Determine search terms	Same + verify multiple Boolean string searches
3	Search databases	Identify journals, test, and export to spreadsheet
4	Select relevant studies	a. Title & abstract review b. Verify interrater agreement, check for drift coding c. Continuous secondary coding of 20% of articles d. Monthly meetings to reconcile coding
5	Assess quality of selected studies	a. Review methods; if needed, review entire manuscript b. Verify interrater agreement, check for drift coding c. When identifying tests, secondary coding of 20% of articles d. When identifying validity evidence, all tests are double-coded e. Monthly meetings to reconcile coding
6	Synthesize findings	Organize in spreadsheet based on coding
7	Report findings	Presentations and publications

Results

The review process produced 239 resources containing validity or reliability evidence. For this study, we focus solely on synthesizing the evidence of validity. At times, multiple resources (e.g., two or more journal articles) presented validity evidence related to a specific test. In total, we identified 480 different instances of validity evidence for 158 of the 255 identified tests.

RQ1: Most and Least Commonly Reported Sources of Validity Evidence

Table 2 presents frequencies for each respective source of validity evidence located during the review process. No particular source was present for at least half (50%) of the tests. Test-content evidence, most commonly reported, was found in relation to 96 tests. Evidence based on consequences of testing was rarely found, such evidence was identified for 8 tests.

Table 2: Summary of Validity Evidence

	Test Content	Internal Structure	Relations to Other Variables	Response Processes	Consequences of Testing
Total Instances	189	126	119	37	9
Instruments for which the source is represented (% of 255)	96 (37.6%)	85 (33.3%)	67 (26.3%)	27 (10.6%)	8 (3.1%)

RQ2: Evidence Related to Claims Underlying the Test-Score Interpretation and Use

Of the 255 total tests we found, 23 tests (9.0%) included test-score interpretation statements, 49 tests (19.2%) described use statements, and 38 tests (14.9%) provided explicit claims or assumptions. Moreover, we found 0 sources of validity evidence for 97 tests (see Table 3). Less than two sources of evidence were found for roughly 70% of the 255 tests, with a median of 1 source of evidence per instrument. At least three sources of evidence were located for approximately 14% of the tests.

Table 3: Sources of Validity Evidence Represented Per Instrument

Zero Sources	One Source	Two Sources	Three Sources	Four Sources	Five Sources
97 instruments (38.0%)	83 instruments (32.5%)	39 instruments (15.3%)	24 instruments (9.4%)	10 instruments (3.9%)	2 instruments (0.8%)

Discussion

Findings from this study suggest validity considerations for measures of mathematics teacher affect and behavior are largely not consistent with guidelines set forth in the *Standards* (AERA et al., 1999, 2014). Particularly, mathematics education research focused on teacher affect and behavior rarely details how test scores are intended to be interpreted and used. There is a preponderance of quantitative research focused on mathematics teacher affect and behavior that (a) omits any consideration of validity, or (b) relies on minimal evidence without explicitly describing the way(s) in which the evidence contributes to the validity of how test scores are intended to be interpreted and used. Some claims can be implicitly recognized—feedback from experts often supports the claim that items align to the construct (e.g., Folger et al., 2023). However, we found that the claims underlying test-score interpretation and use were seldom made explicit in scholarship. Claims help identify what validity evidence sources are important for validation, thereby communicating the importance of collected evidence (AERA et al., 2014).

Researchers may consider modeling their evaluations of validity from robust validity considerations found in current scholarship. For example, Walkowiak and colleagues (2014) used the *Standards* (AERA et al, 1999, 2014) as a framework for a validation study of the *Mathematics Scan*, a measure of mathematics teaching practices; their study presented validity evidence based on (a) test content, (b) response processes, and (c) relations to other variables. Additionally, Bjerke and Eriksen (2016) presented a validity argument supporting the use of the *Self-Efficacy in Tutoring Children in Primary Mathematics* instrument. Examples of validity evidence found in that study include findings from cognitive interviews with test-takers as evidence of response processes, and results of Rasch analysis as evidence of internal structure.

Validation has been described as "equity-forward scholarship" (Bostic, 2023, p. 218). Test-score interpretation(s), and how data are subsequently used, can implicate opportunities for others (e.g., Shepard, 2016). The intended and unintended consequences of test-score interpretation and use are validity concerns (AERA et al., 2014), yet evidence based on consequences of testing is seldom presented in relation to measures of mathematics teacher affect and behavior. We echo Cronbach (1988) regarding the importance of consequential considerations, "tests that impinge on the rights and life chances of individuals are inherently disputable" (p. 6). Consequences of testing warrant meaningful consideration (e.g., Shepard, 2016; Sireci, 2016). This study shines a light on the need for validation practices to improve in the mathematics education community, while also providing examples and clarifying aspects of validity such that researchers may engage in more robust and meaningful evaluations of validity.

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MEASURING CONCEPTUAL UNDERSTANDING: PATTERNS WITHIN THE SPECIAL EDUCATION LITERATURE

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Conceptual understanding is an essential component of mathematical proficiency for all students, including students with disabilities. Yet, in the special education literature, conceptual understanding is a term often used but rarely defined. To gain insight into how conceptual understanding has been measured (and thus defined), we conducted a systematic review of the special education literature over the past thirty years. Using Skemp's (1978) continuum of understanding, we found most tools and activities reflected instrumental understanding (i.e., disconnected bits of discrete information), employed scoring techniques that emphasized accuracy, or did not provide sufficient information to be coded. The findings suggest aspects of the special education literature base that warrant further exploration and highlight the need for increased opportunities for students with disabilities to develop relational understanding.

Keywords: measurement, special education, students with disabilities

Introduction and Background

So that students with and without disabilities have equal opportunities in mathematics, professionals from mathematics education and special education fields must share disciplinary knowledge (Tan et al., 2019; Yeh et al., 2020). Sharing knowledge among interdisciplinary group members involves interrelating diverse views to gain clarity about concepts and terms (Akkerman et al., 2007). Because mathematical proficiency for students with disabilities has historically been limited to procedural fluency (Foegen & Dougherty, 2017; Lewis & Fisher, 2016) as a measurement of understanding, we believe that gaining clarity about concepts and broadening perspectives about students' demonstration of conceptual understanding (e.g., the comprehension of mathematical concepts, operations, and their relation to one another; Kilpatrick et al., 2001), has the potential to contribute to better opportunities for students with disabilities to develop a firm grasp of mathematical concepts. Exploring definitions of conceptual understanding and the various methods educational researchers have used to measure it is vital because how conceptual understanding has been measured and defined indicates the instructional tools, approaches, and orientations toward supporting the students being assessed (Treffinger, 2009). This study aimed to understand how mathematical conceptual understanding has been measured (and therefore defined) in special education literature over the past three decades.

Study Design

To understand the prevailing definition of conceptual understanding in our field, the authors of this study, researchers in special education, conducted a systematic literature review of existing literature in the top 25 special education journals published between 1990 and 2020.

Screening and Coding

First, the authors formed a research team and screened articles by reading through the titles and abstracts of available volumes based on the method section to see if students' conceptual understanding was at least one stated outcome measure. A researcher external to the authors' research team conducted screening reliability on 20% of all issues to increase the comprehensiveness of the initial phase of the search. Second, the research team conducted a second-round screening by reading the method section of each article and determining if conceptual understanding was a dependent variable in the study. The inclusion criterion included if the article's author(s) described at least one outcome measure as assessing students' understanding or reasoning. Given the range of conceptualizations around what counts as conceptual understanding, we relied on authors to explicitly identify understanding or reasoning as an intended outcome of the intervention. Third, the team recorded the name of the tool(s) authors used to measure understanding, the tool's citation (if any), the reported mathematical activity students were asked to do to demonstrate understanding, and how researchers reported scoring the activity. Fourth, we developed codes for these activities based on Lesh and Doerr's (2003) representations for demonstrating understanding, again relying only on the information provided in each article; thus, some activities had multiple codes (e.g., reflection and debriefing or model-eliciting).

Categorizing Codes

To categorize activity scoring, we coded each tool as using an (a) *accuracy scoring framework*, (b) *understanding scoring framework*, or (c) *other scoring framework*. We then used the collected data to create a set of subcodes based on how researchers described scoring mathematical activities. Scoring types included: (a) *not specified*, (b) *scale* (not specified), (c) *computational accuracy*, (d) *process/procedural accurac*y, (e) *classifying mathematical behavior*, (f) *explanation*, (g) *reasoning/understanding accuracy*, and (h) *justification*. Finally, once all activity and scoring codes were assigned, we analyzed the data in two ways: 1) we generated descriptive data about the tools the authors said they used to measure understanding and the scoring methods authors reported using and assigned codes to broader categories; and 2) given that measuring understanding typically necessitates the convergence of evidence (Wiggins & McTighe, 2005), we examined whether a tool included one or more mathematical activities, employing Skemp's (1978) spectrum of understanding (e.g., instrumental or relational), to determine whether the tool was conceptual in nature. If a tool included more than one activity, we applied the *relational* code; if it included one activity, we applied the *instrumental* code; and if the tool was not described with enough detail to know the number of activities, we applied the *not specified* code. Finally, we looked across the codes and applied the majority code to the whole tool. For example, if a tool was coded as relational-relational-instrumental, the final code would be relational.

Findings

We found that slightly more than forty percent of the tools reported using what we characterized as an *Accuracy Scoring Framework*, using at least one scoring procedure that either assessed an accurate answer or an accurate procedure (n = 29; 42.65%). Slightly less than forty percent (n = 26; 38.24%) of the articles did *Not Specify* scoring criteria. A small proportion of tools (n = 13; 19.12%) used what we characterized as one of two frameworks: an

Understanding Framework or *Other Scoring Framework*. The Understanding Framework reported scoring understanding based on students' explanations, justifications, reasoning, or understanding accuracy. The Other Scoring Framework reported scoring using an unspecified scale or classified students' mathematical behaviors. That is, across the three coding categories—number of activities used, whether or not any of the activities were conceptual in nature, and the nature of the scoring framework used—tools in these studies attended to conceptual understanding in ways that we characterized as *Instrumental*. We coded fewer tools (n = 19; 27.94%) as *Relational* and found that there were nearly as many tools coded as *Not Specified* (n = 18; 26.47%). Of the 68 tools used across these 52 articles, only three tools (4.41%) could not be neatly sorted into one of the three analytic categories, given that the tool they utilized earned one instrumental code, one relational code, and one not specified code. See Table 1 for the distribution of tools across the spectrum of understanding.

Discussion

Assessing conceptual understanding is notoriously difficult, especially when accuracy is conflated with understanding. Deep understanding is likely not easily evidenced by a one-time assessment but necessitates a convergence of evidence (Jin & Wong, 2021; van de Walle et al., 2013; Wiggins & McTighe, 2005). Though researchers have developed instruments to measure understanding, these may be confounded by the realities of the measures' administration intensity, required time commitment, and reliability (Jones et al., 2019). To address this challenge and the opportunity gap implicated, this systematic literature review aimed to understand how mathematical conceptual understanding has been measured (and thus defined) in the special education literature over the past three decades. We found that most tools and activities used in the included studies measured instrumental understanding, used scoring techniques that emphasized accuracy, or did not provide sufficient information to be coded. The *kind* of understanding the researchers in special education literature measured was notably oriented to instrumental rather than relational (and more conceptual) understanding.

Our results suggest that conventional notions in special education research about measuring students with disabilities' mathematical capabilities are limited and merit expansion. To support students' mathematical proficiency through exposure to rigorous instruction, we must consider the tools we use to measure their understanding. We contend that the field of special education and students with disabilities would meaningfully benefit from a purposeful expansion of thought around the term conceptual understanding and, therefore, an expansion of our tools. In addition to advances in measurement, we encourage education researchers to expand the type of understanding included in mathematics intervention efforts. This expansion supports disabled students in meaningfully different kinds of mathematical activity and, ultimately, different kinds of mathematical learning. Such a definitional expansion broadens who can be counted as having an understanding of mathematics, including students who perhaps struggle with accuracy or computation but who, in fact, do understand the underlying mathematical relationships at play.

The field of special education is positioned to take up new approaches, tools, and orientations in mathematics toward supporting disabled students in developing robust mathematical proficiency. Taking up an interdisciplinary and intersectional perspective, we look to our mathematics education colleagues, who have already charted some of these pathways. We believe that collaborating with mathematics education researchers has great potential and that paving the way to a more equitable method of measuring understanding may require combining the distinct knowledge of each field to better support and enhance the mathematics learning of students with disabilities.

Table 1: Distribution of Tools Across the Spectrum of Understanding

Instrumental Understanding (*n* = 28)	**Relational Understanding (*n* = 19)**	**Not Specified (*n* = 18)**
Bottge, Toland et al. (2014)	Bottge, Ma et al. (2014)	Aunio et al. (2005) [2]
Burns et al. (2015)	Casa et al. (2017)	Bryant et al. (2011)
Butler et al. (2003)	Foreman-Murray & Fuchs (2019) [2]	Cary et al. (2017) [2]
Fuchs et al. (2004)	Ives (2007)	Clarke et al. (2019)
Gavin et al. (2013)	Jitendra et al. (1999)	Crawford et al. (2019)
Gavin et al. (2018)	Jordan & Hanich (2000)	Dahlstrom-Hakki et al. (2019) [2]
Jitendra et al. (2016)	Liu & Xin (2017) [2]	Doabler et al. (2016)
Jitendra et al. (2018) [1]	Mabbott & Bisanz (2008) [1]	Doabler et al. (2019)
Mabbott & Bisanz (2008) [3]	Milton et al. (2019)	Jitendra et al. (2017)
Mononen et al. (2014)	Niemi (1996) [2]	Jitendra et al. (2018) [1]
Montague et al. (1993)	Pagliaro & Kritzer (2013) [1]	Kritzer (2009)
Morano et al. (2020)	Van Herwegen et al. (2018)	Opitz et al. (2017)
Nunes et al. (2009) [1]	Wang et al. (2019) [1]	Pagliaro & Kritzer (2013) [1]
Parmar & Signer (2005)	Woodward & Baxter (1997)	Proctor (2012)
Powell et al. (2015)	Woodward et al. (2001)	Schumacher et al. (2018) [1]
Rodrigues et al. (2019) [2]	Woodward & Brown (2006)	
Schumacher et al. (2018) [1]		
Sharp & Dennis (2017)		
Van Hoof et al. (2017)		
Van Luit et al. (2011)		
Wang et al. (2019) [2]		
Woodward et al. (1999)		
Xin et al. (2008)		
Xin et al. (2020)		

All Three Codes (*n* = 3)
Helwig et al. (2002)
Nunes et al. (2009) [1]
Parmar & Cawley (1994)

Note. Articles that contained more than one tool are identified by superscripts. Superscripts indicate how many tools, from that article, are in the respective category. For example, the Mabbot & Bisanz (2008) article used four total tools. The article appears in the *instrumental* column and has the superscript "3" to indicate their study used three tools, all of which we coded as *instrumental*; this article also appears in the *relational* column and has the superscript "1" because they also used one tool that we coded as *relational*.

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PORTRAITURE AS A LENS TO PAINT WITH WORDS: MATHEMATICS TEACHERS’ CURRICULAR REASONING DURING PLANNING

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Portraiture is an emerging methodology in mathematics education that could bring new insights and perspectives. This methodology involves capturing a distinctive story that reflects universal themes; the final product is a co-collaboration between the researcher and the subjects meant to inspire the reader. Listening actively *for* a unique story makes portraiture different from other qualitative research forms. For example, in ethnography, the researcher listens passively *to* a story, focusing on identifying patterns to generalize experiences across a culture, while portraiture concentrates on individuals or small group’s experiences to uncover unique, but universal stories that resonate with the readers (Lawrence-Lightfoot & Davis, 1997).

Teachers’ curricular decisions from the mathematical care perspective have not yet been investigated. I define mathematical care as teachers’ holistic support for mathematics learners: mathematical development, emotional well-being, and sense of belonging in the mathematical community. Goodness is a particular feature of portraiture (Lawrence-Lightfoot & Davis, 1997). Exploring the goodness in portraiture is more than finding positive experiences. It is likewise a dialog between the researcher and the subjects “that allows for the expression of vulnerability, weakness, prejudice and anxiety” (p. 141). In exploring goodness as mathematical care, I am “searching for what is good and healthy” (p. 9) in teachers’ curricular reasoning.

This research employs portraiture to showcase how teachers plan their curriculum by capturing into a narrative canvas the uniqueness and universality (Lawrence-Lightfoot & Davis, 1997) of their thought processes during planning. The research question is: *How does teachers' mathematical care influence them in their curricular decisions?* I depict expressions of teachers' mathematical care, that results from their planning. Emphasizing internal, personal, and historical contexts provides a frame for teachers' actions, and this frame is a valuable resource for interpreting the subjects' "thoughts, emotions, and behaviors" (Lawrence-Lightfoot & Davis, 1997, p. 59). Portraiture methodology allows the researcher to accurately portray teachers' reasoning using detailed descriptions by collecting data from interviews, field notes, visuals, artifacts, assessments, reflections, and memos.

In portraiture, the researcher is more visible than in any other research because the relationship between researcher and participants plays a significant role in completing a portrayal research design in terms of “empirical, ethical and humanistic dimensions” (Lawrence-Lightfoot & Davis, 1997, p. 138). I am portraying teachers’ curricular reasoning framing the context, voice, relationships, emergent themes, and the aesthetic whole (Lawrence-Lightfoot & Davis, 1997). I start broadly, placing the school culture within the portrayal. I include the researcher’s context and then gradually zoom into the central aspect of the portrait: teachers’ thought process informed by mathematical care. Emergent themes in the context of curricular decisions result from how teachers negotiate tensions between a fixed curricula and students ‘mathematical needs. The final written portrait balances the voice, the context, and the emergent themes structured into a complete piece that reflects teachers’ multidimensionality of mathematical care.

The potential impact of this study on mathematics education is the focus on teachers as carers (Noddings, 2013, 2017) for the students as learners and individuals and the dynamic of their decisions to respond to students' mathematical needs.

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EXPLORING STUDENTS' APPROACHES TOWARD DIFFERENT REPRESENTATIONS OF FRACTION CONCEPTS

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Keywords: Mathematical representations, Number concepts and operations, Assessment

Despite the growing body of research on mathematics education, concerns persist about students' performance on assessments (Chand et al., 2021). Studies have shown that students 'performance varies on mathematical modeling problems and open-ended questions compared to conventional tests and multiple-choice questions (Kartal et al., 2016; Danili & Reid, 2005; O'Neil & Brown, 1998). Based on Scheiner's (2016) theoretical framework of multiple knowing and learning processes and Lesh et al.'s (1987) representation framework of mathematical problems, this study explores the impact of different mathematical representations on students' performance and reasoning the study examines the impact on students' understanding when a single mathematical concept is presented through various mathematical representations for a particular assessment item.

Research Question: When given a variety of representations, how do 5th-grade students apply fraction concepts?

Scheiner's (2016) theoretical framework of multiple knowing and learning processes in mathematics highlights the cognitive processes involved in concept construction. It further indicates conceptual understanding develops when a learner understands various ideas associated with the different representations of a concept. Whereas Lesh et al.'s (1987) representation framework of mathematical problems. Identifies five methods of representing mathematical problems: real scripts, manipulative models, spoken language, static pictures, and written symbols; these representations can be used in conjunction with each other to solve mathematical problems. Overall, these frameworks emphasize the crucial role of representation in developing a deeper understanding of mathematical concepts. To create the assessment items for the study, five methods of representation by Lesh et al., (1987) were integrated with the representation of concepts in Scheiner's (2016) framework. Through clinical interviews with four fifth-grade students, I explored how participants responded to different mathematical representations for parallel assessment items. The clinical interviews allowed me to learn about student thinking regarding fractions (Hunting, 1997) specifically if the children had difficulty conceptualizing the concept or its representation.

The data analysis established the following findings: Children have difficulty connecting concepts between representations when presented in an algorithm or equation form but do not struggle when presented with static pictures or real-world problems. The study suggests that while children struggle to associate logarithms with manipulatives, they are more comfortable working with static pictures and manipulatives to solve real-world scenarios.

The framework presented in this study is a valuable tool for educators to delve into diverse mathematical concepts. Additionally, it provides insights to standardized test developers, to offer students multiple avenues for expressing their understanding.

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UNCOVERING THE NARRATIVE: A CURRICULAR COMPARISON OF TWO TEXTBOOKS' LESSONS CONNECTING GRAPHS AND SCENARIOS

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Keywords: Curriculum, Algebra and Algebraic Thinking, High School Education

Textbook lessons present information with the aim of guiding students toward a common mathematical goal. How textbooks guide students on that journey, however, can be remarkably different as the ways that mathematics knowledge is presented in textbooks reflect specific choices in how material is revealed. Thus, reading mathematical texts as narratives (Dietiker, 2013) presents a valuable opportunity to better understand how variation in important contextual components among different textbook lessons on the same mathematical topic affects both the logic and aesthetic of mathematics lessons (Dietiker, 2015). The purpose of this project is to compare two Algebra I lessons from two distinct textbooks—the Interactive Mathematics Program (IMP) (Fendel et al., 2015) and Carnegie Learning (Finocchi et al., 2022a)—to explore the idea of quantities and relationships by connecting graphs and their related scenarios.

The IMP and Carnegie Learning textbooks were selected for study due to some concrete differences in their approaches to curricular materials. IMP units are designed to have "a specific mathematical focus…most units are structured around a central problem and bring in topics as needed to solve that problem, rather than narrowly restricting the mathematical content" (Fendel et al., 1997, p. vii). The Carnegie Learning curriculum is rooted in cognitive science research based on the Adaptive Control of Thought-Rational (ACT-R) theory of human knowledge and cognitive performance, which treats "complex problem solving as the coordination, strengthening–and eventual proceduralizing–of a large number of relatively simple knowledge components" (Finocchi et al., 2022b, p. FM-39). Although problem solving and conceptual understanding are emphasized in both textbooks, they approach these topics in rather distinct ways. While the Carnegie Learning curriculum is firmly rooted in the cognitive science tradition—which has a long history of interconnectedness with mathematics education (Schoenfeld, 1987) and problem solving (Silver, 1987)—IMP's design principles are connected to students' personal validation, active involvement, and need for intrinsically-motivating reasons to solve problems (Alper et al., 1997). According to our research, these differences in orientation may have an impact on how the mathematical stories of these quantity and relationship lessons develop through variations in the dynamics between mathematical characters, plot, and sequencing (Dietiker, 2015; Dietiker, 2016).

This investigation examines the relationship between mathematical characters, setting, and plot with a focus on how the orientation of textbooks may be influencing these relationships. It does this by using the theoretical framework proposed by Dietiker (2015) for interpreting mathematical curriculum as story. Initial findings highlight significant variations in the way the textbooks address and prioritize axis scaling on various graphs, as well as how much emphasis they place on students' identification of independent and dependent variables. As we look toward the future of mathematics curriculum, this line of research represents an opportunity to emphasize how different curricular approaches can impact a lesson's narrative structure and therefore the ways that mathematics educators conceptualize story in mathematics curriculum.

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MAPPING ERRORS IN PROBLEM-SOLVING TO MATHEMATICAL PRACTICES

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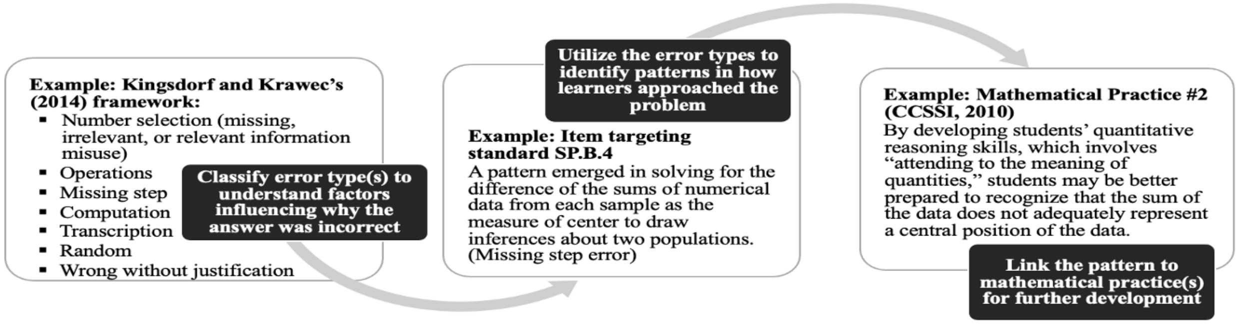
Keywords: Assessment; Research methods

K-12 assessment practices have been identified as needing advancement (Datnow & Hubbard, 2015; Harris et al., 2023). Strategies for using assessment data to inform instruction is a key practice to advance (Wilson, 2018). Careful analysis of students' errors on mathematical assessments in particular has been shown to provide insight into their conceptual understanding (Rakes & Ronau, 2019). In turn, information from incorrect responses is maximized to support teaching and learning (Lannin et al., 2006). Mathematical problem-solving skills are a needed area of study given the continued focus internationally (Mullis et al., 2016) and in the Common Core State Standards - content and practice (CCSSI, 2010). The aim of this poster is to share a process for analyzing incorrect responses to gain insight into targeted areas for development related to mathematical practices. Incorrect written responses (N=2,115) on the seventh grade Problem-Solving Measure CAT prototype items were analyzed collaboratively in coder pairs (≥90% inter-coder agreement). The PSM has substantial reliability and validity evidence (Bostic et al., 2015, 2017, 2024). Fifty-nine items were sampled to represent the content standards. A cyclical approach involving expert (n=5) and practitioner (n=16) feedback through surveying and interviewing informed iterative refinements to the process. Thematic analysis (Braun & Clarke, 2006) of practitioner data revealed the usefulness of describing common errors. Expert data revealed a refinement needed was to re-frame error descriptions to reflect how students approached a problem to adopt a more asset-based lens. This resulted in a three-step process (see Figure). This process contributes to the call for advancements in assessment practices (Harris et al., 2023), namely offering a process for using results to identify targeted areas for learning.

Figure 1: Three-Step Process



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CREATING STUDENT AUTONOMY AND BUILDING FUNCTIONAL REASONING THROUGH AN ENTREPRENEURIAL DESIGN CHALLENGE

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At the core of engaging students in mathematics is having them use their mathematical knowledge to solve personally relevant and authentic problems. We have created entrepreneurial-based design challenges (Authors, 2019) that engage students in rich mathematics. In this paper, we report on 30 students participating in one such challenge. Students were tasked with designing a business that helps users change unwanted behaviors or develop new healthy habits through tracking and visualizing their progress. We present results to show how the challenge provided opportunities for student autonomy in their solutions and in the mathematics they utilized.

Keywords: Curriculum; Algebra & Algebraic Thinking; Affect, Emotion, Beliefs, & Attitudes

Today's K-12 students will be asked to tackle unprecedented environmental, economic, and social challenges (OECD, 2018). They will need to be able to work collaboratively and across disciplines to invent innovative, actionable, and empathetic solutions to messy problems that lack a clear solution path. "Education needs to aim to do more than prepare young people for the world of work; it needs to equip students with the skills they need to become active, responsible and engaged citizens" (OECD, 2018, p. 5). Novel curricular approaches are needed that allow students the autonomy to identify meaningful problems and innovative solution paths, establish connections between in-school learning and students' out-of-school experiences, and engage students in learning and applying targeted disciplinary content knowledge.

Researchers in STEM education have recently begun exploring strategies for leveraging entrepreneurship to connect students' out-of-school knowledge, experiences, and interests to in-school STEM learning (e.g., Authors, 2019; Moore et al., 2017). Given its popular appeal (e.g., the TV show *SharkTank*) and its emphasis on building actionable solutions to real-world problems, entrepreneurship has the potential to support engagement and learning in a STEM setting. The BLINDED project (Authors, 2019) is a novel curricular framework that situates mathematics learning within entrepreneurial pitch competitions. In this paper, we report on a group of 30 students' solutions to the BLINDED TASK, one of 18 BLINDED challenges.

Literature Review

Strategies that support student engagement include: creating a supportive, collaborative, and cognitively demanding learning environment (Lamborn et al., 1992), making content and learning activities authentic (Blumenfeld, et al., 2006), and empowering students to exercise autonomy and authority in relation to the curricular content (Helme & Clarke, 2001; Marks, 2000). Deci and Ryan (1987) refer to autonomy as “supporting choice” (p. 1024) or “encouraging them to make their own choices” (p. 1025). Providing students opportunities for autonomy is important in a mathematics classroom, where students often perceive the subject as disconnected from their cultures, lived experiences, and future aspirations (Boaler, 2002; Gutstein, 2003). Authentic tasks can establish a purpose for learning (Blumenfeld, et al., 2006) and can help students connect the content they are learning in school to situations they find important, relevant, and worth pursuing (Reschley & Christenson, 2012). Authentic activities can also empower students to incorporate their unique out-of-school identities in mathematics (Attard, 2012; Bobis et al., 2011; Helme & Clarke, 2001; Marks, 2000; Yair, 2000), which further builds their autonomy, promotes self-monitoring and persistence (Helme & Clarke, 2001; Leon et al., 2015), and supports students’ “sense of control and self-worth” (Bobis et al., 2011, p. 37). Thus, a learning opportunity that leverages authentic contexts and promotes autonomy could support improvements in students' confidence and growth mindset in mathematics.

The Project Framework and Challenge

Combining features of project-based learning (Krajcik & Blumenfeld, 2006), design-based learning (Kolodner, 2002), and entrepreneurial-based learning (Yuste et al., 2014), the BLINDED project framework (Authors, 2019) was developed to leverage authentic entrepreneurial practices and open-ended design challenges to motivate the learning of specific mathematics content. Students work collaboratively to: 1) define the problem and research the context (Krajcik & Blumenfeld, 2006; Rivet & Krajcik, 2004); 2) build, test, and refine prototype solutions (Fortus et al., 2004; Razzouk & Schute, 2012); 3) demonstrate the actionability of their solutions (Lackeus, 2015; Kolodner, 2002); and 4) deliver 5-minute pitches to panels of judges (Passaro et al., 2017; Krajcik & Blumenfeld, 2006). In the BLINDED TASK challenge, students are tasked with inventing a business that helps users (individuals or companies) set and achieve goals through tracking and visualizing their progress. Students had to build, evaluate, and interpret functions that map changes in performance onto changes in a visualization. BLINDED TASK uses design criteria to connect students’ real-world solutions to math-specific school learning and establish an immediate purpose for building functions. These criteria include: identifying relevant behaviors or habits to address, inventing visualizations to monitor progress towards the goal, and building functions that translate progress in the target behavior to changes in the visualization. These criteria were created to engage students in functional reasoning through the defining, testing, and refining of generalizable relationships between two co-varying quantities (Warren et al., 2006), namely student- defined measures of the target behavior and changes in the student-invented visualization.

Methods

A mixed-methods research convergent parallel design (Creswell & Plano Clark, 2017) was used to explore the students’ experiences with the BLINDED TASK implementation on student autonomy and functional reasoning. Quantitative and qualitative data were collected concurrently throughout the project, analyzed separately, and then merged together to answer the following research questions: *(1) How does the autonomy afforded by the D&P framework manifest in students’ solutions?, (2) How do students demonstrate functional reasoning while participating in the BLINDED TASK challenge?, and (3) How does participating in the BLINDED TASK challenge affect students’ confidence and growth mindset in math?*

Sample and Procedure

The BLINDED TASK challenge was implemented across eight days in a Math 1 class with 30 7th and 8th grade students in an urban setting. The school is 100% male, 28% low-income, and 75% minority. Students participated in the challenge in teams of three or four. In launching the challenge, the teacher focused students on the United Nations Sustainable Development Goals, allowing students the autonomy to identify goals that they found relevant.

Data Collection

To allow for a complete picture of students' experiences, we collected data from a variety of sources (Cohen et al., 2011), including daily written work samples and Pitch Decks (pitch presentation slides, animations, and handwritten prototypes). Students took a pre- and post-survey with items measuring their growth mindset and confidence in math using a 6-point Likert-scale. There were three growth mindset questions from Code et al. (2016), who adapted them from a measure on general intelligence mindset (Dweck, 2008). The survey also had one item on confidence, which was adapted from self-efficacy items in Usher (2007).

Data Analysis

Qualitatively this research draws on student artifacts, specifically their daily work samples and Pitch Decks. We analyzed the documents to describe the products each team designed and the mathematics in which they engaged. We coded student products into broad categories for the type of context they selected for their business. We collected exemplar quotes and pictures from their prototypes to highlight the functional relationships in their solutions.

Quantitatively the survey data were analyzed using paired-sample *t*-tests to determine if there were significant differences in students' growth mindset or confidence after engaging in the BLINDED TASK.

Results

Autonomy in Context

During the BLINDED TASK challenge, students drew on their interests and their experiences with a number of social justice initiatives. Across the ten groups they selected the following contexts: mental health (n=4), school improvement (n=2), food waste (n=1), social media addiction (n=1), school violence (n=1), and health (n=1). The context for three student groups will be highlighted, followed by the functional reasoning for each group.

The **Against Waste** team created a solution "to reduce food waste in schools and to make people more aware about conserving and recycling food," after seeing the amount of daily trash in their school cafeteria. They used a visualization of a trash can to help schools reduce their food waste in the cafeteria. The **Discover You** team created a mental health app, because of the rising numbers they noticed in teen depression. The third team, **PoGo**, focused on school improvement, creating an app that, in their words, "...allows Teachers and Principals to track the percentage of all students in a school who are at an Economic Disadvantage (E.D.)" compared to overall performance. Their hope was that "schools will be able to see that the percentage of E.D. students at their school are struggling and make sure that students are receiving all the things they may need (such as breakfast/lunch, proper transportation, proper school materials)."

Functional Reasoning

Students utilized a variety of strategies for building functions to allow users of their product to visualize data. Below we briefly describe the products designed by the three teams and the mathematics they utilized in their solution, including their functional relationship, input variables, output variables, and rule for connecting their function to a visual representation of their solution. The **Against Waste** group created a functional relationship for food waste and trash can visualization. They used the number of bags of trash for their input and the color and appearance of a trash can via an animation for their output variable. They were able to operationalize their input variables, based on observing data from their school cafeteria. Though they did not describe an explicit rule they had all the pieces, and the context was meaningful to them. They stated, "If you are wasting too much, the visual is heaping out foods and trash, and when you reach your goal, there will be a reward animation." **Discover You** created a functional relationship for depression based on sleep, feeling of worth, and school stress. Their input variables were a measure of worth, happiness, and stress on a scale from 1-10 and number of hours of sleep. These measures come from polling their users in their app. Their output variable was a horizontal progress bar. The rule they created for depression was: [(School Stress)$\times$ -1 +Worth+Happiness+Hours of sleep]$\times$ 10/3. Their equation accounted for the input variables they identified as important factors in depression and multiplying school stress by negative one shows the students knew they had to invert the scale for stress because it is a negative factor of depression. They created a Scratch prototype for their visual, which showed a depression score as the voltage in a battery. The **PoGo t**eam built a functional relationship for the academic achievement for students with Economic Disadvantage. based on input variables from state level School Report Card data. Their output variable was a wheel and thumb and the rule they created was based on regression analysis and the Scratch coding of a visual thumbs up and down based on residual values. In their words,

> Our app uses State School Report Card for middle schools in County to gather our data onto a graph, then it transfers that information on a graph to a wheel that is color-coded based on the subject and then has a thumbs-up emoji that changes color and rotates based on how much above, at, or below the line of regression they are at.

This group used advanced functional reasoning to create their regression analysis and turn it into a convincing visual for their users.

Confidence and Growth Mindset

There were significant increases on students' self reports from pre to post in growth mindset (pre=3.20, post4.29, t=-5.17, p<0.001), and non-significant increases in confidence (pre=4.33, post4.67, t=-1.62, p=0.058). This suggests after engaging in the BLINDED TASK challenge students had a stronger disposition towards a growth mindset.

Conclusion

By providing students with an open-ended task, they had the autonomy to create their product and create their own functions. Across the ten teams, students opted to tackle specific environmental, economic, and social challenges (OECD, 2018) that were both authentic and personally meaningful. The BLINDED framework afforded students the autonomy to identify personally meaningful problems and explore unique solution paths, while also bounding the mathematics content with which they engaged. By equipping students with this autonomy, the framework created opportunities for students to see their place in the math classroom, which improved their confidence and willingness to persist when encountering difficult problems. By allowing students the autonomy to identify personally meaningful contexts, the challenge opened the space for them to actively engage in functional reasoning. Students drew on their experiences with their chosen contexts to identify, operationalize, and define relationships between authentic input variables and their corresponding output variables, namely visualizations that allow users to track progress towards a goal. Findings from this study demonstrate how providing students with the autonomy to create a solution to an authentic problem gave them confidence in their mathematical ideas and generated powerful solutions to emerging issues facing young adults.

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INTRODUCING A REPOSITORY OF QUANTITATIVE MEASURES USED IN MATHEMATICS EDUCATION

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This paper reports on the developments of a repository of quantitative assessments used in mathematics education contexts between 2000 and 2020. The repository is public and freely available, identifies validity evidence of associated measures, and has potential to inform future quantitative mathematics education scholarship., This paper discusses the types of instruments and reports the data analysis regarding the types of validity evidence found for the 1,034 instruments where the validity evidence was categorized, and located across over 1,200 sources.

Keywords: Assessment

A vision for the future of mathematics education scholarship includes researchers having access to high-quality assessments, assessment developers providing robust validity evidence for the interpretation and uses of assessment scores, and users being able to reflect on their assessment options efficiently and effectively without challenges like paywalls and institutional access restrictions. Finding and selecting appropriate quantitative assessments to use for mathematics and statistics education research can be difficult. Institutional access to journal articles describing assessments, as well as limited descriptions of them, are two challenges to finding and selecting appropriate tools (Author et al., 2021). Therefore, a goal of this paper is to introduce scholars to a repository of quantitative assessments and their associated validity evidence. This repository is designed to make it easier for scholars to ascertain if there is a suitable assessment to measure a desired construct and to browse its associated validity evidence.

Many manuscripts describe an assessment but do not provide details about the validity evidence related to the assessment's intended uses or score interpretations. That is, it is unclear the degree to which the assessment accurately and reliably measures what it intends (Author, 2017). Broadly speaking, the *Standards for Educational and Psychological Testing* (*Standards*) characterize validity as the degree to which evidence supports an intended claim (AERA et al., 2014). A goal to fostering quantitative mathematics education scholarship is building a robust knowledge base that uses assessments with strong validity evidence (Authors et al., 2022; Kane, 2013). The authors of this paper built a publicly available database of mathematics education assessments and seek to address this goal. This paper will present the recently launched database of mathematics and statistics education assessments and their associated validity evidence and present some findings about the quantitative assessments found in it. In this paper, the terms assessment and instrument are used interchangeably, whereas measure and test have more narrow implications (Author, 2019).

Literature Review

Prior to 2014, there was scant discussion of validity and quantitative assessment within mathematics education scholarship (Author, 2017). There has been a substantial increase in scholarship exploring the degree to which validity and validity evidence are discussed in mathematics and statistics education as well as scholarship focusing on how those validity arguments are communicated (e.g., Pellegrino et al., 2016; Walkowiak et al., 2019; Wilhelm & Berebitsky, 2019). Validation is a process that broadly includes (a) explicitly describing how assessment scores are intended to be interpreted and subsequently used, (b) identifying the claims, or assumptions, underlying the score interpretation and use, and (c) gathering evidence to evaluate those claims (AERA et al., 2014; Author et al., 2023; Kane, 2013). A *use statement* communicates how to appropriately use an assessment or its results. An *interpretation statement* informs others about how to effectively interpret an assessment's results. Put simply, interpretation and use statements describe what the scores mean and how to use that information (Authors et al., 2023). Claims are "implied by a proposed test interpretation" (AERA et al., 2014, p. 12). They may reflect aspects of an assessment for a given test-score interpretation and use, such as claiming that a test adequately functions as a unidimensional measure of a given construct. Claims may also reflect assertions arising from the test-score interpretation; that a student has achieved some level of mastery (e.g., AERA et al., 2014, Authors et al., 2023; Kane, 2013).

The *Standards* highlight five sources of validity evidence: test content, response process, relations to other variables, internal structure, and consequences from testing (AERA et al., 2014). For example, Rasch measurement or factor analysis can provide validity evidence based on internal structure supporting claims of an instrument's unidimensionality. While the *Standards* do not consider reliability a source of validity evidence itself, it is an important related component (AERA et al. 2014). These sources have been adopted by the three largest educational research, educational measurement, and psychological communities, which suggests it is appropriate for scholars and their associated scholarship to align with these *Standards*. Better alignment with the *Standards* has potential to position quantitative mathematics education research as equity-forward scholarship that embraces and supports diversity, equity, and inclusivity efforts (Author, 2023). With a greater attention paid to the quality of information collected on quantitative assessments in mathematics and statistics education scholarship, there is a pressing need to provide researchers with a means to explore available measures efficiently and effectively for future scholarly use.

Methods

In the February of 2020, 41 participants attended the Validity Evidence of Measures in Mathematics Education (VM^{2}Ed) conference. Participants had previously published or presented on quantitative assessments and were committed to fostering conversations around validity and assessment within mathematics and statistics education scholarship. Participants included mathematics education faculty, researcher scientists, psychometricians, assessment developers, and graduate students. The goal of the conference was to create an understanding of validity within mathematics education contexts and solicit recommendations from experts about information necessary to build a repository of quantitative mathematics education instruments. After building a solid foundation of how validity is conceptualized and operationalized using the

Standards, the committee of experts gave recommendations about a synthesis procedure to identify and categorize validity evidence, interpretation statements, and use statements of quantitative assessments. Following development of the synthesis procedure, participants were divided into six synthesis groups: (1) Elementary (K-6) Tests and Instruments; (2) Secondary (7-12) Tests and Instruments; (3) Undergraduate and Graduate Mathematics Tests and Instruments; (4) Statistics Education (K-20) Tests and Instruments; (5) Teacher Education Tests (content knowledge); and (6) Teacher Education Instruments. Regarding Teacher Education Tests and Teacher Education Instruments, tests were defined as measures of content knowledge whereas instruments included all other constructs (e.g., affect and efficacy).

After the conference, each group searched for instruments and tests that fell within their parameters. The search was standardized across all groups to include searching the top 24 peer-reviewed mathematics journals (Nivens & Otten, 2017). After identifying the assessments, group members used all available scholarship to conduct a wider search for validity evidence, claims about the assessments, as well as interpretation and use statements associated with those assessments. Finally, the claims and evidence for each assessment were coded using the five sources of validity evidence and reliability (AERA et al., 2014) as an a priori framework.

A codebook was developed, based on an analysis of data from the conversations and artifacts from the 2020 conference, to define and characterize different evidence types to serve as subcodes for each of the sources. This resulted in 89 unique evidence types (see Figure 1). Then each piece of evidence that was found received an "evidence source" code (i.e. five sources and reliability) and an "evidence type" subcode. Training was provided to participants to ensure reliability and promote trustworthiness in the coding process across synthesis teams. For example, evidence types for test content include data from experts or literature review. Examples of internal structure evidence types include exploratory factor analyses, item difficulty, and Rasch modeling. If additional types were discovered in the literature, then it was added to the codebook and shared with the other groups.

Consequences of Testing	Internal Structure	Relations to Other Variables	Reliability	Response Process	Test Content
Appropriate cut score	Bayesian Network Models	Alignment with expert opinion of test user (e.g. teacher, therapist)	Alternate form	Cognitive interviews	Alignment with frameworks/standards/theory/learning trajectory
Bias as one consequence of testing	Cluster analysis	Convergent Association or Divergent Association	EAP/PV test reliability	Error related to response patterns/CTT	Construct Definition
Cost-benefit analysis	Factor Analysis- Bifactor	Correlation analysis	Generalizability theory- D-studies	Eye tracking/physiological data	Data from experts
Documentation of unintended behavior changes based on test use	Factor Analysis- Confirmatory Factor Analysis (CFA)	Discriminant validity	Generalizability theory- G-studies	fMRI (functional Magnetic Reasoning Image)	Fairness of content
Explicit intended uses and interpretations and warn against inappropriate uses	Factor Analysis- Exploratory Factor Analysis/Exploratory Structural Equation Modeling	Discrimination power	Inter-rater reliability- Kappa	Focus groups	Field Work
Impact of assessment is similar under clinical and practical implementations	Factor Analysis- Multi-trait Multi-method matrix (MTMM)	Hierarchical Linear Modeling	Inter-rater reliability- Percent agreement	Generalizability-theory (G-theory) related evidence	Literature Review
Item functioning such as DIF - unknown subgroups had to know	Factor Analysis- Not specified	Multi-trait Multi-method matrix (MTMM)	Internal consistency or alternatives- Alpha	Log data	Participant-generated content
Motivational consequences	Factor Analysis - Parallel Analysis	Statistical Testing (e.g., t-test, regression, and chi-square)	Internal consistency or alternatives- IRT or Rasch reliability	Predicted response patterns/processes based on Learning Trajectories	Revision Process
	Factor Analysis- Principal Axis Factoring (PAF)	Structural Equation Model	Internal consistency or alternatives- Omega	Rater agreement/reliability	Standard Setting
	Factor Analysis- Principal Component Analysis (PCA)	Treatment/control study	Internal consistency or alternatives- Raykov	Rater training and calibration	
	Item difficulty	Triangulation/Crystallization with qualitative data	Item remainder correlations	Sorting tasks	
	Item Scale Correlations		Kuder-Richardson formula 20	Student written work	
	Item Response Theory (IRT)		SEM measurement	Studies of respondent's speed of task completion	
	Latent Class Analysis (LCA)		Sensitivity analysis	Think alouds	
	Latent Profile Analysis (LPA)		Split-half reliability		
	Multidimensional scaling		Test - Retest		
	Rasch modeling				
	TETRAD				

Figure 1: Validity Evidence Types

When evidence was found, it was sorted into two categories: those that had an associated claim and those that did not have an associated claim. If there was a claim, then it was noted, and the same classifications of the evidence source and evidence type was linked with the claim. An example of a claim is, “The Cronbach’s alpha, 0.87, was found to be sufficiently high for the measure to be considered internally reliable” (Attridge, 2013, p. 103), and the source was classified as reliability and the type of evidence was coded as: Internal consistency or alternatives- Alpha.

Similarly, it was noted as “yes” or “no” if an article with validity evidence contained an interpretation or use statement for an assessment. If there was an interpretation or use statement then it was noted, verbatim from the source.

Every assessment in the repository was also tagged with key features such as the population being tested, construct measured by the assessment, and type of item in the assessment. Across all groups, each assessment was put into one of four construct bins: knowledge, affect, behavior, or classroom and instruction. The validity evidence coding, claims, interpretation and use statements, and key tagging features comprised the data used to create the repository.

In the following section, broad information about the assessments collected by each group is provided. Then, a detailed analysis of the validity evidence disaggregated by evidence types and interpretation and use statements is provided. This analysis highlights the results regarding the validity evidence collected by all six groups, as well as provides an exemplar of the type of information users can find in the repository. Finally, details about the development and features of the repository are provided.

Results

Categorization and Validity Evidence

As of February 2024, 1,034 assessments were identified for inclusion in the repository. Synthesis groups found validity evidence for 764 of the 1,034 assessments (Table 1). Hence, validity evidence was located for 74% of the assessments. In total, group members reviewed 1,206 different articles, proceedings, or book chapters between Sept. 2020-Aug. 2023 that contained information about a known quantitative assessment used in mathematics or statistics education. As an example, of the 379 instruments identified by the Secondary group, there were 223 instruments (58.8%) for which validity evidence was found from 301 different papers.

Table 1: Counts of Instruments in the Repository

Synthesis Group	Total Number of Instruments	Instruments With Evidence	Number of Papers with Evidence
Elementary	109	105	159
Secondary	379	223	301
Statistics Ed. K-20	65	79	203
Teacher Ed. Instruments	255	158	239
Teacher Ed. Tests	45	44	149
Undergraduate	155	155	155
TOTAL	**1034**	**764**	**1206**

In total there were 3,524 pieces of evidence in the 1,206 papers. Of the validity evidence found across all six groups, the most frequent source of validity evidence was test content (30.5%) (Table 2). Reliability evidence was present in 917 of the 3,524 pieces of evidence (26.0%). Results show consequences of testing had the smallest frequency of evidence in the articles at 1.7%.

Table 2: Count of Each Evidence Type by Synthesis Group

	Conseq. of Testing	Internal Structure	Relations to Other Variables	Reliability	Response Process	Test Content
Elementary	14	82	74	115	29	77
Secondary	23	165	162	268	53	354
Statistics Ed. K-20	11	175	137	158	32	245
Teacher Ed. Instruments	9	126	119	225	37	189
Teacher Ed. Tests	0	62	41	44	4	59
Undergraduate	4	102	48	107	23	151
TOTAL	**61**	**712**	**581**	**917**	**178**	**1075**

Of the 1,206 papers that presented validity evidence, 307 had an interpretation statement and only 345 had a use statement. Therefore, approximately a quarter of the articles with validity evidence contained statements about how to interpret assessment data (307 of 1,206; 25.5%) and 38.6% had statements regarding how to use the assessment (345 of 1,206). Across all six groups, the Secondary group had the highest percentage of use statements (149 of 301, 49.5%) and the Undergraduate group had the highest percentage of interpretation statements (65 of 155, 41.9%). The lowest percentage of interpretation and use statements was found from the Teacher Education Instrument group (10.5% and 11.3% respectively).

Table 3: Percent of Each Evidence Type by Synthesis Group

	Interpretation	Use
Elementary	36	41
Secondary	84	149
Statistics Ed. K-20	78	57
Teacher Ed. Instruments	25	27
Teacher Ed. Tests	19	19
Undergraduate Ed.	65	52
TOTAL	**307**	**345**

Repository Development

From the synthesis group work products, a free online searchable repository was created to house the name, description, citations, interpretation and use statements, claims, and validity evidence associated with the identified assessments. To access the repository, users create a login and briefly acknowledge a user agreement. Once logged in, users can search for available assessments, peruse their associated validity evidence, and participate in training modules about validity. The website was designed to be user friendly and accessible with the help of a User Experience (UX) designer and computer programmer, and to display evidence in a way that is clearly aligned with the *Standards* (AERA et al., 2014). The repository has several important design features explained below: search features, researcher portal, and ease of adding new instruments.

Design Feature: Search Features. The database has a search feature that allows users to input text and search for instruments. In addition, users can refine a search using the key features or tags assigned to the assessments (e.g. population, construct, item type, etc.). See Figure 2 for an image of the advanced search by tagging feature. Tagging features were intentionally created to be consistent across all six synthesis groups, so researchers can easily sort through and search for instruments. The tagging features include: Synthesis Group, Population, Grade Level, Construct, Type of Instrument, Mode of Delivery, and Item Type. The main categories of assessment constructs used across all groups were: knowledge, behavior, affect, and classroom & instruction. Each group has more specific constructs tailored to particular assessments under each of these main categories. For example, sub-constructs of the knowledge category could include geometry, limits, measurement, algebra readiness, etc. The repository provides a page for researchers to peruse more specific subconstructs related to each group and more effectively search the repository for their needs.

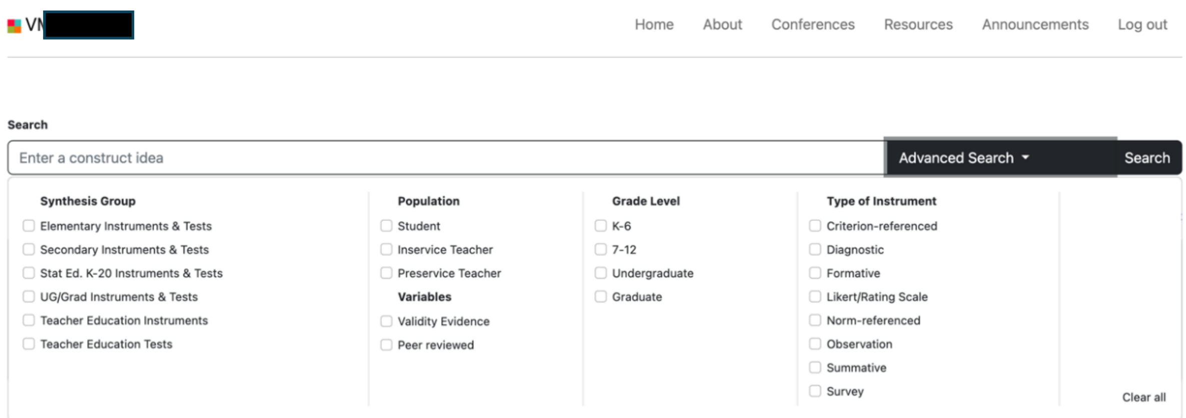

Figure 2: Search Engine and Tagging Features

After a search is submitted, a list of instruments is returned in groups of 10. Additional pages for searches when results include more than 10 instruments. The instrument list page has the names of the returned instruments and contains information related to population, grade level, construct, and a count of the number of validity evidence items found for the instrument.

If a user is interested in a particular instrument, then they can select it from the instrument results page. All the tagging features of the instrument are provided at the top of the page under the instrument name (Figure 3). Then, the articles that were found to contain validity evidence for the instrument are provided, along with a citation, DOI, summary of the validity evidence, and abstract. There is a tab on the instrument results page (shown in the red box in Figure 3) for the main instrument information (described above) and then any source of validity evidence that was found for the instrument. For example, there was evidence found related to test content, reliability, and internal structure for the instrument in Figure 3. Clicking on any of those tabs would open a new page with all the associated validity evidence, evidence types, any potential claims, and any interpretation and use statements. This structure aligns with the coding framework used by the synthesis teams and allows users to several ways to find and explore the validity evidence found for an instrument.

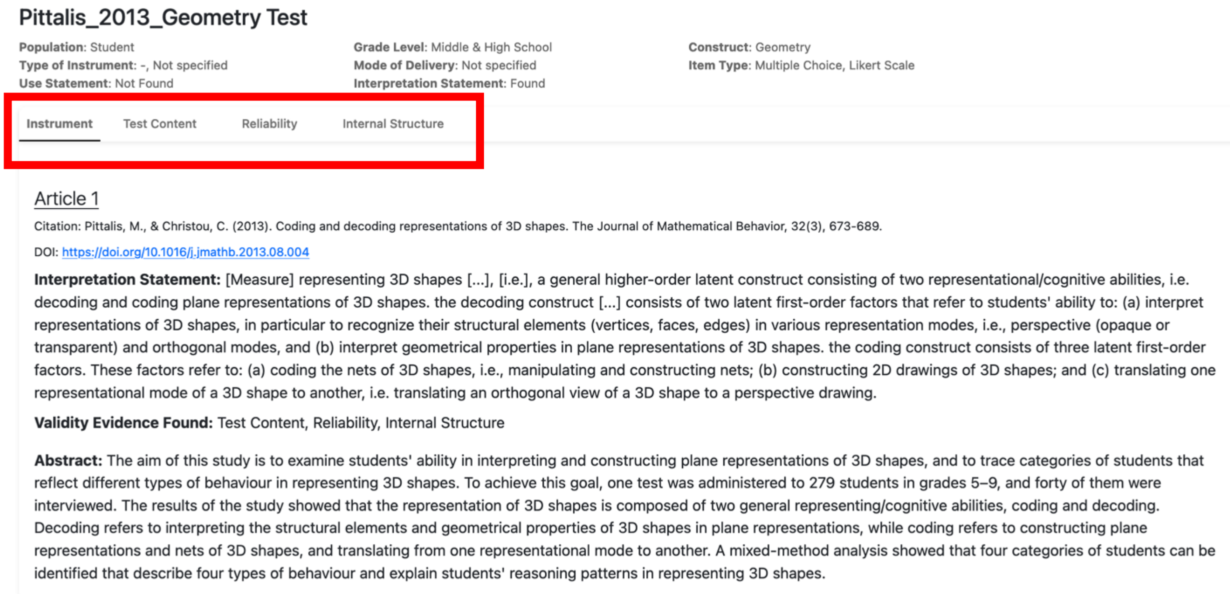

Figure 3: Snapshot of Instrument Results Page

Design Feature: Researcher Portal. There is also a researcher portal with training modules, both written and video based. These provide users with educative content pertaining to validity and validation, the proper use of the repository, and recently added or highlighted assessments the user may want to consider. Once logged into the researcher portal, users can save "favorites" or share results with collaborators. Users can also provide feedback on the results of an assessment or download the search results for a search.

Design Feature: Ease of Adding New Instrument. Initially, assessments were added by the project leaders based on the synthesis groups' work. Presently, users can submit a request electronically to upload an assessment to the repository. Users are asked to tag key features of the assessment as well as to identify and upload literature containing interpretation or statements, validity evidence, and claims associated with the assessment (Figure 4). Once an assessment is submitted, a repository curator vets the materials to ensure the inclusion criteria have been met. Once verified, the assessment is added to the repository. If inclusion criteria have not been met, then the project leader or curator communicates the additional information needed to include the instrument in the repository.

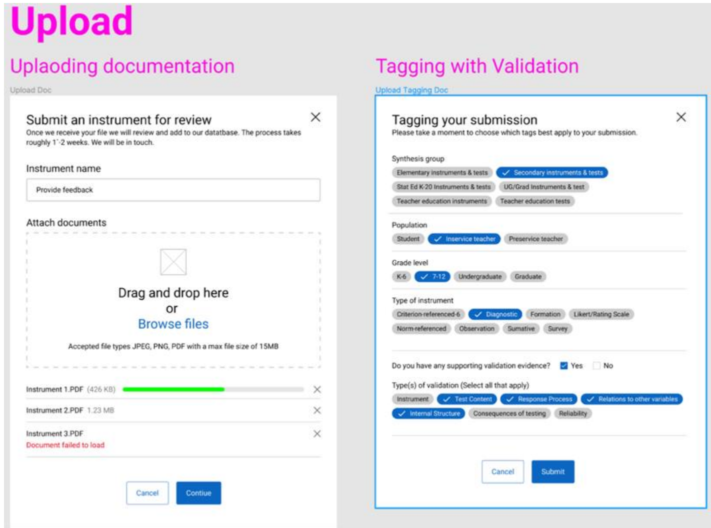

Figure 4: Adding a New Instrument

Discussion and Conclusions

PMENA session participants will explore the features of the repository. A goal of this five-year project was to produce a repository that could be used by scholars seeking a quantitative assessment for use in mathematics or statistics education research. First, it was necessary to identify the quantitative assessments used since 2000 and to gather information about them. This required multiple individuals culling through numerous databases. The second step was to evaluate the information for each assessment and categorize that information within the Standards (AERA et al., 2014) framework. The third step was to develop a working repository, accessible to broad communities containing this information.

The work across the six groups led to the creation of a repository with 1,034 instruments with validity evidence presented from 1,206 papers. Currently, about three-fourths of the instruments in the repository contain validity evidence. Most of the time that validity evidence was found, it was evidence of test content or reliability. There were very few reports of consequences of testing (1.7%) and response process (5.1%), which suggests future test scholarship should pursue these areas, which heeds previous calls (Author, 2023, 2021). In addition, approximately one-quarter of the papers contained interpretation or use statements. Researchers need to be aware of the validity evidence for a measure before they choose to use it for their own purposes. These findings have substantial ramifications for researchers in mathematics and statistics education.

Ultimately, this work makes it easier for scholars to locate assessments for a given construct or to evaluate the viability of assessments considering their known validity evidence and claims. Moreover, this repository allows scholars to gather further or more robust validity evidence for it. Proper validity arguments are foundational to robust quantitative research in mathematics and statistics education which makes this public repository of assessments and their associated validity evidence so beneficial to the research community.

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ASSESSING PRESERVICE TEACHERS' KNOWLEDGE FOR TEACHING UNIT FRACTIONS

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Keywords: Assessment; Mathematical Knowledge for Teaching; Pre-Service Teacher Education

Introduction

Teacher acquisition of robust mathematical knowledge for teaching (MKT) is essential in pursuing a quality education. In the context of knowledge for teaching fractions, work like Steffe and Olive (2010) and Hackenberg et al., (2016) already support teachers' knowledge growth. Nevertheless, less targeted has been the development of instruments for assessing teachers' knowledge for teaching fractions (Van de Walle et al., 2022). Similar to other studies (e.g., Norton et.al., 2015), our study aimed to develop an instrument for assessing pre-service teachers' knowledge for teaching unit fractions using length models through two research questions: (1) What set of tasks can elicit evidence of teachers' knowledge of teaching unit fractions using length models?; (2) How can teachers' responses to the tasks be interpreted to gain insights into their knowledge of teaching unit fractions using length models?

Methods

We used Pellegrino et al's (2001) Assessment Triangle Framework (which involves three elements: cognition, observation, and interpretation) to develop the instrument. As a cognition model, we used Silverman & Thompson's (2008) 5-component framework for MKT. With this framework, we designed the first version of the assessment instrument and rubric, targeting two key developmental understandings (KDU): (a) all the 1/n parts that compose a whole must have *equal length*, and (b) To complete the whole, *n parts* of 1/n length are needed. Regarding observation and interpretation elements, we conducted two phases of data collection. In phase 1, we tested the designed instrument with 6 pre-service teachers, followed by a revision of both the instrument and the rubric. In phase 2, we tested the instrument and rubric again with 20 pre-service teachers. This second round of data collection and analysis allowed us to test further validity and reliability based on intercoder agreement between the two researchers.

Results and Conclusions

In phase 1, we adjusted to the instrument and rubric. For example, we added clarifying sentences and asked for additional student models to ensure gathering sufficient data regarding this component. Additionally, we refined the two KDUs being considered in the rubric. In phase 2, we tested for reliability. We computed Kendall's tau between two raters finding that all five components' tau was 0.676 ($p<.001$), representing positive associations between raters. Furthermore, we also calculated Cohen's kappa (weighted) value as a measure of inter-rater reliability. The two raters' kappa scores were 0.558, which means that the reliability between the two raters is moderate and satisfactory (Landis & Koch, 1977).

Through the process carried out, we were able to respond to the two research questions. Firstly, the set of open-ended tasks elicited meaningful evidence (e.g., drawings, explanations, etc.) of the participants' knowledge for teaching unit fractions using length models. Secondly, key features of the rubric include considering evidence of a specific component, across several items, allowing the raters to assess a bigger picture of the participants' knowledge.

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INVESTIGATING TEACHER APPROACHES TO ASSESSMENT IN A NEW DETRACKED MATHEMATICS CURRICULUM

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This study investigates teacher approaches to assessment in a detracked mathematics curriculum through semi-structured interviews. The detracked curriculum promotes inclusivity by accommodating students of all learning levels in mathematics. Assessments play a crucial role in addressing significant learning gaps. The study highlights leveled assessments as a common assessment practice used by the grade 9 math teachers. Further, the study unveils the various forms of assessments utilized by teachers and illuminates the challenges they encounter, encompassing time constraints, resource limitations, and the need for additional support.

Keywords: assessment, detracked curriculum, student learning, mathematical practices

Purposes and Framework

The responsibility for teachers to uphold the expectations of the curriculum and demonstrate student achievement success can be represented by their assessment practices. The assessment practices of teachers often consider students' wide range of interests, needs, and strengths, as well as personal interpretations of the curriculum (Horn, 2006; Venkakrishnan & Wiliam, 2003). The teachers' assessment choice directly impacts various aspects of the student's learning experience including (but not limited to): student inquiry, achievement, engagement, motivation, and autonomy (Herppich et al., 2018; Lovall-Jones et al., 2014; Quigley et al., 2020). The task of developing assessments can be daunting as the range of students' abilities and preferences vary from classroom to classroom, but one method that may aid teachers in accounting for all types of students is differentiation (Lovall-Jones et al., 2014; Marks et al., 2021). Generally, teachers rely on feedback and formative assessment to support differentiated instruction to adapt to and ensure students' progress (Dayal, 2021; Herppich et al., 2018; Marks et al., 2021). The practices of catering to students' individual needs, often related to differentiated instruction, are crucial in the transition of new curricula, especially with adapting to a detracked curriculum.

In Canada and the US, a detracked curriculum can offer more equitable, inclusive learning environments for student success (Horn, 2006; Tereshchenko et al., 2019). Since September 2021, a new detracked curriculum has been implemented in grade 9 mathematics classes in schools across the province of Ontario, Canada. Since the transition from the tracked to the detracked curriculum involves combining two previously segregated learning level courses, one of the greatest challenges teachers have experienced is closing the larger gaps among student abilities. Therefore, the modification of assessment preparation and implementation has been critical in delivering curriculum content, particularly for varied self-assessments, tests, and grading (Bruce et al., 2010; Dayal, 2021).

Specifically for the subject of mathematics, teachers must consider more particular aspects when differentiating to meet their students' individual needs. These aspects may include adopting a holistic mathematics perspective, making crucial decisions about what mathematics knowledge should be prioritized, adapting both course-based and application-based assessments, and balancing professional and personal decisions (Horn, 2006; Quigley et al., 2020). In addition, during the preparation of assessments, teachers need to gain greater comfort with mathematics curricula content, collaborate with other colleagues, and contemplate the method of measuring the quality of student work and understanding (Horn, 2006; Ulusoy & Incikabi, 2020).

As a result of the collaboration between OISE, University of Toronto and an urban school board in Canada, a multi-year project has been conducted about the experience of grade 9 mathematics teachers with the new detracked curriculum. The project findings included teachers' preparation of assessment, the newfound practices of levelled assessment, and the implementation of various forms of assessment.

The purpose of the project was to investigate the following questions:

- What are the assessment and evaluation practices of teachers in the new de-tracked mathematics curriculum?
- How do teachers differentiate their assessment practices to meet the diverse needs of students in the detracked curriculum?
- How do teachers prepare assessments with respect to the expectations of the new detracked curriculum?

Methods

Participants

The participants from this study consisted of grade 9 mathematics teachers from different schools across the largest urban school board in Canada. The teachers are currently teaching destreamed or have taught destreamed in the past. In this paper, we include the perspectives from three participants who discuss their experiences as current grade 9 destreamed mathematics teachers, each given pseudonyms to maintain anonymity.

Data Collection

In this study, the data collection arose from semi-structured interviews during the Fall 2023 semester. The participants' interviews were transcribed verbatim and qualitatively coded using the *NVivo 12* software. This paper provides a thematic review of the findings that emerged from teacher interviews. There were surveys administered based on *The Attitudes and Practices for Teaching Mathematics Survey* (McDougall et al., 2001), which informs teachers of their own perceptions towards mathematics teaching and learning, as well as their values and comfort level to mathematics curriculum changes.

Interviews

The semi-structured interviews took place individually through Microsoft Teams, where teachers attended remotely from their respective schools. The interviews ranged from 50 to 60 minutes, which were audio recorded and then transcribed by the research team.

Results

Leveled Assessment

A popular assessment practice that the grade 9 math teachers interviewed in this study have been using to meet the varying mathematical abilities in their students is the leveled assessment. An example of how Hazel, a teacher in her third year of teaching, is doing leveled assessments in her classroom is by having a "Level 1-2 and a Level 3-4 version of the [math] question" (Hazel, 2023). Catherine, a teacher with 15 years of teaching experience in various grades and subjects, also uses a lot of leveled assessments in her classroom, in which she says, "I try to do a lot of leveled assessments, and any task or assessments with multiple entry points" (Catherine, 2023).

Among the benefits of using leveled assessment is that it provides students the opportunity to make choices in the assessment process, thereby increasing learner autonomy. As Hazel describes, it allows students to choose the level of assessment that "can show their best understanding because that's what the test or quiz is about" (Hazel, 2023). Leveled assessments also allows the teacher to be more inclusive towards the varying levels in students' mathematical abilities. Catherine shared her experience of leveled assessment as the following: "I think this made it more accessible to more of our students" (Catherine, 2023). Giving students choice in the level in which they wish to be assessed through leveled assessments, further increases student motivation. This year was the first year that Hazel started to implement leveled assessments more regularly. Once she started doing it at almost every evaluation, she witnessed a significant positive change in her students. She explains, "between the first task where I had no leveled assessment and the first or second quiz whenever I started implementing it about a month into school, I did notice kids were not leaving things blank anymore" (Hazel, 2023).

What makes leveled assessments successful is students' ability to self-assess what they know and do not know. Students' self-assessment in determining the appropriate level of assessment they perceive to be suitable for them is, in fact, what makes the leveled assessment a possible assessment practice used in the classroom. In other words, implementing leveled assessments can be challenging if students lack in their self-assessment skills. Having experienced the benefits of implementing leveled assessments, Catherine started this year with many leveled assessments in her classroom. Although this method has worked well for her students in the past years, it was a different case this year. The main challenge, according to Catherine, is that her students "don't seem to have an understanding yet of how to assess what they know and don't know" (Catherine, 2023). She continued by explaining, "And even when we tried a leveled assessment, they wanted to do all of the questions. They didn't know which ones they could and couldn't do. They didn't actually know their own ability" (Catherine, 2023).

Different Forms of Assessment

Using various forms of assessments was another important assessment practice being used by the teachers. These included formative assessments, quizzes, non-testing quizzes, assignments, projects, class participation, math journals, and culminating tasks.

In Rachel's classroom, she uses most of the assessments. Rachel describes how she uses math journals in her classroom: "we essentially outline the practice work that's been assigned for each lesson in that unit. And then we just asked some [students], oh have you completed any? Yes, no, if some, list how much?" (Rachel, 2023). She continues by describing how she asks her students to "then just reflect on if you had to assess your understanding for this topic from one to four, and any additional comments that you want to add. And also share any struggles that you may have had, challenges, etc." (Rachel, 2023).

In another case, small and more frequent assessments that weigh less in marks such as formative assessments and quizzes were regularly used by Catherine and her colleagues. For Catherine and another math teacher at her school, many of their assessments are formative. Two other teachers at her school, as Catherine explains, "do a lot more short little quizzes that are not worth a lot but they do count towards their mark" (Catherine, 2023).

When it comes to bigger assessments such as the grade 9 math exams, some schools are implementing non-traditional assessments. In Hazel's school, aside from the EQAO, a culminating task is used in replacement of grade 9 math exam. According to Hazel, culminating task is "worthwhile and interesting and actually acts as a culmination of all their learning" (Hazel, 2023). Hazel shares her positive outlook: "I like culminating [tasks] better, I see value in them. Because you can complete a test in an hour doesn't mean that you're good at math. It means that you can mimic whatever you did in class" (Hazel, 2023). She continues by saying, "Everyone is good at math in their way…. if you can make connections and understand the meaning of math, that is what is most important" (Hazel, 2023).

Challenges

The interviews further revealed some of the assessment challenges experienced by these teachers, which involved teachers' lack of time and resources.

The issue of having lack of time and resources makes it especially harder for the grade 9 math teachers who teach detracked math courses with such large gaps in students' abilities. Hazel speaks about the limited amount of time she has to prepare: "When you have three preps, it is a lot of work, because even if I were to split my prep 75 minutes evenly it would only be 25 minutes to do math and I definitely take more than that" (Hazel, 2023). Similarly, Rachel agreed how having more time will allow her to develop better assessments.

Moreover, teachers lacked in resources and support in developing assessments that would effectively cover the expectations of the curriculum. For example, Hazel wishes to implement triangulation of assessment which are done in thinking classroom. The triangulation of assessment involves observations, conversations, and products. However, she believes she needs more support to be able to do so. She explains, "I just have not figured out exactly how to do that, and I think that is something I need little bit more mentorship on, a little bit more like collaboration because it is a lot to take on" (Hazel, 2023). Further, more resources are needed when it comes to developing culminating tasks in replacement of final exams. Hazel suggests: "I think it would be worthwhile to have some sort of group of people or something to develop exemplars of what you could do as accommodating that is successful" (Hazel, 2023).

Conclusion

The study explores grade 9 mathematics teachers' assessment practices in Ontario's new detracked curriculum. Leveled assessments emerge as a prominent practice, providing students autonomy in choosing their evaluation level and fostering inclusivity. Despite its benefits, challenges arise when students struggle with self-assessment, emphasizing the need for honing this skill in the curriculum. The study sheds light on the diverse forms of assessment, ranging from traditional tests to innovative methods like culminating tasks and math journals. These varied approaches showcase teachers' commitment to capturing the multifaceted aspects of students' mathematical understanding.

When it comes to challenges, time constraints and insufficient resources emerge as critical obstacles, hindering teachers in their pursuit of more engaging and comprehensive assessments. The study underscores the importance of collaborative efforts, mentorship, and the development of resources to support teachers in adapting their assessment practices to the evolving educational landscape. In navigating the complexities of the detracked curriculum, teachers play a pivotal role in shaping a more inclusive and equitable mathematics education.

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CONCEPTUALIZING LEARNING PATHWAYS PRESENTED BY TEXTBOOKS: INTRODUCING A 4S FRAMEWORK AND ITS VISUALIZATION

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While mathematical textbooks have been extensively analyzed and compared, few studies have explored the divergent learning pathways presented by textbooks and their implications. This study introduces an analytical framework and its visualization to elucidate these pathways, focusing on four key facets of instructional activities: scope, sequence, structure, and scale. We exemplify the use of the 4S framework and its visualization in comparing the learning pathways for congruent triangles in a Chinese textbook and a US textbook. Results suggest that these textbooks offer fairly similar learning pathways for students, particularly in terms of content scope. However, variations emerge in the instructional activities' sequence, structure, and scale. These disparities underscore the affordance of the 4S framework. We conclude by discussing the implications for curriculum development.

Keywords: Curriculum, Learning Trajectories and Progressions, Geometry

Mathematics textbooks are pivotal in connecting the intended curriculum, as defined by national standards, to its practical implementation in classrooms, a subject that has captivated researchers for decades (Valverde et al., 2002). However, despite the extensive scrutiny and comparison of mathematical content in textbooks, there needs to be more exploration into the divergent learning pathways various textbooks offer and the implications thereof. For instance, Fan et al. (2013) noted that approximately 63% of the literature they reviewed in mathematics textbook research studies primarily focused on textbook analysis and comparison, typically yielded descriptive findings from systematic coding using content analysis. While such analyses have made valuable contributions in revealing the similarities and differences between textbooks, they often fall short in providing insights into the unfolding learning pathways within these textbooks, resulting in a gap in comprehension of learning experiences facilitated by these textbooks. As such, this study endeavors to develop an analytical framework for examining learning pathways delineated in textbooks. This analytical framework composes four dimensions: scope, structure, sequence, and scale, which we refer to as the 4S model. According to this 4S model, a visualization of learning pathways is developed. In the following, we first introduced the 4S model. Then, we exemplified the application of the 4S model and its visualization by examining the learning pathways for congruent triangles as presented in a US textbook. Ultimately, we illustrated the affordance of the 4S model by answering the following research questions: what are the similarities and differences between the learning pathways for congruent triangles as presented in textbooks in different countries?

A 4S Framework: Scope, Sequence, Structure, and Scale

An Overview of the 4S Framework

We have adopted the term "learning pathways" from Kim and Remillard's (2020) research on curriculum comparison, where they define a learning pathway as "a sequence of student learning that outlines the development of related concepts in a structured order" (Kim & Remillard, 2020, p. 33). In line with this descriptive definition, learning pathways encompass three key dimensions: the main targeted mathematical concepts covered, which we refer to as "scope"; the developmental order of these mathematical concepts, which we define as "sequence"; and the underlying logical structure(e.g., mathematical structure, content organization structure), denoted as "structure." Since mathematical concepts often evolve through instructional activities, and various types of instructional activities demand varying levels of cognitive effort and time, we propose the inclusion of a fourth dimension, "scale," to broadly capture how different textbooks allocate learning resources to different mathematical concepts. As such, learning pathways can be examined from the following four dimensions: scope, sequence, structure, and scale. We name this framework the 4S framework. In the following, we will exemplify the use of this frame in comparing textbooks with the case of triangle congruency.

Visualization and Application of the 4S Framework

In this session, we demonstrate the application of the 4S framework to analyze learning pathways presented by textbooks and visualize each pathway by comparing the pathways for learning triangle congruency in two textbooks. We chose the topic of triangle congruency for two reasons. Firstly, triangle congruency is fundamental to geometry and is consistently included in upper-level geometry curricula worldwide (Jones & Fujita, 2013), enabling access to textbooks from different countries for comparison. Secondly, multiple conditions of triangle congruency, e.g., side-angle-side (SAS) and side-side-side (SSS), allow for various sequencing possibilities (author, 2023), leading to distinct learning pathways with diverse logical structures. This variability makes triangle congruency an ideal subject for comparing learning pathways in different textbooks. In this illustration, for convenience, we selected the Student Editions of an eighth-grade math textbook by People's Education Press (2013) in China (PEP Math) and "Discovery Geometry" (DG) by Serra (2008) in the US for analysis.

Using the 4S framework, we examined the learning pathways designed by DG and PEP Math across four dimensions: scope, sequence, structure, and scale, and visualized each learning pathway (refer to Figure 1). Given that our goal is to demonstrate the application and advantages of the 4S framework, our analysis focused on the instructional activities employed, including expository texts, explorations, and worked-out examples in the lessons examining different congruency conditions.

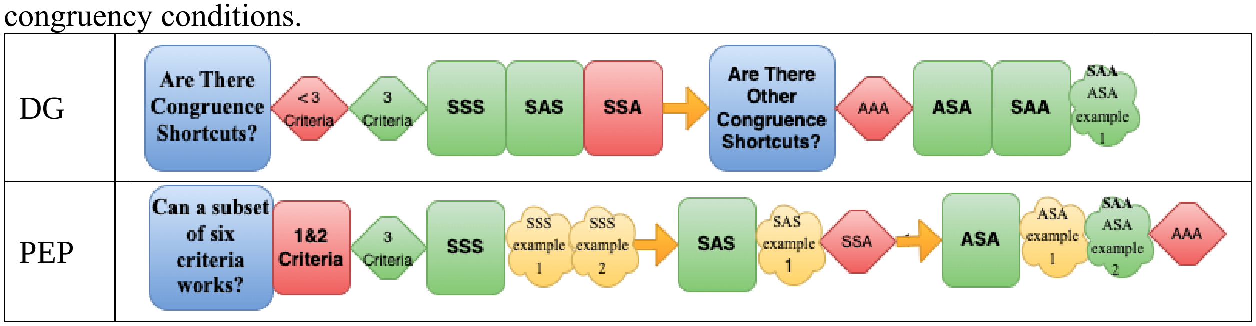

Figure 1: The Learning Pathways Presented in DG and PEP Math.

We considered each lesson an analysis unit and used yellow arrows to connect different lessons. As depicted in Figure 1, the learning pathway presented by DG consists of two lessons, while there are three lessons in the case of PEP. We treated the instructional activities included in each lesson as sub-analysis units, distinguishing them with various shapes, colors, and sizes.

Specifically, a large blue square represents an overarching inquiry question, such as "Are there congruence shortcuts?" in Figure 1. We employed green to indicate combinations of criteria, which we refer to as conditions in the above that work (e.g., SSS in Figure 1), red to signify conditions that cannot guarantee congruency (e.g., SSA in Figure 1), and yellow to demonstrate the application of a condition in problem-solving (e.g., SAS example 1 in Figure 1). We used various shapes and sizes to represent different types of instructional activities. A large rectangle was used for exploratory activities (e.g., using compass-and-straightedge construction to justify SSS can guarantee triangle congruency), while a small hexagon indicated brief instructional activities, such as a brief discussion or providing a counterexample for justification (e.g., use not all equilateral triangles are congruent to justify AAA won't work). Additionally, a cloud shape, sized between a rectangle and a hexagon, represented worked-out examples in the textbooks. In the following sections, we illustrate each dimension and its connection to the visualization within the context of comparing two learning pathways for triangle congruency.

Scope. Scope captures the mathematical content that has been covered in the textbooks, which is presented by the texts in Figure 1. These texts include the overarching inquiry questions and the main foci of the instructional activities. Figure 1 shows that both textbooks examine situations with less than three congruent and exactly three congruent criteria. Regarding exactly three congruent criteria, both DG and PEP math include exploring or discussing the following six situations: SSS, SAS, SSA, ASA, SAA, and AAA. Thus, the scopes of the learning pathways presented in the two textbooks are similar.

Sequence. The sequence shows how a textbook organizes instructional activities to develop a learner's understanding of the topic. Here, we sequenced the main instructional activities linearly covered in DG and PEP. As Figure 1 shows, DG and PEP math sequence the examination of situations similarly. Both textbooks open with an overarching inquiry question. DG asks if there are congruence shortcuts and PEP math inquiry, "Is it possible to identify various subsets of these six conditions (i.e., three pairs of congruent sides and three pairs of congruent angles) that will also make two triangles congruent?" To answer this overarching question, both textbooks start with the situation when only one or two criteria are congruent, then transition to examining the conditions with three criteria. Both textbooks also sequence the six conditions of three criteria similarly. Both start with SSS and then move on to SAS and SSA. However, slightly differently, DG discusses the AAA condition before exploring ASA and SAA conditions, while PEP math discusses the AAA situation at the end. These sequence differences reflect and are influenced by the "structure," a construct we discuss next.

Structure. We assume textbooks sequence instructional activities based on logical structures. The patterns reflected by the sequence suggest the underlying logical structures. DG and PEP math share some similarities regarding the structure of learning pathways. For example, both follow an inquiry mode; both open the topic with an overarching inquiry question and then devote a sequence of instructional activities to answer the question. One distinguishing structural difference between them is that PEP math always follows the examination of a condition with worked-out examples, while DG seldom does. Another structural difference relates to the sequence difference we mentioned earlier: AAA goes before or after the exploration of ASA and SSA. Reading through PEP math and DG carefully, we found that this difference reflects a structural distinction that PEP math goes through the cases systematically with a clear underlying mathematical logic. It starts with three pairs of congruent sides, zero pairs of congruent angles (3S0A: SSS), then 2S1A: SAS and SSA, then 1S2A: ASA and AAS), and ends with 0S3A: AAA. However, DG separates the examination of the six conditions into groups, located in two lessons, with two categorical titles: "Are there congruence shortcuts?" and "Are there other congruence shortcuts?" without explicating the underlying logic.

Scale. Scale is a dimension to capture the educational resources the curriculum developers distribute to a topic or a specific instructional task. It is reflected by the anticipated time spent on the instructional activity, the priority of the instructional activity in the lesson, and the cognitive demands for completing the instructional activities. As shown in Figure 1, PEP covers the content with three lessons while DG only devotes two lessons, reflecting that the overall scale of triangle congruency in PEP is more significant than in DG.

The overall scale difference may be due to the differences between the instructional activities in PEP Math and DG. Firstly, while DG only mentions that a condition with fewer than three criteria won't guarantee triangle congruence, PEP requires students to use compass-and-straightedge construction to test if a condition with one criterion (e.g., one side equivalent or one angle equivalent) or two criteria work. Regarding SSS and SAS, although both DG and PEP require students to engage in exploratory activities, PEP strengthens students' understanding of SSS and SAS conditions by providing worked-out examples immediately after each exploratory activity, resulting in a larger scale of learning resources devoted to the learning of SSS and SAS compared to DG. However, regarding SSA and SAA, DG allocates more learning resources to them than PEP does, as it expects students to engage in an exploratory activity rather than a brief discussion, as PEP does. One thing to mention is that although we assume the scale of exploratory activities in both textbooks is similar, they are different; many details differ, which is beyond the focus of this framework.

Discussion and Conclusion

In this study, we propose a framework for textbook analysis that complements the dominating focus on topic listings or task analyses in this research area. Using the 4S framework to explore four critical dimensions—scope, sequence, structure, and scale—has yielded nuanced insights into instructional activities concerning congruent triangles in Chinese and US mathematical textbooks. While the similarity in the overall content coverage implies a shared foundation in mathematical concepts, the analysis of sequence and structure dimensions highlights differences in content organization, suggesting potential divergences in students' learning trajectories. Notably, variations in scale were identified to impact the level of student engagement with the material, prompting considerations for educators to tailor teaching strategies to match the instructional activity scale. This disparity in learning pathways may help elucidate why eighth-grade students in the US struggled with the TIMSS items intended to assess their understanding of triangle congruency (Author, 2021).

The 4S framework exhibits significant potential in navigating nuanced distinctions within instructional activities related to triangle congruency. We recommend that future researchers apply this framework to various mathematical topics in textbook analysis, further refining it. This framework can also analyze how a textbook evolves version by version (e.g., how DG presents learning pathways differently in 2003, 2008, and 2013 versions). Our analysis using the 4S framework carries substantial implications for curriculum development, emphasizing the importance for educators, curriculum developers, and policymakers to address the breadth or depth of content and the intricacies in sequencing, structure, and scale. Such considerations are needed to construct a shared vision of mathematics curriculum that will equip future students with the mathematics understanding needed to be productive citizens in uncertain times.

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NAVIGATING A NEW DETRACKED MATHEMATICS CURRICULUM: TEACHER APPROACHES TO PROGRAM PLANNING AND ASSESSMENT

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Current research indicates introducing a detracked curriculum can bring about positive educational change, but there will always be challenges at the enacted level. In detracked mathematics courses, teachers must differentiate their students' needs, adopt a connected perspective of mathematics, decide which key mathematics ideas take focus, and distinguish course-based and application-based evaluation. This research project examines the experiences of grade 9 mathematics teachers in the detracked classroom. The findings of this project display that teachers put great effort into acquiring class resources and time, preparing and designing courses, and developing new forms of assessment and evaluation.

Keywords: Detracked curriculum, assessment, mathematics practices

Purposes and Framework

Although a new curriculum can bring about positive educational change, there will always a challenge with implementation at the enacted level: the teacher level. Detracked classrooms have a history of disrupting fluency and routine in the curriculum for teachers and students (Horn, 2006; Quigley et al., 2020; Tereshchenko et al., 2019; Venkatakrishnan & Wiliam, 2003). Furthermore, teachers in detracked classrooms are responsible for making individual interpretations of the curriculum (Herppich et al., 2018; Horn, 2006; Venkakrishnan & Wiliam, 2003). These curriculum interpretations about planning and evaluation are what directly impact students' inquiry, development, and achievement (Herppich et al., 2018; McGee et al., 2013; Quigley et al., 2020).

For countries like Canada and the US, the detracked curriculum encourages inclusivity for teaching and learning by creating more equitable environments for success (Tereshchenko et al., 2019; Horn, 2006). As of September 2021, all schools in the province of Ontario in Canada have recently rolled out a new detracked grade 9 mathematics course. During this transition from the tracked to the detracked curriculum, there are several changes that teachers must consider, primarily, the broader scope of students' mixed abilities. For teachers, meeting the student's individual learning needs can be a great ordeal, as they must prepare/design a multitude of tasks and varied activities (Mellroth et al., 2021; Perkins, 2016).

In addition to differentiating their students' needs, teachers must contemplate many other factors particular to the subject of mathematics. These factors may involve teachers adopting a connected perspective of mathematics, deciding on focused key mathematics ideas, balancing professional and personal decisions, and distinguishing between course-based and application-based evaluation (Gottfried; 2014; Horn, 2006; Quigley et al., 2020). In addition, the Ten Dimensions of Mathematics Education (McDougall, 2004), present key components that contribute to successful mathematics teacher practices and approaches. Specifically, the components of program scope and planning and assessment are prominent areas of change for teaching detracked mathematics (Ferreyro-Mazieres, 2016). For this paper, we draw from the Ten Dimensions of Mathematics Education framework to guide our data analysis and findings.

The Ontario Institute of Studies in Education and the largest school board in Canada have collaborated on a multi-year project with several grade 9 detracked teachers. From this project, our findings suggest that grade 9 mathematics teachers have put great effort into acquiring classroom resources, preparing and designing courses, and developing new forms of assessment and evaluation. Moreover, teachers expressed their hardships transitioning into the new curriculum, including lack of curriculum expectations, lack of preparation time, and the overwhelming amount of curriculum content.

The purpose of the project was to investigate the following questions:

- What do teachers perceive to be the biggest challenges in implementing the new mathematics-detracked curriculum?
- How are teachers preparing their mathematics classes with respect to the expectations of the new curriculum?
- How are teachers approaching assessment and evaluation with respect to the expectations of the new curriculum?

Methods

Participants

Participants for this project were found from sixteen different schools across a large urban school district. All participants were required to be current teachers who were presently or previously teaching the Grade 9 detracked mathematics course. For this paper, we discuss the responses of eleven teachers relating to the themes of program planning, assessment, and teacher support/challenges.

Data Collection and Analysis

The data collected for this project came from semi-structured interviews taking place with teachers, once in the first semester and once in the second semester. Interviews were transcribed verbatim and coded for emerging themes through *NVivo 12* software, guided by the Ten Dimensions of Mathematics Education framework. In this report, recorded themes emerged from the first round of interviews with participants as well as *The Attitudes and Practices for Teaching Mathematics Survey* (McDougall et al., 2001) was administered for interviews but was largely used to inform participants of their *own perceptions* of the importance and their degree of comfort in implementing reform mathematics teaching and learning strategies.

Interviews

Semi-structured interviews took place with participants in their individual schools after the second professional learning session. Interviews ranged in length from approximately 50 to 60 minutes, which were audio recorded and then transcribed by the interviewer.

Results

Program Scope and Planning

Various planning strategies were mentioned among the teacher participants, ranging from spiralling the curriculum, concept-based planning divided into units, as well as daily on-the-go planning. Other considerations were also addressed such as preparing students for the grade 10 academic mathematics course and planning for students with potential foundational mathematics knowledge gaps.

Some teachers described using the spiralling approach for planning the Grade 9 mathematics course. One teacher found this approach very effective and described the process: “Everything weaves together, we go back and forth. There is a sense where all the units come together to the central big idea and having discussions that also go back to previous topics helps with retention” (Teacher 1). Other teachers used the unit-by-unit approach, which covered one concept after another. Teacher 2 explained their reasoning behind using this approach:

> So, for this year, our planning process was by unit. I think we decided that we wanted to try and just go by unit, because in order to spiral, [the students] have to have a certain amount of knowledge or consistency in the other areas before you can spiral things on top.

A few teachers mentioned the challenge of planning day by day or week by week rather than having a long-range plan already in place. For example, one teacher described planning on a “catch-up basis” since she is “a little bit busy these days with three different courses and planning things week by week or month by month” (Teacher 3).

The request to be given time to plan was made by almost every teacher in this project. The teachers value planning time and having the time to collaborate with other teachers to discuss different teaching strategies. Teacher 4 hoped to discuss topics “starting from how you review, how you engage students, and how you use those cooperative learning strategies”. Another teacher from the same school requested for planning time with “more guidance in the curriculum” (Teacher 5). Several teachers reported that the new curriculum was released too late in the previous academic year, leaving them not enough time to prepare. For example, one teacher from Mona Lavender noted: “It was not considerate of teachers to release the new curriculum almost at the end of June, leaving very little time to start preparing. Almost requiring teachers to spend their summer preparing if they wanted to get a head start” (Teacher 6). Another teacher expressed her frustration: “Considering when we got our curriculum – it was six pages, single sided, no exemplars, no examples. There was no structure, no timeline, there was nothing, and it was left to the teachers” (Teacher 7).

Assessment

Assessment is a topic of interest for many teachers and something that they can continuously improve upon. Several teachers mentioned formative feedback as a practice that they used in their classroom. For example, one teacher mentioned that, after the whiteboard group activities, the groups present their solutions to the class. She said that is when “we come back and talk about how they think they did, which parts were good and they receive feedback” (Teacher 1).

Another teacher explained that she is “constantly monitoring, giving feedback, doing formative assessment and figuring out how that can translate into evaluations” (Teacher 8).

Several teachers reported using parallel assessments as part of their practice. One teacher from Mandevilla SS described her process: “I give two different questions. I say you either choose this one or this one. One is out of three, which is easier. One is out of five, which is harder. The students can decide on which question they want to do” (Teacher 7). A teacher from Ninebark SS called this process leveled assessments. She said: “I have learned this idea of leveled assessment from a colleague at our board” and “students can choose which leveled question they want to attempt for them to show their understanding” (Teacher 9).

Project-based or performance-based evaluations were also mentioned by some teachers. Teacher 2 shared her practice:

> They have a little mini project or problem that they have to work on, and they may have to present that to the class or discuss their solution or provide feedback on somebody else's solution to show their understanding. Smaller tasks like that, plus regular contribution to class, and classroom conversations is how we can assess them that way as well.

Challenges with Curriculum Implementation and Teacher Support

The participants reported a lack of curriculum guidelines and specific expectations for each mathematics strand. For example, one teacher said: “We need more guidance in the curriculum. The curriculum is really loose and open, and it does not tell us what they want us to do. We are used to having more specific expectations” (Teacher 5). Another teacher said: “I felt like the Ontario Ministry strands were sometimes vague” (Teacher 1). The participant also reported having an excessive amount of content to cover with the new curriculum guidelines. One teacher honestly said: “It has been hard to fit everything in” (Teacher 10) and another teacher said: “At the moment, we have six sections of grade nine math. None of us will be able to complete the entire curriculum, given the fact that we are doing this for the first time” (Teacher 11).

When asked if they felt supported by the board or education system, participants reported a lack of teacher support. One teacher responded with: “No, I feel pressured to try it out. We have to make this work but without necessarily a good plan of action” (Teacher 10). Another teacher expressed his frustration: “All the new stuff you would really want help with is marked under construction, and well, that is not very helpful because that is the stuff I really need help with” (Teacher 5).

The participants also reported a lack of professional learning sessions. One teacher explained that having the resources themselves was not enough, she would have liked to see how the resources were implemented in a grade 9 classroom. She stated: “I feel like we were kind of thrown into the forest with a bunch of survival supplies and like, go, enjoy, and we were not really taught how to use these supplies” (Teacher 1). Professional learning sessions were requested by several teachers. One specifically noted that a session on coding might be helpful for some grade 9 teachers. She said: I have heard that coding seems to be a first exposure for many educators. So maybe some professional development might be needed there” (Teacher 4).

Conclusion

The literature disclosed the difficulties that students and educators experience during the implementation of the detracked curriculum, especially with regard to the subject of mathematics. As well, prior research recognizes the challenges teachers face when considering the wide range of abilities within the detracked classroom. Our project findings reflect commonalities from these studies, as teachers experienced numerous frustrations and difficulties with curriculum design, course planning, and course evaluation.

Among participants, there was one popular demand: additional time. Teachers expressed the need for more time to adjust to the curriculum, preparation time, and collaboration time. An additional common request was for the school board to provide extra professional learning sessions, as resources were not sufficient. Many teachers also employed newer approaches, including the spiralling approach for program planning and levelled or parallel assessment for course evaluations. The results of the project showed teachers experienced confusion and frustration when interpreting broad-based curriculum expectations, a disconnect from teacher implementation of the detracked course. These findings were consistent with previous literature, as it was notable that there was a lack of specific curriculum guidelines and teacher support.

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MATHEMATICAL TASKS FROM 141 SECONDARY ALGEBRA LESSONS: A PREPONDERANCE OF PROCEDURES WITHOUT CONNECTIONS

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Mathematics educators have written a great deal about cognitively-demanding tasks but this study of 141 lessons across 47 different algebra classes found cognitively-demanding tasks to be rare in practice. Of 2,378 coded tasks, 93% were low cognitive demand, predominantly procedures without connections to meaning. Only 6% were high cognitive demand, predominantly procedures with connections to meaning, not entailing the complex processes of doing mathematics. This article breaks down the cognitive demand of tasks by lesson segments (independent work, group work, homework) and also compares tasks in flipped lessons versus non-flipped lessons. We discuss the need for professional development that is aligned to these curricular realities and note discrepancies between these findings and other research.

Keywords: curriculum; algebra and algebraic thinking; instructional activities

For decades, mathematics educators have drawn attention to the importance of rich problem solving opportunities (e.g., Lesh & Zawojewski, 2007; Schoen & Charles, 2003) and cognitively-demanding tasks (e.g., Lappan et al., 2012; Stein et al., 1996; Stein et al., 2000) in the school mathematics experiences of students. Many textbook analyses (Glasnovic Gracin, 2018; Hwang & Ham, 2021) and other curriculum studies (Arbaugh & Brown, 2005; Jackson et al., 2013) have attended to cognitive demand in various ways, yet the United States and certain other countries are still known to have a predominant procedural focus (Dolores Flores et al., 2020; Hiebert et al., 2005; Lénárt, 2018; Litke, 2020). It is not clear that the cognitively-demanding tasks common in mathematics education professional development programs, preservice teacher education courses, and the books and journals of the scholarly community are common or even easily detectable in typical mathematics classrooms.

As part of a multi-year, non-interventionist project focused on understanding current algebra instruction, we observed 47 secondary algebra classes in a variety of school districts (de Araujo et al., 2017a). Based on three lesson observations in each class (n = 141), we compiled class profiles of the instructional patterns and classroom discourse (Otten et al., 2023). We also collected the student materials from each lesson, including any tasks that students were expected to work on during the class period as well as any tasks assigned as homework. Our research question in this brief report is the following: *What levels of cognitive demand were entailed in the tasks assigned to students in these Algebra 1 classes?* Furthermore, we were interested in how the cognitive demand varied across different parts of the lesson (e.g., in-class independent work time, in-class group work time, homework) or between groups of classes with different instructional models, such as classes implementing flipped instruction versus those implementing non-flipped instruction. Flipped instruction is defined by students being expected to watch videos (or read text or listen to an audio recording) as their homework rather than completing exercises (Bergmann & Sams, 2015; Otten et al., 2021). The broader project contained both types of algebra classes and it is worthwhile to examine potential differences in cognitive demand because some teachers have stated that flipped instruction, by moving lecture to a video outside of class and freeing up class time, can allow for rich tasks or extended problem-solving endeavors to be incorporated into lessons (de Araujo et al., 2017b).

Theoretical Perspective

In our overarching project, we view students' mathematical learning as a sociocultural process whereby they come to participate actively in a mathematical community (Vygotsky, 1978). Thus, the lesson observation protocol (Otten et al., 2021) attended to the nature of the classroom discourse, the interactions of the students with one another and with the teacher, the authority dynamics at play with regard to the mathematical content, and some of the social patterns exhibited by the teacher. Because we view learning as occurring through and being defined by students' participation in forms of mathematical discourse (Lemke, 1990), we identified various structural lesson segments such as whole-class discourse, independent work time, group work time, and non-instructional time. Note that group work here refers to time when the teacher explicitly expects students to collaborate with peers.

It also matters what type of mathematical activities the students are engaging in within their mathematical community (Doyle, 1983). Thus, we used the well-known mathematical task framework (Table 1), which provides a classification based on the thinking required to complete the tasks—what Stein and colleagues (1996) called cognitive demand. Stein et al. (1996) emphasized the importance of students engaging in tasks at different levels of cognitive demand, though much of their work focused on the cognitive demand of tasks in "reform classrooms" because they noted, as we have, that there is an overabundance of low cognitive demand tasks implemented in U.S. mathematics classrooms. Other scholars have traced the difficulty of implementing cognitively-demanding tasks (e.g., Boston & Smith, 2009; Henningsen & Stein, 1997). In the present study, we look not at all the tasks contained in particular textbooks but rather at the tasks selected by teachers to actually pose to students in our observed lessons. That being said, the analysis that follows was of those posed tasks as written, not as actually carried out to completion.

Table 1: Levels of cognitive demand entailed in mathematics tasks (Stein et al., 2000)

Level of Cognitive Demand	Description	Example
Non-Mathematical Activity	Task does not require students to engage with any discernible mathematical idea.	Coloring a picture related to task context
Memorization	Task involves recalling definitions, terms, or formulae but not applying them.	"Which term is the coefficient of x?"
Procedure Without Connection to Meaning	Task involves executing a known algorithm or producing correct answers based on previous instruction.	"Find the roots of this quadratic equation using factoring."

Procedure With Connection to Meaning	Task involves a procedure with explicit connections to underlying mathematical ideas, multiple representations, or an explanation of why the procedure works.	"Explain how you know that [this quadratic equation] has only one root."
Doing Mathematics	Task involves nonalgorithmic thinking and self-regulated explorations or mathematical concepts, processes, or relationships.	Determine end behavior of a new polynomial function and explain why it occurs

For some portions of analysis, Memorization and Procedures Without Connection were combined as "low cognitive demand" and Procedure With Connection and Doing Mathematics were combined as "high cognitive demand."

Method

The study involved 47 Algebra 1 (or equivalent) classes (Grades 8–9) in Midwestern U.S. school districts that ranged from rural to urban contexts and from small to large enrollments. The project focused on classes using flipped instruction ($n = 23$) and non-flipped instruction ($n = 24$), but this report focuses on all classes together and the mathematics tasks assigned to students during class time (independent work or group work) or as homework (tasks expected to be completed outside of class; distinct from those during class).

We conducted three lesson observations throughout the school year for each of the 47 classes (141 lessons) and we collected the tasks, either as problems captured within the class (e.g., recorded from the front board) or as materials from the teacher (e.g., handouts or homework assignments). Tasks were enumerated based on how they were presented to students, though multi-part problems (e.g., a, b, and c) were coded as a single task. If the teacher completely solved the problem, with no expectation for students to devise their own solution (e.g., a worked example), then it was not included as a task for students. The classes used a variety of curriculum materials with the vast majority being either teacher- or school-generated resources or publisher-developed textbook series. By design, the study focused on Algebra 1 courses and so there were no integrated curriculum series. Two team members observed each lesson and independently coded each task for level of cognitive demand (Table 1) based on the task as written, not implemented. After independent coding, coders met and reconciled any disagreements. If multiple levels of demand were discernible in a single task, it was coded at the highest level evident. For the analysis here, we simply report descriptive statistics as a means of summarizing the variability in cognitive demand. We also present findings broken into sub-categories, including the lesson segment in which it occurred (independent work, group work, homework) and the two instructional models observed (flipped instruction, non-flipped instruction).

Findings

Overall, our team coded 2,378 mathematical tasks (16.8 per lesson) and the overwhelming majority involved low cognitive demand (Figure 1), with 91% falling in the category of Procedures Without Connections to Meaning. These low tasks covered a vast array of algebraic topics (linear equations, quadratic equations, plotting graphs, writing functions from word problems) but were all characterized by the fact that the students had been shown a process for completing such problems and were not expected to justify their work or make conceptual connections to other ideas. Across the 141 lessons, there was a grand total of 5 Doing Mathematics tasks (0.2%). Even in this brief report, we could almost print all of them in their entirety, but instead we will just mention that one was an open-ended project in which students made a budget for their hypothetical finances and then provided justifications for certain calculations within that budget, and another Doing Mathematics task occurred when a teacher asked students to try solving a new type of equation before the teacher showed the procedure.

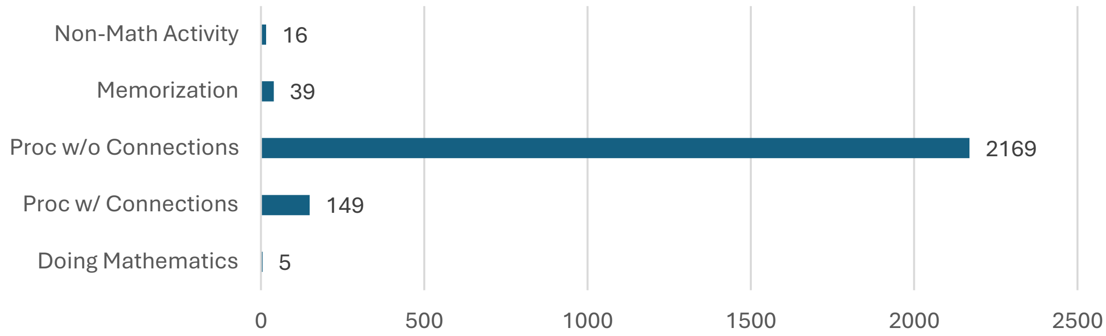

Figure 1: Total number of tasks from 141 algebra lessons, by level of cognitive demand

With regard to lesson segments, 133 of the lessons (94%) had *independent work time*, lasting for an average of 45% of the total class time, though this varied from as brief as 1 minute to nearly the entire period. There were 13 tasks on average that students worked on during independent work time per lesson (if they did not finish, they could complete them after class or next day). These tasks were low cognitive demand 92% of the time and high cognitive demand 7% of the time (1% non-mathematical). *Group work* was less common. Only 31 lessons (22%) contained any group work, and it involved 6 tasks on average. The types of tasks were roughly the same as during independent work, with 91% low cognitive demand and 8% high cognitive demand (1% non-mathematical). *Homework* tasks distinct from the tasks started in class were also relatively rare, occurring in 37 of the lessons (26%), with 13 tasks per homework assignment. We coded 96% to be low cognitive demand and 4% to be high cognitive demand.

Therefore, the breakdown of task types was essentially the same across the different segments of the lessons, and the similarities continue when comparing flipped and non-flipped classes. Both groups had matching cognitive demand profiles for independent work (92% low, 7% high) and for homework (95% low, 5% high), though non-flipped classes were more likely to have assigned homework tasks, as expected. A noticeable difference arose, however, between the flipped classes and the non-flipped classes in the area of group work. They had similar rates and durations of group work but the non-flipped classes had more cognitively-demanding tasks. Within flipped classes, the group work entailed low cognitive demand 96% of the time and high cognitive demand only 3% of the time, whereas non-flipped classes had low cognitive demand 70% of the time and had high cognitive demand 27% of the time (3% non-mathematical).

Discussion

Across a wide variety of school districts and a diverse group of teachers, the Algebra 1 lessons we observed involved an overwhelming proportion of low cognitive demand tasks

(93%). High cognitive demand tasks were extremely rare and Doing Mathematics tasks, in particular, were almost non-existent. This result should give pause to mathematics education innovation efforts that rely on an assumption of readily-available, cognitively-demanding tasks, at least in algebra. Because algebra is perhaps more procedural than other topics areas, these findings do not necessarily extend across secondary mathematics, nor do they extend to states with significant levels of reform-oriented curricula, but these findings are similar to what we have observed in other studies. Thus, it may be worthwhile to design professional development based on an assumption of teachers' comfort with low demand tasks (Otten et al., 2022).

It was surprising to us that, overall, group work was relatively rare (occurring in only 22% of lessons) and the tasks pursued in groups were not of higher cognitive demand than the other tasks. This finding was affected, however, by our over-sampling of flipped instruction. Looking only at the non-flipped classes, group work did post slightly higher rates of cognitively-demanding tasks (though almost never Doing Mathematics). This did not translate to higher student learning scores, though, on the procedural or conceptual measures in our broader study (Otten et al., 2023). As noted in critiques of the cognitive demand framework (Otten et al., 2017), it is wise to be cautious of direct links between cognitive demand and student achievement measures, though it remains important for more diverse forms of student outcomes.

With regard to flipped classes, we did not confirm teachers' stated intentions for flipping (de Araujo et al., 2017). Rather than using the extra time in class for group work or rich tasks, the flipped classes focused on independent work and low demand tasks. Our other analyses have showed that this focus on independent work is associated with learning gains (Otten et al., 2023).

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COMPARISON OF A GRADE 8 TOPIC FROM ONE U.S.-BASED AND ONE INDIA-BASED MATHEMATICS TEXTBOOKS

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Keywords: Curriculum, Middle School Education

In this poster, we present a comparison of a mathematical topic from two mathematics textbooks for Grade 8 to learn how the topic is addressed in each country. The two textbooks were (1) *Connected Mathematics Project* (CMP) (Lappan et al., 2014), a research-based textbook used in the United States, and (2) *National Council of Educational Research and Training* (NCERT, 2022), a textbook used in India by the schools affiliated with the Indian National Educational Board. The mathematical topic was “Inverse Variation”, which was also referred to as “Inverse Proportions” in the NCERT textbook. Through this comparative analysis across two diverse international contexts, we highlight differences between these textbooks and identify their contributions toward rich mathematical learning experiences.

The criteria used for comparison were: (1) flow, structure, and placement of the topic; (2) cognitive demand of the task(s); and (3) presence of realistic and relevant use of context. For the first criterion, we examined the *mathematical story* (Dietiker, 2015) of these topics in both textbooks. The second criterion focused on the cognitive demand (high or low) level (Smith & Stein, 1998) of the tasks and problems in each textbook. Additionally, providing relevant and realistic contexts while teaching mathematics is important so that students relate to these contexts and mathematically analyze them (Boaler, 2015). For this third criterion, we examined how each textbook provided relevant and realistic contexts for students.

To analyze, the first author read in detail the topic of Inverse Variation (or Inverse Proportions, as in the NCERT textbook), applying each criterion to examine every statement, question, problem, and worked example in both textbooks. For this examination, a qualitative analysis was conducted to analyze these components, taking into account each criterion.

Based on all three criteria, the CMP textbook had more positive elements than the NCERT textbook. The mathematical story of CMP enables students to reason through important mathematical concepts and relationships, thus helping them understand inverse variation using multiple dimensions, which was lacking in NCERT. Also, CMP had both low and high cognitive demand problems with more high-level, that is, “procedures with connections” (Smith & Stein, 1998) types of problems. However, the NCERT textbook's problems had lower cognitive demand due to the excessive use of scaffolding and demonstration examples. Lastly, both textbooks included a substantial number of problems in relevant contexts.

While both textbooks serve distinct student populations, we believe that students can greatly benefit from rich mathematical learning experiences when the CMP textbook is used (with the incorporation of a few improvements). To provide similar learning experiences, the NCERT textbook must fill in the gaps and make modifications based on the first two criteria. Thus, this curriculum comparison project highlights the offerings of a research-based textbook from the

U.S. and an Indian textbook prescribed by the Indian Educational Board, to their respective students and teachers, as well as the textbooks' offering to each other, which may eventually contribute to the changing future of mathematical learning in diverse social contexts.

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INVESTIGATING ELEMENTARY TEACHERS' PROFESSIONAL OBLIGATIONS WHEN MAKING MATHEMATICS CURRICULAR DECISIONS

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Elementary teachers are held accountable to multiple professional obligations when adopting curricular resources. We investigated how three elementary teachers prioritized and negotiated individual, interpersonal, institutional, and disciplinary obligations when making mathematics curricular decisions. We found teachers were drawn towards individual and interpersonal obligations more often, suggesting they prioritized their students. Meanwhile, their curricular decisions were influenced by school, district, and state policies, suggesting several nuances of institutional obligations. Teachers only occasionally responded to disciplinary obligations. Institutional and individual obligations often conflicted with each other while individual and disciplinary obligations complemented each other. We illuminate the multiple constraints that teachers navigate along with current educational uncertainties.

Keywords: Curriculum, Elementary School Education, Teachers' Professional Obligations

The current teaching context is remarkably uncertain due to flourishing online resources, Artificial Intelligence, and changing student needs specifically since the pandemic. In addition, over the past decade, there has been a significant increase in teachers' access to curricular materials, including virtual resources (Hodge et al., 2019). As a result, many teachers supplement their mandated materials with additional resources (Francom et al., 2021; Kaufman et al., 2018; Polly, 2017) "to fill needs perceived in their contexts, their students, or themselves" (Silver, 2022, p. 459). Our prior studies showed that elementary teachers utilized up to 14 distinct mathematics curricular materials when preparing to teach (Authors, 2022). In creating, selecting, and implementing those resources, teachers consider multiple factors such as state, school, and district curriculum policies (Authors, 2023a; Remillard & Heck, 2014) and instructional contexts such as students' access to curricular resources and students' learning needs (Mutten et al., 2011; Keiser & Lambdin, 1996; Ormond, 2017).

Herbst and Chazan (2011) sorted these multiple factors into categories of professional obligations, labeling four different kinds of obligations: individual, institutional, interpersonal, and disciplinary. Mutton and colleagues (2011) found that novice elementary teachers, facing limited time, often use existing materials instead of creating new ones that might better meet students' needs. This illustrates the tension novice teachers experience between institutional obligations (limited time for preparation) and individual obligations (addressing students' needs) when making curricular resources.

Other researchers have investigated the complexity of teachers' curricular decision-making. For example, when analyzing curricular materials, teachers evaluate various aspects related to students, discipline, institutions, and context, including whether the materials align with standards, provide cohesive mathematics instruction, facilitate student interpretation and performance, promote mathematical understanding, and adapt to classroom context and content (Dingman et al., 2021). These aspects highlight the complexity of teachers' mathematics curriculum decision-making, encompassing factors related to content (discipline), students (learners), institutions, and contexts (e.g., standards). However, less well studied is how elementary teachers prioritize and negotiate these factors when adopting mathematics curricular resources. We used the lens of teachers' professional obligations to understand this underdeveloped area of research, as described below.

Theoretical Framework

To understand the factors influencing elementary teachers' mathematics curricular decisions, we employed Herbst and Chazan's (2011) Teachers' Professional Obligations framework. Herbst and Chazan proposed that teachers' curricular decisions are influenced by four types of obligations: Individual, Disciplinary, Institutional, and Interpersonal. Individual obligations focus on attending to each student's well-being; identities; and behavioral, cognitive, emotional, or social needs. Disciplinary obligations pull teachers toward consideration of accurate and reliable representations of mathematical knowledge, mathematical practices, and mathematical applications. Institutional obligations involve fulfilling duties to institutions, such as adhering to calendars, examinations, and curricula. Chazan and colleagues (2016) also highlighted that teachers are drawn toward multiple obligations associated with institutions. They urged mathematics education researchers to focus on societies and institutional contexts to acknowledge complex and uncertain situations teachers need to work on that are shaped by society and institutional rules. Finally, interpersonal obligations acknowledge the teacher-classroom relationship, especially focusing on how teachers' decisions need to work in settings with limited resources that need to be shared across many students with different needs.

Herbst and Chazan primarily discuss these obligations in the context of secondary mathematics teaching, while elementary teachers arguably negotiate these obligations in different ways. Elementary teachers make curricular decisions for a variety of subject areas (i.e., not just mathematics but also science, social studies, reading, and writing) to fulfill the needs of one group of students. Considering these differences, we interpreted teachers' interpersonal, disciplinary, and institutional obligations specifically for the context of elementary instruction. We acknowledged differentiation of content as the core of interpersonal obligation, which led to the identification of two dimensions of differentiation, teachers may *select different materials* or a*ssign different tasks from the same material* for students with different needs at the shared time in the shared space. For the disciplinary obligation, we labeled what teachers think is disciplinary rather than researchers or mathematics specialists and experts. For instance, teachers may prioritize teaching math facts as disciplinary, considering the importance of foundational math skills regardless of broader mathematical practices. In addition, when attempting to do other types of obligations, some mathematical practices might not be seen in the classroom even though teachers prioritize those practices (Herbst & Chazan, 2020). Lastly, we incorporated the role of standards (e.g., state and common core standards) as institutional obligations as teachers explained selecting curricular resources to meet those standards.

Aligning with the purpose of this framework that "combined with the personal assets (including knowledge, skills, and beliefs) that an individual teacher brings with them to that position and that role, those norms and obligations can help explain teacher action and decision-making" (Herbst & Chazan, 2011; p. 417), we identified how elementary teachers respond to these four types of obligations when creating, selecting, and implementing mathematics curriculum to optimize their students' mathematics learning in the current uncertain context.

Objectives and Research Questions

We investigated how different obligations influenced elementary teachers' mathematics curricular decisions. The following research questions guided our study: (a) Which types of obligations do elementary teachers consider when making mathematics curricular decisions? and (b) How do elementary teachers negotiate decision-making among their multiple obligations?

Methods

Context, Participants, and Data

This study was part of a larger national project on decisions made by elementary teachers regarding their mathematics curriculum. The larger project included a sequence of interviews with individual teachers. For this study, we selected three teachers from the same Midwestern school —Audrey, Jamie, and Kasie (pseudonyms)—due to their diverse teaching experiences across different grade levels. Audrey had six years of elementary teaching experience, with the last four years spent teaching fourth grade. Kasie and Jamie were third- and fourth-grade teachers, respectively, each with thirteen years of teaching experience. We collected data through a three-phase interview— two individual interviews and a focus-group interview. In the first interview, we asked the teachers what curricular resources they used for mathematics teaching and how much control they felt in adopting those resources. In the second interview, they shared how they created cohesion from a variety of resources they selected. In the third focus-group interview, the teachers shared experiences with curricular coherence and collaboration with each other. The data for this study consists of the transcription of the three teachers' first and second interviews. We purposefully omitted the data from the third interview as the teachers focused on sharing their collaborations more and less about their obligations.

Data Analysis

Initially, we coded our data using a deductive coding method (Saldaña, 2016). Using four pre-defined categories of obligations (individual, interpersonal, disciplinary, and institutional), each author coded a set of data independently. We then conducted weekly meetings to resolve our discrepancies. In those meetings, we had extensive discussions about coding units, types of obligations, and relationships among various types of obligations. Below, we share a transcript from a part of our meeting in which three coders collectively engaged in resolving discrepancies.

Table 1: An Excerpt of Research Team's Conversation to Resolve Coding Discrepancies

Coder	Excerpt
1	Why did you put it under institutional?
2	I don't know if it's an obligation to the [school district] or if it's like they have them because they've been provided by the institution, and they like them, so they use them.
3	Yeah, right. And also I think the other thing is like she [the teacher] is saying pretty good. So you are not sure what she is referring to either, right?
2	Okay, and I put it there because it said [school district].
1	Yeah, that was also my question about this one. I think I wanted to capture if they didn't have if the [school district] didn't give them these resources. It's not clear to me that they would use them.
2	Yeah
1	But you think of it in a way that we just talked about another quote: if the [school district] doesn't provide them then the teacher would complain that because the [school district] does not provide that and they have to buy it, which is expensive, so they have to create their own then that make it institutional. What you said just now is in the previous sentence. We do find materials for all the subjects, and I feel like [she was saying] "We have the freedom overall to find what works best with our kids."
2	They're pretty good, meaning, they must align with the most things they must align with standards. But the main priority is their kids.
1	Oh, okay. So, that made it individual.
2	Cool. We agreed on the individual? Just making sure.

In this excerpt, the three authors discussed a coding unit in which two authors had different codes. One author initially coded as an institutional because the teacher discussed the school district-mandated curriculum. However, after an in-depth analysis of the overall meaning of the teacher's description, we concluded that the teacher indicated individual obligation rather than institutional. Our next process involved counting the number of codes for each obligation and identifying patterns. We looked for patterns of how teachers negotiated multiple obligations, including when two or more obligations complemented and/or conflicted with each other.

Findings

We organize this section by our research questions. In the first subsection, we describe the obligations teachers considered when making mathematics curricular decisions. In the second subsection, we share examples to illustrate how teachers navigated multiple obligations.

Obligations Influencing Teachers' Decision-Making

All three teachers mentioned each of the four obligations: individual, interpersonal, institutional, and disciplinary (Table 2). For Audrey and Kasie, obligations to individuals were the largest category with institutional obligations as the next largest category. Jamie equally prioritized individual and institutional obligations, each at just over one-third of the total responses. For all the teachers, disciplinary obligations were a minor factor, at close to 15% of total responses. Finally, interpersonal obligations were consistently the least important.

Table 2: Frequency and Percent of Obligations Across Three Teachers

	Individual	Institutional	Disciplinary	Interpersonal	Total
Audrey	69	46	23	17	155
	44.50%	29.70%	14.80%	11.00%	
Jamie	34	35	14	13	96
	35.40%	36.50%	14.60%	13.50%	
Kasie	72	56	26	20	174
	41.40%	32.20%	14.90%	11.50%	
Total	175	137	63	50	425
	41.20%	32.20%	14.80%	11.80%	

Individual obligations. Teachers shared many different concerns relative to curriculum and their individual students. They mentioned multiple aspects of students' experiences including enjoyment of activity, engagement, reading levels, understanding, background, and anxiety. When talking about enVision Mathematics (the mandated curriculum), Audrey responded:

> I don't think that it's grade-level appropriate, the words are very big. The book, itself, we have two different workbooks, which are probably three to four inches big. So, it's not useful for little kiddos' hands to be pulling pages out. Our fourth graders can't rip the pages out very easily. So, it's just, trying to use the actual resources that are provided for us. Those are, it's just very difficult. And, as a teacher, when I think, I have to teach enVision today, I have, like, that doom feeling overcome with myself, because it's not fun. It's not fun for the kids, they don't get excited... It's, yeah, it's supposed to, kind of, be integrated, you see the same characters all the way through, from kindergarten to fourth grade, the characters will be on each page and they grow with you. And that's about as inviting that book gets, is that there are characters.

The teacher identified multiple aspects of her obligations to students. She reflected upon the level of vocabulary, students' abilities to physically use curriculum resources, student fun and excitement, and storytelling features that persist throughout the curriculum.

Jamie shared she "lik[ed] BrainPOP because they're quick, concise videos that the kids can watch and understand, see visuals of some concepts, and they just don't take a whole lot of time, but they're fun, and the kids enjoy them." Here, Jamie unpacked her reasoning for liking BrainPoP as those resources were comprehensible and engaging to students. Given that these were grade-level teachers, it is not surprising that they attended to students consistently.

Institutional obligations. Audrey, Jamie, and Kasie shared that their curricular decisions were guided by rules and policies set by their school, school district, and state. These obligations appeared in the form of standardized test preparation, content standards, mandated textbooks, and time constraints for mathematics instructions. Interestingly, when responding to multiple institutional obligations, teachers' decision-making was further complexified. For example, Jamie described why she used multiple curriculum resources in her teaching:

> [As a Midwestern State], we are special, and we have our own standards, so sometimes we'll have a standard that doesn't have a lot of materials available. Sometimes even in the

enVision textbook, it doesn't have [a state] standard, and so that forces me, basically, to seek out other materials, or to create my own.

Jamie had a district-provided curricular resource (enVision Mathematics) that she felt obligated to use, yet she also felt the need to move beyond this resource because her state had standards that the provided textbook did not address.

Kasie also shared multiple aspects of institutional obligations, commenting:

> With their updates that they made with the new enVision that our [school district] adopted when we adopted new textbooks, I was like, okay, we're gonna actually give it a real shot with this. …overall, we've been pretty happy with the pacing guide following the enVision book, and, kind of, the order that it has gone.

Here, Kasie explained how the district-mandated curriculum aligned with the district-mandated pacing guide, allowing her to use the curriculum to meet institutional expectations. In part, teachers' curricular decisions were partly influenced by the extent to which the provided curricular resources adequately addressed institutional obligations across different levels. It is interesting to note that Kasie and Jamie taught in the same grade level and within the same instructional context, yet they interpreted mandated curriculum in different ways.

Disciplinary obligations. The teachers only minimally attended to disciplinary obligations, including which resource would better present mathematics concepts and procedures. Jamie emphasized the importance of using specific vocabulary, commenting:

> I just try to make sure to use those correct vocabulary, like the acute angles, right angles, and we talk about how to line up with the vertex, and things like that, because across all of the curriculum, you usually see that same vocabulary being used, so once the kids are familiar with that, it's usually pretty easy to make the connections.

Jamie's curricular decision-making was guided by students' familiarity with certain terminologies as they would appear throughout the curriculum, thus students would make connections with mathematics taught across different topics and curricular resources. This suggests using consistent vocabulary as one way to better teach mathematics as a discipline.

Kasie also shared similar disciplinary obligations when implementing given resources:

> With the perimeter, we've had, most of it has used enough models that the kids are catching on to the concept. We haven't moved more to the abstract yet, but we're getting to that closer now. So, so far, it hasn't been as challenging, but now that we're moving more to the abstract, or where hey have to make their own models, it's gonna become more challenging.

This quote unpacks Kasie's interpretation that teaching concrete ideas and then transiting to abstract concepts is a good approach in mathematics instruction.

Interpersonal obligations. The teachers responded to interpersonal obligations minimally in the form of differentiation needed for specific groups of students. Audrey shared how a specific curricular resource helped her to meet the needs of students who were at different levels:

> We are trying to meet students at all of these different levels. Where, in kindergarten and first grade, you're usually at a kindergarten or a first-grade level. Very rarely do you have a first-grader that can do fourth-grade math. So, I think that's why, two through four, we really do very similar resources. That's how I found out about Super Teacher Worksheets, was our third-grade team expressing to me how much they enjoyed it.

Kasie shared she used "a little bit of IXL for [her] earlier finishers, but most of [her] students did not finish early and didn't get on to IXL. So, most of them just used enVision Math."

Other obligations. Audrey, Jamie, and Kasie also cited contextual and personal factors in their curricular decisions. For instance, they mentioned changes brought about by the COVID-19 pandemic and personal life events like having a baby, indicating that curricular decisions are influenced by factors beyond the four types of obligations we discussed.

Negotiating Multiple Obligations

All three teachers shared they negotiated multiple obligations, sometimes finding them complementary and other times conflicting, with individual obligations consistently either complementing or conflicting with other types of obligations. They supplemented institutionally mandated resources with other materials to engage students who struggled with the mandated resources. This reason for supplementation suggests a conflict between institutional and individual obligations. Audrey responded to this conflict in the following quote:

> The most challenging [thing] is trying to incorporate the enVision series… I did not like their graphing unit. It didn't focus on exactly what our standards wanted. It was a lot of higher-order thinking, things like that, which can be wonderful, but I have still a lot of students who are not at that level—yet. And so, it's very overwhelming, and a lot of my students open that book and already are frustrated, just opening the book.

In this quote, Audrey highlighted that while the institution-mandated curriculum (enVision) could offer opportunities for higher-order thinking, it did not match her students' current mathematical understanding. In addition, she found it challenging to connect with given standards, suggesting another layer of complexity she faced within the same obligation. As such, she was obligated to use enVision math as it was institution-mandated despite its lack of alignment with another institutional obligation (meeting standards).

Teachers also expressed conflicts between institutional and disciplinary obligations, noting they were challenged to balance students' individual needs with institutional time constraints for mathematics instruction. Kasie highlighted this challenge, stating, "I don't enjoy using [Dreambox], and the kids are burnt out on it by third grade." Kasie's comment illustrates the mismatch between the district-mandated curriculum (Dreambox) and the needs of her students. We found several instances in which teachers expressed multiple obligations complemented each other. When asked how she connected multiple resources, Jamie shared:

> When we use the curriculum maps, we have certain standards that have district-wide assessments, and then other standards, we have to choose whether we want to give a summative assessment on them. So, I use the data from those, then, to basically inform my instruction for intervention, which we have every day for 30 minutes.

Here, Jamie expressed that institutional obligations such as using formative assessment results and standards, assisted her in identifying students who required extra support and specific type of support they needed, "infor[ing her] "instruction for intervention" (interpersonal obligation).

The teachers also provided insights into how their selection of supplementary resources was influenced by individual, institutional, and disciplinary obligations, which were at times aligned and at others in conflict. They explained their rationale for incorporating supplementary resources when the mandated ones failed to meet their disciplinary and/or individual obligations. For example, Jamie explained that she opted to supplement her mandated textbook with resources from Pinterest because activities like "Factor Ninja" on Pinterest were more effective for teaching "factors and multiples," and students "[we]re more engaged."

Audrey discussed how four types of obligations—institutional, individual, interpersonal, and disciplinary—intersected when examining the usefulness of a mandated curriculum, DreamBox:

> I disagree with that app [Dreambox], as a whole. I don't think my students benefit from it. The best thing I could say is the higher-order thinking, it does ask questions that really makes our students think, but …it's to a point where they give up, because it's too hard for them. It's supposed to meet them where they are, and it can meet some of my students who are above average, pretty well, but any student that's on grade level or lower, does not meet them.

Audrey used Dreambox due to institutional requirements and it aligned well with her disciplinary obligations by promoting better student engagement and higher-order thinking. Yet, it did not serve well for her students overall because "it's too hard for them," (individual obligations) and did not address the needs of those students "that [are] "on grade level or lower" (interpersonal obligations). Similarly, Kasie mentioned:

> I use [Teachers Pay Teachers and Pinterest] for more engagement because those topics are ones that in the past kids zoned out on, especially elapsed time…So, telling time and doing elapsed time is a hard concept for them to grab onto and they don't do a whole lot of elapsed time in the younger grades. It's more like an hour or half-hour times not down to the minute, which is what they have to do in third grade. So, finding a more engaging way, made it more fun for the kids and I found that my kids did better with elapsed time than they had done in the past without that kind of engaging project-based learning that we found there.

In this quote, Kasie articulated she opted for Teachers Pay Teachers and Pinterest because she could attain individual (for more engagement), institutional ("they have to do in third grade"]), and disciplinary obligations ("project-based learning"). This was an example of how three types of obligations complemented each other during her curricular decisions.

Overall, our findings suggested multiplicated ways in which Audrey, Jamie, and Kasie's mathematics curricular decisions interacted with individual, interpersonal, institutional, and disciplinary obligations. As evidenced by the examples above, institutional obligations often conflicted with individual and interpersonal obligations, whereas disciplinary and individual obligations typically complemented each other. In the next section, we delve into the potential reasons behind these complementary and conflicting obligations.

Discussion and Implications

In this study, we investigated how different obligations influenced elementary teachers' mathematics curricular decisions. We found that elementary teachers primarily prioritized individual and interpersonal obligations, indicating a strong focus on student needs and interactions. In addition, teachers attended to policies over disciplinary and interpersonal aspects, navigating complex layers of institutional obligations at the school, district, and state levels. For instance, teachers faced challenges in fully implementing a district-purchased textbook within the limited time allocated for mathematics instruction.

These findings may be attributed to the demanding schedule and workload of elementary teachers in the United States, who often teach multiple subjects to the same group of students for an entire semester. Consequently, they prioritize materials that engage students effectively and efficiently prepare them for standards within the allotted time. We anticipate these obligations may differ for secondary teachers, who may focus more on disciplinary rather than individual obligations. This assertion is partly supported by Dingman and colleagues' (2021) framework we mentioned earlier, which emphasizes middle school teachers' consideration of mathematics (discipline) in multiple layers. These layers include anticipating how students might view particular mathematical concepts, understanding the contextual meanings of mathematical concepts, and ensuring the cohesive arrangement of mathematics topics. In our study, teachers described how students engaged with mathematics and whether the content presented in given resources aligned with given standards. However, their focus was not on unpacking mathematics as a discipline when making curricular decisions. Another reason for teachers' minimal attention to disciplinary obligations could be because they trusted the curriculum as written. In another study, we found that teachers used textbook content as given without modification because it was research-based (Authors, 2023b). We recommend providing opportunities such as professional development for teachers to deepen their understanding of disciplinary aspects when evaluating curricular resources.

Minimal reference to interpersonal obligations may suggest that teachers believed differentiation was adequately addressed before making curricular decisions. Alternatively, they may have perceived differentiation as less crucial due to their focus on all students or they may have been influenced by the content of the interview questions. Further investigation is necessary to better understand the reasons for the limited instances of interpersonal obligations.

Overall, our study highlights the complexity of elementary teachers' decision-making as they deal with multiple obligations in uncertain educational times. By highlighting multi-layered complexity within and across four types of obligations, our findings contribute to understanding the nuances of teachers' professional obligations as proposed by Herbst and Chazan (2011). Aligning with Herbst and Chazan (2020), we recognize that teachers may have additional obligations beyond those discussed, such as adapting to online learning during the COVID-19 pandemic. Thus, we acknowledge that teachers' obligations are also dynamic and contextual. In our next study, we will perform another layer of analysis, which will further unpack Herbst and Chazan's theoretical framework (2011) on teachers' professional obligations.

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EXPLORING ELEMENTARY TEACHERS' POST-COVID MATHEMATICS CURRICULAR USE: FOCUS ON CURRICULAR REASONING

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Teachers navigate a range of curricular resources to create mathematics learning opportunities for their students. We investigated how three components of teachers' curricular reasoning (i.e., curricular knowledge, curricular vision, and curricular trust) interact with each other when elementary teachers navigate their available resources. Using a collective case-study design, we identified themes related to the complex relationships among elementary teachers' curricular knowledge, vision, and trust at two U.S. schools. Our findings indicated multifaceted ways in which teachers' curricular knowledge, vision, and trust are interconnected with and influence one another. The two teachers highlighted here acknowledged a long curricular "path" or "road," both within and across grade levels, for learning mathematics. We discuss research and practical implications for the importance of understanding teachers' curricular reasoning.

Keywords: Curriculum, Elementary School Education, Instructional Vision, Teacher Knowledge

The proliferation of curricular materials in the last decade has provided opportunities for teachers to supplement their mandated curriculum with a wide range of curricular resources (Polly, 2017). Following the COVID-19 pandemic (hereafter referred to as "pandemic"), online resources flooded the curricular landscape (Francom et al., 2021), which further challenged teachers to navigate a wide pool of mandated and supplemental resources in this new context (Giorgio-Doherty et al., 2021). Prior studies indicated that elementary teachers used up to 14 mathematics curricular materials during and after the pandemic (Doherty et al., 2022). These developments and the context of uncertain times resulting from the pandemic prompted us to continue to investigate teachers' use of curriculum materials given that little is known about how teachers navigate the ever increasingly complex curricular landscape. In this study, we explored elementary teachers' curricular decision-making through the lens of curricular reasoning (e.g., Breyfogle et al., 2010) guided by the following research question: How do the components of teachers' curricular reasoning (i.e., curricular knowledge, vision, and trust) interact with each other as teachers navigate the post-pandemic curricular landscape?

Conceptual Framework

We frame our study using Breyfogle et al.'s (2010) notion of curricular reasoning, which they define as "the thinking processes that teachers engage in as they work with curriculum materials to plan, implement, and reflect on instruction" (p. 308). They proposed three components of curricular reasoning: (a) curricular knowledge (Grossman, 1990; Shulman, 1986), (b) curricular vision (Darling-Hammond et al., 2005), and (c) curricular trust (Drake & Sherin, 2009). Curricular knowledge indicates familiarity with curricular resources, including the curriculum's philosophical perspective and its horizontal and vertical organization. Curricular vision involves knowing the path of learning—where students have been, where they are now, and where they need to go, along with the knowledge of how to get them there. Curricular trust builds upon, and reciprocally interacts with, curricular vision; it is the belief that a curriculum, as written, will satisfy a teacher's vision. Breyfogle and colleagues developed a model illustrating the potential relationships among curricular reasoning, curricular knowledge, curricular vision, and curricular trust (see Breyfogle et al., 2010, p. 310). In our study, we aimed to continue to investigate these relationships in the context of elementary teachers' curricular decision making. The authors also proposed that curricular changes and increased focus on mathematical standards warrant attention to these components of curricular reasoning; post-pandemic, we add the proliferation and increasing accessibility of a wide range of curricular resources to this justification for attention to teachers' curricular reasoning.

Review of Relevant Literature

Many teachers draw from multiple curricular resources that are complex and layered to serve the diverse learning needs of their students (Doherty et al., 2022; Sawyer et al., 2020). During the pandemic, teachers were required to explore alternative resources (e.g., online practice programs) with the goal of providing equitable learning opportunities (Keldgord & Ching, 2022). For example, during the transition from face-to-face to remote modalities, many teachers considered students' access to technology when selecting online resources (Huck & Zhang, 2021). This suggests that the expanding landscape of curricular resources intensified the complexity of teachers' post-pandemic curricular decision-making (e.g., Francom et al., 2021). How teachers navigate such complexity to create equitable student learning opportunities arguably influences and is impacted by their curricular reasoning. The connections and interactions among the components of curricular reasoning proposed in Breyfogle and colleagues' (2010) framework (i.e., curricular knowledge, curricular vision, curricular trust) are complex because, for example, teachers draw on different categories of mathematical knowledge for teaching (e.g., Ball et al., 2008). We unpack the components of curricular reasoning by reporting on two elementary teachers' post-pandemic mathematics curricular decisions.

Methods

Two elementary teachers (Audrey and Mary, pseudonyms) from a midwestern U.S. elementary school participated in this study. We purposefully selected these teachers to provide representation of different grade levels (i.e., third and fourth) and teaching experiences (6 and 11 years). Using a collective case study design (Yin, 2009), we report a collective understanding of how teachers' curricular knowledge, vision, and trust were interconnected within the broader umbrella of curricular reasoning and are reflected in their curricular decisions. We collected data from a three-phase interview process— two individual interviews followed by a focus-group interview. In the first and second interviews, the teachers shared the curricular resources they used, their level of control in using them, and strategies they used to establish coherence among the selected materials. In the focus-group interview, the teachers reported how they adapted their curricular materials and collaborated with one another. We drew both collective and case-specific meanings from these individual and focus-group interviews. We transcribed all interview data and used a top-down and bottom-up interactive data analysis (Chi, 1997) to explore the complex relationships between teachers' curricular knowledge, vision, and trust.

Findings

Audrey and Mary's curriculum knowledge was evidenced when they shared their level of trust for particular curricular resources and their vision about curricular coherence and student learning opportunities. In using their curricular knowledge, Audrey and Mary were purposeful in which resources they selected and how they implemented them to create students' mathematics learning opportunities, which ultimately influenced their curricular trust. Audrey mentioned that enVision (their primary curricular resource) "didn't focus on exactly what [their] standards wanted," requiring her to supplement with other resources. This finding indicated Audrey's curricular knowledge influenced her curricular trust. As such, the teacher did not fully trust enVision because she had knowledge of the alignment of curricular resources with standards.

Audrey and Mary also shared nuances suggesting how curricular vision and trust interact with each other. Mary mentioned that "having [a] set routine in [her] schedule helps kids make connections in math every day, because they build off of each other." She also shared how she established the curricular map or routine in using enVision, stating, "the first day, [students] have a question [to] test their prior knowledge. Then, you read the question together as a class, and then give the students time to solve it however they choose…then we watch a video." In this quote, Mary's curricular knowledge of enVision is reflected through her vision about how curricular materials should be organized (e.g., checking on students' prior knowledge, and presenting new materials).

Interestingly, both teachers mentioned that after the pandemic they began trusting teacher-created resources from Teachers Pay Teachers more than some of the other available supplemental resources. Mary shared that she often used Teachers Pay Teachers because she "could also look at ratings that other teachers had given to the product to see if a lot of teachers liked it that [she is] more likely to look more closely at it." This quote indicated that Mary's initial curricular trust was influenced by other teachers' curricular use. We also found that the teachers' curricular trust interacted with their vision related to creating appropriate student learning opportunities. For example, Audrey mentioned, "I'll pick a worksheet that is easier to read than enVision; enVison is very hard to do independently, where the Super Teacher Worksheet is very basic."

Overall, our findings align with Breyfogle et al. (2010), in that teachers' curricular knowledge is foundational to their curricular reasoning, which generates their curricular vision, which influences their curricular trust. However, we acknowledge that the relationship among curricular knowledge, vision, and trust is not straightforward and other dimensions of the relationships are being explored in our ongoing studies. As an example of such complexity, we found that there are contextual and teacher-related factors that influence teachers' curricular reasoning. Audrey mentioned, "I and [my colleague] are both moms of young children, and so it [enVision] is easier for us as far as planning and getting materials ready, because it's all already created." This quote indicates that teachers' curricular reasoning is also influenced by their personal circumstances and time constraints.

Scholarly Significance

The current uncertain times in mathematics education impact all contexts in which our work takes place, and our collective effort is needed to support teachers in mathematics classrooms as expectations of them continue to change and increase on many different levels (i.e., local, state, federal) and from many different stakeholders (e.g., school administration, parents). Supporting their curricular decision making by better understanding the curricular context in which they are operating and the relationships among the components of curricular reasoning offers one such effort. A focus on curricular vision in which we strive for equitable and engaging mathematics experiences for all children requires attention to teachers' curricular knowledge and trust. Our findings suggest multifaceted ways in which teachers' curricular knowledge, curricular vision, and curricular trust are interconnected with and influence one another. This investigation of teachers' curricular decision making is significant for mathematics teacher educators, policy makers, teachers, and curricular writers/developers. Understanding teachers' curricular reasoning in relation to knowledge, vision, and trust contributes to knowledge needed to purposefully develop, adapt, and implement mathematics curricular materials in ways that are effective for creating equitable learning environments. Moreover, documenting and disseminating teachers' curricular reasoning and noticing is valuable to acknowledge the extensive time and intellectual effort that teachers devote to mathematics teaching.

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USING HALLUCINATION TO ENVISION THE FUTURE OF MATHEMATICS EDUCATION

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Consistent with this year's conference theme of envisioning the future of mathematics education in uncertain times, and as an extension of recent research presentations related to the implementation of pattern-based assessments in individual classrooms and at scale (Stroup, 2020), this paper will contrast the responses of elementary and middle school students from a statewide implementation of the pattern-based items (PBIs) with the responses from two AIs to the same questions, and where the AIs hallucinate in ways that result in their performance being generally less robust than that of the students. The kinds of questions the AIs are less good at may be seen to point us toward what learners are better at and, consequently, could serve to refocus schools-based mathematics education on advancing meaningful student agency centered on uniquely human capacities for engaging structural patterns, exploring possibility, and modeling.

Keywords: Modeling, Learning Theory, Curriculum, Assessment

This paper developed from reflections on practice-focused discussions with in-service elementary and secondary teachers related to the role generative AIs could have in their teaching. The discussions took place as part of courses with titles like "Algebra for Teachers" or "Geometry for Teachers" included as part of degree programs at a public university. Some of the teachers were already experienced users of generative AIs, especially for tasks like writing short multiple-choice assessments related to a specific state standard, helping to draft the outlines of lesson plans, or assisting in managing their workload relative to authoring reports where "individual education plans" (IEPs) for students were frequently mentioned as an example. For other teachers, they had never "touched" a generative AI prior to enrolling in the course. The introduction of pattern-based questions (PBQs) was new for nearly all the students (some had an instructor for another course who used PBQs in their in-service courses). As part of considering various approaches to supporting classroom-based assessment, participants read a *Report* (Stroup, 2020) analyzing the results of implementing pattern-based questions statewide at scale (N > 400,000). Especially since some of the teachers had used the AIs to generate quizzes to be implemented with their students, the question arose: How would the AIs do on the same pattern-based questions used statewide with elementary and middle-school students? This paper emerges from our efforts to engage this question.

Although not part of the initial conversations with the teachers, the ability of generative AIs to outperform the corresponding late-secondary or adult test takers on many widely administered high-stakes tests (OpenAI, 2023) can serve as a contrasting case with the hallucinations and comparative low scores, relative to elementary and middle school students, the same AIs generated for pattern-based items. The contrast was brought into the conversation as part of realizing that the hallucinated responses from the AIs could be used by the teachers to "both" teach math, by having the students "correct the AI", and teach about generative AIs, by discussing how the "artificial" intelligence would not be able to see any of its specific responses as needing correction. The same probability-driven algorithms that might generate responses that are treated as correct – and that might even be submitted, in other contexts, by students as part of completing homework assignments – could also generate responses to the pattern-based mathematics items that were clearly "wrong" or, at a minimum, in need of correction or editing. On their own, the AIs have no way of determining the differences between one of their largely accurate or correct responses from responses that school-aged students would be able to identify as hallucinations.

New questions to be discussed in relation to specific examples arise. What allows us to recognize mathematics-related hallucinations when each AI that generates the hallucination can't? How might the better results from many school-aged learners on certain types of questions – e.g., ones engaging with pattern, possibility, and modeling – provide an opportunity to move beyond the fear that AIs are better than us, as reinforced by their doing better on the traditional assessment items we currently use to evaluate the effectiveness of education, to focus on what we're better at. AIs, critiqued within a reflective frame provoked by our ability to identify hallucinations, can serve as tools in affirming human agency and might well advance our efforts to support meaningful forms of equitable learning outcomes for mathematics education.

What matters about a hallucination? - Revisiting Chomsky's Green Ideas

As a pioneer in the development of formal linguistics, Noam Chomsky began his career in the late 1950's and 60's by critiquing the sense that the word "grammar" might only denote sets of conventions within a given language (e.g., subject-verb agreement in English) and positioned his "transformational generative grammar" in opposition to probability-based, often behaviorist situated, models of language production, as well as language acquisition. Given this, it is not surprising that, more recently, Chomsky has engaged in a vigorous critique of the over-attribution of significance to the capabilities of generative AIs. "[H]yperbolic headlines" notwithstanding, AIs based on large language models (LLMs) are, for Chomsky and his co-authors, "lumbering statistical engines" that "become increasingly proficient at generating statistically probable outputs" that can, at best, appear "human like" (Chomsky, Roberts & Watumull, 2023). These probability-focused models contrast with Chomsky's longstanding commitment to "linguistic structure" (1956, p. 116) or what Howard Gardner and colleagues summarize as a "mathematically oriented analysis of language" (Gardner, Kornhaber, & Wake, 1996, p. 127).

The shared use of the word "generative" as part of the name for Chomsky's theory and as part of the name for LLM AIs is significant for this paper. For Chomsky it's not just that we can produce, or generate, novel sentences, any one of which might be seen as unlikely to happen by chance (or to be improbable in and of itself), but also that we can recognize a distinction between a phrase that is grammatical and meaningless and a phrase that is ungrammatical and meaningless.

Generative AIs, when they hallucinate, don't generally produce the latter. Instead, hallucinations are text that is grammatical but also incorrect or wrong in what it means. Chomsky's best-known example of a grammatical but meaningless phrase is "colorless green ideas sleep furiously" (1956, p. 116). It matters both that Chomsky *intentionally* created this expression to illustrate aspects of his larger analysis of how grammar works *and* that "an English speaker" would be able to "recognize" this expression "is grammatical" and meaningless (p.116).

Generative AIs, based on existing LLMs, can do neither. These AIs can neither intend to generate a meaningless sentence nor can they, on their own, recognize when they have produced what might be seen by us as a semantic trainwreck. The fact that we *are* able to do both may begin to provide us with a way to think about what can, and possibly should, be the focus of schools-based mathematics education. Specific kinds of mathematical tasks or prompts where AIs tend to produce hallucinations may provide a way for us to begin to recognize, and thereby begin to refocus teaching and learning on, what we, including our students, are "good at."

Methodologically, we need not embrace Chomsky's parsing of linguistic competence into categories of syntax and semantics to see how what we are uniquely able to do should be foregrounded. Moreover, most post-structural analyses and certainly much of equity-advancing critical theory allow us to reject any discussion of activities as vital to our sense of who we are as the production of natural language or the development of mathematical agency "really" work, by starting from a prior assumption that there must be a privileged or ultimately privileging point of view along the lines of a what Chomsky posited as a "universal" grammar.

Avoiding any sense that a neutral, object-ifying, or outside-of-language type place to stand has to exist, we are free to make visible in our analyses of the ways AIs might be deployed, or become part of institutionalized practice in schools, how it is that issues of power, access, and identity are always already present, certainly, in the data used to train these AIs but also in terms of *the who* and *the how* of critiques of the output.

Who gets to say an output isn't right or is a hallucination? Will school-aged students be allowed, or even encouraged, to make such judgments? Or is the possibility that AIs might be wrong to be hidden in ways that will have the effect of helping to preserve privileging, universalizing, and plausibly dehumanizing notions of schools-based curricula and assessment (sometimes advanced as part of what could well have been sincere commitments to improving equitable outcomes or, at the very least, not to leave any child behind)? Are teachers to be empowered to make judgments about how AIs might support in-class, and potentially group-mediated, engagement with mathematics, or will AI-augmented tutoring environments be used to further constrain instruction to be about only the kinds of routines the AIs are able to support? The relatively brief considerations and comparisons that follow are meant only as an invitation for us to envision a future for mathematics education that is more, not less, fully humanized.

Speculation on why pattern-based items might be more likely to cause hallucinations

Although the pattern-based items that are presented in what follows may appear to be similar to standard multi-select assessment items, they are distinct in how they are meant to function, and this may (or may not) have something to do with how the generative AIs tend to hallucinate in attempting to respond to the examples that follow.

Most high-stakes assessment items are developed with the goal of being able to produce a scale score related to a relatively stable (predictable) latent trait. Considerations of whether or not these assessments are effective in measuring what is intended would take us well beyond the scope of this paper. What does matter is that training AIs relative to an assessment paradigm centered on convergence to a latent trait will play to the sweet spot of what generative AIs do well. Stable traits, no matter what they may actually represent, are what make probabilities-driven, large language models capable of doing what they do, including, in this case, endorsing the most likely to be correct response, or set of responses, to latent-trait-focused items.

In contrast, and as the name suggests, pattern-based items are developed, as will be seen in what follows, to use the full combinatoric space of possible selections to four-response (A, B, C, D) items (15 possible combinations if "no response" is not included). The modes appearing in the data (see below and Stroup, 2020) are meant to relate to patterns in how learners might understand or engage with a particular topic. Teachers, including those in the classes mentioned earlier, have found attending to patterns much more informative and useful to them professionally than attempting to make instructional decisions either based on sets of averages or based on difficult-to-interpret scale scores. Their situated knowledge about meaningful patterns in student reasoning also allows them to develop effective pattern-based items for use in their own classrooms.

Although teachers have asked that the modes in responses (e.g., frequencies of AD or BC) be projected onto, and sorted relative to, (one of many possible) a partial credit score (similar, in this case, to the scoring of a True/False assessment), what matters most for pattern-based items is the relative presence of specific modes in the data. The fact that the frequencies (or percent presence) associated with specific modes ("bumps" in the data) are meant to, and do, shift based on changes in students' understandings, makes them distinct from traditional items centers on preserving stability in the ordered probabilities for specific students getting items correct.

Hallucination 1: Seeing Possibility

The item shown in Figure 1 was developed to assess the fifth-grade Texas Essential Knowledge and Skills standard focused on students being able to "classify two-dimensional figures" as this would include reasoning in terms of "the presence or absence of parallel or perpendicular lines, or the presence or absence of angles of a specified size" (TEKS 5.5A). A similar focus at the fifth-grade level can be found in the Common Core State Standards (*cf.*, CCSS Math Content 5.G.A.1).

Rather than ask about a given shape, the question shown in Figure 1 centers on "which is possible" as a description of geometric shapes. In their critique of generative AIs, Chomsky and colleagues make the observations that their "deepest flaw" is the "absence" of what they see as the "most critical capacity of any intelligence: to say not only what is the case ... but also what is not the case" and, indeed in a logical sense related to this specific item, "what could and could not be the case" (Chomsky, Roberts, & Watumull, 2023).

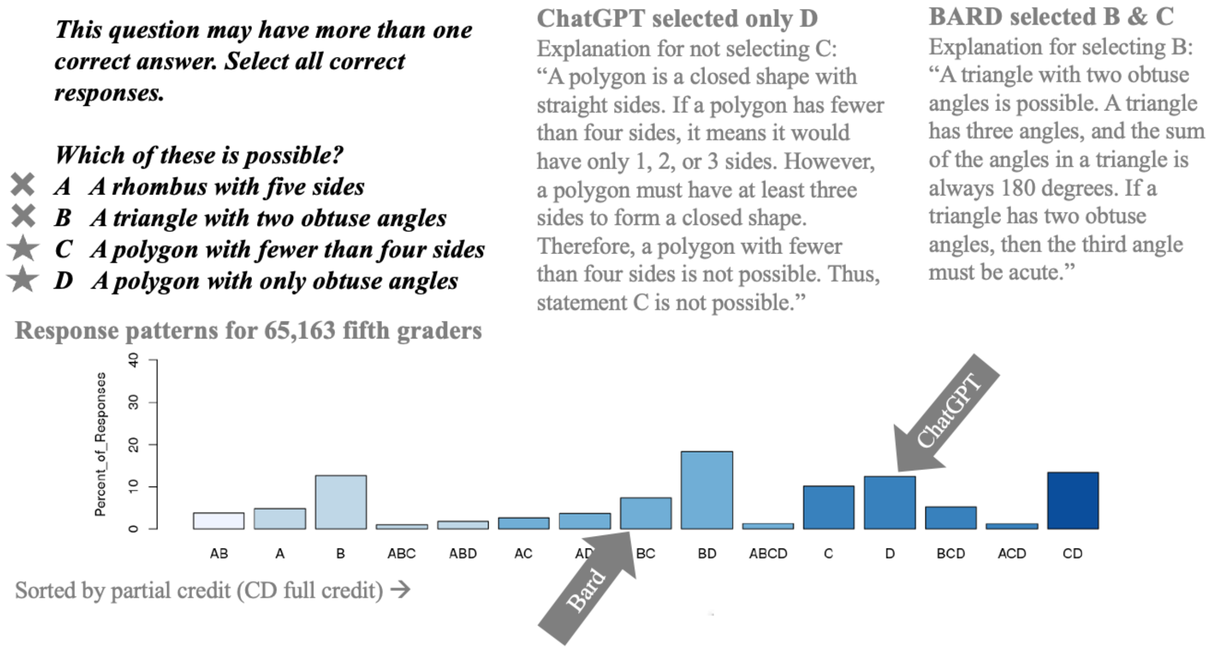

Figure 1: Hallucinations of two AIs and the responses of 65,163 fifth graders

The probabilities-based predictive patterning used by the AIs need not be seen as the same as the structural reasoning about patterns we can use to decide what's possible. Worth noting, the ability to "look for and make use of structure" is listed as one of the Standards for Mathematical Practice in the CCSS (2023).

As discussed previously, the sentences from the AIs included in Figure 1 are grammatical. What is said in some of these sentences, however, can be judged to be false. The output of ChatGPT allows that a polygon with "fewer than four sides" could have "only 1, 2, or 3 sides" but doesn't recognize that a figure with three sides would be a polygon, and consequently concludes a polygon with fewer than four sides, response C, is not possible. ChatGPT selects only D. The assigned (or projected) partial credit, using a F/T-type matching partial credit model, would be 0.75.

Bard (with a successor now called "Gemini") outputs the grammatical but incorrect statement that "A triangle with two obtuse angles is possible." Interestingly, the AI then goes on to conclude that, "the third angle" on this impossible triangle "must be acute." The partial credit for selecting only D would be 0.5, putting it below the overall student results of 0.57 as partial credit and well below the 13% of the students who would receive full credit for selecting CD.

Although the AIs themselves would not, at the time this was done, be able to identify the elements in the hallucinations that are like Chomsky's "green ideas," it is certainly reasonable to expect that fifth-grade students would be able to engage with, and even enjoy (according to the one teacher who reported trying it out in their class), finding the "mistakes" of the AI. The AIs are certainly powerful in what they do, but our ability to think about what's possible, and not just what is inductively probable, may exceed what they can do.

Focusing mathematics learning and teaching on explorations of what's possible, especially when contrasted with the hallucinations of the AIs, might well point us toward greater emphasis on what we, including our students, are good at. Seeing the possible can also be seen as related to modeling, the next contrasting case to be considered.

Hallucination 2: Modeling

The ability to "model with mathematics" as associated with applying the mathematics students know "to solve problems arising in everyday life" is listed, like the ability to look for structure, as one of the "Standards for Mathematical Practice" outlined in the CCSS (2023). Attending to modeling has been a major area of research in mathematics education (*cf.* Lesh & Doerr, 2003) and a particular area of focus for a working group of PME-NA. To the extent that mathematics can be used to develop a "causal explanation" in a way that Chomsky and his colleagues do not see as true of the AIs, mathematical modeling reaches well beyond the mathematics classroom. Moreover, modeling has been a throughline for human cultures for millennia as a way to engage the harmonies and even the dissonances associated with a wide range of arts.

These accounts, of course, can stand on their own as reasons for highlighting modeling. But the hallucinations of AIs relative to simple forms of modeling that are readily managed by many fourth graders might serve to remind us of the sense in which this is something we can do well. This is also something we lose track of if we allow what AIs can do only on their own, or as part of updating tutoring platforms, to serve as a kind of upper bound on what mathematics education should emphasize. Modeling is creative in a way that may be seen as linked to the previous discussion of seeing possibilities.

"Use Mathematical Models" as it is specifically referenced in the Texas fourth-grade mathematics standards (TEKS 4.1.A-D), served as the basis for the development of the pattern-based item shown in Figure 2. Students are asked which equation "can be used" to describe the number and exchanges of cards. Bard correctly notes relative to this item that after "giving away 6 cards" Maria will have "84 – 6 = 78 cards" but then goes on to immediately conclude that response B, "2 x 42 + 6" is a "correct equation" for this exchange (using + 6 instead of – 6 for "giving away" cards). Both A and D are rejected leaving only C, which does, as a response, closely align with the account of "giving away 6 cards" from 84. 0.25 would be assigned as partial credit.

Then, like Bard, ChatGTP also suggests that "multiplying 2 by 42" and then "adding 6" takes into account "the fact that Maria starts off with twice as many cards as Mike" and "subtracts" the cards she gave away. The operation shown is adding six, but it is described as subtracting the cards she "gave away." ChatGPT would also be assigned 0.25 as partial credit.

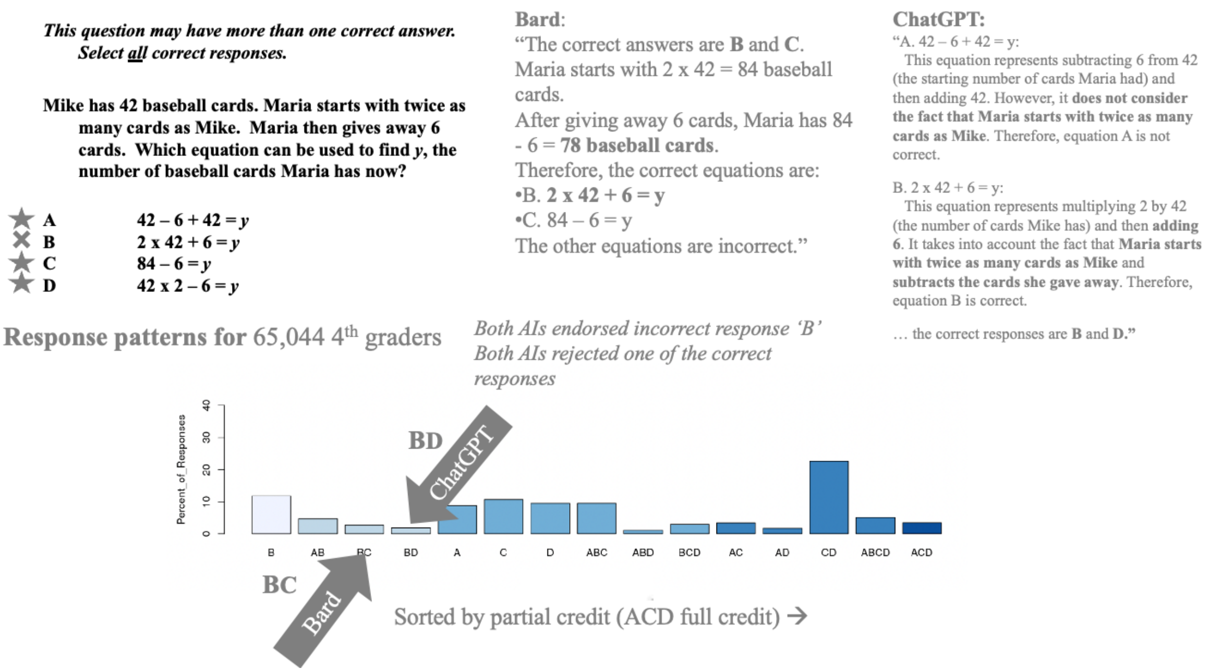

Figure 2: Hallucinations for a modeling item

As with the previous example, the AIs themselves would not be able to recognize the significant inconsistency in, as an example, representing subtracting 6 with "+ 6." It also seems highly likely, especially given the comparatively strong responses of 65,044 students, that fourth graders, as part of a classroom-situated discussion, would be able to recognize the inconsistency in the AIs' hallucinations. Both "the math" of modeling and an understanding of aspects of AIs would be advanced through the critique of the AIs' responses.

Hallucination 3: Reasoning Beyond Rote Procedures

While part of the state standard for "understanding" systems of linear equations does include how the "solutions" to systems of two linear equations in two variables "represent points of intersection of the graphs of the equations" and "the meaning of the solution(s) in the context of the problem" can model " real-world contexts" that can be used to "assess the reasonableness" of solutions themselves (TEKS 8.8.A-D), the actual practice of solving systems, as revealed in part by the hallucinations of the AIs in Figure 3, tends to center on rote sequences of steps associated with phrases like "do the same thing to both sides" or "isolate the x." Any links to modeling and/or to valid steps for solving equations that don't align with the well-rehearsed sequences are, most often, excluded from student sense-making.

Whether or not the "first step" of the rehearsed sequences for solving begins with the constants or begins with terms involving x in moving toward what can be viewed as isolating x on one side of the equal sign, the possibility of "dividing both sides" by a factor that is common to the terms on either side of the equal sign is not considered viable (response A). The reason would tend to be that no matter how one starts in adding or subtracting from both sides, dividing is typically treated as (almost necessarily) the last step.

The responses of the AIs (Fig. 3) reflect the more frequently occurring – the more procedurally probable – sequences for solving found in the data used for training the AIs. Consistent with the AIs' difficulty in reasoning, in what was called a structural sense discussed earlier, about what's "possible," both AIs hallucinated in attempting to evaluate which of the responses "could" be used as a "first step" in solving "the given equation."

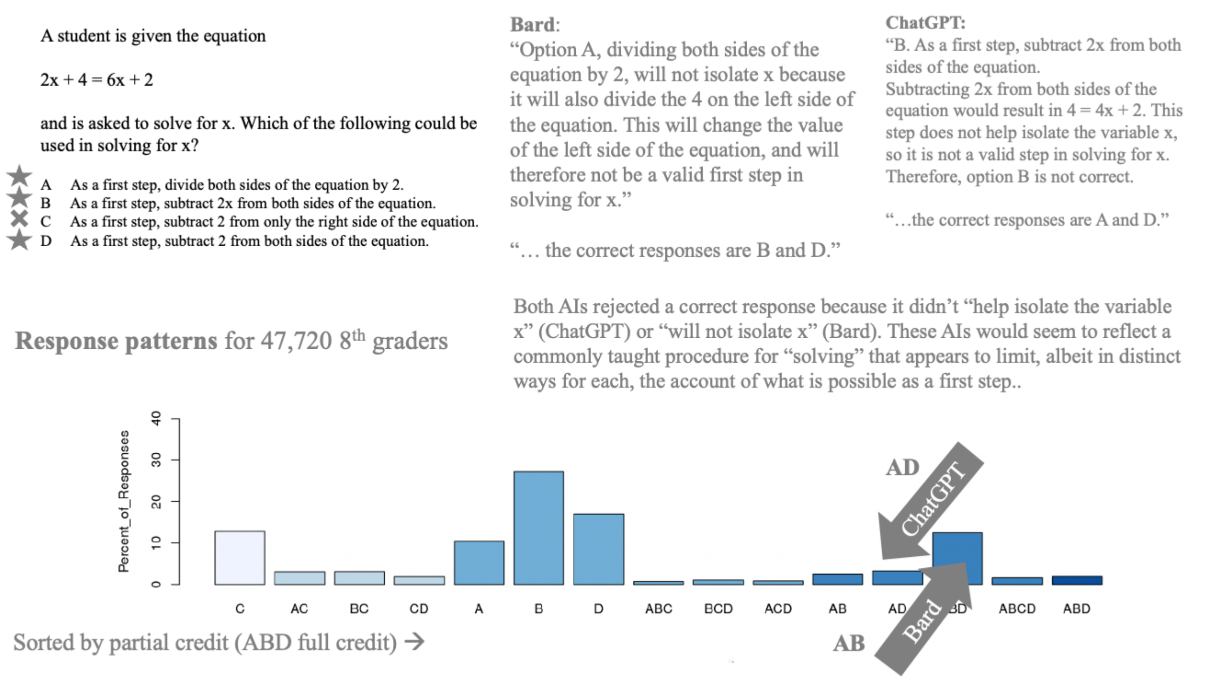

Figure 3: Hallucinating and routines for solving equations

Perhaps reflective of students' training that there needs to be consistency – to the point of near exclusivity – in carrying out a first step, most students chose only one response or the BD response, the response allowing that adding or subtracting from both sides can involve either constant terms or terms with x as a first step. The fact that AIs, in terms of a partial credit projection, did do better than the students on this item, distracts from the sense in which their respective responses used a version of "not isolating x" as a rationale for excluding one of the valid first steps.

Interestingly, the students may want to come to the defense of the AIs because the AIs would be doing something close to what they were taught to do. They might see the hallucination as understandable and may not even want to treat the responses summarized above as hallucinations. We've even had teachers in previous versions of the courses mentioned earlier say this question is "unfair" because starting with division, while certainly valid as a first step, would not be consistent with what the students had (likely) been taught.

Conclusion: Returning to how we can know questions

The hallucinations in the last example might be seen to bring us back to a sense meant to be illustrated across all the examples provided: that the hallucinations of generative AIs can serve both to foreground classroom-situated engagement with mathematics as part of students correcting the AIs and, especially as suggested in the last example, to raise subtle questions related to how the AIs function. It is one thing to say that an AI doesn't know when it is hallucinating. It is a more interesting question to ask if and how we can know. What we might hope for is that activities more fully consistent with our own abilities to do, and create, mathematics will play a central role in engaging these sorts of "How we can know?"-type questions... for the AIs, for ourselves, and, even more importantly, for the futures of our students.

Postscript for the reviewers

Given that some of the issues raised by the thoughtful reviewers may be similar to those raised by our colleagues, here are some very brief follow-on thoughts and pointers to other resources. In addition to the examples provided above, more detailed accounts of pattern-based items, both as implemented (Stroup, 2020; Stroup *et al*., 2023) and as well their connections efforts to develop "constructivist statistics" (Stroup, 1996; Stroup & Wilensky, 2000), are referenced. An account of what is meant by the use of "hallucination" is provided in the text in a way that is largely consistent with normative AI-referencing definitions. For this work, however, the more central questions include *who* decides what is/isn't a hallucination (the generative AIs can't) and, relative mathematics learning, *how* are we, including our students, able to decide. Finally, it is accurate to note that "generative AI" is only one of an array of kinds of AIs, some of which have also been deployed in educational contexts. Given a reviewer's request for more of a theoretical framing, it is worth briefly noting there are deep and longstanding interactions between the application of ideas in (what has come to be called) artificial intelligence (often overlapping with aspects of cybernetics) research to learning-related accounts of the emergence of "intelligence," where the role of creativity is implicated (*cf.* Piaget's [1970, pp. 81-82] discussion of Chomsky's emphasis), as well as structure (including *recursion*, central to Chomsky's transformational grammar as well as to how [perhaps ironically] the generative AIs work, to say nothing of its importance to Seymour Papert's development of LOGO as a LISP-like list processing language that only later acquired a "turtle"), as well as (child) "development" (e.g., Papert [1980, p. 160] notes, "the theory of mother structures {as discussed by Bourbaki} is a theory of learning"). Situated in relation to issues of access and equity – relative to *what* will be seen as mattering in mathematics education, *for whom*, and *how* it will matter – the emergence of generative AIs may provide an impetus for us to re-engage with some aspects of our own *generative* roots.

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AN EXAMINATION OF JAPANESE CURRICULUM MATERIALS THROUGH QUANTITATIVE REASONING: THE TREATMENT OF PROPORTIONAL RELATIONSHIPS

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This study investigated how Japanese curriculum materials represent proportional relationships through the lens of quantitative reasoning. We examined the tasks, questions, and representations in the Japanese elementary and lower secondary level course of study, teachers' guide, and Mathematics International textbook series. Findings showed that proportional relationships in the selected Japanese curriculum materials are intertwined with quantitative reasoning and covariation of quantities. Findings also showed that, starting from 5th grade, perspectives of both multiple-batches and variable-parts have been taken into consideration together with utilizing partitive and quotative division in Japanese curriculum materials. We discuss the implications of findings for teaching and learning of proportional reasoning.

Keywords: curriculum, quantitative reasoning, covariational reasoning, proportional reasoning.

Background

Textbooks significantly influence classroom activities (Valverde et al., 2002) by building a pathway "… between the intentions of the designers of curriculum policy and the teachers that provide instruction in classrooms" (Valverde et al., 2002, p.2). Therefore, examining textbooks provides some possible insights about opportunities for students to learn mathematics (e.g., Son & Senk, 2010) and teacher's teaching and learning (e.g., Son & Kim, 2015).

The treatment of proportional relationships in textbooks is crucial to study. Proportional reasoning, involving core concepts of ratio, rate, and proportion, developed in elementary and middle school is essential for secondary and collegiate mathematics (Izsak & Jacobson, 2017; NCTM, 1989), supporting students' multiplicative reasoning (Lobato & Ellis, 2010; Simon & Placa, 2012). However, students (Modestou & Gagatsis, 2007) and teachers (Pitta-Pantazi & Christou, 2011) often face challenges with proportional relationships. Research shows teachers struggle with understanding the covariation of quantities in proportional relationships (Orrill & Brown, 2012) and differentiating between directly and inversely proportional situations, as well as non-proportional scenarios (Arıcan, 2019). Preservice teachers who struggle to discern different meanings of division encounter difficulties in distinguishing perspectives on ratios and establishing proportional relationships (Ölmez, 2021).

Beckmann & Izsak (2015) introduced multiple-batches and variable-parts perspectives, to expand research on multiplication, division, and proportional relationships. Multiple-batches perspective views 2 apples and 5 oranges as one batch and the covariation between the two quantities as multiples of the original batch. A person having multiple-batches perspective can think of ratio of 2 apples and 5 oranges and use partitive division to find how many apples per one orange or vice versa. This person can also think of the ratio, 2 apples and 12 apples, and use quotative division to find out that six batches are needed for the mixture. Thus, the person engages in partitive division of the quantities in within ratios and quotitive division of the quantities in between ratios (Karagoz Akar, 2010; Noelting, 1980). Whereas variable-parts perspective views 2-to-5 ratio as fixed and the size of each part could vary. A person having variable-parts perspective can think of the ratio, 2 apples and 5 oranges, and use quotative division to determine that 5/2 as much oranges as apples or 2/5 as much apples as oranges are needed for the mixture. This person can also think of the ratio, 2 apples and 12 apples, and use partitive division of 12 with 2 to determine that each part contains 6 apples since the size of the parts can vary. Thus, the person engages in quotitive division of the quantities in within ratios and partitive division of the quantities in between ratios (Karagoz Akar, 2010; Noelting, 1980a). Researchers emphasized the importance of both ratio perspectives for understanding proportional reasoning (Arıcan, 2019; Beckmann & Izsak, 2015). The multiple-batches perspective has garnered attention in mathematics education (Lobato & Ellis, 2010), whereas the variable-parts perspective is less explored (Beckmann & Kulow, 2018).

Taking into account of Thompson and Carlson (2017)'s argument that Japanese curriculum materials have potential to study how quantitative reasoning can be integrated in curriculum standards and textbooks, we investigated "What potential do Japanese curriculum materials in Grades 5 to 7 have in promoting proportional reasoning through quantitative reasoning?"

Conceptual Framework

Quantitative reasoning is "the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships" (Thompson, 1990, p. 12). A quantity is a measurable quality of an object coming into being with a person's conception of a situation by considering the measurable quality of an object (Thompson, 1994). For instance, if a person using 2 apples and 5 oranges to get a special taste of a juice wants to get more juice of the same taste, the person might use the quantities, 12 apples and 30 oranges. The change in the number of apples requiring a change in the numbers of oranges for getting the same taste, which is a simultaneous "bidirectional relationship" change, is called covariation (Thompson & Thompson, 1996). When we consider the taste itself and the measure of it, the multiplicative relationship between apples and oranges remains constant (2:5 and 12:30) and it is called invariance. In this regard, "the result of comparing two quantities multiplicatively" is called a ratio (Thompson, 1994, p.190) and "a reflectively abstracted constant ratio" is rate (Thompson, 1994, p.18). This definition indicates that a rate is a linear function in the form of $f(x) = mx$ (Thompson, 1994). Rate refers to the set of equal ratios (Lobato *et al.*, 2010). Proportional reasoning is the reasoning in situations involving invariant relationships between two covarying quantities (Lamon, 2012).

In juxtaposing Beckmann & Izsak's (2015) perspectives on proportional relationships with quantitative reasoning, a person can understand and use both partitive and quotitive division for quantities in within ratios (e.g., 2 apples and 5 oranges) and between ratios (e.g., 2 apples and 12 apples). It allows the person to recognize quantitative relationships as per-one and as a scale factor. The person can conceive an image of per-one and scale factor relationship operated on simultaneously covarying quantities both in within and between ratio situations regardless of the division type. Thus, a person considers the quantities through quantitative relationships constituting an intensive quantity, i.e., a measurable quality of object (Thompson, 1994).

Method

In this study, we examined one of the six most widely used textbook series in elementary mathematics in Japan (Watanabe et al., 2017) called Mathematics International (MI), along with Japanese course of study (COS) (Takahashi *et al.*, 2008), and the teachers' guides (Isoda, 2010). The publisher of the MI textbooks, Tokyo Shoseki, collaborated with Global Educational Resources in 2011 and published MI in English in 2012. We used the English translated version of MI textbooks in this study. Utilizing the content analysis method (Krippendorff, 2018), we analyzed how quantities were introduced and how the relationships between them were promoted in the statements, how proportion and ratio is defined and presented in the tasks, questions, problem situations and representations in the course of study (COS), teachers' guide (TG), and the relevant units from the textbooks.

Findings

Findings showed that Japanese curriculum materials emphasize both multiple-batches and variable-parts perspectives on ratios with a focus on both partitive and quotative division and build on direct and inverse proportional relationships with a focus on covariation of quantities. In lieu of word limit, we present examples from 5th grade and 7th grade.

In 5th grade, ratio is mentioned for the first time in Quantities and Measurements domain of the COS with the objective stated as "Students will understand the average of measured quantities and the ratio of two unlike quantities." (Takahashi *et al.*, 2008, p. 13). A task about rabbit cages is presented in 5th grade MI textbook to investigate population density (*See* Figure1).

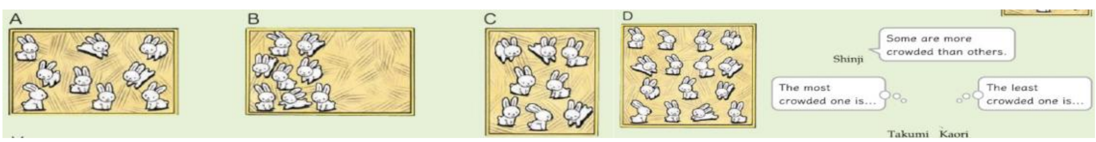

Figure 1. Rabbit cage (Fujii & Iitaka, 2012, Grade 5, p. A93).

Students are asked to decide what quantities are needed to compare crowdedness of cages at first. Then, a table involving area of the cages and number of rabbits in each cage is given (*See* Figure 2) and three different student ideas on crowdedness are modeled (*See* Figure 3).

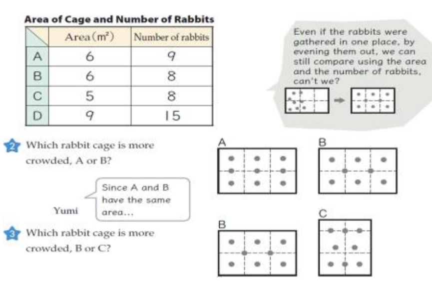

Figure 2. Table of crowdedness (Fujii & Iitaka, 2012, Grade 5, p. A94).

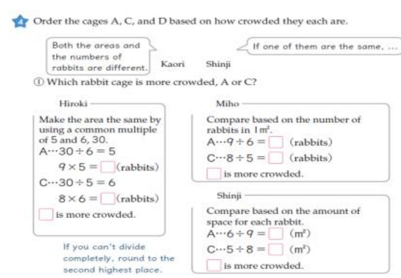

Figure 3. Ideas of students (Fujii & Iitaka, 2012, Grade 5, p. A94).

In this task, the multiple-batches perspective seemed to be used by Hiroki, Miho and Shinji. Hiroki seemed to use quotative division, while Miho and Shinji use partitive division. Notably, thinking of 6 m^2 with 9 rabbits and 5 m^2 with 8 rabbits as associated extensive quantities in groups, one might think of how many 6 there are in 30 and how many 5 there are in 30. Finding out the results as 5 and 6 respectively and thinking that the batches need to be kept the same for the crowdedness to stay the same, one can determine that she needs to multiply 5 with 9 and 6 with 8, resulting in 45 rabbits and 48 rabbits. Miho seems to think partitive division that for the associated group of 6 m^2 with 9 rabbits, for 1 m^2, there are 1,5 rabbits; and for the associated group of 5 m^2 with 8 rabbits, for 1 m^2 there are 1,6 rabbits. These two division types are also emphasized in the teachers' guide that teachers should use them in their teaching.

As a follow up of rabbit cage task, a task involving attempts at a basketball game is presented in 5th grade MI textbook (*See* Figure 4).

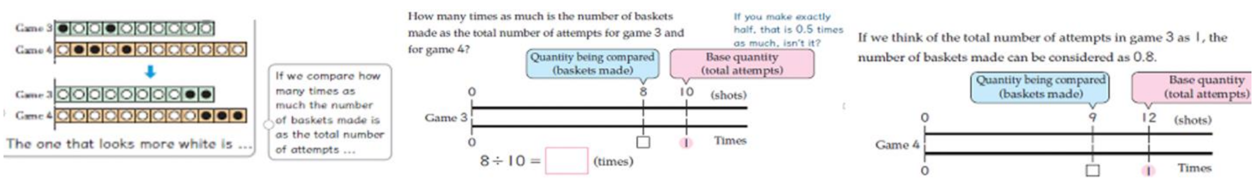

Figure 4: Basketball game (Fujii & Iitaka, 2012, Grade 5, p. B52).

The variable-parts perspective seemed to be employed with quotitive division in the original ratio focusing on part-whole relationship (i.e., 8 baskets made and 10 total attempts). Students are guided to use a number line to calculate per unit quantities. Differently from the previous task, two quantities are classified as base quantity and quantity being compared utilizing quotitive division. The main objective seems to enable students to represent the situation using rates, defined in the textbook as "The number that expresses how many times as much a quantity is compared to the base quantity is called the rate" (Fujii and Iitaka, 2012, Grade 5, p. B53).

In 7th grade MI textbook, direct and inverse proportional relationship between two variables and the continuous nature of their covarying relationship is represented visually (*See* Figure 5).

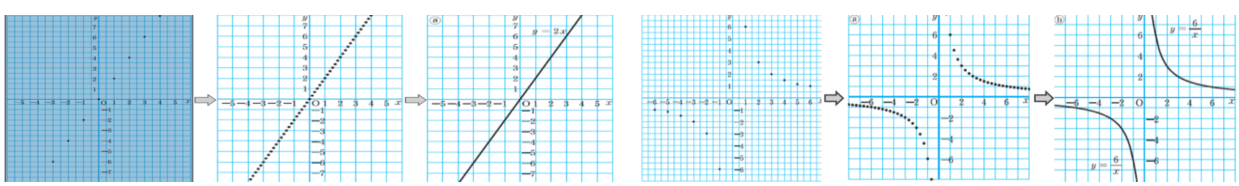

Figure 5: y=2x and $y = \frac{6}{x}$ graphs (Fujii & Matano, 2012, Grade 7, p. 117-118; p. 127-129)

Students are asked to explicitly think about the interval for x values and corresponding y values getting smaller and smaller producing a straight line or a curve.

Discussion and Conclusion

Findings showed that proportional relationships in the selected Japanese curriculum materials are intertwined with quantitative reasoning and covariation of quantities and developed through different perspectives of proportional reasoning and division types (Beckmann & Izsak, 2015). Considering the previous research showing teachers' struggles in proportional and non-proportional relationships and different division types (Arıcan, 2019; Orrill & Brown, 2012; Ölmez, 2021), we argue the sampled Japanese curriculum materials can be used by teachers and teacher educators to study and improve learners' proportional reasoning. Previous research indicates that Grade 2-4 MI textbooks concentrate on the relative size meaning of division, which requires grasping one quantity's measure relative to the other (Thompson & Saldanha, 2003), for both partitive and quotative situations (Karagoz Akar et al. 2022). Although partitive and quotative division do not necessarily require multiplicative reasoning, relative size, as a third model for division, "requires students to reason multiplicatively" and facilitate their understanding of non-integer divisors (Byerley et al., 2012, p. 359). Promoting the relative size in early grades could enhance multiple-batches and variable-parts perspectives of proportional reasoning. Future studies can explore students' understanding of relative size and its relation with/contribution to proportional reasoning perspectives.

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THE ALGEBRA CONCEPT INVENTORY FOR COLLEGE STUDENTS

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There are currently no large-scale assessments to measure algebraic conceptual understanding, particularly among college students with no more than an elementary algebra, or Algebra I, background. Here we describe the creation and validation of the Algebra Concept Inventory (ACI), which was developed for use with college students enrolled in elementary algebra or above. We describe how items on the ACI were administered and tested for validity and reliability. Analysis suggests that the instrument has reasonable validity and reliability. These results could inform researchers and practitioners on what conceptual understanding in algebra might look like and how it might be assessed.

Keywords: Algebra and Algebraic Thinking; Equity, Inclusion, and Diversity; Undergraduate Education; Research Methods

Algebra can be a barrier to degree completion in college (e.g., Adelman, 2006; Bailey et al., 2010), and difficulties that K-12 students have experienced with algebra content has been extensively documented (e.g., Booth, 1988, 2011; Kieran, 1992). Understanding of key algebraic ideas has also been shown to impact college students in higher-level college courses like Calculus (e.g., Frank & Thompson, 2021; Stewart & Reeder, 2017). Algebra courses in college tend to focus on procedures disconnected from sense-making (e.g., Goldrick-Rab, 2007; Hodara, 2011), which may be one reason why college students in higher-level courses still struggle with algebraic ideas. It is important to connect procedural fluency with conceptual understanding (Kilpatrick, et al., 2001), and therefore, there is a critical need to better understand and assess students' algebra conceptions. However, there are not yet any widely-validated assessments to measure college students' algebraic conceptual understanding. Existing large-scale validated algebra assessments exist for K-12 students but focus primarily on computational skills, or only on a narrow subdomain of conceptions. Measures of computational skill are not necessarily valid measures of conceptual understanding, because 1) learners may have robust conceptual understanding, but make computational mistakes, particularly when they have math or test anxiety (e.g., Ashcraft, 2002; Ashcraft & Kirk, 2001; Moran, 2016; Namkung et al., 2019); or 2) learners may have little conceptual understanding, yet produce "correct" answers for the mathematically invalid reasons (e.g., Aly, 2022; Erlwanger, 1973; Leatham & Winiecke, 2014).

This paper describes how we have developed and tested college students' conceptual understanding in algebra using the *Algebra Concept Inventory* (*ACI*), in an attempt to address this gap. This process is ongoing.

Literature Review

While various algebraic proficiency instruments have been created, currently there are no widely-validated instruments that focus on a broad range of topics in algebraic conceptual understanding. TIMMS and NAEP (Mullis, et al., 2020; National Center for Education Statistics, 2023) have been widely validated nationally/internationally but have a broader focus and only contain a limited number of questions aimed at assessing algebraic conceptual understanding. There are also state-wide assessments that contain some items intended to measure conceptual understanding but that primarily focus on computational skills (e.g., Massachusetts Department of Elementary & Secondary Education, 2023; New York State Education Department, 2023). There are a few instruments that have been designed to measure a few specific algebra concepts in elementary or middle school (Ralston, et al., 2018; Russell, 2019; Russell et al., 2009), but these have not been tested with high school or college students, and the different population of interest means that the narrow range of conceptions do not include more complex or abstract conceptions that are critical to secondary and postsecondary mathematics.

Some concept inventories have been developed to assess algebraic conceptions relevant to calculus and other higher-level courses (Carlson, Oehrtman, & Engelke, 2010; Carlson, Madison, & West, 2010); however these instruments are not appropriate for students in lower-level courses such as elementary and intermediate algebra (or Algebra I/II in high school), and their focus is not on some of the core conceptions from these lower-level courses that may be particularly critical to algebraic reasoning. Further, while many of these have been tested extensively qualitatively, they have not to date published results of larger-scale psychometric validation. Recently, researchers Hyland and O'Shea (2022) in Ireland generated a 31-item algebra concept inventory for college students, but includes algebraic objects that would not be familiar to students in a first-year algebra course and has not yet been tested through cognitive interviews or psychometric analysis. Thus, an algebra concept inventory that has been validated in large-scale data collection is sorely needed, particularly one that is appropriate for administration to students at all levels of prior algebra experience, and not just those in higher-level college courses.

Measuring Conceptual Understanding: Sample Item

There is insufficient space to describe the design of the ACI here, but it focuses on assessing specific conceptions of algebraic concepts (e.g., equivalence, syntactic meaning, algebraic properties, variable, function, covariation), rather than other skills like computation. For example, this item was designed to assess whether students can identify the existing syntactic structure of an algebraic expression vs. a procedure one might use to simplify the expression:

Sample item: Which of the following **best** describes the meaning of the expression $(2x + 3)(5x + 1)$ **as it is currently written**?

a. $2x$ is being multiplied separately by $5x$ and by 1, 3 is being multiplied separately by $5x$ and by 1, and these four results are being added together.
b. The result of adding $2x$ and 3 is being multiplied by the result of adding $5x$ and 1.

Method

A total of 402 unique items were developed and tested for the ACI. Items were administered to 7,658 students enrolled in all mathematics classes at the algebra level or above at a large urban community college campus. Data were collected from spring 2019 to fall 2022, in eight waves. Data collection followed a common-item random groups equating design, selected because it allows investigation of a large item pool while allowing simultaneous calibration across multiple forms (de Ayala, 2009; Kolen & Brennan, 2004). For the first wave of testing, the last ten items on each form were anchor items, all taken from the National Assessment of Educational Progress (NAEP) grade 8 item bank. For subsequent waves, six anchor items were included: three were NAEP items and three were ACI items that had performed well during the first wave. Each form had roughly 25 items. Forms were randomly administered within each class to ensure no association between test form and class or instructor.

Just before answering inventory items, students were invited to participate in cognitive interviews, and paid for their time. In total, 135 interviews were conducted. Each was roughly 1-1.5 hours long and structured as a "retrospective think-aloud" protocol (Sudman et al., 1996), which has been shown to reveal comparable information to concurrent think-aloud protocols, and is also less likely to have negative effects on task performance (e.g., van den Haak et al., 2003). Interviews were analyzed qualitatively to assess construct validity of the items, but there is insufficient space to report that analysis here, where we focus on quantitative results.

To prepare data for item-response theory analysis, ACI items were dichotomized into correct/incorrect using the response key. Then, two-parameter logistic models (Birnbaum, 1968) were estimated using marginal maximum likelihood (MML) on each wave, using the R package "mirt" (Chalmers, 2012). Because of the planned missingness data collection design, the default number of model iterations was extended to allow for all models to converge successfully. Based on these models, we examined item parameters (difficulty and discrimination) and item information functions for item analysis, and computed person estimates using expected a posteriori (EAP) factor scores for convergent validity analysis. Reliability estimates were computed directly from IRT models. To investigate model fit, we computed item fit statistics, using the PV-Q1 statistic (and significance test) (Chalmers & Ng, 2017) for each item.

To investigate measurement invariance, we used multi-group IRT models and a model comparison approach. Because of the planned missingness design (and sometimes small observed subsample sizes), we used a piecewise DIF detection strategy (Thissen et al., 1993) that starts from a fully constrained model and drops constraints for each item separately. More specifically, with respect to each examinee characteristic considered, we first estimated a fully constrained model (where, across groups, item discriminations, difficulties, latent mean and variance are constrained to equality). Then, for each item, the same model was estimated, but with unconstrained item parameters (difficulty and discrimination), thus "temporarily" allowing differential item functioning (DIF) for the item. A likelihood ratio test was then performed to test if the model allowing DIF for the item had a better fit than the constrained model. This resulted in a series of tests of the significance of differential item functioning for all items. Because it is a multiple testing strategy, *p*-values were subsequently Bonferroni-corrected.

Validating the ACI

IRT Models: Item Discrimination and Difficulty

Some items were dropped when issues were found during analysis (e.g., typographical errors; multiple correct answers); however, none were dropped due to unsatisfactory IRT parameters. 2PL IRT models were run on all waves (Table 1). We considered 1PL and 3PL models but chose

2PL models because they allow discrimination to vary by item and are more useable for item selection than 3PL models because item coefficients are more interpretable, and less prone to calibration errors due to their lower number of item parameters (San Martin et al., 2015).

Table 1. 2PL Model Coefficients Across all Eight Waves

Discrimination parameter	Proportion of Unique Items
>=0.65 "moderate"[a]	63.4%
>=1.35 "high"	31.3%
>=1.7 "very high"	18.5%
Difficulty parameter	Theta
mean	0.00
1st quartile	-0.85
median	-0.14
3rd quartile	0.63
Total number of unique items in 2PL models	399

[a] Characterizations of categories of discrimination parameters are taken from Baker (2001).

Discrimination is called as "moderate" if ≥ 0.65, "high" if ≥ 1.35 and "very high" if ≥ 1.7 (Baker, 2001). Based on these classifications, 63.4% of all items (253) have moderate or better, and roughly one-third have high or very high discrimination. Table 2 reports item fit for each wave using Chalmers' $PV - Q_1$ test, chosen because it performs better than other fit statistics at controlling Type I error (Chalmers & Ng, 2017). Only 5% of items were significantly misfitted by the 2PL models where $\alpha = 0.05$, which suggests that this is likely due to random variation.

Table 2. Measures of Item Misfit in 2PL IRT Models

	Number of Items With Significant[a] Misfit[b]	Total Number of Items	Percentage of Items With Significant Misfit
Wave 1	1	33	3.0%
Wave 2	5	125	4.0%
Wave 3	4	66	6.1%
Wave 4	3	72	4.2%
Wave 5	8	100	8.0%
Wave 6	5	99	5.1%
Wave 7	2	39	5.1%
Wave 8	0	31	0.0%
Total	28	565	5.0%

[a] Significant at the $\alpha = 0.05$ level
[b] Misfit as measured by Chalmers' Chi-Square Statistic ($PV - Q_1$)

Reliability

In IRT, Theta represents the number of standard deviations above or below the mean an individual is on the measure of the latent trait, and the reliability of an item varies based on values of Theta. Peak information values for all waves (Table 3) have excellent reliability ($R \geq 0.9$). For various waves excellent reliability ($R \geq 0.9$) was obtained for values ranging from $\theta = [-2.7, 2.2]$ (assuming a standard normal distribution of knowledge, this corresponds to satisfactory reliability for ~98% of examinees). In addition, shorter tests can be constructed from a subset of items with the highest discrimination: for example, the 10 items with the best discrimination from Wave 1 yields a test with excellent reliability ($R \geq 0.9$) for $\theta = [-2,1]$.

Table 3. Reliability (*R*) for each wave of item administration of the ACI

	Theta at max info[a]	Info max[b]	R for info max[c]	theta w/ $R \geq 0.8$	theta w/ $R \geq 0.9$	Number of Items Tested
Wave 1	-1.4	26.4	0.96	[-2.8, 0.4]	[-2.4, -0.2]	33
Wave 2	-1.5	37.8	0.97	[-3.0, 2.1]	[-2.7, 0.9]	104
Wave 3	-0.6	24.3	0.96	[-2.3, 1.5]	[-1.8, 0.7]	57
Wave 4	-0.6	30.1	0.97	[-2.4, 2.1]	[-1.9, 1.2]	69
Wave 5	0.7	177.1	0.99	[-2.3, 2.9]	[-1.4, 1.8]	100
Wave 6	-0.6	105.3	0.99	[-1.7, 3.0]	[-1.0, 2.2]	99
Wave 7	-0.1	21.7	0.95	[-1.5, 1.8]	[-1.0, 1.1]	39
Wave 8	0.1	11.3	0.91	[-0.9, 1.2]	[-1.2, 0.3]	31

[a] info = 2PL IRT model information function
[bd] max = information function maximum for 2PL model
[e] $R = 1 - \frac{1}{Info}$
[c] expected reliability in Normal(0,1) ability distribution for 2PL models

ACI Score and Prior Algebra Course Completion: Convergent Validity

To explore convergent validity of the ACI, we explored the relationship between scores on the ACI (using theta scores from the 2PL model) to various measures of mathematics course level. For example, correlation of students' ACI scores with the level of algebra courses they have already successfully completed would be evidence of convergent validity. First, we consider linear regression models with level of student's course (where "level" is defined based on the algebra course pre-requisite requirements of the course) as the independent variable, and ACI score as the dependent variable (Table 4).

Table 4. Regression of course level as predictor of ACI scores (2PL model)

Course Level[a]	Coefficient	SE	*p*-value (vs. low)	*p*-value (vs. high)
mid	0.347	0.014	0.000	0.000
high	0.700	0.017	0.000	

[a]reference group: low; low = no algebra course prerequisite; mid = elementary algebra course prerequisite; high = intermediate algebra course prerequisite; score is Theta score

Differences in scores in Table 4 are significant for all pairwise comparisons ($p < 0.001$). Scores for students in each level course were on average 0.35 SD higher than in the next lower course ("mid" vs. "low"; "high" vs. "mid"), providing strong evidence of convergent validity. We also considered a more nuanced course sequence based on prerequisites (see Table 5).

Table 5. Sequence level of various courses in the sample, based on their prerequisites

Various elementary algebra courses	1
Various 100-level courses with an elementary algebra pre-requisite	2
Intermediate algebra courses	2
College algebra	2
Discrete math with intermediate algebra prerequisite	3
Precalculus	3
Math for elementary teachers with intermediate algebra prerequisite	3
Math for elementary teachers, second term	4
Advanced statistics with precalculus prerequisite	4
Introduction to geometry with precalculus prerequisite	4
Calculus I	4
Calculus II	5
Calculus III	6
Differential equations with Calculus II prerequisite	6
Linear algebra with Calculus II prerequisite	6
Abstract algebra	7

Table 6 shows that linear regression models using this more refined set of levels again reveals a strong correlation between level and ACI score.

Table 6. Regression of more nuanced course level in predicting ACI score,

Course Position in Sequence	Coef.	SE	p-value (vs. 1)	p-value (vs. 2)	p-value (vs. 3)	p-value (vs. 4)	p-value (vs. 5)	p-value (vs. 6)
2	0.504	0.017	0.000					
3	0.623	0.031	0.000	0.000				
4	0.888	0.023	0.000	0.000	0.000			
5	1.059	0.033	0.000	0.000	0.000	0.000		
6	1.232	0.041	0.000	0.000	0.000	0.000	0.000	
7	1.661	0.226	0.000	0.000	0.000	0.001	0.008	0.060

The largest gain (one half SD) in Table 6 is between sequence level 1 and 2, or between students who have/have not satisfied an elementary algebra (Algebra I) prerequisite. This provides further evidence of convergent validity, because the ACI has been designed to focus on concepts relevant to elementary algebra specifically.

Differential Item Functioning: Measurement Invariance and Discriminant Validity

Differential item functioning (DIF) related to irrelevant examinee characteristics was also analyzed, one subtype of discriminant validity (or whether the ACI measures algebraic conceptual understanding and not something else, like English literacy). Each wave was tested for DIF in three separate 2PL models: one each for race/ethnicity, gender, and English-language-learner status. There was no consistent evidence of DIF. Only a negligible number of items had significant DIF for $\alpha = 0.05$ (using a Bonferroni correction for the number of tests within each model and not across models, which is overly conservative). Many items were tested in multiple waves, and none of these had significant DIF in more than one wave, suggesting that significant DIF in one wave but not others for these items was likely due to random variation.

Limitations

The City University of New York (CUNY) where this instrument was tested is not nationally representative, and thus further research is needed to validate the ACI with less-diverse populations in other geographic areas; this research is currently underway with a larger national sample in the US. However, CUNY's diversity does make it a excellent candidate for initial validation with marginalized students who have often been neglected in large-scale assessment validation. Further studies are also necessary to determine whether the ACI may be valid for use with high school or middle school students. Finally, the ACI has been developed to make *diagnostic* judgements about *groups* of students—not high-stakes decisions for individuals—and thus the ACI should not be used alone to make high-stakes individual decisions such as course placement or successful course completion.

Discussion and Conclusion

This study suggests that algebraic conceptual understanding, as conceptualized by the items included on the ACI, is a measurable domain with reasonable validity and reliability. Item response theory (IRT) analysis resulting in large proportion of items with good discrimination parameter estimates, suggesting the ACI can differentiate well between students of various levels. Reliability was also excellent for all waves of data collection, and based on reliability estimates, even shorter tests can be constructed with excellent reliability for a range of levels of algebraic conceptual understanding. Students with higher algebra course prerequisites had higher ACI scores, providing evidence of convergent validity. Finally, differential item functioning analysis demonstrated that the ACI had satisfactory measurement invariance with respect to race/ethnicity, gender, or English-language-learner status.

However, the ACI in its current form is a summative measurement that provides only one measure of students' algebraic conceptual understanding. Future research could expand this to a more nuanced diagnostic tool that provides more detailed information about the specific conceptions that students have and what kinds of instructional approaches may be best adapted to students with different conceptions about various algebraic concepts. This work is ongoing, and includes in-depth qualitative analysis of student thinking to more comprehensively map out in more detail the various conceptions that students may hold of algebra concepts; work with cognitive diagnostic models on ACI items might provide more nuanced diagnostic information; exploration of different curricular materials and teaching techniques and the subsequent impact on the development of algebraic conceptual understanding. Our hope is that the ACI will also enable other practitioners and researchers to explore these questions as well.

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Chapter 2:

Early Algebra, Algebraic Thinking, and Function

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

MATHEMATICALLY MODELING SOLUTIONS TO OCEAN PLASTIC POLLUTION

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Keywords: Modeling; Sustainability; Algebra and Algebraic Thinking

Mathematical models are powerful tools employed by experts to describe, predict and communicate the future course of environmental impacts (Barwell, 2018). I focus on how undergraduate students mathematically model solutions to a specific environmental issue: ocean plastic pollution. Mathematics provides a lens for students to visualize and quantify the rapid rate of plastic entering the ocean, as well as simulate possible solutions to the plastic pollution crisis. I provide a preliminary analysis of three groups of students who critically reflect on their mathematical model of potential global solutions to ocean plastic pollution. In this study, I ask: *How do students reflect on the severity of ocean plastic pollution and the urgency for solutions in the context of constructing a mathematical model?*

Mathematical modeling tasks provide a realistic learning experience that helps students understand real-world applications through mathematics (Keril & Gurel, 2016). Mathematical modeling often consists of an iterative process through a sequence of modeling cycles that students express, test, and revise their interpretations of a problem (Abbassian et al, 2020). The specific type of mathematical modeling employed in this paper involves a socio-critical mathematics perspective, which is derived from Skovsmose's (1994) critical mathematics education (CME) framework. Skovsmose's CME contains the vision that mathematics can be used to help students explore issues such as environmental challenges, leading students to reflect on how to take action against such challenges. Such critical reflections are an integral part of CME and socio-critical mathematical modeling.

This study took place over two 75-minute periods of an undergraduate college algebra course focused on modeling. Students worked in groups of 2-5 to develop mathematical models that represented solutions to ocean plastic pollution. At the beginning of the modeling task, students were given a prompt derived from real data provided by Ritchie and Roser (2018) on the state of ocean plastic pollution since the mid-twentieth century. After learning about the exponential growth of ocean plastic pollution and potential solutions to ocean plastic pollution, students were tasked with developing a mathematical model for their chosen solution. The data in this study consists of the students' final modeling report, which was qualitatively analyzed to determine how students critically reflected on solutions to ocean plastic pollution.

A preliminary analysis revealed that students noted the importance of both individual and collective/global action, as well as the urgency for more severe measures to curtail ocean plastic pollution. In particular, one of the groups acknowledged the importance of individual action, while the other two groups discussed both individual and global action. As an example, one of the groups compared a scenario in which nothing was done to combat plastic pollution to a few other scenarios of enacting global plastic bans of varying degrees, finding that such bans can prevent millions of tons of plastic from entering the ocean over the course of the next several decades. The students in this group noted the need for immediate, large-scale action. Overall, this preliminary analysis provides a window into understanding how students make sense of solutions to ocean plastic pollution. This study is a step in the direction of understanding how students make sense of solutions to large-scale environmental issues.

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FIRST AND THIRD GRADERS' INTERPRETATIONS OF AN EQUALITY INCORRECT WORKED EXAMPLE

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Keywords: Algebra and Algebraic Thinking; Number Concepts and Operations.

Elementary students often only experience arithmetic problems with the unknown to the right of the equal sign, e.g., 3 + 5 = 8. These limited experiences may result in students thinking that the equal sign appears between two numbers or is a signal to complete an operation. Thus, arithmetic problems that do not match such thinking (e.g., 3 + 5 = __ + 4) are tricky because students often solve them inaccurately using their prior experience (e.g., Knuth et al., 2006; McNeil & Alibali, 2002, 2004, 2005). Having students analyze worked examples could expose them to thinking about the equal sign as a symbol of equality. Particularly, an incorrect worked example can draw students' attention to why solving 3 + 5 = __ + 4 as 8 or 12 is not adequate. We explore whether first and third graders identified the incorrect part of an incorrect worked example of 3 + 5 = <u>8</u> + 4 and their reasoning. Twenty-seven first graders and 26 third graders from a public school in the Midwest, United States participated. We showed them the incorrect worked example and asked them whether 8 in 3 + 5 = <u>8</u> + 4 was correct and to explain their reasoning. Our analysis began with identifying whether students thought the answer of 8 was correct or incorrect. If they thought it was incorrect, we further classified their responses based on whether they found the correct answer or not. Next, we analyzed students' strategies by classifying them as having an operational (one side), an operational (two sides), or a relational view of the equal sign, or the equal sign as not relevant. We also coded for how students used the number four in the problem (i.e., ignores 4, notices 4, adds 4, considers 4+4, part of equality). Out of the 53 students, 58% identified that 8 was not correct, but only 40% of students provided the correct answer. Students' uses of the number four in the problem often aligned with a certain interpretation of the equal sign and specific solution strategies. Out of the 53 students, 19 of them (36%) ignored the four on the right side of the equation, using an operational (one side) view of the equal sign. Likewise, these students interpreted the problem as 3 + 5 = __ and thought the answer of 8 was correct. Three students (6%) made some reference to the four on the right side of the equation but did not do anything with it, maintaining an operational (one side) view of the equal sign. Beyond noticing the four, five students (9%) added it on to their total, demonstrating an operational (two sides) view of the equal sign. Three students (6%) mentioned the addition fact 4+4 but did not use it in their final answer. For example, a third grader agreed that 8 would be the correct answer in the blank, explaining, "Yes, because if you have five fingers and you put up three, it's eight." However, she continued without stopping, "Four plus four equals eight." Finally, 23 students (43%) used the equality of amounts on either side of the equal sign to determine that eight would not be correct, suggesting a relational view of the equal sign. Overall, students used similar strategies to those in previous studies (e.g., McNeil & Alibali, 2004, 2005), suggesting that responding to incorrect worked examples can provide similar information about students' understanding as having students solve problems on their own.

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EXPLORING THE IMPACT OF SCAFFOLDED PROBLEM-POSING ON COLLEGE ALGEBRA STUDENTS' INTERESTS

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Problem-posing can enhance math learning, but the effectiveness of different scaffolding approaches is not well understood. In this study, 120 College Algebra students posed problems related to their interests in popular culture (e.g., video games) or STEM careers (e.g., healthcare), using optional scaffolds (video, template, or example problem) for personalized problem-posing. The quality of the problems and measures of math interest were assessed. Results indicate that different scaffolds support different problem-posing outcomes, varying by learner and course.

Keywords: Algebra and Algebraic Thinking; Undergraduate Education

College Algebra is often the first college-level math course for many students. In Texas, many students take corequisite College Algebra courses, simultaneously enrolled in developmental and credit-level math due to not being college-ready based on assessments like the SAT (Texas Higher Education Coordinating Board, 2018). Approximately 60% of college students are in corequisite courses, with 66% being Black and Latinx and around 70% being first-generation college students (Grubb et al., 2021; Brathwaite et al., 2020; Nix et al., 2020).

Making math content relevant is crucial, as students often view it as disconnected from real life (McCoy, 2005) and question its practical use (Chazan, 1999). Relevance interventions are essential in gatekeeper courses like College Algebra (Riegle Crumb et al., 2019). This study combines personalization and utility value (UV) interventions by having students create math problems based on their interests. Personalized learning has shown benefits for corequisite students (Darwin et al., 2022), but research on problem-posing interventions is limited.

Problem-posing, where students generate new mathematical problems through inquiry (Silver, 1994), is enhanced by several approaches such as reviewing others' problems (Brown & Walter, 1990), interacting with an authentic audience (Crespo, 2003), receiving guidance and examples (Walkington & Bernacki, 2015), collaborating with peers (Walkington & Hayata, 2017), and working within familiar contexts and artifacts (Bonotto, 2013; English, 1998). Problem-posing can be free, semi-structured, or structured (Stoyana & Ellerton, 1996), with the most effective strategy combining open-ended tasks with structured guidelines (Wang et al., 2022). Further, it can outperform problem-solving if students solve the problems they create (Kapur, 2015).

Problem-posing is key in mathematics education to promote productive struggle (Cai & Hwang, 2023). Personalizing problem-posing activities, such as allowing students to pose problems about personal interests (e.g., sports, social networking) or career interests (e.g., nursing, IT), shows great potential in enhancing mathematics learning (Walkington et al., 2024; Walkington et al., 2022; Walkington, 2017; Walkington & Bernacki, 2015).

We explored the effects of different scaffolds on students' math interest and problem quality, focusing on corequisite versus traditional students. Our research questions (RQs) were: (1) How does using instructional resources during algebra problem-posing relate to students' (a) STEM career interest and (b) math interest outcomes?, (2) How does resource use during problem-posing relate to the quality of the problems posed?, and (3) How do these effects vary for corequisite students specifically?

Theoretical Framework

Problem-posing involves students creating novel mathematical problems through inquiry (Silver, 1994). Personalized problem-posing allows students to design word problems related to their career (Walkington et al., 2022) or personal interests (Walkington & Bernacki, 2015). Interest, defined as engagement and the tendency to re-engage with specific areas (Hidi & Renninger, 2006), is linked to improved performance and learning (Renninger & Hidi, 2022). Connecting content to students' interests, such as popular culture or career aspirations, can trigger and sustain situational interest. Personalization helps students appreciate and understand math's application in their interest areas (Walkington & Bernacki, 2015).

However, students often find problem-posing challenging, needing to understand problem characteristics like formulating questions and ensuring mathematical relevance (Silver & Cai, 1996; Van Harpen & Sriraman, 2013; Yu et al., 2005). Evaluating the quality of posed problems involves criteria such as originality, complexity (Silver & Cai, 2005), and realistic responses to constraints (Verschaffel et al., 2009). Problem-posing requires students to choose appropriate units, incorporate realistic quantities, and possess sufficient prior knowledge (Yu et al., 2005). Walkington and Bernacki (2015) found that students struggling with algebra problem-posing often had difficulty articulating the necessity of an intercept term, connecting independent and dependent quantities, and devising multiplicative scenarios.

Methods

In this study, n=120 students enrolled in College Algebra courses in the United States completed online activities in ASSISTments where they posed problems related to their interests. Sixty students identified as female, 52 as male, 8 as other/NA; 76 identified as White, 11 as Black, 8 as Asian, 1 as American Indian, 6 as Pacific Islander, 10 as Other/Multiple Races, and 8 NA; sixty-one students identified as Non-Hispanic, 51 as Hispanic, and 8 as NA. Students were enrolled in online and in-person College Algebra courses, covering linear, quadratic, exponential, logarithmic, and power functions. There were 63 corequisite students and 57 students enrolled in a traditional College Algebra course.

Students were randomly assigned one of four conditions (business-as-usual, career-posing [CP], career-solving, pop-posing [PP]). In the CP condition, the students would pose algebra problems based on a STEM career interest (i.e., Business/Finance, Earth/Space/Chemical Sciences, Engineering, Health Care Services, Information Technology, Natural Sciences, Social Sciences, STEM teaching). In the PP condition, students would pose algebra problems based on a chosen popular culture interest (i.e., Shopping, Social Media, Sports, Video Games). These categories were chosen based on surveys and interviews with students. In this study, we focus on the PP (n=55) and CP (n=65) conditions, as these conditions had different scaffolds/resources tested (templates, examples, and videos; see Figure 1) for problem-posing. Students received three types of support: (1) a video showing the application of mathematics in their area of interest, (2) a template to help construct a problem with relevant quantities and units, and (3) an example problem using the unit's function in their interest area. Students then used these scaffolds to pose a math problem. Following each unit, students rated their interest in

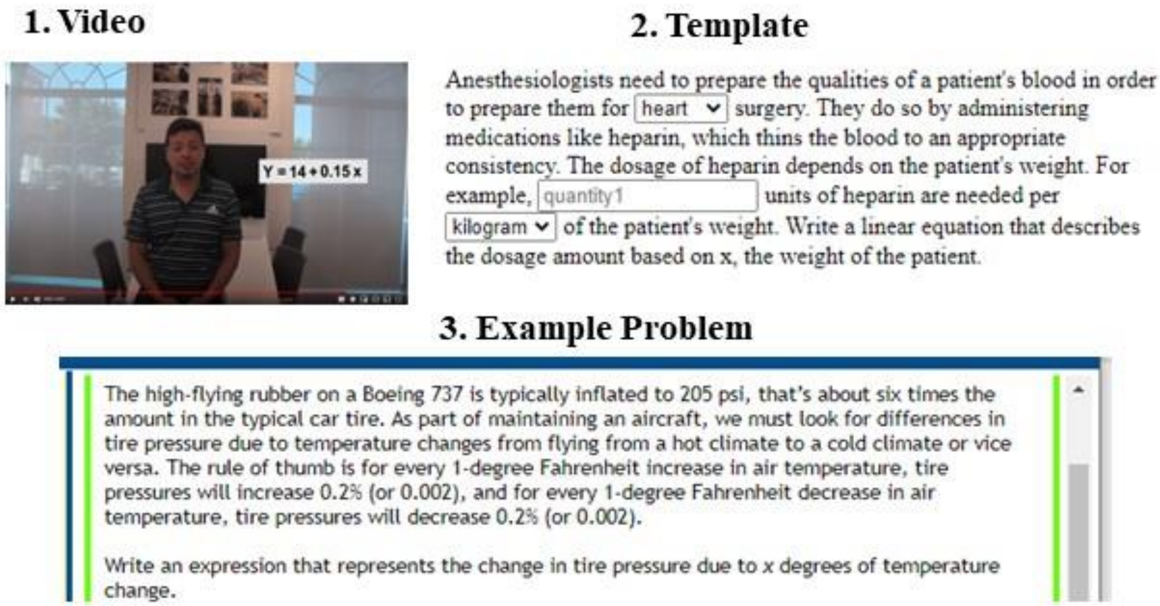

Figure 1: Scaffolds for problem-posing available to students in the present study

mathematics using an eight-item scale (Linnenbrink-Garcia et al., 2010) and their interest in one of eight STEM careers using the CABIN scale (Su et al., 2019). The quality of the problems posed by students was assessed using a rubric from prior studies (e.g., Walkington et al., 2022; Table 1), which included 4 sections scored on a 0-1 scale with multiple indicators per section.

Table 1: Overview of rubric for rating the quality of students' posed problems

Rubric Section	Description
Part A: Quality of Context	Rich and authentic narrative story in posed context
Part B: Mathematical Validity	If posed problem is solvable and consistent
Part C: Mathematical Complexity	Complexity of terms, numbers, and parameters chosen.
Part D: Mathematical Language	Valid and precise mathematical language is used
Part E: Originality	How closely problem resembles scaffolded example

Note. Contact the authors for a full rubric of how these parts were operationalized.

Results

To address RQ 1, when examining all students, regression with a hierarchical linear model showed no overall association between students' scaffold use and their post-unit STEM career interest. However, corequisite students specifically showed an increase in their post-unit STEM career interest per each video watched ($\beta = 0.29$, $SE = 0.11$, $p = .009$), but a decrease per each template used ($\beta = -0.31$, $SE = 0.10$, $p < .001$). For traditional and corequisite students, there was no significant association between scaffold use and post-unit math interest.

To address RQ 2, analyses of problem quality dimensions indicated that, when examining all students, students' use of the example problem scaffold was associated with an increase in the mathematical accuracy of the algebra problems they posed ($\beta = 0.26$ per example, $SE = 0.12$, $p = .037$) and an increase in overall quality of the problems they posed ($\beta = 0.27$ per example, $SE = 0.13$, $p = .032$). After removing the set of posed problems where students directly cut and paste from the provided template, template use was still associated with a decrease in problem originality ($B = -0.42$, $SE = 0.15$, $p = .004$) per template used in both conditions. When looking at corequisite students and the interactions with treatment, we found that when using the template scaffolds, there was an increase in the mathematical complexity ($\beta = 0.40$, $SE = 0.20$, $p = .041$) and mathematical language used ($\beta = 0.46$, $SE = 0.21$, $p = .027$) of problems they posed.

To address RQ 3, a comparison between the corequisite students and how they used scaffolds compared to traditional students can be found in Table 2; generally, corequisite students utilized the scaffolds more frequently, although this only reached statistical significance for the videos.

Table 2: College Students' Engagement with Scaffolds

Variable	Corequisite		Traditional		t(*df*)	*p*	Cohen's d
	M	*SD*	*M*	*SD*			
AU Video	.56	.50	.45	.50	-2.31(499)	.02*	.22
AU Template	.69	.46	.69	.47	-.07(499)	.94	0
AU Example	.64	.48	.58	.50	-1.38(449)	.17	.12
U1 Video	.76	.43	.79	.41	.44(140)	.66	.07
U1 Template	.87	.34	.83	.38	-.64(140)	.52	.11
U1 Example	.82	.39	.77	.43	-.73(140)	.47	.12
U2 Video	.46	.50	.40	.50	-.60(113)	.55	.12
U2 Template	.57	.50	.67	.48	1.10(113)	.16	.20
U2 Example	.50	.50	.47	.51	-.35(113)	.37	.05
U3 Video	.32	.47	.21	.41	-1.09(80)	.28	.25
U3 Template	.50	.51	.53	.51	.24(80)	.82	.06
U3 Example	.45	.50	.42	.50	-.30(80)	.76	.06
U4 Video	.57	.50	.30	.47	-2.78(110)	.01*	.57
U4 Template	.70	.46	.67	.47	-.23(110)	.82	.06
U4 Example	.68	.47	.60	.50	-.82(110)	.41	.16

Note. Engagement was calculated on a 0-1 scale if the student clicked on the scaffold; $*p < .05$. All Units = AU; Unit 1 = U1; Unit 2 = U2; Unit 3= U3; Unit 4 = U4.

Discussion and Significance

This study examined various scaffolds for mathematical problem-posing and their effects on math interest, STEM career interest, and the quality of problems posed by college students. We focused on differences between corequisite students (those not deemed "college ready" in mathematics) and traditional students. Key findings include: (1) Corequisite students used video scaffolds more than traditional students, which increased their STEM career interest; (2) The template scaffold for corequisite students decreased STEM career interest but increased the complexity and use of mathematical language in their posed problems; and (3) Example problems improved accuracy and overall quality of posed problems for all students.

The findings indicate that different scaffolds affect various outcomes: videos promoting relevance boost motivation, while templates enhance mathematical language use. Students with different mathematical backgrounds may need different supports; this intervention showed promise for corequisite students. Subtle differences in scaffold design significantly impact problem posing activities and interests, warranting further investigation in math education.

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INFLUENCE OF GAMIFICATION ON MATHEMATICAL SELF-EFFICACY AMONG ONLINE AND IN-PERSON COLLEGE ALGEBRA COREQUISITE STUDENTS

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Texas higher education is transitioning to a corequisite model, where students take developmental and traditional mathematics classes concurrently. However, little is known about supporting corequisite students, particularly minorities and first-generation college students (FGCS). This study examined the impact of gamification on corequisite students' mathematical self-efficacy (MSE) and games self-efficacy (GSE) in both online and in-person courses. Based on the course survey data, students exhibited gains in MSE and GSE overall, in the online modality, and among FGCS and female students. The results suggest that gamification has the potential to support algebra corequisite students, particularly females, FGCS, and online.

Keywords: Affect, Emotions, Beliefs, and Attitudes; Algebra and Algebraic Thinking; Online and Distance Education; Undergraduate Education

Texas House Bill 2223 transitioned developmental students to the corequisite model, where students take their developmental and traditional mathematics courses concurrently (Texas High Education Coordinating Board, 2018). While the corequisite model provides a framework for student support, primarily broad-based reforms currently exist (Texas Corequisite Project, 2020). Some studies have been conducted to see how interventions aid corequisite students in mathematics (e.g., personalized learning to students' interest; Darwin et al., 2022) or how the instructor may influence student learning (Darwin & Ataide Pinheiro, 2023), but these studies are few. This model deserves attention because 60% of all college students will take a corequisite course (Grubb et al., 2021). Of those students, a disproportionate 66% are Black and Latinx, 70% of whom are first-generation college students (FGCS; Brathwaite et al., 2020; Nix et al., 2020).

Course modality also contributes to corequisite students' academic success. 97% of two-year colleges offered online courses (Community College Research Center [CCRC], 2013), and students deemed not college ready are more likely to fail/withdraw (62%) compared to face-to-face students (43%) for online learning. Further, these students enrolled in online courses have lower academic success compared to face-to-face students (Coleman et al., 2017; Ryu et al., 2022).

One reform-based strategy that has been successful with college-level mathematics students is gamification (Faghihi et al., 2014; Lanuza, 2020), which is the process of using game-design elements in a non-game context (Putz et al., 2020). Gamification is often associated with increases in student motivation, engagement, and learning compared to traditional instructional strategies (Manzano-León et al., 2021) while reducing anxiety (Turan et al., 2016), especially in mathematics (Shyr et al., 2021). While gamification is a promising instructional strategy for college students enrolled in math courses (Faghihi et al., 2014; Lanuza, 2020; Putz et al., 2020), there is little research on the impact of gamification on corequisite courses in mathematics, both in general and specific to course modality. The purpose of this research project is to contribute to that knowledge by examining how gamification influences online and face-to-face corequisite courses in terms of mathematical self-efficacy (MSE) and game self-efficacy (GSE) by pursuing the following RQ: *What is the difference in students' MSE and GSE after participating in a gamified College Algebra in-person or virtual corequisite mathematics course?*

Theoretical Framework

Self-efficacy, an extension of Bandura's (1993) Social Cognitive Theory, measures how students' self-beliefs influence learning and performance (Chan & Abdullah, 2018). Self-efficacy is an individual's perception of their level of functioning in a situated task by making "judgments of their capabilities to organize and execute courses of action required to attain designated types of performances" (Bandura, 1986, p. 1167).

Self-efficacy and motivation are closely related (Bandura, 1993; Morris et al., 2017), particularly in mathematics (Skaalvik et al., 2015). According to Bandura (1989), "people's self-efficacy beliefs determine their level of motivation, as reflected in how much effort they will exert in an endeavor and how long they will persevere in the face of obstacles" (p. 1176). Klassen and Tze (2014) defined motivation as "a set of beliefs that influence people's *movement* towards the attainment of valued goals," which is obtained through choice, effort, and persistence behaviors (p. 1). According to Bandura (1977), self-efficacy is commonly understood as domain and context-specific, meaning one can have different levels of self-efficacy in different domains or for particular situations of functioning. For example, a students' self-efficacy in *mathematics* and towards *games* could be different, warranting the need for each construct (i.e., GSE, MSE) to be measured and explored separately. Therefore, this study examined how a mathematics gamification intervention influenced MSE and GSE.

Methods

Participants were 72 students (48 online, 24 in-person) enrolled in four algebra corequisite courses (two online and two in-person) at a community college in North Texas. Students completed a 36-question pre- and post-survey containing Likert-style questions addressing MSE and GSE at the beginning and end of the course. MSE questions were taken from Florella et al.'s (2021) Mathematics Motivation Questionnaire (MMQ) - a validated measure of intrinsic value, self-regulation, self-efficacy, utility value, and test anxiety (Cronbach's alpha of $\alpha = .93$). The authors designed GSE questions to determine the types of games (educational versus entertainment) and the extent to which students played games in and outside the classroom. The post-survey also included three free-response (FR) questions asking the students to describe their gamification experiences within the course.

Games were implemented weekly and aligned with course material. The games were simple and typically based on common childhood or icebreaker games (e.g., *Red Light-Green Light* and *Two Truths and a Lie*). The authors purposefully opted for what Liberoth (2015) refers to as *shallow gamification* or *framification,* so the games selected were simple to learn, could easily be applied to any content or curriculum, and could be readily adopted by teachers for classroom use. Online students completed the games in teams selected randomly by the professor; in-person students chose their partners. The content and game elements were consistent between online and in-person classes; however, slight modifications were made for the different modalities (e.g., in-person teams worked face-to-face, while online students collaborated through Google Slides).

Results

Overall, 31 students completed both the pre-and post-survey (19 online students and 12 in-person students; see Table 1). A two-way ANOVA revealed a statistically significant increase overall for both MSE and GSE from the pre to post-survey, and students in the online course demonstrated a significant increase in both MSE and GSE post-survey responses.

Table 1: Pre and Post-Survey Data on Student MSE and GSE

Variable (n)	Pre-Survey		Post-Survey		*t*(*df*)	*p*	Cohen's d
	M	*SD*	*M*	*SD*			
MSE							
Overall (31)	3.31	.71	3.53	.66	2.66(30)	.012*	.45
Online (19)	3.19	.68	3.48	.67	2.46(18)	.024*	.51
In-person (12)	3.51	.74	3.62	.70	1.08(11)	.303	.35
GSE							
Overall (31)	3.63	.63	3.87	.60	2.63(30)	.013*	.52
Online (19)	3.55	.58	3.80	.65	2.21(18)	.04*	.50
In-person (12)	3.75	.71	3.99	.53	1.41(11)	.187	.58

Note. MSE = mathematical self-efficacy. GSE = games self-efficacy. MSE and GSE were measured on a 1-5 Likert scale. *$p < .05$.

In addition to modality, an ANOVA was run to detect differences in gender, ethnicity, FGCS, and emerging bilinguals in the survey data. The ANOVA revealed a significant gender difference in the pre-MSE ($F = 5.09$, $p = .032$) and pre-GSE ($F = 9.15$, $p = .005$) surveys, with no significant difference in the post-MSE or post-GSE survey. Additionally, there was a statistically significant difference in the post-MSE survey for FGCS ($F = -5.06$, $p = .032$). No other demographics showed statistically significant differences.

Free-Response Results

Post-survey FRs by participants suggested the students overall enjoyed the intervention. Some online students appreciate the intellectual benefits of gamification (e.g., reinforcement, instructional support, and understanding of the material), as they reported: "We got to use our brains, and practice out problems in a low-stakes environment" and "I feel that it was easier for me to grasp the concepts we have learned via Google slide games as we were able to learn the materials enjoyably." Further, some in-person students appreciated how gamification made the material fun and competitive, mentioning that "it was a fun experience, I learn better doing educational games" and "it helps me think faster since it's a game."

Generally, the in-person students enjoyed the collaborative aspect of the games, focusing on the benefits obtained from the learning experience. For example, "I loved how we were all able to collaborate and put all our brains together to figure out the problem," and "it was something new and productive. It let me make conversations with new people while still learning and listening to new perspectives." However, the online students reported mixed results in the collaborative component online, as there was a lack of consistency among teammate interactions: e.g., "It was ok, not many students this semester were really involved" and "I felt as if it was very interactive where we had to comment on our 3 group mates as to what we think as well. It was also encouraging when you got the answer correct and the 3 agreed."

Discussion

Consistent with the studies done by Lanuza (2020) and Shyr et al. (2021), incorporating gamification in the college mathematics classroom saw an increase in MSE survey results and student enjoyment. Similarly, the results of this study showed an overall increase in survey results for MSE and GSE, suggesting that the students not only strengthened their mathematical knowledge and skills but also cultivated positive attitudes toward their perceptions of mathematics and gamification.

The results of this study differed from the literature in terms of modality. Most studies on online and corequisite mathematics courses show students fare no better and often do worse in online environments compared to face-to-face (Ashby et al., 2011; CCRC, 2013; Ryu et al., 2022). However, this study demonstrated that the online modality significantly increased MSE and GSE, whereas there was no significant difference in the in-person survey data. Therefore, gamification may be an effective intervention for online corequisite students. These results were supported by free-response data, where online students appreciated the support and reinforcement gamification provided, even if the collaborative aspects were not as strong as the in-person environment. Given the little research that exists on how to support online developmental mathematics students (Ashby et al., 2011), this study provides critical knowledge to those teaching online corequisite mathematics courses.

The pre-survey for both MSE and GSE showed a significant difference between males and females, with males scoring higher in their MSE and GSE. This aligns with the research, where females often report less confidence in their ability to do mathematics due to societal factors such as the stereotype threat (Buck et al., 2020), and males are often perceived as better at games (Rice et al., 2015). While the pre-survey data confirmed these findings in the literature, the post-survey data showed no significant difference between males and females for both MSE and GSE, suggesting that gamification can be a powerful tool in helping female mathematics corequisite students become more confident in their mathematics skills and abilities.

An opposite pattern existed in FGCS MSE, with no significant difference between FGCS and non-FGCS in the pre-survey but a significant difference in the post-survey. FGCS became increasingly confident in their MSE at rates greater than their non-FGCS peers. Again, these results contradict the literature, which suggests FGCS have lower MSE and struggle in mathematics corequisite courses (Brathwaite et al., 2020). Yet, in this study, FGCS outstripped their peers in the MSE after the gamification intervention. High failure/withdrawal rates are among the greatest challenges plaguing mathematics students placed in developmental courses

(CCRC, 2013). The results of this study add to a growing body of knowledge that gamification can be an effective strategy for improving student motivation and self-efficacy. As students become more engaged and confident in their abilities to succeed in their mathematics corequisite courses, they are more likely to persist and complete these courses. Corequisite course completion rates are crucial as they are often considered gatekeeping courses to achieving an associate's or bachelor's degree (Brathwaite et al., 2020). While this creates a financial incentive, as college degrees are often considered essential to obtaining high-paying jobs (Crisp et al., 2021), this is also an equity issue as the majority of students in corequisite courses are minorities and FGCS. This study demonstrated that gamification is an effective strategy to increase engagement and MSE, yet more research is needed to see if these gains translate into a reduction in corequisite course failure/withdrawal rates, particularly among marginalized students.

Further, these findings are promising, but additional research is needed on the influence of gamification in mathematics corequisite minority groups and emerging bilinguals. Limitations of this study are the small sample size taken from a single community college.

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PENSAMIENTO FUNCIONAL DE ESTUDIANTES UNIVERSITARIOS AL RESOLVER UNA TAREA DONDE SUBYACE LA FUNCIÓN ESCALONADA

FUNCTIONAL THINKING OF COLLEGE STUDENTS WHEN SOLVING A TASK WHERE THE STEP FUNCTION UNDERLIES

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En el presente artículo se exponen los resultados de una investigación cualitativa relacionada con el pensamiento funcional que exhiben estudiantes universitarios al describir una situación problema cercana a la vida real relacionada con la función escalonada. Se diseñó una tarea y se implementó en un ambiente online, mediante el uso de la plataforma Zoom. La revisión de literatura que se utilizó para el análisis de los resultados se basó en el Pensamiento Funcional. Los participantes en este estudio fueron 14 estudiantes de primer semestre de nivel universitario. Como resultado se observó la evolución del pensamiento funcional de los estudiantes exhibido a través de las representaciones verbales y gráficas, las cuales se fueron refinando y adaptando mejor a la situación problema.

Palabras clave: Resolución de problemas, representaciones matemáticas, precálculo y tecnología.

Introducción

Dentro de los distintos tipos de funciones que existen, podemos encontrar la función escalonada. Varios investigadores (De Villiers, 1988; Kaput y Roschelle, 2013; Smith, 2008; Pittalis, Pitta-Pantazi, y Christou, 2020 y Vargas-Alejo, Reyes y Escalante, 2016) han identificado que los estudiantes -de distintos niveles educativos- presentan dificultades para resolver situaciones problema asociadas a esta función. Algunas de estas dificultades pueden deberse a que los estudiantes no suelen resolver problemas asociados a funciones de carácter no continuo y constante, así como también puede deberse a que necesitan desarrollar su pensamiento funcional [PF]. El objetivo de esta investigación fue analizar el PF que exhiben los estudiantes universitarios durante la resolución de una situación problema asociada a la función escalonada. La pregunta de investigación fue ¿Cómo es el PF que externan los estudiantes al resolver la situación problema “Campeonato Olímpico 2022”? La situación problema incluida en la tarea puede ser descrita mediante una función escalonada. La investigación se realizó en la plataforma Zoom, debido a la pandemia COVID-19.

Revisión de Literatura

Pensamiento Funcional

Aprender matemáticas, con base en el PF, implica que los estudiantes busquen y generalicen patrones y relaciones mediante el uso de distintas representaciones (Kaput, 2008). El PF se define como “una actividad cognitiva que se centra en la relación entre dos (o más) cantidades variables, específicamente en los tipos de pensamiento que se derivan de una relación específica a generalizaciones de esa relación entre instancias” (Smith, 2008, p. 143). Smith (2008) propuso tres tipos de pensamiento funcional [TPF]: a) recurrencia, b) covariación y c) correspondencia.

Panorkou y Maloney (2016), con base en investigaciones de Smith (2008), extendieron esta clasificación a cuatro TPF: a) covariación cualitativa, b) covariación cuantitativa, c) covariación cuantitativa y visualización de la relación como una línea en el plano de coordenadas y d) correspondencia y covariación.

Metodología

La investigación se llevó a cabo con un enfoque cualitativo, ya que interesaba conocer los TPF exhibidos por los estudiantes y cómo estos se refinaron al resolver la situación problema. Los participantes fueron 14 estudiantes, de 18 a 20 años, quienes estaban cursando por primera vez el curso de cálculo diferencial de primer semestre de la Licenciatura en Ingeniería Mecatrónica, Industrial y Química. Por lo tanto, los estudiantes desconocían el concepto de la función escalonada. Los estudiantes fueron organizados en equipos (A, B, C y D). Para este estudio se diseñó una tarea que incluía la situación problema "Campeonato Olímpico 2022" la cual puede ser resuelta mediante el concepto de función escalonada. La duración de la implementación de la tarea fue de 110 minutos aproximadamente. Las fuentes de datos fueron: a) la videograbación de la sesión, b) procedimientos de los estudiantes y c) las notas del profesor. Para el análisis de los datos, se adaptaron los TPF propuestos por Panorkou y Maloney (2016) para describir el PF de los estudiantes (Tabla 2) durante la resolución de la situación problema. Debido al poco espacio en este documento se eligió presentar el análisis del trabajo del equipo D.

Tabla 2: Niveles de Pensamiento Funcional (Adaptado de Panorkou y Maloney, 2016)

TPF	Descripción	Comportamiento
PF1	Variación	El estudiante exhibe pensamiento de variación, es decir, puede identificar el patrón de cambio de una o más cantidades que varian.
PF2	Covariación Cualitativa	El estudiante no solo exhibe pensamiento de variación, sino que identifica y relaciona cualitativamente las variables que están involucradas en la situación problema.
PF3	Covariación Cuantitativa	El estudiante no solo exhibe covariación cualitativa, sino que identifica y relaciona cuantitativamente las variables que están involucradas en la situación problema.

Resultados y Discusión

En esta sección se describen y discuten, por episodios, los resultados del equipo D con base en los TPF que los estudiantes exhibieron (Tabla 3).

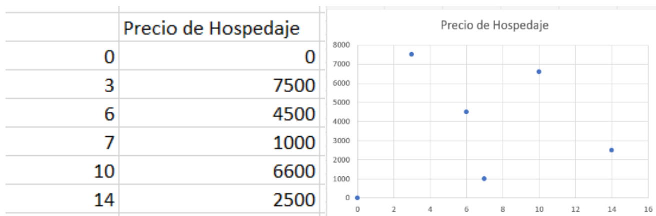
Tabla 3: TPF exhibidos por el Equipo D

Equipos	Estudiantes	Primer Episodio	Segundo Episodio	Tercer Episodio

D	d_k $k = 1,2,3$	PF1 con representación verbal y escrita	PF2 y PF3 con representación escrita y verbal	PF2 y PF3 con representación escrita, verbal y gráfica

Primer Episodio

Durante la resolución de la situación problema planteada, el Equipo D realizó una primera propuesta (Figura 1), la cual se caracterizó por una tabla de Excel y una gráfica de puntos.



	Precio de Hospedaje
0	0
3	7500
6	4500
7	1000
10	6600
14	2500

Figura 1: Primera tabla y gráfico del Equipo D

El equipo D exhibió Variación [PF1] puesto que identificó el patrón de cambio en cada una de las variables: *precio de hospedaje* y *días*. Aunque detectó cierta covariación cualitativa y cuantitativa [PF2 y PF3], su PF fue inestable en esos niveles. Es decir, tuvo dificultades para describir y relacionar las representaciones gráfica y tabular; lo cual puede observarse en la siguiente conversación.

d_2: Si las gimnastas se quedan aquí [Hotel Tokyo Grand Palace], entonces pagan tres días a 7,500 y luego 4,500.
P: ¿Por qué son 7,500 y 4,500?
d_1: Es que son 2,500…2,500…y 2,500 [PF1] en cada día. No cambia.
P: Interesante, pero oigan ¿qué pasa aquí [Señaló el intervalo de cero a dos con el puntero a su gráfico] en el contexto del problema?
d_3: Pues lo que dijo d_2, ah no… ¡Espere! No sé. Algo está mal.

La interacción con el profesor permitió al equipo autoevaluar sus procedimientos y redireccionarlos. Esto coincide con lo mencionado por Pittalis et al. (2020), en el sentido que las primeras respuestas de los estudiantes tienden a tener un PF inestable, pero pueden refinarse.

Segundo Episodio

El equipo construyó una gráfica tipo poligonal (Figura 2) adaptada a su nueva propuesta con otro presupuesto que respondiera a la situación problema.

Figura 2: Carta y gráfico del Equipo D

El equipo exhibió cierta Covariación Cuantitativa [PF3] en la descripción de su carta, puesto que no sólo identificó un patrón de cambio en las variables *costo* y *noches* [PF1], sino que también describió la relación entre las variables de manera cualitativa [PF2] y cuantitativa [PF3]. Parte del PF2 puede observarse en su carta cuando escribió "las siguientes tres noches baja el costo si se hospeda en el hotel dragón" y el PF3 se observa cuando en su carta escribió "sería aumentar a 6,600 las siguientes tres noches y sería $2,400 las siguientes tres noches", además de en la gráfica. No obstante, cuando el profesor le preguntó al equipo "¿qué sentido tiene la unión de los puntos de su nuevo gráfico dentro del contexto del problema?", presentaron dificultades para describirla, mencionaron que "la curva tenía que ser continua, por lo que no habían encontrado otra forma de describirla más que uniendo los puntos". Esto concuerda con investigaciones previas de De Villiers (1988), Kaput y Roschelle (2013) y Vargas-Alejo et al. (2016), que señalan que los estudiantes tienden a resolver situaciones problema con funciones continuas y suelen unir puntos al intentar graficar este tipo de funciones o bien están acostumbrados a hacer gráficas continuas sin reflexionar sobre su significado.

Tercer Episodio

Después de que cada uno de los equipos expusieran sus resultados, el profesor les sugirió tomar en cuenta los comentarios recibidos. El equipo D modificó su gráfico (Figura 3), exhibiendo así Covariación Cuantitativa [PF3].

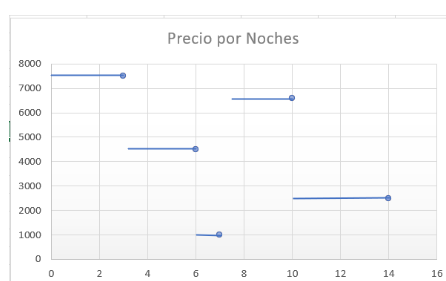


Figura 3: Gráfico del Equipo D

El equipo expuso su nueva gráfica y le explicó al docente que "la función ya no tenía un carácter continuo y, por lo tanto, había intervalos donde la relación entre el precio y la cantidad de noches tenía un carácter de tipo constante".

Reflexiones finales y conclusiones

En respuesta a la pregunta de investigación se observó que, el PF de los estudiantes del equipo D, externado al resolver la situación problema, se fue refinando con base en las distintas interacciones en el aula. En el episodio 1, el equipo D exhibió un PF1. Presentó dificultades para explicar la covariación en su representación gráfica, y darle significado en términos del contexto del problema. La gráfica no se adaptaba a sus descripciones de tipo verbal y escrita. En el segundo episodio, redireccionó su forma de pensar y, no solo logró responder a la situación problema planteada, sino que exhibió un desarrollo de su PF. Finalmente, exhibió un PF3, ya que en el tercer episodio construyó un gráfico que se adaptaba mejor a su descripción escrita y verbal. Se concluye, por lo tanto, que el PF de los estudiantes asociado a la situación problema cambió y se refinó durante el proceso de resolución a partir de la interacción con el profesor y los compañeros.

Reconocimiento

La investigación tuvo apoyo de CONACYT mediante las becas de estudiantes de posgrado.

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THINKING ABOUT EQUALITY AND EQUIVALENCE IN DEVELOPMENTAL MATHEMATICS: A CASE STUDY THROUGH A DISCURSIVE APPROACH

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Mathematical equality and equivalence are critical concepts that span K-12 and tertiary education. Few studies use sociocultural approaches to examine thinking about equivalence in postsecondary education. This case study utilizes a discursive approach to explore how one student graduating from a developmental mathematics program at a community college thinks about equality and equivalence. We analyze this student's word use, visual representations, routines, and narratives as components of her discourse to explore the patterns in her thinking. The results provide the student's rich and nuanced thinking while revealing when and how consistencies and inconsistencies occur in her thinking. Questions allowing students to generate their own examples and definitions can be useful in eliciting their thinking and a discursive approach can have implications in enhancing communication in postsecondary classrooms.

Keywords: Algebra and algebraic thinking, communication, mathematical representations, undergraduate education

Introduction

Mathematical equivalence is a critical concept that spans K-12 and tertiary education (Kaput et al., 2008) during which students form their realizations associated with equivalence and its signifiers (e.g., equal sign). College students' thinking about equality and the equal sign can be consistent with those of middle school students (Fyfe et al., 2020) but the specific ways undergraduate students think about equality and equivalence are relatively unknown. More research is needed to explore how postsecondary students interpret equality, the equal sign, and equivalence (Kieran & Hernandez, 2019). Existing research on equivalence and equality predominantly uses cognitive lenses whereas we extend existing frameworks by using a sociocultural, discursive framework. Our approach reveals the rich thinking of students as well as identify consistencies and inconsistencies in their thinking about equality and equivalence, having implications for enhancing communication in postsecondary classrooms. We particularly focus on equality and the equal sign as aspects of equivalence (Emre-Akdoğan, 2023; Kieran, 1981; Knuth et al., 2006; McNeil et al., 2006) and address the following research question: How does one student who graduated from a developmental mathematics program at a community college think about equality and equivalence in the context of single-variable equations?

Theoretical Framework

We use Sfard's (2008) commognitive framework, which considers mathematics as a sociocultural activity and thinking as a form of communication with one's self. From this perspective, examining participants' thinking is tantamount to examining their communication through their discourses. Sfard (2008) views mathematics as a discourse and identifies the components of mathematical discourse as word use, visual mediators, routines, and endorsed narratives. *Word use* refers to colloquial and specialized vocabulary used for mathematical communication; *visual mediators* are visual tools and signs participants generate and use to realize mathematical objects; *routines* are set of metarules that characterize the patterned actions participants use as they substantiate their mathematical discourses; *endorsed narratives* refer to the narratives that participants consider to be true about mathematical objects and their relationships (Sfard, 2008). Students' endorsed narratives can be idiosyncratic and different from those endorsed by the experts in mathematical communities (Güçler, 2014, 2016).

A metarule of focus in this study is *saming*, through which participants assign "one signifier (giving one name) to a number of things that, so far, have not been considered as in any way "the same" but are mutually replaceable in a certain closed set of narratives" (Sfard, 2008, p. 170). Saming is a critical component of realizing equivalence because a conceptual interpretation of equality and the equal sign requires viewing them relationally as signifying equivalence or sameness (Emre-Akdoğan, 2023; Kieran & Hernandez, 2019; Knuth et al., 2006). We explore our participant's spectrum of characterizations of equality, the equal sign, equivalence and sameness through a discursive perspective in the context of equations involving single-variables.

Methodology

We use a qualitative case study design to examine how one student who graduated from a developmental mathematics program at a community college thinks about equality and equivalence. Rae (a pseudonym) completed her developmental mathematics coursework at a rural community college in Eastern U.S. and hoped to continue her education to become a paralegal. The data collection was based on a semi-structured interview that included open-ended questions and tasks about equality and equivalence to elicit Rae's thinking. The open-ended questions focused on Rae's realizations of equality and equivalence where she was asked to explain what those terms meant to her, generate examples for each, and define the terms in her own words. Due to space reasons, we only focus on the open-ended part of the interview about equality because it provided authentic information about Rae's thinking based on her own discourse and examples. The interview was video-recorded and transcribed verbatim with a focus on Rae's words (what is said) and actions (what is done) (Sfard, 2008).

When analyzing data, we focused on the words and visual mediators Rae used as she communicated about how she realized equality and equivalence. We were particularly interested in her realization and use of the equal sign as a visual mediator. Whereas the visual mediators refer to the visual tools and signs Rae used, *when* and *how* she used them would indicate the routines in her discourse. We also explored whether and how she used *saming* as a routine during the interview. Finally, we explored explicit as well as implied narratives Rae generated about equality and equivalence to elicit her endorsed narratives.

Results

In the transcripts provided, *I* refers to the interviewer and *R* refers to Rae. Rae's actions are provided in parentheses in conjunction with her word use and narratives. At the beginning of the interview, we asked Rae to explain what equality means to her. Her response was as follows:

1. I: What does equality mean to you?
2. R: Being equal, yeah, numbers being. Well, there are solution. Always a solution to a problem. Just equality and being equal, as I said.

3. I: Can you give me an example of equality?
4. R: Um what is it…(laughter) zero equals zero.
5. I: Do you mind writing that?
6. R: (laughter) Yeah that's fine. This is what was in my mind. (Writes what is in Figure 1. Writes 0. Underlines it. Then writes another 0 under the fraction bar and an equal sign to the right of 0/0. Then writes 0 to the right of the equal sign.)
7. I: Can you give me another example of equality?
8. R: Um see. I don't know if this is going to be able to be distributed. Mmm (laughter). I don't know. (Writes what is in Figure 2. Writes 25x and underlines 25x. Then, writes + 7 and scratches it out and writes a 5 instead. Writes a 5 under underlined 25x. Writes = 30 to the right of 25x/5 + 5. Then, underlines 30 and writes a 5 underneath. Writes =6 to the right of 30/5.)
9. I: So you just did a lot of thinking there. Would you mind unpacking a little bit for me what you were thinking about?
10. R: Well I was…I just had the 25 because I said this is what we're learning. Well, the kids are learning, students are learning in class right now. And I just had fives in my head and just wrote 25x. And then I wanted to just add it and I was going to do a number, not five. Obviously, I was going to just do any random number, but then I tried to make it easier on myself and I picked five to make it 30 because five goes into 30. Six equally. I ended up adding the five into the 25x when I was dividing. I think…And then distribute, I tried using…tried making a math problem.

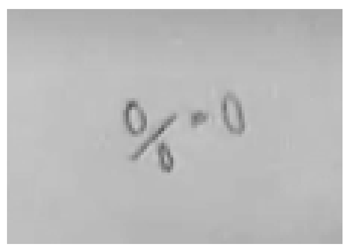

Figure 1. Rae's representation of her first example for equality

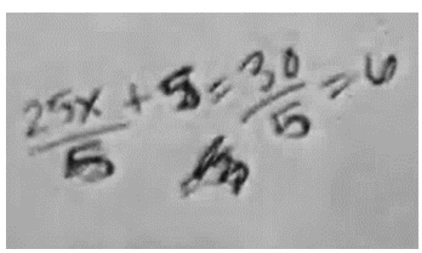

Figure 2. Rae's representation of her second example for equality

When talking about equality, Rae used the words “equal”, “numbers”, “solution” [2] and her word use indicated that she thought about equations when thinking about equality [4]. She also seemed to endorse the narrative that *an equation always has a solution* [2]. In her examples for equality, her visual mediators included numbers, a variable, fraction bars, and the equal sign (Figure 1, Figure 2). In both examples, Rae signified an equality as an equation. Rae’s use of the equal sign as a visual mediator was consistent with an operational rather than a relational approach in that it signified an arithmetic or an algebraic operation (Emre-Akdoğan, 2023; Kieran, 2004; Knuth et al., 2006). Her actions and routine of drawing fraction bars right after writing a number or an expression ([6], [8]) indicated that she thought about factoring the numbers before thinking about the next term on the left-hand side of her equations, before using the equal sign, and before completing the right-hand side of her equations. Although the visual mediator we inferred as a fraction bar could also be interpreted as an underline, Rae’s word use when she said “able to be distributed” [8], “Five goes into 30. Six equally” [10], “I was dividing” [10], “then distribute” [10] indicated that she was using the sign to signify division. Her actions indicated that she was using *factoring*, *finding a common factor across numbers*, and *dividing* as routines in her discourse when thinking about and generating equations. In Figure 2, Rae divided 30 by 5 to reach 6, instead of stopping when she wrote 25x/5 +5=30, which suggested that she interpreted the equal sign as a signal to execute an arithmetic operation (Siegler, 2003). The same action also indicated that she viewed the left-hand side of the equation equal to the right-hand side if they had a common factor. This may also explain why she may have viewed 0 as a factor of 0 to write 0/0=0 in Figure 1. Rae implicitly endorsed the narrative *mathematical objects are equal because they share a common factor* and viewed this as the commonality required to identify the left-hand side of an equation as the same as right-hand side, demonstrating how she realized *saming* as a routine in her discourse. We also asked Rae to define equality and asked her to elaborate on the relationship between equality and equivalence, if she saw any.

11. I: How would you define equality?
12. R: Things…and numbers have to mesh. Coincide with each other or equal each other.
13. I: In your opinion, is there a relationship between equality and equivalence?
14. R: I feel like there is because they are both have the word equal in it…And like I said they have to mesh in order to go into each other numbers…from my perspective.

Instead of defining what equality *is*, and treating it as distinct mathematical object, Rae provided phrases instead [12]. Rae’s word use included terms like “mesh” ([12], [14]), “coincide” [12], “equal” [12] to endorse the narrative that *numbers need to have a common factor in order to be equal or equivalent.* The word use and endorsed narrative in this excerpt were consistent with Rae’s discourse throughout the interview and aligned with her previous word use such as “distribute” [10], “dividing” [10], and “going into (evenly)” [10]. Rae identified the relationship between equality and equivalence through the commonality of the word “equal” in them [14] and the need for numbers to mesh or “go into each other” [14] for both of them. She seemed to consider equality and equivalence as relationships between numbers, where the numbers have a common factor, and one number divides the others evenly with no remainders.

Discussion and Implications

While Rae's discourse on equality and equivalence differed significantly from the discourses of mathematical experts, her discourse was mostly consistent. When she used the equal sign, she did so to signify an operation rather than a relation between mathematical objects, which is a finding that is consistent with the previous literature (Emre-Akdoğan, 2023; Fyfe et al., 2020; Kieran, 2004; Knuth et al., 2006; Siegler, 2003). We suggest teachers pay explicit attention to how their students use words, visual mediators, routines, and endorsed narratives in the classroom to address student difficulties and enhance communication in their postsecondary classrooms (Güçler 2014, 2016). We also recommend, as we have done in our work, teachers to provide opportunities for their students to generate their own examples and definitions of mathematical concepts before introducing these to the students.

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PREDICTORS OF MIDDLE-SCHOOL STUDENTS' PERFORMANCE ON ORDER-OF-OPERATIONS PROBLEMS

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Keywords: Algebra and Algebraic Thinking, Middle School Education, Cognition

Only 31% of 8th graders in the U.S. are proficient in mathematics (Irwin et al., 2023) and many students struggle to progress beyond Algebra I (Kena et al., 2015), potentially due to the increased need to attend to algebraic structure. In particular, many middle-school students habitually solve problems from left to right, often causing inefficiency (e.g., solving 46+72+54 by first adding 46+72 instead of 46+54; Lee et al., 2022) and sometimes violating the rules of precedence, especially as new and varied problem representations prevent solving through left-to-right (LTR) calculations (e.g., incorrectly adding 5+2 in 5+2×3; Gunnarsson et al., 2016).

Whereas prior research has demonstrated general effects of perceptual cues on students' problem-solving performance (Harrison et al., 2020; Landy & Goldstone, 2007), the current study systematically examines the effects of specific cues in math notation to compare the benefits of each (i.e., spacing [4×3 + 5 + 7], color ["4 × 3" highlighted in 4 × 3 + 5 + 7], or no cues) on 6th graders' performance on order-of-operations problems, as well as potential moderators of these effects. We will present analyses from a planned sample size of 600 U.S. middle school students, with 80% power to detect a main effect of $d > 0.20$ at $p < .05$. In a series of regression analyses, we will examine student-level, perceptual-cue-level, and problem-level predictors of performance, with performance indicated by accuracy and response time (RT). Student-level predictors include prior order-of-operations knowledge, perceptual processing skills, math anxiety, and math value. Problems are of a variety of formats, including some in which LTR calculations would be valid (e.g., 5×6–2+10) and some in which it would not (e.g., 4–7×2+9). We hypothesize that students with lower prior knowledge will benefit more (e.g., higher accuracy, faster RTs) from perceptual cues (either spacing or color) when faced with problems in a format in which the LTR process is invalid. Similarly, we hypothesize that students with lower perceptual processing skills, higher math anxiety, and lower math value will benefit more from perceptual cues, particularly on the invalid LTR process problems. This study will provide information about how perceptual processes can be leveraged to make learning mathematics easier for middle-school students through online platforms.

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COVARIATIONAL REASONING ABOUT A LINEAR CONTEXT WITH DYNAGRAPHS

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This case study investigates the thinking a student utilizes while exploring a task involving a linear function application. The study tracks the level of covariational reasoning and uses of slope a student applies while making sense of the situation contextually, on a Cartesian graph, and on a dynagraph. Although the thinking revealed aligns with past research findings that students often have conceptions of slope that are isolated from covariational thinking, the student in this case study demonstrates strong covariational thinking across representations that enables him to use slope for different purposes while never referencing the term "slope" and never displaying any signs of shape thinking.

Keywords: Algebra and Algebraic Thinking, Mathematical Representations, Technology

Background

Most secondary students in North America are required to have at least one algebra course, with linear functions as a key topic of the course (Dolores Flores et al., 2020; Stanton & Moore-Russo, 2012). With a functions-based approach (Yerushalmy & Chazan, 2002), instructors guide students to develop a solid foundational understanding of linear functions and their covariational relationships. In such an approach, there is a focus on understanding relationships and functional behavior before emphasizing symbolic manipulation. For this, students use and translate between different mathematical representations (Lesh, 1979) to understand what it means to be linear.

Research suggests that linear functions and slope are typically introduced by eighth grade (Nagle & Moore-Russo, 2014; Nagle et al., 2022). These topics are often reduced to rote procedures (Stump, 1999) punctuated by mnemonics (Walter & Gerson, 2007), such as "delta *y* over delta *x*" or "rise over run." Reiken (2008) described these student views of slope as number from formula, number from counting, and number in front of *x*; all of which could be performed as memorized procedures without any covariational reasoning. An instructional emphasis on such memorized procedures may be why students develop fragmented meanings for slope (e.g., Dolores-Flores et al., 2019; Hattikudur et. al., 2011; Postelnicu, 2011; Teuscher & Reys, 2010).

Students often fail to see slope as a parameter denoting a constant rate of change between two variables and struggle to work with linear functions in different representations (Adu-Gyamfi & Bossé, 2014; Tanışlı & Bike Kalkan, 2018) or in less familiar contexts (Moore & Thompson, 2014; Zaslavsky et al. 2002). Stump (2001) described physical (e.g., the steepness of a ramp or the pitch of a roof) and functional (e.g., the height of a candle over time) contexts for using slope and found students may be confident working with slope in one context type but not the other. As a result, it is widely acknowledged that the thinking students exhibit around slope may depend on the contexts used to engage that thinking (Byerley & Thompson, 2017). Even teachers can display certain limitations in their treatments of contextualized slope tasks. Paolucci and Stepp (2021) reported their sample of pre-service teachers tended to design contextual tasks that emphasize static, physical uses of slope rather than the dynamic, functional uses Moore-Russo et al. (2011) described. Rivera López et al. (2024) found that practicing teachers are quick to decontextualize applied linear situations focusing on algorithms rather than interpreting their real-world contexts.

Davis (2007) looked at high school students' understanding of *y*-intercepts in linear contextual situations and found that students had difficulty connecting different representations. If students are unable to move flexibly between representations of linear functions, they will have a hard time making sense of regression lines in statistics (Nagle et al., 2017), connecting slope as a measure of steepness with the tangent of an angle in trigonometric contexts (Nagle & Moore-Russo, 2013), navigating the extension of slope to understand situational instantaneous rates of change in calculus (Moore-Russo & Nagle, 2024), and applying linear relations as is frequently required in science classes (Planinic et al., 2012).

Knuth (2000) reported that students often use algebraic approaches to linear tasks even when graphical approaches are more efficient, noting that most manipulate equations to be in the form $y = mx + b$. Adu-Gyamfi and Bossé (2014) found that even when students are able to move from a given representation of a linear function to another, they may still have limitations in how they understand linear functions. Results such as these have led to calls for a more concerted effort in connecting thinking about slope used for different purposes, with covariational reasoning as the link (Nagle & Moore-Russo, 2013).

Dynagraphs are digital representations that use parallel number lines where students manipulate input values to see the corresponding output values. So, input and output variables are depicted separately, rather than in a single coordinate pair (Bailey et al. 2020). Research suggests dynagraphs may help direct students' attention to how inputs and outputs change correspondingly and support covariational thinking (Antonini et al. 2020; Ozen et al. 2021). As a result, dynagraphs may be a powerful tool for exploring students' thinking about covarying quantities and may challenge students to move beyond reliance on shape thinking to determine slope (Moore & Thompson, 2015).

Framing the Study

Lesh (1981, p. 241) stated, "The coordination of a system of ideas is achieved progressively, but its completion is marked by a qualitative 'jump' as the student shifts to a qualitatively higher level of thought. This is portrayed in the APOS-slope framework, in Figure 1, which was first introduced by Nagle et al. (2016) and later more thoroughly vetted (Nagle et al., 2019). The APOS-Slope framework builds on Dubinsky's APOS theory (1984, 2014) to consider how geometric (G), algebraic (A), and functional (F) conceptualizations of slope converge into a linear constant (L) conceptualization as students develop a more robust understanding of slope. The APOS-Slope Framework also highlights the different purposes for which slope is commonly used in algebra (namely, to describe behavior, measure steepness, and determine relationships).

Thompson and Carlson (2017), as well as Paoletti and Vishnubhotla (2023), have suggested hierarchies of covariational reasoning that expand on previous work (Carlson et al., 2002) to define levels of covariational reasoning for modeling dynamic events. We now give brief descriptions of Thompson and Carlson's (2017) major levels, which we use for this study. *No coordination* involves no connection between the change of one variable with another. *Precoordination* of values involves attending to the change in one variable then to the change in another without simultaneous consideration of the two. *Gross coordination* of values involves considering the direction of change for each variable simultaneously, e.g. this output decreases as the input increases. *Coordination of values* involves coordinating magnitudes of inputs with corresponding values of outputs being able to create a discrete collection of ordered pairs. *Chunky continuous covariation* involves simultaneous coordination of the output's value with uniform intervals of change for the input; this involves both direction and magnitude, but the input increments are only considered in "chunks" of a fixed size. *Smooth continuous covariation* involves simultaneously coordinating outputs with input variables using both direction and magnitude but not being limited to fixed increments, rather understanding that within each interval both variables' values undergo synchronized change that is both smooth and continuous.

This study brings the APOS-Slope framework together with the levels of covariational reasoning to analyze a student's thinking about a linear relationship. Past work indicates that students' stages of slope reasoning and underlying covariational reasoning may vary based on the purpose for which they are using slope. For instance, in studies of students' approaches to informally placing a line of best fit (Casey & Nagle, 2016; Nagle et al., 2017), students used *gross coordination* language when justifying whether the line increased or decreased, suggesting they were applying covariational reasoning while reasoning about slope to *describe behavior*. However, the same students struggled to explain how they chose the angle or tilt of their lines of best fit, providing explanations that were void of covariational language as they reasoned about slope in order to *measure steepness*. Similarly, a review of a standards-based secondary mathematics curriculum showed an instructional emphasis on covariational reasoning when using slope to *describe behavior*, but few opportunities to reason covariationally when using slope to *measure steepness* or *determine relationships* (Fisher et al., 2021; Nagle et al., 2022).

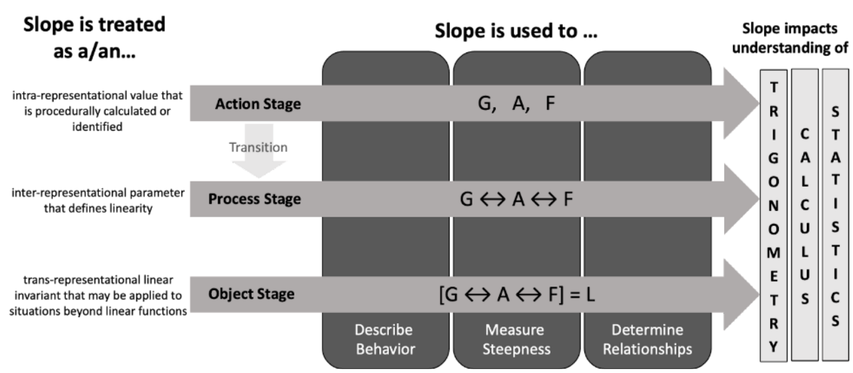

Figure 1: APOS-Slope Framework (adapted from Nagle et al., 2019)

In this study we consider one student's covariational reasoning involving linear functions in a contextual situation as he encounters dynagraphs for the first time along with more familiar mathematical representations (e.g., Cartesian graphs, two-column number tables, and linear equations) he has already experienced in the algebra curriculum at his school. The following research questions guided the study. How does a student who successfully completed a high school algebra class reason about a dynamic, functional context involving a linear relation? What level of covariational reasoning does the student display in this interaction? How does the student's covariational reasoning vary depending on the representation or particular use of slope?

Methods

This case study looks at a task-based interview with Cyrus, a 16-year old male who was enrolled in an Algebra 2 course in a public high school in New York state after completing his Algebra 1 course earning a grade of A. The entire interview lasted over an hour and a half, but we will focus on the section of the interview related to one contextual task. The situation, which follows, involves a functional application of slope as a property of a dynamic linear situation that requires covariational reasoning. *Maddy is a spunky, 5-foot tall, 16-year old girl who goes out and gets a summer job mowing yards in a neighborhood where all the yards are the same size. If each yard takes her the same time to mow, explore the relationship between the number of yards (x) she mows and the total time in hours she spends mowing (y).*

After being introduced to the task, Cyrus was asked to explain the relationship in terms of the real-world context, provide a graph of the relationship on a Cartesian plane, explore three dynagraphs to determine which one represented the same context, and then revisit the Cartesian graph initially presented. The interview was semi-structured in format, with pre-planned probing questions and additional questions to probe for understanding based on Cyrus' responses.

Findings

Initial Contextual and Cartesian Graph Reasoning

When introduced to the problem context, Cyrus demonstrated that he was able to reason about the functional context of the task. He applied *gross coordination* to describe the relationship between yards mowed and total time elapsed (i.e., "mowing more lawns will take more time"), and he translated this reasoning while using slope to *describe behavior* as he sketched a graph of the relationship on a Cartesian plane, starting at the origin. Cyrus explained the graph would be a "diagonal line going up because x increasing means you're moving up the x-axis and y increasing means you're moving up on the y-axis." He was able to label and explain the units for both axes when asked.

When asked to determine the slope of the line he sketched, Cyrus transitioned to *coordination of values*. He recognized a point near (1, 0.9) on his sketched line, explaining "the y-value doesn't go exactly up 1 and the x-value goes up 1." When asked how he would interpret that slope in the context, he confidently explained the unit rate, stating that "for every one yard that Maddy mows, it takes a little less than one hour to mow it." This thinking demonstrates at least *coordination of values* reasoning in the graphical context extended to at least *chunky continuous covariation* reasoning in the real-world context.

When asked to explain how the graph would change if the yards were not all the same size, Cyrus responded it "might be a little more wavy." He added that "it might go up in one spot and then maybe dip back down." When asked to try to sketch this behavior, Cyrus sketched a non-monotonic, non-linear graph. At this point, Cyrus did not seem to be able to apply both *gross coordination* and *chunky continuous covariation* to make sense of changing slope while keeping the increasing relationship between inputs and outputs. A summary of Cyrus' reasoning while engaging with the context and the Cartesian graph during this first part of the interview is provided in Table 1. Notice how *gross* contextual reasoning first translated to *gross* Cartesian graph reasoning, which then supported *coordination of values* in the Cartesian setting that prompted *chunky continuous covariation* to make sense of unit rates in the context.

Table 1: Covariation and Slope Reasoning in Context and Cartesian Graph Settings

	Contextual Situation	Cartesian Graph
Gross Coordination	*Mowing more lawns will take more time*	[The graph will be] *a diagonal line going up because x increasing means you're moving up the x-axis and y increasing means you're moving up the y-axis.* [Describe Behavior]
Coordination of Values		*The y-value doesn't go exactly up 1 and the x-value goes up 1.* [Measure Steepness]
Chunky Continuous Coordination	*For every one yard that Maddy mows, it takes a little less than one hour to mow it.*	

Dynagraph Reasoning

Cyrus had not seen dynagraphs previously. After a brief introduction to dynagraphs, Cyrus was then asked to explore three dynagraphs, representing a decreasing linear function, an increasing linear function, and a constant function, respectively. Cyrus was given time to explore the dynagraphs with the goal of determining which one might represent the context of Maddy mowing yards. Upon exploring the first dynagraph (of a decreasing linear function), Cyrus was moving toward *gross coordination* by explaining that when "moving the lawns mowed to bigger numbers, the total time is getting farther away." But he seemed to be focusing on the distance between inputs and outputs rather than the direction that they were both moving, explaining that this dynagraph "sort of" made sense for Maddy's context because "it kind of shows a relationship between the time and total yards mowed." When prompted to pick two specific points from the dynagraph and explain how the inputs and outputs were changing, Cyrus stopped the dynagraph at an input of 5 and confidently (and correctly) identified this would be the point that is "over 5 on the x-axis and up 1.5 on the y-axis" on the Cartesian graph. He then dragged the dynagraph to an input value of 8 and explained (correctly) that "the *x* would be 8, and the *y* would be 0." After making a connection between the dynagraph and Cartesian graph, Cyrus was then able to apply *gross coordination* to the dynagraph to explain that the inputs and outputs were moving in opposite directions but did not attempt to tie that back to Maddy's context at this point.

Cyrus extended his reasoning when exploring the second dynagraph (of an increasing linear function), immediately, saying that "as the *x* is increasing, the *y* is increasing but at a much slower rate." This thinking implies Cyrus is comparing the amount of change in *x* with the amount of change in *y*, suggesting he seems to be using *coordination of variables* and may also be using *chunky continuous covariation*. Cyrus applied *coordination of values* with proportional reasoning to justify why this dynagraph made sense for Maddy's context, explaining that "if every yard is around the same, then, like, if this is 2 [dragged input to 2] and it's around 1 [signaling to the output value] then if this is 5 [dragged input until output is 5 and signaled to output] then it's around 10 [signaling to the corresponding input value]. It just makes sense because every yard is similar."

On the third dynagraph (of a constant function), Cyrus applied at least *gross coordination* when indicating that "as I move the *x* back and forth, the *y* is staying the exact same." He connects this to the context by explaining that it would mean that "if you mow ten lawns, it would take the exact same time as if you mowed five," which showed *coordination of values*. After exploring all three dynagraphs, Cyrus reported that the second dynagraph was the best match to the context provided. He added that the third dynagraph "didn't make sense at all" and applied *gross coordination* related to the first dynagraph to conclude it was not feasible because "as you increase the total number of yards mowed, your total time isn't going to go down." The researcher asked Cyrus to describe how the dynagraph might act differently if the yards varied in size. Cyrus applied *chunky continuous covariation* while explaining that "as you would move up the *x*-axis, the *y* would increase at different rates." He continued, "one yard would be bigger; so, it (the *y*-value) would increase more because it would take longer." Although he is not referencing slope, this thinking is consistent with using slope to *measure steepness* while applying *chunky continuous covariation*. A summary of Cyrus' reasoning across the context, Cartesian graph, and dynagraph is provided in Table 2.

Table 2: Covariation and Slope Reasoning in Context, Cartesian Graph and Dynagraph

	Contextual Situation	Cartesian Graph	Dynagraph
Gross Coordination	*If you mow ten lawns, it would take the exact same time as if you mowed five.* *As you increase the total number of yards mowed, your total time isn't going to go down.*		*As I'm moving the lawns mowed to bigger numbers, the total time is getting farther away.* *As I move the x back and forth, the y stays the exact same.* [Describe Behavior]
Coordination of Values	*If every yard is around the same, then, like, if this [input] is 2 and it's [the output's] around 1 then if this [output] is 5 then it's [the input's] around 10. It just makes sense because every yard is similar.*	Explains movement is *over 5 on the x-axis and up 1.5 on the y-axis.*	
Chunky Continuous Coordination	*One yard would be bigger; so, it [the output] would increase more because it would take longer.* [Measure Steepness]	*As you would move up the x-axis, the y would increase at different rates.*	*As the x is increasing, the y is increasing but at a much slower rate.* [Measure Steepness]

Revising the Cartesian Graph

At the end of the interview, Cyrus was asked again to graph the relationship between number of yards and total time mowing, where the yards varied in size. Cyrus first drew a "wavy" nonmonotonic graph like he had at the beginning of the interview, and the interviewer asked him how he would interpret one of the waves if *y* represented total time. Cyrus quickly realized the error in his graph, explaining, "so now I'm thinking this is completely wrong because it's [the decreasing segment of the graph] saying that...time is going backwards now which doesn't really make sense." Cyrus was initially unsure as to how to sketch the graph of the situation but said that "different lawns would take a longer time; so, it wouldn't be like a constant line of the *x* and *y* increasing at a constant rate." He continued, "it would be, the *x* increases one yard at a time, so that would be at a constant rate, but the *y* would be more sporadic because if they're different lawns, they're going to take different amounts of time." At this point, we see evidence of *chunky continuous covariation*, where the inputs are changed one yard at a time and the corresponding change in outputs is known to vary since the yards are different sizes. Cyrus then applies this thinking while sketching the graph of this scenario one more time.

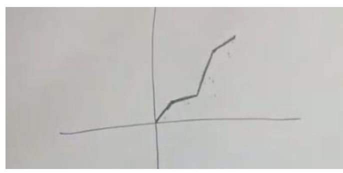

Figure 2: Cyrus' Sketch for the Variable Yard Size Context

Cyrus transfers his *chunky continuous covariation* reasoning to the Cartesian graph, clearly delineating unit intervals of 1 yard and applying thinking consistent with using slope to measure steepness, but never using the word "slope," even when pushed to verbalize what feature of the graph he was looking at. In particular, Cyrus drew the first segment of the graph and said "the first line is like that. It's like a normal yard." Then he sketches the next segment [with a smaller, still positive, slope] and explains that "the next lawn is a really small one; so, the time it takes is not significant. So, the *y* barely goes up, but you still get the lawn done; so, the *x* changes." He then stated, "the next one is a big lawn; so, it goes up pretty significantly [drawing a steep segment as he talks], and you still get the lawn done." He then said, "the next lawn is pretty normal" and explained that its *y* changes like the first one, drawing a line segment that appears to have the same slope as the first line segment he drew. Despite not using the words slope or parallel, through his words and drawing Cyrus displayed thinking consistent with using slope to *determine relationships* while using *chunky covariational reasoning*. He then goes on to explain that in the "normal lines" the *y* might change 1 to 1 with the *x*, points to the steepest segment and says the *y* might change 3 to 1 with the *x* and then points to the least steep line and explains that *y* changes something like ½ to 1 with the *x*. He is able to connect this thinking to the context by pointing to each segment on the graph and accurately identifying whether it is an average-, small-, or large- sized yard. Despite this rich, *chunky continuous covariation* grounded understanding of the context and its representation on the Cartesian plane, Cyrus never referenced the words slope, tilt, or steepness of the segments he drew. Instead, when asked by the researcher how he knew which lawns were larger or smaller, he responded "because, like, the....I don't know what the right word is, but...let's say, like, the angle, I guess, the angle of that little segment of this line."

Conclusions and Future Study

Although Cyrus reported no previous experience with dynagraphs, and despite never using the word slope throughout the interview, he demonstrated a strong understanding of the context grounded in covariational reasoning that enabled him to work flexibly with both the dynagraph and Cartesian graph representations. In fact, Cyrus demonstrated thinking consistent with all three uses of slope: *describe behavior*, *measure steepness*, and *determine relationships*. He was able to reason about a unit rate of change (amount of time it took to mow one lawn) and clearly explained how that unit rate was represented using both the Cartesian graph and dynagraph representations.

By the end of the interview, Cyrus also sketched a piecewise Cartesian graph, where each piece was a linear segment with a different rate of change. The line segments he sketched (see Figure 2) show his attention to a unit change in the number of yards mowed and a strong understanding of the amount of change in the y-values based on the size of a yard. Cyrus clearly explained that a small yard would have little change in y for the set change in x representing one yard mowed, while a large yard would have a greater change in y. He also described yards that were the same size on his graph, applying thinking consistent with *determining relationships*.

Cyrus's thinking stands out considering past research that suggests students often rely on shape thinking (Moore & Thompson, 2015). Shape thinking related to slope is often seen in relation to the three purposes of slope, as outlined in Table 3, and contrasted to the type of reasoning Cyrus demonstrated in this interview. Shape thinking can be the result of learning slope applied to a Cartesian graph, void of covariational reasoning or connection with other relationships. Cyrus demonstrated the ability to reason covariationally about constant rate of change but had not connected that thinking to the concept of slope, despite earning an A in his high school algebra course. Even when pressed at what feature(s) of the Cartesian graph he was focusing on, Cyrus was unsure what to call it, settling on the "angle" of the segments as the best description.

Table 3: Shape Thinking Compared to Cyrus' Thinking of Slope

	Describe Behavior	Measure Steepness	Determine Relationships
Shape Thinking	Lines that go up have positive slope; lines that go down have negative slope; horizontal lines have zero slope.	Steeper lines have greater slope; less steep lines have smaller slope.	Parallel lines have equal slopes; perpendicular lines have negative reciprocal slopes.
Cyrus' Thinking	The graph will be a *diagonal line going up because x increasing means you're moving up the x-axis and y increasing means you're moving up the y-axis.*	*The next lawn is a really small one; so, the time it takes is not significant. So, the y barely goes up, but you still get the lawn done so the x changes.*	This lawn is *pretty normal;* so, the y-value will go up the same amount as the other one since the lawns are the same size.

In this way, Cyrus' case provides a new lens on the well-documented divide between covariational reasoning and the concept of slope. While most research suggests students have learned to apply slope in a particular representation void of covariational reasoning, Cyrus has built a strong foundation of covariational reasoning but does not connect this to the concept of slope (despite completing Algebra I with an A). However, throughout the course of the interview, he was able to apply that reasoning across representations, suggesting that his covariational reasoning forms a strong foundation on which slope can be built. He was particularly successful in making sense of dynagraphs, a brand-new representation for him, likely because the dynagraph made the covariational relationship between inputs and outputs more visible than the static Cartesian graphs he had seen in the past. His interview showed he moved flexibly between the context, the dynagraph and the Cartesian graph, reasoning in ways consistent with all three uses of slope.

Future research should continue to explore the role of multiple representations while individuals seek to understand slope used for different purposes. Are dynagraphs as helpful in building initial covariational reasoning about slope as they were in helping Cyrus apply his existing covariational reasoning?

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TYPES OF REASONING IN COVARIATIONAL FUNCTIONAL THINKING

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In this study, a theoretical model describing covariational functional thinking in 13-to-16-year-old students (Grades 8-10) was formulated and empirically validated in three countries (n=350). The hypothesis posited that students engage in covariational functional thinking through five types of reasoning: correspondence view of function, direction of change, calculation of constant rate of change, discerning varying rates of change, and comparing the intensity of change. A structural model indicated that students' capacity in tasks requiring a correspondence view of function predicts their qualitative understanding of quantities varying simultaneously. Subsequently, the analysis revealed three parallel paths from 'direction of change' to 'comparing the intensity of change': one direct path and two indirect paths through 'calculating the constant rate of change' and 'discerning varying rates of change'.

Keywords: algebra and algebraic thinking; function; functional thinking; covariation

Functional thinking is acknowledged as a key component in mathematics education and a fundamental aspect of algebra due to its pivotal role in understanding essential mathematical concepts and in developing important mathematical skills (National Council of Teachers of Mathematics, 2000). Functional thinking has been generally defined as the process of building, describing, and reasoning with and about functions (Blanton, Brizuela, et al., 2015) and is associated with various conceptions and views of functions (Doorman et al., 2012; Dubinsky & Harel, 1992). Research findings indicate that typical curricula often focus predominantly on the correspondence view (Thompson & Carlson, 2017). This approach mainly suggests a static conception of functions, making it difficult for students to envision that the symbolic expression of functions represents relationships among varying quantities (Stephens, et al. 2017). Conversely, considering function as a dynamic process of covariation, emphasizing the covariation of the dependent variable with the independent variable, has been associated with a more intuitive alternative to a formal, correspondence perspective on function and a dynamic and more general view of the function concept (Johnson, 2012; Paoletti & Moore, 2018).

The main goal of the present study is to provide a flexible description of Grade 8-10 students' covariational functional thinking, by proposing a theoretical framework that describes types of this kind of reasoning. We integrate research on functional thinking (Lichti & Roth, 2019), function learning progression (Arieli-Attali et al., 2012; Graf et al., 2010), and covariational reasoning frameworks (Carlson et al., 2002; Jones, 2022), to unpack the essential characteristics of each type of reasoning. These types of reasoning are operationalized via the construction of a model whose robustness is theoretically founded and empirically tested.

Theoretical Framework

Functional thinking is related to different conceptions and views of functions (Doorman et al., 2012) that are equally important for fully shaping the notion of function and the ability to use it effectively. The views of functions that learners need to develop include: Function as (a) an input-output assignment that often precedes carrying out a calculation process; (b) a correspondence relation, focusing on the particular relation between the independent and the dependent variable; (c) a dynamic process of covariation between the independent and the dependent variables; and (d) a mathematical object, that can, e.g., be examined, compared with or connected to other mathematical objects.

The covariational approach is grounded on Confrey and Smith's work (1995) and includes analyzing, manipulating and comprehending the relations between changing quantities. It refers to the variation of the independent variable x and the resulting covariation of the dependent variable y. In other words, it captures the manner of change in y, if x changes uniformly (Lichti & Roth, 2019). A slightly different perspective is proposed by Thompson and Carlson (2017). They describe *covariational reasoning* as going beyond mere awareness of change, but entailing specific mental actions to conceive situations as composed of quantities and relationships among quantities whose values vary. In this perspective, they claim that "a function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other" (2017, p. 444).

Function Learning Progression

Graf et al. (2019) provided a description of a learning progression for the concept of function that was validated through empirical data, based on a synthesis of previous function understanding models. The first level, *preinstruction*, includes only extending sequences. The second level, *familiarization*, refers to perceiving a function as an algebraic formula or equation. The third level, *making connections*, has a qualitative difference compared to the previous ones as it entails understanding the concept of dependence and an emergent recognition that a function can be captured through different representations. The fourth level, *synthesis*, includes covariational reasoning and the idea of one-valuedness. At this level students are expected to conceive the covariation of variables and attend to global features of graphs. The next level gives emphasis to perceiving a *function as an object* that can be operated upon. The most sophisticated level refers to *function families* that can be perceived as parameterized objects.

Covariational reasoning

Thompson and Carlson (2017) analyzed covariational reasoning by providing an in-depth description of students' progression. Their framework builds upon the covariation framework proposed by Carlson et al. (2002), which was further validated and extended by Yu (2024). Carlson et al.'s (2002) framework investigated students' conceptions of functions using covariational reasoning as an explanatory framework for student reasoning. It identifies the mental actions that students undertake when manipulating the magnitudes or numerical values of covarying quantities in respect to five developmental levels. These mental actions serve as a classification system for the behaviors exhibited by students during covariation tasks. For example, the first mental action involves coordinating the value of one variable with changes in the other and recognize dependence, and corresponding behaviors include labelling the axes with verbal indications of this coordination. The first mental action and behavior describe the level of *coordination*. The second mental action emphasizes coordinating the direction of change of one variable with changes in the other (*direction*). This is manifested in behaviors such as constructing an increasing straight line and verbalizing an awareness of the direction of change of the output value concerning changes in the input. The third mental action focuses on coordinating the amount of change of one variable with changes in the other (*quantitative coordination*). Behaviors associated with action include verbalizing an awareness of the amount of change. The fourth mental action involves coordinating the average rate of change of the function with uniform increments of change in the input variable (*average rate*). Corresponding behaviors include verbalizing an awareness of this rate of change and constructing contiguous secant lines for the domain. Moving to the fifth mental action, it entails coordinating the instantaneous rate of change of the function with continuous changes in the independent variable across the entire domain of the function (*instantaneous rate*). Behaviors related to this action include verbalizing this coordination and constructing a smooth curve with clear indications of concavity changes.

Purpose of the Study

The purpose of this study was to present a theoretical model clarifying the types of reasoning in covariational functional thinking among students in Grades 8 to 10. This grade range corresponds to the formal introduction to functions in middle school and the exploration of various function types and representations during the first years of high school, preceding the study of calculus concepts. In line with Calson et al. (2010), we propose that students in this age group should progress from an action-oriented view (involving arithmetic computations for individual numerical values) to a covariational view of function.

Proposed Model-Aims of the Study

We hypothesized that students' covariational functional thinking can be described by five types of reasoning. We synthesized types of reasoning and mental actions identified in various frameworks, aiming to explicitly describe anticipated abilities based on a dynamic conception of function (Carlson et al., 2002). These descriptions align with school curricula requirements and address the potentials and needs of students within the examined grade range. To make the description of the proposed types of reasoning explicit, we refer to the anticipated actions.

The first type of reasoning refers to perceiving the *correspondence* view of function, reflecting the action-oriented view (Dubinsky & Harel, 1992). It includes conceiving a function as (a) an algebraic formula/equation and as (b) rule, dependence that can be represented in the form of a graph or table of values (Graf et al., 2019). This type of reasoning includes actions such as calculating the output value for a given input value based on a function formula, graphically representing linear functions, interpreting graph representations by understanding input and corresponding output values and identifying/symbolizing the functional relation between two varying quantities expressed in different forms. The second type of reasoning refers to understanding the *direction of change* of one quantity/variable with changes in the other quantity/variable in a qualitative way, as individuals envision the two quantities/variables varying together (Carlson et al., 2002). This includes verbalizing an awareness of the direction of change of the output value due to a change in the input variable, by interpreting different representations of functional relationships, such as algebraic formulas, graphs, or contextualized scenarios. Individuals explain how two quantities involved in a contextualized functional relationship change when the algebraic formula of the relationship is provided (without making any calculations), match real-life scenarios with graph representations or construct a graph based on a scenario that involves co-varying quantities. It should be noted that this type of reasoning involves a basic conceptualization of the direction of change, without discerning nuances in the growth rate, such as linear or nonlinear tendencies.

The third type of reasoning involves *calculating the constant rate of change* - coordinating the amount of change of one variable with changes in the other variable - when the graph, the description, or a table of values of a functional situation involving a linear function is provided (Carlson et al., 2002). Individuals coordinate quantitatively the covarying quantities by calculating the gradient of the linear function. In this case, the change in the dependent variable to every one change in the independent variable is the same. So, individuals may use the coordinates of selected points on a graph or table or calculate the rate of the vertical increment to the horizontal increment on the graph (Hauger, 1997).

The fourth type of reasoning includes *discerning varying rates of change* in functional situations (Johnson, 2012). Individuals verbalize how and why the quantities vary over a continuous domain and reason about how the relationship between the two quantities is represented. Individuals exhibit a well-established representational fluency and stable concept of constant change to differentiate between intervals in which the two quantities vary with a constant or changing change. Johnson and McClintock (2018) showed that when students predicted the type of graph type representing the relationship between covarying quantities, their initial step in discerning variation in unidimensional change was distinguishing between linear and nonlinear patterns. They understand that linearity is a characteristic of functions where the change in constant (Arieli-Attali et al., 2012).

The fifth type of reasoning reflects *comparing the intensity of change* in co-varying quantities within functional situations (Arieli-Attali et al., 2012; Carlson et al., 2002; Jones, 2022). It encompasses calculating, interpreting, and comparing the constant rate of change in complex functional situations, and reasoning about non-numerical varying rate of change (coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function). In instances of varying rate of change, this type of reasoning includes constructing a smooth curve that clearly indicates how the rate of change between the two quantities varies.

Drawing upon a synthesis of frameworks describing levels of reasoning, we hypothesized a structural model that delineates the relationships among the proposed types of reasoning (see Figure 1). Specifically, aligning with Graf et al. (2010), we posited that the ‘correspondence view of function’ reasoning is a prerequisite for understanding a covariational perspective of function. This type of reasoning facilitates familiarization with the concept and the pointwise interpretation by observing occurrences at specific points before discerning overall trends. Consequently, we hypothesized that this reasoning directly influences the ‘direction of change’ reasoning, which is essential for comprehending how one quantity changes concerning the change in another (Carlson et al., 2002). Subsequently, we hypothesized that ‘direction of change’ directly affects the types of thinking involved in the ‘calculation of constant rate of change’ and ‘discernment of constant and varying rates of change’. Finally, we assumed that ‘calculation of constant rate of change’ and ‘discern constant and varying rates of change’ serve as predictors of ‘compare the intensity of change’.

The aims of the study were (a) to investigate whether different functional thinking tasks could be categorized based on the factors of the proposed model, (b) to examine the structure and the relations between student’s types of reasoning in covariational functional thinking by empirically validating the hypothesized structural model

Methodology

Sample-Measures

The participants in this study constituted a convenience sample, comprising of 350 middle and high school students from three countries: Cyprus (n=187), Greece (n=43) and Poland (n=119). Ninety-three were eighth graders, 165 were ninth graders, and 93 were tenth graders. The schools and the teachers involved participated voluntarily.

Test items were adopted or developed based on previous research studies and multiple tasks were used for each presumed construct of the framework. Most of the test items were based on previous qualitative studies, teaching experiments, and intervention studies (see Arieli-Attali et al., 2012; Carlson et al., 2002; Castillo-Garsow, 2012). Thus, we made modifications in the format and wording of the items to meet the needs of a written test. The test comprised 16 tasks.

Correspondence view of function. Four tasks were used to measure this type of reasoning. The first one presented the formula of a linear function and students were asked to calculate output or input values when given the corresponding input or output one (T1). In the second task, students were tasked with constructing the graph of the linear function from the first task and appropriately labelling the axis (T2). For the third task, students were instructed to provide y or x values for a given x or y value for a function based on its graph (T3). The fourth task presented three tables of values and four graphs (T4). Students had to match each table with the corresponding graph.

Direction of change. Four tasks were used to capture the entire spectrum of this type of reasoning. In the first task (T5), a real-life scenario featuring a linear functional relationship and the algebraic formula of the function was presented. Students were required to determine the accuracy of statements describing how the two quantities covary. Similarly, the second task presented the graph of a linear function illustrating a currency change (T6). Students were tasked with evaluating the accuracy of four statements describing the covariation of the two quantities. For the third task, students were instructed to construct a piece-wise graph representing the direction of change in distance covered with respect to time, in a distance-time scenario (T7). In the fourth task, verbal descriptions of six mobile pay monthly plans, each involving different functional relations between monthly cost and talking minutes, were provided along with four graphs (T8). Students were required to match each graph with the appropriate verbal description.

Calculation of constant rate of change. Three tasks were used. In the first task (T9), the graph of a linear function was presented. Students were instructed to find the gradient of the line. Similarly, in the second task (T10), students were required to calculate the rate of change of a linear function by utilizing the coordinates of a set of points belonging to the line, provided in a table of values. The third task (T11) featured a picture of a climbing ramp illustrating the position of a roller skate girl, along with her vertical and horizontal distances from the starting point. A graph depicting the variation of the vertical distance in relation to the variation of the horizontal distance was provided. The task required finding the gradient of the function describing her movement when driving up the ramp.

Discern varying rates of change. Three tasks were used to measure this type of reasoning, focusing on interpreting situations where the rate of change between the involved quantities varies. The first task (T12) involved a boy moving between two points, both forwards and backwards. The distance between the points consisted of both a flat and uphill section. The task required selecting the appropriate graph that represents the horizontal distance covered from the starting point with respect to time. To do so, it was necessary to accept that the rate of change between distance and time can be constant (e.g. on the flat part) and non-constant when climbing up and down the hill. The second task (T13) is a modified version of a problem proposed by Castillo-Garsow (2012) and necessitates interpreting in a real-life scenario, employing a discontinuous function, time intervals where the rate of change is zero and time-points where the dependent variable increases instantly. The third task (T14) presented a piecewise linear graph illustrating the height of water in a vessel in respect to time when water is poured in at a constant rate. Students had to select the vessel made up of two cylinders that corresponded to the graph.

Compare the intensity of change. Two tasks were used to measure this type of reasoning that emphasized on the investigation of variations in the intensity of the change rate between the quantities involved in functional situations. The first task (T15) was based on the Bottle Problem (Carlson, et al., 2002), illustrating two bottles, and requiring students to find the graph of the height in respect to time, given that the water is poured into the bottle at a constant rate. This required an intuitive recognition of inflection points, where the rate of change shifts from increasing to decreasing or vice versa. The second task (T16) involved a distance-time scenario and a corresponding piecewise line graph. Students had to interpret five verbal statements involving the rate of change, by finding the time interval that corresponded to each statement.

Data analysis

The goal of the analysis was to estimate the relative strength of the proposed models. Because we proposed a theoretically driven model, our main interest was in the assessment of fit of the hypothesized a priori measurement model to the data. After this we examined the validity of the hypothesized structural model. We used partial least squares (PLS) techniques to analyze structural equation modelling (SEM) systems, using the SmartPLS software. We adopted a reflective analysis because reflective indicators constitute a representative set of all possible items within the conceptual domain of a construct. To examine the validity of a reflective measurement model, multiple indicators are taken into consideration: reflective indicator loadings; internal consistency reliability; and convergent and discriminant validity. Reflective indicator loadings are expected to exceed .70, while acceptable values are considered over .40 (Hair et al., 2019). In terms of internal consistency reliability, Cronbach's alpha and composite reliability (CRI) values in an exploratory study should exceed .60. Average variance extracted (AVE) greater than .50 is a good measure of convergent validity. To assess the validity of the structural model we established the explanatory and predictive power of the model.

Results

First, we investigated the validify of the proposed measurement model by examining the convergent and discriminant validity. Table 1 shows that all the loadings were greater than .50, giving support to the assumption that all factors were adequately measured by the observed variables. In accordance with out theoretical assumption, all covariational functional thinking tasks were clustered into five first-order factors in the expected factor-loading pattern. The significance levels of the associated t-values were calculated by bootstrapping, using 5000 subsamples with the same number of cases as in the original sample, and all of them were statistically significant (Hair et al., 2017). The average extracted (AVE) of the four of the five first-order factors exceeded the threshold value of .50. Only the AVE value of the factor 'Discern varying rates of change' was marginally smaller than .50. The composite reliability index (CRI) ranged from .71 to .84. (above the threshold value of .60). Our research findings reaffirmed the a priori hypothesized model, confirming that five types of reasoning can explain students' variances in covariational functional thinking situations.

The validation of the structural equation model reaffirmed the general structure of the hypothesized model and showed the existence of a sequential effect among the five factors, encompassing both direct and indirect effects. Specifically, the type of reasoning 'correspondence view' exhibited a strong direct effect on the 'direction of change' reasoning type (r=.50, p<.01). Subsequently, the hypothesized model assumed the existence of parallel paths from 'direction of change' to 'calculation of constant rate of change' and 'discern varying rates of change'. However, the analysis revealed the existence of a third parallel path from 'direction of change' to the 'compare the intensity of change' reasoning type (see Figure 1, the first number indicates the regression coefficient and the number in parenthesis the corresponding t-value). The inclusion of this third parallel path in the model led to an increase of the R^2 of the factor 'compare the intensity of change' from .40 to .49. Consequently, the standardized solution of the final model showed a weak statistically regression coefficient from 'direction of change' to (a) 'calculation of constant rate of change' (r=.20, p<.01), 'discern varying rates' (r=.23, p<.01) and 'compare the intensity' (r=.34, p<.01). Finally, the 'discern varying rates' factor proved to be a strong predictive factor for 'compare the intensity' (r=.71, p<.01), while the 'calculation of constant rate of change' factor exhibited a moderate predictive relationship with 'compare the intensity' (r=.50, p<.01). The model provided very weak R^2 value for 'calculate constant rate of change' (R^2 =.17), 'discern varying rates of change' (R^2 =.27) and 'direction of change' (R^2 =.35) and moderate for 'compare the intensity of change' (R^2 =.50). Finally, the Q^2 values showed medium predictive accuracy of the model for 'direction of change' (Q^2 =.34) and 'compare the intensity of change' (Q^2 =.25) and weak for the other two (Q^2 =.04).

Table 1: Measurement model convergent and discriminant validity indices

Factor/Type of Reasoning	Loadings	AVE	CRI	Cronbach's alpha
Correspondence view of		.58	.84	.75
function	.78			
T1	.72			
T2	.84			
T3	.69			
T4				
Direction of change		.54	.82	.72
T5	.72			
T6	.77			
T7	.61			
T8	.83			
Calculation of constant rate of		.59	.81	.65
change	.86			
T9	.82			
T10	.60			
T11				
Discern varying rates of change		.46	.71	.46
T12	.54			
T13	.58			
T14	.87			
Compare the intensity of change		.66	.79	.53
T15	.67			
T16	.93			

Discussion

The primary aim of the study was to propose a model describing the types of reasoning exhibited by Grade 8-10 students in covariational functional thinking. We developed a model hypothesizing that students within this age range engage in covariational functional thinking through five types of reasoning. The proposed model elaborates on and extends existing frameworks, advancing the related literature (Jones, 2022; Smith, 2008), by identifying and explicitly describing specific types of reasoning whose activation is essential for adequately responding to a variety of covariational functional thinking tasks. This type of research provides a new theoretical lens for integrating research on understanding the concept of function, functional thinking, and covariational reasoning, by explaining how each type of reasoning facilitates specific actions contributing to understanding function. For instance, the types of reasoning underlying the importance of covariational reasoning serve as an explanatory framework to examine the relationship between the quantities involved in the functional relationship by envisioning their simultaneous variation. Understanding how two quantities covary and discerning between constant and varying rates of change makes it possible to interpret their graph representation or explain how a paired set of values of a function can be deduced based on a given pair of values.

The study described a structural model which indicated that students are initially more successful in tasks requiring a correspondence view of function. Thus, advancements in students' correspondence view of function might enhance their further development in qualitatively conceiving the direction of change of the quantities involved in a functional situation by increasing awareness of the relationship between two quantities and the understanding that a variation in one quantity corresponds to a change in the other. This finding aligns with research by Graf et al. (2018), suggesting that perceiving the correspondence view of function facilitates familiarity with the concept, which is essential for more demanding covariational reasoning. The empirically validated model demonstrated the existence of three parallel paths from 'direction of change' to 'comparing the intensity of change': one direct path and two indirect paths through 'calculating the constant rate of change' and 'discerning varying rates of change'. The underlying assumption is that students improve in tasks involving the comparison of intensity of change by further enhancing their understanding of direction of change, calculating the constant rate of change, discerning varying rates of change, or in two or all types of tasks. This structure illustrates the catalytic role of direction of change as it explains both direct and indirect effects and the importance of coordinating covarying quantities both qualitatively and quantitatively to better understand and accomplish the comparison of intensity of change. This finding empirically validates Carlson et al.'s (2002) and Thompson and Carlson's (2017) covariational reasoning frameworks, suggesting that qualitative coordination precedes quantitative coordination. Finally, the model provided compelling evidence for the importance of 'discerning varying rates of change,' which primarily models students' capacity to distinguish intervals with constant or varying rates of change and how this distinction is reflected in the graphical representation of quantities. This type of reasoning proved to be the strongest direct predictor of students' capacity to compare the intensity of change, as they associate linearity with a constant rate of change between changing quantities and non-linearity with varying rates of change.

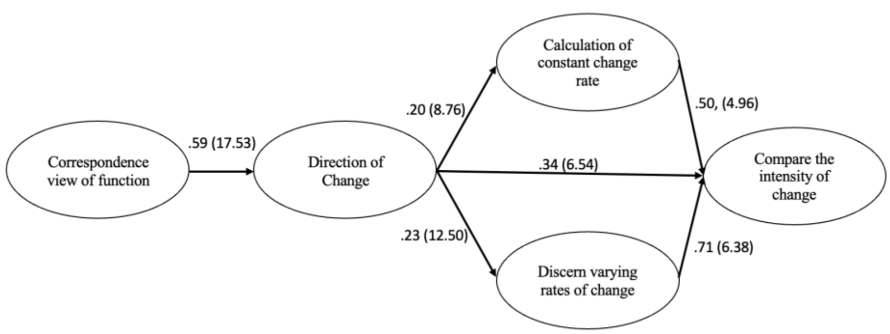

Figure 1: The Structural Model

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A FRAMEWORK FOR UNDERSTANDING THE EQUAL SIGN IN MIDDLE SCHOOL MATHEMATICS

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Keywords: Algebra and Algebraic Thinking

Research on the teaching and learning of the equal sign has largely focused on two different meanings for the equal sign—operational and relational—and the difficulties students experience in reasoning about equations when their understanding is restricted to an operational meaning (Carpenter et al., 2003; Kieran, 1981; Knuth et al., 2005; Matthews & Fuchs, 2020). Many researchers have focused on solving students' overreliance on the operational meaning of the equal sign by designing instructional interventions to help students replace their operational meaning for the equal sign with a relational meaning (see Hornburg et al., 2021 for a summary of instructional approaches). Scholars have also developed and tested frameworks for describing progressively sophisticated understandings of the equal sign to capture students' transition from operational to relational reasoning (c.f., Rittle-Johnson et al., 2011). However, much of the research on equivalence has left out two additional meanings for the equal sign—assignment and substitution (Jones & Pratt, 2011; Prediger, 2010). The framework presented in this poster for meanings of the equal sign is designed to address this hole in the literature.

Our interest in the four meanings for the equal sign led us to conduct a textual analysis of four middle school mathematics curricula. We sampled four units from the 7^{th} and 8^{th} grade textbooks of each series and coded the equations in those sections to identify which meanings of the equal sign were being used. We found that all four meanings of the equal sign were present in every grade in each series. Our results led us to reject models that hypothesize expert understanding arises from the gradual replacement of the operational meaning with the relational meaning of the equal sign. Rather, our finding that all four meanings of the equal sign are consistently used in middle school mathematics suggests that expert understanding develops by adopting additional meanings for the equal sign and becoming increasingly adept at using the contexts in which equations are embedded to determine which meaning of the equal sign is being used.

After identifying the meanings for the equal sign in each equation, we expanded our unit of analysis from a single equation to a single problem, which often involved multiple equations. We did so to see if there were common patterns in the use of equal sign meanings that students would need to recognize. We found that a substitution meaning for the equal sign was only used in problems where a relational or assignment meaning had already been used. We also noted that the assignment meaning was rarely used if the substitution meaning was not also used later in the problem. While Jones et al. (2012) posit that the substitution meaning is part of a relational understanding of the equal sign, our data suggests that the substitution meaning is also strongly connected to the assignment meaning of the equal sign. Thus, we suggest that an expert conception of the equal sign in secondary mathematics consists of three different understandings: an operational understanding based on the operational meaning, a relational understanding that is comprised of the linked use of the relational and substitution meanings, and an assignment understanding that is comprised of the linked use of the assignment and substitution meanings.

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EXPANDING SLOPE UNDERSTANDING: THE COMPOSED UNIT RATIO

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This study investigates slope as a composed unit ratio, offering an alternative understanding that diverges from a multiplicative comparison meaning. Through the lens of a teacher's unique interpretation, we bridge the gap between the different meanings of slope and the varied understandings of ratio, uncovering a nuanced meaning of slope as a ratio. Our findings suggest that viewing slope as a composed unit ratio offers accessible and meaningful pathways for learners, by highlighting a productive understanding. We advocate for further exploration into this meaning to enrich pedagogical strategies and support the development of robust mathematical understandings.

Keywords: Algebra and Algebraic Thinking, Teacher Knowledge

The slope concept is foundational for mathematical learning. Traditionally, slope has been understood and taught through several lenses, including rise over run, algebraic formulas, ratios, and as a measure of line steepness. These conceptions, as outlined by Nagle and Moore-Russo (2013) and other researchers (e.g., Byerley & Thompson, 2017; López et al., 2024), are pervasive in educational research and practice, offering valuable insights into the different ways teachers conceptualize slope. However, these frameworks often consider slope-as-ratio to mean a multiplicative comparison—an understanding recognized for its depth but noted for its scarcity among both teachers and students (Cho & Nagle, 2017; DeJarnette et al., 2020). In this study, we propose that understanding slope as a composed unit ratio may provide more accessible and equally rigorous pathways to deep mathematics.

Despite extensive analysis of the challenges surrounding the teaching and learning of slope, the exploration of composed unit reasoning as a conceptual foundation for slope has been relatively underexamined. Our research addresses this gap through the lens of a teacher, and specifically her interpretation of slope as a composed unit ratio. By investigating the intersection of the bodies of research regarding the different meanings of slope and the various interpretations of ratio, we illustrate that the teacher's conceptualizations of slope, while not aligning with a multiplicative comparison meaning, encompass other rich, nuanced understandings of ratio. The introduction of this alternative meaning of slope not only broadens the conceptual repertoire available for teaching slope but also has the potential to support the development of productive mathematical meanings among learners. Thus, this manuscript seeks to answer the question: What characterizes a ratio-as-composed-unit meaning for slope, and what are the affordances and constraints of this meaning?

Background and Theoretical Framework: Teachers' Meanings of Slope

Researchers have characterized a number of different meanings teachers hold for slope, many building from Nagle and Moore-Russo's (2013) 11 conceptualizations of slope. These meanings include slope as a geometric ratio, an algebraic ratio, and a physical property, among others. Synthesizing the literature base as a whole (e.g., Byerley & Thompson, 2017; Coe, 2007; López et al., 2024; Stump, 1999), three of the most prevalent teacher meanings for slope are (a) slope as an index of the steepness of a line, (b) slope as rise over run, and (c) slope as ratio. We discuss each of these in turn.

Slope-as-steepness meaning entails conceiving of a line as a physical object and making perceptual associations between its steepness and a numerical value; Nagle and Moore-Russo (2013) called this conception "physical property" (p. 3). When holding this conception, one might conclude that the slopes of two lines are the same if they have the same steepness (as determined visually), even if they are graphed in coordinate systems with different scales. This is a fairly common conception for both pre-service teachers (Avcu & Biber, 2022; Paulucci & Strepp, 2021; Tasova & Moore, 2018) and in-service teachers (Byerley & Thompson, 2017; López et al., 2024; Stump, 1999). For instance, Tasova and Moore (2018) found that one pre-service teacher's meaning of slope as a measure of steepness hindered her ability to recognize consistency across graphs in different coordinate orientations.

Slope-as-rise-over-run meaning entails thinking about slope as a procedure for moving up and over a specified number of units on a Cartesian coordinate plane, what Nagle and Moore-Russo (2013) called "geometric ratio" (p. 3), or when determined by the slope formula, $\frac{y_2-y_1}{x_2-x_1}$, "algebraic ratio" (p. 3). In Nagle and Moore-Russo's study, these were two of the most common conceptions. Multiple researchers have found that most teachers' meanings for slope include the slope formula (e.g., Byerley & Thompson, 2017; López et al., 2024; Stump, 1999; 2001). Although teachers can articulate the slope formula as a ratio, Byerley and Thompson (2017) showed that for many teachers this conception is non-multiplicative, as there is no attention to the change in one quantity compared to the change in the other quantity.

Slope-as-ratio meaning is a consequence of comparing the changes in two quantities multiplicatively to create an emergent quantity (Ellis, 2007). This entails understanding slope as a measure of one quantity's variation with respect to the variation of another quantity. For instance, one can conceive of speed as an emergent quantity through the multiplicative comparison of change in distance to change in time (Sherin, 2000). Few teachers refer to slope in this manner, and they can struggle to explain the use of division in the slope formula (e.g., Avcu & Biber, 2022; Byerley & Thompson, 2017; Coe, 2007; Talib et al., 2023). At the same time, slope-as-ratio meaning "is particularly powerful in that it supports one's ability to make sense of slope in a variety of situations" (Diamond, 2020, p. 166). A slope-as-ratio meaning is taken as evidence of a deeper understanding of slope and is critical for making connections between a slope value and the constant rate of change in a linear function (DeJarnette et al., 2020; Dolores Flores et al., 2020; Lobato & Siebert, 2002; Talib et al., 2023). Understanding slope as a ratio supports the ability to conceptualize the invariability of slope (Deniz & Kabael, 2017), to transfer the slope concept to other contexts (Hoban et al., 2013), and to use algebraic manipulations to determine slopes effectively (Cho & Nagle, 2017).

Two Ways to Understand Slope as a Ratio

Research addressing slope as a ratio typically considers ratio to mean a multiplicative comparison (Hoban, 2021; Talib et al., 2023). However, it is also possible to think of a ratio as a *composed unit*. A composed unit is created by joining two quantities to create a new unit, such as 5 cm: 2 sec (Lamon, 1994; Lobato & Ellis, 2010). One can then iterate or partition the created composed unit, maintaining the simultaneity between those quantities while creating new units (Jacobson et al., 2018). For instance, a person developing speed as a ratio can think of an inchworm crawling 5 centimeters in 2 seconds. The composed 5 cm : 2 sec unit could then be iterated to create other equivalent ratios, such as 10 cm : 4 sec, 20 cm: 8 sec, and so forth. This unit can also be partitioned to create, for instance, a unit ratio of 2.5 cm: 1 sec or 1 cm : 2/5 sec. Although some researchers consider the composed unit to be pre-ratio reasoning (e.g., Lesh et al., 1988), others point out that it can be used in combination with other concepts to develop a robust understanding of proportionality (Ellis, 2013; Lobato & Ellis, 2010).

In contrast, a *multiplicative comparison* entails considering how many times larger one quantity is compared to the other (Kaput & Maxwell-West, 1994; Lobato & Ellis, 2010). To continue the above example, this means understanding that the inchworm travels 2.5 cm for every second, or the number of centimeters traveled is always 2.5 times as large as the number of seconds. Regardless of whether one creates a composed unit or makes a multiplicative comparison, both ways of reasoning entail keeping the ratio of one quantity invariant to the other as the numerical values of both quantities change by the same factor (Aydeniz Temizer, 2022).

We found only one study, by DeJarnette and colleagues (2020), that distinguished between ratio as composite unit and ratio as multiplicative comparison in relation to slope meanings. The authors claimed that interpreting ratio as a multiplicative comparison, which they called a single value, is necessary for a sophisticated understanding of slope. However, we suspect that limiting the slope-as-ratio meaning strictly to multiplicative comparisons might miss instances in which teachers (or students) are beginning to build a multiplicative understanding by iterating and partitioning composed units. Given the documented difficulties teachers have with reasoning with slope as a multiplicative comparison, it may be fruitful to consider instances in which teachers are understanding slope as composed units and reasoning with such units in order to build notions of invariance. In our study, we present a case of a teacher whose interpretation of slope as a ratio of composed unit showcases a quantitative and productive understanding, thereby enabling a meaningful engagement with various scenarios.

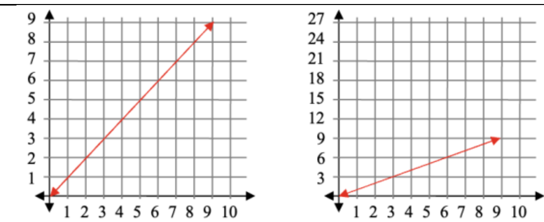
Methods

This study is part of a larger investigation aimed at understanding how teachers support mathematical generalizing (e.g., Ellis et al., 2024). Within this broader project, we identified Ms. R, a sixth-year high school algebra teacher, for an in-depth case study due to her insights into the teaching and understanding of slope. We adopted an investigative and descriptive case study approach (Merriam, 1998; Yin, 2009) to explore Ms. R's meanings of slope. This paper reports on findings from three semi-structured clinical interviews (Ginsburg, 1997) with Ms. R designed to probe her conceptualizations of and MKT related to slope. The 90-minute interviews focused on her understandings of slope and ratio, her insights into how students develop these understandings, and her strategies for supporting its development. This paper concentrates on Ms. R's personal meanings of slope, and our analysis of her broader MKT_{slope} is reported elsewhere (Tasova et al., 2024).

To investigate Ms. R's meanings of slope, we employed a set of questions inspired by Diamond (2020), such as Ms. R's spontaneous associations with the term "slope," contexts or situations she associates with slope, and her interpretation of specific slope values such as ½ and −1. Further, we introduced Ms. R to tasks, such as the "Five Students Problem" (adapted from Diamond, 2013; see Figure 1a), designed to reveal the extent to which her various meanings of slope. We asked Ms. R about what she would say or do with each of these students (see Figure 1a) to support them in developing a desirable understanding of slope and why. Additionally, we presented "The Hypothetical Student Situation" (adapted from Diamond, 2020; see Figure 1b), where a student questions the consistency of slope's meaning upon observing a function appearing steeper on one set of axes compared to another and solicited Ms. R's response to this confusion. To further explore Ms. R's nuanced understanding of slope as a composed unit ratio, we crafted follow-up questions. These questions were instrumental in highlighting the benefits of conceptualizing slope as a composed unit ratio.

Five students are discussing the meaning of slope in a linear context. Student A says that slope is $\frac{y_2-y_1}{x_2-x_1}$. Student B says that slope is the steepness of the line. Student C says that slope is rise over run. Student D says that slope is the rate of change of the line. Student E says that slope is the number m.

(a)



(b)

Figure 1: (a) Five Students problem and (b) The Hypothetical Student Situation

Our analysis process involved a qualitative approach, initially conducting a conceptual analysis to understand Ms. R's verbal and non-verbal explanations, thereby constructing viable models of her mathematics (Steffe & Thompson, 2000). Our analysis relied on aforementioned characterizations of teachers' meanings for slope and meanings for ratio in order to identify Ms. R's meanings. We then attempted to connect these categories that we identified and seek potential implications of those meanings in Ms. R's classroom teaching.

Results

We structure our analysis around two main themes. Firstly, we identify Ms. R's meanings of slope and her concerns regarding traditional slope understandings, highlighting her preference for slope-as-ratio meaning. Secondly, we focus on the exploration of slope as a composed unit ratio, exemplified by Ms. R's pedagogical approach and its impact on student learning. Through this exploration, we offer evidence as to the potential effectiveness of viewing slope as a composed unit for fostering a deeper understanding of mathematical relationships.

Ms. R's Various Understanding of Slope

Analysis of the interviews suggested Ms. R's understanding of slope was multifaceted, encompassing several meanings: slope-as-steepness, slope-as-rise-over-run, slope-as-formula, slope-as-*m*, and slope-as-rate-of-change[1]. She also understood and articulated limitations of

[1] We adopt Ms. R's terminology, using "rate of change" to describe the "ratio" understanding of slope, despite our awareness of the conceptual differences between the two terms.

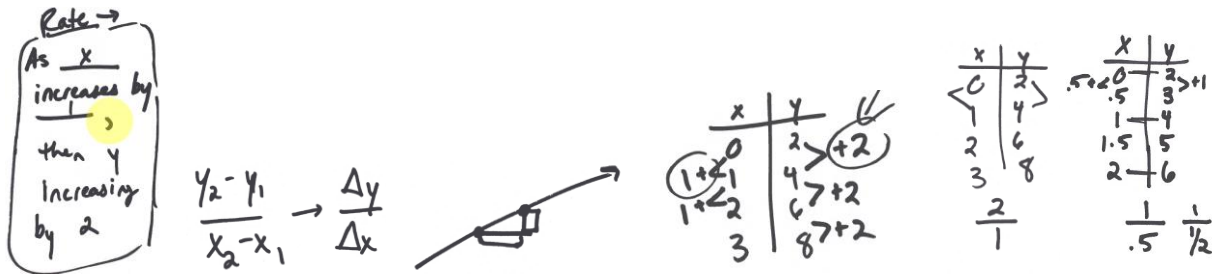
meanings for slope that did not include a rate of change understanding. For instance, in considering the Five Students problem (Figure 1a), Ms. R expressed concern about slope-as-steepness meaning in terms of its potential to mislead students into thinking that lines have the same slope based on their visual resemblance, regardless of axis scale or orientation. When asked The Hypothetical Student Situation task in which the same linear function is graphed on two coordinate axes with different scales (Figure 1b) and a student stated that the two functions had different slopes, she stated "steepness, that's probably where they're getting lost, because it'd be better to talk about change. I mean these lines, they don't initially look like they have the same slope…but they are the same graphs."

When considering slope as rise over run, Ms. R noted that students may rely on rise-over-run meanings because "that is their big middle school focus." Referencing the Five Students problem, she also suggested that students may rely on the slope formula only as a memorized fact: "this kid [Student A] spits out the slope formula because the teacher told him to over and over." Ms. R considered the slope formula to be more useful than the rise-over-run meaning due to its broader applicability to other scenarios, such as the arithmetic mean, and she saw both the rise-over-run meaning and the formula meaning to be superior to viewing slope as the number "m". Ms. R explained, "they know they are supposed to look at the number glued to the x [referring to $y = mx + b$], and that is it." Ms. R then wrote the formula $-2x + y = 3$, and explained that a student who viewed slope as the number "m" would get confused by an equation in this form: "They'll be like, what the heck happened in my graph, my equation?" Collectively, we had evidence from the interviews that Ms. R not only held these various meanings but was also able to position them against each other to discuss their productivity. Next, we illustrate her meaning of slope as a composed unit ratio.

Ms. R's Slope as Ratio Meaning: Composed Unit

Ms. R emphasized that she privileged Student D's meaning: "if this one actually knows what, like, rate of change of the line means, I like that one the best." She viewed Student D's meaning as versatile and applicable across various representations, such as table, equation, and graph, because it was not limited to specific formula or procedure. However, Ms. R's meaning was unclear. While she said she valued the rate-of-change meaning of slope, we were interested in exactly what her rate-of-change meaning was and, hence, what the meaning she valued involved. Did she construct a ratio as a multiplicative comparison, which she could view as a unit rate? Or did she have an alternate meaning for slope as a ratio or a rate of change? To gain insights into Ms. R's meanings, we asked her to create an example to describe how she would facilitate her students' development of the rate-of-change understanding. We hoped that by drawing attention to student development, we would not only gain insights into the meanings she could enact to solve problems but also those aspects of her meanings that she was consciously aware of.

Ms. R described an example with a speed of 25 miles per hour and explained that she could break this down into "for every 1 hour, the car is going 25 miles." Ms. R then explained that she could help students create similar phrases with a template: "As ___ increases/decreases by ___, then ___ increases/decreases by ___." Ms. R then clarified that students could use this template to generate equivalent ratios with different numbers, for instance, "for every two hours, the car is going 50 miles," which represents "the same slope." Ms. R's attention to the connection between the number of miles and the number of hours, combined with her understanding that this unit can be iterated to create other equivalent ratios, suggests a composed unit understanding of ratio.



(a) (b) (c) (d) (e)

Figure 2: (a) Slope as ratio, (b) Slope as formula, (c) Slope as rise over run, (d) Slope as rate of change in a table, (e) Determining the change in *y* for a 0.5-unit change in *x*.

Moreover, Ms. R did explicitly describe slope as a rate of change between two variables. She stated, “When I hear the word slope, I think rate of change…like two variables are changing.” As she continued, her ongoing commentary suggested a composed unit ratio meaning: “as x increases by 1, y increases by 2.” In order to demonstrate this meaning, Ms. R connected it to four representations (Figure 2a-d), pointing out that “they all connect, like they mean the same thing. But they just look different.” Figure 2a shows the composed unit 2:1. Ms. R could also interpret this composed unit via the slope formula (Figure 2b), and she could also demonstrate that meaning visually on a graph that was not drawn to scale. In describing the table in Figure 2d, Ms. R noted, “it [referring to the change in y-values] goes up 2 for every 1 x.”

Based on this response, we hypothesized that Ms. R’s meaning of slope involved ratio as composed unit, but it was unclear whether she could also consider the slope ratio to be a multiplicative comparison between changes in quantities. We therefore pressed Ms. R on her examples in Figure 2, particularly about the meaning of the value of “2”. Ms. R responded, “Technically, it’s 2 over 1, but we like to simplify it to just 2 for some reason I don’t know.” We then asked her what “2” would mean if x changed by a value other than 1. With this, we aimed to determine if Ms. R understood the slope’s value as an indicator of a multiplicative relationship, illustrating how many times the change in y is larger than the change in x. To answer our question, she created a new table (Figure 2e) and concluded that for a change in x of 0.5, the change in y should be 1. We took her actions and descriptions to suggest that Ms. R viewed her original ratio, 2:1, as a composed unit and partitioned it to create an equivalent ratio (1:0.5). This is a contraindication of slope as a multiplicative comparison as she did not multiply 0.5 by 2 to determine the change in y, nor did she ever appeal to a constant multiple between the two.

To further probe the extent Ms. R’s slope-as-ratio meaning was consistent with a composed unit or multiplicative comparison, we also asked her to explain the division in the slope formula, i.e., to explain why one divides to calculate a slope such as 2/1 or 1/0.5. Ms. R struggled to provide a clear explanation, and she interpreted the vinculum (division bar) as a means for matching changes in quantities: “That is communicating 2 is matching with 1.” Ms. R also explained, “to me, the division is this little comma [pointing to the comma in her phrase in Figure 2a].” These responses were further contraindications that Ms. R’s meaning of slope-as-ratio entailed a multiplicative comparison. Despite our attempts to explore division in different contexts, she could not provide a substantial rationale beyond following the formula's instructions, remarking, "I'm trying to see how division comes in the way, and I'm not seeing it. I don't really know why I even divide, beside because the formula said so."

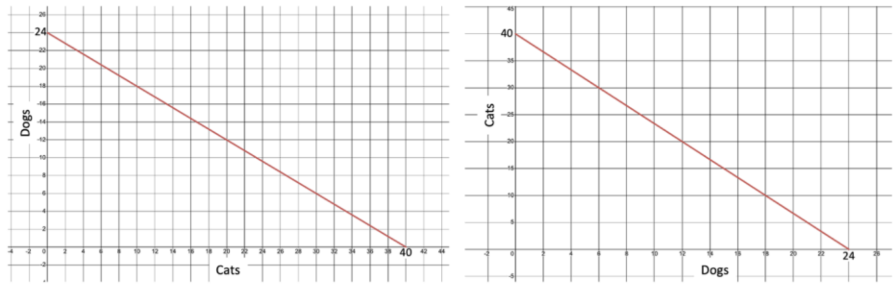
The Affordances and Constraints of Slope as Composed Unit

Ms. R's meaning of slope-as-ratio as a composed unit but not as a multiplicative comparison meant that she could develop equivalent ratios through iterating and partitioning the unit, but she could not directly determine the change in *y*-values for any associated change in *x* through multiplication. She also could not explain why slope involved division. This is consistent with the limitations identified in the literature, in which teachers struggle with conceptualizing slope and division as expressions of relative size, often viewing them non-quantitatively (Byerley & Thompson, 2017). These challenges hinder their ability to connect division with proportional reasoning (Coe, 2007), complicate the interpretation of points on graphs (Thompson, 2013), and hinder the recognition of slope as a consistent ratio of change (Stump, 2001).

Despite these limitations, we argue that there are also affordances to the meaning of slope as a composed unit ratio, beyond solving textbook problems. Ms. R could articulate meaningful connections based on a statement about slope as changes in *y*-values tied to corresponding changes in *x*-values; she could describe what that statement meant in terms of a function's graph, in terms of an associated table of values, and in terms of the formula for determining slope. Ms. R also had meaningful ways to support her own students in developing these connections. For instance, referring to the statement in Figure 2a and the graph in Figure 2c, she noted, "I would like for them to draw, like, the little triangle [drawing a triangle similar to that seen in Figure 2c], show me this part [referring to the run] and this part [pointing to the rise]."

Moreover, the meaning of slope as a composed unit ratio enabled Ms. R to conceive equivalence in the form of an invariant relationship across two graphs representing the same relationship in two different orientations (see Figure 3), which is a very sophisticated and productive way of thinking about graphical relationships (Moore et al., 2022). It also enabled her to respond in a meaningful way to a student thinking. We designed a task building on an activity she had implemented in the classroom, which referenced a pet-sitting business: A pet sitter can spend up to 8 hours each day feeding animals. Each cat requires 12 minutes per day, and each dog requires 20 minutes per day. We presented Ms. R with two graphs of the maximum numbers of dogs and cats that a pet sitter can feed, one with dogs on the *x*-axis and one with cats on the *x*-axis (see Figure 3), and we asked her which of the students' responses she agreed with, if any. Ms. R stated that Student D's response was the most correct, and therefore the slopes of the two graphs must be identical, despite their different visual representations and reciprocal values.

Five students are comparing the slopes of these graphs (see the figure below) in Pet Sitter Task context.



$d = 24 - \frac{3}{5}c$ $c = 40 - \frac{5}{3}d$

Student A says that slope is $\frac{y_2 - y_1}{x_2 - x_1}$. She chooses two points in each graph and calculate the slope for each graph. So, the slope of graph on the left is $\frac{24-0}{0-40} = -\frac{24}{40} = -\frac{3}{5}$ and the slope of graph on the right is $\frac{40-0}{0-24} = -\frac{40}{24} = -\frac{5}{3}$. So, the slopes are different. In fact, the slopes are reciprocals of each other.
Student B says that slope is the steepness of the line. Both graphs look like they have the same steepness. Therefore, the slopes of these graphs are the same.
Student C says that slope is rise over run. Therefore, the slopes are different.
Student D says that slope is the rate of change. So, both graphs show that "For every 3 dogs you are done feeding, you can feed 5 more cats". So, the rate of change in both graphs are the same. Therefore, the slopes are the same.
[*Student D also knows that inputs can be represented on the y-axis and outputs can be represented on the x-axis*.]
Student E says that slope is the number *m*. So, the slopes are different.

Figure 3: Dog and Cat Feeding Time Functions and Graphs

We then presented Ms. R with another hypothetical student answer, which claimed that slope is a rate of change, but the two slopes were different because one graph showed that for every 5 cats fed you can feed 3 more dogs, and the other graph showed that for every 3 dogs fed you can feed 5 more cats. Therefore, the rates of change and thus slopes must be different. Ms. R maintained her original stance, arguing that the numerical values in the two slopes symbolized the same quantities, and thus represented the same rate of change. She elaborated, "I would be like, you said for every 5 cats, the variable after 5 is *cats* [emphasis added] in both of your sentences, you know, like, the variable after 3 is *dogs* [emphasis added]." Given that both expressions conveyed an equivalent meaning, Ms. R reasoned that "So, I would say, like, your rate of change is the same." She argued this was akin to reordering a sentence in different contexts. This highlights Ms. R's relational and quantitative understanding of slope viewed as a composed ratio. Her conceptualization enabled her to interpret the slopes of the two graphs as the same, and it allowed her to understand the meaning of slope in terms of coordinated changes in quantities as represented in tables, graphs, and equations.

Discussion and Conclusion

In this study, we have explored a nuanced meanings of slope as a ratio, proposing a new perspective that transcends the algebraic or geometric ratio conceptions identified by Nagle and Moore-Russo (2013), and yet does not reach the level of multiplicative comparison. While the multiplicative comparison conception of slope is recognized for its depth of understanding, it remains a challenging achievement for both teachers and students. This insight aligns with the broader literature, which has long noted the difficulties inherent in grasping slope from a multiplicative standpoint (Cho & Nagle, 2017; DeJarnette et al., 2020; Dolores-Flores et al., 2020). A potential implication of our research is that understanding ratio as a composed unit may serve as a more accessible—since it is not as cognitively complex as a multiplicative meaning—and meaningful foundation for developing deeper insights into the concept of slope.

The case of Ms. R illustrates that benefits traditionally associated with understanding slope as a multiplicative comparison are still attainable through the lens of a composed unit ratio. Ms. R's slope meaning allowed a fully quantitative understanding, enabling her to effectively interpret and apply the concept of slope in various contexts. She correctly used algebraic manipulations to determine slopes and connected a slope value to the constant rate of change in a linear function (Cho & Nagle, 2017; DeJarnette et al., 2020; Diamond, 2020; Dolores-Flores et al., 2020; Lobato & Siebert, 2002; Talib et al., 2023). This underscores the flexibility and effectiveness of the composed unit ratio approach in fostering a comprehensive understanding of slope.

To clarify, our stance is not to undermine the importance of understanding ratios as multiplicative comparisons or their relevance in understanding the concept of slope. Instead, we propose prioritizing the development of an understanding of ratios as composed units as a foundational step. This strategy involves encouraging the development of equivalent slopes as ratios of changes that leverage values smaller than 1 and values with "messy numbers," thereby enhancing learners' understanding of the invariant relationship between changes in *y*-values and their corresponding *x*-values. For instance, Ms. R's ability to conceptualize an equivalent slope of 1/0.5 from an initial slope of 2/1 exemplifies the potential of this method to deepen understanding. We could ask similar questions to encourage other equivalent slopes, such as 14/7, 9/4.5, 0.5/0.25, or 0.4/0.2. Asking learners to reflect on what is invariant across all these different ratios could encourage attention to the fact that regardless of the increase in *y*-values, the increase in *x*-values remain twice as large.

We must acknowledge that our insights are based on the experiences of a single teacher, Ms. R, providing a compelling case that it is feasible to conceptualize slope as a ratio in effective and impactful ways without necessarily incorporating the notion of multiplicative comparison. Ms. R's example shines a light on the viability of comprehending slope through the lens of a composed unit. However, further research is needed to assess the prevalence and efficacy of this approach among broader populations, including pre-service teachers and secondary students. We believe that such investigations will contribute significantly to the mathematics education field by offering alternative pathways to understanding slope, thereby enriching pedagogical strategies and student learning experiences.

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PENSAMIENTO ALGEBRAICO Y COMPUTACIONAL: UN ESTUDIO EXPLORATORIO CON UNIVERSITARIOS

ALGEBRAIC AND COMPUTATIONAL THINKING: AN EXPLORATIVE RESEARCH ON COLLEGE

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Un problema para la educación de alumnos universitarios es la desconexión entre los temas vistos en sus cursos de matemáticas y las aplicaciones en su práctica profesional. Para el caso de un ingeniero de software, es necesario desarrollar habilidades que le permitan modelar y solucionar problemas en contextos de programación, en el cual pueden intervenir tres tipos de lenguaje que se interrelacionan: el vernáculo, algebraico y computacional. Por tanto, surge el interés de investigar cómo los estudiantes de ingeniería afines a esta práctica relacionan estos tres tipos de lenguaje. Para ello, se diseñó un cuestionario con problemas de álgebra básica y programación. Como resultado se encontró que los estudiantes tienen facilidad para describir en lenguaje vernáculo, códigos que están escritos en lenguaje computacional. Sin embargo, enfrentan dificultades para expresar esos códigos en lenguaje algebraico.

Palabras clave: pensamiento algebraico, pensamiento computacional, resolución de problemas

Introducción

Los cursos de matemáticas para estudiantes de ingeniería suelen tener un enfoque generalizado, con herramientas que de forma tradicional han sido seleccionadas como todo aquello que un ingeniero debe saber emplear. Además, como menciona Devlin (2001), estos cursos suelen ser tradicionales, utilizando el modelo “recipiente que vierte”, donde se vierten los contenidos de un curso sobre los estudiantes como si fueran una jarra vacía, esperando que retengan toda esta información. Al final, esta práctica educativa puede ocasionar que los estudiantes de ingenierías no vean valor en sus cursos de matemáticas.

En particular, el rol del ingeniero de software requiere del desarrollo de habilidades específicas que le permitan ser capaz de modelar y solucionar situaciones problema de su práctica (Parnas, 1999). Por ejemplo, un estudiante de programación debe ser capaz de encontrar errores que el compilador pueda demostrar, tales como errores gramaticales, de sintaxis y semántica; así como errores lógicos y deficiencias de calidad de software.

Hay antecedentes de enseñanza del álgebra mediante la programación para el desarrollo del pensamiento algebraico y computacional (Kilhamn & Bråting, 2019; Bråting & Kilhamn, 2021). Tanto el álgebra como el software son representados mediante un lenguaje. Bråting y Kilhamn (2021) exponen las diferencias entre ellos, así como su testimonio añadiendo el pensamiento computacional mediante la programación en la currícula para el bachillerato.

Marco Conceptual

Sibgatullin et al. (2022) define que en el pensamiento algebraico se incluyen 5 categorías: (a) generalización de la aritmética, (b) manipulación y transformación de igualdades mediante operaciones inversas, (c) análisis de estructuras matemáticas, (d) relaciones y funciones y, por último, (e) lenguaje algebraico y su representación.

Por otra parte, el pensamiento computacional se refiere a los procesos de pensamiento involucrados en la formulación de problemas y la representación de sus soluciones de una manera que pueda ser realizada por una computadora (Proctor, 2022). Es una metáfora universal del razonamiento utilizada tanto por humanos como por máquinas, que abarca un amplio espectro de razonamiento a través del tiempo y las disciplinas (Henderson et al., 2007). El pensamiento computacional implica resolver problemas utilizando una base de evaluación lógica y, a menudo, matemática, y es lo que hacen los profesionales de la informática cuando analizan y diseñan sistemas (Walden et al., 2014). Mejora la integración de las tecnologías digitales con las ideas humanas y enfatiza habilidades como la creatividad, el pensamiento lógico y el pensamiento crítico (Zacharis & Niros, 2019).

De acuerdo con Bocconi et al. (2018), existe un gran debate sobre el concepto de pensamiento computacional, donde existen 2 corrientes principales: a) una habilidad que va más allá de la programación y que engloba otras habilidades como la resolución de problemas, pensamiento lógico y la creatividad; b) un enfoque más orientado a la tecnología, donde se busca que se desarrollen competencias de los empleados en el sector de las TIC que finalmente resolverán problemas sociales. El enfoque a tomar en este trabajo será orientado a la primera idea, aunque no existe ninguna contradicción con que, al desarrollar las habilidades descritas no se mejoren las competencias, la fuerza de trabajo y se tenga un impacto en la sociedad.

El lenguaje, es característica de ambos tipos de pensamiento, algebraico y computacional. Así, para llevar a cabo la investigación, se propone una serie de problemas en un contexto de programación. Los problemas diseñados bajo este marco promueven el uso de 3 lenguajes: (a) vernáculo, (b) algebraico y (c) computacional.

Duval (2017) afirma que existen distintos signos para representar los objetos matemáticos. No se debe confundir el signo con el objeto. Por ejemplo, la suma no es el signo más (+), sino que, es la idea de esta operación y cualquier interpretación en línea con un proceso de adición. Cada objeto matemático tiene una o más representaciones. Los registros de representación se refieren a un sistema que permite representar varios objetos. Existen varias representaciones para un mismo objeto en diferentes sistemas de representación.

Material y método

Este estudio de carácter exploratorio se implementó con 22 estudiantes de ingeniería en computación y 9 de informática, en una universidad pública de México. De los 31 estudiantes, sólo 2 manifestaron haber programado y 6 mencionaron haber tenido pocas experiencias con ese tipo de tareas. La implementación se hizo en un curso de precálculo de primer semestre, pero a la par esos estudiantes cursaban la materia de "fundamentos de programación". Para el estudio se diseñaron 5 problemas en un contexto de programación. Los fragmentos de código utilizados son en lenguaje Python. A continuación, en la Figura 1 se exponen los problemas, mientras que su propósito se discute junto con las respuestas dadas por los estudiantes en la sección de resultados.

Problema 1

(a) Describe en tus propias palabras y en una sentencia corta qué hace el siguiente programa:

```
numero1 = int(input("Ingrese el primer número entero: "))
numero2 = int(input("Ingrese el segundo número entero: "))
resultado = numero1 + numero2
print(f"El resultado es {resultado}")
```

Responde las siguientes preguntas:

(b) ¿Qué pasaría si al asignar la variable resultado, se alterna el orden de las variables numero1 y numeros2 en la suma?

(c) ¿Cómo escribirías en su forma algebraica la misma expresión?

Problema 2

(a) Describe en tus propias palabras y en una sentencia corta qué hace el siguiente programa:

```
numero = int(input("Ingresa un número entero: "))
resultado = numero * numero
print(f"El resultado es #{resultado}")
```

(b) ¿Cómo escribirías en su forma algebraica la expresión de la segunda instrucción del código?

Problema 3

(a) Describe en tus propias palabras y en una sentencia corta qué hace el siguiente programa:

```
 estatura = int(input("Ingresa tu estatura en centímetros: "))
peso = float(input("Ingresa tu peso en kilogramos: "))
resultado = peso / (estatura * estatura)
print(f"El resultado es #{resultado:.2f}")
```

Responde las siguientes preguntas:

(b) ¿Qué pasaría si al asignar la variable resultado, se eliminan los paréntesis de la instrucción. Justifica tu respuesta.

(c) ¿Cómo podrías reescribir la asignación de resultado sin utilizar paréntesis?

(d) Escribe una expresión algebraica equivalente a la instrucción donde se asigna la variable resultado.

Problema 4

(a) En el sistema inglés, se utilizan libras como medida de peso y los pies para medir la altura. Utilizando colores o subrayando/encerrando, separa las instrucciones que convierten las unidades y las que realizan el cálculo del índic de masa corporal.

```
peso_ingresado = float(input("Ingrese su peso en libras: "))
altura_ingresada = float(input("Ingrese su altura en pulgadas: "))
peso_kilogramos = peso_libras * 0.453592
altura_metros = altura_pulgadas * 0.0254
imc = peso_kilogramos / (altura_metros * altura_metros)
print(f"Su índice de masa corporal (IMC) es: {imc:.2f}")
```

Responde las siguientes preguntas, justifica tus respuestas:

(b) ¿Qué pasaría si en lugar de float se utiliza int?

(c) ¿Qué pasaría si se altera el orden de los operadores en la división?

Problema 5

Utilizando las operaciones suma, resta, multiplicación y división, responde las siguientes preguntas:

(a) ¿En qué operaciones se altera el resultado si se cambia el orden de los operandos?

(b) ¿Qué utilidad tienen los paréntesis cuando se utilizan varias operaciones en una misma expresión? ¿Cómo influyen estas propiedades de las operaciones en la programación?

Implementación

Los problemas se aplicaron en 2 ocasiones, la primera aplicación fue individualmente, la segunda se hizo en grupos de 3 a 4 alumnos con el fin de promover la socialización de las respuestas, formando en total 8 grupos. Al final, con ayuda del profesor se revisaron las respuestas en plenaria y se explicaron conceptos involucrados en los problemas.

Figura 1: Problemas planteados en la experimentación

Resultados

Tabla 1: Análisis de la aplicación de los instrumentos

	Respuestas individuales	Respuestas grupales
1.a	Suma de 2 números (28). Describe el programa (2). No respondió (1).	Todos respondieron de forma correcta.
1.b	No cambia (23). Depende del caso (4). Sí cambia (2). Error de programación (2).	Únicamente un equipo respondió de forma incorrecta mencionando un cambio de orden en el que aparecen las variables.
1.c	a+b con diversas literales (16). x+x (5). numero1+numero2 (4). Expresión que contenía un producto (2). Otros (4).	Todos los equipos respondieron con expresiones de la forma a+b.
2.a	Multiplicación (13). Elevar al cuadrado (4).	Sólo un equipo respondió de forma

	Multiplicación por sí mismo (3). Potencia 2 (1). Multiplica 2 números (7). Descripción línea por línea sin mencionar la operación como en 1.a (2). Múltiples respuestas (1).	equivocada “multiplica 2 números”. Algunos mencionaron potencias, multiplicación, multiplicar por sí mismo o elevar al cuadrado.
2.b	x*x (7). a*b (7). x^2 (5). ab (2). -x*y (2). $n1^2$ (1). numero*numero (2). Valores concretos (2). Sin respuesta (2).	El mismo equipo que respondió mal (2.a) utilizó una expresión de la forma a*b. Las respuestas fueron acordes a (2.a).
3.a	Términos como masa, masa corporal o IMC (13). Descripción de las operaciones sin hacer una interpretación real de la fórmula (7). Descripción errónea de las operaciones (5). Otras (6)	6 equipos mencionaron IMC, los otros 2 describieron la fórmula a nivel algebraico sin darle un significado.
3.b	Da resultado incorrecto (14). Hay error de programación (7). No cambia el resultado (3). Sin respuesta (2). Respuestas erróneas divergentes (5).	6 equipos respondieron de forma correcta. Un equipo dijo que sólo se hacía más grande el código y otro escribió la fórmula: estatura*estatura/peso.
3.c	Sin respuesta (10). Invierte: estatura * estatura / peso (6). Uso de potencias (5). Agrupa usando comillas (2). Divide la expresión en dos (1). Respuesta no relevante (7).	3 equipos dejaron la pregunta en blanco, un equipo la respondió separando en 2 instrucciones y otro utilizó potencias. El resto escribieron expresiones erróneas.
3.d	Escribe expresión algebraica correcta con paréntesis o exponentes (11). Invierte la división (7). Asigna valores concretos (4). Omite paréntesis (3). Otras (6).	Sólo 3 equipos respondieron a/(b*b), el resto cambiaron el orden de los argumentos de la división u omitieron los paréntesis.
4.a	Separación correcta de instrucciones (10). Separación de las operaciones de entrada/salida y de los cálculos (7). Sin respuesta (6). Otras (8)	4 equipos separaron de forma correcta las instrucciones. Un equipo no respondió. Los otros 3 equipos resaltaron de forma errónea.
4.b	-Sin respuesta (14). No mostraría los decimales (5). Resultados enteros (5). Cambio de tipo en las variables (2). No cambiaría en nada (2). Error de programación (1). Otro (2).	6 equipos respondieron de forma correcta al decir que no habría decimales o que se mostraría sólo el resultado entero. Un equipo mencionó que el resultado no cambia y un equipo mencionó un error de programación.
4.c	Resultado cambia (18). Sin respuesta (9). Nada (3). Error en algunos casos (1).	Un equipo escribió la fórmula p * a / a, el resto respondió de forma correcta.
5.a	Resta y división (7). Sin respuesta (7). Multiplicación y la división (5). División (5) Todas las operaciones (1). Otras respuestas (6).	Sólo un equipo respondió de forma equivocada, mencionando a la multiplicación y la división
5.b	Jerarquía, orden (16). Separación de	Todos los equipos mencionaron ordenar.

operaciones (7). Sin respuesta (6). Otras (2).

Tabla 2: Éxito obtenido individual y en equipos

	Individuales		Equipos			Individuales		Equipos	
P	A	%	A	%	P	A	%	A	%
1.a	28/31	90,32%	8/8	100%	3.c	6/31	19,35%	2/8	25%
1.b	23/31	74,19%	7/8	87,5%	3.d	11/31	38,71%	3/8	37,5%
1.c	16/31	51,61%	8/8	100%	4.a	10/31	32,26%	4/8	50,%
2.a	21/31	67,74%	7/8	87,5%	4.b	13/31	41,94%	6/8	75,%
2.b	13/31	41,94%	7/8	87,5%	4.c	18/31	58,06%	7/8	87,5%
3.a	22/31	70,97%	8/8	100%	5.a	7/31	22,58%	7/8	87,5%
3.b	14/31	45,16%	6/8	75%	5.b	23/31	74,19%	8/8	100%

Conclusiones

Los estudiantes que inician sus estudios en carreras de ingenierías afines a la programación ingresan con un pensamiento computacional limitado, incluso nulo, lo cual podría ser comprensible, pues es ahí en donde se espera desarrollar ese tipo de pensamiento. Sin embargo, también se pudo constatar que los estudiantes han desarrollado un pensamiento algebraico limitado, a pesar del prolongado tiempo de estudio del álgebra. Entre las concepciones erróneas encontradas se puede mencionar que, los estudiantes no conocen las reglas del lenguaje algebraico, pues no reconocen la utilización y asignación de las variables, así como el significado de algunos operadores y propiedades de las operaciones aritméticas. Desde el marco de Duval (2017), los estudiantes no conocen las representaciones de los objetos, y como vimos en los trabajos de Sibgatullin et al. (2022) y Proctor (2023), el lenguaje para describir objetos es importante para el pensamiento algebraico y computacional.

Los problemas que se propusieron para el estudio implicaron una reflexión y uso de tres tipos de lenguaje en distintos sistemas de representación: vernáculo, algebraico y computacional. Al respecto, hay evidencia de que los programas sencillos en un lenguaje de programación pueden ser explicados por los estudiantes en el lenguaje vernáculo, pero con dificultades para expresarse en el lenguaje algebraico. Además, conforme se introdujeron otros signos propios de la programación, los estudiantes mostraron mayor dificultad para interpretar el código.

Estos resultados conducen al diseño de una investigación más amplia en la que se pretende desarrollar el pensamiento computacional y mejorar el pensamiento algebraico de los estudiantes que ingresan a una ingeniería de software. A fin de que puedan encontrar relación y utilidad de sus conocimientos matemáticos para realizar tareas de su práctica como futuros profesionales.

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THE TEACHING AND LEARNING OF ALGEBRA: SIX ELEMENTS THAT BUILD ALGEBRAIC FLUENCY FROM CONCEPTUAL UNDERSTANDINGS

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This work presents six research-based elements that align with building algebraic fluency from conceptual understandings in the teaching and learning of algebra. The six elements are: symbol sense, processes/relationships of algebra, process as an object, anticipating solution strategies, anticipating solution formats, and relationships among representations.

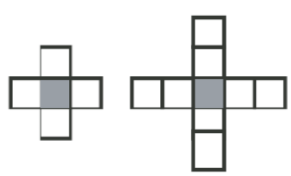
Keywords: Algebra and algebraic thinking, algebraic fluency, conceptual understanding

The purpose of this work is to describe how six research-based elements contribute to students building algebraic fluency from conceptual understandings. First, we use Kaput's (2008) two core aspects of algebra to illustrate the difference between algebra readiness and conceptual understandings of algebra. Second, we provide a characterization of algebraic fluency from conceptual understandings of algebra. Third, we describe six research-based elements of algebraic fluency and illustrate connections to the teaching and learning of algebra.

Core Aspects of Algebra, Algebra Readiness, and Conceptual Understandings

Kaput (2008) defines algebra using two core aspects. First, "[a]lgebra as systematically symbolizing generalizations of regularities and constraints" (p. 11). One example of a regularity of our number system is the commutative property of addition. We note that 7 + 3 = 3 + 7 and can generalize this regularity symbolically as a + b = b + a. Another example of symbolizing generalizations of regularities is finding a rule for the n^{th} term in a visual pattern like in Figure 1.

Examine the shapes below.



1. Draw the next two shapes in the pattern.
2. Count both the shaded and the unshaded squares. Without drawing the shapes how many squares will be in the:10^{th} shape? 25^{th} shape? 100^{th} shape? n^{th} shape?

Figure 1: Visual Pattern, Growing Squares

The first core aspect aligns with an algebra readiness perspective (Feikes et al., 2022). Algebra readiness involves helping students see regularities, describe generalizations from the regularities, and represent these generalizations symbolically. The growing squares pattern (Figure 1) illustrates this as students can generalize and represent a regularity symbolically by expressing the n[th] shape as "$4n + 1$".

However, algebra entails more than just generalization and symbolic representation. Kaput's (2008) second core aspect is: "algebra as syntactically guided reasoning and actions on generalizations expressed in conventional symbols systems" (p.11). We look at this second core aspect as the processes, properties, procedures, and symbolic generalizations which allow for the abstract manipulation of algebraic objects. This characteristic of algebra allows for the modeling of real-life situations, the creation of abstractions, the manipulation of algebraic objects, and the application of abstractions to real-life situations. An example consistent with this core aspect of algebra is the derivation of the quadratic formula by completing the square (See Figure 2).

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Figure 2: Quadratic Formula

The derivation of the quadratic formula by performing actions on variables *a*, *b*, and *c* through completing the square is a process which becomes a mathematical object. This object can then be used in other algebraic work or manipulations. The core aspects of algebra are what allow for the manipulation of algebraic generalizations.

Kaput's second core aspect of algebra aligns with our perspective of developing conceptual understandings of algebra. We understand conceptual understandings in a way that is consistent with the National Research Council (NRC) (2001) and National Council of Teachers of Mathematics (NCTM) (2014) descriptions of students having an integrated and functional grasp of mathematical ideas. Opportunities to develop conceptual understandings of algebra occur when students are provided problems where they can develop and manipulate symbolic generalizations of regularities in meaningful ways (Feikes et al., 2021; Feikes et al., 2022).

Algebraic Fluency from Conceptual Understandings of Algebra

Our perspective of algebraic fluency is based on the NCTM (2023, p. 1) position statement on procedural fluency. Our work defines algebraic fluency as the ability to apply algebraic processes, properties, and procedures with efficiency, flexibility, and accuracy; to transfer algebraic processes, properties, and procedures to different problems and contexts; to build or modify algebraic processes, properties, and procedures from other processes, properties, and procedures; and to recognize when a particular algebraic process, property, or procedure is more appropriate than another. Algebraic fluency entails understanding how to carry out procedures, why procedures can be performed, and which is more appropriate. Research on procedural fluency is relevant to our perspective on algebraic fluency. Efficiency, flexibility, and accuracy involve the knowledge of multiple strategies and the ability to apply them in different contexts (Star, 2005), including procedures and processes in algebra.

Algebraic fluency builds from conceptual understandings so that students become skillful in using procedures appropriately, flexibly, and efficiently, considering different representations, using reasoning to apply these representations to different purposes, and producing accurate answers (NCTM, 2014). Students who build conceptual understandings are more likely to remember and use topics, concepts, and procedures without error due to reasoning and understandings of mathematical relationships (e.g., Fuson et al., 2005; Hiebert & Carpenter, 1992; Hiebert et al., 1997). Alternatively, mindlessly manipulating symbols or learning tricks like memorizing formulas or mnemonics are typically applied to specific problems, likely to be misused in different mathematical problems, and often quickly forgotten (NRC, 2001).

To develop fluency, students need to practice strategies and procedures to solidify their knowledge (NCTM, 2014, p. 45). For example, multiplying two binomials or two larger polynomials (like a binomial and a trinomial) can be taught as an application of the distributive property. This would be a conceptual way to teach and learn this skill because it presents this skill as part of the coherent whole of algebra. As students practice the skill to develop proficiency, they can develop fluency by being asked about patterns and strategies when multiplying the polynomial expressions.

The following example illustrates algebraic fluency with conceptual understandings of algebra when a student recognizes properties of a given equation and related procedures. To solve for x in $\frac{1}{2}(x+4)-1=5$, students could consider a variety of mathematical concepts. They may recognize that the equation represents $6-1=5$, such that $\frac{1}{2}(x+4)$ should equal 6. The distributive property could be used to create an equivalent expression for $\frac{1}{2}(x+4)$ or each term could be multiplied by 2 so that $\frac{1}{2}$ is no longer part of the equation. Students who have developed algebraic fluency from conceptual understandings could consider the benefits and drawbacks of different ways of finding a solution.

Six Elements that Algin with Building Algebraic Fluency from Conceptual Understandings

We have identified six research-based elements which build algebraic fluency from conceptual understandings of algebra and allow students to comprehend algebraic notation or symbols and operate within the processes, properties, and procedures of algebra.

1. Developing *symbol sense* by learning the constructs that algebraic symbols convey.
2. Understanding *processes/relationships of algebra* and how to express these with symbols, e.g., $2n$ is “$n + n$” or “$2 \times n$”; $y = 3x$.
3. Conceptualizing a *process as an object,* often called process-object duality or procept.
4. Understanding, anticipating, and being proficient with *solution strategies*.
5. Anticipating *solution formats*, like solutions as a single number, a graph, or a function.
6. Noticing and expressing *relationships among representations,* like relationships between algebraic expressions, tables, and graphs.

Discussion of the Six Elements

The six elements that align with building algebraic fluency with conceptual understandings have important implications for the teaching and learning of algebra. For example, Arcavi et al. (2017) describe *symbol sense* as giving meaning to symbols and expressions and connecting the symbols to underlying concepts (p. 94). Students need to have an understanding of the constructs conveyed with symbols. Becoming fluent with the abstract ideas represented by the symbols and examples that help these representations become transparent is a key to helping students learn to think with algebra (Kieran, 2007).

Understanding *processes/relationships of algebra* is significant because it helps students express the generalizable from the particular. Mason and Sutherland (2002) emphasize that algebra is about processes and understanding them through the symbols used to represent the processes. Students need to learn to "see through" the symbols by being aware of the processes and applying them (Wheeler, 1989).

Algebra includes processes that are represented with symbols. A conceptual leap occurs when students begin to see and act on *processes as objects* (c.f. Arcavi et al., 2017; Kieran, 1992). Sfard (1991) describes recognizing processes as objects as reification. Warren et al. (2016) have suggested that one of the benefits for students seeing a process as an object is when algebraic objects become accessible which leads to identifying algebraic structures. Further, students should be able to unpack an objectified process into objects related by processes (Tall & Gray, 1994). Instructors of algebra need to encourage students to see processes as objects by providing examples, discussing how processes act as objects, and working on examples that both compress processes into objects and decompress objects to processes (Tall & Gray, 1994).

Understanding, anticipating, and demonstrating proficiency with *solution strategie*s and anticipating *solution formats* require more attention during instruction and work on mathematical problems. Students need to anticipate aspects of solution formats so that possible strategies can be considered (Boero, 2001; Booth, 1988). To help students develop fluency with algebraic solutions formats and strategies, instructors should discuss possible solutions, different ways the solution could be conveyed, different solution strategies, and examples where when a solution does not meet the anticipated expectation.

The final feature of *relationships among representations* is central for creating meaning in algebra (Kieran, 2007, p. 712). Encouraging multiple representations provides opportunities for students to make sense of algebra as represented in different types of thinking and allows students more ways to express their algebraic understandings (Kieran, 2006). We need to help students analyze multiple representations, encouraging them to notice what is similar and what is different about each (Jacobs et al., 2010).

The six elements are interrelated and build upon each other. For example, Kieran (1992) notes, "the development of algebraic symbolism … allowed the symbolic forms to be used structurally as objects" (p. 391). This statement relates to the *process as an object* feature and shows the importance of *symbol sense* in having a structural perspective of algebra.

Additional Considerations and Conclusion

We have identified six elements that align with building algebraic fluency from conceptual understandings of algebra to address theoretical and practitioner needs. Each one of the six elements are research-based and together they form a unique framework for examining the teaching and learning of algebra. Building algebraic fluency should position students as capable, using reasoning and decision-making to improve skill and understanding (NCTM, 2023). We suggest that student learning of algebra can be improved by developing skills and understandings around the six elements. This can occur in classrooms by discussing conceptual aspects of algebra and providing opportunities for students to build understandings about algebraic symbols, processes, properties, and procedures. Research is needed to understand how the six elements of algebraic fluency impact student achievement, equitable classroom practices, and the development of assessments for algebra.

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Chapter 3:

Equity and Justice

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

TOWARDS ARTICULATING CRITIQUE TO ALGORITHMS WITH POSITIONING THEORY

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Information and communication technologies transform our personal and social practices. The demand for highly skilled labor for the specialized technology-intensive market has risen dramatically. For example, we can notice a visible ethical demand that schools should produce efficient programmers who learn programming (algorithmic actions) from early years at schools to develop a "right" technical mindset and dispositions for living and making out in technology-intensive societies. Digital programs or platforms operate with a variety of algorithms regulated by mathematical procedures, principles or relations. Algorithms are codes or structures that regulate computation objects and digital environments. They are 'a series of steps undertaken to solve a particular problem or accomplish a defined outcome' (Diakopolous, 2015). Algorithms have four functions: prioritization, classification, association, and filtering. These rules act as a sort of moral order that sets the stage where actions of prioritization, classification, association, and filtering take place. Often, these processes operate within value grids that actors/institutions/agencies establish for a particular purpose. For example, digital platforms such as Facebook create particular cues for users based on viewing histories of particular content or patterns of likes or dislikes initiated by users. This necessitates paying attention to particular discursive acts/ storylines that algorithms generate for people to adopt or resist within the interactive nexus of rights and duties. Positioning theory (PT) studies dynamic ways in which human actions are constituted within changing moral fields where social actions are regulated through the assignment of rights and duties to actors (see Harré, & Van Langenhove,1999a)). Dooley and Grimes (2023) identified how an imaginary interaction between preservice teachers and students in mathematics classroom situations can identify complex configurations of dynamic positions that students can take to resist dominant positions ascribed to them under the moral discourse of mathematics. They used Goffman's theory of two-face behavior to highlight how different moral discourses set dynamic constraints on students and teachers to show their performance of positions under the positive face. Here, a positive face can be characterized as "a desire to be appreciated and valued by others, a desire for approval", and a negative face as "concern for freedom of action, a desire to be unimpeded" (pp. 48–49). Similar situations can be observed in ways Facebook algorithms direct the patterns of likes in particular directions. This theoretical critique on algorithms invites the mathematics education community to see algorithms' actions "as behavioral nudges" for guiding human conduct in a particular direction for creating particular human beings with serious implications for social justice and injustices.

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THE PURPOSES OF SCHOOL MATHEMATICS FOR SECONDARY MATHEMATICS TEACHERS

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Keywords: Teacher Beliefs; Instructional Vision; Equity, Inclusion, and Diversity

With its three-part *Catalyzing Change* series, the National Council of Teachers of Mathematics (NCTM) challenged educators and leaders to broaden the purposes of school mathematics beyond college and career readiness. At the secondary level, they suggested that mathematics should help students to "(1) expand professional opportunity; (2) understand and critique the world; and (3) experience wonder, joy, and beauty" (NCTM, 2018, p. 9).

The NCTM's (2018) proposals are consistent with arguments made by critical mathematics education scholars in prior decades, who wrote about the aims of teaching mathematics while asking *whose* social contexts were being taken up in schools (e.g., Ernest, 2000; Ernest, 2010; Greer & Mukhopadhyay, 2003). But those scholars did not focus solely on potential (individual) benefits of school mathematics; they also argued that school mathematics privileges some students and perpetuates patterns of marginalization, suggesting that mathematics has historically served as a gatekeeper and tool for students' stratification (Ernest, 2018; Louie, 2017; Martin et al., 2010). These more "hidden" purposes of school mathematics are notably absent from the NCTM reports. Consequently, so too is an argument for how the proposed purposes and strategies would help to counter the potential harmful practices of school mathematics. Alongside the idealistic purposes to which we aspire, we must, as a field, consider the realistic purposes that school mathematics has historically served.

To understand the prevalence of these various purposes of school mathematics among the "frontline" of mathematics education, this study investigated the purposes of teaching mathematics in schools asserted by middle and high school mathematics teachers. This poster will present interview-gathered perspectives of 20 randomly selected middle and high school mathematics teachers from the midwestern U.S. state of Missouri. The teachers' subject areas ranged from sixth grade mathematics through AP Calculus, and years of experience ranged 2-30 years. The interview protocol consisted of various questions related to purpose including, "what do you think teaching mathematics accomplishes with respect to society?" and "is there anything that school mathematics accomplishes that you wish it did not accomplish?" The interviews were coded using inductive methods and consolidated using thematic analysis.

My findings reveal that teachers primarily think about the purposes of school mathematics with respect to its benefits at an individual and social level; however, they also described critical aims. All 20 respondents described one purpose of school mathematics to be related to NCTM's (2018) "expanding professional opportunities." But some suggested additional purposes that were not included in NCTM's (2018) list, including fifteen responses related to students' development of transferable skills (e.g., communication, collaboration, logical thinking) and, more critically, five teachers suggested that school mathematics serves as a tool for students' stratification. Given the prevalence of responses related to this critical purpose of school mathematics that is absent from NCTM's (2018) reports, this study indicates a need to expand the frameworks beyond the benefits of teaching mathematics in school to consider the harmful byproducts of mathematics education.

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SPATIAL EQUITY CONSIDERATIONS FOR CALCULUS AND STATISTICS AS SECONDARY MATHEMATICS ENDPOINTS

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This paper considers geospatial equity implications relative to the calculus versus statistics secondary mathematics endpoint debate. Using two proprietary datasets from the Advanced Placement® (AP®) program— (1) the population of 2015-'19 AP® course audit data and (2) the population of AP® examination scores received by U.S. public school students over the same period—the geographic availability of and achievement in AP® Calculus AB and AP® Statistics are examined. Results indicate that availability and achievement vary across space and subject. On average, rural students face the largest locale-based achievement disadvantage in both subjects. Further, AP® Statistics is shown to be least available in rural schools and to rural students. Implications for locale-based achievement and availability supports are discussed, particularly in the context of ongoing debates.

Keywords: High School Education; Calculus; Data Analysis and Statistics; Equity, Inclusion, and Diversity

Current debates concerning the optimal endpoint of secondary mathematics, be it calculus, statistics, or calculus *and* statistics often employ equity arguments to support each possibility (Bland et al., 2024; Burdman et al., 2018). Secondary calculus is fraught with inequities in access, enrollment, and outcomes. Students who are either of Color or from low-SES backgrounds—or both— have fewer opportunities to take calculus than their white and/or more affluent peers (Kolluri, 2018; Oakes, 1990). Additionally, structural racism, tracking, low expectations, and a lack of calculus-specific teacher professional learning impact Black, Latinx, and low-SES students' opportunities to experience success in calculus (Bressoud, 2020; McGee, 2020; McGee & Martin, 2011). In response, many suggest that statistics and data science suffer less from issues of inequity and may offer all students more opportunities to have positive experiences in higher-level mathematics (LaMar & Boaler, 2021).

Conversely, advocates for preserving calculus as the ultimate course in secondary mathematics have shown that secondary calculus can have benefits in the college admissions process as well as in post-secondary calculus courses and beyond (Bressoud, 2020; Ferrini-Mundy & Gaudard, 1992). Calculus proponents argue that denying opportunities for secondary calculus also denies or delays post-secondary STEM opportunities for students who would otherwise enroll in the course (Bressoud, 2020). Further, statistics also has equity concerns; there is clear racism and sexism in statistics and data science, as datasets and algorithms often reflect the white, male power structures that exist and persist in society today, creating false binaries, actively discriminating, and perpetuating harm (D'Ignazio & Klein, 2020; Noble, 2018).

Thus, weighing the calculus versus statistics debate is complex, with both subjects having documented equity concerns across several intersecting dimensions of identity—specifically race, gender, and class. As the field collectively grapples with inequities along these dimensions and their implications for secondary mathematics curricula, I argue that a critical fourth dimension is missing from the conversation—that of spatial equity, which should be considered in conjunction with gender, race, and class.

Drawing from critical spatial theory, which calls for the interrogation of "the intersections of space, power, and knowledge in order to expose geographies that perpetuate or disrupt inequities in both processes and outcomes" (Annamma, 2017, p. 4), this study examines how students' geographies shape their opportunities to enroll and succeed in advanced calculus and statistics. Using licensed data from The College Board®, I use regression analysis and spatial data science to address the following questions: (1) How does geography shape the *availability* of advanced secondary calculus and statistics across communities in the United States? (2) How does geography shape student exam *achievement* in advanced secondary calculus and statistics across communities in the United States?

Methods

Data and Measures

This study makes use of two primary datasets, both of which are licensed from The College Board® and contain data about their Advanced Placement® (AP®) program. AP® data is particularly useful for this analysis given that the Advanced Placement® program is a primary means of secondary calculus and statistics delivery across the country, reaching hundreds of thousands of students annually (Bressoud, 2020; Lee & Harrison, 2021). Data set one is the full population of all U.S. public schools with AP® Calculus AB and/or AP® Statistics course audits[2] approved by The College Board® in at least one of the four years prior to the onset of the Covid-19 pandemic (*N* = 18,557 schools). Data set two is the population of all AP® Calculus AB and AP® Statistics examinations taken by U.S. public school students in the same four-year window (*N* = 1,733,822 exams). Both data sets have a selection of student/school population demographic variables and, importantly, a National Center of Education Statistics (NCES) locale variable that communicates whether the school is in a rural, town, suburb, or city area. The audit data has been joined with the balance of U.S. public schools which were not approved to offer AP® Calculus AB and/or AP® Statistics, and the geographic location of all schools was obtained.

Analytic Strategy

To consider the geographic landscape of AP® Calculus AB and AP® Statistics course *availability* (RQ1) across the U.S., I conduct exploratory spatial data analysis (ESDA). Using ESDA, data is examined concerning both the geographic locale and state political boundaries to reveal how space shapes course availability.

To consider the geographic landscape of AP® Calculus AB and AP® Statistics student exam *achievement* (RQ2) across the country, I consider the following model for each subject:

$$Y_{ij} = \beta_0 + \beta_1 City_i + \beta_2 Town_i + \beta_3 Rural_i + \beta_4 RaceEth_i + \beta_5 Female_i + \beta_6 SESDecile_i + a_j + \epsilon_{ij}$$

In this model Y_{ij} is the predicted standardized AP® exam score, ranging from 1 to 5, for the *i*th student in the *j*th state by year fixed effect. The suburban locale serves as the comparison group.

[2] In this study, I use AP® audit data as a proxy for course availability. However, it is possible that schools had syllabi approved through the audit process but ultimately did not offer the corresponding course.

Results

Course Availability (RQ1)

Over the four-year period, of 21,139 public high schools, 11,094 schools (52.5%) offered AP® Calculus AB and 7,463 schools (35.3%) offered AP® Statistics. This equates to 5,328,661 students attending schools without AP® Calculus AB and 10,025,690 students in high schools without AP® Statistics. This unavailability is also distributed disproportionately across space.

Relative to AP® Calculus AB, AP® Statistics is less available in schools and to students across all four primary NCES locales. In city locales, 666 less schools (*representing 1,176,509 students*) offer AP® Statistics as compared with AP® Calculus AB, 686 less *(1,246,204 students)* in suburban areas, 749 less *(957,120 students*) in towns, and 1,530 less schools *(1,318,196 students*) in rural areas. That is, both in terms of school and student counts, rural areas face the largest disparities in AP® Statistics availability as compared with AP® Calculus AB.

Rural (un)availability varies by state, region, and subject. The proportion of rural schools offering AP® Calculus AB is strongest in much of the Northeast, with over 70% of rural schools in Maryland, Connecticut, Vermont, Massachusetts, New Jersey, Delaware, and New Hampshire offering AP® Calculus AB. Conversely, the availability of AP® Calculus AB is generally weakest in the Midwest—less than 15% of rural schools in Nebraska, North Dakota, South Dakota, Missouri, and Kansas offer the course.

Table 1: Geographic Availability of AP® Calculus AB and AP® Statistics Courses

NCES Primary Locale	Total # Schools	**AP® Calc AB Availability** # Schools (%)	**AP® Stats Availability** # Schools (%)
City	5,110 (24.2%)	2,782 (25.1%)	2,116 (28.4%)
Suburb	5,186 (24.5%)	3,773 (34.0%)	3,087 (41.4%)
Town	2,876 (13.6%)	1,518 (13.7%)	769 (10.3%)
Rural	7,967 (37.7%)	3,021 (27.2%)	1,491 (20.0%)
Total	21,139	11,094	7,463

The same regional patterns hold for rural availability in AP® Statistics, although availability is far more limited. Only five states, all within the Northeast, have 50% or more of their rural schools offering AP® Statistics—Maryland, Delaware, New Jersey, Connecticut, and Massachusetts. Concerningly, fifteen percent or less of rural schools in twenty states offer the course: eight states are in the Midwest; nine are in the West; and three are in the South.

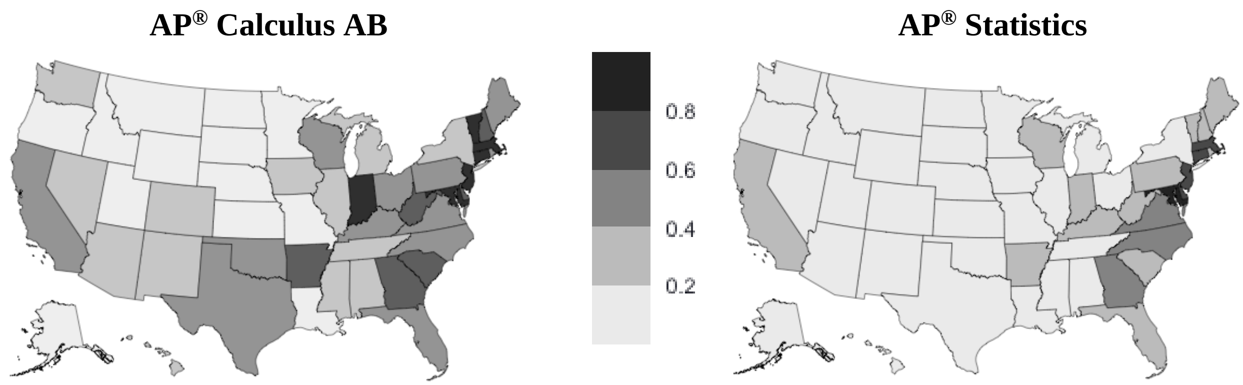

Figure 1: Proportion of Rural Schools Offering Each AP® Course by State

Course Achievement (RQ2)

Spatial inequities also exist related to course achievement in AP® Calculus AB and AP® Statistics. Controlling for documented inequities experienced by students of Color, women, and those from low socio-economic backgrounds, the regression coefficients on geographic locale as predictors for AP® Calculus AB and AP® Statistics exam achievement are given in Figure 2.

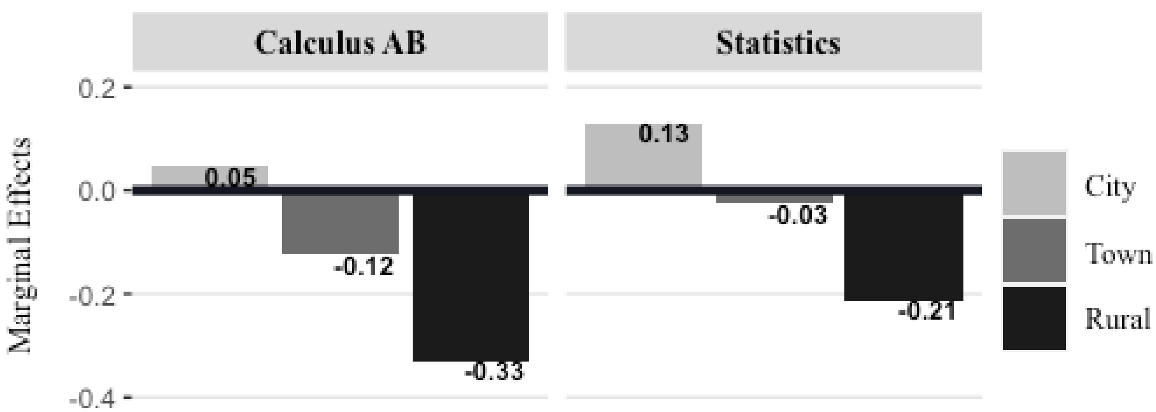

Figure 2: Marginal Effects of Locale on AP® Mathematics Exam Achievement

Compared to students in suburban schools, rural students, on average, receive lower AP® exam scores in both AP® Calculus AB (-0.33 pts.) and AP® Statistics (-0.21 pts). The models also show a slight advantage for city-located students (AP® Calculus AB: 0.05 pts.; AP® Statistics: 0.13 pts.) in comparison to suburban-located peers. The rural disadvantage is consistent across both exams but is largest in AP® Calculus AB. Notably, however, the improvement in rural disadvantage in AP® Statistics is closely mirrored by an increase in city advantage, thus, city-rural gaps improve only slightly when moving from AP® Calculus AB to AP® Statistics.

Discussion and Conclusion

Rural disadvantage exists in both the availability of AP® mathematics coursework and AP® mathematics achievement (see also Wolfe et al., 2023). While AP® Calculus is more widely available nationwide, of the four geographic locales, AP® Calculus AB is least available, proportionally, in rural schools. This unavailability worsens in AP® Statistics, with rural schools and students having the largest absolute disadvantage. There is also spatial inequity in both AP® Calculus AB and AP® Statistics achievement. Across both examinations, rural students, on average, receive the lowest exam scores across all locales. The suburban-rural achievement gap is wider in AP® Calculus AB (0.38 pts.), while the city-rural achievement gap is similar (+/- .04 pts.) across both subjects.

As the field wrestles with whether and how to alter secondary mathematics pathways in service of modern career requirements, equity—and ultimately— students, this paper presents an additional spatial dimension to broaden discussions around these choices. Sizable rural disadvantages in achievement span both typical secondary mathematics endpoints, necessitating support and resources from both the field and policymakers, regardless of the outcome of current debates. Further, this paper demonstrates that a choice to pivot from AP® Calculus AB to AP® Statistics would be spatially exclusionary without concurrent initiatives to expand availability to the latter, particularly in rural areas of the United States.

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REHUMANIZING MATHEMATICS THROUGH MULTIMODAL MICROANALYSIS

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We, five math teacher educators (MTEs), share how multimodal microanalysis has the potential to contribute to rehumanizing mathematics education by expanding what counts as evidence of students' knowings. We define multimodal microanalysis as the detailed examination and interpretation of minute interactions between various sensory modalities and mathematical representations at a granular level. Using mathematics cognitive interview videos, we share our research design and analysis processes and discuss how multimodal microanalysis can contribute to rehumanizing mathematics teaching and learning.

Keywords: Teacher Noticing, Cognition, Equity, Inclusion, and Diversity

Background and Theoretical Perspectives

As five critically conscious mathematics teacher educators (MTEs) with diverse areas of expertise and lived experiences, we acknowledge that social, cultural, and political factors influence all aspects of our work. Our diversity enhanced this research project because our different perspectives, experiences, and insights fostered creativity, innovation, and a more comprehensive understanding of the complex issues we studied. Before this project, three of us exclusively attended to only what our students said or wrote. After learning about embodied cognition and working with the other two members of the research team, we realized our limited noticing neglected to acknowledge valuable embodied utterances. Drawing upon the literature on embodied cognition and multimodal noticing, our research reveals how multimodal microanalysis can contribute to rehumanizing mathematics education. We define multimodal microanalysis as the detailed examination and interpretation of minute interactions between various sensory modalities and mathematical representations at a granular level.

Rehumanizing Mathematics Through Multimodal Microanalysis

Noticing. Teachers engage in noticing when they attend, interpret, and respond (AIR) to learners (Jacobs et al., 2010). Noticing skills enables teachers to catalyze critical moments and optimize students' learning trajectories (Stockero & Van Zoest, 2013). Teachers attend to and interpret a myriad of evidence of students' thinking but must decide which evidence is pivotal and how to respond to these critical moments (Rotem & Ayalon, 2023). The sociopolitical framings of a teacher impact whether the nature of their noticings leans towards a deficit or anti-deficit model (Louie et al., 2021). Moreover, teachers with multidimensional noticing skills consider how past and future events shape and are shaped by their dispositions and instructional practices (van Es et al., 2022).

Embodied cognition. Embodied cognition research centers on students' use of "body-based resources to make meaning and to connect new ideas and representations to prior experiences"

(Nathan, 2022, p. 4), including gestures, body forms, simulations, and the use of materials, such as manipulatives. Our thinking and understanding of the world are closely linked to our physical bodies and experiences. Moreover, both teaching and learning are dialogic, multimodal activities. Our understanding of abstract concepts, such as mathematics, is rooted in our sensory experiences and bodily movements. As such, we want to begin with and continuously provide students with playful, concrete experiences (Abrahamson et al., 2020).

Multimodal noticing. Walkoe and colleagues (2023) connected embodied cognition and noticing in their research on multimodal noticing. Multimodal noticing adds to the noticing literature by including noticing nonverbal evidence. It is critical for teachers to develop multimodal noticing skills because gestures and actions play a critical role in student thinking (Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2012; Walkoe et al., 2023).

Rehumanizing mathematics. The three dimensions of Rochelle Gutiérrez's (2018) rehumanizing mathematics framework that are most relevant to our multimodal microanalysis include: (1) emotions and body, (2) participation and positioning, and (3) broadening mathematics. Teachers skilled in multimodal noticing consider students' embodied utterances as evidence of students' participation and knowings. Students may exhibit these embodied utterances, such as facial expressions, gestures, or finger counting, before providing their knowings verbally or in writing (Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2012). The broadening of what counts as evidence of mathematical knowings practiced by teachers using multimodal noticing contributes to their positioning of each and every student as a knower and doer of mathematics. This expansion of teachers' ideologies and pedagogies can serve as micro and macro affirmations that promote a rehumanizing of mathematics (Abrahamson et al., 2020).

As seen in Figures 1 and 2, Bondurant and colleagues (2023) layered Gutiérrez's (2018) rehumanizing mathematics framework as a lens over the AIR framework and found that multimodal noticing contributed to the rehumanizing of mathematics. Despite the promising potential of multimodal noticing, most novice teachers' noticings focus on general impressions and lack connections to any evidence from critical events (Bondurant et al., 2020; Moss & Poling, 2019). Although experienced MTEs notice specific pivotal moments, unless they are prompted and reminded to focus specifically on embodied utterances, they privilege students' verbal or written work as evidence of students' knowings (Bondurant et al., 2023). We attribute this to the plethora of stimuli the noticer must attend to at a given moment. We embarked on this study to see if microanalysis can serve as a vehicle for nuanced noticings of embodied utterances.

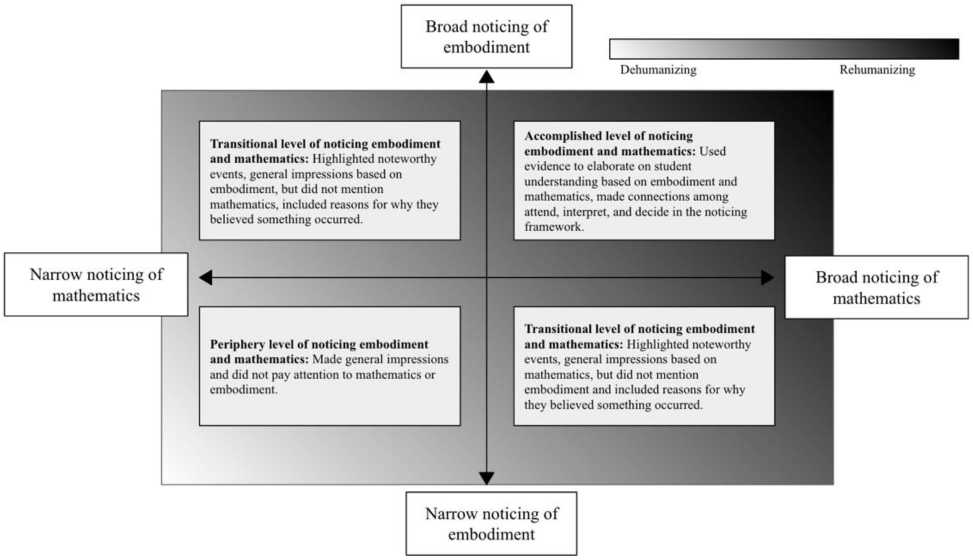

Figure 1: Rehumanizing Noticing through Embodiment

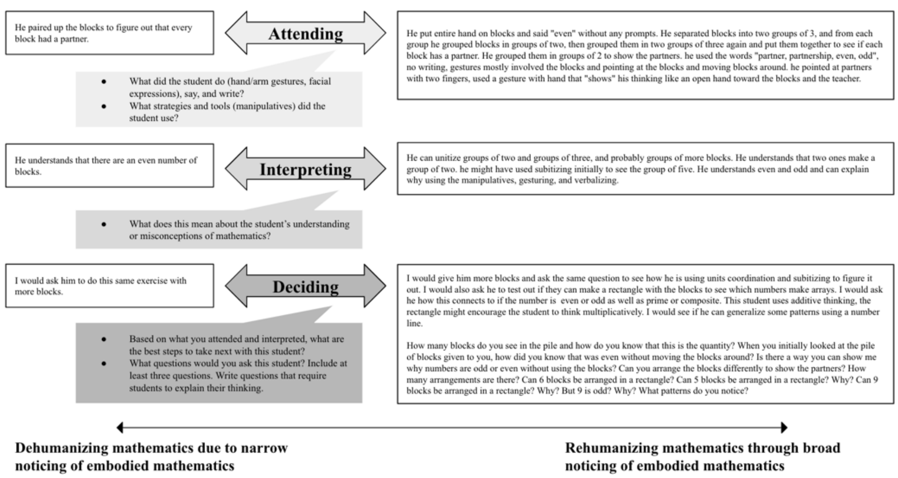

Figure 2: Examples of Noticing of Embodiment and Mathematics Methodology

Microanalysis

Analysis of students' spoken, and written words only captures the end products of their learning, which may not provide the researcher a complete story of the students' thinking or knowings. Microgenetic approaches accentuate fine-grained processes of learning and change that occur "at the smallest observable time scales" (Parnafes & diSessa, 2013, p. 7). The researcher seeks a "moment-by-moment explanatory account of learning in particular contexts" and "conceptual resolution" that yields "very fine distinctions in meaning" that must be tracked (Parnafes & diSessa, 2013, p. 7). Microanalysis involves closely observing and understanding small-scale alterations as they happen, which provides insight into broader developmental changes over time (Calais 2008). The major advantage of using microanalysis is that it captures in-the-moment processes of learning. An in-depth examination of small-scale educational artifacts provides the researcher with nuanced insights into specific phenomena. Researchers employing microanalysis often focus on detailed observations, interactions, and individual experiences to uncover subtle patterns and meanings. We used microanalysis to intentionally foreground and focus on embodied utterances as evidence of students' knowings.

Data

We used ELAN a free, multimodal annotation tool for digital audio and video media created by the Max Planck Institute for Psycholinguistics, The Language Archive, Nijmegen, The Netherlands, and available at https://archive.mpi.nl/tla/elan (Lausberg & Sloetjes, 2009). We viewed a 40-second segment from a cognitive interview video from the course resource *Mathematics for Elementary Teachers: A Contemporary Approach* by Musser and colleagues (2013). In the video segment, a second-grade boy is given six blocks and asked, "Is the number of blocks even or odd?" We chose this video because it includes a manipulative and the video frame includes the student's face and body, providing us with opportunities to attend to embodiment in our noticings. In the video, the student first quickly provided the end product (the answer) but then struggled to explain the process. The student quickly declared that six is even without any interaction with the blocks, suggesting a unitizing strategy. However, when asked how he knew, he initially organized the six blocks into two rows of three, stating they were "even." When asked to explain, the student used a different method, likening an even number to having "partners" and combining the blocks into three groups of twos. Ball and Bass (2003) reported that third-grade students might perceive numbers like six as both even and odd because of their reasoning that the number of groups of two is odd. Further exploration showed that third-grade children may use fair sharing, groups of two, or the alternation of even and odd numbers on the number line strategies to determine if a number is even (Bass, 2005).

Procedure

To foreground embodiment and evidence of student knowings, we intentionally sequenced our noticings as follows: (1) student gesture, (2) student words, (3) instructor gesture, and (4) instructor words. When focusing on gestures, we muted the audio. By focusing on words after gestures we were able to uncover the instructor's and our missed opportunities to notice student's embodied utterances. For each of the four noticing focuses, we watched the video segment multiple times at full, 0.75, and 0.5 speeds. We met via Zoom to analyze the data, discuss noticings, conduct open coding, and consolidate microanalysis notes (Corbin & Strauss, 2014).

Findings

Through microanalysis, we uncovered a plethora of nuanced embodied utterances that we neglected to notice previously. Regarding the student, we uncovered student-embodied utterances that provided valuable evidence of the student's knowings. For example, we uncovered that the student had memorized the fact that six is even. The student immediately stated, "My teacher taught us that…," positioning the teacher as an undisputed authority of knowledge. However, when he had to explain why he was initially uncertain about whether six was even since there were an odd number (three) of "partners" (groups of two blocks), we found that using the blocks and gesturing helped him develop his understanding of why six is even. Moreover, the student's larger gesturing, which moved away from his body, suggested a possible growth in the student's confidence (Cuddy, 2015). Regarding the instructor, we noticed the instructor's embodied utterances and how they connected to evidence of the student's knowings. For example, when the instructor was stoic for extended periods, the student did not receive any scaffolding, which challenged the student to demonstrate his understanding. We also noticed when the instructor's gestures mirrored and echoed the student's gestures. Mirroring gestures are when one individual initiates gesturing about a task, while another simultaneously mimics or matches those gestures, allowing for a real-time physical representation of the first person's reasoning, while echoing gestures involve a sequential reproduction of gestures with a noticeable time gap between them as both individuals explain their reasoning (Walkington et al., 2018).

Based on our review of the literature, expanding our noticings to include embodied utterances through microanalysis contributes to rehumanizing mathematics by providing opportunities to highlight and leverage every student as a brilliant mathematician (Abrahamson et al., 2020; Gutiérrez, 2018).

Discussion

In this microanalysis we set out to explore how embodiment-focused microanalysis of student-teacher interactions in elementary mathematics expands research mathematics educators' professional noticing and contributes to a rehumanizing of mathematics teaching and learning. The microanalysis process provided several "aha" moments for us. We realized that although our previous work had explicitly and intentionally focused on embodiment, microanalysis was needed to uncover a more complete understanding of students' knowings. We found the student initially relied on the teacher as an authority figure but as the student engaged with the manipulatives and reasoned through the explanation their gestures became bigger and more confident. Furthermore, we noticed a point we missed before where the student appeared to be processing thought while tapping and dragging their finger on the table, in the middle of arranging the blocks as both two groups of three and three groups of two. Based on our findings, we consider microanalysis a critical tool to avoid missed opportunities that can acknowledge students' knowings.

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TEACHER IDENTITY AND COMMUNITY KNOWLEDGE THROUGH DIGITAL MATHEMATICS STORYTELLING IN INDONESIA

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This project explored the use of Digital Mathematics Storytelling (DMST) with Indonesian mathematics teachers, focusing specifically on mathematical identity and the connection to community funds of mathematical knowledge. Traditionally, mathematics teaching in Indonesia marginalizes local knowledge in favoring of a Eurocentric outlook. DMST challenges this by facilitating the integration of community-rooted mathematical narratives into educational spheres by enhancing teacher identity and fostering a broader understanding and appreciation of mathematics. Engaging in a three-day DMST workshop prompted teachers to recognize and harness the mathematical community knowledge in their communities and forge a richer, more inclusive, humanizing disposition towards their mathematics teaching.

Keywords: Teacher Beliefs; Equity, Inclusion, and Diversity; Research Methods; Technology

Background

Mathematics learning throughout the world is often dominated by traditional, academically-focused, Eurocentric perspectives, dissuading mathematics students and teachers from seeing people like themselves as mathematical beings, from recognizing their family and community as holders of strong mathematical knowledge, and from using modern communication tools such as social media and online video to engage in mathematical discourse (Chao, 2018; D'Ambrosio, 1985; Joseph, 2011; Powell & Frankenstein, 1997; Star et al., 2014; Vakil, 2014; Zulkardi et al., 2020). This research project explored the ways that Digital Mathematics Storytelling (DMST), a technology-based teaching tool that draws on the ancient practice of storytelling, impacted mathematics teaching identities when used with mathematics teachers in Indonesia. Using qualitative and design-based research methods, this project explored the ways that mathematics teachers shared and discussed connections to family and community-based mathematics through creating short videos within a 3-day digital mathematics storytelling clinic. This research project also fostered global collaboration between researchers in the USA and Indonesia around cutting-edge mathematics teaching pedagogy that utilized widely available mobile technology, particularly engaging mathematics teachers in ways that connect learning technology with research-based pedagogy.

Theoretical Framing

Children throughout Southeast Asia live mathematically rich lives, yet their academic mathematics achievement is more correlated with family wealth and access to tutors, rather than the ability to connect out-of-school mathematics knowledge to in-school mathematics knowledge

(Ginting et al., 2018; Saleh et al., 2017; Trung & Nguyen, 2020; Zulkardi et al., 2020). This disconnect, in turn, creates gaps in mathematics proficiency throughout Southeast Asia, where many adolescent children decide that mathematics is not something that they identify with and subsequently end up removing themselves from future STEM oriented careers (Sheffield et al., 2018; Vuong et al., 2020; Yonezawa et al., 2016). One equity-oriented approach to solving this problem is through connecting out-of-school mathematics with in-school mathematics through a *funds of knowledge* approach, which honors families, communities, and the knowledge they bring to classroom mathematics (Aguirre et al., 2012; Civil, 2014). Additionally, a *storytelling*-based approach draws on historic norms, connecting the way storytelling is used by communities to share important knowledge between generations and define a community history (Lambert, 2013; Prusak et al., 2012).

Identity is not only embodied within the stories a person tells about themselves, but also encompasses the actual act of narrating or storytelling (Sfard & Prusak, 2005). Identity is a verb, made and remade through the act of storytelling. Our stories are not merely descriptions of a static reality, but rather dynamic constructs that can change over time and context. Our narratives serve as constructs that embody our range of experiences, characteristics, and expectations, thereby defining the creation and evolution of our personal and social identities. Even more important than telling a story to explore our identity is the way that identities are reified and endorsed through the acceptance, validation, and re-telling of our narratives. Simply put, our stories are our identities.

Counter-storytelling, therefore, involves sharing stories and experiences that challenge existing dominant (and oppressive) narratives and stereotypes (Solórzano & Yosso, 2002). Counter-storytelling is a tool for individuals in marginalized communities to highlight their experiences and perspectives, and challenge destructive narratives that perpetuate harmful stereotypes. Through counter-storytelling, individuals and communities reclaim their own narratives and thereby their own identities.

When people tell narratives about their out-of-school mathematical experiences, they position themselves and their communities as mathematical. They tap into the power of authorship to counter stigmas that mathematics must only used academically for school–they enact the truth that mathematics is community oriented (Aguirre et al., 2013; Langer-Osuna & Nasir, 2016a). And today's children, particularly in Southeast Asia, are growing up in a world where video-sharing platforms like TikTok, WhatsApp, and Instagram allow for easy sharing of personal stories (Rideout, 2017; Yue et al., 2019). Therefore, this research project explored the ways that teachers can use Digital Mathematics Storytelling, a mechanism in which videos, photographs, and audio come together, to share mathematically-rich narratives from families and communities connected their out-of-school mathematics and community knowledge with their in-school mathematics teaching practices.

The DMST technique involves telling a personal story, one that centers on experiences, conflicts, and growth. These are stories one would share around a meal or family gathering, tales that can be told and retold. These stories often do not follow the Eurocentric three-act framework propagated by Western media, but instead draw upon local community storytelling archetypes involving folktales and family histories (Levy, 2000; Osman, 1999; Tacchi, 2009; Thang & Mahmud, 2017). DMST, therefore, does not involve just creating videos of simplistic mathematics situations, such as trying to figure out the most economical option when buying fruit at the market. Rather, DMST shows mathematics as it really exists within the community, showcasing not only mathematics, but the beauty of the community itself. For example, a digital mathematics story might involve a teacher exploring the ways her Islamic faith guides her continued understanding of mathematics and mathematics pedagogy, drawing connections between passages in the Quran and how she used this knowledge to reflect not only on her knowledge of linear equations, but on her own community responsibilities as a teacher.

Research Methods

The Digital Math Storytelling clinics at the heart of this research project involved 3 sessions over one week hosted at a university on the island of Java in Indonesia. Utilizing a Participant Design-Based Research methodology (Amiel & Reeves, 2008; The Design-Based Research Collective, 2003; Vakil et al., 2016), participants themselves gave continual feedback towards the development of the protocols used within the study, so that it aligned with the cultural and societal norms of mathematics teaching in Indonesia. The Participant Design-Based Research structure allowed our team to update the existing Digital Mathematics Storytelling Protocol (Author, 2019) through feedback and rapid iterations with the research participants themselves. For instance, based on early feedback from participants, the daily schedule of the clinics were revised to incorporate daily prayer times.

The hypothesis grounding this research project was that, when mathematics teachers engage in a DMST workshop situated in their community, they will: (1) develop a stronger sense of their mathematics identity as connected to their family and community identities and (2) tell stories about the rich mathematics examples that come from their own communities and connect these examples to their own teaching.

And while the analysis for this study is still ongoing, the outcomes are being measured through two instruments: **a)** Pre- and post-clinic questionnaires about mathematics identity and, **b)** the Digital Mathematics Stories themselves, which showcase unique storytelling affordances unique to each participant's culture and community, along with the ways they position themselves towards mathematics.

The Clinic

The Digital Math Storytelling clinic comprised three sessions, each lasting four hours, over the course of one week. The workshops focused on eliciting mathematical identities and community mathematical knowledge through storytelling. Each session revolved around a *storycircle* (Lambert, 2013), in which research participants shared stories in small groups in order to elicit constructive feedback. These storycircles allowed each storyteller to evolve and hone their stories and often consisted of a mixture of oral storytelling and sharing video footage or images of the emerging stories. The final session of the workshop revolved around a community screening, in which participants shared their final video stories and then engaged in a whole group discussion about the ideas, emotions, and connections to teacher identity evoked through the creation of, sharing, and community watching of each video. Overall, the makeup of each session evolved based on the feedback and needs of the actual participants, who were positioned not only as research subjects but as equal partners in planning and developing the research plan.

The Participants

The participants consisted of sixteen mathematics teachers enrolled in either a Master's degree program (10) or a Doctoral degree program (6) at an Indonesian university focused on teacher education. The participants varied in their mathematics teaching experience, from pre-service teachers who were just beginning their mathematics teaching careers to teachers with more than 15-years of experience. Teachers were solicited from a pool of current students in the mathematics teacher education program and were invited to participate in a research study focused on creating digital video stories to explore their identities. Participants were also given a small stipend to help offset the costs of their transportation to the workshop.

Data Sources

Two measures of stories and interviews, outlined in the Table 1, were be used to measure potential changes in participant's mathematics identities, digital literacies, and dispositions towards community mathematics. These measures were integrated into the research methods as follows. First, the research team used a Participant Design-Based Research method with storytelling, meaning that the research design and questions shifted based upon the participants' feedback. While the primary source of data was the final digital stories that the participants created, we also collected pre and post-workshop questionnaires focused on mathematical identity, digital literacy, and dispositions towards community mathematics. Additionally, the research team collected field notes, recorded the whole group discussions from the final screenings, and recorded their own debrief conversations after each session.

Additionally, the Participant Design-Based Research method also meant that each participant had opportunities to submit continual feedback to the research team through the daily editable online agendas. This feedback was taken up directly by the research team to substantially evolve the day-to-day structure of each session as well as focus on what the participants felt was important. For instance, originally, each session was conducted in English since each participant indicated that they were comfortable using English as the common language. But after the first session, the research team received feedback that the level of academic discourse in English was beyond the comfort level of several participants. Therefore, the rest of the sessions were conducted in Bahasa Indonesian, with English being used minimally.

Data Analysis

To analyze the data, we first transcribed all the video and audio data in Bahasa Indonesian, then translated them into English. Because the research team consists of an international collaboration between the United States and Indonesia, we were able to engage in the analysis in both languages and take care to understand the cultural meanings of the words the participants chose.

We used constant comparison analysis (Corbin & Strauss, 2008) and narrative inquiry (Clandinin & Connelly, 2000) to compare existing measures of (1) participant's mathematics identities as connected to other social identities as detailed in the work of Aguirre, Mayfield-Ingram, and Martin (2013) and Langer Osuna and Nasir (2016b) and (2) digital literacies as detailed by The International Society for Technology in Education's (ISTE) seven standards for digital literacy: empowered learner, digital citizen, knowledge constructor, innovative designer, computational thinker, creative communicator, and global collaborator (ISTE, 2016). The constant comparative and narrative inquiry-based analysis allow the research team to analyze the video behind the participant's digital mathematics stories. In this way, the storytelling emerges, but is compared with or "measured" against an already known item of the existing construct. Third, the researcher used these tangible findings to construct a tool for future data collection: a Digital Mathematics Storytelling protocol using a Participatory Design-Design Framework created specifically for mathematics teachers in Indonesia (Bang & Vossoughi, 2016; de Jager et al., 2017; Lambert, 2013). Findings for each teacher were written up in a short, 1-page format and sent to each teacher for a member check and to ensure that they had a voice in the research process.

Table 1: Data Sources, Analysis Methods, and Theoretical Framework

Data	Analysis Methods	Theoretical Framework
Digital Mathematics Stories	Constant Comparative Method Narrative Inquiry	Math Identity: Aguirre, Mayfield-Ingram, and Martin (2013) Math Identity: Langer Osuna and Nasir (2016b) Digital Literacy: ISTE Digital Literacy Standards (2016)
Pre/Post Questionnaires	Constant Comparative Method	

Results

Initial results from our analysis of the teachers' data shows some emerging findings that we hope to process and reflect on with the community during the conference. First, the teachers engaged heavily with the construct of the counter story. Of the sixteen stories presented, fourteen of the stories specifically involved creating counter narratives to the ways each teacher was positioned. These stories explored the positional identities of gender, social economic status, ethnicity, age, and parental status. Generally, the teachers used the digital mathematics storytelling workshops to explore the ways in which they were positioned in mathematics as learners, how that affected the ways they saw mathematics, and how they actively fought against this positioning in their teaching practice. For instance, in Figure 1, a participant shares an emotional story about a family death and its effects on the way she perceives the world and the way she teaches.

Figure 1. A Story Exploring Life After the Death of a Family Member

However, these counter stories were often the second, third, or even fourth stories that the participants told. The initial stories were often shallow, as the participants felt that the connection to everyday mathematics had to be the focus of the story. For instance, one of the participants' initial stories revolved around the various risks and advantages of monetary investment, with the story concluding that investing in gold was the smartest long-term solution. Two other initial stories focused on the mathematics behind skin care regiments and the amount of product and frequency of application needed for a healthy skin care routine. In all of these cases, the storytellers completely abandoned these storylines during the first storycircle when it became obvious that their stories could be more connected to their various identities and counter narratives.

Second, the stories allowed ways for teachers to share and explore aspects of their identities that they rarely had the chance to share with other teachers. Three of the stories involved the struggles of raising a family while also working as a teacher. Five of the stories involved the difficulty of working as a professional mathematics teacher while also supporting parents and family members economically.

One of the teacher's stories delved deeply into the ways she felt pressure from her intersecting role as a mother of three children within Islamic Indonesian society and her role as a mathematics education leader in her community. Her story delved into the duality of both joy and frustration. She expressed gratitude for the opportunity to raise three children, engaging them in fun and critical mathematics play from an early age. But she also expressed frustration with the expectation with having to serve as caretaker and coordinator of her children's schedules while she felt that her male counterparts were able to completely focus on their professional identities and not take care of the endless minutia of parenting. This story pushed back on the frustrating gender norms existing the storyteller felt.

However, another narrative painted a counter story to this narrative. A male teacher's story focused specifically on the mathematics play that he and his young children engaged in on a daily basis, focusing on how much time and effort he spent in creating playful and educational experiences for his children so that his female partner did not have to shoulder the burden of caretaking entirely on herself. Both these stories, in parallel, showcase the unique ways that this storytelling experience allows multiple narrative perspectives on parenting to emerge. And the ensuing screening opened discussion on gender roles and our responsibility as educational leaders to push back on norms that felt oppressive.

Third, the ensuing discussions after each story opened pathways for teachers to connect their mathematics teaching to humanizing practices. Every single post-survey result mentioned some form of how this practice allowed them to engage in mathematics teaching from their heart, as opposed to the robotic and technical practice that they often felt positioned to do as mathematics teachers. For instance, in Figure 2, a participant shares a story exploring her own evolving Islam faith and the ways that her daily prayer and study are connected to how she sees mathematics and her role as a mathematics teacher in her community. In this example, the storyteller wrestles heavily with her own emerging identity as a mathematics teacher and the purpose of mathematics in society. She also wrestles with her growing Islamic faith and uses mathematics, religion, and her own personal experiences to connect how passages from the Quran are connected to how mathematics is learned and understood. The discussion that ensued from this story focused heavily on the role of religion in mathematics education practice, particularly in ways that were culturally relevant to the students that these teachers would work with.

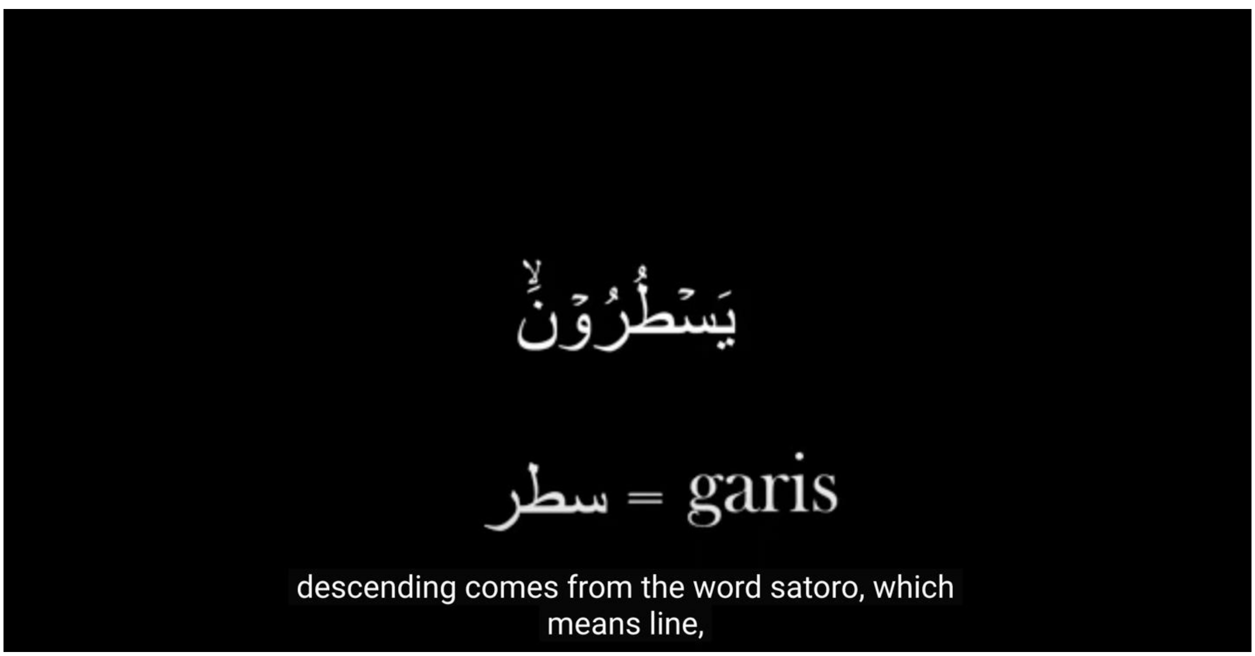

Figure 2. A Story Exploring Islamic faith and Community Responsibility

Finally, supporting a finding from our prior studies using digital mathematics storytelling in US-based communities (Chao et al., 2021, 2022), we once again found that this final community screening and discussion became a crucial space for the stories to live. This space, to share and comment on each other's stories, particularly at the end of the Digital Mathematics Storytelling experience became a cathartic and transformative space where the teachers clearly see who they are and how it connects to their practice as mathematics educators. And, in our own reflection of the data, we found that it is this conversation during this screening that was filled with the richest feedback and personal connections. The teachers spoke of their own family trauma, their own experiences with mathematics, their own hopes as teachers, parents, siblings, and community members, their own frustrations, and finally, ways to find their own joy within their practice. In fact, during this last three-hour screening and discussion, one of facilitators noted that it was time for prayer. But the participants decided to keep going with the screening and discussion and push their prayer back until after the discussion was finished.

Discussion

Overall, the Digital Mathematics Storytelling experience was successful in eliciting narrative-based identity stories from these mathematics teachers in Indonesia, not only by offering opportunities to create and share their counter stories, but also engage in artifact creation that humanized their practice. These results show the global impact that a practice like this can have on mathematics teachers, particularly in an environment like Indonesia which has a deep history of Arabic, Indian, Dutch, and Chinese pedagogical influences (Patahuddin et al., 2018). While the original intent of the project was to elicit specific Indonesian national and regional mathematics teaching practices, the research team learned that what the mathematics teachers most resonated with through this practice was the opportunity to forge and share stories about their own mathematics identities as connected to their family, community, and culture.

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TRANSFORMING HOW WE COME TO KNOW (THROUGH) MATHEMATICS

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Mathematics shapes how we see ourselves, each other, and the world. Therefore, how we come to know mathematics has consequences for how we relate to one another and to the world around us. For example, when we rely on a mathematics that we come to know as fixed and static to conceptualize social identities such as race, gender, and class, it is easy to interpret these identities as fixed and static. By re-reading empirical examples from existing literature on Cartesian geometry and the mathematics of space in conversation with theoretical provocations, this conceptual paper explores how transforming how students come to know mathematics has the potential to transform how they come to know each other and the world around them.

Keywords: Equity, Inclusion, and Diversity; Social Justice; Instructional Activities and Practices

Recently, one of us observed a high-school classroom in which students were given a paper with multiple straight lines on a Cartesian grid. They completed sentences such as "two lines are parallel when (the slopes are the same)" and "two lines are perpendicular when (the slopes are negative reciprocal)." In this and many other classrooms, the Cartesian grid is presented as existing a priori, with students' primary mathematical tasks being to plot prescribed lines and perform calculations about those lines. Thus, students come to know mathematical structures like the Cartesian grid as absolute: an objective external truth "discovered" by an external authority and handed down over time. This essentialist approach to mathematics (Skovsmose, 2020) is dehumanizing to students (Gutiérrez, 2018) and leads them to experience mathematical properties as arbitrarily defined. Students often think that a right triangle must be oriented in a certain way lest it become isosceles (Vogelstein et al., 2019), for example, or a square must not be tilted lest it become a diamond (Clement et al., 1999). Deprived of the opportunity to engage with the creative potential of mathematics (de Freitas & Sinclair, 2013), students come to know mathematics as fixed, rigid, and absolute, rather than as subjective, flexible, and expansive.

What we come to know mathematics as, and how we come to know mathematics, matters because mathematics structures society. Critical mathematics scholars have illustrated how the formatting power of mathematics shapes technologies, naturalizes the classification and ordering of humans, and makes material some abstractions over others (e.g., Borba & Skovsmose, 1997; Bullock & Meiners, 2019; Chronaki, 2018; Diaz, 2021). Given the imperial and colonialist histories of the mathematics that is most commonly taught in U.S. schools today, mathematics typically materializes abstractions that maintain inequitable relations of power (Appelbaum & Stathopoulou, 2020; Bishop, 1990; Martin, 2009). When students learn mathematics in school, they are also learning about what mathematics is and learning from mathematics about society.

In line with the conference theme of envisioning the future of mathematics education, we build on the efforts of these critical mathematics scholars who warn of the consequences of students coming to know mathematics in restrictive ways and ask: how might transforming how students come to know mathematics *also* transform how they come to know the world?

Theoretically, we draw on theories of learning and theories of society that treat knowing and social life as relational, situated, and interactionally produced (e.g., Erickson, 2004; Lave & Wenger, 1991), which indicate that how students learn math in school matters for how they know and act in society. Methodologically, we plug empirical examples from the mathematics education literature and conceptual provocations from the social studies of mathematics into each other (Jackson & Mazzei, 2013), reading for possibilities for more "expansive and insurgent ways of learning, being, and acting" in the world with mathematics (Warren et. al, 2020, p. 278).

Coming to Know Space

Consider different understandings of the relationship between identity and power. Popular understandings often take identity as a set of fixed characteristics that affect people in predictable ways; men have more privilege than women, or people of color are subordinated to white people (e.g., Cheng, 2020). A person's identity and how they experience the world can be described by the intersection of their locations on axes of privilege or domination. This explanation draws on a mathematical model that many people are familiar with because they have been taught, since early childhood, to plot points on number lines and Cartesian planes, to find intersections of lines, and to compare areas of a plane as being greater than or less than. With this geometric model in mind, comparing privileges based on fixed identity markers seems perfectly logical. It also leads to reductive notions of hierarchy and oppression.

By contrast, feminists of color have long recognized identity as a complex, shifting, and situated phenomenon (e.g., Moraga & Anzaldúa, 1983). The concept of intersectionality, for example, was developed to explain how people's vulnerabilities to racism, sexism, and other forms of oppression not only differ from each other but also depend, dynamically, on context (Crenshaw, 1991). For example, in Leila Fernandes' geography of a jute mill, two mill workers experience conflict (Barad, 2007). How the conflict unfolds, however, cannot be predicted just based on the workers' caste, class, and other identity markers. Instead, the workers sometimes connect based on caste and sometimes on class; when and how their caste and class matter shift as the conflict unfolds. Tracing how different boundaries between caste and class are (re)produced throughout this conflict illustrates how identities and power relations are constantly (re)configured through interaction. Barad argues that this careful accounting of identity (re)formation in the jute mill could be better described by topological than by geometrical representations because of topology's attention to connectivity and boundaries and change in intensive space compared to Cartesian geometry's treatment of locations and positions as fixed against a pre-existing grid. We take this as an example of how the mathematical models we choose– dare we say mathematical ontologies– draw attention to different sets of relations in the world. When Cartesian geometry is a popular way of understanding spatial relations, it is unsurprising that identity might be treated as if it can be fixed, categorized, and ordered, or that people's experiences of power and oppression might be visualized as existing along particular lines. If topology were as accessible and familiar to the public as Cartesian geometry, might different explanations of intersectionality become more popular?

In other words, could unsettling the dominance of certain mathematical models unsettle ways of thinking that stifle more expansive relational possibilities? We examine this question by thinking with mathematical models of space and specifically, three empirical examples that explore making Cartesian geometry less consequential in how learners conceptualize space. We note that space is not the only way to investigate this question, nor is Cartesian geometry is not a “problem” that needs to be “solved” with topological or other models; first, Cartesian geometry can also be useful, and second, substituting one deity for another leaves untransformed the deification we seek to challenge. Rather, we focus on Cartesian geometry because its dominance has already been challenged in the literature, allowing us to wonder with these other scholars what kinds of knowledges or what kinds of relations might be made possible by alternatives.

Animating Multiplicitous Conceptions of Space

Dominguez and colleagues (2023) describe a philosophical conversation in which fifth graders considered different conceptions of space. After some open questions about what moves and what does not move, the conversation turned to students sharing an expansive range of ideas about what space is in relation to the universe, to the world, and between themselves and each other. They considered, for example, whether a rock or a seagull had a better sense of space, and parried that into an interrogation of “often-unquestioned anthropocentric arrangement[s] of space” (p. 1162). Dominguez then introduced a toy airplane and a set of cardboard Cartesian planes, and the students worked together to chart the airplane’s movement using Cartesian coordinates. The initial conversation about movement supported students’ knowing of space as first and foremost dynamic, created by motion, and dependent on perspective. As a result, students responded to the introduction of Cartesian coordinates by refusing to accept space as discontinuous and by animating motion along the z-axis in relation to imagined topological features such as cliffs, caves, or volcanos. Dominguez and colleagues propose that this philosophical conversation shifted students’ mathematical attention in their coming to know space: from space “as a set of static properties [to] a process that features prominently the idea of possibility” (p. 1155), or from essences to multiplicities. We highlight that it made possible different relations between students, non-human relatives, and the world. The fifth graders in this example, however, had already been introduced to Cartesian planes. In fact, Dominguez was invited to lead this conversation because their teacher was concerned that “some students are confused about the order of x- and y-coordinates for plotting points” (p. 1163), perhaps because they– like the students in our opening anecdote– had once been handed printed Cartesian grids with the x-axis and y-axis already determined for them. Our next example shows how students might engage in earlier encounters with space in ways that can also (un)fix the fixity of Cartesian geometry as the predominant, predetermined, and predetermining way to conceptualize space.

(Way)finding New Relational Possibilities.

In Lehrer and Pritchard (2002), students in a third-grade classroom produced maps of their school’s playground. Over multiple trips to the playground and multiple revisions, students negotiated measures of lengths and changes in direction, and decisions about origin and scale until the configurations of play structures in the maps corresponded to those in the playground– something that students agreed “good” maps should do. The teacher encouraged students to reflect, in journals and in conversations, about how their maps changed over time. These reflections helped children construct identities as mathematical doers and knowers, in contrast to mathematical identities that emphasize efficiency and rote procedures (e.g., as “human calculators”). In this example of children coming to know their familiar spaces differently, we glimpse how transforming how we come to know (through) mathematics can transform relational possibilities. First, the children came to have a different relation with an "absolute" frame of reference; through mapmaking, they came to understand that what is fixed on a map is a negotiable choice as opposed to being dictated by a teacher, a textbook, or some other absolute authority. Second, because these negotiations took place publicly, students were driven by a desire for their maps to be legible to each other; their mapmaking was oriented by a concern for collective and not just individual sensemaking. Third, beginning with their own everyday wayfinding offered these third-graders the opportunity to see themselves and each other as people who can make decisions about configuring origins, measures of distance, measures of direction, and scale; Vossoughi and colleagues (2021) refer to this as the "cultivation and experience of capability" (p. 136). Finally, the students also worked with their parents to make a map of their home or neighborhood spaces. Multiple parents expressed that their children showed them new ways of navigating space and more flexible ways of interpreting represented space, which hints at how adults' coming to know space, too, might be transformed.

Attuning to Both Mobile and Grid Epistemologies

Taylor (2020) focuses on adults, analyzing two episodes from participatory community planning meetings where local residents and professional urban planners gather over a table-sized map of the neighborhood. In one episode, a local resident– a "longtime resident of an aging African- American community"– retells how an interstate highway was "carved intentionally through" the community when he was a child and asks how the community can recover; the young White woman planner redirects the conversation to a questionnaire that they are trying to complete (p. 407). In the other episode, a similarly positioned resident shares a similar personal experience. Through parallel tracing of his story with hands in the air and her fingers and stickers on the map, he and a similarly positioned planner together construct what Taylor calls "a new text that layers together Cartesian notions of space with corporeal realities of space" (p. 419). This new text, Taylor argues, learns from both the embodied and dynamic ways that the resident has come to know his neighborhood over time and the more static ways that the planner has come to know the neighborhood through a map. The relational attunement between this resident and planner made their ways of knowing commensurable. Importantly, it also made possible new spatial imaginaries for the neighborhood, as the newly created text became part of the final document on record at the city planning office, and new spatial epistemologies, as the planners later implemented "'walking charrettes' as a means of highlighting the racial and cultural histories of places too easily hidden by easily accessible representations" (p. 424). Taylor's work suggests that bringing grid epistemologies into conversation with mobile epistemologies can create new ways of understanding and acting on the world.

Discussion

Borba & Skovsmose (1997) suggest that the formatting power of mathematics can be challenged by challenging the "ideology of certainty" in classroom practice, creating opportunities for multiplicity, provisionality, uncertainty (p. 22). Our three examples do just that. We caveat, however, that they are not simply examples of "student-centeredness" or "dialogue," because pedagogical shifts alone do not prohibit the possibility that students come to know Cartesian planes as the only legitimate frame of reference for conceptualizing space; transforming epistemologies can be a mechanism for transforming ontologies, but may be insufficient if the mathematics students are coming to know is not also transformed. Instead, transforming how students come to know (through) mathematics must be paired with transforming the mathematics they come to know. For example, to return to Barad's observations about identity, it is the mathematical conception of space as intensive and dynamically produced (as the fifth-graders conceive of it), the recognition of mathematical agency and responsibility in articulating how they and others are positioned and oriented in relation to one another (as the third-graders recognize), and the attunement to both mobility and fixity (as the residents and planners attune), that can lead to models of identity and relations that offer more expansive possibilities than Cartesian geometry. Dialogue is not enough.

Space, of course, is just one thing whose reimagining has the potential to transform our ontological and relational orientations towards each other and our non-human relatives. We continue to wonder how else transforming how students come to know (through) mathematics might transform how they come to know each other and the world they live in.

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EXPLORING PRESERVICE TEACHERS' MATHEMATICAL IDENTITIES ACROSS ETHNIC BACKGROUNDS: INSIGHTS FROM MATHEMATICAL AUTOBIOGRAPHIES

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This qualitative study delves into the complex concept of mathematical identities among preservice teachers of diverse racial and ethnic backgrounds. Analyzing mathematical autobiographies of three (Black, Asian, and Hispanic) elementary preservice teachers, we uncover distinct challenges and experiences that influence their mathematical identities. The themes emphasize the importance of culturally relevant pedagogy, stereotype challenges, and linguistically sensitive interventions to foster inclusive mathematics education. Our findings contribute to a deeper comprehension of mathematical identity formation and advocate for inclusive educational support.

Keywords: Culturally Relevant Pedagogy; Diversity, Equity, and Inclusion; Preservice Teacher Education

Purpose of Study

In mathematics education, the concept of mathematical identity has evolved into a multifaceted construct, deeply interwoven with an individual's cognitive dimensions, emotional attributes, and life trajectory. As delineated by Boaler and fellow scholars (2000), mathematical identity is not a fixed entity but rather a dynamic and socially constructed phenomenon. It emerges as a complex interplay of individual experiences, beliefs, attitudes, and self-beliefs regarding one's mathematical ability and affinity for the subject. Narratives, like mathematical autobiographies that chronicle individuals' encounters with mathematics in and out of the classroom, enrich the fabric of mathematical identities (McCulloch et al., 2013). Fundamental to developing mathematical identity is an acknowledgment that it manifests through narratives: accounts that not only reflect but also shape our self-conceptions in relation to mathematics (Aguirre et al., 2013). Rooted in personal life narratives, these accounts simultaneously reflect the socio-cultural contexts that influence their identity formations (Drake et al., 2001; Sfard & Prusak, 2005). Within this paradigm, mathematical identity emerges not as a uniform entity but as a mosaic that is intricately crafted from the intersections of personal experiences, societal expectations, and cultural backgrounds.

As preservice teachers embark on their journey to becoming educators, their mathematical identities assume heightened significance, shaping not only their understanding of mathematics and pedagogical practices but also their engagement with students (Lutovac & Kaasila, 2014). However, despite the growing body of research on mathematical identity development (e.g., Beijaard et al., 2004; Bishop, 2012; Black et al., 2019; Goldstein, 2018; Heyd-Metzuyanim, 2015; Kaspersen et al., 2017; Sfard & Prusak, 2005), there exists a gap in understanding how racial and ethnic backgrounds intersect with these processes among preservice teachers.

Our study seeks to address this gap by documenting the development of mathematical identities among preservice teachers of various racial and ethnic backgrounds. Specifically, we focus on three individuals from Black, Asian, and Hispanic heritages within predominantly white educational institutions. By examining the intricate interplay between racial and ethnic identities and mathematical self-concepts, this research aims to illuminate the nuanced ways in which sociocultural factors shape individuals' engagement with mathematics and their trajectories within the field of education. Through this study, we, as mathematics educators, endeavor to contribute to a deeper understanding of the complex dynamics underlying mathematical identity formation and to foster more inclusive, equitable, mathematics education practices.

Framework and Perspectives

Understanding preservice teachers' mathematical identities is essential for the improvement of mathematics instruction and for effective teacher education programs. We embrace the sociocultural view of learning (Lave & Wenger, 1991) as the theoretical framework to investigate the selected preservice teachers' mathematical identities. Learning is "an integral part of generative social practice in the lived-in world" (Lave & Wenger, 1991, p. 35), and individuals' mathematical identities are formed through the process of sharing experiences within their communities. Thus, we highly regard preservice teachers' mathematical learning experiences based on their race, ethnicity, culture, language, and stereotype.

We adopt the perspective that identity is a collection of stories shared by individuals within various social constructs (Holland & Lave, 2001; Sfard & Prusak, 2005). These stories, encompassing both actual and designated identities, shape individuals' perceptions, actions, and future mathematical aspirations. Mathematical identity, therefore, consists of the narratives that reflect individuals' past experiences, present engagement, and future expectations related to mathematics and mathematical teaching. Actual identities encompass personal experiences and achievements in mathematics, while designated identities involve anticipated roles and attitudes toward the subject matter (Sfard & Prusak, 2005). For preservice teachers, their mathematical identities are shaped by a combination of early, formal, and informal mathematical experiences both in and out of the classroom, and societal perceptions of mathematical abilities. To investigate the selected preservice teachers' mathematical identities, we employ autobiographical narratives as the methodological approach. Autobiographies provide a rich source of data, allowing the preservice teachers, in their own words, to reflect on their mathematical learning experiences and to articulate the significance of various events and interactions.

Methodology

This study utilized a qualitative approach. More specifically, mathematical autobiography assignments were the primary data source. The participants, who were enrolled in teacher education programs in three separate institutions across the U.S., reflected on their mathematical experiences, beliefs, and influences. The data analysis, involving thematic coding, focused on racial and ethnic identities and mathematical experiences.

The total number of mathematical autobiographies that the participating elementary preservice teachers submitted was 227. Among them, 214 (94.3%) self-identified as White, 8 (3.5%) as Black, 3 (1.3%) as Asian, and 2 (0.9%) as Hispanic. In this report, we focused on the traditionally underrepresented groups in teacher education programs. Specifically, we identified three distinct cases involving preservice teachers from the Black, Asian, and Hispanic heritages. We selected these individuals due to their compelling narratives, shedding light on how their racial, ethnic, and cultural backgrounds have influenced their development of mathematical identities.

Results and Discussion

In this section, we present compelling narratives from Nicole, Emily, and Maria (pseudonyms) that illustrate the complex relationship between personal backgrounds and academic journeys. Through their first-hand accounts, we explore the landscapes marked by systemic challenges, cultural pressures, and language barriers. These mathematical autobiographies contribute towards the ongoing dialogue on educational equity and inclusivity and offer valuable insights into the diverse experiences, including their shortcomings, within our educational system.

Nicole's Story

Nicole, a junior Black female preservice teacher, provided a story reflecting upon her upbringing within the environments that did not prioritize academic achievement. In her recollection, she depicted a scarcity of educational resources at home and within her neighborhoods. Compounding this, there was a dearth of encouragement and support towards higher educational aspirations. She shared, "In my neighborhood, no one talked about going to college... Math seemed irrelevant to me, and I did not enjoy doing math work."

In her mathematical autobiography, Nicole highlighted the transformative influence of Miss J, her 10th-grade geometry teacher. Despite Nicole's initial struggles with the content, she vividly recalled Miss J's unwavering support and encouragement. Recognizing Nicole's intellectual potential, Miss J played a pivotal role in fostering a profound sense of self-belief and confidence within Nicole. Reflecting on Miss J's impact, Nicole affirmed, "She always said that I would be a good teacher because I explain my thoughts clearly and understandably." This excerpt encapsulates the profound influence individual teachers can have, particularly those who possess the insight and dedication to empower students from historically underserved communities. Furthermore, Miss J emphasized the importance of fostering inclusive learning environments that affirm the potential of every student.

Nicole's introspection stands as a compelling testament to the transformative power of mentorship and the significance of role models of the same race. She recalled, "I always sought advice from Miss J not only for mathematics but for my college choices and other things too. Because she is also Black, I could trust her better, and she became my role model." Nicole's journey highlights the imperative for educators to recognize and challenge systemic barriers while actively advocating for the holistic development and empowerment of all learners. In particular, her story reminds us of the profound responsibility inherent within the field of education—and among educators—to serve as a catalyst for social change and in achieving equity.

Finally, this mathematical autobiography underscores the pervasive systemic inequities that profoundly influence the educational trajectories of similar Black students. Nicole's account resonates deeply within the context of contemporary discussions surrounding educational equity and sheds crucial light on the multifaceted challenges facing marginalized communities. Furthermore, this narrative accentuates the need for culturally relevant pedagogy and the creation of supportive learning environments that acknowledge and address the detrimental, prejudicial experiences and circumstances of students like Nicole.

Emily's Story

Emily, a junior Asian American female preservice teacher, recalled her childhood experiences and the challenges she had faced in learning mathematics. Her immigrant parents, who had arrived shortly before she was born, strongly equated academic success as life's success. This expectation led to Emily attending after-school tutorial sessions and weekend Korean language lessons. Reflecting on her early childhood years, Emily described a sense of disconnect between the academic expectations ingrained by her family-cultural norms and her aptitude in mathematics. She resented the "achieve at all costs" mentality and questioned, "Why I had to do more than my friends. Also, my parents used teaching methods different from those my teachers used, making it even harder for me to grasp the mathematical concepts."

Despite the common belief that Asian students, in general, are naturally talented in mathematics, Emily challenged this by discussing her own difficulties. She delved into the nuanced challenges she had faced as a Korean American. For example, she met the pervasive misconception that her cultural background inherently predisposed her to excel in mathematics. In her mathematical autobiography, Emily reflected, "People always assumed I was great at math because I'm Asian... But the truth is I struggled just like everyone else." Hence, for Emily, there has existed a gap between the people's expectations and the actuality, revealing why it is crucial to reconsider the stereotype about Asian—or any other— students and their mathematical abilities. The disconnect between this assumption and her actual struggles with the subject became even more pronounced as she shared instances of teachers singling her out to highlight her work. Due to the association between her most common Korean last name, "Kim," and her Asian appearance with an unwarranted expectation in mathematical ability, Emily felt disdain towards learning mathematics.

These narratives collectively underscore the imperative to challenge stereotypes and recognize a diverse range of mathematical abilities among Asian students. Emily's experiences serve as a reminder for a more empathetic understanding of individuals' capabilities that are unshackled from the constraints of societal expectations based on cultural or ethnic stereotypes.

Maria's Story

Maria, a female junior, grew up in Wyoming and in a Hispanic household where Spanish was the primary language of communication. Reflecting on her early childhood years, she recalled the learning obstacles beginning in kindergarten. "That's when my struggles first began," she reminisced. Maria's mathematical autobiography revealed complex and often abrupt transitions to learn English in the academic context. Moreover, the expectations to form unfamiliar social bonds and to absorb incongruous knowledge posed added barriers.

The convergence among the diverse expectations and challenges contributed substantially to shaping her early educational impressions. In her narration, she expressed, "I always found story problems to be the most difficult growing up. I never knew why I struggled with them so much. Now, looking back, I believe vocabulary was a significant barrier I had to overcome, coupled with not knowing where to begin when faced with a story problem." Furthermore, Maria, recalling her experiences as an elementary school mathematics student, had difficulty articulating useful problem-solving strategies. In sum, linguistic obstacles and cognitive challenges appeared as the pivotal factors contributing towards her struggles, and these daunting expectations significantly affected her early educational journey.

In contrast to her own experience, Maria envisions her future classroom as a space where all students, especially those learning English as a second language, "are supported with more culturally relevant word problems that they can connect themselves with" and "can read the problems in their own languages." This desire stems from her mathematical identity, the one she has built through her own experiences, and by embracing students' diversity, Maria is committed to creating culturally relevant and inclusive classrooms.

Conclusion

The mathematical autobiographies from the preservice teachers, Nicole, Emily, and Maria, offer compelling insights into the complex interplay among racial, ethnic, and mathematical identities. Through their unique life stories, we sense the multifaceted challenges faced by students from nondominant backgrounds. These challenges, ranging from systemic inequities to cultural expectations to language barriers, highlight the urgent need for a comprehensive, empathetic understanding of the factors influencing mathematical learning.

Nicole's journey exemplifies the transformative power of a dedicated educator. Her story underscores the pivotal role teachers can play in empowering historically underserved students and shows the profound impact of supportive learning environments on individual trajectories. Emily's narrative challenges stereotypes about Asian-American students' mathematical abilities, emphasizing the importance in recognizing individual learners free from cultural biases. Specifically, her experience calls for teacher training that overcomes preconceived biases about learners. Maria's struggles with language ability and mathematical understanding spotlight the relationship between linguistic and cognitive factors in shaping her mathematical identity. Her story stresses the necessity for linguistically sensitive, culturally relevant, educational interventions tailored to meet the diverse needs of students.

These mathematical autobiographies highlight the need for educators and the education system to embrace inclusive, culturally responsive, linguistically sensitive approaches to mathematics education. Nicole's experience with systemic inequalities, Emily's encounter with stereotype challenges, and Maria's struggle with language barriers are pervasive challenges faced by diverse learners. By proactively addressing them through inclusive instructional methods, educators can deconstruct barriers, overcome stereotypes, and create a learning environment where every student feels valued and empowered to seek mathematical excellence. Additionally, the mathematical autobiographies of these three preservice teachers emphasize the pivotal role of teachers in driving social change and equity and formulating a more inclusive, supportive educational framework.

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CAPTURING INTERSECTIONALITY?: EXPLORING A METHOD USED TO STUDY THE MATHEMATICS LEARNING EXPERIENCES OF BLACK GIRLS

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There has been a rise in the use of intersectionality theory for understanding the complex experiences of mathematics learners with multiple marginalized socially constructed identity markers. This increase calls for an astute examination of the methods of data collection and analyses used to capture such complexity. This study examines a method used to study an aspect of mathematics learning for Black girls to ascertain its viability for characterization as an intersectional approach. While several conditions were met, the well-intentioned design of this research method fell just shy of the classification as an intersectional approach. The results of this examination emphasize the importance of consideration for future intersectional research in mathematics education.

Keywords: Intersectionality, Gender, Research Methods

Introduction

Intersectionality is a construct widely used in social sciences and humanities to understand the complexity of the human experience. Several mathematics education scholars call for it to be taken up in our field to avoid the erasure of the experiences of, for example, girls of color (Bullock, 2018; Gholson, 2016; Leyva, 2017). However, taking up this construct should be done thoughtfully and responsibly and requires attention to our current methods or the development of new or supplemental methods of data collection and analyses. In this paper, I use insights from Crenshaw's conception of intersectionality alongside the guidance from several intersectionality scholars (i.e. Bowleg, 2008; Collins & Blige, 2016; Hancock, 2016) to determine how well a research method that I created meets the challenge of intersectional research of Black girls' mathematics learning experiences.

Intersectionality and its importance as a construct in mathematics education research

Intersectionality is a term coined by Kimberlé Crenshaw, a law scholar, interested in the ways that single axes thinking about race, gender, class, ability, sexual orientation and other socially constructed identity markers can serve to hide or erase the experiences of those at the intersection of these markers. Intersectionality, as an analytical tool, broadens our conception of the social complexities of the human experience where conditions must be understood by many factors in mutually influencing ways (Collins & Bilge, 2016). Crenshaw pays homage to nineteenth century scholars like Sojourner Truth and Anna Julia Cooper who wrote and spoke about the Black woman experience at the intersection of race and gender. Crenshaw states that intersectionality is "about how structures make certain identities the consequence of and vehicle for vulnerability (Southbank Centre, 2016)." An example of this vulnerability is when a company is called to diversify along a single axis such as race or gender and the company meets the requirements by hiring more men of color and more white women effectively making women of color vulnerable. In mathematics education, a parallel example is the failure to disaggregate mathematics achievement data by race x gender assuming that all Black children (race only view) or all girls (gender only view), for example, experience mathematics learning the same and hiding or ignoring the mathematics learning experiences of girls of color when, in fact, scholars studying the mathematics experiences of Black girls have demonstrated this not to be true (Gholson, 2016, Gholson & Martin, 2014; 2019; Joseph et al., 2017).

Intersectional Methods of Data Collection and Analyses

A methodological issue to consider is how one captures or "measures" intersectional experience and could also be how one uses a method to elicit an intersectional experience during data collection. Adequate treatment of intersectionality considers what practices, policies, and institutional structures play a role in contributing to the exclusion of some and not others. Bowleg (2008) takes up this methodological question of measuring intersectionality as she considered what she initially referred to as the "triple jeopardy approach" to studying the stress and resilience of Black lesbian women. Bowleg (2008) grappled with the Black + Lesbian + Woman or an additive approach to the analysis of the data as the additive approach should be replaced by conceptualizing the layered intersectional experience as multiplicative. Reiterated by Wing (1997), "multiply each of my parts together, 1 x 1 x 1 x 1 x 1 and you still have one indivisible being (p. 31)."

Both Hancock (2007) and Bowleg (2008) provide guidance for consideration of a multi-method intersectional approach to research. It is on these dimensions that I will later examine whether or not the card sort is a successful method for studying the intersectionality of Black girls' mathematics experiences.

1. An intersectional approach to research considers the role of socially constructed identity markers alongside individual and institutional factors and should recognize the dynamic interaction between them should be reflected in the analyses (Hancock, 2007).
2. More than one socially constructed identity marker should be examined and each should matter equally, though the relationship between them is an open empirical question (Hancock, 2007).
3. Any questions asked in interviews, surveys, or questionnaires should tap into the interdependence and mutuality of the socially constructed identity markers avoiding any implications that they are separate and able to be ranked (Bowleg, 2008; Hancock, 2007).
4. There should be a focus on meaningful constructs such as stress or discrimination rather than focusing on socially constructed identity markers alone (Bowleg, 2008).
5. And multiple methods are necessary and sufficient for data collection and analysis (Hancock, 2007).

The Identity Card Sort Method and Analysis

Mathematics education has been conceptualized as a racial project (Martin, 2013) and, even further, as a white, patriarchal space that makes explicit how interlocking systems of racism and patriarchy shape intersectional oppression and resistance (Levya, 2021, p. 121)." My primary research interests investigate the interplay between socially constructed identity markers and individual or personal identity markers for Black girls' and women whose socially constructed identities are in stark opposition to the construction of mathematics education as a white, patriarchal space. To this end, I conducted a research study with multiple components and data methods to investigate the racialized gendered experiences of six young Black women (see Table 1).

Table 1. Study participants. All names are pseudonyms. ***PWI = predominantly white institution**

Name	Age	Level in School	Current school	Current math course	College major/career aspirations
Courtney	18	First year university student	Large, public *PWI	College Algebra and Trigonometry (Fall semester)	Political science/Pre-Law
Elissa	18	First year university student	Large, public PWI	Survey of Calculus 1	Pre-med, Human Biology
Janet	18	High school senior	Small, urban, predominantly black, public charter	Precalculus Honors	Veterinary medicine
Kristen	17	High school senior	Mid-sized, suburban, public PWI	AP Statistics	Middle school teacher
Riley	18	First year university student	Large, public PWI		Interdisciplinary Studies in Social Science
Shannon	17	High school senior	Small, single-sex, public, urban predominantly Black, public	Precalculus	English teacher

I devised a method to accompany a semi-structured interview that, at the time, I perceived to be a method to elucidate the function of the young women's intersectional socially constructed identities across space (home, school, and math class) in an effort to ascertain the role race and gender played in mathematics learning for young Black women. That extent to which this method is, indeed, rises to the level of an intersectional approach, however is at question. I wondered, to what degree is this method, the identity card sort, adequate for elucidating the girls' intersectional mathematics learning experiences. So I ask: How, if at all, is the identity card sort representative of an adequate intersectional research method? How, if at all, is the analysis of the identity card sort data representative of an adequate intersectional research analysis? Where might it fall short?

During individual semi-structured interviews, I employed a variation of an activity known as *Diversity Toss* (Jilk, 2010, Nieto & Bode, 2008; Teaching Tolerance, 2014) which required the young women to assign salience to aspects of their socially constructed and personal identity markers as they see themselves or as they interpret others seeing them across various social spaces. Each young woman was given five index cards to record their race, gender, name, a hobby or interest they strongly identify with, and their religious or spiritual affiliation or belief. Once completed, each young woman was asked to place the cards in order based on degree of salience to them when in a particular space (at home, at school, in math class). Cards were placed vertically as indicators of greatest to least salience and could be placed horizontally indicating equal salience at a particular level in a space. Cards could be removed, altogether, if that aspect of their identity had no salience for them in a space. One purpose for this activity was to highlight the multiple dimensions of each girl's identity and the importance of self-identification in identity work (Kemple, Harris, & Lee, 2015; Teaching Tolerance, 2014). It was also used to determine the extent to which young Black women are thinking about the gendered - racialized aspects of their socially constructed identities differently across spaces. During and after card placement, participants were asked about the rationale for their orderings.

Analysis of the identity card sort data included creating graphic depictions of the shifting salience of the various aspects of each young woman's identities across the various spaces (Figures 1 & 2). Since gender and race are dominant structures around which individuals self-identify and co-construct identity (Jilk, 2010) and were the focus of the study, this data was also analyzed through the construction of charts depicting shifting salience of gender and race across spaces (Figure 3 and 4). In the space remaining, I use Hancock's (2007) and Bowleg's (2008) guidance to determine the classification and efficacy of the identity card sort as a multi-method intersectional approach to research.

Identity Card Sort as an Intersectional Research Method

The first two conditions are quite easily satisfied having used multiple methods as the identity card sort was embedded into a semi-structured interview and there being more than one socially constructed identity marker (race, gender, religion) examined all mattering equally. I made the decision to ask the young women to assess the salience of each marker across various spaces and in relation to one another (can be horizontal for equal salience, can be removed if no salience) which Hancock (2007) says is an open empirical question. This aspect of the method and analysis was particularly important in discerning the role of race and gender across space where I found that race was lower in salience (level 2 through level 5) at home for all of the young women and increased in salience for all of the young women in math class. Its salience was at level 1 or 2 for everyone except Janet (See Figure 4). The same was true for gender, though it began with high salience (level 1) for some when at home and maintained or increased in salience across school and math class spaces for five of the girls. For Shannon, gender was not as salient in math class because she attended a single-gender school (See Figure 3).

It is not as clear as to whether the interdependence and mutuality condition is satisfied. I asked the young women about their experiences of being a Black girl in various contexts/spaces. Shifting salience or removing a marker can be interpreted as a treatment of markers as mutually exclusive or independent even if both Elissa and Riley chose to keep their race and gender salience at the same levels in school and in math class. Shifting or removing, however, does not mean it is no longer a part of who they are, it merely means that this aspect of their identity has less (or no) salience for them in these spaces. This notion of salience draws on the situative perspective of identity in relation to activity asking who am I here, who am I here versus there, and who can I become (Hand & Gresalfi, 2015).

The question of ranking is another open question. An explicit question that would violate this condition would be to ask a participant whether racism or sexism played a role in their discrimination or marginalization. The young women were not asked to choose between any of the markers. At any time, a young woman could have chosen to put all five cards horizontally at any level. Elissa and Riley were the only two that chose to use the horizontal function during the activity. And it is important to note that while hobby or religion cards were removed by some of the young women, race and gender cards were never removed. What remains, however, is the question of whether this method rises above ranking when there is evidence that gender and race salience do not both shift together for all participants. It begs the question of what the results of a card sort would be if one card has "Black woman" written on it. Would participants ask about splitting the card to indicate varying salience of race and gender in particular contexts.

The final two conditions to consider are how the method and analysis recognize the dynamic interaction between individual and institutional factors and avoid focusing solely on the markers but on a meaningful construct related to the markers (Bowleg, 2008; Hancock, 2007). While the identity card sort highlights the racialized-gendered nature of mathematics learning spaces for young Black women and focuses on salience of markers in context, specifically the shift from home to school to math class, it fails to adequately address institutional factors. It is very individual focused and does not, in and of itself, implicate any institutional practices or policies of mathematics education complicit in the heighted salience of race and gender for the young women.

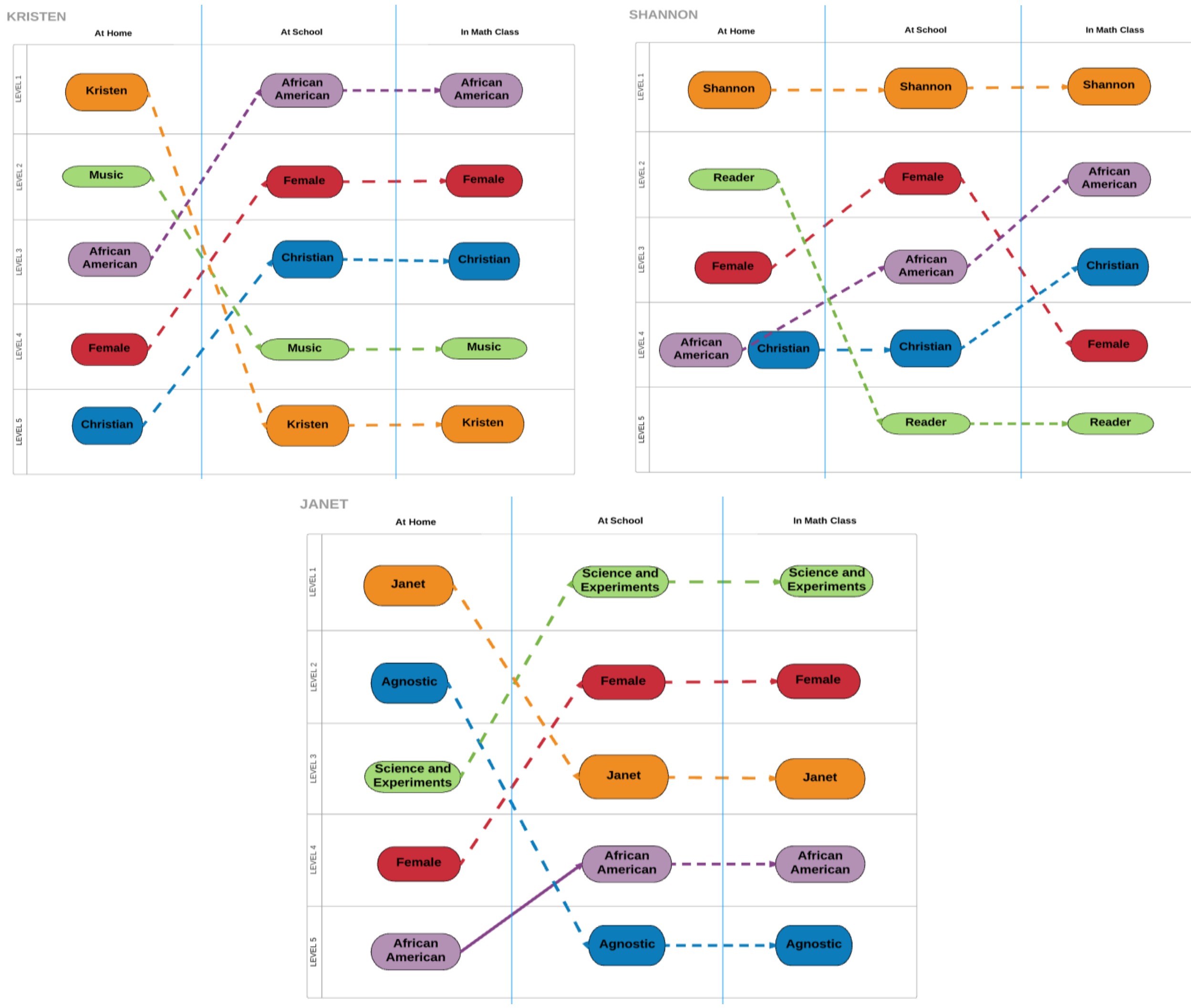

Figure 1: Heterogeneity of the Card Sort Responses for High School Girls

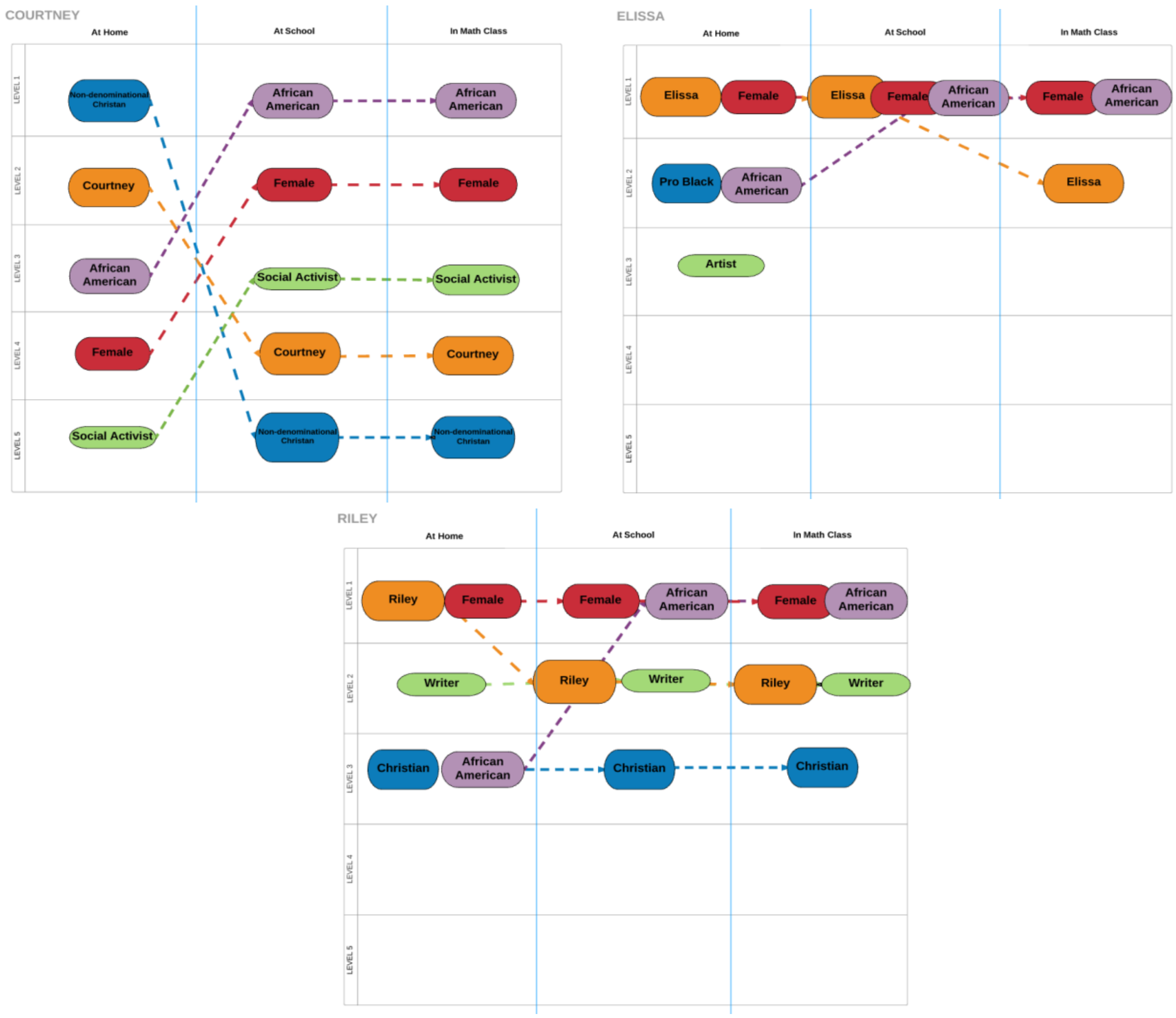

Figure 2: Heterogeneity of the Card Sort Responses for University girls

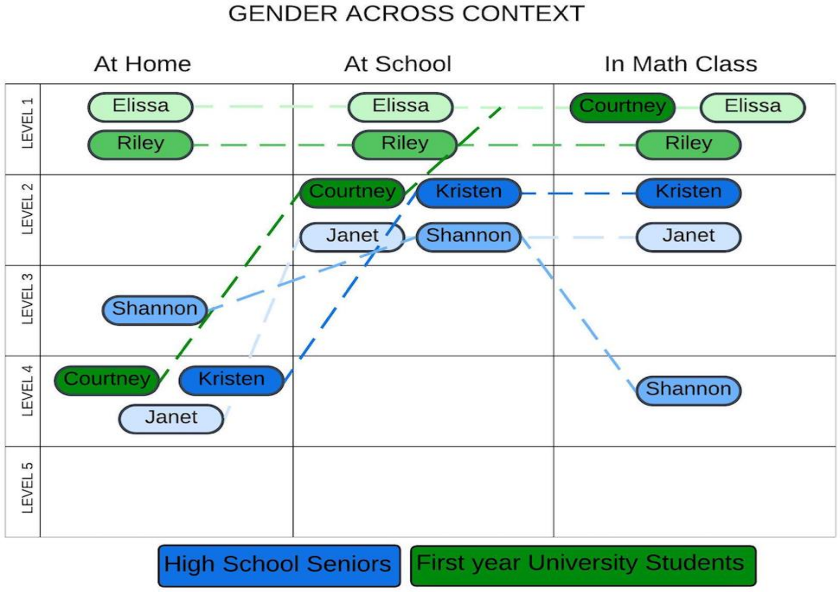

Figure 3: Gender Salience Across Space for all six young women

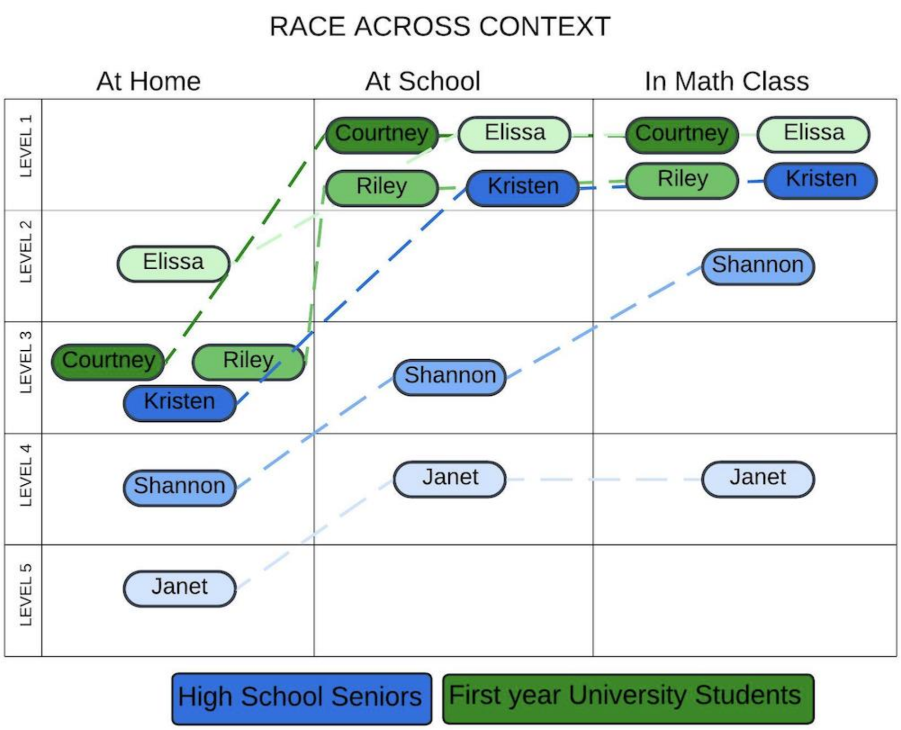

Figure 4: Racial Salience Across Space for all six young women

Discussion and Conclusion

Social theories like intersectionality used to understand the complexity of the human experience, especially those intended to answer complex questions of inequality and justice require careful methodological consideration (Hancock, 2016). It can be difficult to fully hold all of the complexities in singular methods of data collection and analyses which is why scholars call for a multi-method approach. Though progress has been made, the development of research designs and methods to effectively capture/measure or even methods for eliciting intersectionality during data collection remains underexplored (Hancock, 2007) and many researchers proceed labeling methods and analyses intersectional without fully considering to what extent conditions such as those made by Bowleg (2008) and Hancock (2007) are being met. Bowleg's piece, coincidentally, is one for which she examines her own methodological approaches for how well they meet the challenge.

The identity card sort was also, admittedly, conceptualized and employed without prior consideration for its adherence to various aspects of intersectional theory. And this post hoc analysis revealed that the method used in my study fell short concerning ranking socially constructed identity markers as well as lack of consideration for institutional factors. Additionally, a more robust analysis is needed to determine whether the identity card sort in conjunction with or situated within a semi-structured interview would be sufficient. Beyond whether or not the identity card sort qualifies as intersectional research, there is a question about the stability of the arrangements, for example, would a participant respond the same a day or two or even a week later. Nevertheless, there are a myriad of uses for the card sort as a research method, an activity to be done with teachers during professional learning to discuss the role of socially constructed identities in schools, or as a reflective activity for qualitative researchers exploring their positionality. If nothing else, I hope this examination of the identity card sort beacons the mathematics education research community to be more intentional about our approach to intersectional research as we answer the call to explore the mathematics learning experiences and make critical discoveries and changes towards inequality and justice for our most vulnerable learners.

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The Limits of Interventions in Promoting Gender Equity in Undergraduate Mathematics Advising

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Keywords: gender, systemic change, design experiments, undergraduate education.

We use the term *mathematics advising* to describe academic advising programs for incoming undergraduates offered by mathematics departments to guide students toward a decision about which mathematics course(s) to register for. Unlike traditional academic advising, mathematics advising sessions are very short (five to ten minutes) and the advisor and student have not met previously (Grites, 1979). Advisors frequently have little to no training, often unwittingly producing gendered inequities (Gholson et al., 2021; Margolis et al., 2023). We report on the development and implementation of math advisor training designed to disrupt interactional patterns of gendered discrimination. The training curriculum consists of four lessons: *Why Mathematics Advising Matters, The Work of Mathematics Advising, Essential Practices in Equitable Advising, Understanding Your Mathematics Advising Ecology*.

Methods

We address the research question: Which aspects of a mathematics advising curriculum were implemented with fidelity? This work fits within the methodological approach of Design-Based Implementation Research (DBIR) (Fishman & Penuel, 2018). The broad goal of DBIR is to "address differences (both positive and negative) between innovative interventions as designed and as they are actually implemented in practice" (McKenney & Reeves, 2020). We describe the development of the mathematics advising curriculum, outline the curriculum and its implementation, and our fidelity of implementation (FOI) process. To analyze the FOI, we used an integer scale from zero to thee. Zero indicated subsections that were skipped entirely, and one to three indicated content that was implemented with low, medium, or high fidelity, respectively.

Findings and Discussion

We found that out of a total of 11 sections spanning four lessons, there were no sections with a high fidelity of implementation (i.e, FOI of three). The majority of sections (seven) were not implemented at all (FOI of zero). Three sections had a low fidelity of implementation (FOI of one), and one section had a medium fidelity of implementation (FOI of two). Three out of the four sections which included "gender" or "women" in their title were skipped completely (FOI of zero). Despite the commitment of the co-developers from the mathematics department, lessons relating explicitly to gender discrimination were not taken up by the facilitator. We attribute this to a few issues: time constraints in the design process for deep engagement with the curriculum materials, historical time constraints for training sessions within the mathematics department, and aversion to facilitating discussions related to gender discrimination and inequities. Only one lesson, *Lesson 3: Essential Practices in Equitable Advising*, was implemented entirely albeit with low fidelity. These findings suggest that the practices were valued over the motivating rationales for engaging in *appreciative advising* due to gender discrimination. Broaching the topic of gender discrimination is not typical in mathematics department trainings and future iterations of the advice for facilitation need to support discussions of social science research that may be less familiar to mathematics faculty.

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AN EXAMINATION OF EQUITY-BASED TERMINOLOGY IN PMENA PROCEEDINGS 1981-2000

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This study investigates the frequency and themes appearing in studies involving a collection of equity-based terminology (EBT) in 14 PMENA proceedings from 1981 to 2000. The percentage of EBT waxed and waned, but it was much higher from 1991 through 2000 than from 1981 through 1990. The highest percentage of EBT occurred in 1995 when the conference organizers decided to focus on diversity. Four themes emerged in research reports where EBT appeared: nature of a shifting sample; expansion in equity – 1991; teachers and equity; and policy and project influences. For instance, the sample where EBT appeared shifted from a focus on diversity to identify achievement differences to a focus on minority students to boost their achievement using culture-based interventions. Equity became much more nuanced and subtle in 1991 with an awareness that other factors such as societal racism influence student achievement.

Keywords: equity, inclusion, and diversity

The research trajectory focusing on equity in mathematics education has seen significant shifts over the decades, with an increased emphasis in recent studies. The current analysis specifically examines the integration and evolution of equity-based terminology (EBT) within the proceedings of the North American Chapter of the Psychology of Mathematics Education from 1981 to 2000. This study utilizes a dual approach: quantitatively tracking the fluctuation of EBT as a percentage of the total pages in the examined proceedings and qualitatively identifying prevalent themes and trends concerning EBT. This approach is set against a backdrop of broader trends in mathematics education research and equity-focused studies reported in other scholarly works.

In their broader historical analysis, Inglis and Foster (2018) explored the thematic evolution in prominent mathematics education journals since their inception around the late 1960s and early 1970s, noting a decline in Euclidean geometry studies and a rise in sociocultural theories from the 1990s onward. This period also witnessed a diversification of theoretical frameworks, including semiotics and embodied cognition, and a move away from experimental methods that dominated the 1970s. Hanna and Sidoli (2002) documented shifts in the focus areas of the *Educational Studies in Mathematics* journal, highlighting the growing importance of problem-solving, cognitive issues, and social factors, alongside a variable emphasis on gender and ethnicity research across the decades.

Extending this perspective to more recent developments, Vithal, Brodie, and Subbaye (2024) reviewed equity research in mathematics from 2017-2022, observing that conceptualizations of equity have notably broadened to emphasize identity, power, recognition, and representation. They highlighted that a significant volume of studies now focuses on the mathematical practices and teacher actions within specific classrooms and schools aimed at addressing inequality. Additionally, they noted that international studies have shed light on how societal inequalities impact and are reflected in student achievement within school systems. The narrative of shifting research priorities is further supported by the findings of Gökçe and Güner (2021), who analyzed over a thousand articles to identify evolving research interests from problem-solving initially to later emphases on technology, teacher training, and equity, particularly from the 2000s onward. This shift is also seen in the resurgence of equity as a focal research theme, encompassing issues of motivation, attitude, and demographic impacts on educational outcomes.

Meanwhile, Parks and Schmeichel (2012) identified significant obstacles in addressing race and ethnicity within the domain, pointing to systemic issues like the marginalization of race discussions and the oversimplification of racial categories in research settings. Two key research questions emerge from this historical context: How does the frequency of equity-based keywords vary in 14 proceedings from 1981 through 2000? What themes appear in research studies involving EBT in these proceedings?

Methods

The research focused on analyzing equity-based terminology (EBT) within the proceedings of the Psychology of Mathematics Education North America (PMENA) meetings from 1981 to 2000, where electronic texts were available for searching. Specific annual meetings selected for study included years from 1981 through 2000, with a few exceptions due to the availability of searchable texts. The terms examined encompassed a wide array of EBTs such as "African," "equity," "socio," "race," "ethnic," "justice," "disab*," "gender," "diversity," "equality," "cultur*," "identity," and "ethnomathematics." Certain proceedings required conversion into searchable PDF formats due to their initial unsearchable state, facilitating a comprehensive textual analysis.

For the purposes of this study, the analysis unit was a single page within the proceedings documents. EBTs were only counted if they appeared in the narrative sections of the texts, such as plenaries, research reports, and discussion groups, which involved original research or theoretical synthesis. Mentions in the table of contents or reference sections were excluded from the count. The methodology involved calculating the total number of pages featuring EBT and then establishing the percentage of these pages in relation to the total number of relevant pages in the proceedings, adjusting for non-English sections and introductory content. Instances of multiple EBTs on the same page were treated as distinct occurrences, while repeated mentions of the same term on a page were not double counted.

Each identified page was then coded for thematic analysis. The coding process started from specific sentences containing EBTs and expanded outward to include broader textual contexts. Although pages featuring terms related to gender and culture were included in the quantitative count, they were not thematically coded. The resultant codes were compiled into documents for each proceedings year, facilitating the examination of broader thematic trends across the analyzed periods. This detailed approach provided insights into the evolution and focus of equity discussions within the field of mathematics education over the studied years.

Results

Quantitative

The frequency and prevalence of equity-based terminology (EBT) in PMENA proceedings show notable fluctuations from 1981 to 2000. Beginning in 1987, researchers started focusing significantly on cultural, gender, equity, and racial issues, with these topics gaining momentum and peaking in interest in 1991. This peak aligns with broader research trends identified by Inglis and Foster (2018), indicating a growing interest in cultural aspects within the field. The year 1995 marked the highest emphasis on EBT, particularly terms related to equity, race, gender, and culture. This surge was influenced by the conference's theme centered on diversity and varied educational settings, leading to a considerable presence of these terms in the academic discourse. However, the focus shifted away from diversity in subsequent years, with a notable decrease in the mention of these terms by 2000, reflecting a shift in thematic focus within the proceedings towards other aspects of mathematics education research.

Qualitative

Nature of a Shifting Sample

The collection of proceedings documents predominantly highlighted the use of diverse samples in educational research to examine various aspects of mathematics education, ranging from students' math capabilities to the effectiveness of specific interventions. These samples, often detailed by race, gender, SES, and less frequently by language or disability labels, were instrumental in studies such as Bell & Burns (1981) and Izsák & Fuson (2000), aiming to understand phenomena or test educational models. However, despite the diversity of the samples, it was less common for results to be disaggregated by these diverse groups, with studies like Armstrong (1989) not exploring the underlying causes of differential achievements noted by Brown & Wheatley (1990). The 1990s saw a refined focus on particular student groups, including minority and low SES students, to address specific educational disparities or enhance understanding of their mathematical capabilities, as seen in the works of Presmeg et al. (1995) and Mousley (1990). This period also noted a cultural integration into educational strategies, aiming to leverage students' backgrounds for educational gains, exemplified by Ayers et al. (1998) and Neves & Fraga (1987). Interestingly, while student diversity was frequently acknowledged, the diversity of teachers was seldom addressed in these studies.

Expansion in Equity – 1991

The 1991 academic proceedings indicate evolving themes regarding educational equity, particularly in the context of Black students' achievement. Researchers like Ajose (1991) highlight the multifaceted influences beyond familial factors, including the mathematics curriculum, teacher biases, and classroom dynamics, which shape Black students' educational outcomes. Campbell, Benson, Bamberger, and Hutchinson (1991) counter deficit narratives by emphasizing minority students' valuable informal knowledge applicable to real-world problem-solving. The impact of societal racism on school processes and student achievement is underscored, drawing from Eisenhart’s (1991) anthropological insights into cultural effects on education. Additionally, there's growing awareness of how research might inadvertently normalize deficits, with Kamii (1991) cautioning against overgeneralizing her findings across socioeconomic statuses, while Campbell et al. recognize early educational assets in minority kindergartners, laying groundwork for future equity-focused research that includes considerations of Black students' identities.

Teachers and Equity

The 1995 proceedings and subsequent research highlighted the pivotal role of teachers in promoting equity in mathematics education. Studies from the mid-1990s to early 2000s, including those by Tharp and Lovell (1995), Santa Cruz and McLeod (1996), and others, observed how preservice and practicing teachers conceptualize and implement equity. Tharp and Lovell identified four stages of development among preservice teachers' understanding of equity, noting a concerning trend where only a small fraction reached the highest stage of recognizing equity as a complex, interactive process vital for enhancing students' mathematical understanding. Similarly, Becker, Pence, and Pors (1995) and De La Cruz (1995) examined the impact of policy initiatives and teacher preparedness, finding gaps in readiness and implementation, particularly with non-Spanish-speaking teachers working with Spanish-speaking students. Later research by Fuson, Sherin, and Smith (1998) and Hodge (1998) developed models linking equity and pedagogy, pointing out that equity concerns are magnified under reform-oriented teaching practices. Edwards (1999) and Riggs (2000) further examined the dynamics within bilingual and minority-focused educational settings, emphasizing the need for pedagogical support and role models to effectively foster equity. Collectively, these studies underscore the challenges and importance of equipping teachers to advance equity in mathematics education.

Policy and Project Influences

The period from the 1990s to the present has been influential in shaping mathematics education through the lens of equity, marked by the introduction of the *Curriculum and Evaluation Standards for School Mathematics* (hereafter referred to as *Standards*) by the National Council of Teachers of Mathematics (NCTM, 1989). Starting with these standards, which emphasized equity alongside educational accountability, subsequent research and educational projects began to incorporate these principles. Notably, it took several years for equity-focused studies to emerge in academic discourse, as evidenced by their inclusion in the PME-NA proceedings from the late 1990s. Research during this period, such as Tharp and Lovell's (1995) examination of preservice teachers' perceptions of equity, Kaplan's (1997) study on language minority students, and Presmeg's (1997) exploration of cultural influences on classroom mathematics, underscored the ongoing impact of the *Standards*. Moreover, initiatives like Equity 2000 and the San Diego Project Leadership Institute further propelled equity to the forefront of educational reform efforts, inspiring studies and curriculum developments that integrated these values at both state and national levels.

Discussion

The study analyzes the presence of evidence-based teaching (EBT) in the PME-NA Proceedings from 1981 to 2000, highlighting a noticeable increase in discussions related to equity, especially in the 1990s compared to the 1980s. During the earlier decade, EBT constituted a smaller fraction of the proceedings, peaking at 40.8% in 1995, a year particularly focused on diversity. Discussions during this period nuanced the use of race in research, initially employed to assess human capabilities and curriculum viability, and later to explore differential academic achievements among students. Interestingly, race was frequently mentioned at the beginning of studies to describe student samples but was seldom used to describe teachers, indicating a narrow focus on student outcomes without considering how teacher identity might affect educational practices. Additionally, while race often surfaced in concluding sections through the disaggregation of achievement data, it was rarely integral to the analytical processes or explanatory discussions within the studies.

Further complicating discussions around race in the 1991 Proceedings, research began to challenge deficit narratives associated with minority students by considering broader influences on their mathematical achievement, such as teacher attitudes and societal contexts. This shift was part of a broader trend during the decade, as seen in national policy initiatives and projects like Equity 2000, which influenced studies on instructional strategies that align with students' cultural backgrounds. This period also saw researchers like Ayers et al. (1998) investigating the integration of students' funds of knowledge into their mathematical learning, indicating a growing commitment to addressing equity in mathematics education, setting the stage for further exploration into these themes in the new millennium.

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TWO COMPETING STORYLINES THAT INFLUENCED STATUS DURING SMALL GROUP PROOF ACTIVITY

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Dominant narratives about what it means to write and do mathematics influence who typically gets viewed as competent in proof activity. In our ongoing work to explore interaction patterns that maintain (legitimate) or disrupt (delegitimate) mathematical status hierarchies while students collaborate on proof-related tasks, we applied positioning theory as an analytic lens. From this analysis we observed two storylines about what it means to engage in proof activity that influenced status: the Informal Language and the Formal Language storylines. This brief report contributes an image of how these storylines influenced mathematical status among three students while they worked on proving a statement in an intro-to-proof course over Zoom.

Keywords: Undergraduate Education, Equity, Inclusion, and Diversity, Reasoning and Proof

Dominant narratives about what it means to write and do mathematics influence who typically gets viewed as competent in proof activity (Weber & Melhuish, 2022). Proof-based courses are predominantly taught by (mostly white) men and the majority of students who take them are men (Blair et al., 2013), making these spaces vulnerable to dominant narratives such as 'doing mathematics' reflects 'doing masculinity' (Jaremus et al., 2020; Leyva et al., 2017; Mendick, 2006) and privileges whiteness (Battey & Leyva, 2016; Martin, 2019) and Eurocentric perspectives (Rowlands & Carson, 2002). Such narratives can influence who gains higher status and power as students engage in proof activity, which can (re)produce inequitable mathematical status hierarchies in proof-oriented classrooms. By using positioning theory in our ongoing efforts to identify interaction patterns that maintain (legitimate) or disrupt (delegitimate) mathematical status hierarchies while students collaborate on proof-related tasks (e.g., Ellis & Alzaga Elizondo, 2023; Ellis et al., 2024), we have observed two storylines about what it means to engage in proof activity that seem to have an effect on mathematical status (defined in the next section): the Informal Language and the Formal Language storylines. This brief report contributes an image of how these storylines influenced three students' relative mathematical status while they worked on proving a statement in an intro-to-proof course.

Theoretical Perspectives

Perceptions of status can impact how students communicate, and thus learn, while working in groups (Cohen & Lotan, 2014; Esmonde, 2009). Generally, 'status' can be viewed as a relative position that is widely accepted as advantageous; status is not fixed or inherent, yet exists within a relative ranking system (Cohen et al., 1999; Ridgeway, 2018). In classroom contexts, 'academic status' refers to perceptions of who is "smart" (e.g., answers questions in class) and 'social status' refers to perceptions of social standing (e.g., popularity). Then, 'diffuse status characteristics' exist within and beyond classroom contexts and refer to discernable identity features by others, such as race (via skin color) or language use (via intonations or accents), gender expression, and ability status. We use the term 'mathematical status' to mean a student's relative position among peers in the context of doing mathematics, which likely involves a complex combination of academic, social, and diffuse characteristics. Interaction patterns that maintain/disrupt mathematical status hierarchies are difficult to identify; therefore, we turned to positioning theory for analytic tools to investigate such phenomenon.

Positioning theory describes social interactions in terms of communication acts, storylines, and positions as rights and duties (Harré, 2012; Herbel-Eisenmann et al., 2015). *Communication acts* include the interpretive meanings conveyed by verbal and nonverbal discourse. Interactions in human life "tend to follow already established patterns of development" (Herbel-Eisenmann et al., 2015, p. 188) called *storylines*. Storylines are continual repertoires that can be commonly shared (such as cultural norms, beliefs, and values) and can reflect expectations and conventions that relate participants according to available positions, or rights and duties. *Positioning* is thought of as a fluid unfolding of communication acts (via discourse) that organize social structures by situating people in dialogue as participants in mutually constructed storylines (Davies & Harré, 1999; Herbel-Eisenmann et al., 2015). Power relations fluctuate based on interactions and available positions, and so situations will never be 'status-free' or fully 'equitable'. However, we assert that situations in which status hierarchies are legitimated likely amplify inequities, while situations where status hierarchies are delegitimated likely attenuate inequities, with *in*equities defined as situations that prevent access to resources needed for learning (Shah & Lewis, 2019).

Methods

The data for this study comes from a larger project that designed and implemented inquiry-oriented intro-to-proof curricular materials. Given that issues of status and power likely arise in inquiry classroom contexts (Battey & McMichael, 2021), this project provided a rich context to study how status hierarchies impact student-student interactions. Data from an intro-to-proof class taught in a U.S. urban public university synchronously over Zoom was used for the present study. Justin (white man), Abigail and Alison (white women), worked together to complete a proof of the statement: An element of a group appears at *most* one time in each row of its Cayley Table. Students used a shared Google Doc while they discussed ideas verbally over Zoom. Video screen recordings simultaneously captured students' Zoom and Google Doc activity.

Screen recordings were used to create multimodal transcripts (Hoffman, 2019), coordinating verbal and nonverbal Google Doc activity. This was critical for our analysis because students often used their Google Doc activity to supplement their verbal discourse (Alzaga Elizondo, 2022). The multimodal transcripts were parsed into utterances and exchanges, in which utterances were taken as complete thoughts distinct from each other based on content, intonation, and/or pausing. Exchanges were the primary unit of analysis, defined when at least two participants were interacting at the 10^2 (seconds to minutes) timescale (see Herbel-Eisenmann et al., 2015 for Lemke's timescales). A transition between exchanges was based on topic shifts. Then, to apply positioning theory constructs, we collaboratively rewatched video segments of exchanges and recorded observations of possible meanings underlying discourse (communication acts), storylines, and available positions (rights/duties).

Preliminary Results

Abigail, Alison, and Justin's mathematical proving activity in this episode was collaborative overall (Alzaga Elizondo, 2022), yet our positioning analysis revealed a mathematical status hierarchy that ultimately positioned Justin higher relative to Abigail and Alison. While several processes contributed to this positioning, in this preliminary report, we focus on the observed evidence of two competing storylines in relation to mathematical status: the Formal Language storyline reflected a "right" or "proper" way to "do proofs" that required validation by an expert authority, while the Informal Language storyline reflected the value in creativity and problem solving in "doing proofs." The excerpts presented here highlight the emergence and influence of these storylines regarding the students' relative mathematical status.

Exchange 1 – Justifying a Line in Their Proof

Prior to this exchange, the students drafted a proof that read "AQ=B and AW=B, where Q and, W, and A are symmetries, then AQ=AW, then Q=W." The instructor commented that they should justify the last line of the proof. Justin made the following suggestion:

> Justin: I'm not sure if he'd *allow* us this but I think we might be able to do *this* (starts typing "A^-1" on a line above their proof) […] but if we do A inverse, does that *show* that it's supposed to be the opposite of A?
>
> Alison: (smiling) I feel like that's getting ahead of where we're at
>
> Justin: yeah that's also (laughing) kinda why I didn't want to do it (deletes "A^-1")

In lieu of using a "formal" inverse rule, Abigail drafted an informal justification for why AW=AQ implies W=Q, writing "if you perform an identical symmetry as a first step you start with an identical symmetry." Meaning, AW and AQ are identical composite symmetries that begin with performing the symmetry A. Since the first step is identical, the second step (i.e., performing W and Q) must also be identical (i.e., W=Q). Abigail followed with the disclaimer:

> Abigail: (descending tone) I don't know how to do anything formally I'm just so, (cross talk) I don't know, if this was my proof that's probably what I'd put but I know it's not formal.
>
> Alison: I think we're not doing a super formal proof right now anyway so it's alright.
>
> Abigail: (smiling) yeah I don't think so either.

This exchange provided evidence of how the two storylines emerged from the small group interaction. Before suggesting a formal rule, Justin expressed the concern that it might not be allowed ("I'm not sure if he'd allow us this"), which signified a reliance on an external authority to validate their justification. In contrast, Abigail offered an informal justification, explicitly stating that she knew it was "not formal" and that she doesn't "know how to do anything formally." Alison acknowledged that they were "not doing a super formal proof" and Abigail agreed, potentially uplifting the value of using informal language in proofs.

Exchange 2 – Justin Positioned As Expert

Prior to the next exchange, the instructor hinted to the entire class that they should come up with an inverse rule (i.e., AA^-1 = I) to complete the proof. Upon re-entering the breakout room to continue working in their groups, the following exchange occurred:

Justin: I'm annoyed (cross talk) It's inverse [...] the first thing I said was right. (laughing) we're like "oh we don't wanna do that, it's too complicated" go back "hey (cross talk) you wanna use the inverse"

Alison: no we didn't say it was too complicated. I thought it was gonna be too early (types "then (A^-1)AQ=(A^-1)AW")

Justin: that's what I meant though (laughing) (under breath) it's just like (laughing) uh I had it right at first. (cross talk) uh we need to add the rule.

Abigail: so we just have a rule?

Justin: (speaking quickly) the rule is super easy to make it's A to the power negative one times A equals (inaudible) and then you (inaudible cross talk) use the 1S equals S which is equal to S1 and you get S equals S.

This exchange provided evidence of the Formal Language storyline influencing the emerging mathematical status hierarchy among the students. Justin referenced that he had the "right" idea originally, which evoked the Formal Language storyline: there is a "right" or "proper" way to "do proofs" that can be validated by an expert authority. We interpreted this communication act as implicitly undervaluing their previous informal justification. He then communicated that their prior apprehension around adding an inverse rule was "too complicated" yet the instructor confirmed "you wanna use the inverse." Aligning himself with the "right" way based on the instructor's direction positioned Justin with expertise, increasing his mathematical status relative to Alison and Abigail. Almost interrupting Justin, Alison corrected him by countering that they did not think his idea was "too complicated," they were unsure whether they could use it. We interpreted this communication act as a reference to the inquiry nature of the course where students and the instructor built up mathematical tools (i.e., the inverse rule) to use. Justin agreed with Alison and continued to express that he "had it right at first." Abigail and Justin both mentioned they needed to "add the rule" which Justin quickly responded with how to write the rule symbolically. A possible interpretation of the immediacy in his speech is that adding the rule was trivial or "easy" ("the rule is super easy to make"). This exchange highlights the competing nature of the Formal and Informal Language storylines as Justin repeatedly stated he had the "right" answer (elevating his status) while Alison negotiated a lower status positioning by defending their informal strategy.

Discussion

This preliminary report contributes evidence of two competing storylines about what it means to engage in proof activity that impacted three students' relative mathematical status as they collaborated on drafting a proof. The Formal Language storyline reflected a "right" or "proper" way to "do proofs" that required validation by an expert authority. In contrast, the Informal Language storyline reflected the value in creativity and problem solving in "doing proofs." We argue that the Formal Language storyline likely influenced mathematical status by elevating Justin's "right" idea and momentarily lowering Abigail's and Alison's mathematical status. We do not want to imply that the Informal Language storyline is better than the Formal Language storyline; however, we do want to point out that "doing proofs" (engaging in proof activity) includes both formal and informal language, yet over privileging formal language may maintain imbalanced mathematical status hierarchies (e.g., Adiredja, 2019). We also acknowledge that all three students in the episode identified as white. Further analysis may evidence gendered differences in the relative positioning between the students, and future analysis of different episodes should take into account whether and how positioning relates to racialized hierarchies of mathematical ability (e.g., Martin, 2009).

One implication of this work relates to what it means to "do proofs" and how this meaning is communicated to students in proof-based courses. Some research has shown that students' perceptions of and approaches to proof can be both logical and creative and meaningful for them, especially when they experience problem-based proof courses (e.g., Smith, 2006). Communicating the value in both informal and formal language in proof activity may serve as a way to disrupt mathematical status hierarchy formation arising from these competing storylines.

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THEORIZING MATHEMATICS EDUCATION AS A PUBLIC THING: A CALL FOR RESEARCHING MATHEMATICS IN ADJACENT LANDSCAPES

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In this conceptual paper, we position mathematics education as a public thing (Honig, 2017) and explore the consequences of acknowledging and investing in mathematics education as such. We propose that in embracing and insisting on mathematics education as a public thing, researchers must understand mathematics education as a joint investment in the human activity of mathematics and in democratic practices of living together within the sociopolitical sphere as formatted and shaped through mathematical activity. Drawing on an empirical example of organizing against school closures in central Texas, we call for mathematics education researchers to attend to mathematics in adjacent landscapes.

Keywords: Research Methods; Systemic Change; Equity, Inclusion, and Diversity; Policy

In this paper, we position mathematics education as a public thing (Honig, 2017) and explore the consequences of acknowledging and investing in mathematics education as such. We propose that in embracing and insisting on mathematics education as a public thing, researchers must understand mathematics education as a joint investment in the human activity of mathematics and in democratic practices of living together within the sociopolitical sphere as formatted and shaped through mathematical activity (Skovsmose, 1998). As such, mathematics education researchers are positioned as people who (must) care for and deliberate over mathematics education as a public thing. Drawing on an empirical example of parent and community organizing against school closures in central Texas (Gómez Marchant et al., 2023a), we call for mathematics education researchers to attend to mathematics in adjacent landscapes–landscapes of public life that exist beyond school or classroom walls, such as those of civic engagement.

Democracy, Civic Engagement, and 'Public Things'

Honig (2017) argued "democracy is rooted in common love for, antipathy to, and contestation of public things" (p. 4). Public things are public goods or objects (e.g., roads, libraries, utilities, universities, parks, railways, schools) with which individuals in communities construct relational attachments of care and concern that define the use and maintenance of these objects. *Civic engagement* then describes participation in communal life entailing the negotiation of use and relations with public things. Honig (2017) proposes democracy cannot exist without public things because public things create a space of coming together and deliberation. In this sense, people's civic engagements are over public things. Public things are part of the 'holding environment' of democratic citizenship; they furnish the world of democratic life…. Also constitute us, complement us, limit us, thwart us, and interpellate us into democratic citizenship. (Honig, 2017, p. 5). For example, memories of a childhood street, neighborhood, parks, or cities are imbued with affective and identity defining aspects. If the public things one civically identifies with are sufficiently threatened, then one may be motivated to stronger degrees of civic engagement. Knight Abowitz (2018) demonstrated how public education is a public thing when Betsy DeVos' nomination (and eventual confirmation) as Secretary of Education was seen as an attack on public schools because of her history of advocacy for privatizing education. The possibility of education no longer being a public thing became more likely. Consequently, DeVos' nomination activated many to protest and civically engage to a degree not seen before for a nomination to Secretary of Education (see Brown, 2017; Garcia, 2017; Knight Abowitz, 2018). Recognizing schools as public things also provides explanatory power to why school closures have been described as experiencing a social or civic death (Johnson, 2012). A school being closed destroys a community's relational attachments to a public thing.

Mathematics Education as a Public Thing in a Landscape of Racialized Contestation

Public things, including schools, are not immune to inequities. Instead, Honig (2017) warns, "Any successful public thing presents us with this problem: the public things that constitute the demos exclude some and privilege others. In the United States, what is called 'public' is sometimes white, sometimes black; it is rarely both" (p. 24). Bell (2004) and Justice (2023) demonstrate how a (white) public education system became a public thing accessible to Black learners after *Brown v Board of Education* required the desegregation of public schools. Both authors emphasize how schools remained white and violent towards Black learners. San Miguel (2001) showed how public education as a public thing was limited for Latinx children, particularly through the use of language as a proxy for race (Cervantes-Soon, 2014; Garcia & Otheguy, 2017). The history of public education as a public thing for Black, Indigenous, Latinx, and other minoritized populations in the United States warrant Honig's (2017) claim: "when public things *are* democratized, the response of the powerful is often to abandon them. White flight is not just from the urban to the suburban; it is from the public to the private thing" (p. 24, emphasis in original). Public things then are surveilled and policed asymmetrically to the benefit of white people (Keyes et al., 2023). These acts are warranted through discourses of access, maintenance, concern, and care for schools (see Justice, 2023; Keyes et al., 2023).

Schooling as a public thing is part of the United State's racial project (see Gutiérrez, 2001; Justice, 2023). As Justice (2023) argued, the public in public goods describing schools and other social goods was for the benefit of whites: "It can be misleading when historians today identify points in the past when writers intoned words like 'the common good' or 'the public good' as a way to suggest that schooling was for everyone's equal benefit. It wasn't" (p. 157). When the relational attachments become strained or points of tension, then individuals are activated to participate in the destruction or maintenance of the public thing. Esparza (2023) provides an example of a Mexican and Mexican-American community in Texas that developed their own community public school district (public thing) in resistance to the white schools they were not allowed to attend. The Mexican and Mexican-American community worked together to organize, create their own relational and affective nodes with a dedicated school space, to develop policies and regulations to maintain their public thing. Eventually, however, policies developed by the greater governing body of the city and state education agency destroyed the Mexican and Mexican-American community schools and maintained white's ownership of public things.

Mathematics education researchers (see Bullock, 2019, 2023; Gutiérrez, 2018; Martin, 2003, 2019) have highlighted parallels to the 'goods' Justice described within *mathematics for all* statements. Mathematics and mathematics education is complicit in the racial project schools engage in (see Bullock, 2019). Mathematics as a public thing "press[es] us into relations with others" (Honig, 2017, p. 6). As such, the 'for all' discourses provide a false promise of access and public-ness to mathematics education. Policy and regulations benefiting those most adjacent to whiteness surveil and police who gets to learn mathematics (see Morton & Riegle-Crumb, 2019), when is someone doing mathematics (see Dowling, 1991; Lundin & Christensen, 2017), and what is mathematics (see Thanheiser, 2023). We continue our thought experiment by acknowledging that mathematics education as a public thing has always been contested through the emphasis of academic mathematics—a sphere of mathematics defined by (white) gatekeepers (e.g., standards; Eurocentric ways of thinking about mathematics). We call for mathematics education researchers to care for mathematics education as a public thing through pursuing sociopolitical mathematics activity and research in adjacent landscapes.

Mathematics as Human Social Practice and Formatting Democratic Life

Whereas we treat mathematics education as a public thing (a noun), we understand mathematics itself as an activity (a verb) describing human social practices related to organizing and modeling space and time, number, pattern, magnitudes, etc., in manners both concrete and abstract, particular, and generalizable (Dowlings, 1991; Skovsmose, 1998; Thanheiser, 2023). Specifically, we draw on a social theory of numeracy (Craig & Guzmán, 2018), extending it to mathematics broadly, wherein mathematics is understood as a human social practice embedded in, responding to, and organizing social-political-historical landscapes. "Numeracy is not a mathematical ability, but instead, the social ways in which people engage that ability. Numeracy is therefore motivated by broader concerns for social goals and mathematical ability is strategically employed because numeracies are purposeful" (Craig & Guzmán, 2018, p. 14).

Skovsmose (1998) described how mathematical activity underlies social and political life, through its role in structuration (Giddens, 1984), calling this the "formatting power" of mathematics. From a US historical perspective, for example, Cohen (2003) traces how the original US Constitution defines the conditions of democracy through approaches to counting and distribution of representation that structure access and participation including the 3/5th rule that counted enslaved individuals as less than full humans but used their presence to increase the political representation of the enslavers. Hacking (2015) traces the emergence of statistics and related possibilities of census collection for population management as creating categories of personhood individuals are then interpolated into. The post/modern state treats individuals and communities as a resource to be managed through counting and sorting (Scott, 2020). Today, even as "science" is under popular contestation, policy discourse continues to call on data and numbers to lend legitimacy to argumentation (Mudry, 2009). Even in equity discourse, mathematical models of equity (Tate et al., 1995) are likely to dominate, even or especially, when they may silence evidence of human relations and the human condition (Gómez Marchant et al., 2023a).

Investing in mathematics education as a public thing means deliberating over the content (what is un/seen as mathematics; see Gutiérrez 2018; 2023) and contexts (where and when mathematics is happening; see Gómez Marchant et al., 2023b). Understanding the public-ness of mathematics education entails attending to the mathematical activity that formats, structurates, and drives public life and deliberation, if often in invisible ways. Skovsmose refers to this type of research as a form of “mathematical archaeology” (1998), an excavation of those mathematical activities that are or can become “frozen” into technologies, decision-making practices, or other aspects of public life not often recognized as explicitly mathematical. We refer to research examining the mathematical activity of life beyond schools as mathematics education research in adjacent landscapes. Next, we share an example from our own current and ongoing research.

An Example of Research in Adjacent Landscapes: Organizing Against School Closures

At the December 2022 Sunny Field Independent School District (SFISD) school board meeting, the administration presented seven proposed plans to close two to three elementary schools due to the changing property values and inflation, decreased enrollment, and state legislative failure to increase the basic allotment per learner. A group of mostly white parents and community members began to organize to push back on the district's proposals for closing their predominantly Latine and Black schools. Two members of the research team acted as participant observers (Spradley, 2016) in the organizing efforts of the parents, collecting audio and fieldnotes at community meetings (Emerson et. al., 2011), and conducted oral history interviews (Portelli, 2018) with several members of the group.

In describing and reflecting on the district’s handling of, and their own participation in organizing against, potential school closures, parents highlighted multiple forms of mathematical activity including 1) the district’s mis/management of data to explain and justify the need and targeting of schools for closure, 2) parental re/narrating of the school closure story through the surfacing and circulation of different data, 3) parental critique and rejection of numbers and data as meaningful or sufficient in representing children, family, and community relations with schools, and 4) parental awareness and critique in relation to the exclusion and underrepresentation of Latine voices in district hosted meetings and in parental organizing. Parent, Fiona (pseudonym), evidenced each of these, as she explained the motivation and consequence of using *Freedom of Information Act* requests to obtain data from the district.

> [The district] had <u>data that supposedly encapsulated</u> the fact that they were going to take equity into account. And then that was later used in a very different way–the word equity … Like if you're being moved to another school, then we'll make sure that <u>it's kind of the same socioeconomic status</u>…but the <u>only things that was talked about was maps, capacity</u>…none of it was like any of the stuff <u>that we later collected and put on a giant spreadsheet and was like, let's get a real look at the schools</u>…they had this list of reasons why you were being closed, but then <u>they didn't actually have data that matched that</u> (Fiona, Interview, 0:33:14)

The spreadsheets parents created gave them the opportunity to interpret the data the district was using in their decision-making processes and provided them a different picture of the schools set to close which they later shared with others at community meetings.

Conclusion

Positioning mathematics education as a public thing centers the sociopolitical nature of mathematical activity in our everyday lives; consequently, placing the responsibility on mathematics education researchers to care for and be concerned for the public nature of mathematics education. Mathematics education has been a part of laws, policies, and regulations that have harmed and oppressed (see Bullock, 2019; Cohen, 2003). As such, mathematics education researchers are complicit in a sordid history. To show care for and recuperate, our work as mathematics education researchers must intentionally, respectfully, and cautiously transgress the boundaries of our comfort within educational landscapes. Mathematics education researchers must themselves become activated to the political and care for our public things.

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SONICALLY CONSTRUCTING THE MATHEMATICAL GEOGRAPHIES OF TWO ELEMENTARY LATINÉ LEARNERS

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The authors share their experience creating sonic constructions representing the school mathematics experiences of two Latiné learners. Antonio and Juanita are 3rd grade learners in two different rural schools in the southeastern US. An interview with each learner was edited such that the interviewers were removed. The remaining snippets of the remaining audio were curated to share Antonio's and Juanita's testimonios. Through the sonic constructions, deeper explorations of how Antonio and Juanita navigate racial and linguistic spaces can be conducted.

Keywords: Equity, Inclusion, and Diversity; Research Methods; Elementary School Education

Sound waves reverberate around us helping to engage, define, and make sense of space. Although sound frequencies are only one source of information used to make sense of the world, sound is ubiquitous, and many messages pass through our auditory existence (Gershon, 2011; Gershon & Applebaum, 2020; McLoughlin, 2023). Hence, there is an important relationship between sound and the construction of space. Teachers and researchers embark on sonic journeys to engage with the sound of mathematical discourse, the classroom environment, an interview, a conversation between learners, and the stories about navigating schools and mathematics being told. The construction of knowledge at schools is dependent on the privilege and dependency the educational environments have placed on sonic constructions in classrooms (Gershon & Applebaum, 2018). Sound studies in education (see Gershon, 2011, 2013; Gershon & Applebaum, 2020; Wargo, 2018, 2019) deeply consider the role sonic layers play in our educational institutions, including the multimodal storytelling capabilities of learners.

The politics of sound (Kangieser, 2015; McLoughlin, 2023) forefronts questions of who is heard, who listens, how one is heard, and what one does with sound. Our journey through and with the world incorporates the sounds of an ecosystem and our own sonic additions to it, even if we cannot or choose not to hear life's reverberations (McLoughlin, 2023). "Sounds are therefore necessarily educational in nature, sensual data so rife with information that the listener can render often disparate-seeming sounds into embodied meaning systems" (Gershon, 2011, p. 66). Our meaning-making, particularly of the social other, is dependent on how the other is heard. Researchers treating sound as apolitical audio frequencies in space is is prohibitive to our being able to reach deeper understandings of our environments (Littlejohn, 2021).

Theoretical Framework – Geographies of Despair/Hope

Hidalgo (2017) describes spaces and soundscapes for Communities of Color as geographies of hope and geographies of despair. In her project, *predatory landscapes*, Hidalgo (2011) worked together to "resist spatial domination through creative and inventive appropriation of space" (Hidalgo, 2017, p. 80). Geographies of hope are the appropriation of a dominant space. Hidalgo's (2011; 2017) exploration also provided further insight on the geographies of despair the participants were navigating. Geographies of despair are spaces where dominant narratives maintain racial hierarchies. Hidalgo (2017) described the larger number of fringe financial services available in the southern area of Phoenix where immigrant populations have historically been located. These geographies of despair demonstrate the long-standing history of white supremacy through laws and policy targeting immigrant populations. By mapping the geographies of despair, Hidalgo (2011; 2017) is engaged in mapping phenomena through sounds, visuals, and other media to make visible the invisible and disappeared (see Popovski & Young, 2023; Russell & de Souza, 2023). Insight into the tensions between geographies of hope/despair can provide powerful understandings of Latiné childrens' experiences navigating school. Hidalgo's work influenced us to recontextualize explorations of learners' geographies of hope/despair. Our focus in this paper was on the audio recorded testimonios of Antonio and Juanita to create a sonic construction where they share their geographies of hope/despair.

Positionality Statement

The research team is dedicated to disrupting the white mechanisms of educational institutions for the flourishing of Latiné children. As a collective, we weave and vulnerably share our stories as Latinx, Iranian, white, multilingual, immigrant, queer, cisgender, children in poverty, and having invisible health and mental issues. It is important to share openly to create a counterspace for us and our partners. We push ourselves to work in different mediums to disseminate the valuable knowledge gifted to us by our partners. We recognize our position in the academy—a cruel (white) apparatus—requires our continued adjacency to whiteness. We work together to maintain as much of our former selves as possible.

Methodology: Sonically Constructing Antonio and Juanita's Testimonios

We wanted to maintain the sound of their voices through the process of constructing their testimonios. Hence, working through the medium of sound was important. As Wargo (2018) described, "Sound is not merely a descriptive experience but a set of social relations. Sound operates as a worldview. It is a symbolic system and form of composing" (p. 14). Using the video/audio file, the first author listened repeatedly to Antonio's and Juanita's interviews ranging 16 to 20 minutes. First, it was important to remove the voice of the interviewer. We used Final Cut Pro to separate the audio of the video (to depend only on the sonic landscape—the totality of the sounds record) and edit the audio file such that only the voices of the learners were left. This included as much as possible removing the interjecting sounds made by interviewers to suggest the interviewee is being heard (e.g., yes, mhm, hmm). From these files, the first author listened for Antonio's and Juanita's possible testimonio. It was important to listen for descriptions of their experiences (e.g., who they are, what brings them joy, what silences them, the emotionality of their experience in schools as Latiné learners). Furthermore, listening specifically to how these experiences culminate in their mathematics learning. In general, we wanted to understand how two Latiné learners navigated their experiences in school and the mathematics classroom. Littlejohn (2021) warns this kind of editing is creating a fictitious version of Antonio and Juanita, but that it is not a limitation. “Like written ethnographies, sonic ones are fictions. This does not mean that they are untrue or do not index reality. Rather, they are ‘partial truths’ that both reflect the specific engagements between ethnographers, people and place” (Littlejohn, 2021, p. 36). Therefore, like counterstorytelling (Solórzano & Yosso, 2002), we ground Antonio and Juanita’s testimonios in the empirical data provided, our own experiences navigating school mathematical spaces, and the literature to contextualize their experiential knowledge through the sonic construction.

From this foundation, the first author was able to curate together seven-minute clips of each learner. Then, each audio curation was further edited by listening closely to each learners’ stories and trying to create an audio experience for the listener showcasing the experiences of Antonio and Juanita. It helped to think of Antonio and Juanita as giving a tour of their experiences. This aided in constructing two-minute “tours” of their geographies of hope/despair. The first author then shared these with the others in the research group and, with their advocacy for Antonio and Juanita, worked together to edit, modify, and maintain the partial truth of their testimonios (Littlejohn, 2021). The group concluded with two sonic constructions. From these constructions, we moved on to mapping their experiences by creating animations to go along with their testimonios. The animations were created by the first author. Feedback was discussed with the research group until consensus was reached.

Antonio and Juanita’s Testimonio Tours

In this section, the transcripts of Antonio’s and Juanita’s sonic constructions are presented. During our presentation, we will take time to listen to and watch the sonic constructions. The goal of including portraits and our mappings of their experiences is to make Antonio and Juanita present for the reader, to humanize them and remind the reader of the children speaking to them. Please read their words carefully, as the fragile gifts they are (Gómez Marchant & Aguilar, 2023). Their sonic constructions are manifestations of their counterstories, helping us in understanding how they navigate school and the mathematics classroom.

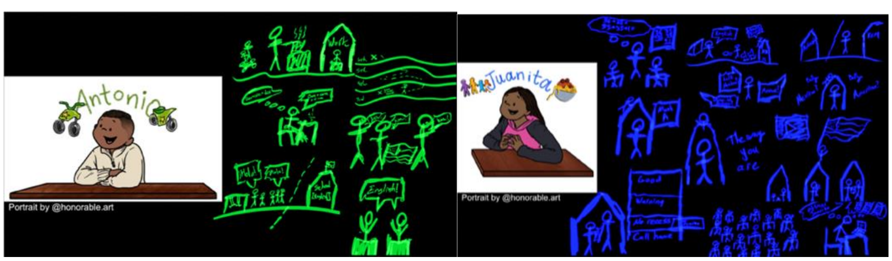

Figure 1: Portraits of Antonio and Juanita adjacent to their school experience maps

Antonio: My favorite color is green and uh I can drive a um four wheeler like with my brother in the back. My mom is really sweet and my dad, he cooks sometimes when it's in the summer and my mom's at work. Because I want to challenge myself to see how far I can get before I go to fourth grade. And then when I go to fourth grade I'm going to see if I can get all the way to sixth grade, like skip fifth and go to sixth. When I'm adding, sometimes I just um, go in my head and I sometimes do it in Spanish and then I try to say it in English. And once it's like wrong I um, do it, the whole thing in Spanish and then if it's wrong still I do the whole thing in English and it's right then um I'll like feel like I got it right just with only one try but I only got it right in like three tries sometimes. So I did in my head, uno plus uno equals dos. And then um, I wrote it down, one plus one equals two. And then I got it right and then I got a treat. Sometimes I don't really like to do it in Spanish but mostly all the time I do. I would think they would be impressed by that I'm trying–I'm kind of embarrassed to say it. Because like, because I only have American teachers. Like mad because all the time me and Jocelyn would speak Spanish and um they said no secrets in class And we would say it in English and then they'd be like oh wait that's what they were saying so they got kind of mad that we were um, doing secrets but we weren't. I only do it at recess because my teachers won't let me speak in Spanish. It makes me feel kind of sad that I can't speak Spanish in class. And I want to speak Spanish in class. I did it one time at recess when they were trying to speak to some, a little bit of a Spanish guy and they tried to say hello in Spanish and they couldn't say it so I helped them say hola and they said um hola. And then I felt like um special that I helped.

Juanita: I love having fun. My favorite color is blue. My favorite food is spaghetti. My dad works–he builds houses. My brother plays soccer. I have two dogs, Levi and Daisy. I don't speak Spanish. I understand a little bit of Spanish but not a lot of Spanish. Well I'm new. I just came to this school. I came from McK. The bus wouldn't pick me up from McK so we had to come to a different school. I just feel lonely here. So they are like, are you Mexican? Are you half Mexican? Are you White-ish Mexican? And I just–I just don't answer them a lot because that makes me feel bad. And my dad just tells me don't let anyone bully you. Like he–my dad would tell me not to worry if I'm Mexican or American. I sometimes do say words wrong. I mean my dad's girlfriend is in Mexico and she makes me happy. She doesn't really care if I'm Mexican or not. She just cares of who I am and I just like it. So I watch YouTube. And there is a lot of Americans on YouTube. I wanted to be an American and I didn't want to be Mexican. Like my dad he cheers me up all the time. He would make–he would probably make me food. Hug me. Maybe buy me something. I get kind of scared, not all the time but she usually makes me clip down. My teacher probably would look at me a lot because I don't–I don't answer a lot of questions there Because like I'm scared if I get them wrong and I feel bad that I don't want anyone looking at me. I probably know a lot of things and I help my–my classmates all the time with math because I do get a lot of good grades. I got one B on my report card and four A's and my dad was proud. Cause I want to be a police when I grow up and I like–I was watching videos about it and you like learn math in there. Like they be writing things and you have to find like the answer to kind of like math. When I get like–we do this math mashup for our homework, when I get all those answers right. I feel proud of myself.

Discussion and Conclusion

Antonio’s and Juanita’s testimonios emphasize the tensions of their geographies of hope and geographies of despair (Hidalgo, 2017) within their differing learning environments. One hears and sees how their mathematics, linguistic, and racial identities within differing contexts construct these landscapes of educational and mathematical possibilities. Antonio’s and Juanita’s mappings is a first step in showing how mathematics, language, and race collide within their lived experiences at school. Focusing on the audio medium for meaning making helped in creating a portrait of the tensions their racial, linguistic, and mathematical identities face. This process also maintained Antonio's and Juanita's knowledge of racialized spaces, thereby requiring the research team to attend to Antonio and Juanita differently. Antonio's landscapes were more heavily influenced by his experiences with language. Juanita's with her racialized experience. Both are emergent within the other. As tools for listening to learners, they both prompt conversations of the relationship between language and race, specifically in the mathematics classroom. This helps in further recognizing how the politics of sound (Kangieser, 2015; McLoughlin, 2023). Antonio's and Juanita's auditory mappings provide insight on their joy but also on their oppressed experiences.

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MATHEMATICAL MODELING AND SOCIAL JUSTICE IN K-12 MATHEMATICS EDUCATION: A SYSTEMATIC LITERATURE REVIEW

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Mathematics educators have recently shifted towards prioritizing social justice within the classroom, reflecting a broader recognition of the importance of addressing societal inequities and fostering inclusive learning (Buell & Shulman, 2019; Gutstein & Peterson, 2013). There is additional consensus that mathematical modeling can help students connect their experiences outside the classroom with big ideas in K-12 mathematics (Ball et al., 2005). These perspectives underscored the urgency of addressing themes such as cultural diversity, sociopolitical topics, and environmental issues relevant to learners in diverse mathematics classrooms worldwide (Aguirre et al., 2019; Jung & Magiera, 2023; Felton-Koestler, 2020; Rosa et al., 2022).

Mathematical modeling and social justice research approaches are rapidly expanding, encompassing diverse perspectives and epistemologies worldwide (e.g., Barbosa, 2006; Jung & Brady, 2023; Orey & Rosa, 2023). It is important to summarize these views and understand the different approaches, designs, and methods to guide informed research lines that adds to this emerging field. With this goal in mind, we embarked on a systematic literature review to analyze the current state-of-the-art of mathematical modeling and social justice development research. In this poster, we present one research question that has guided our first efforts to summarize this field of research: *What are the predominant themes at the intersection of mathematical modeling and social justice across the research in mathematics education?*

Methodology. In this study, we employed a systematic literature review (Torres-Carrión, 2018; Snyder, 2019) to explore the integration of social justice into K-12 mathematical modeling. We searched six relevant academic databases (Web of Science, Scopus, Springer, JSTOR, Taylor and Francis, and Eric) using keywords (e.g., social justice, equity, culturally responsive, and mathematical modeling). A multi-stage screening process ensured methodological rigor and minimized bias (Moher et al., 2015). At first, two independent reviewers categorized titles as “relevant,” “irrelevant,” or “maybe” using Covidence, a website for multi-layer screening. Reviewers did this process by ensuring each paper was related to mathematical modeling, social justice, and K-12 education. Discrepancies were resolved by a third reviewer. This process reduced the initial pool from 5,685 to 539; we kept papers classified as “relevant” and as “maybe”. After abstracts were similarly reviewed, we conducted full-text analyses of the remaining studies.

Summary of results. Our ongoing process has completed the initial screening phase, where we evaluated the titles and abstracts of the papers. As we transition to the full-text screening stage, preliminary discussions among the research team have illuminated the potential development of several thematic codes: ethnomodeling, teacher knowledge of social justice and mathematical modeling, culturally responsive pedagogical approaches on mathematical modeling, and the evolution of mathematical modeling and social justice in education. In this poster session, we will provide a summary of the interaction between modeling and social justice by sharing the relevant themes, subthemes, details, and recommendations for future work.

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A DIVERGENT PERSPECTIVE OF AN ADHD LEARNER IN COLLABORATIVE MATHEMATICS

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Education research literature consistently frames ADHD learners as a collection of deficits running counter to their own academic success in their educational trajectories. These deficit orientations to ADHD learners contribute to widespread assumptions that they lack the ability to contribute meaningfully in educational settings. We explore the hypothesis that one ADHD learner's participation in collaborative mathematics served to support and facilitate joint attention within his peer group. This hypothesis serves as an explicit counter narrative to deficit orientations held against ADHD learners, in literature and in the classroom. ADHD learners bring valuable assets to the success of their peer groups. This work challenges linear understandings of human development that can function to pathologize human cognitive and cultural diversity rather than embrace it as valuable to our collective continuance.

Keywords: Special Education, Students with Disabilities, Equity, Inclusion and Diversity

ADHD: Contesting Deficit Framings

Attention-Deficit Hyperactivity Disorder (ADHD) is one of the most widely researched neurodevelopmental disorders worldwide (Cooper, 2001). ADHD learners, present in nearly every K-12 classroom, are defined by the Diagnostic and Statistical Manual of Mental Disorders (5th ed.) as having a deficit in the area of attention (American Psychiatric Association, 2013).

While we (the authors) acknowledge the biological differences in neurodevelopment and impairments experienced by ADHD individuals, we also maintain that these impairments exist within an unjust, social framework that imposes normative standards of functioning on individuals who experience ADHD and other types of neurodivergence (Brown, 2014; Jurgens, 2020).

Overwhelmingly, the extant literature on ADHD learners in K-12 settings constructs them as poor students, who are incapable of academic success as a result of their identifications with ADHD. When students are constructed as fundamentally deficient in their capacities for learning, it is imperative that we re-examine the ways we investigate the learning of ADHD students. Critical and divergent perspectives of ADHD and other learning disabilities are necessary to disrupt deficit perspectives in research and ensure proper accessibility and accommodations for ADHD learners in K-12 learning contexts.

ADHD Learners and Collaborative Mathematics

Collaborative mathematics is understood as a critical site for agency and authority in mathematics learning (Langer-Osuna, 2016). Research shows disabled learners often have limited access to opportunities for agency and authority in mathematics learning (Tan et al., 2019). With collaboration understood as a highly valuable configuration for learning mathematics, critical analyses of access and participation in collaborative mathematics is crucial to the development of equitable educational opportunities for disabled students in mathematics.

Despite the abundance of research that has been conducted around ADHD, an initial review of literature identified zero (n=0) studies investigating the participation of ADHD learners in collaborative mathematics (Hertel & Gargroetzi, 2023). Examination of these cross-sections of the literature (ADHD learners in mathematics and ADHD learners in collaborative groups) evidence a pattern of constructing ADHD students as deficient. They are framed as deficient in their mathematical abilities with concern for on-task time, work completion, calculation accuracy, and problem solving (Zentall et al., 1994; Lucangeli et al., 2006). Literature regarding the collaborative activity of ADHD students is either approached from a position of concern about managing negative social interactions presumed common for ADHD learners or as a prescriptive intervention to mitigate poor academic outcomes (Saunders et al., 1996; Watkins et al., 2008). In one article, it is noted that collaborative groups with ADHD learners "surprisingly" had better mathematical problem-solving outcomes than those without ADHD learners as members of the collaborative group (Zentall, 2011, p. 38). This analysis speaks to the profound and harmful silence in the literature about the assets and contributions of ADHD learners to mathematical collaboration–a critical site for powerful mathematics learning.

Disability Justice and Counter Narrative

Drawing on Disability Justice and other critical frameworks we present the findings and discussion of this analysis as a counter narrative (Solorzano & Yosso, 2011), that challenges the majoritarian story of deficit orientations from which the mathematics and collaboration of ADHD learners is currently researched. A counter narrative that provides an asset-based framing of the contributions to mathematics collaboration by an ADHD student is prudent and speaks back to the deficit notions in both bodies of literature.

To do so, we share findings from an interaction analysis (Jordan & Henderson, 1995) of a Latino, ADHD, 9th grade student, Sammy, working with his peers in collaborative mathematics problem solving. Through this analysis, we challenge the deficit framing of ADHD learners. Specifically, we notice that Sammy plays a key role in the coordination of his group's collective attention to tasks, ideas, and objects that are important to their overall success as a group, a construct referred to as joint attention in the extant literature on collaboration and group work (Barron, 2003). While Sammy's identity as an ADHD student constructs him as having a deficit in the area of attention, his ability to coordinate joint attention, and the skills and assets required to do so are of particular interest in carrying out this analysis.

In order to investigate the affordances of Sammy's contributions to collaboration with his peers in mathematics, we asked with regard to Sammy during collaborative mathematics problem-solving, 1) What are Sammy's contributions to collaboration with his mathematics peers? And 2) What are the outcomes of Sammy's contributions to collaboration for the other members of his group?

Methods

The findings reported in this paper represent one minute of fine-grained interaction analysis (Jordan & Henderson, 1995) as an early phase in progressive refinement of hypothesis (Engle et al., 2014) where we examine the case of Sammy's participation in collaborative mathematics problem solving activities. Progressive refinement of hypotheses describes the ongoing and iterative process of observing and analyzing a phenomenon of interest within and across data sets. Here we explore the hypothesis that one ADHD learner's participation in collaborative mathematics served to support and facilitate joint attention within his group, serving as an explicit counter narrative to the fundamental construction of ADHD as having a deficit in the area of attention. As students with ADHD are often positioned as deficient in the capacity of their attention, Sammy's case as an ADHD and Mexican/Hispanic student who effectively coordinates joint attention, disrupts the pervasive deficit narratives of ADHD students and constructions of racially marginalized students in mathematics as "problematic."

In this analysis we use interaction analytic techniques to investigate human activities, such as talk, nonverbal interaction, and the use of artifacts and technologies, identifying routine practices and problems and the resources for their solution as indicative of social ecologies of participation (Erickson, 2005; Jordan & Henderson, 1995). This method of analysis affords identifying regularities in the ways in which participants utilize the resources of the complex social and material world of actors and objects within which they operate (Jordan & Henderson, 1995).

Context of the Study

This analysis is situated within a two-year ethnographic study of the co-construction of mathematics and social identities in a predominantly Latine serving California public high school (Gargroetzi, 2020; 2023). For one year, data collection focused on one mathematics class and eight focal students of whom Sammy was one. During that time the second author served as a consistent participant observer (Spradley, 2016) in the focal mathematics classroom and occasionally accompanied the focal students to their other classes and activities. The instruction and learning opportunities in the focal classroom were structured around collaborative problem-solving activities where students worked in groups of three or four.

Sammy, a 9th grade student in the year of study, identified himself as Hispanic and Mexican, male, and an "ADHD kid." Sammy reported that mathematics was his favorite subject, but that both his ethnoracial and disability identities complicated this. He had encountered the notion that Mexicans could not be good at mathematics (see Gargroetzi, 2023) and he explained that he believed being an "ADHD kid" meant he couldn't get an A in mathematics. For the year of data collection in Sammy's class he acted as a co-researcher. Four years later, Sammy, has volunteered to participate in future rounds of video co-analysis (not yet reflected in this paper).

Sammy's peers in the collaborative work represented are Selina (10th grade, Latina, repeating the course), Gisela (10th grade, Latina, with an IEP, repeating the course), and Gabriel (9th grade, Latino). The ethnoracial and ability contexts of the school and these students' relationships with mathematics, schooling, and their classroom community are explored in other work (see Gargroetzi, 2023; 2024). All student names are self-selected pseudonyms.

Data and Analysis

Data for this analysis comes from a larger set of ethnographic data including field notes and artifacts of student work, student interviews and focus groups, and video of two student groups participating in collaborative mathematics over the course of one unit (7 days, 700 minutes). Microanalysis examines one minute of collaborative mathematics selected from video from the first day of the recorded unit. In the first phase of analysis, the authors and colleagues created content logs documenting major activities in 5-minute intervals across the 7-day unit. In a concurrent second phase, as we developed our initial hypothesis, we participated in collaborative viewing sessions (Jordan & Henderson, 1995, Erickson, 2012) with two different groups of researchers representing varied expertise in interaction analysis, Disability Studies, mathematics education, special education, and learning theory, working to generate and refine our hypotheses and analytic lens in relation to the empirical record.

In doing so we began to document and define the types of contributions, any verbal or nonverbal activity that took place within the context of the collaboration captured on video whether directed toward another peer or not (see Table 1), that were made to the collaborative activity. In mapping this landscape, we identified a variety of types of contributions including those often explicitly documented as contributions to collaborations such as asking or responding to a question, as well as others that are not often considered as a potential contribution. For example, students in the focal group often engaged thinking aloud - a verbalization of their own questions or ideas while problem solving. Observing gaze, posture, and tone of voice, it was inferred that these questions or ideas were not necessarily directed at any particular student or even the whole group of students. Rather, they were externalizations of students' thought processes, or in simpler terms, "thinking aloud". While some analytic lenses may not attend to these personal externalizations as part of the collaboration, we included these in our analysis based on the hypothesis that this type of contribution to the collaborative space might afford opportunities for initiating joint attention, collaboration, and peer sensemaking.

Table 1: Types of Contributions to Collaboration

Type	Definition
Thinking aloud (not directed at a peer)	When a student is talking out loud either about their thoughts or questions regarding the problem they are working on or a related mathematical idea.
Collaborative Sharing	Share or propose mathematical idea, reasoning, observation, approach to problem solving, or solution
Talking while Gesturing	Gesture accompanies verbalization (hands or head)
Gesturing	Gesture without verbalization (hands or head)
Touch materials	Touches own materials or materials that belong to or are sitting on a peer's desk.
Pointing	Point to diagram or worksheet
Questioning (math-related)	Ask a mathematical question
Responding to peer (math-related)	Respond to a peer's mathematical question
Clarifying question (task-related)	Ask question about task expectations
Provided clarification (task-related)	Respond to a task expectations question or expand on
Endorse a peer's idea (math-related)	Agree with or promote peer's contribution/solution
Provide affirmation/confirmation	Confirm/affirm a peer's contribution/solution

Celebration (math-related or task-related)	Celebrate completion or correctness of solution (self or peer)

After using the larger data set to generate a set of contribution types, we focused our analysis on one minute of fine-grained interaction analysis based on a detailed transcript of the video record from the first day of the unit (minutes 59:00-1:04). The transcription documents both verbalization and nonverbal activity such as gaze, gesture, posture, writing, and other movements. Transcription conventions follow modified approaches drawing on Jefferson (1984) and Ochs (1979). Video was reviewed repeatedly to refine the transcript and produce detailed analytic memos for each conversational turn (Sacks et al., 1974). We selected this video segment based on it representing the first episode of the peer group engaged in a teacher-assigned collaborative task in a new collaborative group. The segment is bounded by initiation of the collaborative work amongst the group at the beginning and the collective uptake of a shared solution at the end, marking a shift in attention within the group.

Using the detailed transcript we coded for Sammy's contributions (n=10) and mapped the outcomes for peers of each of Sammy's contributions (See Table 3 in Findings). Outcomes were identified and analyzed based on examining the activity immediately following the contribution (3-5 seconds prior). As is common for social arrangements, an outcome in one moment was also often a contribution for the next. Findings report on Sammy's contributions and subsequent outcomes for his collaborative group.

Locating the Researchers

The researchers are an ADHD researcher (PhD student) and neurotypical faculty advisor who are jointly invested in examining power, privilege, and marginalization in mathematics learning with commitments to humanizing representations of minoritized learners that privilege their voices and perspectives. As two white, non-binary but cis-presenting researchers we recognize our privileged positions as a form of property and we work to unseat those investments in ourselves. The data shared in this paper represents insight into the lives of youth with whom the second author has been in close relations over the course of multiple years. The first author brings unique insights and commitments to this data in relation to ADHD learners and learning. We invite Sammy and other ADHD learners to join us in this analytic work and the work to build a counter narrative oriented to assets over deficits.

Findings

The segment of video-based interaction analysis included 21 talk turns where Sammy contributed nearly half of these turns (n=10). Sammy's contribution types during this time included verbalizations such as collaborative sharing (n=8) and providing confirmation (n=2) as well as movement such as pointing (n=3), touching materials (n=3), and often both at the same time: talking while gesturing (n=4). In examining the immediate outcomes within the group following Sammy's contributions, we noted three categories of outcomes (see Table 2). The outcomes of Sammy's contributions included (1) initiate or sustain joint attention, (2) initiate or sustain a pattern of sharing ideas, (3) a peer or the group utilizes Sammy's contribution as a resource for sensemaking or solution pathway. On occasion no explicit outcome was observed if there was no evidenced attention (physical or verbal) to Sammy's contribution. Notably, on three occasions Sammy's contributions initiated either joint attention or a new pattern of collaborative sharing, both fundamental processes for collaborative problem solving (Barron, 2003). On other occasions his contributions supported the continuation of these critical configurations within the group. Examples are provided for each type of outcome. We further unpack one instance of Sammy's contribution providing resources for the participation and problem-solving activities of his peers and the collaborative group.

Table 2: Types of Outcomes for Peer Following Sammy's Contributions

Type (# of instances)	Definition	Example
Initiate sharing (n=2)	After a contribution to collaboration has been made, peers begin to share their own ideas, questions, or solutions in response.	When Sammy and Selina are talking about the number of squares needed to form a box with a lid, Selina begins to share her own observations about the problem.
Sustain pattern of sharing (n=6)	Peers continue sharing ideas, questions, or solutions back to back, maintaining the flow of collaborative contributions.	Gisela, Selina, and Sammy all share mathematical ideas at the very beginning of their collaboration.
Initiate joint attention (n=1)	A contribution to collaboration is followed by two or more students attending to the same idea, question, or object.	Sammy points to a diagram that he thinks can be folded to form a box with a lid and Selina responds by asking a reaffirming question, "it does?" Sammy and Selina then begin discussing what is necessary in the diagram in order to ensure it will fold into a box.
Sustain joint attention (n=6)	Joint attention is maintained for multiple talk and nonverbal turns among peers.	Sammy and Selina discuss how many squares are needed in a diagram for it to fold into a box with a lid. (attending to the idea or conversation about the problem)
Peers utilize Sammy's activity as a resource for sensemaking (n=4)	Sammy's verbal or nonverbal actions are utilized by peers either in individual sensemaking or as an inroad to collaboration.	Gisela erases her answer for the second diagram directly following Sammy pointing to the second diagram and sharing his answer.
Group utilizes a solution pathway offered by Sammy (n=1)	Sammy's shared solution is utilized by peers in their own sensemaking or solution pathway.	Gabriel's gaze shifts to Sammy's paper while Sammy responds to and explains his reasoning to Gisela.

No explicit outcome observed (n=1)		Collaborative Sharing

Vignette Analysis: Gisela Utilizes Sammy's Contribution to Access the Mathematics and the Collaboration

We selected a vignette to share that explicitly challenges the deficit-oriented framing of ADHD learners in mathematics education research. ADHD student talk and movement is often depicted in the literature as disruptive to their teachers and to the learning of their classmates. In contrast, this vignette demonstrates how, in the context of collaboration with his peers, Sammy's talk and movement contribute to the collaborative space providing an alternative resource for sensemaking for his peer, Gisela. This inroad to sensemaking was utilized by Gisela for her own individual problem solving and subsequently as an entry point for dialogue with her group.

During this interaction, representing 18 seconds of the analyzed video segment, Sammy and his peers consider which of a group of nets would make a box with a lid (see Table 3).

Table 3: Transcription of Vignette

Line	Name	Verbal	Nonverbal
1	Selina	Um- there's one box- so how many boxes make it- one, two, three, four, [five, six	
2	Sammy	This one would!] (.....)	Pointing at the specific diagram that he thinks would work. Sammy's gaze lingers over to Selina's desk.
3	Sammy	This one would	Points again at the same diagram Gisela starts erasing her previous answer and then waits.
4	Selina	It does?- [Six?-	Selina's gaze is directed toward her paper.
5	Sammy	Yeah]	Gaze shifts down to his paper
6	Selina	[What you notice is that both of them] are six	Selina points to two of the diagrams that have 6 six squares.
7	Sammy	Cuz you have to have four sides-]	Gaze shifts back to Selina Gesturing with his hand and pointer finger, tracing four sides of the box.
8	Sammy	Yeah- cuz they have to have four sides and then one on the bottom- one on the top/	Gesturing with his hand and pointer finger, tracing four sides of the box and then pointing to where the bottom and top would be.

9	Gisela	/Should we check?	Gisela gestures with her hands in somewhat of a shrugging motion, shifting her body to her right.

Initially, while everyone else is processing and answering the first question on their worksheets individually, Sammy verbalizes his solution to the question on multiple occasions, pointing to the second and last diagrams on question one, which are in fact the only two correct answers, but receives no explicit response from others. Sammy says, "I know the last one will, it'll create a lid," as he taps his pencil on his face. Then Sammy momentarily shifts his gaze toward the middle of the group, then back to his paper. The group is silent for a few moments, Gisela writes on her paper, Selina is leaned over her paper, and Gabriel's gaze appears to be on something beyond Gisela's desk. Sammy leans back in his seat and then talks while gesturing, "well, it would be difficult to...," and uses his hands to visualize what the final folded net diagram would look like. Immediately following, Selina begins to speak and gesture (line 1). Though her talk does not suggest she has taken up Sammy's ideas, it does shift the group into a pattern of sharing such that the outcome of Sammy's contribution was to initiate a pattern of sharing. Sammy again states his answer to the first question, excitedly stating, "this one would," while tapping on the specific diagram on his paper using his index finger (line 2). He looks over at Selina's desk, where she continues to lean into her worksheet and does not verbally respond to his answer. Just as Sammy is about to repeat his answer, Gisela starts to flip her pencil in her hand and begins erasing something that is written on her paper in the same location on her paper as to where Sammy was just pointing on his own paper. A few more seconds pass, and Selina raises her head slightly and Sammy's gaze shifts toward her face and then back toward his paper. Again, Sammy says, "this one would," this time more reserved, tapping on his paper (line 3). This time Selina responds verbally, "it does?" (line 4). Sammy and Selina continue to exchange ideas (lines 5-8), with Gisela now joining the discussion by asking if they should check their answer (line 9).

While there was no verbal response to Sammy's contribution in line 2, the nonverbal activity that occurred in the following five seconds where Gisela revised her thinking and then eventually entered the conversation with a new request to check their work together demonstrates one example of how Sammy's contributions in verbalizations and gesture contributed to supporting peers and the collaboration by providing an alternative resource for sensemaking. By sharing his answer even when his group mates were not attending to a shared conversational space and using behaviors such as pointing, gesturing with his hands, and changing the tone of his voice, Gisela was able to silently utilize Sammy's shared answer as an opportunity to reconsider her first answer to the question, attend to the conversation between Sammy and Selina, and eventually join with her own suggestion for a collaborative next step.

Discussion and Conclusions

This initial micro-analysis of Sammy's contributions and the outcomes for peers in his collaborative mathematics group suggest that activity such as externalization (thinking aloud), talking while gesturing, and movements such as modeling with hands, pointing, and tapping can provide inroads critical to collaborative work such as initiating and sustaining joint attention, establishing and sustaining patterns of sharing within the group, and furthermore can provide sensemaking resources that are utilized by peers for their own problem solving and to pave the road for access to the collaborative dialogue. These findings are of particular note because these same activities are frequently pathologized as problematic, distracting for others, and indicators of the inattention of ADHD learners.

Sociocultural theories of learning (Vygotsky, 2012) that understand externalization of thinking through talk and gesture as providing critical resources for learning undergirds our understanding of the potential value of collaborative mathematics activity. However, these same sociocultural theories of human development and learning also suggest a hierarchical understanding of human development wherein advanced development is marked by moving much of this thinking to internal speech (Vygotsky, 2012). Along with the Disability Justice movement (Baglieri et al., 2011; Sins Invalid, 2015), and building on existing critique of such hierarchical developmentalism in some of Vygotsky's work (i.e. Bang, 2017), we suggest that linear understandings of human development can function to pathologize human cognitive and cultural diversity rather than embrace it as valuable to our collective continuance. We hope that this micro-analysis of Sammy's contributions and the outcomes for his peers provide one example of the potential value of neurodiversity and of varied ways of being and participating that can serve as an asset for mathematics learning and collaboration.

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EMPOWERING VOICES: DIGITAL MATHEMATICS STORYTELLING FOR MUSLIM IMMIGRANT GIRLS

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Keywords: Equity, Inclusion, and Diversity; Technology; Gender

Immigrant students, including Muslim girls, encounter significant educational challenges, such as cultural and language barriers, leading to isolation and diminished self-esteem and academic performance (Cerna et al., 2021). Addressing these challenges requires dismantling structural barriers and fostering inclusive educational environments that value diverse identities (Nishina et al., 2019). In mathematics education, mathematics identity is pivotal for student engagement, achievement, and resilience, underscoring the need to recognize and affirm diverse identities (Aguirre et al., 2013). Digital mathematics storytelling has proven effective in enhancing mathematics identity by connecting mathematics concepts with real-life applications and boosting students' interest in the subject (Chao et al., 2021). Nonetheless, research on digital mathematics storytelling among students with intersecting identities remains limited. This study seeks to explore the role of digital storytelling in reconstructing mathematics identity and how intersecting identities affect connection with mathematics.

In this poster, we integrate mathematics identity (Nasir & De Royston, 2013), intersectionality (Crenshaw, 1989), and funds of knowledge (Moll et al., 1992) as theoretical frameworks to explore the identities of Muslim immigrant girls in the U.S. We employ a qualitative narrative inquiry (Connelly & Clandinin, 1990). Participants, five young female immigrants, created videos that reflect their journey in learning mathematics, integrating it with their daily life experiences connected to their families and communities. This research involved four-week storycircles (Lambert, 2018) where participants shared and received feedback on their developing work. Initially, we introduced the concept of mathematics storytelling, prompting participants to create their own stories and discuss their mathematics learning experiences in the U.S. and their home countries. As the sessions progressed, they explored and shared their videos related to gender, religion, and immigration within the context of mathematics education.

Initial findings show that digital storytelling plays important role in reconstructing mathematics identity of Muslim immigrant girls, helping them adapt to a new educational system while incorporating their cultural experiences into mathematics learning. These participants navigate and bridge the gap between their home country's mathematics curriculum and that of the U.S., cultivating a sense of belonging in mathematics. Their intersecting identities shape their connection with mathematics, merging the values and practices learned from family and community with formal education. This process not only fosters inclusivity and engagement in mathematics but also challenges stereotypes, serving as a powerful tool for participants to combat racism, sexism, and Islamophobia.

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TEACHERS' POSITIONING AND ITS IMPACT ON MULTILINGUAL STUDENTS' LEARNING OPPORTUNITIES

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Using positioning theory, we examined mathematics lessons led by two monolingual teachers in multilingual elementary settings, focusing on learning opportunities. Despite employing research-based strategies, teachers often do not position multilingual students as capable problem solvers and independent learners. Their emphasis on equal participation inadvertently limited learning opportunities. We advocate for analyzing teacher discourse and practices through the positioning framework to uncover hidden biases and complexities. While the response of South Korean teachers to the increasing number of multilingual students remains unclear, our findings illustrate the general inconsistency in supporting and positioning culturally and linguistically diverse students in mathematics classrooms.

Keywords: Equity, Inclusion, and Diversity, Classroom Discourse, Teacher Beliefs, Elementary School Education

In today's diverse educational landscape, educators worldwide grapple with meeting the needs of multilingual students, a challenge accentuated in the U.S. where mathematics classrooms are reflective of increasing cultural diversity (García et al., 2008). Despite serving approximately five million English learners (ELs) in U.S., PK-12 schools, a persistent difference in mathematics assessment scores remains between ELs and their English-proficient peers (Howard, 2010; Soland & Sandilos, 2021). Research-validated instructional approaches are essential to ensure academic success for all students (Culatta et al., 2006), including linguistically diverse students. This study focuses on emergent bilinguals (EBs), denoting students still acquiring instructional language fluency, and acknowledge the dynamic nature of language acquisition, while valuing linguistic and cultural assets (García et al., 2008).

Our investigation delves into the intricate political and cultural contexts shaping EBs' learning experiences in the local community. We examined how two Korean teachers positioned students, including EBs, in multilingual mathematics classrooms, exploring the impact on students' learning opportunities. This research is unique as it focuses on South Korea's evolving multilingual student population, a phenomenon challenging traditional perceptions (from 1.91% in 2017 to 3.5% in 2023). Using a positioning theory framework, we aim to enrich our understanding of interactions in multilingual classrooms and their implications for students' learning opportunities.

Theoretical Perspectives

Disparities in learning opportunities persist, disproportionately affecting low-income, culturally diverse students, often placing EBs in lower-level mathematics courses (Umansky, 2016). Problem-based curricula and supportive environments enhance access to mathematics, crucial for educational achievement (Stein et al., 2007). Cognitive demand distinguishes learning opportunities, with routine memorization considered low and conceptual understanding high. Teachers' instructional practices influence students' access to learning opportunities, impacting participation and access to classroom discussions (Martin-Beltran, 2010). Social dynamics and power relations further shape access to opportunities in mathematics (Kayi-Aydar, 2019).

South Korea lacks specificity regarding multilingual students, amid the recent increase in racially and linguistically diverse populations. Research highlights disadvantages faced by multicultural students, including insufficient support, deficit positioning by teachers, and inadequate teacher preparation (Authors, 2022; Cho et al., 2006; Song et al., 2011). Song et al. (2011) found elementary teachers overlooked EBs' language backgrounds, providing inequitable learning opportunities. Similarly, authors (2014) noted Korean teachers' limited recognition of the cognitive demands of teaching mathematics to EBs.

Framework: Positioning Theory

Positioning theory, emphasizing moral dimensions of social roles, offers insights into interactions in bi/multilingual contexts (Harré & van Langenhove, L, 1999; Kayi-Aydar, 2019). Grounded in social constructionist perspectives, it views learning as constructed through interactions and power dynamics, making it a valuable framework for analyzing classroom discourse. This theory helps elucidate the relationships among power dynamics, competence, positional identities, and language learning experiences. Employed as both theoretical framework and methodological tool, it comprises positions, storylines, and speech acts, visualized in a triangular framework (Kayi-Aydar, 2019; Warren & Moghaddam, 2018). Within mathematics discourses, changes in any element due to challenges in rights and responsibilities allocation lead to modifications across the framework (Harré, 2012). Specifically, teachers' positioning of EBs can significantly affect their access to interactional opportunities (Pinnow & Chval, 2015). Positioning theory, focusing on moment-to-moment interactions, allows us to analyze how teachers' views expand or restrict learning opportunities for EBs, guiding our examination of teacher-student interactions in multilingual mathematics classrooms.

Methods

Participants and Settings

We observed two fourth-grade teachers, in South Korea, both native Korean speakers. Teacher A, with 18 years of experience, taught in a metropolitan city with a diverse immigrant population, while Teacher B, a novice teacher, worked in a city with diverse ethnic communities. Teacher A's class had 5 EBs from various ethnic backgrounds, born in South Korea but lacking proficiency in academic discourse. Teacher B's class had 17 multilingual students, some identified as EBs, receiving special language support. Despite teaching in different schools, both teachers covered the same mathematics topic using a government-developed textbook, focusing on real-life data collection and bar graph construction.

Data Collection and Analysis

We extracted a subset of data from a larger international study (Authors, 2019) conducted in South Korea, consisting of class observations, lesson plans, surveys, and interviews. Two teachers stood out for their different approaches to teaching the same mathematics concept in multilingual classrooms. We analyzed their acts, positions, and storylines, especially concerning

EBs' learning opportunities, using video recordings, transcripts, lesson plans, and interviews. Our analysis, rooted in Korean transcripts due to all authors being native speakers, focused on how teachers positioned themselves and students, particularly EBs, in mathematics classrooms. Utilizing Kayi-Aydar's (2019) positioning analysis framework, we identified 32 episodes where teachers positioned themselves or others regarding learning opportunities. Throughout the analysis process, we coded the repeated patterns and construct storylines that best capture each teacher's positioning. Our analysis aimed to reveal how teachers' acts positioned EBs and non-EBs and provided them with learning opportunities in language and mathematics. Finally, we reflected on the implications of each teacher's positioning.

Findings

We analyzed how classroom discussions and the cognitive demands of the tasks manifest in two teachers' lessons, as well as how teachers provided learning opportunities for participation. We found that two Korean teachers positioned both EBs and non-EBs as equal participants in the classroom community. However, the ways in which the teachers supported EBs' mathematical learning by engaging discussions and challenging tasks varied. Consequently, this led to different positions of students during class discussion and varying storylines about learning opportunities.

Teacher A focused on equal participation through hints and simple tasks, positioning herself as the sole decision-maker, while Teacher B emphasizes student autonomy while being a facilitator. Accordingly, Teacher A lowered cognitive demand, whereas Teacher B maintained it. There is also a disparity in student-initiated discourse between two classrooms; in Teacher A's classroom, the students did not have opportunities to ask questions or initiate discussions. Although Teacher B allowed students to speak up with questions and reactions, they were constrained by a sentence frame, limiting detailed mathematical reasoning.

Notably, Teacher A positioned students as passive learners by directing all tasks to ensure equal participation with easy tasks and unison answer. While her teaching methods initially seem engaging, they limit students to simple tasks in both language and mathematics by asking simple questions for choral responding and not asking for any further questions after student response. Teacher A avoided high-level questions, assuming EBs may struggle, and preferred predefined responses over exploring student ideas. In interviews, she defended simple activities, believing EBs cannot handle complex tasks. Despite using everyday language, Teacher A restricted both EBs and non-EBs to one-sentence roles to ensure equality, rather than challenging EBs. Also, she often lowered task complexity, underestimating students' abilities, and encourages copying over exploration. As a result, students in Teacher A's class may struggle to develop a strong sense of capability or confidence in mathematics, as her teaching methods tend to foster passive learning and limit opportunities for exploration and critical thinking.

In contrast to Teacher A, Teacher B positioned students as independent learners, fostering equal participation through rigorous tasks. She empowered students to solve problems by encouraging open-ended discussions and justifications. By refraining from correcting incorrect answers and redirecting questions to peers, she maintained the cognitive demand of mathematical tasks. For instance, Teacher B avoided labeling tasks as "easy," emphasizing the inherent challenges. She fostered a collaborative environment where student contributions were valued, positioning them as competent problem solvers. Overall, Teacher B positioned herself as a facilitator, guiding students to co-construct mathematical understanding through inquiry-based learning, promoting active engagement, and reasoning skills development. As a result, students in Teacher B's class may be more likely to see themselves as capable mathematics solvers and approach mathematics with greater confidence.

Discussion

The impact of teacher positioning on students' participation and access to learning opportunities in mathematics and language is evident (Kayi-Aydar, 2019). Both teachers utilized various supports to scaffold EBs' and non-EBs' learning, aligning with prior research recommendations (Celedón-Pattichis & Ramirez, 2012; Chval & Chaves, 2007; Moschkovich, 2010). However, they framed EBs as individuals needing linguistic support, extending supports to all students, perhaps reflecting cultural storylines valuing collective learning experiences or ethical principles emphasizing fairness. While both aimed for equal participation for EBs and non-EBs, Teacher A positioned students as dependent learners, resulting in passive participation, whereas Teacher B fostered a student-centric approach, promoting active contributions (Bossér & Lindahl, 2019).

The research findings suggest that, despite efforts to ensure equal learning opportunities for both EB and non-EB students, disparities in learning opportunities exist based on how teachers perceive the duties and rights of them and their students. The way teachers view themselves as educators can ultimately shape how they structure tasks and classroom discourse, potentially limiting students' learning experiences. This study highlights the need for teachers to undergo professional development that challenges their perceptions of their role and promotes the use of rigorous tasks in mathematics education.

Our data analysis, informed by positioning theory, revealed nuanced insights into classroom dynamics. Initially engaging, Teacher A's lesson hid deficit views despite diverse activities, while Teacher B's seemingly ordinary lesson empowered students through inquiry-based methods. This underscores the importance of positioning theory in evaluating instruction quality (Bossér & Lindahl, 2019). Integrating this theory into teacher education can address deficit views and promote equity over equality (Bossér & Lindahl, 2019). Reflecting on classroom episodes and adopting research-based strategies can mitigate deficit views and support EBs effectively.

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Rehumanizing Mathematics Education for Multiracial Children: MultiCrit and Mathematics Education

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Keywords: Social Justice; Equity, Inclusion, and Diversity; Ethnomathematics

In the recent calls for rehumanizing mathematics, researchers and educators call for a mathematics education that centers sociopolitical contexts (Gutiérrez, 2013) and students' funds of knowledge (Moll et al., 1992). Within these sociopolitical contexts, many researchers seek to understand how different communities of students learn and engage with mathematics. I argue that these agendas often take a monoracial approach that leaves out the voices and experiences of multiracial students. As a multiracial educator, I seek to use this poster to center the eight critical multiracial theory tenets (MultiCrit) within mathematics education (Harris, 2016, p. 800).

Tenet 1: A challenge to ahistoricism. This tenet encourages mathematics educators and researchers to analyze the historical context of issues specific to the multiracial community. For example, how has mathematics been used historically to condemn the multiracial community as the end of the pure white race? (Watkins, 2001).

Tenet 2: Interest convergence. Mathematics educators and researchers should consider how multiracial students are positioned as desirable only when they meet the needs of K-12 institutions built around white ideologies (Cross, 2005). How are multiracial students simultaneously oppressed and used by institutions to evade culpability for a lack of diversity?

Tenet 3: Experiential knowledge. Highlighting the experiences of multiracial students provides counterstories and counternarratives to dominant ideologies. In mathematics spaces, these stories shed light on how the mathematical funds of knowledge for multiracial students come from multiple communities, which may combine, conflict, and compete.

Tenet 4: A challenge to dominant ideologies. Mathematics educators should seek to elevate the experiences of multiracial students. Ethnomathematics should be embedded in curriculum and include the voices of students who come from multiple cultures and ways of knowing. In addition, mathematics educators should highlight multiracial mathematicians who have contributed to the field of mathematics.

Tenet 5: Racism, monoracism, and colorism. Multiracial students experience racism, monoracism, and colorism, but often lack the resources to navigate and talk about these experiences (Yong, 2020). One example is how multiracial students are often lumped into one category of "two or more races". This racial category devalues the voices and experiences of multiracial people by assigning them to a broad category that lacks statistical power.

Tenet 6: A monoracial paradigm of race. Racism is often discussed in monoracial terms, which excludes the experiences of multiracial people. Multiracial students should be provided opportunities to explore how their multiple identities fit into mathematics spaces.

Tenet 7: Differential micro-racialization. Multiracial people are often transracialized (Alim, 2016) as various races in different settings. Investigating how multiracial students are transracialized in mathematics classrooms may unveil the power dynamics these students resist.

Tenet 8: Intersections of multiple racial identities. Multiracial students navigate intersectionality not only through nonracial identities but also through multiple heritages. Students should be provided space in mathematics classrooms for this identity exploration.

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BETWEEN THE WORLD, MATHEMATICS, AND ME: BLACK FATHERS' PERSPECTIVES ON MATHEMATICS SOCIALIZATION DURING THE COVID-19 PANDEMIC

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This paper reports findings from a broader study that examines and documents the role of K-12 mathematics education amid the remote-schooling experiences of Black families (U.S.) during the COVID-19 pandemic. Moreover, the paper focuses specifically on perspectives of Black fathers. Given then-unfolding social restrictions during the stay-at-home period of the pandemic, the research design centers on semi-structured individual-family and multi-family interviews conducted via secured online video-sharing software. Here, we discuss a central theme from our analyses: tensions regarding mathematics homework as a cross-generational bridge.

Keywords: Affect, emotion, beliefs, and attitudes; Equity, Inclusion, and Diversity; Informal Education; Instructional activities and practices

> You must never look away from this. You must always remember that the sociology, the history, the economics, the graphs, the charts, the regressions all land, with great violence, upon the body (Coates, 2015, p. 10).

During the opening months of 2020, the United States government initiated a disjointed, state-by-state series of social and economic mitigation measures in response to the quickening spread of the COVID-19 coronavirus (hereafter pandemic). Across the country's diverse subareas—regional, rural, urban, tribal, suburban—parents and guardians suddenly faced numerous immediate challenges and decisions regarding everyday personal and professional routines, including the first multi-state emergency stoppage of in-person schooling. Most schools initially adopted a remote or online approach, requiring teachers and students to continue their classes by using video-conferencing software, file-sharing platforms, and course-management systems. There was also a massive effort (though with little to no federal guidance) to provide computing and internet-access resources to teachers and families so that they may participate in remote teaching and learning. Newly emergent studies reveal varying access and participation rates with this new form of remote schooling across the country—disturbing patterns that align with and exacerbate existing societal inequalities.

Purpose and Framing of the Study

This study aimed to document and critically analyze the role of K-12 mathematics education amid the remote-schooling experiences of Black families with school-aged children—i.e., familial networks of home- and community-based caregivers, including parents and guardians—during the pandemic. The specific focus on mathematics socialization among Black families (Martin, 2000; McGee & Spencer, 2015) contributes generally to a growing and needed area of research (also see Gholson & Wilkes, 2017; Washington, 2019; Walker, 2016). Based on related literature, we characterize mathematics socialization conceptually as the interaction of roles, practices, behaviors, and perspectives (see Saleem, Howard, Schmidt-Temple, Langley, &

Howard, 2024). Moreover, this study is timely given its attention to the unique societal moment of the pandemic—with conditions that temporarily upended conventional notions of work, school, home life, and intersections thereof.

A broader goal for this research is to develop a collaborative research agenda with other mathematics education researchers who are currently studying closely related phenomena (e.g., families, mathematics socialization, identity, race)—in order to further the study of how Black people conceptualize and actualize their roles as *mathematics socialization agents* (Martin, 2000; 2007; also see Larnell, 2016). Representing part of a broader, this paper focuses on the question: How did Black fathers support their children's mathematics education during the pandemic? Figure 1 frames broader areas of concern and research questions within the full research study.

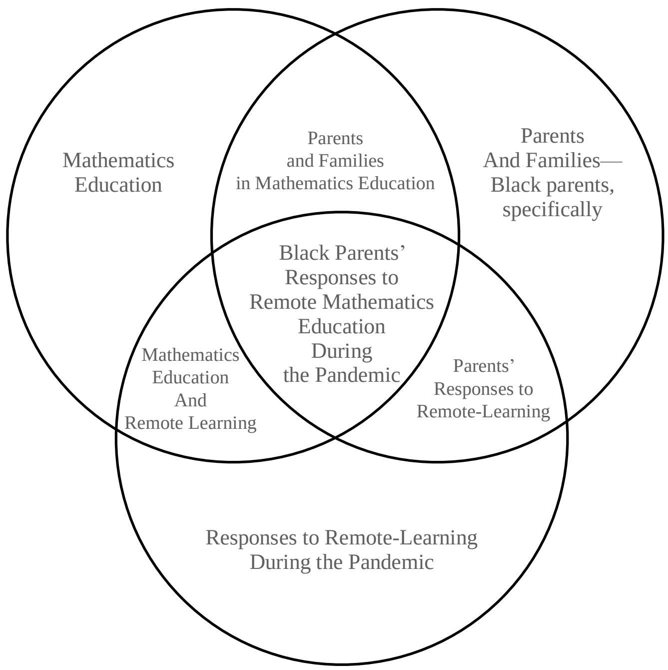

Figure 1: Black Family Mathematics Socialization Conceptual Areas

According to Washington (2019), "Mathematics is an important front on which to wage the fight for full participation in [this] society for African Americans because it has been used systematically to disenfranchise African Americans from upward social mobility and political power" (p. 2). Echoing Washington's overall sentiment and specific centering of Black families (also see McGee & Spencer, 2015), our specific goal for this paper is to contribute to characterizing the roles that Black parents and guardians construct for themselves as well as their perspectives on mathematics education at large and during the pandemic, specifically.

Based on the broader study's questions, the theoretical framework combines elements of Martin's mathematics socialization framework (specifically, beliefs about differential treatment, motivations, relationships to school, mathematics-dependent goals, and expectations for children) and Washington's (2019) framing of supportive practices that extend from those socialization goals (also see Civil, Planas, & Quintos, 2005). The focus on supportive practices is intentional and not regarded to be biased; rather, given the damaging preponderance of narratives about the underachievement of Black learners (see Martin, 2003, 2019 for related critiques), examining the nature of support (and presuming foundationally that such support exists) signals a potent and much-needed epistemic counternarrative. Moreover, this paper's focus on Black fathers supporting their children's mathematics education is also a purposeful choice and an opportunity to fortify such a counternarrative.

Design, Data, and Methods

The study's design was organized to document and critically analyze parents'/guardians' perspectives by way of transcribed interviews (individual families) and focus groups (multiple families). Given the pandemic and the need to avoid group gatherings, all research activity with participants was conducted and recorded virtually using university-secured video conferencing (i.e., Zoom) software. The broader study includes 6 dual-parent or individual-led families; in this paper, we focus on the contributions of three fathers across individual-family interviews and multi-family focus groups. All interview prompts and protocols were developed based on and adapted from other recent studies on Black family socialization in mathematics (e.g., Washington, 2019) and related studies of mathematics socialization and mathematics identity among Black learners (e.g., Author 1, 2016; Martin, 2000, 2006). Conceptually, the interviews centered on how these parents regarded (a) their roles as mathematics socialization agents, (b) their education-supporting practices with school-aged children in their families, and (c) the kinds of resources that they put into practice with and on behalf of their school-aged children.

Participant Descriptions

Lawrence, James, and Oscar are (cisgender male) fathers who participated in individual interviews and a multi-family focus group during the data collection phase of the study; excerpts of their transcribed exchanges during those interviews were included based on coding described in the next section. Each identified themselves and their families as working class; Lawrence is a father of five children and works as an education professional who works in schools, James is a father of four children and works in the banking industry, and Oscar is a father of one school-aged child and works as a human resource professional. Across the three fathers, the grade levels represented by their children span grades K-12.

Summary of Analysis Methods

The interviews were recorded on the video conferencing platform and transcribed professionally. Recordings and transcripts were stored on a secure cloud platform. Coding schemes were developed using a framework of socialization factors based on existing literature. The transcribed text was analyzed primarily using NVivo. Additionally, we viewed and discussed video recordings and excerpted transcripts to develop emergent themes. Analytical memos were developed for the excerpted transcripts presented in this article; these memos were reviewed and discussed, which led to the presentation of the findings in this paper. (Note: We will discuss their positionality in the presentation, due to space limitations here.)

Perspectives on Remote School Mathematics during the Pandemic: Two Themes

In this section, we briefly discuss a central theme (due to space constraints) from our current readings and analyses of data collected from Black fathers: Tensions regarding mathematics homework as a cross-generational bridge. Across all participants in the study, there were consistent and frequent references to homework assignments as a key nexus for parental support (see Schnee & Bose, 2010).

Decades after their own school experiences, each of these three fathers expressed similar perspectives across grade bands, from elementary to high school. Lawrence, particularly, discussed the concern that homework was being deemphasized in his children's mathematics classes—and that parents were left to assemble supplemental opportunities to interact to support their children's at-home mathematics education.

Lawrence: …I'm gonna be honest with you…like we work on math. They don't give homework like we got homework when we were kids. I mean, they don't bring any homework home. Like we have to do stuff. We have to find things for them to do, you know, at home.

As Lawrence explains, a key shift in contemporary school mathematics homework is a broader reliance on technologies (e.g., devices, software, online platforms) as basic elements for mathematics teaching and learning and not merely ancillary. Relatedly, James discusses this as a tension in relation to his perceptions of support for his children learning mathematics.

James: My kids have [Apple] iPads. I'm not an iPad person (chuckle), but they have iPads. And everything…information is fast. You know, you go to your computer and find an answer…When it comes to homework, my kids have no computers. You're gonna go to pencil and paper. Now, once you understand that concept…you can use a computer. You can use a calculator. You can use, you know, things of that sort. But they're gonna understand the foundation because, uh, it's kinda like the foundation teaches them how to really think about what they're doin'. Not a quick answer.

For James, the practice of helping children with mathematics homework is primarily a paper-and-pencil exercise, but during remote schooling, those processes were communicated virtually and across platforms and programs. For Oscar and Lawrence, differences in how mathematics was taught at school versus home settings were also common concerns.

Oscar: Um, I know with my daughter, um, it was a problem, um, that I decided to step in myself and do it, uh, but again, it was countered always as the differentiation between what was taught in the classroom versus what is being taught at home.

Lawrence: So, um, now they got—if they're struggling with something, and-and we talking, and we communicating with the teacher about what they're struggling with, um, we make sure there's that communication. And we'll bring stuff home, and-and we'll work on things, you know. But, um, yeah. That's kind of like pretty much how we handled it.

Discussion

Again, the focus on Black fathers' perspectives and the findings reported here represent a portion of the broader, ongoing study focused on Black families' experiences with school mathematics during the pandemic. The central theme of homework, though prompted by our interview protocol and framework, were elaborated and revisited multiple times across multiple participants. Our hope is that this research will contribute to our broader understandings of families roles in school mathematics—and toward redressing the dearth of mathematics education scholarship that includes and specifically focuses on Black communities.

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WOMEN DOCTORAL STUDENTS' VIEWS OF THE ROLE OF THE ASSOCIATION FOR WOMEN IN MATHEMATICS

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Keywords: Gender, Equity, Doctoral education

Despite improvement in representation at the undergraduate level, women are still underrepresented in doctoral mathematics with only 28% of doctoral degrees awarded to women in 2018 (National Science Foundation, 2021). In addition to inequitable representation, women in doctoral mathematics have experienced overt prejudice, feelings of invisibility, insufficient or negative advising experiences, and a lack of belonging (Herzig, 2004; Miller, 2015). Furthermore, women's reports of success are often mitigated by expressions of struggle centering around academic support (Ataide Pinheiro, 2021). In contrast to a demonstrated need, there is little research investigating interventions supporting women in doctoral mathematics or the efficacy of student organizations for women at the graduate level, such as the Association for Women in Mathematics (AWM). However, there is evidence that undergraduate science, technology, engineering and mathematics (STEM) student organizations may improve support for students from underrepresented groups by enhancing personal contact with faculty members (Mwaikinda & Aruguete, 2016).

In this poster I investigate the perspective of five women doctoral students with respect to their participation in a graduate student organization, a chapter of AWM. I answer the research question: *How do women mathematics doctoral students, as AWM members, view the role of AWM in their mathematics department?*

Methods

Participants included five women enrolled in a doctoral mathematics program at a research-intensive (R1) university in the western U.S. The women were selected using a stratified purposeful sampling (Patton, 1990) with respect to years' experience in the program. This included one woman enrolled in coursework, two women in early research, one woman in advanced research, and one woman taking a leave of absence after completing coursework. Additional sampling criteria included participation in the local graduate chapter of AWM. Data included one-hour narrative interviews (Mueller, 2019) from each of the five participants. Analysis included the first two steps of the of the thematic analysis methodology (Braun & Clarke, 2006) – familiarizing myself with the data and open coding meaningful passages. One code that emerged across interviews was *building a sense of community*.

Preliminary Results

All five women described AWM and its related activities as a way to build a sense of community in their mathematics department, meaning they saw AWM as a way to bring people together in a shared space and build connections. A second-year student described the group's weekly gatherings, saying "it builds a sense of comradery and helps connect people, especially if they aren't in your research area or your classes who maybe you wouldn't know otherwise and just helps connect people in the department." Interestingly, participants expressed varying positions about how this community building supports the women in the department. As part of the poster presentation, I will further explore this tension of community building and gender.

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AN ELABORATION ON MASTER NARRATIVES IN MATHEMATICS AND HOW UNDERGRADUATES RELATE TO COUNTERNARRATIVES

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Because the master narratives about mathematics in the US often play an exclusionary role in students' educational experiences, educators have sought to integrate counternarratives into instruction that might disrupt these effects. As part of a larger project to develop a research-informed curriculum for undergraduate introduction to proof courses, we gathered author stories from a diverse set of mathematicians for students to read and reflect upon. To study student responses to these author stories, we synthesized a framework of the master narrative of mathematics in the US and identified how the author stories countered elements of this narrative. We then analyzed 80 student reflections from one introduction to proof course to identify whether and how students either endorsed or countered the elements of the master narrative. Our findings point to a positive, yet modest capacity for these stories as counternarratives.

Keywords: Undergraduate Education, Equity, Inclusion, and Diversity

In recent years, counternarratives have become not just a means for research, but a potentially transformative tool to incorporate into instruction. Stories both about ourselves and society provide a means through which to organize our understanding of the world. Culturally shared narratives, or dominant/master narratives, provide a framing to compare or integrate personal experiences. These master narratives often disadvantage marginalized groups, such as women and people of color, in fields like mathematics (Berry III et al., 2011; Leyva, 2017; Adiredja, 2019). Narratives portraying mathematics as neutral, individualistic, and meritocratic reinforce hegemonic norms, dismissing alternative ways of learning and working that align with diverse identities and cultures (McBride, 1994; Cobb & Russell, 2015; Cervia, 2019). Counternarratives (e.g., Solórzano, D. G. & Yosso, 2002) can serve to challenge dominant narratives by not only disrupting storylines about who can succeed in mathematics (e.g., Berry III et al., 2011; Langer-Osuna et al., 2016; Leyva, 2016; McGee, 2009) but also redefining what it means to engage with the subject. However, as cautioned by Cervia's (2019) exploration of scientists, siimply showcasing successful individuals from minoritized groups can inadvertently reinforce existing master narratives, especially if these narratives align with traditional norms. Disrupting master narratives requires challenging not only demographic stereotypes but also the perceived traits and norms associated with mathematicians and mathematical activity.

In this study, we analyze student reflections on mathematician biographies paired with proofs in an introduction to proof course. These narratives, tailored for the intended audience of undergraduate mathematics students, aim to challenge dominant narratives about who can succeed in mathematics and how mathematics is practiced. By synthesizing literature on mathematics and mathematician storylines, we develop a framework to examine how counternarratives resonate with students and align with or challenge dominant narratives in the United States. Our research questions are twofold: What elements of mathematician counternarratives were salient to a group of undergraduate mathematics students? How do these elements align (or not) with dominant narratives of mathematics in the United States?

Theoretical Framing

Broadly, we adopt McLean and Syed's (2016) narrative distinctions. Master narratives are culturally shared stories that guide behavior and identity within a culture. They dictate who excels in math and define the essence of being a mathematician. Counternarratives, on the other hand, challenge these dominant storylines. Personal narratives involve negotiating between societal expectations and individual identity, influenced by both master and counternarratives.

Drawing from various research areas, we identified themes in master narratives about mathematics. These themes encompass beliefs about mathematicians, descriptions of mathematicians, countered narratives, and theoretical discourses. While these categories overlap, they collectively shape the perception of the American Mathematician. In Table 1, we present these narrative elements alongside quotes from key studies that inform our understanding.

Table 1: Mathematical Master Narrative Elements and Storylines

Element/Storyline	Reference
Mathematics is done by privileged white men.	"[P]opular discourses overwhelmingly construct mathematicians as white, heterosexual, middle-class men" (Moreau et al., 2008, p. 25).
Mathematics is done by those who are brilliant, but socially inept.	"Mathematicians were often portrayed as socially inept nerds" (Di Martino et al., 2023, p. 11).
Mathematics is done in isolation.	"Mathematicians always work alone in my mind. They're always like those, those hobbits that live in their own little room… Like this would be perfect, lined with grease boards and chalkboards" (Female Math Major, Piatek-Jimenez, 2008, p. 638).
Mathematics is dry and not fun.	"Mathematics is full of rules and formulas to be remembered. Mathematics is dry, it does not leave room to feelings. Mathematics does not make sense, the aim of learning certain things is not clear. In mathematics, there is no room to express one's own ideas" (Di Martino, & Zan, 2011, p. 477).
Mathematics ability is innate, and mathematics is easy for those with high ability.	"Instead of seeing learning math as an ongoing process, learning at my own pace, it became "either you get it or you don't." I put math in the, 'it's just not for me' category" (College student, John et al., 2022, p. 9).
Mathematics is colorblind and neutral.	"The mathematical model of equality constructed intentionally perpetuates the myth of mathematics as neutral and objective to maintain white institutional spaces" (Gómez Marchant et al., 2023 p. 12).
Mathematics is individualistic and a meritocracy.	"A prominent feature of education in the United States is the widespread endorsement of an achievement narrative, which links individual motivation and effort to academic achievement." (Zavala & Hand, 2019)

We emphasize that these storylines together support greater myths such as the existence of a racial hierarchy of ability (Battey & Leyva, 2016). All of these elements serve a hegemonic role in preserving exclusionary standards and the white patriarchal space of mathematics (Battey & Leyva, 2016; Battey & Marshall, 2023; Leyva, 2017; Martin, 2013; McGee, 2020).

Methods

We present data from one implementation of this curriculum in an introduction to proof course at a large research university in the United States. As part of this curriculum, interesting proofs are paired with author stories from modern mathematicians. In this study, students wrote reflections on six author stories (authors included two Black men, one Latinx woman, one Pacific Islander one, one white-nonbinary person, and one Latino) The authors told their stories both as mathematicians and outside of mathematics, obstacles and struggles, and images of what doing mathematics looked like (See Melhuish et al., in press). Twenty-one students were enrolled in the class and were listed in university documentation as having the following ethnicities: 11 White, 2 African-American, 6 Hispanic, 1 Asian, 1 Unknown. There were 7 women and 14 men.

The focus of this submission is on the written reflections of sixteen student participants on the assigned author stories. Students submitted a mean of 5 reflections, for a total of 80 student reflections. Student reflections varied in length from three sentences to two handwritten pages. Prompts for the reflections included questions such as: "What stuck out to you about Dr. ____'s story?" "How did Dr. ____'s story resonate with your own story? How did it differ?"

For the purpose of this study, we identified passages of the reflections where students noted components of the master mathematical narrative were either countered or endorsed in the author stories. Two authors independently coded each of the 80 reflections for counters and endorsements, resolving discrepancies through discussion. Overall, we identified 127 instances of countering the master narrative elements and 15 instances of endorsing them.

Results

Student reflections revealed how they internalized the author stories. Some students endorsed master narrative elements by highlighting them in the stories or by opposing counter elements. Others recognized counternarrative elements, emphasizing their significance. We present Table 2 to summarize the narrative elements reported by students. Additionally, we introduce two new categories: linear trajectory in becoming a mathematician and family and cultural expectations. Students expressed surprise at the non-linear career paths of mathematicians, echoing narratives of academia's linear progression. Similarly, they were surprised by authors facing family discouragement from pursuing mathematics rather than a more lucrative career.

Table 2: Mathematical Master Narrative Elements Found in Student Reflections

Master Narrative Element/Storyline	# Countering	# Endorsing
Mathematics is done by privileged white men.	6	1
Mathematics is done by those who are brilliant, but socially inept.	2	0
Mathematics is done in isolation.	5	2
Mathematics is dry and not fun.	13	0
Becoming a mathematician is a linear trajectory.	13	0
Mathematics ability is innate, and mathematics is easy for those with high ability.	15	4
Mathematics is colorblind and neutral.	5	0
Mathematics is individualistic and a meritocracy.	0	3
Family and culture would hold positive views on choosing mathematics as a career	5	0

A Look at How Students Reflected on Elements of Master Narratives

Mathematics is done by privileged white men. None of the author stories were written by white men. We found that some students explicitly referenced this master narrative. Students further identified themselves in the stories with one woman explaining, “being a Latinx woman is personally inspiring, as I too am a Latinx woman.” Such comments also extended to economic and educational privilege with several students noting things like relating to coming from a “rural” or “blue collar” town without great public schooling. We see each of these comments implicitly showing awareness of the master narrative about white, privileged men as mathematicians, but focusing on the counter to it and the ways the counter aligns with their personal narratives.

Mathematics ability is innate, and mathematics is easy for those with high ability. Every author stories included times of struggling with mathematics, in the academia, or in school. This was the most commonly attended to element in the student reflections. Several students reflected that they related to these stories because their own “identity of being ‘good’ at school” had been challenged. One student explained, “even people who have earned their degree in mathematics have struggled with math. Even though I struggle on some topics, it doesn’t mean I’m not geared for math.” These comments reflect a negotiation of their personal narratives where students seemed to take the counter story elements and expand their idea of who can be a mathematician and whether it is necessary to always have things come easy.

Becoming a mathematician is a linear trajectory. A number of the author stories represented atypical paths to mathematics including nearly failing out of school, leaving the academy and returning, and cycling through many majors before arriving at mathematics. Students often reflected on these elements with some noting elements such as “detours.” One student reflectied, “I had a pre-conceived notion that mathematicians must be persistently passionate about math. Dr. [W] proved that notion incorrect.” These elements served to expand out who can do mathematics (not just those who are always passionate about math) and how one can arrive at being a mathematician.

Mathematics is dry and not fun. The author stories also contained ways that the authors loved mathematics and how they went about doing mathematics. Some students contrasted the standard approach to school mathematics such as one student notoing the importance of the idea of “learn[ing] in a way you are changed and not in a way that you memorize.” Students further linked ideas of mathematics being fun with the challenge.

Other Story Elements. A major theme in author stories was the collaborative nature of mathematics. However, students rarely mentioned this in their reflections. Some noted the need to connect with others, but most focused on professors and mentoring rather than peer collaboration. Ideas of individualism and meritocracy were mixed. One student countered the idea of innate ability but endorsed meritocracy by stating that effort leads to success. Others recognized the support from people in their lives, challenging individualism.

Discussion

In this report, we provided two fundamental contributions. First, we stitched together a dispersed literature base to create a master narrative of mathematics with attention to: who does mathematics (mathematicians), how one becomes a mathematician, how one does mathematics, and what mathematics is. Second, we explored how author stories that challenge these narrative elements were perceived by students. The student reflections provided strong evidence that the author stories were successful at challenging several elements of the master narrative around mathematics: mathematics being dry and not fun, the path to a career as a mathematician being linear, and mathematics ability as being innate and mathematics being easy for those with high ability. The master narrative that white men dominate math was less disrupted. Only a few students, particularly women of color, mentioned race. Meritocracy and individualism were either not addressed or reinforced by students' reflections. Stories often highlighted overcoming barriers, emphasizing individual effort. Further research should explore which stories can disrupt which narratives and how to effectively engage students in reflecting on their personal narratives.

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ADVISOR PERSPECTIVES: SERRVICES AND PROGRAMS FOR AFRICAN INSTERNATIONAL STUDENTS IN HIGHER EDUCATION INSTITUTIONS IN THE UNITED STATES

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The globalization of higher education in the United States has led to a significant increase in the enrollment of international students, particularly from Africa, thereby enhancing the cultural diversity of academic institutions (Hendrickson et al., 2011; Lee & Rice, 2017). African international students face unique challenges due to their diverse academic backgrounds and specific cultural and social needs (Hendrickson et al., 2011; Lee & Rice, 2017; Irungu, 2013). While these students enrich the academic environment and contribute financially to the U.S. economy, their academic journey, especially in mathematics education, is fraught with difficulties such as language barriers, cultural adjustments, social integration issues, and academic pressures (Irungu, 2013). Existing orientation programs and support services often fail to address the specific needs of African students, hindering their academic success (Smith & Khawaja, 2011).

Higher learning institutions have developed support systems and programs to assist international students, including academic advising, tutoring, mentorship programs, and mental health services. However, African international students encounter difficulties accessing these services due to cultural, language, and financial barriers (Smith & Khawaja, 2011). Tailored support structures are essential to effectively address these challenges and ensure the academic success and well-being of African international students in U.S. higher education institutions (Jin, 2019).

A multiple-case study approach was employed to explore the experiences of African international students in mathematics education in U.S. institutions, focusing on the perspectives of international student advisors (Altbach & Knight, 2018; Poyrazli & Grahame, 2007). The study aims to assess the effectiveness of existing support services and programs for African students and understand how advisors perceive these services (Olson & Banjong, 2016). Preliminary findings indicate that while resources are available, there are significant challenges and gaps in the support systems, emphasizing the need for culturally responsive approaches and tailored support for African international students (Irungu, 2013; Poyrazli & Grahame, 2007).

This study underscores the significance of addressing the unique challenges faced by African international students in U.S. higher education institutions and calls for refined strategies to create a more inclusive and supportive environment (Mwangi et al., 2019; Smith & Khawaja, 2011). By investing in culturally relevant programming, fostering collaboration among stakeholders, and continuously evaluating support strategies, institutions can enhance the academic success and well-being of African international students, contributing to a more equitable and enriching educational experience (Smith & Khawaja, 2011).

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EXAMINING EQUITY-ORIENTED INSTRUCTIONAL PRACTICES IN A DEVELOPMENTAL ALGEBRA PROJECT TO SUPPORT UNDERGRADUATE STUDENTS' EQUITY OUTCOMES

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The Thinking With Algebra (TWA) project, which consists of a developmental algebra curriculum and supporting professional development, has been integrated into diverse algebra courses to support undergraduate students' success in college algebra (Feikes et al., 2024). This paper examines the equity-oriented instructional practices integrated into TWA to support strengthened equity outcomes among students. This project draws on sociopolitical equity scholarship (Rubel, 2017), including a framework consisting of achievement, access, identity, and power (Gutiérrez, 2012). Instructor and student data indicate positive changes in student achievement and identity outcomes. Future directions include expanding the integration of equity-oriented instructional practices and investigation of student equity outcomes. Applying an equity lens to TWA is critical given the barriers students, particularly minoritized students, face in algebra.

Keywords: Equity, Inclusion, and Diversity; Instructional Activities and Practices; Undergraduate Education; Algebra and Algebraic Thinking.

Background

Ample evidence indicates a national crisis related to students successfully completing algebra coursework, which is extremely concerning given that algebra serves as a gateway to higher level mathematics and science courses (Domina et al., 2015; Long et al., 2012), to postsecondary success at colleges and universities (Spielhagen, 2006), and to STEM degree pathways (Loewenberg, 2003). Despite efforts to improve the teaching and learning of algebra, many students are still struggling with the subject and are not persisting in their mathematics coursework (Greenes, 2008). Additionally, when students do not successfully pass algebra coursework, aspects of their mathematical identities are likely to be negatively impacted, such as developing negative attitudes and beliefs about mathematics. Even more concerning is the significant underrepresentation of minoritized students, including Black, Latinx, and low-income students, in the student population successfully completing algebra coursework (LaFave, 2019), attaining STEM degrees (Fry et al., 2021), and pursuing STEM careers (Fry et al., 2021).

This paper draws on sociopolitical mathematics education equity scholarship (Rubel, 2017), a framework within this literature consisting of four dimensions: achievement, access, identity, and power (Gutiérrez, 2012), and prior research by the authors (Oppland-Cordell et al., 2024) to examine how the instructional practices of the TWA project, which is funded by the National Science Foundation Improving Undergraduate STEM Education: Education and Human

Resources (IUSE: HER; DUE 2021414), align with equity-oriented instructional practices appearing in this scholarship. The research questions are: (1) *How do TWA instructional practices align with equity-oriented instructional practices?* and (2) *How does students' participation in TWA impact their mathematical achievement and identity development?* Such findings will expand knowledge about how to create equitable mathematics learning environments in developmental academic settings that support students' mathematical success.

Theoretical Framework

Gutiérrez's (2012) equity definition consists of four dimensions: access, achievement, identity, and power. Drawing on this definition, this project defines the four equity dimensions in relation to the design, instructional practices, and professional development components of TWA. Table 1 includes these definitions.

Table 1: TWA's Equity Dimension Definitions Adapted from Gutiérrez's (2012) Framework

Equity Dimension	Design, Instructional Practices, and Professional Development
Achievement	TWA supports students' strengthened mathematics achievement outcomes as measured by participation rates, persistence rates, and student self-perceptions of achievement.
Identity	TWA supports students' strengthened mathematics identity development, including how they co-construct this identity with their other identities. Measures include student self-perceptions and perceptions of how other view them as mathematics learners.
Access	TWA mandates professional development to support high quality mathematics instructors. Through group work and whole class discussions, TWA encourages a learning environment that supports student participation in and out of class.
Power	Through small group and whole class discussions, diverse voices and alternative notions of mathematics knowledge are embraced in the mathematics learning context.

While Gutiérrez's equity dimension framework provides guidance for defining equity for the TWA project, we also wanted to investigate how specific instructional practices integrated into TWA align with equity-oriented instructional practices appearing in emerging sociopolitical mathematics education equity scholarship. Drawing on Gutiérrez's (2007) equity research, Rubel (2017) identified and organized four "equity-directed instructional practices from four models of progressive pedagogy," of which three are highlighted here: standards-based mathematics instruction (SBMI), complex instruction (CI), and culturally relevant pedagogy (CRP) (p. 69). Rubel also highlighted a specific instructional practice from each model that collectively revealed how these pedagogies are interconnected and build upon one another: *SBMI*: teaching for understanding; *CI:* multidimensional participation; and *CRP*: connecting mathematics instruction to students' experiences. While the SBMI and CI examples strongly align with the access and achievement dimensions in Gutiérrez's work, the CRP example closely aligns with the identity and power dimensions (Rubel, 2017). This study draws on these specific examples to inspect overlap and growth opportunities for the equity-oriented instructional practices employed in TWA. In particular, this study explores how TWA currently teaches developmental mathematics in ways that contributes to strengthening students' conceptual understanding, supporting their multidimensional participation, connecting mathematics instruction to their experiences, and applying mathematics in critical ways to navigate their lives and worlds.

Methods

TWA Context and Participants

TWA is a developmental algebra curriculum that supports undergraduate instructors in teaching algebra in a way that conceptually and procedurally prepares students for success in college algebra (Feikes et al., 2021; Feikes et al., 2024). TWA also provides a faculty workshop to support college instructors with understanding and implementing the curriculum with undergraduate students. This study focuses on TWA implementation at a community college in Illinois designated as a Predominately Black Institution (PBI) and an emerging Hispanic Serving Institution (HSI) and a public university in Indiana designated as an HSI. Preliminary findings related to the achievement and identity dimensions are provided for student populations in both of these contexts, which reflect general student demographic data at the respective institutions.

Data Collection and Analysis

The authors created a collaborative inquiry community (Larrivee, 2000) to explore TWA instructional practices. A critical component of the data collection process included biweekly professional development meetings where the team discussed TWA instructional practices that aligned with Rubel's (2017) framework. Such discussions centered on central themes of the TWA curriculum (e.g., distributed practice) and TWA classroom organizational approaches (e.g., class discussions, small-group work) contained in the TWA instructor textbook: *Thinking with Algebra (TWA) Success in Algebra and Beyond* (Feikes et al., 2023).

We also provide preliminary findings on student outcomes related to aspects of the achievement and identity equity dimensions to provide evidence of TWA's positive impact on students. Such student outcomes were explored using mixed-methods, including quantitative and qualitative methods, which this project plans to expand on in the future by collecting and analyzing data related to all four equity dimensions. The quantitative data collected and analyzed included pre- and post-survey student responses that focused on mathematics self-efficacy (Bandura et al., 1999). For six instructors who used TWA and implemented equity-oriented instructional practices during the first year of the project in 2020, the quantitative and qualitative data collected and analyzed included survey and interview data, which revealed how instructors believed TWA impacted student identity development.

Preliminary Findings

Table 2 indicates how TWA instructional practices mapped onto Rubel's (2017) equity-directed instructional practice examples, which are organized within four pedagogical models.

Table 2: TWA Instructional Practices Aligning with Rubel's (2017) Equity Examples

Progressive Pedagogical Models (Rubel, 2017)	Questions Explored for TWA Related to Rubel's (2017) Examples	TWA Instructional Practices
Standards-based mathematics instruction (SBMI)	How does TWA teach students developmental mathematics in ways that supports their conceptual understanding?	* Curriculum design directly supports conceptual understanding * Curriculum integrates distributed practice * Students engage with the physical format of curricular materials
Complex instruction (CI)	How does TWA support students' multidimensional participation?	* Integration of small-group work and whole class discussions in mathematics learning contexts * Students are encouraged to discuss personal understandings or misunderstandings, diverse solutions, and methods * Connect/build on prior math knowledge
Culturally relevant pedagogy (CRP)	How does TWA connect mathematics instruction to students' experiences?	* Building strong relationships with students * Embracing multiple solution strategies that reflect students' experiences and culture

Preliminary findings on achievement and identity student outcomes based on both quantitative and qualitative student and instructor data indicate positive changes in both of these dimensions. For example, statistical analysis of the mathematics self-efficacy survey data showed emerging confidence (n=39, p=.12) in students' ability to succeed in their next mathematics class with a small (d =.33) effect size. The authors plan to collect more data to test at a smaller, $\alpha = .05$, significance level. Feedback from instructors through interview and survey responses indicated that students are constructing strengthened mathematical identities (e.g., confidence) in relation to their engagement with the TWA curriculum.

Conclusion and Future Directions

Emerging mathematics education equity research supports applying broader sociopolitical equity definitions to mathematics education research because this theoretical lens can provide additional knowledge regarding how and why students and minoritized students attain mathematical success. This research addresses a gap in existing mathematics education scholarship by drawing on emerging equity research to identify equity-oriented instructional practices integrated into the TWA developmental mathematics curriculum that can serve to support strengthened equity outcomes among students, broadly defining such equity student outcomes using a sociopolitical equity framework, and providing evidence of improvements in student achievement and identity outcomes based on student and instructor data. Future directions include collecting additional quantitative and qualitative data to further investigate student experiences and outcomes related to the four equity dimensions, investigating such experiences and outcomes for minoritized learners, and integrating additional equity-oriented instructional practices that can support improving such experiences and outcomes. Examples of additional equity-oriented instructional practices include integrating culturally responsive and social justice-oriented materials into the TWA curriculum. Importantly, this project expands knowledge about the equity-directed instructional practices that can be integrated into mathematics learning contexts, including developmental settings, to support students' mathematical success.

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PAYDAY POWER-UP: LET'S MAKE A RAISE DISTRIBUTION

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This study illustrates a problem-solving activity reflective of a real-life situation wherein high school students articulate their reasoning while contemplating the effectiveness and fairness of distributing raises. Study data were gathered via teaching experiment methodology in problem-solving sessions at a mathematics camp for secondary students. Examination of the data indicated that students sought entry points for problem-solving by making assumptions and pinpointing crucial variables. Framing mathematics within the concept of fairness encouraged students to review their work, engage in active listening, and find purpose in mathematical reasoning.

Keywords: Equity, inclusion, and diversity; Social justice; Communication; Problem-solving

The literature suggests that addressing real-life problems prompts students to tap into their funds of knowledge (Hunter et al., 2022; Jung & Magiera, 2023) and everyday experiences (Civil, 2018) in constructing an understanding of mathematical concepts. Recent research underscores the importance of enhancing mathematical sense-making through communication, particularly when addressing issues of equity and inclusion, helping students to comprehend and challenge injustice in real-life contexts (Berry et al., 2020; Kokka, 2020; Ozturk, 2023). This study extends prior research by examining students' mathematically diverse approaches to distributing a raise fairly.

Conceptual Framework

The current study drew from Gutstein's (2016) use of mathematics to contemplate individuals' life realities and to examine (un)fairness in society. We adapted Jung and Magiera's (2023) framework to study students' approaches to understanding and challenging inequitable situations in realistic problem contexts while examining the distribution of resources and power among cultural and socioeconomic groups (Figure 1):

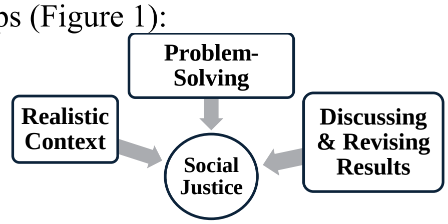


Figure 1: Elements of Social Justice-Oriented Problem-Solving (Adapted from Jung & Magiera, 2023)

Within the framework, students solve problems using real-life knowledge (directly related to the problem context), which is typically integrated into devising problem-solving strategies instead of relying solely on formal mathematical knowledge. The students then develop models and strategies that relate to the problem situation, marking the initial stage of mathematization. Next, students shift their focus from the problem's contextual details to the mathematical aspects, applying mathematical skills and strategies to arrive at a solution. During class discussions, students present their work and actively listen to each other while conversing about identifying sources of equitable opportunities and analyzing systems of power and resources that impact different societal groups. These discussions link broader social justice issues to students' personal experiences and mathematical knowledge, enhancing their understanding of society via lesson contents (Gutstein, 2016). The adapted framework guided data analysis, illuminating students' communication related to developing a fair, effective method for distributing a raise.

Research Setting and Participants

This paper's data emerged from a 6-week after-school math program for high school students, meeting twice a week for 2 hours. The research explored secondary students' application of their knowledge in mathematics to solve real-life problems related to equity and social justice (Ozturk, 2021). The mathematical content focused on using quantities and their units for problem-solving. Participants, including six 10th-grade students, two observer-researchers, and one teacher-researcher, are pseudonymized in the article.

Methods and Data Collection

The class followed a three-phase cyclical process drawn from McClain's (2002) teaching experiment approach: (a) performing group problem-solving; (b) presenting solutions to the whole class; and (c) revising solutions in groups. The chosen method facilitated peer feedback, critical listening, and reflection on diverse reasoning methods. Each 2-hour class involved two problems. Data collection encompassed video and audio recordings and transcriptions, students' written work, and observer researchers' notes as supplemental data used to organize video recordings and transcripts during initial data analysis.

Data Analysis, Sample Student Dialogues, and Students' Written Work

The data for this report comprised a 1-hour-long compilation of video excerpts wherein students solved a problem and communicated their solutions to the class. The criterion for selecting video excerpts was the portrayal of students engaged in identifying potentially important variables and using them to find a mathematical method for determining raises. We focused on the role of communication in helping students critically evaluate the fairness of each method while addressing the problem.

Data analysis featured a thematic analysis approach (Braun & Clarke, 2012) comprising three steps: (a) documenting instances of students' problem-solving; (b) employing a conceptual framework (Figure 1) to categorize students' application of everyday and mathematical knowledge, and detail students' utilization of models when supporting their communication through mathematical representations; and (c) inductively creating themes to classify evidence of students' engagement in developing mathematical methods for distributing a raise, with a specific emphasis on fairness. While ongoing analysis aims to uncover the impact of social-justice-focused classroom discourse on collective mathematical sense-making in another paper, this research report specifically aims to illustrate one theme related to the groups' initial approach to identifying key variables and using them to explain the fairness of their raise distribution. The following discussion includes examples of students' work and discussion excerpts.

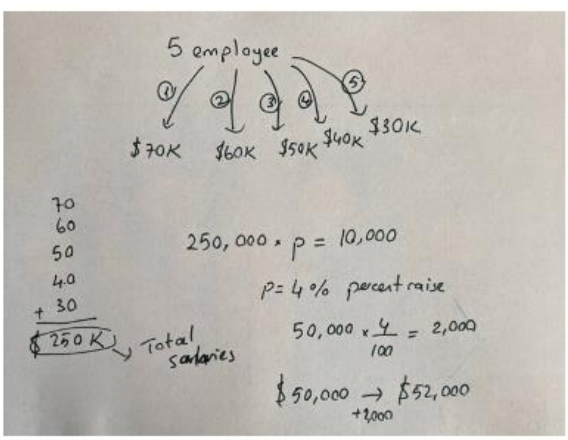
Identifying key variables. Within this theme, students collaborated in pairs throughout the problem-solving process. Groups began by brainstorming and determining variables to consider and developing a mathematical method to solve the problem. The Distributing Raises Problem (Figure 2) tasked students with identifying crucial variables to establish a mathematical approach for determining raises. Students examined variables such as an employee's educational background, years of experience, working hours (full-time vs. part-time), job position, and salary. Group A decided to allocate raises based on employee's salaries (see Table 1).

> Imagine you're the boss of a small company with $10,000 to give raises to the five employees. How would you decide who gets what? Share your method for distributing the raises in a way that's not just effective but also fair!

Figure 2: Distributing Raises Problem (Adapted from Illustrative Mathematics, 2016)

Group A took considered that employees would hold different full-time positions and receive varying salaries. While working hours and educational background potentially differed among the five employees, salaries emerged as the most crucial variable in Group A's method of distributing raises to them. According to Group A's approach, an employee with a higher salary should receive a larger pay raise than an employee with a lower salary.

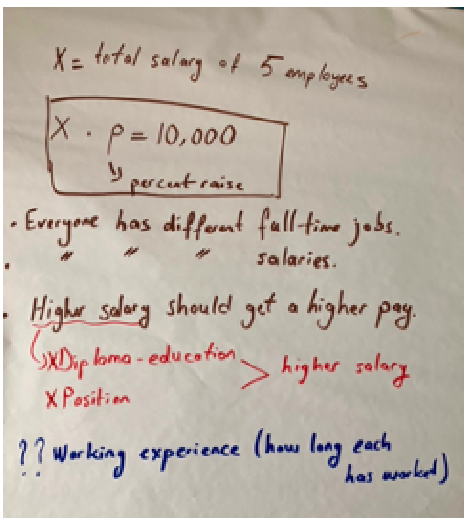
Table 1: Group A's Written Work and Small-Group Discussion

<table>
<tr><th>Students' written work:</th><th>Students' dialogue (group discussion):</th></tr>
<tr><td>
</td><td>Ann: We can make up salaries for now. I want to see how a system may work first. Ok, if we assume $70,000, $60,000, $50,000, $40,000, and $30,000, with a total sum of $250,000, we can calculate the percentage raise for each person.
Amari: Let me write it down.
$250,000. p = $10,000.
Ann: Yeah! Then, we calculate the increase for each employee.
Amari: Like, $50,000 will become $52,000. Salary is the most important thing here.</td></tr>
</table>

The fairness of this method is based on each person receiving the same percentage raise rather than the same amount of money, ensuring equity by acknowledging that individuals with different salaries should obtain proportionate increases. Otherwise, giving all employees the same raise amount regardless of salary or work hours could result in those with lower salaries or fewer work hours receiving a significantly higher percentage raise than their coworkers with higher salaries or longer work hours.

When Group A explained their method, other students questioned the chosen variables for distributing raises. The class assumed that each employee's work impact varied, making a uniform raise unfair. Sydney's observation (see Table 2) highlighted the unfairness of dividing \$10K by 5. Miles's real-life example comparing a waitress to a restaurant manager was supported by Amari's explanation of Group A's method development, accounting for diverse job positions and full-time work. Group A began by assigning salaries to the employees to test their distribution idea alongside the raise percentage, then developed an equation to represent their system (see Table 2), which Anna justified by demonstrating how one quantity (raise percentage) depended on another (total salaries), emphasizing that fairness required the distribution to address employees' salary differences. Thus, Group A's method aligned the calculation of raises with individual salaries, helping the class go beyond computation to interpretation in context.

Table 2: Group A's Revised Written Work and Whole-Class Discussion

Students' written work:	Students' dialogues (whole-class discussion):
	***Sydney:*** Why hasn't anyone divided \$10K by 5? Shouldn't everyone get an equal raise? ***Miles:*** Not fair! What if they have different jobs? \$2K might be significant for a waitress but not for a restaurant manager. ***Amari:*** Right! That's why we're going with the idea of giving the same percentage raise—based on salaries. We keep the same percentage increase, but we also think someone with a higher salary should receive a larger pay raise than someone with a lower salary, assuming all workers are full-time here. ***Tom:*** Why does this method work? ***Anna:*** Because we have an equation. Even if the salaries and their total change, we can still calculate the raises accordingly. ***Tom:*** Why aren't job positions or education important? ***Anna:*** They are important, but we're assuming that higher salaries mean better education and job positions; no need to include them as additional variables here.

Tom's inquiry led to reassessing why education and job positions were omitted as variables, whereupon Anna explained that higher salaries reflected better job positions and education. Instead of treating the salaries separate variables, Group A summed the salaries as a single variable in the equation. Later, the class concluded that work experience could also be a significant variable. Revising Group A's method and including work experience (Y) as a secondary variable would make the equation more complex, thus better reflecting employee qualities such as loyalty—years of experience at the company. This approach would improve fairness by ensuring raises are not solely based on salary; for example, a longer-retained, medium-paid employee could receive a larger raise than a higher-paid employee. In that manner, the raise distribution system would become increasingly attuned to the employee's qualities.

Results and Conclusion

This study describes secondary students' problem-solving process mirroring a real-life scenario in terms of communicating their reasoning while considering fairness in distributing raises. Preliminary analysis revealed the students' search for problem-solving entry points through making assumptions and identifying key variables, thus contextualizing conditions and explaining the reasoning behind various distribution methods. Justifying methods to peers and explaining their mathematical validity engaged the entire class and encouraged students to refine solutions. The findings align with previous research (e.g., Gutstein, 2016; Jung & Magiera, 2023; Ozturk, 2023), confirming that contextualizing mathematics with reference to effectiveness and fairness led students to double-check their work and listen critically to peers while inspiring them to employ mathematical reasoning.

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SENSE OF BELONGING OF MINORITIZED FEMALE STUDENTS IN DIVERSE INTRODUCTORY MATHEMATICS CLASSES

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Sense of belonging is a critical factor in supporting female minoritized students' persistence, motivation, and positive outcomes in STEM education. However, the current research base is primarily focused on singular identities, and situated at the campus or department level, rather than the classroom level, at Predominantly White Institutions (PWI), Historically Black Colleges and Universities (HBCU), or selective research universities l. In this paper, I present the quantitative results from a mixed-methods study that examined belonginess of students in college algebra and precalculus classes at an open-access, diverse, minority-serving institution. Results indicate that by the end of the semester, there was no difference in belonging based on race or gender. However, students with higher mathematics affinity and higher expected final grades had a higher sense of belonging in their mathematics courses.

Keywords: Belonging, Intersectional, Gender, Race, Mathematics Classrooms

Introduction

Higher education institutions are becoming increasingly diverse in their undergraduate populations, but minoritized female students, especially Black and Latina women, continue to be underrepresented in almost all STEM fields (Hatfield, 2022; Ong et al., 2016). Although minoritized female students often begin college with a strong interest in STEM, they are more likely to leave the STEM major (Rainey et al., 2016). Extant literature suggests that introductory mathematics courses serve as obstacles for minoritized female students, as they navigate experiences of isolation, bias, racial and gender microaggressions, stereotype threat, and lack of belonging due to their intersecting racial and gender identities (Johnson et al., 2007; Leyva et al., 2020; McGee & Bentley, 2017; Ong et al., 2011; Museus et al., 2011).

Researchers argue that sense of belonging at the classroom level, is a key factor in supporting minoritized student persistence, and it may even improve participation, student engagement and academic performance (Kirby & Thomas, 2021; Wilson et al., 2015; Strayhorn, 2019). Female minoritized students are less likely to report feeling a sense of belonging in STEM and more likely to report a decrease in their sense of belonging throughout the semester (Rainey et al., 2018). There are limited studies on mathematics classroom-belonging that consider the intersection of race and gender in Predominantly White Institutions (PWI), highly selective universities, or a few Historically Black Colleges and Universities (HBCUs), but not in diverse, open access institutions (Battey et al., 2022; Leyva et al., 2021; Johnson, 2012; Perna 2010). Open-access institutions are colleges that are nonselective in their admission standards and provide increased access to higher education for diverse populations (Anderson, 2015), especially Black and Latinx students (Rendon, 2020). Furthermore, most belonging studies in postsecondary settings consider sense of belonging at the campus or departmental level, rather than at the classroom level.

Research Questions

The purpose of the mixed methods study which this report draws from, is to understand how Black and Latina female STEM students experience sense of belonging in their introductory mathematics courses, college algebra and precalculus, at a diverse, open-access, four-year public institution in the Southeast region. For this paper, I present the quantitative results to the following research questions.

1. How do Black and Latina female students' sense of belonging in the college algebra and precalculus classrooms compare to students in other racial and gender groups?
2. How does Black and Latina female students' sense of belonging in college algebra and precalculus change from the beginning to the end of the semester?
3. Which factors influence Black and Latina female students' sense of belonging in college algebra and precalculus classrooms at the end of the semester?

Theoretical Perspectives

I used two conceptual frameworks: sense of belonging (Strayhorn, 2019) and intersectionality (Crenshaw, 1991). The importance of sense of belonging as a conceptual framework has been well established in the literature (Ostrove & Long, 2007; Strayhorn, 2019). For this study, I use Strayhorn's (2019) definition of sense of belonging: "Students' perceived social support on campus, a feeling or sensation of connectedness, and the experience of mattering or feeling cared about, accepted, respected, valued by, and important to the campus community or others on campus such as faculty, staff, and peers" (p.4). My second framework, intersectionality, is related to one of the core elements of Strayhorn's (2019) theoretical model of belonging: social identities intersect and affect students' sense of belonging, and students experience belonging in different ways. I use Collins and Bilge's (2020) working definition of intersectionality: "Intersectionality investigates how intersecting power relations influence social relations across diverse societies as well as individual experiences in everyday life. As an analytic tool, intersectionality views categories of race, class, gender – among others – as interrelated and mutually shaping one another" (p. 2).

Methods

Participants & Context

Participants in the quantitative phase were students enrolled in college algebra and precalculus during the Fall 2023 semester, at a diverse open-access public college in the Southeast with a student population of about 11,000 (12% Asian/American, 32% Black, 27% Latinx, 24% White/ 59% Female and 41% Male). Approximately 40% of the entering freshman class are first-generation students and over 50% are eligible for a Pell-grant. The college is designated as a Minority Serving Institution (MSI): Asian American and Pacific Islander Serving Institution (AAPISI) and Hispanic Serving Institution (HSI).

Data Collection

The survey participants were students enrolled in 58 sections of introductory mathematics courses during the Fall 2023 semester (seven sections of college algebra with support, 37 sections of college algebra and 14 sections of precalculus). College algebra is typically the first mathematics course in the mathematics sequence (college algebra, precalculus, and calculus) that

STEM majors take at this institution. During the first three weeks of the Fall 2023 semester, 1135 students completed the pre-survey (13% Asian/Asian American, 31% Black/African American, 30% Latinx, 21% White, 5% Other/ 52% Female and 48% Male). Students' ages ranged from 15 to 49 years of age (M=19.08, SD=2.59). The majority of the students reported freshmen standing (76.4% Freshmen, 15.1% Sophomore, 5.7% Junior, and 2.9% Seniors). At the end of the 15-week semester (during weeks 11, 12, and 13), 639 students completed the post-survey.

The pre- and post-surveys consisted of a demographic questionnaire and an adapted version of the Math Sense of Belonging Scale, which Good, Rattan and Dweck (2012) validated and established as a new measure of sense of belonging to mathematics. The demographic questionnaire included questions that asked for students' gender, race/ethnicity, age, major, final grade they expect to earn, and enjoyment of mathematics. All items in the belonging scale are preceded by the statement, "When I am in my college algebra or precalculus class..." For each item, participants rated their agreement on a 6-point Likert-type scale ranging from 1 (strongly disagree) to 6 (strongly agree). The belonging measure includes statements such as "I feel that I belong in the class," and "I feel excluded."

Data Analysis

First, I used SPSS to calculate the descriptive statistics for sense of belonging based on across different racial-gender groups and whether there was a difference in pre and post belonging scores. Next, I conducted a multi-factor analysis of variance (ANOVA) to determine whether there was a statistically significant difference in mean belonging scores between different racial and gender groups. For post-survey results I conducted multi-factor ANOVA to determine whether pre-belonging score, race, gender, gender*race, math affinity, and expected grade influence students' sense of belonging at the end of the semester.

Findings

Pre-Survey Results

The mean level of sense of belonging for Black female students was 4.396 (SD=0.714) and for Latina students was 4.367 (SD=0.695), which are similar to the belonging scores of Latino (4.3732), White male (4.4160), and White female (4.401) students. Black male students had the highest mean belonging score (4.609) while Asian male students had the lowest (4.310). At the beginning of the semester, there was no statistically significant difference between the belonging scores based on students' race or gender alone. However, there is a statistically significant difference in presurvey belonging scores for the interaction effect of race and gender (p=0.025). The only statistically significant difference occurs between Latina students and male Black students ($p=0.034$) where Latina students' belonging was lower. Moreover, belonging scores statistically differed based on students' affinity for mathematics ($p<0.001$) and their expected final grade ($p=0.010$); students with higher mathematics affinity or higher expected grade have a higher sense of belonging to their mathematics class.

Post-Survey Results

End of semester belonging scores for both Black female (4.494) and Latina students (4.399) were slightly higher than at the beginning of the semester, although the change was not statistically significant. Black male students had the highest sense of belonging (4.563) while male Asian students (4.297) had the lowest sense of belonging. However, sense of belonging in these introductory mathematics courses did not significantly differ across gender or race.

Furthermore, the interaction effect of race and gender did not have a statistically significant effect on belonging, contrary to the pre-survey results. However, there was a statistically significant difference in belonging between students with different levels of mathematics affinity ($p<0.001$), expected grades ($p<0.001$), and pre-belonging scores ($p<0.001$). That is, students who have higher pre-belonging scores, higher mathematics affinity, or higher expected final grades have a higher sense of belonging to their mathematics classes.

Table: ANOVA Results of Post-Survey for Sense of Belonging

Dependent Variable: Post_belongingscore

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	251.678[a]	122	2.063	8.254	<.001
Intercept	1097.326	1	1097.326	4390.304	<.001
Gender	.162	1	.162	.646	.422
Race	.798	4	.199	.798	.527
Gender * Race	.197	4	.049	.197	.940
Pre_belongingscore	148.619	105	1.415	5.663	<.001
post_expected_grade	8.476	4	2.119	8.478	<.001
post_mathaffinity	11.273	4	2.818	11.276	<.001
Error	125.721	503	.250		
Total	12654.876	626			
Corrected Total	377.399	625			

a. R Squared = .667 (Adjusted R Squared = .586)

Discussion/Conclusion

This study is part of an explanatory sequential mixed method study that explored sense of belonging of Black female and Latina students at the mathematics classroom level, considering both gender and racial identities at an open-access, racially diverse, minority serving institution. By the end of the semester, there was no statistically significant difference in mathematics classroom belonging based on gender, race, or its interaction effect. This finding runs contrary to the extant literature that report female minoritized students have the lowest sense of belonging in STEM departments (Rainey et al., 2018; Good et al., 2012). Moreover, being a female minoritized student has been found to be negatively correlated to one's sense of belonging in STEM (Johnson, 2012). In this study, sense of belonging of Black female and Latina students did not differ significantly compared to other race and gender groups. That is not to say that their racial and gender identities do not matter, but perhaps they are less salient in a context in which Black female and Latina students are not minoritized, as the study was situated in classrooms in which minoritized students are the majority. In addition, students' enjoyment of learning and doing mathematics and their self-reported expected final grade were positively related to sense of belonging, both at the beginning and at the end of the semester. This finding resembles results from Zumbrunn and colleagues' (2014) study, in which students who felt more capable of succeeding in the course tended to be more engaged in their classes, which may also have positively impacted their sense of belonging.

Although quantitative methods are useful and important, they are limited because they do not provide a comprehensive picture of how students experience belonging in their introductory mathematics classrooms. Therefore, as the next step, I will explain my quantitative findings with qualitative methods using interviews and mathematical autobiographies to gain a broader and deeper understanding of belonging. The strength of mixed methods is that it elaborates and enhances overall interpretations (Greene, 2007). The aim of this work is to contribute to the research base of understanding female minoritized students' sense of belonging and experiences in the introductory mathematics classroom context, particularly given the critical need to better support female minoritized students' undergraduate mathematics classroom experiences.

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LEARNING TO TEACH EQUITABLY: A LOOK AT HOW PRESERVICE TEACHERS INCORPORATE EQUITABLE MATHEMATICS TEACHING PRACTICES

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To better understand how to prepare preservice teachers to teach diverse student populations, this study focused on how preservice mathematics teachers implemented equitable mathematics teaching practices in their lesson plans during a year-long secondary mathematics practicum course. Findings from this study indicate that preservice teachers primarily focused on instructional practices that established classroom norms for participation, attended to students' mathematical thinking, and supported the development of a sociopolitical disposition. However, the preservice teachers did not explicitly incorporate their students' cultural strengths in their lesson plans. The results of this study will further the field's understanding of topics to consider when preparing future teachers to teach mathematics to diverse student populations.

Keywords: Equity, Inclusion, and Diversity, Culturally Relevant Pedagogy

Equitable Mathematics Teaching

Meeting the needs of diverse learners is one of the most critical challenges facing teachers today (Abdulrahim & Orosco, 2020; Gay, 2018). Research on incorporating culturally diverse teaching strategies into the mathematics classroom is growing among mathematics educators, along with a focus on access and equity (Bartell et al., 2017; Bartell, 2013; Gutstein, 2003; Seda & Brown, 2021). Mathematics researchers have studied equitable teaching practices, including incorporating students' community and cultural knowledge into the mathematics curriculum (Civil, 2007), engaging students in controversial topics (Noddings & Brooks, 2017), integrating students' culture into the classroom and curriculum (Gay, 2002; Ladson-Billings, 1995), and using mathematics to address real-world problems and injustices (Frankenstein, 2012; Gutstein, 2006; Wager & Stinson, 2012). However, limited research exists on preparing preservice teachers to incorporate equitable teaching practices in their lessons (Bartell et al., 2017). While mathematics educators have acknowledged the need for teachers to build on students' cultural and mathematical backgrounds to teach mathematics to all students (Turner et al., 2012; NCTM, 2008; White et al., 2016), many preservice teachers are graduating from teacher education programs ill-equipped to meet the needs of culturally and linguistically diverse student populations (Banks, 2015). In response to the need for research to better understand how to prepare preservice teachers to incorporate equitable mathematics teaching practices (Abdulrahim & Orosco, 2020), I followed three preservice teachers during their year-long practicum and examined how they included equitable mathematics teaching practices into their lesson plans.

Theoretical Framework and Related Literature

Mathematics teacher educators and researchers are still learning how to meet students' needs and teach a curriculum that represents every student. A focus on "equity in mathematics education should be one of the most important concerns of teachers, administrators, policymakers, mathematicians, and mathematics educators" (Strutchens et al., 2012, p. 2).

Professional organizations in mathematics education have agreed upon a need for equitable frameworks that challenge the inequities students face in schools today (AMTE, 2022; NCTM, 2014, 2018). To support mathematics teachers in this work, researchers have developed frameworks including equitable teaching strategies to engage all students in mathematical learning (Boaler, 2016), a lens for educators to identify where they are failing to meet the needs of every student (Seda & Brown, 2021), and the dimensions of equity including *access, achievement, identity, and power* (Gutiérrez, 2009).

For this study, equitable mathematics teaching practices are defined by Bartell and colleagues (2017). These "core equitable mathematics teaching practices" provide teachers with guidelines that support all students in learning mathematics (p. 10). The nine equitable mathematics teaching practices include *drawing on students' funds of knowledge*, *establishing classroom norms for participation*, *positioning students as capable*, *monitoring how students position each other*, *attending explicitly to race and culture*, *recognizing multiple forms of discourse and language as a resource*, *pressing for academic success*, *attending to students' mathematical thinking*, and *supporting the development of a sociopolitical disposition* (pp.11-12). In this study, I use these nine teaching practices to answer the research question: How do secondary mathematics preservice teachers implement equitable mathematics teaching practices within their lesson plans?

Method

This study was situated within a larger study in which three preservice teachers participated in a year-long mathematical methods and practicum course. The fall course focused on mathematics methods that prepared preservice teachers to develop a secondary mathematics curriculum. During the course, preservice teachers were introduced to each of the nine Equitable Mathematics Teaching Practices (Bartell et al., 2017) and learned to write lesson plans. The spring course included a teaching practicum that allowed the preservice teacher to observe and teach in a secondary mathematics classroom for two 45-minute math class periods.

Participants for this study included three secondary preservice mathematics teachers in the third year of their education program, each enrolled in a required mathematics practicum course. Beth, a White female, taught AP Calculus and Algebra 1 at Brooks High School. At the same school, Shelby, a White female, taught Algebra 1. Ryan, a White male, taught Pre-AP Algebra 2 at Nolan High School. The majority of the student population at Brooks High School was Hispanic (55.3%) or Black (32.2%), and 88.5% of the students were economically disadvantaged. At Nolan High School, most students were White (55.5%), with 27.1% economically disadvantaged students.

Data collection occurred during the fall and spring semesters and included lesson plans, interviews, class observations, and weekly reflections. The three preservice teachers wrote a combined total of 31 lesson plans during the school year. In this study, the lesson plans were analyzed using nine equitable mathematics teaching practices (Bartell et al., 2017) to better understand which equitable mathematics teaching practices preservice teachers implemented in their lesson plans after studying them in their methods courses. Three researchers coded three lesson plans together, discussing the codes and ensuring agreement upon their use before coding individually. After coding together, each researcher individually coded six additional lesson plans, two for each of the other preservice teachers. In situations where there was a lack of agreement between the three researchers, I considered the explanation given by the researchers to make a final decision. After coding nine lesson plans collectively, I coded the remaining 22 lessons based on the discussion around the codes. Then, I calculated each code's frequency to describe how the three secondary mathematics preservice teachers implemented equitable mathematics teaching practices within their lessons.

In some lesson plans, activities addressed more than one equitable teaching practice. For example, in one of Beth's lesson plans, she divided her class into half, and each half became an expert on one mathematics problem. Then, the groups mixed and taught what they learned to students from the other group. This activity allowed all students to hold the knowledge that another student needed and positioned them to teach their classmates. Additionally, it required all students to actively participate in the learning, so this activity was coded as *monitor how students position each other* and *establish classroom norms for participation.*

Results and Discussion

The data analysis revealed that the three preservice teachers implemented equitable mathematical teaching practices 320 times across their 31 lesson plans. Most often, they incorporated *established classroom norms for participation* in their lesson plans. The next two most frequently used equitable mathematical teaching practices were *attending to students' mathematical thinking* and *supporting the development of a sociopolitical disposition*. Additionally, results from the data analysis showed that the preservice teachers did not attend to *incorporate culture and race* in any of their lesson plans. Table 1 below displays the overall percentages of how each preservice teacher included equitable mathematics teaching practices in their lesson plans. The following sections provide an overview of the four equitable mathematics teaching practices most often used and how the three preservice teachers implemented them into their lessons.

Equitable Mathematics Teaching Practice	**Beth**		**Shelby**		**Ryan**		**Overall**	
	n	%	n	%	n	%	n	%
Draw on students' funds of knowledge	11	6.43	9	8.49	1	2.33	21	6.56
Establish classroom norms for participation	26	15.20	21	19.81	12	27.91	59	18.44
Position students as capable	20	11.70	7	6.60	6	13.95	33	10.31
Monitor how students position each other	23	13.45	18	16.98	5	11.63	46	14.38
Attend explicitly to race and culture	0	0.00	0	0.00	0	0.00	0	0.00
Recognize multiple forms of discourse and language as a resource	22	12.87	12	11.32	2	4.65	36	11.25
Press for academic success	15	8.77	6	5.66	2	4.65	23	7.19
Attend to students' mathematical thinking	32	18.71	11	10.38	10	23.26	53	16.56

Support development of a sociopolitical disposition	22	12.87	22	20.75	5	11.63	49	15.31

Note. Bartell et al., 2017, pp. 11-12

Overall, the preservice teachers most often *established norms for participation* in their lesson plans. All three preservice teachers incorporated tasks in their lesson plans to engage their students in learning and encourage lesson involvement. Beth, for example, included in one lesson a Tic-Tac-Toe board filled with problems where students could select the three questions they wanted to work on, and students could collaborate with their shoulder partner if desired. Allowing her students to choose their problems in her lesson encouraged them to participate in the activity. Shelby varied her pedagogical practices to engage students who were often uninterested in learning mathematics by having them create comic strip word problems and summarize a unit with a poem. Additionally, Ryan used interactive simulations such as a Rock, Paper, and Scissor tournament to involve his students and encourage participation when teaching about growth and decay. Each preservice teacher incorporated tasks where the students contributed to the learning environment instead of passively observing the lesson.

Attending to students' mathematical thinking appeared second most frequently among the equitable mathematical teaching practices. Each preservice teacher incorporated tasks for their students that challenged their thinking and built on their previous mathematical knowledge. For example, Beth used hands-on activities and interactive mathematics programs to build on her students' mathematical knowledge. Shelby most often facilitated meaningful mathematical discourse and posed purposeful questioning to observe and respond to her students' mathematical thinking. In her lesson plans, she included questions such as, "What do we do to finish the equation? What tools might help you?" and "Why do you think so? Would graphing this help?" Shelby's questions helped her recognize her students' mathematical understanding and respond appropriately. The three preservice teachers incorporated instructional practices to support their students' mathematical understanding and ensure they were developmentally ready for their lessons.

The third most often implemented equitable mathematical teaching practice by the preservice teachers focused on *supporting the development of a sociopolitical disposition*. The three preservice teachers primarily incorporated this teaching practice by encouraging their students to consider how to use the mathematics learned to solve problems in an authentic environment. For example, Shelby's goal for one lesson when teaching growth and decay stated, "Students will be able to write exponential functions to describe mathematical or real-world situations." Her lesson included mathematics problems with exponentials that each described a real-world scenario. Similarly, Ryan used Three-Act Tasks (Meyer, 2013) in his lessons to connect the learning for his students to situations they might have experienced previously. Each preservice teacher considered ways to engage their students in the lesson by providing opportunities to solve authentic math problems.

The findings from this study provided insight into how preservice teachers implement equitable mathematics teaching practices in their lessons and highlighted a continued need for instruction that supports preservice teachers in explicitly attending to students' race and culture. While an introduction to equitable teaching practices led the three preservice teachers to include equitable teaching practices in their lesson plans, the findings showed emergent connections (Turner et al., 2012) that did not include activities that challenged power structures in the classroom, empowered students from diverse backgrounds, validated students' home language, or challenged stereotypes. For those preparing mathematics preservice teachers, I encourage ongoing conversations throughout mathematics education courses around the use of equitable teaching practices with an emphasis on how to connect mathematics topics specifically to students' race, culture, language, and personal experiences, as well as support preservice teachers in incorporating meaningful connections for their students. Providing these equitable opportunities in mathematics classes is crucial for students from all backgrounds to develop positive mathematical identities (NCTM, 2018) and a deeper understanding of the mathematics content (Abdulrahim & Orosco, 2020).

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PROPORTIONAL REASONING AND CRITICAL MATHEMATICS CONSCIOUSNESS IN ONE COLLEGE LESSON

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Developing students' Critical Mathematics Consciousness (CMC) is one goal of teaching mathematics for social justice (TMfSJ), but little is known about TMfSJ in post-secondary settings. This study analyzes student reflections on one TMfSJ lesson designed to support students in learning proportional reasoning skills in the context of studying representation of Alaska Native peoples in Congress in a capstone course for college seniors. Our research question was: Which aspects of CMC do students demonstrate in their reflection on the Alaska Representation lesson? Out of 10 students, 7 demonstrated all 6 components of CMC, with another 3 demonstrating 5 components.

Keywords: Equity, Inclusion, and Diversity; Rational Numbers & Proportional Reasoning; Social Justice; Undergraduate Education.

As a human activity, mathematics can never be neutral (Frankenstein, 1983; Ukpokodu, 2007; Wager et al., 2021). Mathematics has always been developed to meet the changing needs of people and their communities (Joseph, 2011). So, we can use mathematics for the social justice projects that matter to us. There are some social justice-oriented lessons available for college settings (e.g. Karaali & Khadjavi, 2019, 2021), but little research has focused on teaching mathematics in social justice contexts in post-secondary settings. Mathematics without meaning or context sends messages on how the world is and how it should stay (D'Ignazio & Klein, 2020; Rubel et al., 2021). Undergraduate students *want* to make the world a better place and they want to know how math can support that goal (Rodriguez et al., 2020). Consequently, we push social justice change agents out of mathematics when we do *not* help them see how mathematics can serve social justice (Lord, 2020). Post-secondary students need opportunities to see how mathematics can influence social justice.

Our study took place in a capstone course for college seniors at a public university in the Pacific Northwest. Some of the course goals were to examine and communicate about social justice issues of race and its intersections with other forms of oppression through data representations. We investigate student reflections on one single teaching math for social justice (TMfSJ) (Gutstein, 2006) lesson designed by Robinson (lead author on this paper) to help students explore proportional reasoning in the context of congressional representation, specifically, in Alaska. The context of Alaska was chosen because 1) the context was not familiar to any of the students so they would experience using mathematics to understand issue that impact communities to which they do not belong, 2) a news article had recently made claims about representation of Indigenous peoples in Congress in light of the first Alaska Native elected to office (Rep. Mary Peltola) so the context was timely, and 3) Robinson had a strong connection to the state.

Background and Framing

The project of social justice broadly “refers to reconstructing society in accordance with the principles of equity, recognition, and inclusion,” (Bell, 2016, p. 4). To understand mathematics as a tool for social justice requires first understanding what we mean by mathematics. Thanheiser (2023) suggests three possible frames for mathematics that allow mathematics researchers to make their stance in their work clear to others. We frame mathematics within the second and third frames. The second frame is mathematics as “a contextual, ever present… lens or language to make sense of the world,” (p. 4). The third frame is mathematics as “a verb (not a noun), a human activity, part of one’s identity,” (p. 5). Mathematics is a human activity and as such is inherently tied to the social context of *who* is doing mathematics and for which *purposes*. Humans can use mathematics to pursue social transformation for justice.

Social Justice through Mathematics

It is not only possible to use mathematics for social justice, but it is impossible to separate mathematics from social justice. Mathematics permeates ideas of fairness, especially at large scale, as well as the decision making that structures laws and distribution of resources. Kimberlé Crenshaw (1988) advocated for legal theories that attended to intersectional identities because labor discrimination suits brought by Black women were being dismissed based on mathematical arguments. Judges in three pivotal cases cited sample size and proportional reasoning arguments to invalidate the discrimination faced by Black women (Crenshaw, 1988). Not only is proportional reasoning central to litigation, but also to distribution of resources. Proportional reasoning in apportionment theory can and has been used to distribute everything from seats in the House of Representatives to voting booth locations to funds for schools (US Census Bureau, n.d.; Verstegen & Knoeppel, 2012). Yet, many students learn about proportions in decontextualized ways that do not support them in transferring their proportional reasoning skills to social justice issues (Simic-Muller, 2015). Mathematics, and especially proportional reasoning, are embedded in social justice projects. So, students need opportunities to learn about topics like proportional reasoning in authentic social justice contexts.

TMfSJ is one framework for situating teaching and learning mathematics in social justice projects. The goals of TMfSJ are for students to both read the world (make sense of the world and analyze power structures) and write the world (change the world for the better in ways that matter to students themselves) with mathematics while developing positive social and cultural identities (Gutstein, 2006). But, as with any human activity, TMfSJ can be co-opted to reinforce, rather than dismantle, oppression. People that hold privileged identities often fall back on oppressive narratives and perspectives when engaging with TMfSJ learning (Esmonde, 2014; Harper et al., 2021). Kokka (2020) built on Gutstein’s (2006) TMfSJ to develop a framework for critical mathematics consciousness (CMC) of privileged students.

Critical Math Consciousness

Kokka (2020) studied the development of CMC for privileged students, specifically white affluent students. While students in our study were not all white or affluent, all students were outside of the Alaska Native community we sought to better understand in the lesson and therefore experienced oppressive systems like racism, colonialism, and urbanism in different ways than Alaska Native communities themselves. Kokka positions mathematics classrooms as settings for developing privileged students’ critical consciousness, positioning mathematics classrooms as one site for social justice activism. From this perspective, Kokka’s CMC has three components: sociopolitical understanding, taking action, and critical civic empathy. The first two components draw on Gutstein's (2006) reading the world with mathematics and writing the world with mathematics, respectively. The last component refers to a concept described by Mirra (2018) in the context of literacy. In the context of CMC, critical civic empathy is "seeing another's perspective while also engaging in structural analysis of power and privilege to take civic action for social transformation," (Kokka, 2020, p. 782). Together, the taking action and critical civic empathy components focus this CMC framework on action, either for oneself or for others.

Stephan and colleagues (2021) developed a CMC framework that explicitly differed from Kokka's (2020) in that they focus on the role mathematics plays in oppression and liberation. Their conception of CMC also has three components that center mathematical awareness (MA): sociopolitical MA, ethical MA, and communicative MA. These CMC components do not map directly to Kokka's (2020) CMC components. Sociopolitical MA is awareness that "Mathematics is used to model and interpret the real world and can be used to make decisions both at the individual and systemic levels that may be oppressive or liberatory," (Stephan et al., 2021, p. 516). Notice that the focus is not only on making sense of the world with mathematics as in Gutstein's (2006) framework, but on how that sense-making can lead to oppressive or liberatory impacts. Ethical MA is awareness that "Human beings do mathematics; thus, there are potential ethical dilemmas and implications of mathematical work; mathematics may be neutral but humans doing mathematics are not," (Stephan et al., 2021, p. 516). Humans doing mathematics can never be neutral. Communicative MA is awareness that "Mathematical communication has the power to educate and mis-educate society and encourage the masses to act in certain ways," (Stephan et al., 2021, p. 516). Notice that mathematics here is positioned as a communication tool.

Both framings of CMC are necessary for this study. Kokka's (2020) framing centers mathematics classrooms as sites for critical consciousness development of students who hold privileged identities. Hence, this framing of CMC is salient for this study in which students held privileged identities relative to the communities impacted directly by the topics in the lesson. Kokka's (2020) framing situates math as the implied vehicle for CMC development, while Stephan and colleagues' (2021) framing focuses CMC explicitly on understanding mathematics as a tool for oppression and liberation.

For this study, we frame CMC as having the six components mentioned above: Kokka's (2020) sociopolitical understanding, taking action, and critical civic empathy, and Stephan and colleagues' (2021) sociopolitical MA, communicative MA, and ethical MA. We define learning in the context of this TMfSJ study as development of CMC. The definition in this project of mathematics as a human tool for social transformation includes two parts: humans use math for specific purposes and those purposes can be liberatory social justice projects. Stephan and colleagues (2021) bring an explicit focus on how mathematics is used as a tool for oppression and liberation, while Kokka (2020) brings a focus on taking social justice action. We use these two complementary CMC frameworks together to help us answer our research question: Which aspects of CMC do students demonstrate in their reflection on the Alaska Representation lesson?

Methods

Context and Positionality

Our identities, especially our racial identities, impact how we understand and experience the world (Crenshaw et al., 1995; Davis & Jett, 2019; Esmonde, 2017; Ladson-Billings & Tate, 1995). In particular, our racial identities impact how we understand and react to the content of lessons in "The Mathematics of Racism," like the Alaska Representation lesson. Robinson developed the lesson with feedback and support from Thanheiser, and led the implementation of the lesson. Robinson is a monolingual white woman and Mathematics Education PhD graduate student. She was born and raised in Alaska, though now lives in the Pacific Northwest. Growing up around but not within Alaska Native communities, she knew Alaska Native people as friends, classmates, teachers, and community leaders. Yet, Robinson has only ever lived in places where her race and ethnicity match the majority of the population and representatives in both houses of Congress. Thanheiser is a Hungarian, German, Jewish immigrant to the United States and Professor of Mathematics Education. Roman is a white cisgender woman, former middle and high school math teacher, and Mathematics Education PhD graduate student.

As mentioned before, none of the ten students were familiar with the Alaskan context. None of them shared that they had spent any significant time in Alaska or that they knew much about Alaskan history or politics. Early in the course students had an opportunity to reflect on and share about their identities through community building activities. During those activities, four students either self-identified as white or omitted reference to race while six students self-identified as holding a racially marginalized identity. Two students identified as Black, one of which further identified as Nigerian-American. Two students identified as Asian, one as Vietnamese and one as half-Indonesian. Two students identified as Latine, one as Mexican-American and one as Columbian American.

The senior capstone course took place over 11 weeks in the Spring of 2023. Students were juniors and seniors who self-selected into the course and majored in a wide variety of disciplines. Students majored in several physical and social science disciplines, mathematics, and creative arts. The lesson in this study occurred over a two-hour class meeting during Week 2 of the term. The lesson started by having students consider and discuss as a whole group the following quote from an article:

> With [Rep. Mary Peltola's] recent swearing-in, it became official for the first time in more than 230 years: A Native American, an Alaska Native and a Native Hawaiian are all members of the House — fully representing the United States' Indigenous people for the first time, according to Rep. Kaiali'i Kahele of Hawaii. Now, there are six Indigenous Americans who are representatives in the House. (Diaz, 2022)

Students were provided with estimates that approximately 9.7 million people or 2.9% of the 2021 US population was American Indian/Alaska Native (AI/AN; *American Indians and Alaska Natives: Key Demographics and Characteristics*, 2023) and roughly 165,000 people or 0.05% of the 2020 US population was Alaska Native (Goto et al., 2004). Students worked in groups to make sense of percentages as proportions and make claims about what fair representation might mean. Next, students learned about some of Alaska's history and elections. Students again worked in groups to compare the proportion of Alaska's population that is AI/AN to other proportions, like the proportion of AI/AN people ever elected to any seat in either house of Congress for Alaska. Students considered which forms of proportions (fractions, decimals, percents, visualizations) were best for 1) creating quickly, 2) comparing two proportions, and 3) communicating a message to others. The lesson ended by learning about and discussing current issues facing Alaska Native communities, like the food and water scarcity and historically low salmon populations, and discussion as a whole group of the implications of underrepresentation in Congress. After the lesson, students were asked to respond to the following prompt in a FlipGrid video:

> Reflect on your learning from the Alaska Representation lesson in 3 minutes or less. Be sure to share about each of the following: What did you learn about mathematics? What did you learn about racism? What do you want to know more about?

For this small study, we use only students' reflections after the lesson. We make no claims about what components of CMC students may have been able to demonstrate *prior* to the lesson. We therefore do not make claims about what CMC components *developed* as a result of the lesson because that implies change over time. Instead, we report on what components of CMC students *demonstrated* when asked to reflect on their learning from the lesson.

Data Collection and Analysis

To answer our research question, we analyzed students' reflection videos. Students recorded their responses as videos in FlipGrid, which automatically generated transcripts that were subsequently cleaned up. These videos and transcripts of student responses serve as the data sources for this study. The 10 responses varied in duration between 1:22 minutes and 2:57 minutes with a mean of 2:08 and median of 1:57. Some students responded to the three prompts in order and referred directly to the language in the prompts, while others did not.

We used MAXQDA software to analyze the data using deductive coding for the six aspects of Kokka's (2020) and Stephan and colleagues' (2021) CMC frameworks. First, Robinson watched all videos and coded any part of a student's response that suggested one of the six CMC codes with the corresponding code. Portions of the transcript could be coded with more than one CMC code, and each student response could include multiple occurrences of a single CMC code. Next, all coded student response segments for each individual CMC code were reviewed together to ensure consistent coding. If a coded response segment *did not* fit the CMC code, then the code was removed from the segment. When a student response included at least one occurrence of a CMC code after both initial coding and subsequent verification then that student was considered to have demonstrated CMC for the corresponding code. Thus, we share the total number of *students* who demonstrated each CMC component as evidenced by at least one coded segment, not the number of coded items. For example, one student's response initially had two occurrences of the sociopolitical MA code: "representation of, for example, Indigenous people in Congress is very… underrepresented according to the proportions of Native and Indigenous people in the US, their population" (Segment 1) and "structural racism and systemic racism prevents representation from dismantling these systems because just bringing um, like, Representatives like Mary Pratola into Congress isn't going to stop racism or dismantle those systems that are keeping us oppressed" (Segment 2). Upon review, Segment 1 retained the sociopolitical MA code because it showed an understanding of the sociopolitical context through proportional reasoning. Segment 2 shows understanding of the sociopolitical context but there is not compelling evidence that mathematics was a tool for that understanding. So, after review this student's response had only one segment coded for sociopolitical MA, which was sufficient to support a demonstration of sociopolitical MA. We share descriptions of each code, examples, and total number of students in the results in Table 1.

Results

Kokka CMC

All three components of Kokka's (2020) CMC framework appeared in every student's response (see Table 1). With respect to the sociopolitical understanding code, students said things like "I also had no idea that there was so little representation of Alaskan Natives and American Indians in Congress" and "with a racist society, you're not going to have equal representation at the high levels where they're actually making the laws, yet everybody's going to be affected by the decisions being made. So that was pretty eye opening." Given that the prompt specifically asked students to reflect on their learning about racism, we hoped that most students would demonstrate sociopolitical understanding in their responses. We also expected many students would demonstrate critical civic empathy in their responses. Because no students held

Table 1: CMC Component Code Descriptions and Totals

	CMC Component	Description	Example	Total Students
Kokka CMC	**Sociopolitical Understanding**	Student learned about a sociopolitical issue	"[Racism] even appears in our government offices where certain groups of people may not only receive representation through the House and Senate"	**10**
	Taking Action	Student expressed a desire for something to change or wondered what action could be taken.	"I just, I wonder, you know what ways we can to get help to get their voices heard"	**10**
	Critical Civic Empathy	Student expressed that representation of Alaska Native people was unfair, referenced structural forces, power, or priviledge	"[there is a] the historical track record regarding these people being silenced often in the political sphere"	**10**
Stephan et al. CMC	**Sociopolitical Mathematical Awareness**	Student shared mathematics as a way to make sense of and represent situations of fairness or unfairness	"math can be a powerful tool in putting into perspective how unfair representation can be for different races and ethnicities that are not white"	**10**
	Ethical Mathematical Awareness	Student suggest doing mathematics is non-neutral and one can make choices about how to do mathematics for specific purposes	"even though math is about numbers and that it could be objective in nature, it's still subject to subjectivity in how that math is interpreted or represented using like visual representations. "	**8**
	Communicative Mathematical Awareness	Student references or describes communicating with mathematics, including to persuade others	"there's different ways that you can write things, and so you want to carefully choose how you're representing your data to make it really understandable"	**9**

an Alaska Native identity, the context of the lesson provided an opportunity to learn about the perspectives of others. In occurrences of the critical civic empathy code, students expressed that the issues of Alaska Native communities are important and referenced historical oppression of Indigenous peoples to put underrepresentation in context. Students said things like “we know throughout time that white people hold the majority of representation for all states pretty much throughout our history” and “in a state that has a racial composition of 21.9% Alaska Native and American Indian, you know, one representative in 63 years of statehood. Not looking so great.”

The goal of the lesson was not yet to take action directly to impact underrepresentation, but to learn more about the issue and how mathematics could be used to understand and communicate about it. Even though the reflection prompt and the lesson itself did not specifically mention taking action, the code came up for every student in terms of discussing action. Students talked about the need for change, their concern or frustration with inaction, or wondering about possible action. Students said things like “there's this common problem of this lack of representation and we seem to be missing it and we acknowledge it, but yet there has been no really like solution” and “I wanna learn more about… what can we do, for example, to help people in Alaska fight climate change, to bring prices for basic necessities down and…yeah, how to advocate.”

All of the occurrences of the taking action code were coded as critical civic empathy as well. One student’s response highlights why: “I am more aware of the fact that racism and segregation is still an ongoing issue and something that we need to go ahead and address whether we may be aware or if we experienced first-hand.” This last quote demonstrates both 1) taking action with “go ahead and address,” and 2) critical civic empathy because the collective “we” that should take action includes people who do not experience the issue directly. All references to action similarly talked about change for others (not the students themselves), specifically, Alaska Native and American Indian peoples.

Stephan et al. CMC

Most students demonstrated all three Stephan and colleagues’ (2021) CMC components. Specifically, all 10 students demonstrated at least two codes, and 7 students demonstrated all three codes. All students demonstrated sociopolitical MA, sharing how their learning about Alaska Native communities was tied to mathematics by using “proportion” or referencing unfair representation. Students said things like “learning about proportions, ratios and percentages are always important to, you know, an individual's ability to critically analyze data, translate and present it is really linked to understanding those concepts” and “in terms of Alaska Natives, they’re very… underrepresented in Congress if you look at their population number.” Students tied their sociopolitical learning directly to their understanding of the mathematics involved in the lesson.

A total of 8 of the 10 students indicated ethical MA. Responses focused on humans doing math for intentional purposes and how that math activity can influence the ways social issues are understood. One student said, “I think that the issues that affect people, say, of color, aren’t necessarily being advocated for or may not be advocated for, especially if there’s no representation in places where policy and laws are made.” This response references the social (in)justice implications of underrepresentation, tying the student’s understanding of the mathematics in the task to their understanding of related ethical implications. Another student shared "when sources or… relevant population data is missing, we question whether the data is an accurate representation of data or the point that it's trying to get across. So, it's important to make sure everybody's voice is heard." The last sentence in this response suggests there are ethical reasons to include people's voices and data sources.

A total of 9 students demonstrated communicative MA, including the previous response. The student described "the point [the data] is trying to get across," suggesting mathematics as a tool for communication. Other instances of communicative MA described how humans make choices to communicate about math, and that these choices influence how others understand representation of Alaska Native peoples. One student shared a concern, saying that "it's sometimes difficult for me personally to see how data can be conveyed concisely and pragmatically and still express the devastating realities that make up these figures." This student suggests that some ways of communicating with data can *obscure* the impact on people's lives. Another student shared that "when you're trying to tell a story about…the data you're trying to convey, you want to tailor it to the basically groups of people you think it's got… you or, you want it to reach the most." This student clearly identifies that communicating with mathematics is about choosing how to best convey your message.

Discussion and Implications

With regard to our research question "Which aspects of CMC do students demonstrate in their reflection on the Alaska Representation lesson?" we found that all 10 student responses demonstrated all three components of Kokka's (2020) CMC framework. Additionally, all 10 student responses demonstrated at least two components of Stephan and colleagues' (2021) CMC framework, with 7 indicating demonstrating all three. This suggests that the lesson was successful in supporting students to learn about proportional reasoning in a context that also promoted social justice. Students learned that math can be used to investigate claims and communicate about social justice topics.

One limitation of the study is that the lesson in the study did not provide an opportunity for students to see themselves as taking action. While all students discussed social justice action, as indicated by the occurrence of the taking action code, the lesson did not appear to support students in actually *taking* action. No student shared an action that they have taken or could take. Also, the lesson did not draw attention to the work Alaska Native communities are already doing to address the topics that arose in the lesson. One small change to the lesson that could potentially address both these limitations is including additional reflective prompts for class discussion, like: "What are Alaska Native communities already doing to advocate for their representation? Find one resource that we didn't talk about in class and share. What steps could you and others take to amplify the work these communities are doing?" A future study could focus on action steps in post-secondary contexts, especially from student perspectives. Part of Gutstein's (2006) writing the world with mathematics is both seeing oneself as able to take action and actually taking action. Hence, to understand TMfSJ in post-secondary contexts we also need to understand how students conceive of and experience social justice action.

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SUPPLEMENTING RADICAL CONSTRUCTIVISM WITH AN ANTI-DEFICIT PERSPECTIVE: CENTERING THE SENSEMAKING OF THE OPPRESSED

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The purpose of this theoretical paper is to describe how an anti-deficit perspective can supplement radical constructivist research programs by deepening their commitment to equity. I discuss how these research programs align with an anti-deficit perspective by treating students' mathematical ways of reasoning as a resource. I also discuss how an anti-deficit perspective can supplement these programs by promoting the construction of sensemaking models that challenge deficit discourses related to students from historically marginalized groups. To illustrate, I present a model highlighting the powerful reasoning of a Chicana undergraduate student.

Keywords: Cognition; Equity, Inclusion, and Diversity; Systemic Change

Radical constructivism (von Glasersfeld, 1995) is a theory of knowing that has been adopted by many researchers to investigate students' cognitive structures in relation to mathematics (e.g., Hunt et al., 2019; Steffe, 2001; Thompson, 1994a; Tillema, 2018). A central goal of researchers working within this paradigm is to challenge what counts as mathematics by positioning students' own mathematical ways of reasoning as the mathematics that should be honored and respected (Hackenberg et al., 2023). Steffe and Thompson (2000) refer to students' *own* mathematical ways of reasoning as *students' mathematics*, and researchers' (or teachers') interpretations of these ways of reasoning as *mathematics of students*. Some researchers (e.g., Steffe & Wiegel, 1996) use the term *second-order model* when referring to *mathematics of students* because it implies that it is an observer's interpretation of what they believe is going on in a student's brain. In this paper, I use the term *model* for convenience.

Given that radical constructivism honors students' mathematics, researchers adopting this paradigm are well-equipped to address dominant equity issues (see Gutiérrez, 2007, 2009) connected to access and achievement in mathematics. For instance, Steffe's research program (e.g., Steffe, 2001; Steffe & Kieren, 1994) provides students access to mathematics because it is their mathematics that they are accessing, which in turn allows them to achieve in mathematics by refining and developing their own mathematical knowledge (Tillema & Hackenberg, 2017). Some researchers, however, might dismiss Steffe's work, and hence radical constructivism, based on the argument that it implicitly perpetuates colorblind ideologies by failing to *explicitly* address issues related to race and culture, among other things (gender, ethnicity, etc.) (Tillema & Hackenberg, 2017). While I agree with the argument that radical constructivist research programs can implicitly perpetuate colorblind ideologies, thus failing to challenge the status quo, I also agree that completely disregarding these programs based on this argument "is far too dismissive" (Tillema & Hackenberg, 2017, p. 57). Instead, I argue that researchers should work towards *extending* (adding onto the framework) or *supplementing* (combining with another framework or perspective) radical constructivism in order to begin challenging the status quo by addressing equity issues specific to students from historically marginalized groups.

Recently, Hackenberg et al. (2023) took on the task of extending radical constructivism to attend to models of students' *social identities* (related to how students associate themselves with respect to race, gender, ethnicity, etc.), in addition to models of students' mathematics. As an example, they described the gendered interactions between two preservice secondary teachers in a past teaching experiment. The female student (at least initially) doubted her own knowledge and paid close attention to the male student's mathematical reasoning despite the fact that she was a powerful math reasoner. Conversely, the male student exhibited confidence in his ways of reasoning, but struggled to understand the mathematical reasoning of the female student. The teacher-researcher chose to disrupt these gendered interactions by asking "each participant to work on the other person's strength relative to themselves—the female participant to work on being more confident in her mathematical thinking, and the male participant to work on understanding his partners' thinking" (p. 27). By attending to the students' gendered interactions, the teacher-researcher was able to construct gender identity models that allowed them to challenge inequitable participation structures throughout the teaching experiment.

The purpose of this theoretical paper is to present another approach that radical constructivist researchers can leverage to enhance their commitment to equity, specifically with respect to historically marginalized students. More to the point, the approach that I am proposing requires researchers to adopt an anti-deficit perspective as they construct models of marginalized students' mathematics. An *anti-deficit perspective* works to challenge deficit perspectives about students from historically marginalized groups by treating their in-school and out-of-school experiences (or funds of knowledge) as resources for learning, and centering their strengths instead of their weaknesses (Adiredja, 2019). Rather than working within the radical constructivist paradigm to extend the types of models constructed (e.g., the social identity models proposed by Tillema & Hackenberg, 2017), this approach works to supplement radical constructivism by blending it with an anti-deficit framing. Ellis (2022) utilized a similar approach focused on constructing anti-deficit (or asset-based) learning trajectories, rather than anti-deficit models.

My aim for this paper is twofold. First, I want to add to the ongoing conversation about enhancing the commitment of radical constructivist research programs to equity-related issues (e.g., Ellis, 2022; Hackenberg et al., 2023; Tillema & Hackenberg, 2017), specifically with respect to the mathematical sensemaking of students from historically marginalized groups. Second, I want to engage math education researchers (both within and outside of this paradigm) in a discussion about how to bridge cognition and equity research more broadly, especially given that these areas have often been studied separately in our field (Adiredja, 2019, 2021). In alignment with the PMENA conference theme, I envision this discussion leading to a future where math education researchers from different methodological and theoretical paradigms can work together to achieve important equity-related goals.

In what follows, I begin by describing the central tenets of radical constructivism and their implications for constructing models of students' mathematics. Next, I elaborate on the meaning of an anti-deficit perspective, and explain how an anti-deficit perspective aligns with and supplements radical constructivist research programs. To showcase the benefit of supplementing radical constructivism with an anti-deficit perspective, I model the powerful sensemaking of a Chicana student based on her work involving a ratio-based math task. Prior to showcasing this model, however, I situate it in a brief review of the history of Latin*[3] students and education in the United States (U.S.) in order to highlight how their historical oppression has contributed to deficit discourses about their inferior ability in mathematics. It is my hope that my modeling will serve to challenge the deficit discourses about the math reasoning of Latin* students more broadly.

Central Tenets of Radical Constructivism

There are two central tenets to radical constructivism: (1) "knowledge is actively built up by the cognizing subject" and (2) "cognition serves the subject's organization of the experiential world, not the discovery of an objective ontological reality" (von Glasersfeld, 1995, p. 51). Taken together, these tenets imply that knowledge does not exist in an objective reality, nor is it an innate trait that certain individuals are born with. Rather, knowledge is something that is actively and constantly being constructed by each individual based on the cognitive adaptations they make to the experiential constraints they encounter. Put differently, knowledge is not static; it is a dynamic construct that is constantly changing from moment to moment in each individual.

In the context of constructing models of students' mathematics, a radical constructivist perspective implies that a student's math knowledge is not something that can be definitively obtained by a teacher-researcher. Thus, the models that a teacher-researcher constructs are *hypothetical* models (von Glasersfeld, 1995) in that they are based on the teacher-researcher's interpretations, and hence, are falsifiable. That is, the models act as hypotheses (similar to scientific models) that can be proven false based on continued interaction with students as they engage with math tasks. While models of students' mathematics can never be proven to depict students' true mathematical realities, they can, over a prolonged period of teacher-student interactions, become more viable (via testing for falsifiability) with these realities in the sense that there are no contradictions with students' explicit ways (e.g. gestures, written work, spoken words) of reasoning mathematically (Steffe & Thompson, 2000; von Glasersfeld, 1995, 2000).

Enhancing Radical Constructivist Research Programs via an Anti-Deficit Lens

Elaborating on an Anti-Deficit Perspective

An anti-deficit perspective works to challenge deficit perspectives that fail to acknowledge students' backgrounds and out-of-school experiences as assets, and instead position the students as the "problem" (Adiredja, 2019; Peck, 2021). With respect to students' sensemaking, an anti-deficit perspective works to challenge deficit perspectives that frown upon informal language, inconsistent and/or ambiguous reasoning, and critical (rather than fast) thought processes (Adiredja, 2019). Thus, there are two important aspects of an anti-deficit perspective. First, an anti-deficit perspective centers the strengths in students' reasoning, even when their reasoning shows signs (from the perspective of the observer) of informality, inconsistency, and/or ambiguity. The goal is to learn *from* students in order to leverage their cognitive resources in learning (Adiredja & Louie, 2020). Second, an anti-deficit perspective views students' funds of knowledge as a valuable resource in the learning process. Here, I define *funds of knowledge* as the knowledge and skills that are important to a student's out-of-school experiences and interests,

[3] Salinas (2020) introduced the term "Latin*" to highlight "the fluidity of social identities" (p. 164). The asterisk is meant to encompass a myriad of ways that people of Latin American descent identify (e.g., Latina, Latino, Latinx).

as well as their household and community culture (cf. Moll et al., 1992). While an anti-deficit perspective is beneficial to all students in the context of mathematics, it is more beneficial for marginalized students (e.g., students of color) as these students have historically experienced obstacles that have deprived them access to high-quality mathematics instruction (Martin, 2009).

How an Anti-Deficit Perspective Aligns with Radical Constructivist Research Program

Given that radical constructivist research programs treat students' mathematics as a resource, there is an anti-deficit element already built into them. In the context of building models of students' mathematics, the teacher-researcher must set their own first-order knowledge aside in order to learn new mathematical ways of reasoning *from* their students (Hackenberg, 2005, 2010). The teacher-researcher may observe inconsistency, ambiguity, and/or informal math language in a student's ways of reasoning, but these are leveraged as tools that can support them in achieving viability in their model of the student's mathematics.

The implicit, anti-deficit nature of radical constructivist research programs allows them to address equity-related issues by challenging common discourses related to what counts as mathematics (cf. Tillema & Hackenberg, 2017). Influenced by Gutiérrez (2013), I define *discourses* as the words, actions, beliefs, norms, systems, institutions, and historical events/factors that determine how power (the degree of privilege one has) is distributed in a given situation. The fluid nature of discourses implies that the distribution of power within them can constantly be challenged (Gutiérrez, 2013). By centering students' mathematics as a resource, for instance, radical constructivist research programs challenge traditional views of student-teacher power relations by giving students, rather than teachers or researchers, the "power to determine what counts as knowledge" (Tillema & Hackenberg, 2017, p. 58).

How an Anti-Deficit Perspective Supplements Radical Constructivist Research Program

While challenging discourses related to what counts as mathematics is important in the equity realm, it is not sufficient in addressing the status quo that favors the needs of white men in mathematics (Adiredja, 2019). In order to begin to challenge the status quo, thus deepening their commitment to equity, researchers leveraging radical constructivism need to be more explicit about *who* their mathematical models originate from. Rather than just learning about students' mathematical ways of reasoning, there must be a deeper commitment to learning who they are as holistic beings with identities, cultures, and experiences outside of mathematics. Inasmuch as an anti-deficit perspective valorizes students' funds of knowledge, it can serve to supplement radical constructivist research program by helping researchers attend more robustly to their participants. In the context of building models of students' mathematics, this would require a teacher-researcher to first learn about their students' funds of knowledge (via surveys, interviews, informal conversations, etc.), and then leverage this knowledge in a way that informs their models. One way the teacher-researcher can accomplish this is by creating math tasks that are based on their students' funds of knowledge. Engaging students with tasks that are relevant to them has the potential to influence their ways of reasoning (Adiredja & Zandieh, 2020), which can in turn aid the teacher-researcher in constructing more viable models of their mathematics.

Sensemaking models that are informed by students' funds of knowledge can help radical constructivist researchers address the status quo, and hence attend more deeply to equity-related issues, by challenging deficit discourses about historically marginalized students. Influenced by Adiredja and Louie (2020), I define *deficit discourses* as the subset of discourses that frame students' academic and cognitive setbacks (from the perspective of an observer) in terms of their personal, family, and/or cultural "deficiencies." To illustrate how these models can challenge deficit discourses, consider the work of Adiredja and Zandieh (2020). These researchers leveraged the lived experiences and resources (consistent with funds of knowledge) of eight undergraduate women of color in order to generate a counterstory that highlighted their powerful sensemaking with respect to the concept of basis in linear algebra. For instance, Stacy, an undergraduate Latina woman, connected the spanning role of a basis to her mom telling her and her siblings "to do all the chores" in the house. This counterstory worked to challenge deficit discourses that have historically positioned Latin* students as academically inferior to their white counterparts (San Miguel & Donato, 2009). In a similar fashion, I argue that models of students' mathematics that leverage students' funds of knowledge can also challenge deficit discourses in relation to historically marginalized students. In what follows, I illustrate an anti-deficit model that showcases the powerful sensemaking of an undergraduate Chicana. Prior to illustrating this example, however, I situate it in a brief review of the oppressive schooling history in the U.S. that has contributed to the deficit discourses faced by Latin* students.

A Brief Review of the Oppressive Schooling History Faced by Latin* Students in the U.S.

Historically, Latin* students have been targeted by deficit discourses that have positioned them as academically inferior to their white counterparts (Contreras & Valverde, 1994; Nieto, 2004; San Miguel, 2011; San Miguel & Donato, 2009). San Miguel (2011) noted that Latinos (I remain consistent with language used by the authors) have been perceived as people who do not value education. San Miguel and Donato (2009) noted that the dominant view that Latinos were inferior to whites (specifically during the first half of the twentieth century) caused school systems to ignore the needs (both linguistic and cultural) of Latino children and view them from a deficit perspective. The main intention of the school system during this era was to "fix" Latino children, and to help them assimilate to white culture. Nieto (2004) noted that prior to the *Brown v. Board of Education* ruling in 1954, the segregation of Latinos in schools was justified by perpetuating deficit discourses about Latinos linguistic skills (e.g., Latinos are deficient in English, and thus, must be taught separately). The teachers in segregated schools were primarily white, and use of the Spanish language in these schools was often met with punishment.

In addition to assimilation and segregation, researchers have documented other ways that deficit discourses have historically impacted Latin* students (e.g., MacDonald & Monkman, 2005; Valencia, 2011). For instance, according to Valencia (2011), Chicano students have been negatively affected by schooling conditions such as (among other things) the suppression of their language and culture, poor school and student funding, lower amounts of highly qualified teachers, and their overrepresentation in developmental and special education classrooms. These conditions have led to the association of Chicano students with school failure via low academic achievement, high rates of grade retention (having Chicanos repeat grades), high pushout rates, underrepresentation in higher education, low performance on standardized tests, and high stress levels in school. Taken together, these historical insights highlight not only the negative impact that deficit discourses have had on the academic, and hence mathematical, success of Latin* students, but also the need for researchers to advocate for Latin* students by intentionally challenging these discourses.

Example of an Anti-Deficit Model: Sensemaking of an Undergraduate Chicana

Background Information, Positionality, and Important Definitions

This example model comes from a teaching session that is part of a teaching experiment (Steffe & Thompson, 2000) I conducted in fall 2023: I was the teacher-researcher (TR), and my doctoral advisor was the witness-researcher (WR). The aim of the teaching experiment was to showcase the rate of change development of two Latina undergraduate students, Yari and Jocelyn (pseudonyms), who were recruited from a developmental math course I taught in the fall. Yari, a first-year public health major and a Chicana whose family is from Oaxaca, Mexico, is the focal student in this example; she agreed to participate in the teaching experiment because of its focus on challenging deficit discourses against Latin* students. The research focus in my teaching experiment is connected to my positionality. I am a Puerto Rican man who spent most of my schooling career distancing myself from my own culture (via assimilation), because of the shame I felt toward my own family—a shame that was influenced by deficit discourses that I bought into (e.g., Puerto Ricans don't value education). My graduate school experience has allowed me to situate these discourses within the system, rather than within my own family. Now I aim to advocate for Latin* students in my research by challenging these deficit discourses.

The teaching session that I focus on is based on Yari's engagement with a ratio task that involved pitchers of *Agua de Tuna* (see Figure 1), a drink in her culture that involves a mixture of water and tuna fruits. My goal during this session was to learn about how Yari reasons about intensive quantities in order to understand her conception of *ratio*—a multiplicative relationship between two quantities (Thompson, 1994b). A *quantity* is a conception of a measurable attribute of an object (Olive & Çağlayan, 2008). An *intensive quantity* is a quantity that (1) cannot be explicitly or directly measured, (2) measures the intensity, or degree of presence (Stroup, 2002), of an attribute, and (3) expresses a relationship between two quantities (Schwartz, 1988). As students reason about quantities, they engage in *quantification*, a process that "involves conceiving of an attribute of an object, conceiving of a unit of measure for the attribute, and forming a relationship between the attribute's measure and the unit of measure" (Johnson, 2015, p. 65). For instance, quantifying the pinkness of a wall would require one to conceive of "pinkness" as an attribute of the wall, conceive of a unit of measure for the wall's pinkness (a relationship between red and white paint), and form a relationship between the unit of measure and the wall's pinkness: a 6:2 (red:white) paint mixture has a higher intensity of pinkness than a 4:2 mixture because it has more red paint for the same amount of white paint.

The tuna flavor of "Agua de Tuna" can be measured as a ratio of cups of water to the number of tuna fruits. Consider the two pitchers of agua de tuna below.	
Pitcher A	**Pitcher B**
4 cups of water 3 tunas	6 cups of water 4 tunas

Figure 1: Agua de Tuna Task (adapted from Johnson's (2015) Hot Chocolate task)

Yari's Engagement with the Agua de Tuna Task: Teaching Session 2 of 13 (October 26th)

Midway through the second teaching session, I gave Yari the Agua de Tuna task and asked her to compare the tuna flavor between both pitchers. Initially, Yari argued that pitcher A would have a higher tuna flavor because it contained less water. Based on this initial argument, it might be tempting to conjecture that Yari did not have a robust conception of ratio because she did not consider how the number of tuna fruits would affect the flavor level; her focus was solely on the amount of water. Yet, it is likely that Yari was drawing from her experience of making Agua de Tuna at home as she noted that her answer was based on "instinct." After all, if Yari were making two Agua de Tuna pitchers at home and wanted to determine the flavor level of the tuna for each pitcher, she might not try to find a relationship between cups of water and tuna fruits. Instead, she might taste both drinks and determine that the one with a stronger tuna flavor has less water compared to the one with a milder tuna flavor.

After Yari provided her initial response, I asked her how she would justify that pitcher A had a higher tuna flavor. She responded by giggling and saying that she would "make it," supporting my initial conjecture that her thought process was linked to her experience of making Agua de Tuna. I giggled with her and asked her to assume that she was not able to make it. Yari responded by saying that she would divide, and eventually divided 4 by 3 (the values for pitcher A) using a calculator to get a new value of "1.33." When I pressed her to explain the meaning of "1.33," Yari stated that, to her, it represented the "flavor level." I then pressed Yari to determine the units associated with the "1.33." To assist Yari in her thinking, I encouraged her to think about her work on the *Filling Bottles* task from the first teaching session. This task (adapted from Johnson, 2015), which depicted a graph of the relationship between the volume (in ounces) and height (in inches) of soda in a bottle, involved a similar division process (9 ounces per 4 inches of water corresponded to 2.25 ounces per inch upon division). Her initial guess was that the units would be "tuna fruits per cup of water." I conjecture that Yari mentally reversed the order of the quantities because having the number of tuna fruits come first in the unit of measure would correspond nicely to her interpretation of "1.33" representing the flavor level of *tuna*.

My goal in this exchange was to get Yari to quantify the tuna flavor so that I could better understand her conception of ratio. Up to this point, I knew that Yari conceived of the tuna flavor as an attribute of Agua de Tuna that could be measured via the relationship "tuna fruits per cup of water." This suggests that the tuna flavor represented an intensive quantity for Yari. But it was not yet clear whether she was using this knowledge to compare the intensity of tuna across the two pitchers. Thus, I asked her to divide the two values for pitcher B (6 and 4). After getting a value of "1.50," I pressed Yari about the units. This time, her response leveraged her work on the previous Filling Bottles task. After noticing that she maintained the order of the units (ounces per inch) when dividing 9 by 4 in the previous task, she wrote "1.50 cups per tuna" on her paper.

WR: Does that make sense to you? One point five cups per tuna?

Yari: (*2 second thinking pause*) Umm … (*3 second thinking pause*) well I don't, I don't like, It doesn't (*pointing to the units after "1.50"*), like it makes sense, but, umm, like if you asked me why (*giggles*), I wouldn't be able to explain (*giggles*).

TR: Well, it might help if you actually do the other one (*points to the "1.33" under pitcher A*). So like, write the units down for this one to … (*Yari says "Okay" and writes "cups per tuna" after the "1.33"*) And now like, try to also like, not get too, like, invested in like the actual math. Cause I think it's also part of like, the math anxiety, when we see the math, it's like everything shuts down. Imagine you, you know, you're talking to your, your mom or something, like at home right? And you're making these pitchers of Tuna with her, and you guys both figured out that one of them has about 1.33 cups of water per tuna (*Yari nods and says "Mmmhmm"*) … and then the other one has 1.5 cups of water per tuna, per one tuna fruit (*Yari nods and says "Mmmhmm"*). So based on that … which one would you say is more tunaey.

Yari: (*responds instantly*) This one (*points to pitcher B*), the pitcher B.

TR: Why Pitcher B?

Yari: Because, umm, oh actually hold on sorry … (*raises hands in front of her and giggles*) Uhh, Mmm, it would still be, pitcher A … (*TR asks "Why" as Yari points to the "1.33 cups of water per tuna" under pitcher A*) Because you're using less cups per water, or per yeah, less cups per, per water, or per tuna.

Reflecting on her work from the Filling Bottles task prompted Yari to maintain the order of the units in the Agua de Tuna task (cups of water per tuna fruit), thus modifying her initial guess (tuna fruits per cups of water). Yet, when the witness researcher asked Yari if the "one point five cups per tuna" made sense, she confessed that she "wouldn't be able to explain" even though it did make sense to her. Leveraging that I knew Yari had firsthand experience with making Agua de Tuna, I took her focus away from the math in front of her by asking her to imagine this situation occurring in her own home with her mother. This triggered a turning point in Yari's confidence as she was able to instantly provide a response to my initial question ("which pitcher was more tunaey?"). At first, Yari pointed to pitcher B, but then quickly shifted to pitcher A as being the pitcher that was more tunaey. When asked to explain, she concluded (stumbling a bit over her words) that pitcher A would be less tunaey because it contained "less cups of water … per tuna."

This exchange is important as it highlights that it is possible for teacher-researchers to leverage a student's funds of knowledge in order to bring out the true potential in their math reasoning. In this case, it wasn't until Yari thought about the situation as an at-home experience that I was able to realize that she was (at least in this context) quantifying the tuna flavor as an association between cups of water and tuna fruits. That is, Yari was able to compare the tuna flavor across both pitchers by associating the cups of water in each pitcher with *one* tuna fruit. Conceiving of a ratio as an association between quantities is consistent with an *identical groups conception of ratio* (Heinz, 2000; Simon, 2006). This contrasts a *ratio as measure conception* (Simon & Blume, 1994), which requires one to conceive of a ratio as a multiplicative relationship that remains invariant. An important note is that Yari's final response (i.e., pitcher A was more tunaey because "you're using less cups of water … per tuna") matched her initial response, where she argued (based on "instinct") that pitcher A would be more tunaey because it contained less water. This suggests that students' intuition, connected to their funds of knowledge, can be a powerful tool in their mathematical ways of reasoning.

Discussion

This anti-deficit model of Yari's ratio-based reasoning, which was viable across future teaching sessions, showcases the powerful, mathematical sensemaking of a Chicana undergraduate student. This is especially significant given that, historically, deficit discourses have positioned Latin* students as academically inferior by (among other things) suppressing their cultural value (Valencia, 2011). One conjecture for why Yari did not leverage her experience of making Agua de tuna until I prompted her to do so is that she has been trained (via deficit discourses and traditional teaching modalities) to see mathematics as disconnected from her culture. Yet, as this model highlights, Yari's culture played a powerful role in bringing out her math reasoning.

There are two ways that an anti-deficit model differs from a traditional radical constructivist model. First, an anti-deficit model centers the sensemaking of students from historically marginalized groups (e.g., women, neurodiverse learners, students of color). When applied specifically to students of color, it challenges colorblind ideologies that assume uniformity in students' needs and experiences. Second, an anti-deficit model valorizes students' funds of knowledge in the sensemaking process. Rather than focusing primarily on students' engagement with math tasks to make sense of their math reasoning, teacher-researchers constructing anti-deficit models go one step further by leveraging students' culture and out-of-school experiences to make sense of (and/or bring out) their math reasoning (e.g., connecting Yari's initial response that pitcher A would be more tunaey because it contains less water to her home experience).

Circling back to the purpose of this paper, I have proposed an approach that radical constructivist researchers can leverage to challenge the status quo and deepen their commitment to equity; an approach that requires supplementing radical constructivism with an anti-deficit perspective to promote the construction of anti-deficit models, specifically with respect to students from historically marginalized groups. The benefit in utilizing this approach is that it has the potential to challenge deficit discourses that have historically positioned these students as academically and culturally inferior to students from the dominant group. I encourage researchers working within this paradigm to consider this approach when inquiring about how to better attend to equity-related issues. More broadly, and in alignment with the PMENA theme, I encourage math education researchers to continue discussing approaches to bridge cognition and equity work in order to promote a future where we work collaboratively across different paradigms to achieve our equity-related goals.

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INCLUDING VOICES THAT MATTER: STUDENT PERSPECTIVES ON COMMUNITY IN ONE SOCIAL JUSTICE MATH CLASSROOM

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Building classroom community with students is important to implementing social justice math lessons. This study investigated students' conceptions of community in a social justice high school math class. Data included two anonymous surveys that were analyzed using thematic analysis. The first finding showed three aspects of community in students' definitions: a group of people, sharing something in common, and group dynamics. The second finding showed that by Survey 2, more students felt comfortable speaking to their peers, and there was an increase in students believing the class was a community. These findings suggest that building community helps students feel more comfortable sharing their ideas, which can in turn help them feel more comfortable discussing sensitive topics (i.e., racism, sexism, in their math classes).

Keywords: Equity, Inclusion, and Diversity; High School Education; Social Justice.

Building community is an important aspect of implementing social justice mathematics (SJM) lessons (Bettez & Hytten, 2013; Conway IV et al., 2022; Thanheiser & Koestler, 2024). Not only is it important, but it "is also essential for SJM exploration that is rooted in and respects all people's humanity" (Conway IV et al., 2022, p. 36). Students have been trained to see mathematics as neutral (Gutiérrez et al., 2023; Thanheiser, 2023). Therefore, they might not feel comfortable talking about SJM topics (e.g., racism, sexism) in their mathematics classrooms. Being in a community offers students "connection, interdependence, and belonging" (Bettez & Hytten, 2013, p. 52). Hopefully, imbued with a sense of belonging, students feel they can share their ideas and thoughts in a space "intentionally built around principles of justice and anti-oppression" (Conway IV et al., 2022, p. 36). Given that students make up a majority of the classroom community, their perspectives matter (Wilkerson, 2021). Thus, we need to understand students' thoughts and feelings on building community, as well as whether they feel it has been built. Consequently, we explore students' conceptions of classroom community in a SJM high school class to in turn support teachers in building community in ways that can support discussions of SJM topics in their classrooms.

Literature Review

Researchers have offered approaches to build community in the classroom (e.g., Conway IV et al., 2022) that can be leveraged by teachers working within social justice curriculums. These approaches involve teachers (1) reflecting on their own biases (Conway IV et al., 2022a), (2) implementing activities (e.g., icebreakers) (Taylor et al., 2022), assigning math autobiographies (Conway IV et al., 2022), and (3) having participation structures in place (e.g., complex instruction, Featherstone et al., 2011) so that students can feel valued no matter how they participate. Yet, with these ways of building community, how does a teacher know that community was built in their classroom? Are they basing it on their own perceptions (e.g., Taylor et al., 2022), or are students' perceptions included? Id-Deen (2024) reminds us that we need to consistently gather feedback from the community to check in regarding their needs. As Wilkerson (2021) states, "A powerful, essential part of our mathematics community is our students…Their voice must be heard and they must be included in conversations, decision making and actions," (para 4). Thus, we need students' continuous feedback to know whether they believe community has been built in the classroom. Little research, however, shows what this looks like in practice (see Adamian, 2022 for exceptions). In this paper, we aim to better understand what is important to high school students in a SJM class when defining community and determining whether they believe community was built. Our hope is that through building community, the students feel safe to discuss sensitive topics because community creates a sense of belonging, and this belonging allows students to share their true feelings. We studied efforts to build community in one high school SJM math course. The research questions are as follows: (1) What definition of community do students have? (2) Did students feel community was built, why or why not?

Methods

Data was collected during the 2023-2024 school year at Forest High School (FHS, pseudonym) in the SJM class. FHS is a public high school located in the Pacific Northwest. The SJM class had a total of 30 students enrolled in the first semester. The class met every other day for 90 minutes. Mr. L (white man of Ashkenazi Central and Eastern Jewish descent, high school teacher) and Roman (white cisgender woman, graduate student) co-taught the course. The class consisted of predesigned TMfSJ lessons (PTMfSJLs, e.g., Berry III et al., 2020) as well as community builders (e.g., significant circles, Esteban-Guitart & Moll, 2014). During the first month of the class, approximately 45 minutes to an hour was dedicated to building community in each class period while the rest of the time was focused on the implementation of a PTMfSJL. After the first month, 20 to 30 minutes was spent on community building each class period.

Two anonymous community surveys were given during the first semester of the SJM class–one in the beginning of October (Survey 1, $n = 28$) and the other in the beginning of January (Survey 2, $n = 25$). There were two types of questions. The first type pulled from practical measures (Jackson et al., 2016), which are quick survey questions that gauge students' feedback on specific instructional strategies. Sample Likert scale phrases in the surveys included "I felt comfortable sharing my mathematical thinking" and "I felt comfortable sharing my views on race, racism, and society." The second type were short response questions aimed at gauging students' perceptions of community. Survey 1 had one short response question, "Do you feel this class is a community? Please explain." Since some of the students in Survey 1 said this depended on their definition of community, in Survey 2, the students were also asked to define community.

We used thematic analysis (Braun & Clarke, 2006) and bottom-up coding to analyze the students' definitions of community on MAXQDA. Roman read the students' definitions and created initial codes to capture what was included in their definitions (definitions could be assigned more than one code). For example, a student's definition of community was, "A group of people that share experiences and work through challenges." Roman's initial codes were, "1. Group of people. 2. Share experiences. 3. Work through challenges." She then read these initial codes to create initial themes. For example, anyone that mentioned working together or working through challenges was coded with the theme "working together." Initial themes were then collapsed into broader themes. For example, "working together", "comfortable with each other", and “getting along” were all collapsed into the broader theme, “Group Dynamics” to depict how members of the community worked with one another.

Results

Research Question 1: How Do Students Define Community?

In the second survey, 24 out of 25 students provided definitions for community. We found three themes: (1) a group of people, (2) sharing something in common (like a goal or interest, etc.), and (3) group dynamics (how the group worked together. The largest theme was “ a group of people (mentioned in 83% of definitions). For example, two students wrote “people together” while another student wrote “a group of people sharing a common interest.” The students also thought that community members must share something in common (mentioned in 58% of definitions), like ideas, beliefs, or goals. As an illustration, students wrote, “a group of people with a similar end goal” or “a group of people sharing something in common, like a location, ideas, and beliefs.” The last major theme was “group dynamics” (mentioned in 37.5% of definitions). For example, students wrote, “a group of people that live in the same place and have some of the same interest[s] by work[ing] together and supporting each other” or “a group of people that share experiences and work through challenges.” To them, working together and supporting each other was important for building community. This makes sense as the students worked on projects and community builders in groups. The way the class was run could have influenced their definitions.

Research Question 2: Did the Students Feel Community Was Built, Why or Why Not?

To answer this question, we compared the students’ responses (yes, no, and in between) in Survey 1 ($n = 28$) and Survey 2 ($n = 25$) (see Figure 1). The number of students who said yes to the class being a community increased from 11 students to 17 students (39.3% versus 68%) from the first to the second survey. Applying the themes from the students’ definitions (from RQ 1) (see Cobigo et al., 2016), students in both surveys tended to focus on sharing something in common, like taking the class, “yes because we all have something similar. We all take this class” (Survey 1); or being in the same place, “yes, we are all in the same class, in the same school, so technically we would be counted as a small community” (Survey 2). In Survey 1, three (out of 11) responses focused on group dynamics, like “we are all working towards the goal of learning social justice math and helping and uplifting each other when we need help.” Group dynamics were mentioned more in Survey 2, where 8 of the 17 “yes” responses shared feeling more comfortable discussing with one another and working together. A student shared,

> yes i feel this class is a community because there is never a time where we have to learn how to find an answer to something by ourselves, we will always work together to figure it out. it just makes us feel more comfortable about sharing ideas and asking questions.

Another student shared, “yes, i feel open to discussions with the whole group and i'm not afraid to show my ideas.” Thus, group dynamics seemed to play a bigger role in why the class was considered a community in Survey 2 versus Survey 1.

The number of students who had in between yes and no responses (e.g., “maybe”) decreased from six students to three students (21.4% versus 12%). Reasons for students’ uncertainty were linked to (1) how community was being defined, (2) sharing something in common, and (3) group dynamics. For example, a student wrote, “Maybe? It depends on how you define a community, because we're all just people that want a math credit for the most part” and another student wrote, “i feel like it's sorta a community but i would still feel judged by my answers or responses by the class.” In Survey 2, of the 3 responses, only one student explained, “Somewhat, we're all here to pass and graduate.” These highlight how the shared class setting and common goal–getting a math credit or trying to pass and graduate–did not automatically create a sense of safety in the learning process. The number of students who said “no” to the class being a community stayed the same ($n = 3$) in both surveys. Reasons included students feeling that (1) the class did not operate as a community and (2) sharing something in common did not make them a community. For instance, a student wrote, “nope, we're just here to get a grade, yes some of us are friends but we in a whole do not communicate or interact like a community” (Survey 1). Another student wrote “naw we're just kids trying to pass” (Survey 2). Thus, even though community was intentionally built in this class, it did not align with *all* the students' understandings of community.

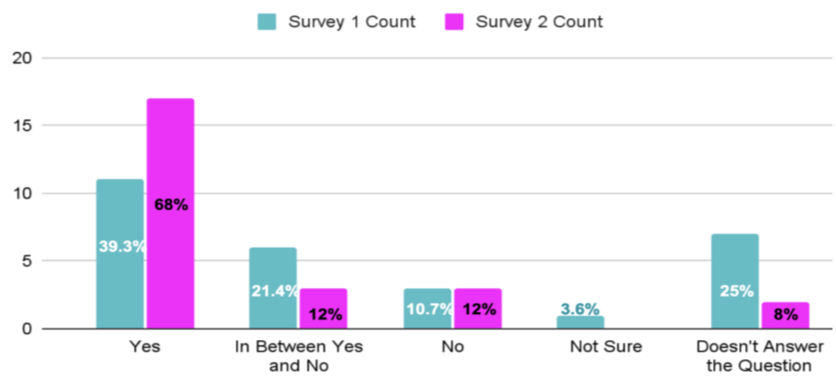

Figure 1: Survey 1 and Survey 2 Count of Whether the Class Was a Community

Discussion

Literature on social justice often talks about the importance of community for implementing SJM lessons (e.g., Thanheiser & Koestler, 2024) and ways to build community (e.g., Taylor et al., 2022), but often omits students’ perspectives from the discussion. In this study, we add to the literature by asking for students’ perspectives on community (RQ 1) and whether they believed community was built (RQ 2) in their SJM class. In response to RQ 1, the three aspects of community in students’ definitions were a group of people, sharing something in common, and group dynamics. In response to RQ 2, we found that the proportion of students who believed the class was a community increased from 39.3% to 68% from Survey 1 to Survey 2.

These results suggest that teachers who intentionally incorporate community building as part of their practice can help students feel a sense of community in their SJM classes. As illustrated by students’ responses to Surveys 1 and 2, this sense of community helped them feel more comfortable sharing their ideas. In Survey 1, students tended to worry about being judged and did not say much about group dynamics, whereas in Survey 2, group dynamics were mentioned more. A student said, “I like the community builders because I actually know the people in this class, my other classes I don't know anyone and don’t feel like I can share my opinion” (Survey

2). This suggests that intentionally building community can foster a safe space where students feel comfortable sharing their thoughts. This has implications for the way teachers encourage discussions around sensitive social justice math topics. If students do not feel comfortable about these topics, then the conversations may not be productive. Thus, intentionally building community may be one way for students to feel more comfortable speaking about these topics.

This shows the importance of intentionally building community and getting students' perspectives. They felt they could share their opinions in this class. If we truly want teachers to be able to discuss sensitive topics in their math classes, then we need students to feel more comfortable speaking about these topics as well. Intentionally building community and knowing what is important to students when it comes to building community is one way to do that.

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PRESERVICE SECONDARY SCHOOL MATHEMATICS TEACHERS' PERCEPTIONS TOWARD STUDENTS WITH DISABILITIES: A PRELIMINARY INVESTIGATION

PERCEPCIONES DE FUTUROS DOCENTES MATEMÁTICOS DE SECUNDARIA HACIA ESTUDIANTES CON DISCAPACIDADES: UNA INVESTIGACIÓN PRELIMINAR

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Better understanding preservice teachers' current perceptions toward students with disabilities will allow mathematics educators to create specific strategies for helping students to develop perceptions promoting inclusive classroom environments. To access these perceptions, we developed an online survey that asks respondents about their knowledge of disabilities, their experiences with people with disabilities, and decisions they would make based on classroom scenarios that involve students with disabilities. We gave this survey to 14 preservice secondary school teachers (PSTs). Key findings include five PSTs presented an inclusive perception toward students with disabilities, seven PSTs presented an ambiguous perception and the perceptions of two PSTs remained unknown. All but the latter two PSTs provided at least some evidence of their willingness to fully include students with disabilities in their mathematics classrooms.

Keywords: Equity, Inclusion and Diversity; Students with Disabilities; Teacher Beliefs; Preservice Teacher Education.

Purpose of the Study

History reflects a progression through four views about the participation of students with disabilities in classrooms: exclusion, segregation, integration, and inclusion (Sónia, 2012). Currently, the goal is classrooms with an inclusive environment for students (Radd et al., 2021), which means that students with and without disabilities are considered contributing members in the learning process. This requirement is what makes inclusion different from integration, as integration—the current norm—is satisfied merely by the presence of students with disabilities in the classroom without regard to the nature of their participation. Mathematics teachers, in their role as instructional leaders, affect the movement from integration to inclusion in their classrooms. Specifically, teachers' perceptions toward students with disabilities can affect, either positively or negatively, the creation of an inclusive environment in their mathematics classrooms. Better understanding teachers' perceptions is essential to being able to support them in making the transition from integration to inclusion, and being able to support preservice teachers to develop an inclusive perception at the beginning of their career will accelerate the change process. To move toward this understanding, our preliminary investigation used a classroom-scenario-based online survey to access preservice secondary school mathematics teachers' perceptions toward students with disabilities.

Theoretical Framework

Papadakis & Kalogiannakis (2020) defined teacher perceptions as "[t]he thoughts or mental images [that] teachers have about their professional activities and their students, which are shaped by their background knowledge and life experiences and influence their professional behavior" (p. 339). Following their definition, we use the phrase *mathematics teachers' perceptions toward students with disabilities* to mean "the thoughts or mental images that influence teachers' interactions toward students with disabilities" and assume that these perceptions are formed from teachers' background knowledge about disability and life experiences with people having a disability. The same definition applies to preservice teachers with respect to their future students.

To have a better understanding of teachers' perceptions toward students with disabilities, it is first important to understand their perceptions toward disabilities. This is why our framework (see Figure 1) addresses both types of perceptions. Perceptions toward disabilities focus on two things: (1) the medical, social, and revolutionary models of disability described in Tan et al. (2019; see Figure 2 below) as a way to categorize the conceptualizations teachers have of disability, and (2) which health conditions teachers recognize as disabilities. Our conceptualization of teachers' perceptions toward students with disabilities focuses on teachers' practice in the classroom around (1) their considerations toward students with disabilities in comparison to students without disabilities, and (2) their application of fairness, justice, equity, and human rights from the perspective of disability.

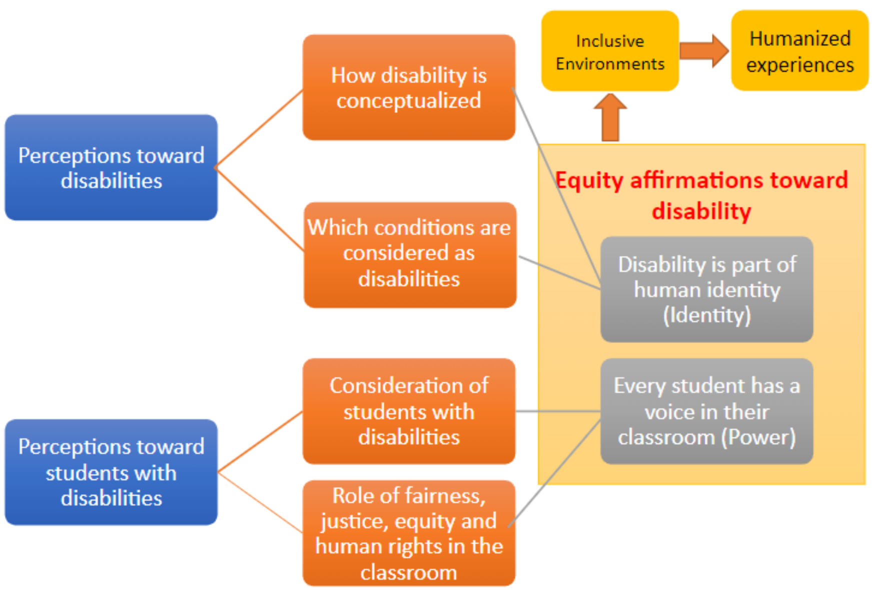

Figure 1: Conceptualization of Teachers' Perceptions toward Disabilities and Students with Disabilities (adapted from Romero Castro & Van Zoest, 2023)

We consider that teachers' perceptions toward disabilities or toward students with disabilities can be identified as inclusive if they promote inclusive environments in classrooms. Otherwise, they are identified as non-inclusive. The right side of Figure 1 illustrates two equity affirmations toward disability that promote inclusive environments and increase the possibility that students will have humanized experiences in their mathematics classrooms. We take the perspective that (1) in order to have an inclusive perception toward disability teachers need to embrace the fact that *disability can be part of human identity,* and (2) to have an inclusive perception toward students with disabilities they need to hold that *every student has a voice in the classroom* (for more details, see Romero Castro & Van Zoest, 2023). These two facts are considered affirmations of equity toward disability, and they are related to Gutiérrez's (2012) two dimensions of the critical axis of equity: identity and power. Particularly, alignment of these equity affirmations with the critical axis of equity is the necessary condition for inclusion toward disability, whereas the relationship between teaching practice and the dimensions in the dominant axis of equity—access and achievement—is a sufficient condition for integration (Romero Castro, 2023). This can be explained by considering that the active participation of students with and without disabilities in the learning process (which requires students' identities to be considered) is what makes an environment inclusive rather than integrative. On the other hand, access and achievement will guarantee that students are present for all classroom activities, which satisfies the definition of integration.

Methods

An anonymous classroom-scenario-based online survey was designed to assess the current perceptions of mathematics teachers. We gave this survey to 14 preservice teachers (PSTs) at the end of a secondary mathematics methods course at a mid-western U.S. university. Before accessing the survey, the PSTs were asked to accept the conditions of a consent form appearing on the first screen of the survey link; all 14 people enrolled in the course accepted. The survey was created using the software Qualtrics (2020) and has three components. The first component assessed which conditions PSTs considered as disabilities. Although other studies have asked preservice teachers to explain their concept of disability (e.g., Mason & Connor, 2022), for our purposes, it seemed more useful to draw on the existing legal documents in the United States that require the integration/inclusion of students with disabilities that belong to certain categories. Specifically, we drew on the categorization of disabilities from the Individuals with Disabilities Education Act of 2004 (as cited in Radd et al., 2021). The PSTs were provided with a list of 12 health conditions and asked to indicate whether each was or was not a disability, or if they were "unsure." The health conditions listed were cerebral palsy, blindness, hearing impairment, leprosy, panic disorder, autism, sleep-wake disorder, Tourette syndrome, spinal cord injury, Down syndrome, oppositional defiant disorder (ODD), and depression; leprosy and sleep-wake disorder are the only conditions on this list that are not considered disabilities in IDEA. For each of the conditions in the list, teachers were also asked to identify whether they have the condition, they have met someone inside/outside the school with the condition, or they have not had exposure to the condition. They also were provided with the opportunity to enter other disabilities and describe their exposure to them. Babik and Gardner (2021) established that exposure since childhood to people with disabilities is one of the factors that positively affect perceptions toward them.

The survey's second component asked the PSTs to describe the nature of their experiences with people with disabilities. There are two questions related to this: one focused on their interactions in their educational career (with teachers, school staff, classmates) and the other focused on their interactions with people outside of school. In each of these two questions, PSTs may choose if they have had only good experiences, at least one complicated experience, or no related experience in the respective environment. If they choose the second option, then they are encouraged to describe why it was complicated. In addition, they were asked to determine whether they felt “totally,” “enough” or “not” prepared to be the teacher of a student with a disability. All these questions help to understand PSTs’ backgrounds, which inform their conceptualization of disability and their perceptions toward students with disabilities in the classroom (see Figure 1).

The third component of the survey positioned the PSTs in two classroom scenarios and asked them to respond to mathematical contributions from students with disabilities during a classroom discussion. The first scenario involves Sam, a student with cerebral palsy who, after working individually on a geometric problem the teacher provided in class, publicly gives an incorrect answer. The second scenario involves Chris, a student with a mathematical learning disability who, after working with a group on an algebraic problem the teacher provided in class, shared the group’s correct answer. As described in Romero Castro & Van Zoest (2023), each scenario is followed by a series of three questions, with each question having three to four choices and the option to write one’s own response if it is not captured by one of the options. Question 1, “What first comes to your mind?” is intended to access the extent to which teachers embrace the complex nature of disabilities as being part of students’ human identities (Figure 1: Equity Affirmation #1). For this question, they are asked to write their response before selecting the choice that best fits their response. Question 2, “What would you do next?” is intended to access the extent to which students with and without disabilities have a voice in the classroom (Figure 1: Equity Affirmation #2). The response to Question 1 informs the response choices for Question 2. Question 3, “Why did you choose that response?” is designed to access their justifications for their answer in Question 2 through a view of the student’s identity.

For the analysis stage, responses to Question 1 were aligned to the models of disability’s conceptualization described in Tan and colleagues (2019). Figure 2 shows a brief description of each model and example choices for Question 1 to illustrate how the response options in the survey support identifying a respondent’s model based on their choices.

Model of disability	Description (Tan et al., 2019)	Example Response Choices for Question 1
Revolutionary	considers disability as part of a person’s humanity and not something to be fixed “posits that mathematics belongs to all students, assuming that all learners, without distinction, are creative thinkers and doers in their multifaceted everyday experience of mathematics” (p. 39)	(Scenario 1) I need to figure out what Sam is thinking because there are multiple ways they could have gotten that wrong answer. (Scenario 2) I need to figure out what Sam is thinking because there are some reasoning ways they could have answered.
Social	“points to the deficiencies within the environment that contributes to the construction of the disability or impairment” (p. 37) “addressing disability issues becomes more a matter of social change rather than individual fixing or even curing or correcting biological or	(Scenario 1) Maybe this problem is too hard for Sam, so I should modify it to support Sam to get the right answer. (Scenario 2) I wish I knew more about Chris’s mathematical learning disabilities so I would know how to help Chris more effectively.

	functional impairments" (p. 39)	
Medical	"locates the deficit, impairments, or disability solely on the individual" (p. 34) "the mathematics learning environment and socially acceptable classroom rules, expectations, and circumscribed ways of knowing and doing mathematics are thought of as adequate and are unquestioned" (p. 36)	(Scenario 1) That's a common answer, even a normal student might come up it because everyone comes to the class with their own conceptions or misconceptions. (Scenario 2) It feels risky to have Chris present the solutions because their mathematical learning disabilities prevent Chris from making sense of it.

Figure 2: Models of Disability and Their Manifestation in for Survey Response Choices

Based on the first equity affirmation we mentioned—*disability can be part of human identity*, the revolutionary model reflects an inclusive perception toward disability whereas the medical and social models reflect a non-inclusive perception. The analysis for the responses to the first question was set up to determine the respondent's perception toward disability in the following way:

- Inclusive when the revolutionary model is shown across both the written and selected responses to both scenarios.
- Non-inclusive when the medical or social models are shown across both the written and selected options respond to both scenarios.
- Ambiguous when the responses vary within or across scenarios.

Responses to Question 2 and Question 3 were analyzed to determine whether PSTs would consider people with disabilities for participating in class, from which it is possible to infer their perceptions toward students with disabilities (see Figure 1) in the following way:

- Inclusive when there is consistent evidence the respondent considered students with disabilities for class participation.
- Non-inclusive when there is consistent evidence the respondent would not consider students with disabilities for class participation.
- Ambiguous when there is inconsistent evidence about the respondent's consideration of students with disabilities for class participation.
- Unknown when there is no response to Questions 2 and 3 for both scenarios.

The survey responses were imported and organized into a Microsoft Excel spreadsheet by respondent number to support the analysis process.

Results & Discussion

Here we briefly discuss key findings for each of the three survey components. In the first component, cerebral palsy and blindness were the only health conditions identified by all 14 preservice teachers (PSTs) as disabilities, even though only five and nine of them, respectively, indicated past interactions with people having those disabilities. Autism and Down syndrome were identified as disabilities by 13 PSTs; one PST was unsure about Autism and said Down syndrome was not a disability. In general, the PSTs identified legal disabilities as disabilities, or were unsure. For example, depression is legally identified as a disability (ADA, 2018) and eight PSTs identified it as such, four were unsure, and only two said that it was not a disability. Interestingly, one of the two PSTs who said depression was not a disability identified themself as having depression and the other as knowing someone who did. Eight PSTs in total identified themselves as having one of the listed health conditions: one with hearing impairment, two with panic disorder, and five with depression[4]. In addition, one PST entered that they had post-traumatic stress disorder (PTSD). Regarding interaction with others, 13 PSTs reported exposure to someone with autism, 10 for Down syndrome, and 9 for hearing loss. Interestingly, although five PSTs in the class reported having depression, eight of their classmates reported not having been exposed to anyone with depression, highlighting the invisible nature of depression as a disability. In addition to the health conditions on the list, one PST each entered that they had exposure to someone with the following: attention-deficit/hyperactivity disorder (ADHD), obsessive-compulsive disorder (OCD), multiple sclerosis, muscular dystrophy, hip dysplasia, and epileptic seizures.

For the second component, seven PSTs (50% of the sample) declared they had all good experiences interacting with people with disabilities *within* school relationships, while the other 50% declared having had at least one complicated experience. Among the reasons given by the PSTs who declared to have at least one complicated experience, the lack of preparation or knowledge about disabilities was the most common. For example, PST 10 wrote, *It was one of my first times working with a student who had a disability [autism], and I felt ill-prepared in helping them.* A different reason is illustrated by PST 6: *How the teacher thought about the student was a negative.* This PST did not identify any contact with a particular disability in the first component of the survey, but their answer implies that they could identify a teaching practice with a non-inclusive perception toward the student with a disability. PST 13 distinguished among different types of disabilities: *Sometimes, students who have disabilities can (not often) be disruptive to the class (especially those with neurodivergent ones). Disabilities that are purely physical are usually handled well by myself, classmates, and teachers*.

Similarly, six of the PSTs declared they had all good experiences interacting with people with disabilities *outside* school relationships, seven declared at least one complicated experience, and one had no experience. PSTs described complicated experiences around family issues and fluency in daily conversations. For example, PST 13 wrote: *Being friends/family with someone who has a disability can be hard. Of course, I do not downplay the dignity of that person. However, loving them takes effort on their hard days* and PST 8 expressed: *Sometimes I struggle with navigating conversations with people and worry about what I might say, so I get worried about saying the wrong thing to someone with a disability*. In the same survey component, one of the PSTs declared to feel totally prepared to be a teacher of a student with a disability, while twelve of them felt somewhat prepared and one of them not really prepared.

The results of the third survey component are organized in Table 1. We first discuss the model of disability that the PSTs' responses mapped to for each scenario and their resulting

[4] It is important to clarify that PSTs were not allowed to select more than one choice in the determination or the exposure parts. This is a design feature that we will modify in future uses of the survey so that respondents will be able to indicate both their own experience and exposure to others.

inferred perception toward disabilities. We then discuss the PSTs' consideration of students with disabilities and their inferred perceptions toward students with disabilities.

In Scenario 1, ten PSTs responses mapped to the revolutionary model, none to the social, three to the medical, and one presented as ambiguous. In Scenario 2, five mapped to the revolutionary model, three to the social, three to the revolutionary, and three presented as ambiguous. Table 1 shows that of the ten PSTs that mapped to the revolutionary model in Scenario 1, four mapped to the revolutionary model in Scenario 2, two to the social, one to the medical, and three presented as ambiguous. These differences in models across the two scenarios could be affected by the differences between the scenarios: type of disability (physical or non-physical), evaluation of the student's contribution (correct or incorrect), and type of work (individual or group). Considering the results, we conclude that four PSTs presented an inclusive perception toward disability, two PSTs presented a non-inclusive perception, and eight PSTs presented as ambiguous. The fact that only four PSTs presented an inclusive perception toward disabilities suggests that there is much work to be done to prepare teachers to promote an inclusive classroom. Furthermore, even PSTs who have an inclusive perception may need

Table 1: Analysis of preservice teachers' modeled perceptions toward disabilities and students with disabilities

PST #	Model of disability reflected (Q1)		Perception toward disabilities	Participation of students with disabilities (Q2 & Q3)		Perception toward students with disabilities
	Scenario 1	Scenario 2		Scenario 1	Scenario 2	
4	Rev	Rev	Inclusive	Yes	Yes	Inclusive
7	Rev	Rev	Inclusive	Yes	Yes	Inclusive
11	Rev	Rev	Inclusive	Yes	Yes	Inclusive
13	Rev	Rev	Inclusive	Yes	Yes	Inclusive
5	Rev	Med/Rev	Ambiguous	Yes	Yes	Inclusive
10	Rev	Med/Rev	Ambiguous	Yes	Missing	Ambiguous
2	Rev	Med/Rev	Ambiguous	Yes	Missing	Ambiguous
12	Rev	Med	Ambiguous	Yes	Missing	Ambiguous
14	Med/Rev	Med	Ambiguous	Yes	Missing	Ambiguous
9	Rev	Soc	Ambiguous	Yes	Missing	Ambiguous
3	Rev	Soc	Ambiguous	Yes	Missing	Ambiguous
6	Med	Rev	Ambiguous	Missing	Yes	Ambiguous
1	Med	Soc	Non-inclusive	Missing	Missing	Unknown
8	Med	Med	Non-inclusive	Missing	Missing	Unknown

additional opportunities to better understand disabilities. For example, PST 13 wrote in the blank box for Question 1 in Scenario 1: *The first thing that comes to my mind, without even reading the numbers yet, relates [to Sam's] perspective to the problem. Without solving and analyzing the error, I would wonder if Sam's experience with wheelchairs informs the solution*. Since

Scenario 1 does not mention anything about Sam's experience with wheelchairs, PST 13 may have assumed that Sam uses a wheelchair because he has cerebral palsy. The reality is that above 50%-60% of children with cerebral palsy can walk independently (CDC, 2023).

Five PSTs' responses to the second and third questions provided evidence that they would consider further participation of the students with disabilities in the scenarios, and thus they were inferred to have an inclusive perception toward students with disabilities. Seven PSTs' responses to one of the scenarios provided evidence that they would consider further participation, but because they were missing responses to Questions 2 and 3 for the other scenario, they were coded Ambiguous. Two PSTs' perceptions toward students with disabilities remain unknown because they did not provide any responses to Questions 2 and 3. It is interesting to note that these two were the same PSTs who had non-inclusive perceptions toward disabilities.

To illustrate, for Scenario 1, nine PSTs chose *Ask Sam to explain their result* (seven of them *to better understand the root of [their] misconception* and two *to give other students a chance to notice whether they share Sam's argument*), and one PST chose *Ask someone else who had the same answer to share their thinking* to *promote discussion in the classroom.* PST 14 chose to create their own response: *Have Sam take me through step by step so I can understand their thinking and build from it*, and their reason behind their response was: *Sam is a person with a disability. He is more than the disability.* All these PSTs were willing to include Sam in the conversation and seemed to value Sam's contributions. For Scenario 2, five PSTs chose *Ask Chris to give their reasoning for their answer* (four of them *to promote discussion in the class* and one *to deepen Chris's mathematical reasoning*). In addition, PST 11 wrote: *I would first ask Chris to explain their reasoning and then solicit help from the group if he struggled* and chose as their reason *to deepen Chris's mathematical reasoning.* Four of these six (PSTs 4, 7, 11, and 13) demonstrated an inclusive perception both toward disabilities and toward students with disabilities. PST 5 demonstrated an inclusive perception towards students with disabilities and an ambiguous perception toward disabilities because they had one response that seems to reflect a medical model of disability. The fact that there was more missing data for Scenario 2, where the student was identified as having a mathematical learning disability, raises the question of whether PST 13 was correct in assuming that PSTs are more comfortable responding to students who have physical disabilities. However, Scenario 1 involved a student's own contribution and Scenario 2 involved a student representing their group's answer, which could also be a reason for the difference.

Conclusion

This preliminary investigation provides a glimpse into preservice secondary school mathematics teachers' perceptions toward disabilities and toward students with disabilities. Our results suggest that PSTs have a fair amount of experience with people with disabilities, and much of it has been positive. Furthermore, all but two PSTs provided at least some evidence of their willingness to fully include students with disabilities in their mathematics classrooms. Despite these encouraging signs, there is much work to be done to support teachers in achieving the goal of creating an inclusive environment in their mathematics classrooms. Better understanding PSTs' current perceptions toward students with disabilities will help teacher educators create specific strategies for helping them to develop perceptions promoting inclusive classroom environments from the beginning of their careers.

Having only two scenarios for PSTs to respond to meant that we could not clearly identify which aspects of the scenarios were responsible for the differences in the models of disability their responses reflected and inconsistencies between their consideration of students with disabilities as contributors of mathematics. Expanding the survey to additional scenarios would allow us to better identify nuances in their responses (e.g., physical vs. intellectual disability, individual vs. group work, accuracy of the contribution).

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PROMOTING EQUITABLE SCHOOL-FAMILY COLLABORATION THROUGH A CULTURALLY RESPONSIVE MATH ROUTINE

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Keywords: Early Childhood Education; Equity, Inclusion and Diversity; Culturally Relevant Pedagogy; Professional Development

This exploratory, qualitative study looked at a mathematical routine, focused on family-provided photos, that elicited children's mathematical and general observations and inquiries and engaged caregivers in mathematical communications. As such, the study of this routine prioritizes family engagement in early mathematics learning through the theories of funds of knowledge (González et al., 2005) and parents as intellectual resources (Civil & Andrade, 2003). As intellectual resources, parents have knowledge and experiences that serve as funds of knowledge to strengthen children's development of positive mathematics outcomes and dispositions, both at home and in school. (Vélez-Ibañez & Greenberg, 1992; Moll, et al., 1992). Set in three kindergarten classrooms serving a diverse, multilingual community in the northeast United States, teacher interviews utilized photo elicitation to study ways in which implementation of this culturally responsive, family-inspired mathematics routine revealed parents' intellectual resources and family and children's mathematical funds of knowledge.

Research-Based Design

This study followed three project teachers after one year of PD and lesson enactment to explore how they were expanding on what they learned from the PD. The focus of the follow-up was to learn how teachers engaged families in mathematics, particularly by amplifying family voices and learning ways to elevate families as assets and intellectual resources. In framing our work on Funds of Knowledge and Parents as Intellectual Resources, we added a component of "Family Inspiration" to a Mathematizing the World Routine (MWR). The original MWR was a routine that teachers engaged in, during year one of our project, focused on developing students' observational and mathematical questioning skills related to some phenomenon, prompting students to notice, wonder, and pose mathematical questions Centered on this revised routine, our study focused on the following research question: What does the culturally responsive, family-inspired MWR reveal about family and children's math knowledge, assets and practices?

Findings

Photo elicitation interviews allowed us to revisit the routine enactment as well as related teacher-family communications. We asked teachers to select the most memorable photos in terms of math talk and family engagement. Recurring themes fell into three categories: 1) Linking family practices with math practices; 2) Making connections among diverse families, peers and educators; and 3) Increasing communication and participation of diverse families. The family-inspired MWR offers a unique integration of mathematical content and family engagement.

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Korean Immigrant Parents' Ways of Providing Support for Mathematics Understanding at Home

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Asian parents are often described as tiger parents in the U.S. media which is a monolithic representation of them. However, it does not consider cultural differences among subgroups in Asian populations. Korean mathematical expressions differ from the U.S. curriculum, and some Korean immigrant parents directly support mathematics at home. This research explores Korean immigrant parental involvement in home math support during the ongoing COVID-19 pandemic and remote schooling. My analysis reveals how these parents utilize culturally relevant approaches to help their children understand U.S. mathematics to negotiate differences in curriculum, cultural artifacts, academic language, and mathematics identities.

Keywords: mathematical processes and practices; equity and justice; culturally relevant pedagogy

Parental Involvement in Education

The importance of parental involvement in children's learning has never been in doubt" (Sénéchal & LeFevre, 2002). This quote emphasizes the positive relationship between parental involvement and their children's learning. Parental involvement in education has three dimensions: involvement at school, involvement at home, and academic socialization (Wang & Sheikh-Khalil, 2014). Among these, home-based involvement happens at home and in local communities such as monitoring schoolwork and progress, supervising homework, setting time or location to do homework, and exploring local places, libraries, or museums. Academic socialization is communication with their children, sharing parental expectations about schoolwork and the importance of education, encouragement of educational goals, and making plans and preparations (Wang & Sheikh-Khalil, 2014).

Parental involvement at home fundamentally influences children's academic success (Jeynes, 2003). Hyde et al. (2006) examined mother and child interactions while working on pre-algebra equivalence problems, such as $3 + 4 + 5 = 3 +$___, and found children understand mathematical concepts better with parental support from home. Not only does it help in understanding concepts, but parental involvement at home also has advantages in shaping identities. According to Vukovic et al. (2013), parents help reduce mathematics anxiety, leading to progress in children's mathematics achievement in higher-order mathematics such as word problems and pre-algebra.

Korean Immigrant Parental Involvement in Education

Asian parents are often described as tiger parents in U.S. media, which presents a monolithic representation of them (Juang et al., 2013). Although there may be commonalities in Asian parenting, parental approaches also vary across subgroups. Specifically, Korean cultural customs, ideas, and lived experiences—such as the devastation of the Korean War leaving the country impoverished for over a generation, living under the North's nuclear threats, and having hostile superpowers close by—surely influence parenting in unique ways. Contrary to the common belief toward Asian parents in general, Kim et al. (2018) found that only 11 percent of Korean immigrant parents in their study provided academic support among the 141 participants. This suggests that a small portion of Korean immigrant parents directly interact with their children's education in the U.S.

Culturally Relevant Practice at Home. Some Korean immigrant parents face difficulties due to a lack of English proficiency when supporting their children's academic progress (E. Kim, 2002), as they are not familiar with the U.S. curriculum and content jargon. For example, in mathematics education, Korean immigrant families encounter differences in the U.S. mathematics curriculum, including cultural, historical, and political contexts, as well as different learning styles, content, mathematical word problems, and so on. However, there has been limited attention given to interactions between Korean-born parents and their American-born children (H. Kim, 2019). To address this gap, the study reported here specifically focuses on how first-generation Korean immigrant parents support their children's mathematical understanding at home by negotiating these differences. The data were collected during the middle of the pandemic in the summer of 2021, a period when such support was particularly salient.

This article employs the theory of *Culturally Relevant Practice* (Ladson-Billings, 1995) as a lens to examine the efforts of parents in negotiating differences and enhancing their children's understanding of U.S. mathematics. The study delves into the interactions of five Korean immigrant parents with elementary-aged children in the 4th, 5th, and 6th grades who are actively involved in their children's mathematics learning at home. Narrative inquiry is utilized to comprehend the diverse experiences of the five participants' families through observations and interview sessions. Due to the literature gap on Korean-American family education in the U.S., this paper specifically focuses on the observational data that highlight Korean families' mathematical interactions. The present article examines these interactions as the key methodology in this analysis. With this in mind, the research question for this study is as follows: *How do first-generation Korean immigrant parents support their elementary-aged children's understanding of mathematics across different cultures, languages, and identities?*

Methods

Narrative inquiry investigates the story itself and subjective experiences (Riessman, 1993), which also include collective narratives among people to discern authentic meaning and value in conversation. The narrative inquiry method aligns well with this study, as it emphasizes the parents and their children's collective narratives to discuss mathematical understanding.

Participants

I recruited five first-generation immigrant families with elementary-aged children via a Korean online community and Korean communities in the U.S. Since this study aims to explore mathematics support at home, participating parents must provide direct mathematics support at home. All participants' names are pseudonyms, chosen by the participants themselves. Except for Seong Chan's family, who stayed for 4 years, the other four families have resided in the U.S. for more than 10 years. Participating children were expected to enter 4th, 5th, and 6th grades, and the data collection period from June to August 2021. Seong Chan was the only father who taught math at home, and his wife Hyeon Ji also occasionally participated in the interview sessions. Participants had varying immigration years, ages, occupations, and numbers of children.

Data Collection and Analysis

The results reported here are derived from observations, which included authentic conversations between Korean immigrant parents and their children regarding mathematical understanding and how to navigate mathematical expressions within two different cultures, languages, and identities. Observation sessions occurred twice, with each session lasting 40 minutes. Follow-up interview sessions lasted about 20 minutes, resulting in a total observation time of 1 hour for each interaction. All sessions were transcribed using the Artificial Intelligence transcribing program, Naver Clova Note (https://clovanote.naver.com/). Each transcript underwent multiple checks, as the technology sometimes did not capture the correct narratives. Additionally, the transcripts were scrutinized for themes that participants highlighted on how to negotiate differences in culture, language, and curriculum. When a person used two languages, Korean and English, I translated the interview verbatim into English to aid the audience of this study. I inserted the translation in italicized fonts for the Korean parts. English-spoken parts were marked with bold fonts to indicate the code-switching from Korean.

Results

Korean immigrant parents and their children jointly 'negotiated' cultures, identities, and language differences in their interactions to enhance their mathematical understanding. Although the parents are not education experts, they endeavored to bridge differences in culture and language. Negotiation is the process by which two or more parties attempt to resolve perceived incompatible goals (Brett, 2000; Carnevale & Pruitt, 1992). Brett (2000) discussed intercultural negotiations involving cultural preferences, norms, priorities, and negotiation strategies. These communicative preferences meet in the middle of a negotiation, dealing with both similarities and differences.

In alignment with Ladson-Billings's culturally relevant practice (1995), I address *cultural negotiation for mathematics understanding*. This negotiation stems from interactions between Korean immigrant parents and their children, where mathematical meaning, concepts, and mathematics identities are jointly explored, revisited, revised, and negotiated in terms of different cultures, languages, and identities. This also provides insight into how immigrant parental involvement either strengthens or impedes children's learning when navigating various cultural, national, and generational differences during the learning process. Parents' immigration with changed social location influences their children, exposing them to two languages at home. If immigrant parents attempt to teach academic skills to their children, their learned curriculums would differ due to generational gaps and diverse cultural backgrounds. However, all participating parents made efforts to connect their learned curriculum with their children's U.S. mathematical concepts. Table 2 presents themes reflecting the differences parents observed and the ways they engaged in cultural negotiations to address disparities in curriculum, artifacts, academic language, and identities.

Table 2: Cultural Negotiation for Differences in Mathematics Understanding

Differences	Cultural Negotiation
Curriculum	Curriculum differences in Korea and the U.S.
Artifacts	Culturally relevant revision
Academic language	Cultural negotiation in language differences
Math identities	Sharing expectations

Discussion

This study explored how Korean immigrant parents support their elementary-aged children's mathematical interactions at home by addressing differences. In alignment with Ladson-Billings's culturally relevant practice (1995), I address *cultural negotiation for mathematics understanding* based on observations of Korean immigrant parental involvement in four ways: negotiation in curriculum, artifacts, languages, and identities during mathematical interactions. While not experts in culturally relevant pedagogy, they provided cultural revisions to enhance their children's mathematical understanding. Parents sought to connect their learned curriculum with the U.S. curriculum and endeavored to revise unfamiliar cultural terms (Ladson-Billings, 1995) in word problems to help their children grasp the mathematical meaning. Regarding languages, when a child made an error using a Korean expression, parents explained the U.S. way. For instance, when a child said, 'Twelfth One,' the parent corrected him by saying, 'One Twelfth.' Beyond fostering understanding, parents also shared their mathematics identities and values in problem-solving to yearn for children's academic success. Interestingly, their emphasis on the use of mental math varied, challenging common myths about Asian parents as a homogeneous population.

Although some Asians have been perceived as model minorities inherently good at mathematics, findings reveal that Korean immigrant parents and their children's narratives still need support in language and cultural differences in their mathematics schoolwork. This study uncovers Korean immigrant parents and their children putting effort into understanding U.S. mathematics, a challenging task. It is a winding road that must navigate many challenges, including cultural differences in mathematics terms and meaning, as well as the interweaving of parents' and children's mathematics curricula. This implies educators need to invite all students, regardless of immigrant backgrounds, cultures, identities, or linguistic assets. Furthermore, educators should conduct more professional development sessions to better understand students' backgrounds and incorporate them into their lessons. They should also invite parents to share their family narratives and culture.

Limitation

One limitation of this study is that it includes only narratives from five families. The observation data, along with my interpretation, may not be a direct representation. However, the follow-up interview session somehow covers the reasoning behind participating parents' approaches.

Conclusion

This study specifically uncovers Korean immigrant parents and their children's cultural negotiation to build mathematical meaning at home during the COVID-19 pandemic. It illustrates that language and culture are interrelated, influencing conceptual differences in mathematics. Immigrant families put in extra effort to understand U.S. mathematics. The study delves into Korean immigrant parents' and children's mathematical interactions, revealing their additional efforts. Future research is needed to understand how Korean immigrant students negotiate mathematical meaning in their classrooms.

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THEORIZING REFUGEE YOUTHS MATHEMATICAL IDENTITIES AND EXPERIENCES: A CALL FOR ASSET-BASED PEDAGOGY FOR EQUITABLE MATHEMATICS INSTRUCTION

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Keywords: Refugee youth, Urban and Rural Refugee, Mathematical experiences, Identity, Equitable mathematics instruction

This poster explores high school refugee students' mathematics learning experiences and identities using the Math-trivium framework (Umeh, 2023) and calls for support for Central Missouri high school teachers in adapting to asset-based pedagogical practices in mathematics to meet the academic needs of refugee youth. Central Missouri schools are seeing an increase in refugee students. The most recent available data shows that the percentage of Multilingual learners more than doubled (from 3.41% to 6.99% between 2009 and 2022 in Columbia Public Schools (CPS) (DESE, 2022), and the Democratic Republic of Congo, Afghanistan, Eritrea, Syria, Burma, and Sudan are the most common countries of nationality (RPC, 2023). While schools are expected to facilitate the reintegration of refugee youths into a new academic and social culture, they often lack the readiness to address these students' specific learning needs and identities (Hos, 2016). Thus, there is a need to provide adequate support for high school teachers to understand the refugee experiences and identity as assets for equitable mathematics instruction.

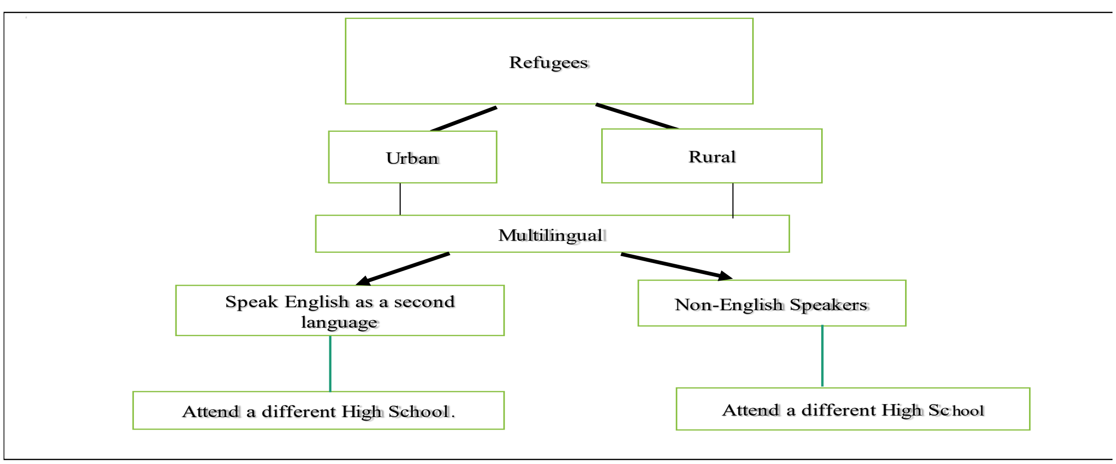

Figure 1: Shows an Identity Map of Refugee Youth.

Therefore, this poster explores the mathematical identities of refugee youth within the context of urban and rural refugees living in Central Missouri, United States of America. Although they have the same refugee status, their mathematics identities and experiences differ.

There is a need to prepare globally competent high school refugee teachers and mathematics teacher educators for more equitable mathematics instruction for refugee youths.

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IDENTIFYING STORYLINES WITH INDIGENOUS AND NEWLY MIGRATED MATHEMATICS STUDENTS

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In this methodological paper, we describe our approach to engaging middle school students in identifying storylines at work in their experiences of mathematics learning. Storylines are an important part of the theorization of positioning but they are underexplored. Our focus is on students who identify (or are identified) with groups that have often been marginalized, namely Indigenous students and students who are newly arrived migrants. It was important for us to garner the trust and also the interest of students while guiding them to conceptualize storylines so that they are the ones identifying the storylines that impact their experiences. In the presentation we will share more data.

Keywords: Equity, Inclusion, and Diversity; First Nations and Indigenous Cultures

Student experiences of mathematics learning have been investigated using the theorization of *positioning*. Indeed, it is important for educators to understand how students understand themselves as learners and how they understand their actions. Students are positioned by teachers, classmates, media, community, organizations, language constructs, and families, through current and historical practices. We are particularly interested in the positioning of students who identify (or are identified) with groups that have often been marginalized, namely Indigenous students and students who are newly arrived migrants.

A focus on storylines

Most research using theories of positioning focuses on the positioning (e.g., Drageset, 2024, Sengupta-Irving, 2021; Tait-McCutcheon & Loveridge, 2016). However, the tradition referred to as positioning theory sees a triad at work in human interaction, including the three elements of positioning, storyline and speech/communication act (e.g., Harré, 2012; Herbel-Eisenmann, 2015). A storyline is a story known to a participants in an interaction. The people in the interaction are associated with positions in the story, which guides the people's choices about how to interact. Storylines are important because they provide the repertoires for action for mathematics students. The storylines available (known) to students and their teachers make certain positions possible, and they exclude other positions from possibility.

As far as we know, the research that focuses on storylines in mathematics education research investigates common stories that appear in public discourse (e.g., Andersson et al., 2022; Chorney et al., 2016; Herbel-Eisenmann et al., 2016; Rodney et al., 2016)—namely news and entertainment media. In our research project we work with Indigenous and new migrant students and aim to identify the storylines at work in their mathematics learning environments. In this paper, we describe how we have gone about doing this, noting the principles we embrace, the challenges we experience, and results of our choices. This is a methodological paper.

We do not yet have detailed data sufficient to share in this paper, but we expect to have data to share for the conference. In this paper we describe our methodological choices and how they manifest in our interactions with participants. The research aim is to identify storylines, animated with accounts of student experiences. We will next share these with teachers and work with them to develop alternative pedagogies that build on student strengths and experiences. (We have some detailed data already including accounts of student experiences, but to honour our anonymity promises, we cannot yet publish any of the accounts of experiences because that would make it possible for some people to identify the students who told us of their experiences.)

First context: Conversations with Indigenous students

Our research project is situated on the traditional unceded and unsurrendered territory of Wolastoqiyik and Mi'kmaq peoples (often marked on maps as part of eastern Canada). Since colonization, the majority of people living here have been settlers, and there has been a recent surge in settlement with the general upswing in global human migration. Given the history of terrible experiences at the hands of settler people and colonizing institutions, it takes time and care to develop a relationship of trust with Indigenous students.

In the first context of our research project, Dave (the first author) worked with a teacher who, for more than a decade, has worked as a support teacher for the Indigenous students from a local Wolastoqiyik community. We note that this teacher's endorsement and collaboration accelerated the students' willingness to speak openly about their experiences (i.e., to trust). In other words, trust can be shared, to some extent, from an already trusted person. Of course, the teacher was willing to endorse and collaborate with us based on our past interactions (also a development of trust). We also drew on the teachers' knowledge of the students to decide on an approach to interaction, and thus decided to work for some time with a group of students (about 8 middle school students) in a series of six biweekly one-hour meetings. The group felt safe when they were all together, so dividing them up probably would have undermined that sense.

In the first interaction Dave briefly described to the students what we hoped for in the research, and taught the students a game he learned from a Mi'kmaq student in a previous research project. The game play would develop relationship, and the Indigenous source of the game showed students that Dave had interacted with other Indigenous students before, attentive to their cultural heritage. Once we started the interactions focused on identifying storylines, we needed to find a way for students to understand the concept of storyline. How to do that was not straightforward, given that Dave and others in a sister research project (with some of the same research team members, also with a similar focus on storylines associated with Indigenous and newly migrated students, but in Norway) have been finding it challenging how to describe a storyline, as discussed in Simensen et al. (2022). The theory work on positioning has not been very helpful, and the paucity and inconsistency of examples of identified storylines in research has led to questions about how to describe storylines and how to name them. Should they be full sentences? Can they be described with a few words like a theme? Those questions persist.

In the interactions, the approach that we settled into as productive (in terms of garnering quantity and quality sharing from the students) had Dave asking what has happened recently in math class or relating to the math they did in math class. After a student identified something that happened, then Dave probed with questions about what is "really happening" when things like that happen. Using the triad of positioning theorization, the happening is a communication act or a series of communication acts, and the description of what is really happening is the storyline. We discussed questions like "Why would [the person] do that?" Dave probed for other similar experiences. It was very productive to compare the happening to parallel happenings outside of school math. For example, when students talked about feelings of success or failure it was helpful to consider feelings of success or failure in other contexts—which tended to be sports contexts, probably because of the passions of the most readily vocal among the students. Or, when talking about feelings within interactions with people in math class, it was productive to consider how that was different from interactions among friends.

After this series of group conversations, the next step was for us to synthesize the stories and dialogue over the hours of interaction and bring them back to the students. The teacher arranged for each student to meet Dave individually, but it was hard to decide how to conduct those interviews. After Sacha and Dave discussed the deep knowledge emerging from the recorded interactions, Dave synthesized the foci with the following storylines:

1. We have to try hard to learn math.
2. Teachers respond to students differently based on the students' reputations.
3. Math is important.
4. The math we do in school is not interesting.
5. We are more worried about failure than attracted by success in math.
6. How I feel about my math performance depends on my expectations for myself.
7. Math teachers don't understand native people [i.e., Indigenous people].
8. Math should be done in silence.
9. Something about parents… <not sure what>

In the interviews with individual students Dave told them he made a list of storylines from the conversations we had had. He said he would read them one at a time, and ask for example stories that illustrate those storylines. For #9 he said that there was a lot of talk about parents but he could not identify a storyline succinctly. He asked each student if they could say what the storyline might be. The fourth of the students (the most shy among them) provided a storyline: "Parents expect you to do well." After this, Dave replaced #9 with the student-provided storyline. Also with the first few students it became clear that #5 and #6 were not like the others. Students said they were not sure about them. Emerging from those conversations, Dave learned that it was helpful to ask them if they thought the statement is true, or if they thought the statement is something people say or think. For #5 and #6 almost all the students said the statements were true (not surprisingly: these emerged from our earlier interactions) but that they were not statements that people think or say. Thus, we see that they are not storylines: they are not known stories. This reminds us that storylines do not have to be true to be well known. And their power is independent of their truth. The students provided rich accounts of the storylines, some of them repeated from earlier interactions. We look forward to reporting on the storylines illustrated by particular experiences. As noted, to protect anonymity we cannot do this yet.

Following the principle of OCAP (ownership, control, access, possession), "First Nations have control over data collection processes, (…) they own and control how this information can be used" (https://fnigc.ca/ocap-training/). Thus, for the learning from our interaction with the Wolastoqiyik students, they themselves and their community are the most important people to report to. Sacha (the second author) identified a knowledge keeper (a.k.a., Elder) from the students' community as a person to report to first, and the teacher endorsed this choice. Dave spent a couple hours with the knowledge keeper to report on what happened thus far in the research conversations, including the specific storylines discussed most recently in individual interviews. Dave did not tell the knowledge keeper about the experiences shared by students, but the knowledge keeper already knew the general nature of those experiences (knowledge keepers know things!). We invited the knowledge keeper to join the next gathering of the student research group. The knowledge keeper told the students of their own experiences as a student and as an adult dealing with Indigenous and settler peers and with "authorities". The students asked questions and told the knowledge keeper about some of their experiences in and related to school. That interaction reinforced our prediction—people tell different stories depending on whom they interact with. Dave closed by asking the students if they wanted to be kept in the loop—to be kept informed and to be given the opportunity to guide the future steps with the knowledge they generated and shared in our conversations. They said yes.

Second context: Conversations with new migrant students

The conversations with the Wolastoqiyik students informed our approach to newly migrated students. Again, the first challenge with addressing these students is garnering their interest and trust, but we did not have the benefit of the same kind of teacher ally with longstanding relationships because the students are newcomers. We also know that many newcomers are highly cautious about what they say about their experiences of migration because they are in the process of qualifying for residency status. Many of them come from situations that prompt them to be extremely careful of their information getting into the wrong hands.

Dave and a graduate student met with some classes to tell them about the research and invite participation in interviews. He said that we are interested in interviewing any students but especially interested in the experiences of newcomers. In the first session, he used the approach he had used with the Wolastoqiyik students—asking for examples of things that happen in math class, and following up with questions about what is really happening when these things happen—but the students seemed bewildered by this. The dynamics of being in a larger group of people and also some linguistic barriers probably impacted their ability to focus on questions that are unlike the questions they are accustomed to in school. For the subsequent groups, Dave switched to an approach informed by the interviews with the individual Wolastoqiyik students.

Dave said he would read some statements, and ask students to say if the statement is true, and then to say if they think others think the statement is true. He said that the interviews would go like this, but in the interviews the interviewer would also ask for examples of things that happen because of people thinking the given statement is true. He said that the interviewer would then go vice versa, and ask for accounts of things the students particularly liked or disliked in math learning. With the story of the liked or disliked thing, the student with the help of the interviewer would try to figure out what statements (storylines) people believed in order to get them to do the thing they were doing in that situation.

For this set of storylines, Dave mixed together some storylines from the Wolastoqiyik students and from the research contexts in the sister project described above. So, Dave would say, for example, "When I do math, I need to use my first language" (i.e., a version of a storyline from the sister project) and ask students to put up their hands if they thought it was true (fewer than half of the students said it was true). And then to put up their hands if they thought some people think it is true (almost all the students thought that some people think it is true). The students were laser focused on these questions, locking eyes with Dave in a way he has not experienced in a classroom before. The struggle to decide whether to put their hands up or not was evident on students' faces (even students who have been in Canada less than a month). In other words, the students were invested in these storylines. The students recognized importance of these storylines—the impact of these storylines on their experience of mathematics learning.

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ENVISIONING CRITICALLY CONSCIOUS MATHEMATICS ENGAGEMENT USING A CULTURALLY AFFIRMING TASK ABOUT HISPANIC HERITAGE

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This report presents a case study of one teacher's impactful implementation of a culturally affirming pre-calculus task, "Being Hispanic: Heritage or Self-Perception?" We observed a sustained engagement by the students and the teacher with the mathematics and the social context within the task. Through an analysis of the classroom discussions and the teacher's reflection on the lesson, we identified a new critical consciousness resource (CCR) called criticality of racialized labels. Our findings highlight the moments and ways in which this CCR was activated. We propose that synergy of the teacher's own resources and excellence, the culturally affirming curriculum, and an anti-deficit math workshop setting contribute to critically conscious engagement with mathematics. Such engagement promotes meaningful experiences for students and teachers thus supporting a more just and equitable mathematics education.

Keywords: Equity, Diversity, and Inclusion; Curriculum; Culturally Relevant Pedagogy; Precalculus

The question of what is meaningful to teachers and students while engaging with people and curriculum in mathematics classrooms is valuable for unpacking what the future of mathematics classrooms looks like. As culturally relevant (Ladson-Billings, 1995), responsive (Aguirre & Zavala, 2013), and affirming (Lozano, 2023) mathematics teaching practices continues to be a sought-after approach to creating more meaningful experiences for all students, in particular traditionally minoritized students, there is a need for growing our understanding of *why* and *how* these practices make an impact in the lives of students and teachers.

In this case study, we describe how critically conscious engagement with mathematics emerged and was sustained by a college instructor, Sarah, (Author 4), at a large public Hispanic Serving Institution (HSI) in the Southwestern U.S. By illustrating what the classroom experience was and how it impacted Sarah and possibly her students, we elucidate specific factors that contribute to shaping the critically conscious engagement in the classroom, and afterwards.

Our analysis centers around one Culturally Affirming (CulturA; Lozano, 2023) precalculus task implemented by Sarah during a weeklong summer bridge program. We analyze how the mathematics and the context of the "Being Hispanic: Heritage or Self-Perception?" (*Being Hispanic*) task, as well as its setting and implementation contributed to the emergence of critically conscious engagement. This window into a Culturally Affirming (CulturA; Lozano, 2023) mathematics classroom serves as a possible model for intentionally co-creating with students critically conscious mathematics learning experiences that transcend the walls of the classroom and the moments when a lesson was enacted.

Prior studies identified opportunities and challenges linked to implementations of mathematics curriculum promoting critically conscious mathematics engagement. Some prior work acknowledges potential “pitfalls” (Harper, 2019) in implementing Teaching Mathematics for Social Justice (TMSJ) curriculum (e.g., Bartell, 2013; Brantlinger, 2013). Others (e.g., Gutstein, 2016; 2003) portray the “promise” (Harper, 2019) of these curricula by highlighting how students were able to see themselves in the curriculum and use it to “read and write the world with mathematics” (Gutstein, 2003). These studies provide a starting point for our discussion of what an impactful implementation of CulturA curriculum entails, by identifying some key contributing factors arising in this implementation of the Being Hispanic task.

The purpose of this paper is to offer a proof-of-concept for how an impactful implementation of a culturally responsive mathematics curriculum can support the co-creation of critically conscious engagement by valuing, eliciting, and building upon students’ voices. This study also contributes to understanding the role curricula like CulturA can play in activating specific critical consciousness resources (CCRs; Witt, Lozano & Anhalt, in press), and the role these can have both within and beyond the classroom. We focus our investigation on the question: *How might a culturally responsive mathematics task engage students and teachers’ critical consciousness?*

Frameworks

CulturA

The task featured in this study was created through Project (blind for review) along with over 80 others designed to enhance relatability and connection through robust, authentic mathematics in the core precalculus curriculum (Lozano, 2023). We operationalize culturally responsive mathematics teaching (CRMT; del Rosario Zavala & Aguirre, 2023) through the implementation of CulturA curriculum. CulturA is a place-based, affirming curriculum mirroring the identities and strengths of the local area by, “…center[ing] cultures, peoples, and identities traditionally absent from standard curricular content, keeping an asset-based lens” (Lozano, 2023) and thus embodies the three strands of CRMT described by del Rosario Zavala and Aguirre (2023). By design, CulturA aims to center knowledges and identities through its tenet of affirming identity and strengths. It also promotes rigor and support by being authentic while also being mathematically robust. Finally, in being place-based, CulturA makes space for leveraging local students’ everyday experiences to enable power and participation.

Critical Consciousness Resources and Critically Conscious Engagement

Developing critical consciousness (CC) with students is one of the three core components of culturally relevant pedagogy (Ladson-Billings, 1995). This goal is vitally important to equipping future generations with the agency to initiate change in their lives and their communities, also in addition to its being associated with various positive well-being outcomes (Jemal, 2017). In this study, we take the perspective that ideas abstracted from our experiences, identities and backgrounds (see also Jemal, 2017; Watts, Diemer, & Voight, 2011; Freire, 1970) inform whether and how people critically engage in affecting change in the world around them. We define *critical consciousness resources* (CCR), “ideas that elevate a person’s awareness or sense of agency to ‘engage the world and others critically’ (Ladson-Billings, 1995, p. 162)” (Witt, Lozano, & Anhalt, in press). We are interested in moments when these CCRs are made public, collectively shaped, and possibly transformed. We describe these moments as critically conscious engagement in mathematics which can lead to increased awareness and agency in social issues. In general, the activation of CCRs stems from perceiving affordances and constraints of present situations (e.g., within CulturA tasks) that may relate to sociopolitical contexts. Over 10 different CCRs were identified previously, and Witt, Lozano, and Anhalt (in press) documented nine prospective mathematics teachers eliciting these resources while engaging with the CulturA curriculum. We hypothesized that investigating CCRs can reveal teachers' strengths and shape the student experience in specific ways. The present study identifies and examines one new CCR, critically of racialized labels, activated through engagement in the Being Hispanic task.

Methods

Data Context and Sources

The data from this study comes from a professional development project for university instructors with a focus on developing anti-deficit teaching with minoritized students (Adiredja, Civil, & Jarnutowski, 2024). One component of this project is a weeklong summer bridge program for local high school students in the Southwest U.S. to transition into the local university. Also inspired by Ladson-Billing's (1995) notion of culturally relevant pedagogy, students' voice and inquiry become the focus in the teaching as participants also engage in critical conversations about race, gender and mathematics.

Sarah was an instructor participant in the project and implemented five CulturA tasks in her pre-calculus section. She had six students in the section, all of whom identified as Hispanic. In the classroom, the teacher sat shoulder to shoulder in a circle with students, at times holding a portable whiteboard to write down and share ideas that were raised during the discussion. Students had on average 5-10 minutes to work on a task, then she would ask students for their thoughts and possible solutions or strategies.

Sarah voluntarily brought up the Being Hispanic task in a debrief discussion with Authors 2 and 3 as "the most interesting because [she] kept thinking about it." Figure 1 shows the entire task, but the first three of the seven questions are most relevant to this study. Her classroom implementation and a debrief/reflection interview occurred two weeks afterwards were video recorded. The transcripts for these two recordings are the main two data sources for this paper.

Being Hispanic: Heritage or Self-Perception?

People living the in United States (US) who acknowledge having Hispanic heritage may or may not self-identify as Hispanic. According to 2017 survey[2], the percentage of US adults with Hispanic heritage who self-identify as Hispanic, H, can be thought of as a function, f, of generation, g, where $g = 1$ means being "first-generation Hispanic," or being born in a Hispanic country, $g = 2$ means being "second generation Hispanic," or having parents who were foreign-born, and so on.

(a) What does the expression $f(3) = 77$ mean in every day terms?

(b) According to survey data, $f(1) > f(2)$. What does this expression tell us in everyday terms?

(c) Suppose f is an increasing function of generation. What would this mean in every day terms? What would it mean for f to be a decreasing function of generation? Do you expect f to be increasing, decreasing, or neither? Explain.

(d) Given the context of this problem, which of the following could be the domain of f? Identify all possible answers and explain.

(I) $[1, 4]$ **(II)** $[0, 10]$ **(III)** $\{1, 2, 3, 4, 5\}$

(IV) $[0, \infty)$ **(V)** $\{0, 1, 2, 3, 4\}$

(e) Tables 1 and 2 are based on survey data. One table shows values of f. The other shows the percentage of US adults with Hispanic heritage who self-identify as non-Hispanic, N, as a function of g. Which table is which? Identify each by filling in the name of the dependent variable.

g	1	2	3	4
	3	8	23	50

Table 1

g	1	2	3	4
	97	92	77	50

Table 2

(f) The function represented in Table 1 increases at an increasing rate. Why is this true? What does this mean in terms of generations and Hispanic self-perception?

[1]Work funded in part by Project Outreach FAMILIA,US DOE, Title V Award P031S20008
[2]2017 Pew Research Center, Report: Hispanic Identity Fades Across Generations as Immigrant Connections Fall Away. Retrieved from: pewresearch.org/hispanic/2017/12/20/

Figure 1. "Being Hispanic: Heritage or Self-Perception?" [reprinted with permission from Lozano et al. 2021]

Analytic Approach

We began our analysis by identifying instances when the teacher (Sarah) identified with the task from the curriculum, or what we call "I-statements" (e.g., "I keep thinking about [the task]"). This approach was similar to that of Witt et al. (in press) but was adapted to better suit the nature of our current data sources. We coded for this type of statement (I statements) and parsed out those statements as they are related to issues of identity, power, access, and achievement (see also Gutierrez, 2009) and relative to the perceived level of agentic disposition they convey. These statements can offer insights into ways that teachers are provoked by or intrigued by the issues raised in the task, whether it is mathematically or socio-politically. In the current paper, this pointed us to the Sarah's implementation of the Being Hispanic task.

We then conducted a thematic analysis (Braun & Clarke, 2006) to identify sociopolitical theme(s) that were raised during the class and interview discussions. We examine the extent that these themes coalesced into a critical consciousness resource. We then returned to the full transcript of the recordings to identify times when this CCR was activated during the classroom implementation and reflection. We situated these moments in context to better understand what may have prompted these ideas to emerge. For more details about how to identify CCRs and their associated contexts see Witt et al (in press).

Findings

Criticality of Racialized Labels

The Being Hispanic task offered an opportunity for Sarah and the students to draw from their personal experience to make sense of the mathematics in the task and prompted a rich discussion about heritage. Our analysis revealed a broader theme related to the students and the teacher questioning, describing, and contrasting racial and ethnic identities. For example, one of the students, José, after realizing that the graph of f was decreasing in the task, questioned what people would identify as if they identified less with Hispanic. This moment in class also led to Sarah discussing the differences between Latina, Chicana, and Hispanic that came up when talking to a friend. We identify this broader theme as a critical consciousness resource called criticality of racialized labels.

This new CCR, criticality of racialized labels, was made explicit in the transcripts in five key moments across two interactional contexts (during Sarah's reflection interview and during Sarah's classroom implementation of the Being Hispanic Task). Three moments occurred during her interview reflection and two during implementation of the task. After sharing the details of these moments, we summarize the ways by which the CCR was activated as a way to operationalize critically conscious engagement.

Criticality of Racialized Labels in Sarah's Reflection

Moment 1. In the quote below, Sarah is recalling how José's question expanded the classroom discussion into one about how a decrease in people self-identifying as Hispanic suggests an assimilation of Hispanic identities occurring as generations go on.

> "I'm really trying to quote [José] as best as I can, [paraphrasing José]'but then you're correlating an ethnic identity, and replacing it with a nat–, like a country identity, which are not the same thing…what is then an American identity? Because it's not ethnic, but if it's not ethnic, is it kind of secretly white—in its ethnicity? And are you just kind of all melting into whiteness?'"

Moment 2. Seconds later Sarah described how her conversations in class became prominent in her conversations with friends. She continues,

> "....a friend was reading poetry by a leading Hispanic author, maybe poet?…[the author] identifies as Latina, or Chicana but not Hispanic, because it's the word that the community didn't make and I had never heard that before. So I keep having conversations as a result of that–what my students said, which was so, man, it was really cool."

Moment 3. Approximately 15 minutes later, Sarah was asked how she felt navigating sensitive topics with her students. She states,

> "…It was intimidating in the best way possible…seeing his–the gears turning and also the disappoint–like the sadness in his face of what it would mean then to lose cultural identity in

favor of white culture...I was like, Oh, this is a lot of feelings in this group. But also man, you said something really profound."

These moments show how this new CCR was evident when she recalled her students' words (Moment 1), when she describes how the conversation extended into her personal life (Moment 2), and when she reflected on how she felt about facilitating this conversation (Moment 3).

Criticality of Racialized Labels in Sarah's Implementation

Moment 4. During her implementation, as Author 4 and her students collectively interpreted what the Being Hispanic task must mean in everyday terms and the following exchanges took place.

> "Sarah (S): So, do we think it's going to be increasing or decreasing? What do you say?
> [inaudible from student]
> S: Well, no, in general, do you think people will [inaudible], generation-wise get away from it, the more or less you identify as Hispanic?"
> [multiple students respond, some say "More", some say "Less"]
> S: I don't know, it's it's completely subjective, what do you think?
> Student: Well, from (b), [inaudible] it's already less,
> S: Somebody picked up on context clues! So yeah, would that be a decreasing or increasing function?
> Student: Decreasing.
> S: Decreasing. So the further you get away, you identify less?
> [nods from students]
> José(J): [inaudible] Then what do you identify more as?
> S: What if you identify more?
> J: Well, what replaces it?
> S: Yeah! Yeah, what *do* you identify more as then?
> Student: American
> J: How does a genealogical background get replaced by a national [background]?
> S: Yeah, that's a really good question my friend.
> Student: What did he ask?
> J: So like Hispanic is like genealogical, [inaudible] but national is like a country?
> S: So an ethnic root versus a country identity. Well yeah, and then you also wonder at what point do you also lose track of your family as to what the heck the history of it was. How many generations does it take to be like, 'I don't know anymore!'"

This exchange initiated a broader discussion where other racial identities were discussed. José continued, *"But then like my question is, is the default that when you lose [said identity], you just become white?"*

Moment 5. Sarah then extended the conversation to how "hyphenated" identities (e.g., Mexican-American, Italian-American, African-American, etc.) may change over time and made a connection to what she knows regarding the term "Black."

> "Yeah, why is the default just white? I mean no one's ever asked me where I'm really from…despite the fact that my mom is-I'm actually first gen on one side, but no one's asked

> and you're very right, because I am white and so people won't ask…No one would call me English-American, I've never heard of one. But also, what happens with an African-American? … from what I've read and what I've been told, is they're reclaiming the word Black because, it does describe the answer that, you don't know, because you couldn't know because it was taken from you…"

These moments showcase how this CCR was initiated by José's engagement with the task (Moment 4) and as a result an opportunity for critically conscious engagement occurred in students and the teacher. These episodes also show how Sarah's affirming disposition and actions in the classroom allowed for her to inform her students of related ways in which racialized labels are critiqued (Moment 5).

Discussion

We argue that an impactful implementation of the CulturA task, Being Hispanic, supported the emergence of critically conscious engagement centered around the CCR, *criticality of racialized labels*. This engagement was initiated by a student, José, noticing and questioning the implications of what it means for self-identification as Hispanic to be a decreasing function of generation (see Figure 1). This prompts him to wonder whether this implies that people who are choosing to adopt a national identity, such as American, are then trending towards "becoming white." This supports a theme in Harper's (2019) qualitative metasynthesis of TMSJ research highlighting,

> "Four cases (Aguirre et al., 2013; Bartell, 2013; Turner & Font Strawhun, 2013; Turner et al., 2009) provided evidence that the students themselves focused on race and racism in relation to the social justice topic without direct prompting from the teacher…" (p. 286).

This is impactful because race and racism have the potential to emerge in many mathematics tasks coming from authentic contexts (e.g., Gutstein, 2003; Lozano 2023; Turner et al., 2022) and the ability to navigate these conversations in a safe and affirming way presents the potential for stronger relationships with students.

José's wondering and Sarah affirming his perspective enabled him to share with the rest of the class some of his background knowledge on the nuances between national and "genealogical" identities. This conversation appears to have been meaningful to Sarah in such a way that she felt comfortable discussing this idea with friends who then informed her more about the nuances between Hispanic, Latina, and Chicana identities. Thus, the critical consciousness engagement we describe extended through the walls of the classroom and instilled a sense of *critical awareness and agency* (Witt et al., in press) with respect to criticality of racialized labels.

When Sarah describes the conversation as, "intimidating in the best way possible," and "really cool" José's words as "poignant" and "profound," one could argue that the critically conscious engagement may have been supported by an *anti-deficit perspective* (Adiredja, 2019) that she embodies and is in accordance with CulturA's *affirming of identity and strengths* tenet (Lozano, 2023). This is evident in the way she frames the interaction in a positive light, focusing on student strengths and the value of this discussion. By acknowledging the care that one needs to take navigating conversations about race and ethnicity and the value of José's voice, Sarah is bringing an awareness of the sociopolitical context of mathematics spaces inhabited by historically minoritized students (Gutiérrez, 2013). In addition to this awareness, one could argue that critically conscious engagement afforded the opportunity for emotional connections to be made between the teacher and students which is a key aim of the curriculum.

Many might argue that our intuition leads us to believe that teaching can "feed one's soul." Teachers often reflect on how they learn just as much from their students as their students learn from them, but additionally in this case, we see how the CulturA curriculum coupled with the professional development setting accentuated the teacher's strengths as an affirming person, allowing her to leverage student thinking to co-construct an impactful space for learning. This was impactful to Sarah's self-reported learning about the nuances of Hispanic identities but also potentially impactful to her students who were encouraged to critically interpret the implications of the mathematics in the task. This vision for an experience that encourages an introspection that grows from thinking critically about mathematics is one that may be critically important for building positive dispositions towards one's mathematical and racial identity.

Our initial examination of this experience yielded *criticality of racialized labels*, being a salient theme in the teacher's experience. Extensions of this work may more carefully examine how this work may connect with other established framings of equity in mathematics education (e.g., Aguire, Mayfield-Ingram, and Martin; 2024). Given the prominence of Hispanic identity in the task and analysis above, future work may benefit from making stronger connections to a LatCrit framing (e.g., Rolón-Dow & Davison, 2021; Solórzano & Yosso, 2001). This case may be analyzed trough a *microaffirmations* (Rolón-Dow & Davison, 2021) lens or considered a counter-story (Solórzano & Yosso, 2001) to narratives regarding the lack of enthusiasm that marginalized students may experience when engaging with critical mathematics curriculum (e.g., Brantlinger, 2013).

Conclusion

Given the potential "pitfalls" (Harper, 2019) of teaching a critically conscious mathematics curriculum (e.g., Bartell, 2013; Brantlinger, 2013), we present this implementation of the CulturA task as a counter story to those that have encountered issues such as: problematic negotiation of mathematics and social justice goals (Bartell, 2013), and resistance from students in having racial and political discussions in class (Brantlinger, 2013). We presented how engagement with CulturA curriculum, the affirming implementation of it, and the flexible setting, served as resources for a new CCR to be activated and thus critically conscious engagement. Further examinations into similar curricula, settings, and implementations may center the critical potential of other mathematics tasks and the role that professional development fostering an anti-deficit perspective plays in supporting the development of personal connections between teachers, students, mathematics, and their lives.

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HIDDEN STORIES: USING NARRATIVE INQUIRY TO INVESTIGATE THE EXPERIENCES OF WOMEN IN INQUIRY-BASED, PROOF-BASED COURSES

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Keywords: Research Methods, Gender, Undergraduate Education, Equity, Inclusion, and Diversity

While there is evidence to suggest that inquiry-based instruction leads to positive student outcomes (Freeman et al., 2014; Laursen et al., 2014), there have been several studies that state otherwise (Johnson et al., 2020; Reinholz et al., 2022). Thus, inquiry learning does not guarantee equitable student outcomes. Therefore, it is critical to identify the classroom conditions and aspects of inquiry that can impede or bolster equitable outcomes. Thus far, much of the research on student outcomes in inquiry classrooms focuses on achievement results (e.g., grades) or survey-based affective measures rather than students' lived experiences (Melhuish et al., 2022). Therefore, it is imperative to ascertain methods for delving deeper into the human experience within inquiry courses.

Purpose & Methodological Perspective

This poster demonstrates how narrative inquiry can help illuminate women's experiences in inquiry-based courses. Understanding women's experiences may reveal conditions and factors that detract from or support equitable outcomes. I answer the research question: *How might narrative inquiry provide insight into women's experiences in an inquiry-based, proof-based course?* Narrative inquiry has been gaining popularity within mathematics education research. Much of the mathematics education research incorporating narrative inquiry involves accentuating racialized, gendered, or religious identities (Allaire, 2018; McGee, 2011), possibly in hopes of revealing the stories of those whose voices generally go unheard. Narrative inquiry is a qualitative research methodology that focuses on studying and understanding the lived experiences of participants through the analysis of narratives or stories (Clandinin & Connelly, 2004). These narratives provide rich data that reveal how individuals perceive and make sense of their experiences, identities, and the social and cultural contexts in which they are situated. Hence, I employ narrative inquiry to gain insight into women's experiences in inquiry-based, proof-based courses.

Study Context

The participants are four undergraduate women across two inquiry-based proof-based courses taught in Fall 2023 and Spring 2024. Data collected includes two student interviews and weekly journal submissions. Additionally, weekly classroom observations were conducted to witness the classroom climate and assess the nature of the inquiry implemented.

Results

The four female participants' narratives varied considerably, yet each elucidated their experiences in their inquiry-based, proof-based course. Therefore, narrative inquiry is an appropriate methodology to answer the call by many (e.g., Adiredja, 2017) for additional research on equity issues in undergraduate education, particularly investigating perceptions of their inquiry learning experiences beyond survey results and achievement outcomes.

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PLACE MATTERS: MAPPING CONSEQUENTIAL GEOGRAPHIES THROUGH PLACE INQUIRY AND SPATIAL METHODS IN BILINGUAL MATHEMATICS EDUCATION

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The spaces we inhabit provide the context in which we learn about ourselves and make sense of who we are; they shape our identities, our relationship with others, and our perspectives of the world. We introduce Mathematics Education Journey Mapping as a critical methodology to center teachers as co-designers in the educational design and research process. We first provide a socio-spatial analysis to mathematics education. Next, we introduce mathematics education journey mapping, a critical qualitative method adapted from Annamma's (2018) Education Journey Mapping. Finally, we analyze Mathematics Education Journey Maps from ten bilingual mathematics educators to highlight the "geography of opportunity" (Butler & Sinclair, 2020) within the context of language, mathematics, and identities and the physical and ideological sites of resistance that bilingual teachers and communities inhabit.

Keywords: Elementary School Education; Equity, Inclusion, and Diversity; Professional Development.

Introduction

The spaces we inhabit play an intricate part in our life experiences. They provide the context in which we learn about ourselves and make sense of who we are; they shape our identities, our relationship with others, and our perspectives of the world (e.g. Lefebvre, 1974). In this paper, we introduce mathematics education journey mapping as a critical methodology that centers bilingual teachers in the educational design and research process. We first provide a socio-spatial analysis to mathematics education. Next, we introduce Mathematics Education Journey Mapping, a critical qualitative method adapted from Annamma's (2018) Education Journey Mapping, to examine how location, locale, and sense of place shape one's mathematics experience as a learner and teacher. Finally, we analyze the Mathematics Education Journey Maps (MEJMs) of ten bilingual educators to showcase how place and space shape the "geography of opportunity" (Butler & Sinclair, 2020) for teacher and student experiences at the intersections of language, mathematics, and identities and the physical and ideological sites of oppression and resistance that bilingual teachers and their communities inhabit.

Theoretical Framework and Literature Review

Here, we draw on the definition of space by political geographer, John Agnew (1987), in which he describes it as spaces that people have made meaningful or have attached to in some way. Agnew's (1987) definition of space has three fundamental aspects: (1) a location - a specific fixed point in space where a place exists, (2) a locale - the material setting in which people conduct their lives at that location, and (3) a sense of place - the meaning attached to a particular location or locale. We build on Agnew's (1987) definition of space, alongside Soja's (2010) concept of "spatial justice" to highlight the connection between space, power, and resistance. Justice and injustice have spatial expressions or, what Soja (2010) has called, "consequential geography" (p.1). The concept of consequential geography highlights how justice and injustice shape - and are shaped by - the localized set of changing social, political, and historical conditions. Thus, the pursuit of educational justice requires a critical understanding and examination of space not as a background to our social lives but as active agents in shaping lived experiences and institutions, including mathematics education.

There has been a growing call for a spatial turn in mathematics education (Larnell & Bullock, 2018; Rubel & Nichole, 2020; Poling & Weiland, 2020). This body of work draws from scholarship in critical geography, urban education, and critical mathematics education to examine the role geography (space) have to generate, sustain, and disrupt inequalities. Larnell & Bullock (2018) offered a theoretical framework for thinking about urban mathematics education scholarship with an explicit focus on spatial logic in addition to the sociopolitical. Rubel and Nichole (2020) draw from frameworks of place, including indigenous ways of knowing, to include four thematic categories "geographies of human opportunity, mapping, human mobility, and land relations and obligations" (p. 6). Recent work on space highlights the potential to teach mathematics for spatial justice or using mathematics to identify power relations in and through space. For example, Rubel and colleagues (2016) investigated issues of social inequalities in New York City math classrooms by interrogating the structures of the lottery and paycheck loans. Poling & Weiland (2020) showcase the use of Common Online Data Analysis Platform (CODAP) and data from Public Use Microdata Areas (PUMA) to explore the "geographies of opportunities"that shape geographic, economic, and educational development across spatial and ideological boundaries, such as cities and suburbs. These studies highlight the potential in examining the arrangements of spatial geography and the distribution of social opportunity as well as levels of scale (e.g. classroom, school, state, etc.) in the scope of spatial consideration. Our project builds from this body of work that argues for considering the spatial dimension of mathematics education and extends the work to consider how the examination of space and the use of mapping can serve as a methodological and pedagogical tool for teachers to examine consequential geographies – the dynamic role that space plays in shaping justice and injustice – require critical examination since social processes and space influence each other (Annamma, 2018; Soja, 2010).

Research Approach

This project builds on an existing partnership in a dual language school in Texas. With funding from the Elementary and Secondary School Emergency Relief, Texas schools were encouraged to address educational inequities through tier three pull-out mathematics interventions consisting of drills of previous grade level skills to "catch up." School leaders noticed disproportionate representation of Black, Latinx, and refugee students from Afghanistan and Central America in the pull-out intervention—and the intervention was not leading to improved outcomes. Building on the belief that competence should be presumed for every student, the burden was on us - as educators and school leaders - to transgress and dismantle curriculum, pedagogy, and assessment practices that were leading to labeling, sorting, and separating of students across race, language, and class. The hope was to identify change within classrooms and also within the research process itself. Building from community engaged research (Warren et al., 2018), this project uses a co-design approach (Ishimaru et al., 2018), a participatory approach to designing solutions in which the school community are co-designers in the collaborative research inquiry and design for educational change. The project follows Ishimaru and colleagues' (2018) four stage co-design process for collaborative research inquiry: a) relationship building & theorizing; b) designing/developing tools to support new relationships and theories of change; c) enacting theories and practices in classrooms and schools; and d) analyzing and reflecting on our process for continued learning and innovation.

Partnership activities require careful attention to historicity, power, and relationality (Warren et al., 2019). The Mathematics Education Journey Mapping served as a tool for the second stage of the co-design process using mapping to collectively develop theories for change that leveraged the historical experiences and unique understandings that we each bring to the project. The research team consisted of three bilingual teachers (Monique, Anath, and Patty), the two school math coaches (Jorge and Elyse), school interventionist (Lydia), and four university researchers (Catalina, Lorie, Frankie, and Danica). Understanding the potential for mapping to stimulate participants' reflections about their identities across space and time, we adapted the techniques of Annamma's Education Journey Mapping to create a Mathematics Education Journey Mapping. Specifically, we wanted the codesign team of researchers and teachers to think critically individually and collectively about the ways in which mathematical opportunities, identities, and languages were shaped by space and place. We adapted the framework specific to mathematics teaching and learning with the following prompt:

> Use words, drawings, pictures to tell your personal journey of mathematics teaching and learning. Reflecting: What are four or more critical teaching and learning moments? Looking forward: What are your hopes and goals? Use different colors, images, the math tools, and resources to highlight experiences, show different feelings, and to share your story. These are just suggestions. Be as creative as you like. We will later share our journey / number line with each other.

After sharing the prompt visually, the first author read the prompt out loud and shared her own Math Education Journey Map (MEJM) understanding that identity exploration is a vulnerable process. We wanted to honor that vulnerability by opening up about our own process of identity formation as part of the process of mutual sense-making. After participants created their MEJMs, we each shared our narratives out loud to each other one-by-one. Participants had 15 minutes to work on their individual journey maps and group sharing lasted about 45 minutes.

Data Analysis

Primary data includes the video recorded session and its transcriptions as well as participants' drawn MEJMs. To analyze the data, we relied on Agnew's (1987) three fundamental aspects of space: location, the locale - the material setting at that particular location, and sense of place and alongside Soja's (2010) concept of spatial justice and followed the analytic procedures for mapping from Knigge and Cope's (2006), grounded visualization. To look for particular instances as well as general patterns in the MEJMs, we used Knigge and Cope's (2006) entry-level code categories for coding and solution driven theme-building – conditions, interactions, strategies and tactics, and consequences – to identify important connections within and between maps.

Findings

We organized the preliminary findings into three themes that aligned with the conceptual framework of space. Each theme connects to Agnew's (1987) three fundamental aspects of space: location, the material setting at that location, and sense of place, and Soja's (2010) concept of spatial justice- the connection between space, power, and resistance.

Although the prompt for the MEJMs focused on the participants' personal journey teaching and learning mathematics, math was only explicitly mentioned twice throughout the group sharing of MEJMs. Both mentions of math were in reference to math classes that served as crucial deciding points for future STEM careers. Instead, participants focused on aspects of language, identity, and culture while sharing their journey teaching and learning mathematics.

Findings 1: Systems of Power in Spaces

Throughout their MEJMs, participants identified the distribution of opportunity and the ways in which place not only contains patterns of (in)equality but shapes - and is shaped by - patterns of inequality and systems of power and domination. For instance, Lydia drew a number line in which 0 represented her birth (see Figure 1).

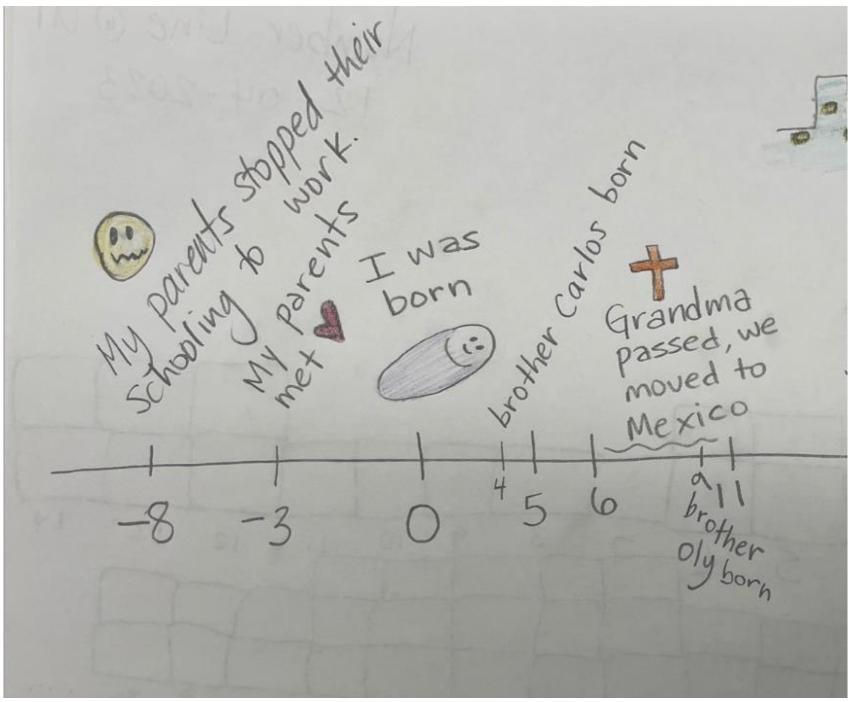

Figure 1: The start of Lydia's Mathematics Education Journey Map

Lydia starts her education journey map with her parents' educational journey:

> This is me, resembling Lydia, when she was born. I didn't put years; I decided to use integers on my timeline. So, I was born (which is 0 on the timeline) but eight years prior, my parents interrupted their schooling to start working and helping - helping their families. My mom is from a large family. They came from a larger home. So, she was always helping with the little ones. As soon as you could start working, she started working and then supporting the

family. My dad, his family migrated up north (to the U.S. to work). So from even before, like way before I was born, I think because they were not able to, you know, continue their education, I knew education – bilingual education and mathematics education - were important.

Lydia then continues to describe moving between the borders of Mexico and the U.S. throughout her K-12 schooling and how the movement between borders shape her bilingual and bicultural identity. Similarly, Anath discussed border crossing:

> I was born in the late 1900s. I did preschool and kindergarten in Mexico. That's where my family's from. Spanish was my first language. Then when I turned six, we moved to Texas and started first grade in the U.S. with basically no English. Back then, I mean, it's still the same now, but it's mostly sink or swim, right? You got it or you didn't. Yet, it really helped that I had Spanish, my first language, which was very well built and that helped me grasp on. Then, there was the pull-out when I was 8. It wasn't beneficial because it was like cat is gato. I already know this. That's also the start of the shift of speaking English only (in classrooms) because that's the only way you're allowed to learn. That's the only way you're gonna survive – English only. Then when I turned 10, that's when 9/11 happened. There was this huge shift again, towards people who don't speak English. Again, a force to speak English as much as you can, especially when you're crossing the borders, so they know that you belong here.

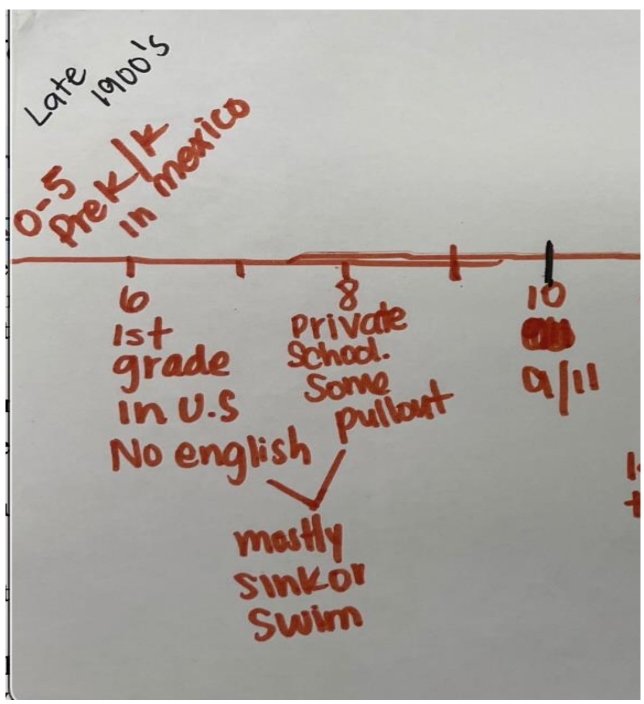

Figure 2: Anath's early school years on Mathematics Education Journey Map

Both Anath and Lydia described space as both physical and ideological (e.g. English hegemony; watered-down curriculum; and the role of compliance in mathematics classroom) and were naming the conditions and interactions within and across locations that perpetuated inequities and exclusion. Note how they both weaved and flowed between stories of their family, the classroom, to the state-national borders to highlight the ways which space is fluid and includes multiple levels of scale (e.g. classroom, school, state, nation). Soja (2005, 2010) describes the process of loosening of territorial boundaries created in terms of political power and cultural identity as processes of deterritorialization and reterritorialization, or debordering and re-bordering, as consequences of the globalization of capital, industry, and labor. This hybridization of spaces occurred throughout most of the teachers' stories in which they, their families, and their students are contending with an assortment of geopolitical, economic, cultural, and oppressive factors across state-sanctioned racialized borders separating countries, classrooms, and families.

Findings 2: Space as Sites of Resistance

A repeated theme across MEJMs was how educational spaces such as schools are not neutral. Boundary lines (e.g., city, state, nation) are traditionally thought of as political dimensions of maps. Yet, the teachers provided expansive notions of boundaries; they name out the boundaried aspects (e.g., English-only instruction; carceral schooling, compliance education) of mathematics education systems. Yet, they also used counter-cartographies to identify, tear down, and reimagine boundaries. Anath did this counter-cartography:

> I switched from psychology to education when I was 19. I thought I was going to teach in the English as a Second Language classroom. And then luckily, my cooperating teacher, she was like, why are you not teaching bilingual? You speak Spanish. And I was hesitant. I am glad she put me in a Spanish class. The class helped me reconnect with my Spanish and not be scared and hiding my Spanish, learning in Spanish and teaching other kids Spanish.

In this short description, Anath highlighted the collective nature of identification and disruption, naming a part of the re-boundarying in her mathematics educational journey that allowed her to honor herself. In mathematics education, where even access to one's native language is a privilege that is often withheld, most of the mathematics teachers' counter-cartographies demonstrated re-boundarying of their present, giving access to biliteracy and biculturalism for themselves and their students. Their responses challenge the neutrality in which mathematics education is often perceived as numbers only and decontextualizes from language and identities.

Findings 3: Space as Spatial, Historical, and Communal

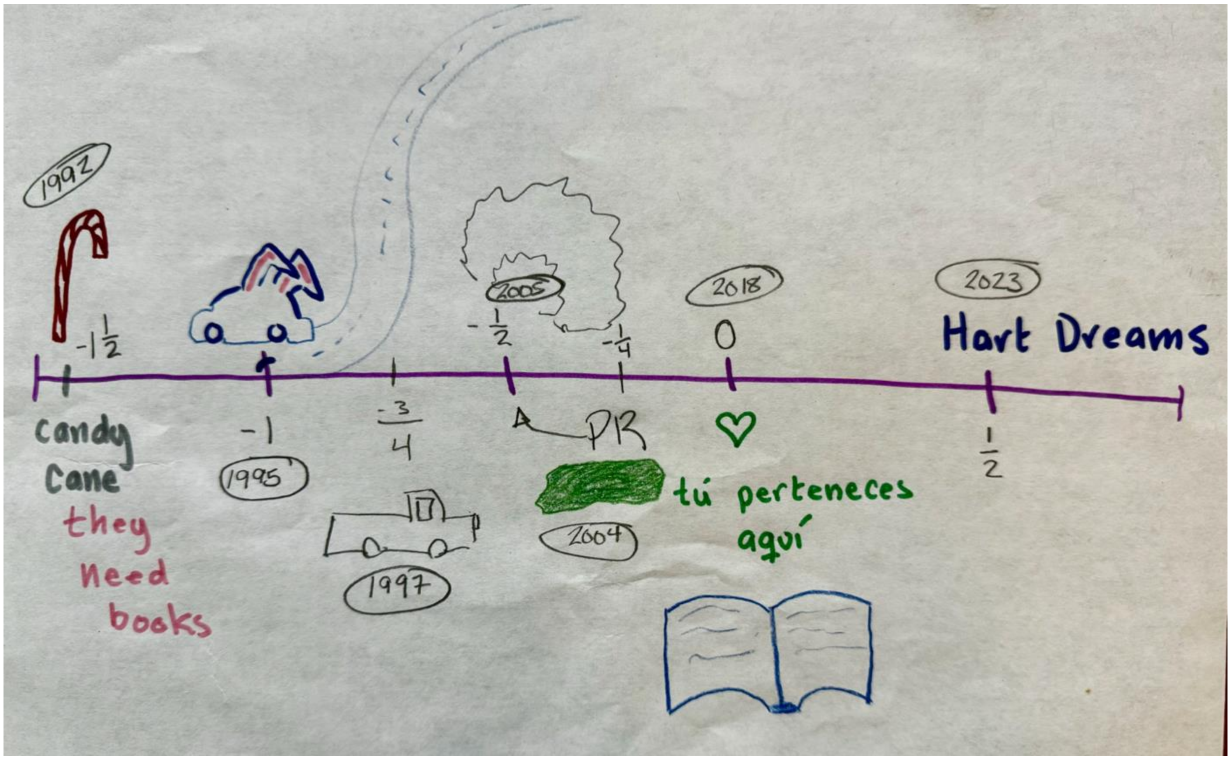

Figure 3: Frankie's Mathematics Education Journey Map

Frankie provides a timeline that marks 0 at "tú perteneces aquí." Below is their description of that moment.

> This is Race and Ethnic Relations (class). She's teaching us about the history of Texas. One of the first things she says in much more words —tú perteneces aquí— y en la manera en lo que lo hace, nos explica que la gente que vivió aquí… en este estado en lo que se conoce como tejas ahorita, la gente que ha vivido en México, siempre han viajado entre los dos libremente y siempre ha habido gente ahí, entonces la gente mexicana, o parte indígena, o latinoamericana, pertenecen aquí en estados unidos diga lo que diga la ley. That was probably the strongest moment that I knew that I have the right to be here. Now, do I have a right to be here? I don't know if I see that politically anymore. I feel like I have a responsibility to be here and to view my education differently. And to also when I speak with other people say that very same tú perteneces aquí, whenever I'm working with children or working with others.

Here, Frankie highlights the consequential geography of justice in which language justice is communal justice. Similar to Lefebvre (1996) arguments that all people, especially those from historically marginalized backgrounds, have a "right to the city," Frankie is calling on the right for linguistic justice in mathematics education.

Discussion

The project builds from a key premise that we - as an education community - arrive at better understandings and solutions to classroom and school-based inequities when those most impacted can influence key processes and decisions. Using the Mathematics Education Journey Map as a methodology allowed us to excavate individual and collective educational experiences in order to understand the particular historical experiences and unique understandings that the co-design team of teachers and researchers bring to the project. In this section, we briefly describe how these experiences can be utilized in the process of moving towards co-design professional learning spaces.

Attending educational justice as a personal endeavor, bilingual educators hold understandings of how dominant constructions of language within mathematics education perpetuate systems of oppression. Their personal experiences conveyed great cost and sacrifice to assimilate. Dominant narratives about newcomers, biliteracy, and English hegemony remain deeply pervasive in educational institutions, including mathematics education, that it can be difficult for even bilingual teachers to perceive its insidious effects. The bilingual teachers drew on their experiences to critique the centering of English as power and potentially challenge it in their own teaching.

Interactions within mathematics classrooms that reflected broader politics of transnationalism, immigration, and monoglossic language ideologies and policies emerged in the maps as a significant aspect of the material conditions of locations. The teachers described feelings of shame and othering and noted the immense power that teachers and schools had to impose a deviant identity upon them. Their MEJMs highlighted cultural tools such as standardized testing and remediation practices (e.g., labeling, separate classrooms) as stigmatizing. At the same time, they resisted these identities in the ways they authored their maps and stories. The mapping process of one's journey as students to teachers allowed participants to be understood as whole, resilient beings navigating difficult experiences and refusing to accept identities of deficiency. They authored their own identities by critiquing interactions that positioned bilingualism and multilingualism as less than and articulated the creation of safe spaces for themselves and their students. In doing so, they insisted upon their robust competencies, right to belonging, and their humanity.

The goal of the MEJMs was to provide a process for us to engage in discussion around theories of change. It was a space to identify our strengths, our goals, and our needs. These stories – our individual and collective stories – serve as a foundation to build together. As stated by Danica in explaining the last mark on her MEJM, "I just kind of put a pencil here because I feel like my story is not yet fully written." The struggle for spatial justice is not an individual endeavor but requires the exercise of collective power to reshape the process of ownership. The MEJMs as a critical methodological tool allowed individual and collective examination of the physical and political dimensions of educational experiences with the goal of learning through these mappings to restructure spaces of learning. Our next step is to bring in families into the co-design.

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ASSIMILATING TO BELONG: A CRITICAL REVIEW OF SENSE OF BELONGING RESEARCH IN MATHEMATICS EDUCATION

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The purpose of this paper is to offer a critical review of the research surrounding sense of belonging to mathematics through a lens that exposes assimilatory perspectives and uplifts the fluidity and beauty of students' ways of being. Guided by Anzaldúa's (1999) concept of living and being in the Borderlands, as well as the mestiza consciousness this review resulted in identifying two types of sense of belonging research: work that aims to alter the student to fit the current constraints defining academic mathematics and work that considers how the system can be changed to best suit the student as they are. Implications from this review are discussed by envisioning what future sense of belonging to mathematics research might look like that would disrupt whiteness in mathematics education and offer space for healing.

Keywords: Systemic Change, Equity, Inclusion and Diversity

In a now frequently cited study that set out to understand member (under)representation in mathematics courses, degrees, and careers, Good, Rattan, and Dweck (2012) identified *sense of belonging to math* as a significant predictor to women's desire to pursue mathematics, academically and professionally. *Sense of belonging to math* was defined as "one's personal belief that one is an *accepted* [emphasis added] member of an academic community [mathematics] whose presence and contributions are valued" (p. 711). Although this study focuses on women, other sense of belonging studies echo the implications that the mathematics classroom can transform into a site for fostering belonging. In the literature on sense of belonging to mathematics, many researchers point to their studies to recommend changes within mathematics classrooms to encourage a greater sense of belonging for marginalized students which would in turn lead to greater representation in mathematics as an academic discipline (e.g., Barbieri & Miller-Cotto, 2021; Bjorklund, 2019; Jaworski & Walker, 2023; Rattan et al., 2012). So logically, starting from Good et al.'s (2012) definition, if we were able to increase a student's sense of belonging to mathematics, a feeling defined by *acceptance*, then we would see an increase in historically marginalized students pursuing mathematics courses, degrees, and careers. Academic mathematics however, is "rooted in appeals to White nationality and White benevolence" and "is a colonizing form of education" (Martin, 2015, p. 21). So, is acceptance truly what these students desire? Acceptance into the field that is used as a weapon (one of the many) to maintain and perpetuate white supremacy (Marchant et al., 2023)? Even if we were to accept this definition of sense of belonging, are there changes that can be made within the four walls of a classroom that can actually combat the oppressive intentions of academic mathematics?

In discussing belonging research, Rochelle Gutiérrez (2022) questions "when or how do we capture the complexity and contradictions in belonging research?" (p. 382). I believe part of the contradiction in belonging research is that either the goal of the work is to find ways to alter the students' perspective (see growth mindset) on mathematics or somehow alter an instructional component within the mathematics classroom that offers them a place of belonging to mathematics. Both goals still attempt to assimilate students as they want children to value a very specific definition of mathematics. This is not to claim instructional practices like cooperative learning, inquiry-based learning, or student-centered instruction are harmful to students, but rather, can these practices lead to a student feeling an increased sense of belonging to their mathematics classroom when the mathematics valued in this space has been curated by whiteness (Battey & Levya, 2016).

The purpose of this paper is to offer a critical review of the research surrounding sense of belonging to mathematics through a lens that exposes assimilatory perspectives and uplifts the fluidity and beauty of students' ways of being. This review is guided by Anzaldúa's (1999) concept of living and being in the Borderlands, as well as the *mestiza consciousness*. Drawing from her experience living between the Mexico-United States border, Anzaldúa theorizes, "an emotional Borderlands which can be found anywhere where there are different kinds of people coming together and occupying the same space or where there are spaces that are sort of hemmed in by these larger groups of people" (Urch et al., 1995, p. 77). She capitalizes Borderlands to distinguish this emotional state from the physical borderlands of the Mexico-United States border. I provide more detail about these concepts in my theoretical framework, but wish to note here that throughout this study, I refer to all students who do not feel they belong to mathematics, as living within the Borderlands. In doing so, I am aim to be inclusive to the diverse group that the racist, sexist, and ableist structures of white supremacy oppress, while also not essentializing the experience of marginalized people. I claim that students who do not feel a sense of belonging to academic mathematics do so for good reason. These students live within the Borderlands of places they are told they belong, navigating the push and pull between the varying worlds in which they experience and perceive mathematics.

Theoretical Framework

"If we have been gagged and disempowered by theories, we can also be loosened and empowered by theories" (Anzaldúa, 1990, p. xxvi).

Borderlands and Positionality

This review is guided by Gloria Anzaldúa's *Borderlands/La Frontera: The New Mestiza* (1999). In this work, inspired by her experience as a Chicana, lesbian, and activist, Anzaldúa examines the in-between spaces of the physical, cultural, and spiritual realms. Living on the Mexico-United States border, she describes a "border culture" where there are "two worlds merging to form a third country" (Anzaldúa, 1999, p. 3). In this borderland, that feels neither fully American nor fully Mexican, "is a vague and undetermined place created by the emotional residue of an unnatural boundary. It is in a constant state of transition" (p. 3). Existing in the in between is not a split 50/50 identity. Rather it is a fluid connection to the various cultures and lands that the borderland runs between. These lands are more than just a physical space, as they also represent something that is spiritual, cultural, and part of her identity. A capitalized, Borderlands, is used when referencing the spiritual, emotional, and cultural experience of being in-between spaces, identities, and categories (Anzaldúa & Keating, 2015).

Anzaldúa (1999) not only defines the Borderlands as a place of being but demands that its residents be seen as they are, in this place; no longer pushed or pulled into categories that are uncomfortable and demand an assimilation into a permanent identity. Referring to herself, and other Borderland people, Anzaldúa (1999) explains that we, “no longer feel that we need to beg entrance [into society], that we need always to make the first overture,” but rather “we ask to be met halfway,” on the *puente* (bridge) where acceptance and understanding reside (p. x). The existence of the *puente* resists the binary of us or them, in or out and instead allows for a middle ground that is not forcing labels or identities on anyone. *Mestiza consciousness* is understanding that the *puente* brings healing and that it is a place where transformation can occur and develop in a way that can have influence beyond the bridge. Anzaldúa further defines *mestiza consciousness* in *Borderlands/La Frontera: The New Mestiza* (1999),

> It is work that the soul performs. That focal point or fulcrum, that juncture where the *mestiza* stands, is where phenomena tend to collide. It is where the possibility of uniting all that is separate occurs. This assembly is not one where severed or separated pieces merely come together. Nor is it a balancing of opposing powers. In attempting to work a synthesis, the self has added a third element which is greater than the sum of its severed parts. That third element is a new consciousness–a mestiza consciousness…(p.85)

Through my experience, as a Chicana of indigenous descent in the field of mathematics, I fully accept and claim as fact that White supremacy is woven into the hierarchical structure of academic mathematics (Battey & Leyva, 2016; Martin, 2015; Martin, 2008; and many others). Those who do not fit the mold that academic mathematics serves live in the Borderlands, being pulled into assimilation to “succeed” in academic mathematics while simultaneously being pushed out because of their various identities that are positioned as “other.” This literature review is driven by the desire to not only recognize those experiencing and existing within these Borderlands but also to provide a path in future research for *mestiza consciousness*. This concept recognizes the pain that exists within the Borderlands but asserts that “its energy comes from continual creative motion that keeps breaking down the unitary aspect of each paradigm” (Anzaldúa, 1999, p. 85).

In my family, we often explain to people that “we didn’t cross the border, the border crossed us.” My ancestors are Genízaros, stripped of their tribal affiliation through enslavement and assimilation. As a result of generations of oppression and discrimination, assimilation became a means of survival. My sister and I were born into a lineage of borderland people but from the second we took our first breath we were pushed to find belonging within the United States culture of whiteness. Now, as a researcher in the field of mathematics education, one would assume my own sense of belonging to academic mathematics to be resoundingly positive. Although I have taught in the mathematics classroom and accomplished the highest academic achievements in this field as a student, I do not feel as though I belong to it. I do have an immensely strong connection to my identity as a mathematician, but so much of this connection stems from mathematical thought outside of academia and the relationships that mathematics has provided me. The mathematical thought that is behind every motion of my mother’s crochet hook connects to each stroke of hair color I painted on my client’s hair during my career as a hairstylist, which connects to each sprinkle of pepper and dash of allspice in my father’s secret family lasagna recipe. These are the types of mathematics that I identify with and feel as though I belong to and have a connection to, not logarithms and the rational root theorem. So, although I have fought through the battlefield of whiteness and stand on the other side of the border within the academic mathematics community, I do not feel as though I belong. I am alone amongst my peers in positioning my journey through mathematics as my healer, as something of beauty. I do not live on the borderlands my ancestors belong to, but I feel their embrace and wisdom forgiving me for abandoning our home. So, while I stand on the white side of the border, I have made a choice. Rather than ensuring I leave the door open for others in the Borderlands, I desire to help open other doors that lead to paths yet unknown. Paths that do not require distance from the body and home or a compromise to belonging.

Review of Research

Sense of belonging literature focuses on students who live within the Borderlands of belonging to their educational institution or to the STEM field as a whole. In reviewing the literature, there are two contradictory positions I found on how to increase a sense of belonging to mathematics. I will first review the most common theme in this literature, which is a focus on how to increase a sense of belonging within the current constraints of academic mathematics. This field of thought aims to alter the student to fit the system. As Gutierréz (2022) has pointed out, "when we study students and try to capture what prevents them from feeling they belong (and supports them to feel included), we are expressing our own desires that we want that for them" (p. 382). To put this more directly, this type of work approaches belonging by encouraging assimilation of Borderland inhabitants into the walls of academic mathematics without regard to where students already feel a sense of belonging. I will then review literature that takes a more critical approach to investigating a sense of belonging to mathematics and accepts Borderlands as a place of being for students, letting them exist how they are. This approach allows space for, and invites, a *mestiza consciousness* for students, instructors, and researchers in the field.

Assimilating to Belong

The following studies seek to understand a sense of belonging to mathematics without regard to how the narrow framing of what is valued in academic mathematics could be acting as a deterrent to those who do not seek assimilation into whiteness. This work envisions a future in this field without a *puente* (bridge) that could help embrace mathematical ways of knowing and being outside of what is defined as academic. That is, it assumes a permanence in this field which smothers the beauty, diversity, complexity, and spirituality that exist throughout mathematics.

Sense of belonging research has pointed to ethnicity/race as a predictor of sense of belonging in educational institutions and STEM as a field. Examining this connection, as early as middle school, Barbieri and Miller-Cotto (2021) look into the significance of a student's sense of belonging in relation to other self belief measures of self concept, importance, and entity view. In this study, they find that "not only is sense of belonging to mathematics a significant predictor of middle school students' learning, but it was the only significant predictor of the motivation and belief measures taken" (p. 7). These findings specifically call attention to underrepresented racial and ethnic minority students (URM), as these students "reported a markedly lower sense of belonging than non-URM students" (Barbieri & Miller-Cotto, 2021, p. 1). When investigating the cause for this racial gap in belonging, Morales-Chicas and Graham (2021) found that racial/ethnic makeup of a student's math class played a significant role in a student's sense of belonging. That is, "Latino and Black students in advanced math reported a greater sense of belonging in math when they perceived more similar racial/ethnic peers in class than in school"

(p. 9). These findings present major implications for districts that participate in math tracking that often results in Black and Latino students being underrepresented in advanced courses (Tyson, 2013).

Some sense of belonging studies focus on students of color who are succeeding academically in an effort to understand how to replicate their success. Strayhorn's (2008) study for example showed that there was a positive correlation between Latino college students earning a higher grade and their sense of belonging. But in a sample of 589 Latino and White students, descriptive statistics show that on average, Latino students receive lower grades in their courses than their White counterparts. To a similar result, Bjorklund (2019) found that "while more latinas/os are taking AP exams than in previous years, their passing rates have remained lower than other groups" (p. 109). So even when students of color are being 'pulled out of their Borderlands' and achieving some form of success, as defined by academic mathematics, they still remain othered in academia from their White counterparts. Furthermore, Bjorklund (2019) claims that the Latina/o participants in their study "leveraged their lack of sense of belonging to bolster their resilience and motivate themselves toward transformative resistance to succeed and prove others wrong" (p. 123). Although transformative resistance might seem like a positive step towards an equitable education for these Latino students, I wonder if we can create a system that allows students of color to succeed for themselves and their ancestors, rather than to prove their White counterparts wrong.

Results from a nine-country survey showed that school sense of belonging is a significant predictor of how students value advanced math courses (Smith et al., 2021). When surveying women in the United States, Thoman et al. (2014) found that women "feel pushed out of STEM when they feel a low sense of belonging" (p. 246). These results echo findings by Good et al. (2012), that show that a low sense of belonging for women in STEM predicts a decrease in desire to pursue math in the future. Women in math environments receive social messaging that women have a lower math ability than men, which further leads to low sense of belonging in math (Good et al., 2012). Rogers et al. (2021) attempts to investigate a more specific association with gender discrimination, school connectedness, and math/science achievement motivation. In a sample of 295 adolescent girls, they found that "gender discrimination was uniquely associated with girls' lower sense of school connectedness" (p.417) along with being a predictor of below average math achievement motivations (but not science).

Taking an intersectional approach, Rodriguez and Blaney (2021) investigate how Latina STEM majors develop their sense of belonging in academia. They find that "white and male classmates unjustly question the abilities and belonging of Latina students in STEM," and that their participants found other spaces on campus to feel supported (p. 452). In these studies, we see the struggle of Borderland women trying to live "between *los intersticios*, the space between the different worlds she inhabits" and being told this place cannot be home (Anzaldúa, 1999, p.32).

Belonging to Borderlands

In accepting the Borderlands as a place of being and a place of home, it is important to not examine how Borderland students succeed in mathematics in an effort to recruit into the field, but rather use this examination towards a futurity praxis, for "students and teachers to live out a future to which they want to belong" (Gutiérrez, 2022, p. 383). We need to highlight that when

Latina students pursue STEM degrees it is driven by their sense of belonging and commitment to their families and communities (Acevedo et al., 2021). Accepting those in the Borderlands also requires partnerships, *puentes.* In their study with Black women in college, Cook-Sather and Seay (2021), demonstrate how "pedagogical partnerships respect Black, female students' intellects; affirm their experiences and expertise; and make them feel they belong in conversations about teaching and learning" (p. 745). Positioning these women as experts of their own sense of belonging is key to acceptance in the field. Acevedo et al. (2021) also acknowledge changes required of the system to truly affect the sense of belonging to STEM. They recommend requiring an Ethnic Studies STEM course, "which confronts the issue of STEM reproducing white supremacy, whiteness and sexism" (p. 74).

Another way to accept Borderlands as a place of home is to recognize that the current system is not a place where Borderland people belong. I want to be clear that this is not to say that they do not belong because they *cannot* belong but rather that forcing a sense of belonging would be an act of assimilation and colonization. In examining how ableism affects students' sense of belonging in learning environments, Nieminen and Pesonen (2022) assert, "inclusive learning environment design cannot be a responsibility of individual teachers practicing 'pedagogies of care', as structural ableism needs to be challenged through systemic solutions from HE [Higher Education] institutions" (p. 2030). Furthermore, "while waiting for such radical changes, it might be more desirable – and safer – for disabled students not to belong to the learning environments" (p. 2031). In a similar argument, when investigating what effect Culturally Responsive Pedagogy might have for Australian students, researchers came to the conclusion that in order to actually improve sense of belonging, they would need to "conceptualize the role of education outside the desire to fit students into the mainstream culture of the school and its society" (Harrison & Skrebneva, 2020, p. 23). Stokes (2023) further highlights the problematic nature of most sense of belonging research as it represents "white supremacy as a normative construct for which people of color must adopt" (p. 30). Stokes continues by emphasizing a future that creates a new "us" which is in line with Anzaldúa's (1999) *mestiza consciousness*, where those in the Borderlands reject the pushing and pulling into systems in which they do not find belonging.

Envisioning a Future

This review of sense of belonging to mathematics research demonstrates the need to envision a future for mathematics education that centers students who are currently othered by academic mathematics and intentionally aim for a *mestiza consciousness* in this field. In order to actually address the member (under)representation in mathematics that Good et al. (2012) sought out to understand, we must emphasize "healing us," not "helping others," by targeting the whole system as opposed to just specific parts (Gutiérrez, 2022, p. 383).

Throughout this paper the word academic was placed in front of mathematics to emphasize that mathematics is so much more than what is valued as mathematical in academia. If we truly want students to find belonging in academic mathematics, we need to disrupt the hierarchy that whiteness has created in this field (Battey & Levya, 2016). As highlighted in the previous section researchers have begun this disruption, but we can build upon and continue this work by centering student perspectives. That is, rather than studying what negotiations or hurdles students must overcome in order to discover a sense of belonging to academic mathematics, instead respect youth as the experts of their own sense of belonging to guide us in defining this feeling.

Rather than research serving the current system it should serve Borderland students and bring their voice into the conversation. By engaging with students that live within the Borderlands of classroom mathematics and in examining sense of belonging while also rejecting a desire to recruit into a broken system, we can embrace a "relational understanding of knowing, existing, and healing that center values and ethics" (Gutiérrez, 2022, p. 381). Let us practice what we preach to our pre-service teachers and in professional developments and meet students where they are, on the *puente*.

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EXPLORING THE NARRATIVES OF WOMEN MATHEMATICS EDUCATORS THROUGH A DIALOGICAL SELF APPROACH

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We employed a dialogical self-approach to delve into the narratives of a group of seven women mathematics educators, including both graduates of the same doctoral program and their advisor. Through this lens, we gained insights into how we described ourselves, recounted personal experiences, and articulated our personal histories within the figured worlds. Three dialogical models emerged, including dialogues between past and present positions, dialogues between internal and external positions, and dialogues between self and societal norms. These models serve to offer insights into how intricate dynamics shape women mathematics educators' perceptions, interactions, and self-identities over time. By examining these dialogical interactions, we gain a deeper understanding of the multifaceted factors influencing the professional and personal development of women in mathematics education.

Keywords: Dialogical Self, Identity, Narratives, Women Mathematics Educator

Women mathematics educators often navigate the complex intersection of mathematics and education, experiencing a dynamic landscape where they simultaneously encounter both distance from and alignment with societal expectations (Zhou et al., 2023). On one hand, they are often stereotyped as extraordinary or exceptional, attributed to their significant involvement in mathematics. On the other hand, they also conform to societal expectations associated with the traditional female role as educators, nurturing individuals within society. This inherent contradiction shapes their professional journey, influencing their self-recognition. This study addresses the critical need to amplify the voices of women in mathematics education, highlighting the unique challenges and perspectives they bring. By examining these dialogical interactions, we gain a deeper understanding of the multifaceted factors influencing the professional and personal development of women in mathematics education, ultimately aiming to inform policies and practices that support their growth and retention in the field. Specifically, the following research question leads to the inquiry: *What dialogical models emerge to interpret the identity of women mathematics scholars over their life-long journey?*

Theoretical Perspectives

Building on Bakhtin's dialogism, Holland and colleagues (1998) articulated the dialogical self approach to elucidate individual identity through the act of authoring oneself within figured worlds (for an overview, see Holland et al., 1998). The concept of the dialogical self is characterized by its social nature, not merely involving interactions between an individual and external people but encompassing a multitude of voices within the self (Hermans, 2001). Hermans additionally (2013) articulates the dialogical self within educational settings, emphasizing that dialogues can occur between different positions within the self, emerge between internal and external positions, or take shape between two external positions of the self. Thus, the dialogical self-approach promotes internal and external discourse, providing educators with a lens for authoring figured worlds and narrating multiple, affordably accessible I-positions.

Akkerman and Meijer (2011) advocate for the adoption of a dialogical self-approach to investigate teacher identity, suggesting a threefold reconceptualization: viewing it as both unitary and multiple, continuous and discontinuous, and individual and social. So far, there are only a few studies in mathematics education which use a dialogical self approach as the analytic tool and theoretical framework to understand teachers' identity.

In Pipere and Micule (2014)'s exploration of three teachers' lifelong relationships with mathematics, emergent dialogical models in the teachers' mathematical identity, intertwined with perceptions of students' mathematical inclinations. This provides valuable insights for practical implementation of exploring mathematics educators' identity. Similarly, William (2011) delves into two mathematics teachers' narratives in the figured world of mathematics teaching and learning, and discovered that their cultural resources such as family, background, school, university, and teaching experiences, provide insight into how they author selves in the teaching mathematics. With the aim to explore the extent of reflexivity in shaping new identity spaces within mathematics, Solomon's (2012) study from the self-authored narratives of two undergraduate women in the figured world of mathematics, revealed the persistent challenge of altering the positioning of women in this field. Despite reflective efforts, the perception of mathematics as a gendered and predominantly masculine domain persists.

Drawing upon methodologies in the reviewed literature and responding to the call made by Lutovac and Kaasila (2018) for a more balanced psychosocial theoretical perspective in exploring identity in mathematics education that equally considers both individual and social dimensions, we have embraced the dialogical self approach to investigate the narratives the women mathematics educators.

Methods

In our study, ongoing monthly meetings of the group, spanning two semesters, have become a dynamic space for intentional narrative construction and self-authorship through carefully crafted prompts. First, we discuss the prompts, and then we spend 30-45 minutes writing our own narratives or stories to respond to them. Finally, we come back together to share our stories, which leads to an evolving inquiry in the group.

The prompts serve as catalysts for rich and dynamic internal and external dialogues. These prompts intricately weave dialogues within self and between self and others, fostering a holistic exploration of personal narratives. Internally, the prompts elicit dialogues within the individual, encouraging members to engage in reflective conversations with their past selves. As we delve into significant memories and episodes, these prompts facilitate an ongoing dialogue with the various I-positions that have shaped our identities across different life stages. Externally, the prompts spark dialogues among group members, creating a shared space for collective reflection and exploration. The prompts encourage open discussions, allowing us to share our diverse perspectives, experiences, and I-positions. Instead of taking on the role of critical friends as self-study (e.g., Hamilton & Pinnegar, 2000), the members primarily engage in active listening and understanding, fostering a strong sense of community and further promoting the group's collective narrative.

Fueled by these prompts, the monthly meetings provide a structured yet open platform for the group members to engage in a continuous process of self-reflection and narrative development. The individual narratives and recordings of group discussions serve as data sources for this exploration.

Findings from Interpretation of Dialogical Narratives

Exploring the question of what dialogical models emerged to interpret the identity of women mathematics scholars over their lifelong journey, we intentionally sought the dialogical voices from the narratives, who they talk to, what they talk of. Three dialogical models have emerged: Dialogues between Internal Past and Present Positions, Dialogues between Internal and External Positions, and Dialogues between Self and Societal Norms.

Dialogues between Internal Past and Present Positions

When addressing the prompts, we naturally initiated dialogues among our different I-positions, especially across the dimension of time. Engaging in a dialogue with our past selves from today's mathematics educator perspective can illuminate numerous challenging experiences, critical moments, and pivotal choices. Many past struggles can then be better explained and understood. Below is an example from Jill, sharing her nadir experience during her dissertation proposal defense with a reconciling perspective：

> I was not prepared for the high-stakes, high-pressure moment of my defense. I don’t remember any details about the meeting and any particular questions I was asked, but I sure do remember the feeling afterward - crushed, embarrassed, defensive, etc. I had passed, but with lots of critique and revisions and thinking to do… In hindsight, of course, I am grateful, but the memory is far from warm and fuzzy.

A different example from Hyunyi involves her revisiting herself as a 13-year-old girl to express her earlier thoughts on teaching and mathematics back then:

> When my mom asked me what I liked and what I wanted to be in the future. As a 13-year-old girl, this question made me think about experiences that had made me happy. I told my mom that my friends in the classroom asked me about their homework and test prep problems, and I felt happy when they thought my collaboration was helpful to them. I continued to explain to my mom that I want to be a literature or mathematics teacher. Since many of my classmates think math is more difficult than literature, I thought it might be better for me to become a math teacher as I would be able to help more people.

These dialogues explain the past challenges and perplexities and reconstruct experiences that align with current identities. By examining internal positions, past selves, and current selves, we aim to understand decision-making processes, such as the choice to pursue a PhD in mathematics education and a commitment to mathematics.

Dialogues between Internal and External Positions

As women mathematics educators recall experiences imprinted in their memories, dialogues emerge between themselves and others. In the example below, Brooke expresses surprise at being called a leader, a role beyond her own self-identification.

> I recall being at my 20-year high school reunion. I had just finished my PhD within the last year and was a lecturer at a Midwestern University. One of my classmate's husband was there, and he was 3 years older than me and the current high school principal. He said "you're a leader why don't you get things started." It surprised me because I did not see myself as a leader. I just saw myself as being me.

In another dialogical context, Ricki was questioned as a female pursuing mathematics.

> I received an interview for graduate school for mathematics. Upon entering the room for my interview, I was greeted with "Oh, you're female." The discovery of my femininity cast a shadow on the interview, disregarding the qualifications that I held. I didn't get into graduate school for mathematics; it left me feeling "not good enough" in mathematics.

The dialogues between internal and external positions confirm identity from others' perspectives. While some dialogues positively contribute to identity development, as seen in Bailey's example, others might negatively impact identity, as illustrated by Ryan's experience. Moreover, dialogues between internal and external positions indicate power inequity. These dialogues reflect not only how individuals perceive themselves internally but also how external factors, such as societal norms or institutional structures, may exert influence and contribute to power imbalances within personal narratives.

Dialogues between Self and Societal Norms

Dialogues between self and societal norms, such as being a woman or living in a specific culture, also emerged. In these dialogues, women mathematics educators author themselves in broader social contexts with collective identities, often grappling with struggles and uncertainty. Below are dialogues from Bima and Lindsay respectively, exploring aspects of culture, gender, and the intersection of both, shaping who they are.

> In the culture where I was raised, we do not usually say "no" when someone in a higher power asks you to do something. For example, when your senior colleagues interpret your ideas in a different way, I still cannot take a stance and say "that was not what I was thinking." But I have begun realizing that is not going to serve me well. I need to be more strategic and defend my ideas.
>
> Another noticing is that things that challenge all these women's identities - in one circumstance or another - is the desire to please others. We feel that tension from time to time, and it sounds like with age we are getting better at managing it, but it appears to be a challenge that can creep up on us again at times and interfere with our choices of how to divide our time between our personal and professional roles.

These narratives exemplify the complex interplay between individual identity construction and external societal influences. As we delve further into these dialogues, we gain valuable insights into the nuanced ways in which personal and social identities intertwine within the narratives of women mathematics educators.

Discussion

Using a dialogical self approach, we facilitate both internal and external discourses, exploring personal histories, narratives, and the development of evolving identities. Our findings indicate that exploring personal histories provides women educators with a deeper understanding of self within complex social relations. Encompassing both individual and social dimensions, the three identified dialogical models provide a foundation for future exploration of educators' identity. These models, in particular, inform designing interview questions, discussion prompts, or survey inquiries. By keeping these dialogical models in mind, researchers can create spaces for participants to respond authentically and author selves in the figured worlds.

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Chapter 4: Geometry and Measurement

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ELEMENTARY PRESERVICE TEACHERS' ANGLE MEASURE APPROACHES GIVEN A CIRCULAR CONTEXT

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Curricular standards emphasize understanding angle measure as a fractional amount of a circle. However, learners struggle to connect angle measure to circular context. In this report, I explore 65 elementary preservice teachers' (PSTs') strategies as they engaged with a circular context. Thematic analysis of PSTs' drawings and written responses indicate that a little over a quarter of the PSTs utilized angle measure to complete the task. These findings suggest that despite the connection between angle measure and the circle concepts, such connection might not be spontaneous for learners. I conclude with implications and future considerations.

Keywords: Geometry and Spatial Reasoning, Measurement, Preservice Teacher Education.

Measurement is a critical mathematical domain for preservice teachers (PSTs) enrolled in teacher education programs (AMTE, 2017). Within this domain, the concept of angle and its measure are critical topics in school mathematics (Barabash, 2017; Thompson, 2008; Thompson et al., 2007). However, the concept is illusive, as learners (including PSTs) struggle with quantifying angle measure (Sinclair et al., 2017). Conventionally, angles are measured in relation to a circle centered at the angle's vertex. The angle is measured using its subtended arc as a specific, yet arbitrary, fractional amount of the circle's circumference. When measuring angles in degrees, the subtended arc is measured in units that are 1/360th of the circumference. This is known as the arc approach to angle measure (Moore, 2013), and is reflected in the *Common Core State Standards for Mathematics* (CCSSM), where angle measure is introduced in fourth grade:

> An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles. (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010, p. 31).

Despite the connection between the concepts of fractions, circles, and angle measure, researchers have reported that students do not relate angles to the circle context (Hardison, 2018; Hardison & Lee, 2019; Moore, 2013), and that students and teachers struggle with understanding angle and its measure (Crompton, 2017; Devichi & Munier, 2013; Keiser, 2004). However, while researchers have explored and reported PSTs' and in-service teachers' mathematical knowledge of various quantities (e.g., length, area, and volume), little is known about how PSTs and in-service teachers quantify angle measure (Smith & Barrett, 2017).

To provide PSTs with opportunities to develop a coherent understanding of mathematics (AMTE, 2017), as well as opportunities to use and connect mathematical representations (NCTM, 2014), it is important to explore PSTs' approaches to solve a problem situation that involve a circular context. In this report, I present a task that would provide PSTs with an opportunity to utilize angle measure given a circular context, summarize my analysis of their quantitative reasoning approaches (a construct I elaborate on in the following section), and provide insights on the implications of the findings.

Theoretical Framework

I build on principles of quantitative reasoning, which involves an individual's mental actions for "conceptualizing an *object* and an *attribute* of it so that the attribute has a unit of measure, and the attribute's measure entails a *proportional relationship*...with its unit" (Thompson, 2011, p. 37, emphasis added). Given this description, quantification involves coordinating three components: an object, its measurable attribute, and a quantification (measurement) process.

Given the conventional approach to angle measure described earlier, the angle is considered the object; its measurable attribute of openness is quantified as a multiplicative relationship between the angle's subtended arc and the circumference of the circle containing the arc. The arc approach to angle measure is beneficial, especially for the study of trigonometry (Moore, 2013, 2014), and inverse trigonometry (Paoletti, 2020). However, students have applied other approaches to angle measure, such as iteration and partitioning (Hardison, 2020; Mullins, 2020). For example, ninth-graders described a 1° angle by attending to the space between the angle's rays as the measurable attribute of angle, and utilized iteration (repeatedly producing 1° angle copies, where 360 copies would form a full circle) or partitioning (successively dividing a known angle into equiangular parts to produce a 1° angle) to quantify the attribute (Hardison, 2020).

The CCSSM's description of 1° angle as a *turn through 1/360 of a circle* suggests the amount of turn as a measurable attribute of angle. Clements and Burns (2000) have argued that understanding angle as a turn is important for students' ability to estimate angle measure. However, conceptualizing angle dynamically as a turn does not guarantee that the angle is viewed quantitatively as *an amount* of turn. Mitchelmore suggest that the difficulty in this conceptualization could be because, "young students do not *spontaneously* conceptualize turning (as found in rotation and hinging situations) in terms of angles" (1998, p. 278, emphasis added).

Since the quantification process being mental actions, where different individuals may have different quantification approaches for the same object, learners quantify angle measure using the length of the angle's subtended arc (Moore, 2013) or the space between the angle's rays (Hardison, 2020) as the angle's measurable attributes. Additionally, researchers reported that learners often spontaneously attend to other measurable attributes of angle (Crompton, 2017; Devichi & Munier, 2013; Keiser, 2004; Lehrer et al., 2012; Thompson, 2013). For example, students associate angle measure with the length of its rays (Keiser, 2004), or the linear distance between the angle's rays (Thompson, 2013). Additionally, students associate angle measure with the angle's orientation, where a change in the angle's orientation (e.g., from the horizontal) is conceptualized as causing a change in the angle's measure (Crompton, 2017; Devichi & Munier, 2013; Lehrer et al., 2012).

Approaches to angle measure that are different from the conventional arc approach indicate that learners may not spontaneously consider the angle's subtended arc as a measurable attribute of angle, suggesting a need for research that explores and acknowledges learners' spontaneous approaches to angle measure (Hardison, 2019, 2020). I address this need by exploring PSTs' spontaneous strategies to solve a problem situation that involves a circular context.

Methods

Participants and Context

The participants for this study are 65 PSTs who were enrolled in an elementary teacher preparation program at a Midwestern university in the United States. The PSTs were near the end of their program: They had already completed all the required mathematics for elementary school teachers courses and participated in this study as part of in-class activities in their elementary mathematics methods course.

Task

PSTs were given a task imbedded in an integrated science, technology, engineering, and mathematics (STEM) context, which involves designing a rocket. Through the design process, students learn that rockets need three fins to fly stably. Using rulers, tape measures, and waxed string to complete the task, the PSTs were given a table with two circles representing the body of two rockets from above to draw their fin placements (Figure 1). PSTs drew their representation of a rocket with three evenly spaced fins, and a second rocket with three randomly spaced fins.

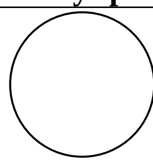
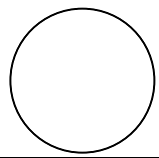
<table>
<tr><th>Rocket 1
(with 3 evenly spaced fins)</th><th>Rocket 2
(with 3 randomly spaced fins)</th></tr>
<tr><td></td><td></td></tr>
<tr><td colspan="2">- What mathematical concept did you use to evenly space the fins?
- Using the previously mentioned concept, what was your strategy to evenly space the fins?</td></tr>
</table>

Figure 1. Placing Three Fins Around a Circular Rocket Model Task

While the task does not explicitly refer to angle measure, the circular context builds on the convention of measuring angles in relation to circles. The task aimed to discern the measurable attribute the PSTs would *spontaneously* attend to, and to answer the research questions: (1) What proportion of participating PSTs would associate the circular context to angle measure? (2) Given a circular context, what measurable attribute do PSTs (spontaneously) attend to?

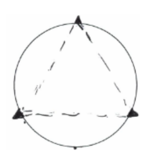
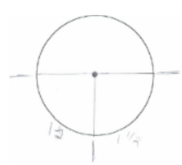
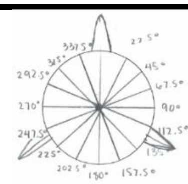
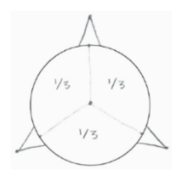
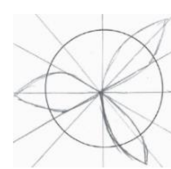
Data and Analysis

The data for this report are PSTs' drawings and written responses to the previous task (Figure 1). I initially sorted the PSTs' drawings based on accuracy using a transparent template (Figure 2) to assess if PSTs' fins placement falls within an acceptable range (Zolfaghari, 2023).

Figure 2. Transparent Template of Acceptable Range for the Fins Placement

PSTs' drawings and written responses were then categorized using thematic analysis (Saldaña, 2013), with attention to the solution strategies and measurable attributes the PSTs attended to when completing the task. Specifically, some PSTs explicitly referred to using *angle measure* to complete the task, while other PSTs used *partitioning*. Additionally, since quantifying angle measure involves using the length of the angle's subtended arc (Moore, 2013) or the space between the angle's rays (Hardison, 2020) as the angle's measurable attribute, the initial categories were attending to *circumference* or *circle interior* as the measurable attribute. Table 1 shows the categories used to organize the data, which were not mutually exclusive. For example, a PST who operated on the circle's interior could have also utilized partitioning.

Table 1: Categorization of PSTs' Written Response to the Task

Category /Code	Accuracy		Solution Strategy		Measurable Attribute	
	Within acceptable range	**Outside acceptable range**	**Angle Measure**	**Partitioning**	**Circle's Interior**	**Circumference**
Description	Evenly spaced fins within an acceptable range based on the template in Figure 2.	Fins placed outside the acceptable range of the template in Figure 2.	Explicitly referring to angle measure in the response.	Breaking a whole into different parts. Breaking the circle into three $\frac{1}{3}$ fractions.	Attending to the interior of the circle to complete the task.	Attending to the circle's circumference to complete the task.
Examples						

Findings & Discussion

In this section, I highlight the frequency of PSTs' responses categories, and relate the findings to existing literature. Curriculum standards emphasize understanding angle measure as a fractional amount of a circle (NGA & CCSSO, 2010), suggesting a connection between fractions, circles, and angle measure concepts. Using and connecting these mathematical representations is considered an effective practice for learning and teaching mathematics (NCTM, 2014). However, only 26% of the PSTs connected the circular context to angle measure, suggesting that the connection between fractions, circles, and angle measure might not be spontaneous for learners (Hardison, 2020; Hardison & Lee, 2019; Moore, 2013).

Table 2. A Breakdown of Strategies and Measurable Attribute Applied by the PSTs

		65 Participating PSTs completed the task	
		Within Acceptable Range 57	**Outside Acceptable Range** 8
Solution Strategy	Angle Measure	16	1
	Partitioning	23	2
Measurable Attribute	Circle's Interior	43	7
	Circumference	14	1

While seventeen PSTs utilized angle measure to evenly space the three fins, only one explicitly referred to angle measure and used a string to evenly space the fins (circumference). The remaining PSTs who used angle measure operated on the circle's interior. Fifty-seven of the participating PSTs (88%) correctly spaced the three fins evenly around the circular rocket representation. Out of those fifty-seven PSTs, forty-three operated on the circle's interior, while fourteen used the circle circumference. Regardless of accuracy, fifty PSTs (77%) attended to the circle's interior as a measurable attribute, while fifteen PSTs (23%) utilized the circumference to complete the task. This suggests that understanding angle measure as a fractional amount of the circumference might not be supported (Moore, 2013; Thompson, 2008).

Conclusion

While measurement is a critical concept for (PSTs) enrolled in teacher education programs (AMTE, 2017), educators in such programs need to attend to and build on PSTs' existing and spontaneous approaches. Moore (2013) utilized the conventional approach to angle measure by emphasizing the relationship between the arc length subtending the angle and a benchmark associated with the circle containing the arc and is centered at the angle's vertex. Moore (2014) and Paoletti (2020) have demonstrated the benefit of the arc approach to angle measure for the study of trigonometry and inverse trigonometry, respectively. However, the arc approach was not spontaneous for participating PSTs, as most of them operated on the circle's interior. Although the fins would be placed physically around the circle (i.e., circumference), the PSTs partitioned the circle's interior as a solution strategy, reflecting that majority of the PSTs attend to the circle's interior as the measurable attribute given a circular context. This finding is not meant to describe PSTs' thinking as lacking. Instead, I would like to highlight PSTs' ability to successfully complete the task using various sophisticated strategies without needing to connect angle measure to the circular contexts (reflected in the few examples provided in Table 2). This finding suggests that understanding angle measure in relation to circles could be supported by building on PSTs' spontaneous attention to the circle's interior.

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CONSTRUCTION AND COORDINATION OF SPATIAL UNITS IN TWO DIMENSIONS: THE CASE OF JAKE

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We report on a collaborative research study that coordinated two prominent theoretical perspectives (units coordination and spatial-numerical structuring) to build second-order models of students' geometric enumerations and measurement. We focus on one prospective elementary teacher, Jake, and his solution to a geometric enumeration task in two dimensions. This work contributes an in-depth explanatory account of Jake's reasoning in terms of his available units-coordinating actions and spatial-numerical structuring processes.

Keywords: Geometry and Spatial Reasoning; Number Concepts and Operations; Cognition; Learning Theory

Much research has investigated students' development of concepts and operations within geometric enumeration and measurement (e.g., Battista 2007; Barrett et al., 2017; Smith & Barrett, 2017). Spatial structuring, defined in the next section, is critical to students' spatial and geometric reasoning. Spatial structuring occurs via coordinations of mental actions such as iterating, partitioning, disembedding, and distributing—all mental actions involved in students' constructions and coordinations of numerical units (Hackenberg & Sevinc, 2024; Steffe, 1992)—as well as spatial mental actions such as rotating, translating, and reflecting. Meaningful measurement reasoning entails a *spatial-numerical linked structuring* (SNLS; Battista et al., 2018). Although units coordination and SNLS have similar constructivist roots and draw on a set of common mental actions, few studies (e.g., Antonides & Battista, 2022; Wheatley & Reynolds, 1996; Zwanch et al., 2023) to date have explicitly coordinated these perspectives.

In this paper, we coordinate the units coordination and SNLS perspectives to offer an in-depth theoretical account one case-study student's reasoning: Jake, a prospective elementary school teacher. We specifically focus on Jake's solution to a non-routine geometric enumeration task within an area context. Consistent with our constructivist epistemology (Beth & Piaget, 1966), we frame Jake's solution in terms of his available mental actions and their coordination. In summary, we seek to answer the following research question: *What spatial and numerical mental actions do students use and coordinate to solve non-routine area measurement problems?*

Theoretical Perspectives

Units coordination is a theory that explains students' capacities for constructing and coordinating multiple levels of quantitative units; for thorough overviews of the theory, see (Hackenberg & Sevinc, 2024; Ulrich, 2015, 2016). In brief, students at Stage 1 can reliably act on units of 1, through mental actions such as *iterating* and *partitioning*, to construct composite units (or two-level units)—collections of units conceptualized as units themselves. Students at Stage 2 can act on units of one using the mental act of *disembedding*, and they can actively iterate composite units to construct composites of composite units (three-level units). Student at *advanced* Stage 2 (Hackenberg & Sevinc, 2022; Tillema & Antonides, 2024) can also flexibly switch between different two-level units in their problem solving, such as switching between three twos and six ones as measures of the same quantity. Students at Stage 3 can bring three-level units into problem situations without needing to actively construct them.

We use the term *spatial structuring* in two senses. On one hand, spatial structuring refers to the mental process (a coordination of mental acts) of constructing a spatial organization or form for one or more objects (Battista & Clements, 1996). On the other hand, we refer to the "spatial structuring perspective" to refer to the body of research literature that frames students' spatial enumerations and measurement in terms of their structuring of spatial objects (e.g., Battista, 1999, 2007; Battista et al., 1998; Barrett et al., 2017; Clements et al., 2018; Cullen et al., 2018). Within this perspective, researchers have long investigated students' application of numerical concepts and strategies for enumerating spatial objects. Battista et al. (2018) introduced the term *spatial-numerical linked structuring* (SNLS) to capture reasoning that links spatial structuring and numerical structures; we view units coordination as a potentially powerful way of explicating students' SNLS.

Methodology

Jake was a student enrolled in the first author's content course focusing on geometry and measurement. Toward the start of the semester, we engaged Jake in one 45-minute semi-structured interview (Clement, 2000) in which we asked him to "think aloud" about his reasoning. We chose geometric enumeration tasks (including length, area, and volume) that we hypothesized would evoke SNLS and units-coordinating actions, without evoking procedural responses (e.g., finding area by multiplying L×W). We focus on Jake's solution to one enumeration task in an area context.

Pattern Tiles Task: (Part 1) How many green triangles would it take to cover Pattern 2 with no gaps or overlaps? (Part 2) How many green triangles would it take to cover the other patterns? [See Figure 1.]

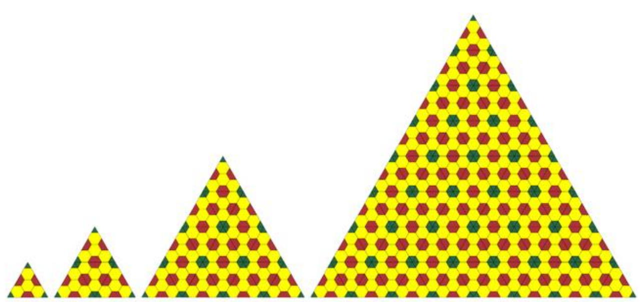

Figure 1: Image presented during the Pattern Tiles Task.

Using guidelines for networking theoretical perspectives (Kidron & Bikner-Ahsbahs, 2015; Prediger et al., 2008), we developed a plan for analyzing data and integrating our analyses. The four researchers split into the "spatial team" (Antonides and Battista) and the "numerical team"

(Zwanch and Norton). Each team developed a codebook consisting of mental actions suggested by existing research literature. Certain codes were common to both teams (e.g., unitizing, iterating, partitioning and subdividing), while other codes were particular to the spatial team (e.g., decomposing, rotating, transposing) or numerical team (e.g., disembedding). Codes appear in our Findings section as italicized mental actions. Each researcher first independently coded Jake's reasoning using their team's codebook, followed by team meetings to discuss and resolve discrepancies. We then engaged in a multi-stage process of collaboratively writing and integrating explanatory accounts of Jake's reasoning. For the Pattern Tiles Task specifically, Antonides wrote an initial analysis of Jake's reasoning from a spatial structuring perspective, then Zwanch wrote an analysis from the units coordinating perspective. Battista then wrote an integration of the two analyses, which Norton further developed and refined.

Findings

Data Excerpt. Jake was posed with Part 1 of the Pattern Tiles Task (triangles in Pattern 2). Three excerpts from his response are provided. The interviewer (Antonides) asked only clarifying questions aimed toward better understanding Jake's reasoning with minimal intentional influence.

> *Jake:* There's six triangles in each hexagon. … And also same with the trapezoid one, it's still a hexagon. It's two trapezoids, though. So, then that would be six triangles as well for each hexagon, as well as each trapezoid that's in there. So, but, some of them are cut off. So, there'd be 1, 7, because I counted those six. 10. 13. 16. Um. 22. Yeah. Um, 28. 34. 40. 43. [pause] 49. 52. 58. 64. 72. 76. Um. That'd be 82. 88. 91. 92, 93. 99. 105. 111. 117. 123. 129. [continues uttering number words inaudibly.] 144.

Analysis. We infer that Jake's solution of 144 green triangles was the product of his coordination of multiple spatial and numerical mental actions in an additive SNLS. To facilitate this SNLS, Jake seemed to treat each triangle, trapezoid, and hexagon as single entities (suggesting *unitization*) that he could *iterate*. Jake also established numerical relations between each shape by spatially *subdividing*, and numerically *partitioning*, each trapezoid into a unit of three triangles and each hexagon into a unit of six triangles. That is, each trapezoid and hexagon represented, for Jake, a two-level structure: one shape that can be subdivided into multiple congruent triangles. Moreover, Jake seemed to flexibly switch between two-level structures—six triangles to one hexagon, three triangles to one trapezoid, and two trapezoids to one hexagon—in his enumeration. This suggests Jake was operating at advanced Stage 2 of units coordination, though the image in Figure 1 may have provided material for facilitating Jake's switching between unit structures.

Data Excerpt. A moment later, Jake made comments relating Patterns 1 and 2.

> *Jake*: Pattern 2, the triangle is double the size of Pattern 1. … It just seems like the tip of Pattern 1 is like in the middle [of Pattern 2]. … So it, at first glance, it looks like it's double. [pause] But it also may not be because in here [Pattern 1], there's only four of the hexagons. And there's a lot more than eight in that [Pattern 2].
>
> *Int*: Okay. Yeah, what do you think is the relationship between Pattern 1 and Pattern 2?

Jake: There's four of Pattern 1 in Pattern 2. … 'Cause I can see the, the lines in it where you can break it up into the smaller Pattern 1 triangles.

Jake [a moment later]: I guess I could divide that [144] by four to find that answer. … Actually no, I think you might divide by three, because you still want to keep one of them. So, if you divide it by three, that essentially eliminates those three triangles. … No, you divide by four because then you're finding out how many is in each one.

Analysis. We infer that Jake's initial focus was not on the relative areas of each pattern, but rather on their relative heights, given his assertion that Pattern 2 is "double the size of Pattern 1." Jake spatially *subdivided* (numerically *partitioned*) the height of Pattern 2 into two parts, each equal to the height of Pattern 1. He quickly re-conceived the relation between Patterns 1 and 2, shifting his attention from length to area. He first did this by *disembedding* and *iterating* hexagons within each pattern, and using the number of hexagons as an indicator of the patterns' relative areas. Then, he *unitized* Pattern 1 and *partitioned*/*subdivided* Pattern 2 into parts that were each congruent to Pattern 1, articulating a four-to-one relationship between their areas.

Having constructed a multiplicative SNLS relating Patterns 1 and 2, Jake anticipated he could divide 144 by some number to find the number of green triangles needed to cover Pattern 1. However, he experienced uncertainty about the appropriate divisor: three or four. We infer Jake experienced a moment of transitioning between additive and multiplicative thinking: removing all but of Pattern 1 (divide by 3) versus finding the number of green triangles within one copy of Pattern 1. This kind of transitioning is characteristic of Stage 2 students since they need to build multiplicative structures in activity. Jake ultimately decided dividing by four made more sense, though he did not actually carry out the computation 144÷4.

Data Excerpt. Jake was then posed with Part 2 of the Pattern Tiles Task. He first used a highlighter to subdivide Pattern 3 into units of Pattern 1.

Jake: So, you just multiply 144 times four. Because this one [Pattern 2] has four [of Pattern 1] right here, and then this one [Pattern 3] has 16 [of Pattern 1]. So that's four times four is 16. … Same way I did Pattern 2 and made the individual triangles and realized that there's one, two, three, four, five, six, seven on that first row. [Counted silently] Five on this row, three, and then one.

Int: Is that 16?

Jake: I think. [Counted subdivisions within Pattern 3] Yes, it does. So, in my head, I'm thinking it's a multiple of four.

Analysis. We infer Jake constructed a multiplicative SNLS by spatially *subdividing* (numerically *partitioning*) Pattern 3 into units of size Pattern 1, with the aid of the image. He also said that he could subdivide Pattern 3 into units of Pattern 2, but he seemed to need to engage in the act of counting units of Pattern 1 within Pattern 3 to conclude a four-to-one relationship between Patterns 2 and 3—indicating Jake *may* not have constructed this multiplicative relationship prior to counting. After constructing this relationship, however, Jake was able to use it to determine the number of green triangles needed to cover Pattern 3. We infer Jake constructed Pattern 3 as a three-level unit structure through his spatial and numerical activity: a unit of 16 units (Pattern 1), each containing some number of green triangles.

Discussion and Conclusion

Jake applied and coordinated several spatial and numerical mental actions to solve each part of the Pattern Tiles Task. Both types of actions were critical to Jake's constructions and coordinations of units. Our findings underscore the importance of researchers attending to both spatial and numerical mental actions in building second-order models of students units coordinating activities, especially in tasks embedded within spatial contexts. Although this work has clear limitations—e.g., working with a single student, and presenting data on only a single task—we view this work as an important step toward future research that incorporates both spatial structuring and units coordinating perspectives. We suggest two potential avenues for future research. First, researchers could further examine and explicate the role of students' units coordinating structures in their SNLS (Battista et al., 2018). In particular, researchers could examine younger (e.g., Stage 0 and Stage 1) students' SNLS, particularly their spatial structuring and units coordinating actions. Second, researchers could extend our work focusing on length, area, and volume, to the context of angle measurement (e.g., see Hardison, 2024; Mullins, 2020).

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SURFACE EXPLORER: INVESTIGATING 3D SOLIDS THROUGH LARGE-SCALE SURFACES IN VIRTUAL REALITY

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Keywords: Geometry and Measurement, Technology and Learning Environment Design, Mathematical Processes and Practices

Currently, most of the research projects in math education involving virtual reality (VR) have its users interact with smaller-than-human scale geometrical objects. These projects leverage the immersive/interactive nature of the VR environments to provide spatial alternatives to teaching 3D geometry through two-dimensional silhouettes (Palatnik & Abrahamson, 2022). While these initiatives are exciting, our goal has been to find ways VR allows its users to experience mathematical ideas that can't be replicated in another medium.

This pointed us towards the gap in the literature that involves exploration of geometric objects that are larger than what we can observe from a single vantage point. At these large scales, the defining features of geometric shapes will be lost to our senses. As an example, standing on top of a large rectangular prism is indiscernible from standing on top a large cube when the horizon extends far enough that the defining features of these solids- the shape of their faces- can't be known trivially. Our goal with this project is to find out how these macro-scale geometric objects could be used in an exploration task.

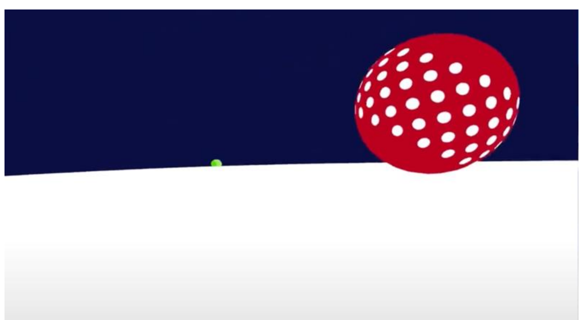

Figure 1: 2 rolling balls, green one disappearing off the horizon, red one is closer.

With these considerations in mind, we designed Surface Explorer as a playful experience that allows students from all backgrounds to engage with a mathematical task in a way that doesn't confront them with mathematics. The environment allows users to roll virtual balls on various surfaces to figure out the shape of the surface they are on. We argue that this novel experience enables students to make mathematical arguments using both mathematical and non-mathematical content knowledge through verbal and non-verbal language. Furthermore, the students' ways in orienting themselves in the space and describing what is happening around them provides information about how they think about 3D geometry outside of a classroom setting.

Through a more extensive study we aim to understand the potential value of investigating macro-scale geometries in classroom settings. We hope that investigating geometrical questions through a novel lens will create new learning opportunities for a wider range of students.

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CHANGES IN STUDENTS' DESCRIPTIONS OF SPATIAL MOVEMENTS ON A GRID

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Precision in language plays an important role in spatial thinking as well as computational thinking. We present results of a study where 26 first and 25 third graders provided verbal instructions for how a character should move along a path to reach a target on two occasions: at the beginning of a study and after playing a programming game with a peer for three, 20-minute sessions. Some of the students also analyzed worked examples of programs at the beginning of their sessions. Results suggest that analyzing worked examples of programs supported students in using more specific spatial language and articulating the number of movements.

Keywords: Geometry and Spatial Reasoning; Instructional Activities and Practices; Cognition; Computational Thinking

Language plays a key role in spatial thinking (e.g., Clements & Sarama, 2009; Hallowell, 2020; Owens, 2015). In line with the mathematical practice of attending to precision in language (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), spatial tasks may involve students communicating and explaining specifics of direction, distance, and location using correct mathematical language (e.g., up, left, next to, etc.). This mathematical practice also plays a role in computational thinking practices that are aligned with programming (Wing, 2006). For example, a key computational thinking practice aligned with mathematics is algorithmic thinking. Algorithmic thinking involves being able to represent and make sense of steps of a process, whether it is interpreting equations in mathematics or programming a robot to complete a task in programming. Students might not realize the need for precision in their use and choice of language if they work alone to program characters to move; however, playing with a peer or analyzing worked examples of a program could help them see the need for precision in their language. The purpose of this study was to investigate how analyzing worked examples of programs and playing with a peer could support students in clearly explaining movements of a character on a grid as opposed to playing with a peer but without analyzing worked examples of programs.

Theoretical Frameworks: Computational and Spatial Thinking

Spatial Structuring, Orientation, and Language

Many early programming games for young students involve controlling movements of a robot or character on a grid (e.g., Coding Awbie™, ScratchJr, Code & Go® Robot Mouse). Programming movements typically requires students to input a direction and number of movements (and sometimes indicate a type of movement). However, to make sense of movements on a grid, students need to understand the columns and rows structuring of the grid (Clements & Sarama, 2009). Students who are developing their spatial structuring of arrays might double-count squares as they switch directions from counting along the column to a row (or vice versa; Battista et al., 1998). In programming, Kocabas et al. (2019) found that some first and third graders double-counted corner squares when programming a character to move on a grid, and they also found that sometimes students double-counted the character's initial square. Shumway et al. (2021) found similar results for kindergarteners who counted the initial grid space when programming a robot on a grid. To move a character on a grid, students need to have an external-based reference system for considering position and movement (Clements & Sarama, 2009). Such positioning may be particularly confusing because different games handle directional movements in contrasting ways (i.e., depending on the character's point of view versus the character's overall position). For example, to move the Code & Go® Robot Mouse to the left, a student would need to program it to turn until it is facing left and then program it to move forward. However, in Coding Awbie™, the student would only need to play a move left block (regardless of where the character Awbie is facing). Students must navigate these interpretations to create algorithms (the series of steps) for the program to execute.

Aside from making programs, students, especially if they are working in groups or talking to others, also need to be able to verbally explain how the character should move. In other words, they need to be able to communicate the algorithms they create or want to create. Typically, children's use of the words *up* and *down* develops early, followed by words such as *beside* and *over*, with horizontal directions *left* and *right* causing more difficulty (Clements & Sarama, 2009). Compared to adults, Lloyd (1991) found that 10-year-olds tended to rely on landmarks, when sufficient, instead of directional words when describing a route based on a simple map. In another route-describing task, six-year-olds relied on language such as *over there* or *forward* to describe movements, only rarely using specific directions, such as *right* (Blades & Medlicott, 1992). In the same study, eight-year-olds used left and right more often, but only used them correctly about a third of the time. Shumway et al. (2021) found that kindergarteners sometimes used *right* and *left* but more often used language such as *here*, *there*, and *forward* (often accompanied by gestures) when describing how to move a robot. Although Clements and Sarama (2009) suggest avoiding the use of words like "over" that are not specific enough in bi-directional spaces, there is little work suggesting what conditions help students take up more specific language.

Cohen and Emmons (2017) used the coding system developed by Cannon et al. (2007) to describe students' language about space in block-building tasks that included several categories such as spatial dimensions and features. Aspects of the framework that align well with describing movements on a grid include language related to *location and direction* and *continuous amounts*. Kocabas et al. (2022) used a modification of this framework in their work where students had to identify and fix discrepancies between a Lego manual and a Lego structure. They further categorized references within the language categories as specific versus generic. We build on this work by focusing on the location and direction aspects of spatial language in describing how a character should move on a grid within a programming study.

Worked Examples

Students need explicit experiences connecting mapping experiences with math (Clements & Sarama, 2009). One way to do this is through the analysis of worked examples. When students analyze worked examples, they can identify the important problem features and learn about solution steps (Booth et al., 2015; Durkin & Rittle-Johnson, 2012). Studying worked examples can promote effective problem-solving strategies and precise use of mathematical language when accompanied by opportunities for self-explanation or guided practice (Booth et al., 2013; Lang et al., 2014). Research has shown students' conceptual knowledge improves when analyzing worked examples, particularly for novice learners, and when those worked examples are incorrect (Durkin & Rittle-Johnson, 2012). This way, students can make sense of common errors in a particular incorrect worked example.

Current Study

Although there is extensive research on the use of worked examples in learning mathematical concepts, particularly in algebra (e.g., Booth et al., 2015), less is known about its role in developing students' mathematical practices and spatial language. In this study, using the context of the Coding Awbie™ programming game, we explored the role of analyzing worked examples of programs on students attending to precise mathematical language to explain movements on a grid. We focus on the following research questions: What is the role of analyzing worked examples of programs on students' explanations of Awbie's movements on a grid? What patterns arise in their explanations?

Method

Participants and Design

For this paper, we analyzed data from 26 first and 25 third graders who were from a US midwestern elementary school. The school's population included 45% economically disadvantaged students and 11% English Learners. In the study, students first took a pretest in the form of an individual interview. After the pretest interviews, students were randomly assigned to a play-group condition ($n = 27$) or an explain-group ($n = 24$) condition. Each condition lasted for six, 20-minute sessions during which students worked with partners to play a programming game where they controlled the character Awbie's movements on the iPad using tangible programming blocks. Students who were in the play-group condition played the Coding Awbie™ programming game in pairs for three sessions, took a midtest, and then in the next three sessions, they explained worked examples of programs, corrected incorrect worked examples, and played the programming game in pairs. Students in the explain-group condition started the first three sessions by explaining worked examples of programs and playing the programming game, then they took a midtest and participated in a presentation about programming applications, and for the last three sessions, they only played with the Coding Awbie™ programming game. After students completed their six sessions, they took a posttest, which followed the same procedure as the pretest (see Figure 1). For this paper, we focus on the pretest, first three sessions, and midtest, because changes from pretest to midtest reflect the difference between only playing the game (play-group condition) versus playing the game *and* analyzing worked examples (explain-group condition).

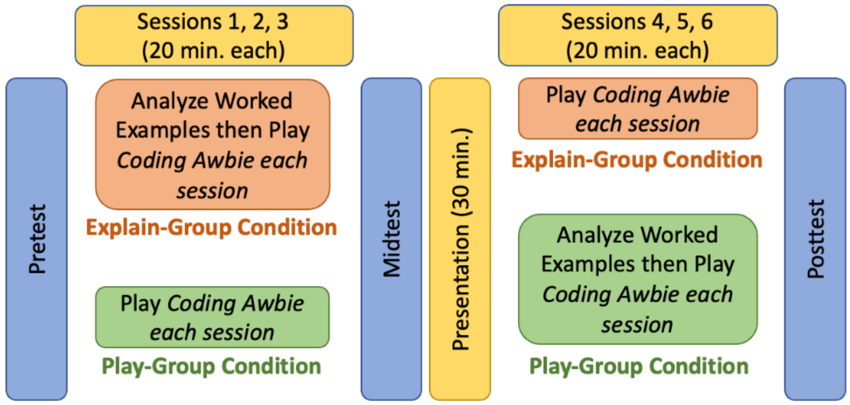

Figure 1: Study Design

Materials

In the individual pretest and midtest interviews (taken about a month apart), students responded to problems such as explaining Awbie's movement, debugging a program, and making a program. We focus on one of the problems that required students to explain how a character should move on a path (as an informal way of creating a program), which was given on the pretest and midtest (see Figure 2 below). The path was rotated 180º on the midtest so that students would not be able to use their explanations from the pretest.

Pretest Instructions: Awbie (point to character) can jump over flowers, bushes, or small rocks but not trees. Tell us a story about the movements Awbie made to get to the Red strawberry (point to it) using this highlighted path (point to path). We will tell your story to another student without showing the path and see if they can guess which strawberry you had Awbie get, so make sure your story only works for getting this *red* strawberry (point to it again).

Midtest Instructions: Tell me a story about how Awbie moves on the yellow path (point to path) to get to this strawberry (point to strawberry). Make sure you use enough details that I could figure out how Awbie moved, even if my eyes were closed.

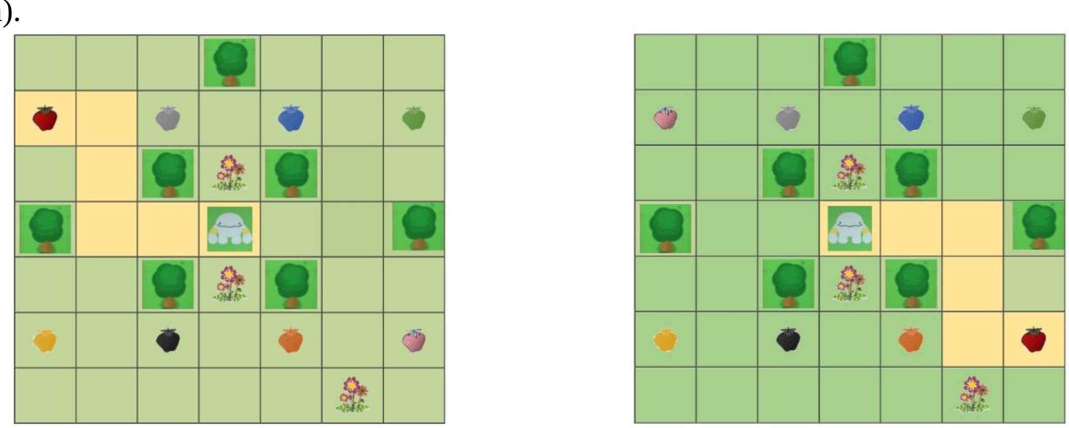

Figure 2: Explaining a Character's Movement Task on the Pretest and Midtest

In each of the three sessions when students in the explain-group condition analyzed worked examples, they analyzed one correct worked example of a program and one incorrect worked example of a program. During the second session, they also analyzed and completed one incomplete worked example of a program, for a total of seven worked examples across the three sessions. The worked examples showed scenes from the game and tangible programming blocks organized into a program to control Awbie's movements. As part of analyzing the worked examples, they also had to apply information from the worked example to a new program. The interviewer read the program to the students and then had the student pairs discuss their answers. Rather than giving feedback on their answers, the interviewer helped facilitate a conversation between the students, such as asking, "Do you agree with your partner? What do you think?" Figure 3 shows one of the worked examples of programs we used in the sessions. Importantly, in this incorrect worked example of a program, students were exposed to the directional words *down* and *left* and had to fix a double-counting error in the program (i.e., the correct program should be walk left (if no number is used, the default it 1), walk down 2, both of which are repeated 2 times).

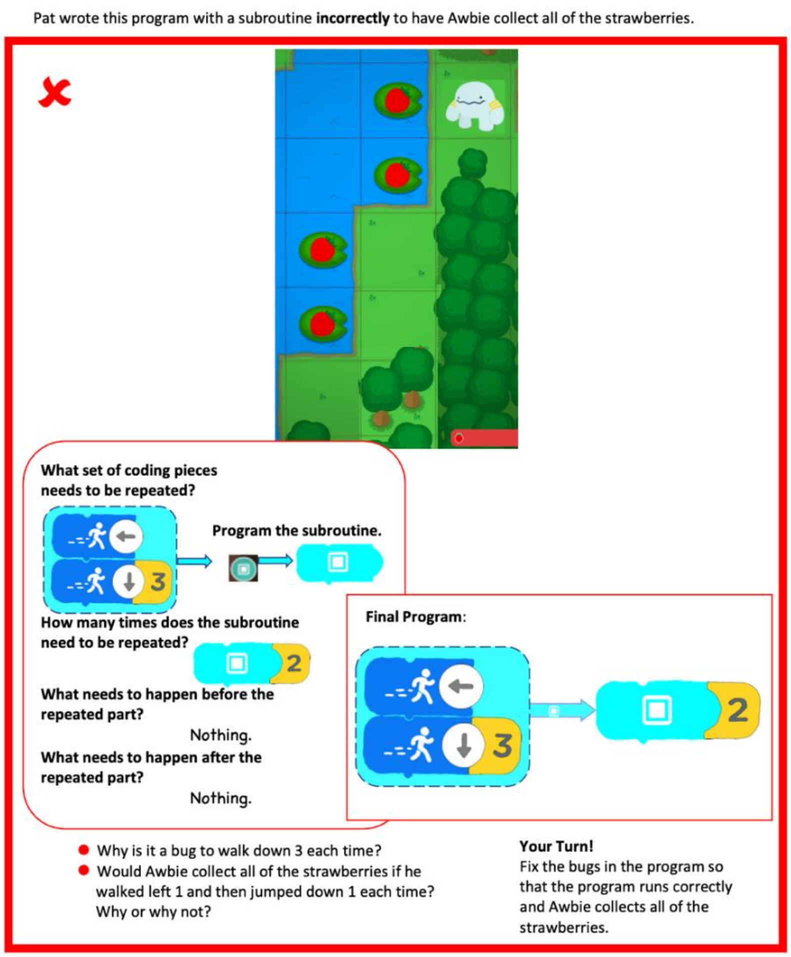

Figure 3: An Incorrect Worked Example of a Program in the Sessions

Analysis

We coded students' language by using a modified form of the framework described previously (Cannon et al., 2007) with the sub-category *specific* versus *generic* distinction used in Kocabas et al. (2022). Further, because our work involves situations where the character could move both up and down and left and right (i.e., specific language), in our sub-categories, we wanted to account for language that specified a horizontal movement but was not specific enough to describe which horizontal direction (e.g., "over"). We classified words that fell into this sub-category as semi-specific language. The location or direction category includes students' use of words indicating where Awbie is or would go. The number category includes students' descriptions of how far Awbie should move. The movement category includes descriptions of how Awbie should move (see Table 1 for examples). We ended up adding a descriptive sub-category for the location or direction category to better capture the meaning of some students' explanations. After coding the categories, we further tallied which words students used within the *location or direction* category.

Table 1: Spatial Language Examples by Sub-Category Codes

Spatial Language	Specific	Semi-specific	Generic	Descriptive
Location or Direction	right, left, up, down	sideways, next, over, straight	here, (over) there, on the trail, this way	easy, hard, long, short
Number	number words	another	a little bit	n/a
Movement	walk, run, jump	turn	move(s), go(es)	quickly

Results

Location or Direction Language

Overall, the play-group condition started out on the pretest using the directional word *up* more than students in the explain-group condition (74% of students versus 33% of students). If students improved on language related to the vertical dimension, we would expect them to use the word *down* on the midtest (recall that the path went down instead of up on the midtest). We found this to be the case with 93% of students in the play-group condition and 79% of students in the explain-group condition using *down* in their explanations. The explain-group saw greater growth in using specific terms for the vertical dimension than the play-group (46% more versus 19% more); however, the play-group also had less room to make gains since more used this language on the pretest. As an example of students' language changes, one first grader (ID: Duck4) in the explain-group condition explained the pretest path, "He has to walk *over here* and then go to *here*." On the posttest, he said, "Go two. Go two more *down*, then over."

The results for language related to horizontal movement, however, add more nuance. Typically, students who used the incorrect term (e.g., *right* instead of *left*) also self-corrected and used the correct term (see Table 2, e.g., *left & right* column); however, a couple of students did use the incorrect horizontal term exclusively (see Table 2, e.g., the *right* column percent is less than the *left & right* column percent for pretest). For example, one third grader (ID: Horse7) explained on the pretest, “He’s going to have to move twice to the right. Yes, twice to the right (points to the left).” Once again, the play-group condition used the correct term *left* more on the pretest than the explain-group condition. However, the play-group condition did not gain in the number of students using *right* in their midtest explanations; on the other hand, more students in the explain-group condition used *right* by the midtest and overall compared to students in the play-group condition. One third grader (ID: Horse5) from the explain-group condition had a dramatic change in explanation. On the pretest, she merely said, “He can walk.” On the midtest, she explained, “So he’s going to go over to the *right two times*. And then he’s going to go *down two times,* then over again to the *right* to get the strawberry. He will be walking.”

Other interesting patterns in terms of directional language included that students who did not use the specific terms *left* or *right* were most likely to use semi-specific terms *to the side*, *sideways,* or *over* on the midtest as well as *forward* and *straight* on either test. They also used non-specific terms *this/that way* and *over/right there*. Often, students mixed terms. For example, a first grader in the play-group condition explained on the midtest, “So he goes *this way* twice, *down* twice, and *over* one” using a non-specific, specific, and semi-specific term. On the pretest, two students (one from each condition: ID: Sheep5 and Sheep7) even described the movement as a *zigzag*, finding a way to characterize the set of movements. Another first grader (ID: Duck7) just called this “making a Z.”

Table 2: Percent of Students Using Specific Horizontal Direction Terms and Gain

	Pretest			**Midtest**			**Gain**		
Group	**Left**	Right	Left & Right	Left	**Right**	Left & Right	**Correct term**	Incorrect term	Left & Right
Play	**33%**	19%	37%	11%	**26%**	37%	**-7%**	-8%	+0%
Explain	**8%**	8%	13%	0%	**42%**	42%	**+34%**	-8%	+29%

Note. Bold indicates percent of students using the correct term to describe the direction on the path.

Number and Movement Language

In terms of students’ use of numbers to help describe the movements, there were similar trends as with the directional language. The play-group condition had more students designating the number of movements on the pretest than the explain-group condition (20% versus 13%). However, on the midtest, the play-group condition had fewer students designating the number of movements than the explain-group condition (48% versus 58%). This means that 45% of students in the explain-group condition improved in using numbers in their explanations compared to 28% of students in the play-group condition.

Although some students used numbers, they double-counted and stated an incorrect number of movements. For example, eight students double-counted a corner square. On the midtest, a third grader (ID: Goat5) said, "I think he would do one, two to this way (points to the right) and *one, two, three* down and then one across. Five students also double-counted the initial square. For example, the same third grader (ID: Goat5) on the pretest said, "So you would go *three* and then stop, and then turn and go two, and then turn and get to the strawberry." Another third grader (ID: Horse8) double-counted both the initial square and both corner squares, describing, "He can go from this by counting from one – it's like *one, two, three*. And then three, you can go down *one, two, three*, and then over one – over *two*." A third grader (ID: Rabbit5) explained the movements and revealed how some might consider double counting the initial square, "So he moved two, but he technically moved three because he's on the third one. But he would move two." Another first grader (ID: Goose8) explained the difficulty in a similar way, "Right, it's three, *but he's on the first one*. I think that's *go two…*"

Exclusively on the pretest, four students used individual movements to indicate the total distance to cover rather than using numbers. For example, one third grader (ID: Horse6) explained, " It would be left, left, up, up, left. And then he gets the strawberry." Another five students described how they needed to move two spaces on the horizontal and the vertical dimension but then used alternative words to indicate movement of one space on the final horizontal dimension. For example, one first grader (ID: Duck4) on the midtest said, "Go *two*. Go *two* more down, then over."

Discussion

As with prior research (Blades & Medlicott, 1992; Clements & Sarama, 2009), students in this study had an easier time using specific words to describe movements on the vertical dimension (i.e., up, down) than the horizontal dimension (i.e., left, right). However, by the midtest, the students in the explain-group condition used the term *right* more than found in prior research, suggesting that analyzing worked examples where designating direction is important might help them see a purpose in using more specific language themselves. Such modeling might be especially important if in their typical spatial tasks, students are working together with the same vantage point and are used to assuming others know what they are talking referring to and can see where they are pointing. They need more experience considering situations where others might do a similar task later. Focusing on specificity of language tasks such as these could serve as a helpful foundation for considering precision in language for other areas of mathematics. For example, the class could consider what they mean by *zigzag* and come up with a definition for how to indicate the size or dimensions of a *zigzag*. This could lead to students adopting horizontal and vertical language or coming up with a class-accepted precise definition for *zigzag*. In turn, such discussion could support inquiry around the definitions of shapes.

Interestingly, some students, even in the explain-group condition, double-counted after playing the programming game for three sessions. Students in the explain-group condition also analyzed a worked example that involved double-counting. It is possible that the double-counting needed to be more explicit, such as was the case when kindergarten students programmed a robot, Shumway et al. (2021), to encourage students to consider the implications of double-counting. For example, in a revised version, students could compare a worked example where a student made a program that involved double-counting versus a worked example where a student did not use double-counting and then could discuss which one works and why. Surprisingly, only third graders double-counted. One reason may be that first graders used numbers slightly less often. Yet, it is surprising that there was not even a single instance of a first grader double counting. The tension between seeing three spaces but only needing to move two spaces (because the character was on one of the three spaces) may be the reason that students double-counted the initial space. It is possible they were considering the initial space as where the character should be placed to start. Students might have also reverted to double counting the corner pieces if they were used to finding the area of arrays (where double counting the corner is needed to find the length of each side).

Overall, the students in the explain-group condition appear to have benefited from the opportunity to analyze worked examples beyond what the play-group condition got from only playing the game. These results provide additional evidence for the benefit of using worked examples to support students' learning of new concepts but also provide initial evidence for worked examples' potential for supporting students' precision in language use. Using worked examples to support mathematical practices is an area that needs to be explored further.

Acknowledgments

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FORMAL AND INFORMAL PROOFS IN HIGH SCHOOL GEOMETRY

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Keywords: Reasoning and Proof, High School Education, Geometry and Spatial Reasoning

Are students who have been taught proofs more likely to respond to proof tasks correctly? In the study reported here, we describe differences in responses to proof items in tests used in a large-scale curriculum comparison study. The conceptual framework for this study was based on the proof schemes framework by Harel & Sowder (1998, 2007). In this framework, *proving* is "the process employed by an individual (or a community) to remove doubts about the truth of an assertion and includes two subprocesses: ascertaining (removing one's own doubts about the truth of an assertion) and persuading (removing others' doubts about the truth of an assertion).

The data reported here were collected as part of a longitudinal study in U.S. high schools, the COSMIC study (Grouws et al, 2013; Tarr et al, 2013; Chávez et al, 2015). This longitudinal study examined the impact of different content organizations on student learning. As part of a curriculum comparison study, we included items involving proofs in geometry (Chávez et al, 2011; Sears & Chávez, 2015). One of the items was designed to elicit an informal argument, for first-year high school students (n = 2508); the second was for students in the third year of high school (n = 1936) who had taken either a geometry course or an integrated course that included topics of proof in geometry. Given the longitudinal nature of the study, the students who took the second test had taken the first two years earlier.

For the first problem, of 2508 first-year students, 29% gave a complete correct answer, 25% a partially correct answer, 31% gave an incorrect response and 9% did not attempt the problem. For the second problem, of 1936 students, 12% did not attempt the problem and 56% gave an incorrect answer. Only 28 students, less than 2%, gave a complete correct answer.

As expected, first-year students did not attempt "formal" proofs. The answers given suggest that students relied on the pictorial representation and made reasonable assumptions that could be the basis of a correct proof. It is important to note that in their responses, students explained how they knew they had a correct answer. A vast majority of third-year students, more than half, did not give an incomplete or partially correct answer.

It seems that the third-year students were not reasoning about the question but instead trying to remember how proofs are supposed to be written. In contrast, more first-year students felt reasonably confident about figuring out an answer for the first problem precisely because it was an unfamiliar problem. Students with little or no instruction in formal proof are more likely to attempt an informal justification. Students who have been taught proofs seemed to have developed an external conviction proof scheme, relying either on the form of the proof (e.g., 2-column proofs) or on the authority of the teacher or textbook to determine what is a valid proof. This may explain why so many students did not write a correct or partially correct proof for the second problem.

Our results suggest that we should give students opportunities to explain how they know, focusing on the ascertaining process of the act of proving and helping them to make this process explicit. For many of the students in this study, writing a proof may have become a recall exercise not an opportunity to explain or justify.

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INVESTIGATING PRESERVICE MATHEMATICS TEACHERS' SPATIAL REASONING IN PROBLEMATIZED TASKS

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Keywords: Geometry and Spatial Reasoning, Preservice Teacher Education, Problem Solving

Learners engaged in reflective thinking often encounter a perplex, difficult, or hesitating situation and need to search for a solution that is not immediately apparent. Piaget (1960) described learners' struggles as a process of resolving disequilibrium and developing new understanding. Recently, Bjork and Bjork (2011) stated that tasks containing "desirable difficulties" can "optimize long-term retention and transfer" (p. 57). Little is known about individuals' internal struggles outside of whole-class discussion settings (Santagata, 2005). This study aims to use individual think-alouds and investigate how preservice elementary mathematics teachers' (PSTs) navigate cognitive struggles or uncertainties through problematized spatial tasks. The theoretical framework builds on Warshauer's (2015) categorization of learners' struggles and Park et al.'s (2022) phases of sensemaking amid uncertainties: (1) phenomena representation where learners interpret and represent the tasks and generate initial thoughts; (2) exploration where learners are engaged in problem solving by analyzing existing and new information and ideas, and generate explanations and reasoning; (3) application and evaluation where learners apply the generated reasoning to the represented phenomena, evaluate the potential solution(s), and possibly revise the initial thoughts.

This study is a part of a large study that explores the effects of technology and tangible manipulatives on PSTs' spatial reasoning. PSTs are enrolled in teacher preparation programs in a public university. Preliminary results report 13 elementary PSTs' spatial reasoning and learning struggles in eight 3D block building tasks. Using the think-alouds methods, PSTs' spatial reasoning processes were recorded. PSTs used multi-link cubes, virtual 3D Block Builder, or both to make 3D structures. All 3D block building tasks are problematized in ways that contain ambiguous or missing information and make multiple solutions possible. Data are coded into three phases of sensemaking. PSTs' certainties (e.g., confidence about an interpretation or a strategy) are coded along with uncertainties (e.g., confusion about some spatial vocabulary).

Results show that only two PSTs provided alternative 3D structures whereas the rest thought of only one 3D structure. In the *phenomena representation* phase, PSTs likely interpreted the problematized task hints as close-ended rather than open-ended. For example, task#2 only provided information about green and orange blocks in a three-block tower; PSTs considered the third block as either green or orange. In *exploration*, about one third of PSTs' spatial reasoning instances (104 in total) began with partial structures and then analyzed possible placements of other blocks. PST#6 built a two-block structure made of a green and a blue block before placing two other red blocks in task#5, which turned to be effective. In *application and evaluation*, some PSTs encountered an impasse when their initial ideas conflicted with given information; they tended to stick with their initial ideas without revisions when they applied their reasoning to build 3D block structures. These findings imply PSTs' early certainties about block quantity, colors, or spatial relationships in the phenomena representation phase could cause uncertainties in later phases. Investigating PSTs' certainties and uncertainties in mathematical sensemaking sheds light on instructional scaffolds that teacher educators could provide.

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TOWARD DECOMPOSING THE CIRCLE: LEARNING TO JUSTIFY THE AREA FORMULA FOR A CIRCLE

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In this study, we studied the impact of a one-time unscripted interaction with an applet on Preservice Teachers' (PSTs') justification of area formula of a circle. After just one classroom exposure including individual exploration, PSTs improved their understanding of this formula. However, for most PSTs, this one experience was not enough to overcome the epistemic gap between justifying the formula as an approximation of the area and a true deductive argument.

Keywords: Technology, Geometry and Spatial Reasoning, Instructional Activities and Practice, Preservice Teacher Education

Background and Literature

Being able to give an informal derivation between the circumference and the area of a circle is a common middle-grades standard (CCSSO, 2010), yet the curved boundary lends challenge to students used to working with polygons. A growing number of online applets and DGS sketches utilize dragging actions to transform geometric figures and provide students with opportunities to explore and act on figures while forming and verifying conjectures about embedded mathematical concepts (Leung, 2011). Using Geogebra, Or (2012) developed one such applet based on the mathematical storyline depicted in Figure 1, to help students bridge the epistemic gap between interpreting the area formula of a circle as an approximation and an exact value.

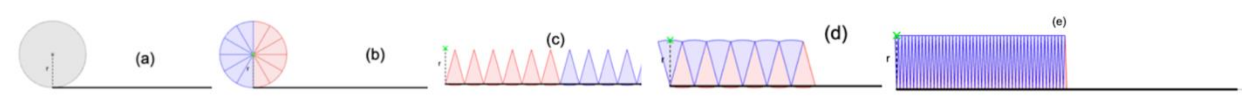


Figure 1: Illustrating the Storyline of Or (2013)'s Applet

Using a slider, the applet leads students through subdividing a circle into an increasing number of pieces. This subdivision is visible as a full circle (Figure 1b), an "unwrapped" circle (Figure 1c) and all positions in between. Once the circle has been unwrapped, continuing to drag the slider rearranges the sectors so that they form an approximation of a parallelogram (1d and 1e) that, if taken to infinity, would be a rectangle with dimensions r and πr. The area of which, we deduce, can be measured as πr^2.

In this study, we studied the impact of a one-time unscripted interaction with this applet on Preservice Teachers' (PST) ability to justify the area formula for a circle. After giving PSTs time to explore the applet in small groups, we brought the whole class together for a discussion of those experiences. Collecting data before and after this event, we asked, *to what degree does engagement with the applet change the way PSTs justify the formula, bridging the epistemic gap between interpreting the area formula of a circle as an approximation and an exact value?*

Methodology

This study was conducted in a Midwestern university. All participants (n=70) were enrolled in one of two sections of a course on geometry for preservice elementary teachers taught by Author 1. The data used in this study were collected at the end of a unit on measurement in which significant work had been done toward a deep understanding of linear and area measurement. This work included activity related to the meaning of perimeter and area in the context of simple and composed polygons. It was our hope that prior experience with decomposition of polygons would support their engagement with circles.

Instructional Sequence

Or (2012)'s applet, was chosen as the primary lesson delivery system to assist participants in applying the technique of decomposition to circles. Participants had used other pre-designed sketches in a unit on classifying special quadrilaterals. Like this experience, participants were asked to engage with applets in small groups prior to whole class discussion about that activity.

Following the applet experience, a whole-class discussion was facilitated by the instructor. During that discussion, the applet was projected at times on a pull-down screen and at times on a white board where PSTs could choose to draw, record predictions, or annotate images and locations where they found mathematical meaning. The goal was not to reteach or to make a demonstration, rather, to focus on specific interactions participants had with the applet and allow for reflection on the significance of those interactions.

Data Collection

Pre- and Post-test Data were collected to better understand how the applet is taken up by PSTs as a tool in justification. Prior to the applet exploration, PSTs were asked to write individual justifications for why the area of a circle could be found using a specified formula. We collected 69 hand-written justifications on blank white paper. Following the instructional sequence, we gave PSTs the same open-ended task of justifying the formula without specifically mandating or even suggesting the use of the applet as a tool. Specifically, PSTs were asked to submit work that "1) Shows that you understand the meaning of radius, diameter, circumference and area; and 2) Provides reasoning about the area formula for circles in your own words."

Data Analysis

We analyzed the pre-justifications first which generated the framework summarized in Table 1 along with our findings. We read through the set of all justifications looking for similarities and differences. Once an original set of categories were identified, we used an iterative process of reading through the data and refining the categories, such as with the category "dimensional analysis" which is described in more detail in a previous paper (Cox & Lo, 2019). When a response seemed to span two categories, we refined the categories when possible. However, when the category did not warrant subdivision, we placed each response according to the most sophisticated reasoning provided. We then applied this framework to the set of post-justifications and were able to use this data to refine the category of "decomposition." This refinement is summarized in Table 2 along with our findings.

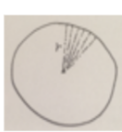
Findings

Exposure to the applet changed the way students thought about the formula for the area of a circle. These shifts are apparent in the type of attempted reasoning in the post-justifications compared to the pre-justifications (Table 1). Prior to exposure, 43.5% (n=30) PSTs sought to justify the symbolic formula through a process of symbolic deconstruction aimed at associating the dimensions of the circle with the symbols and operations within the general formula. We referred to this type of reasoning as *Dimensional Analysis.* In most cases, this meant finding meaning in the isolated terms r, r^2, and π. While r was always defined as the radius of the circle, there was more variation in the ways that PSTs found meaning in r^2 and π. Brandon was one who did reference *π*. He wrote, "*Because radius is half of a circle in order to get the full circle, you must square the radius. You then multiply by π because π is a measurement used in circles.*" Katie wrote, "*We do πr to find the distance around the entire circle. Then, you must multiply that number r again to account for all the area from the edge of the circle to the center of the circle.*" After exposure to the applet, only three PSTs used this type of reasoning.

Table 1: Categorization of Pre- and Post-Justifications of the Area Formula for a Circle

Strategy	Description	Pre	Post
Non-Justification	No attempt at justification.	17	1
Discrete Fact(s)	Brief attempt to connect measurements within the circle to the task of justifying the area formula, but the response doesn't exceed a true, discrete statement that does not support a justification.	9	6
Empirical	Proof by example (Harel & Sowder, 1998).	1	0
Dimensional Analysis	Symbolic deconstruction aimed at associating meaning with the symbols and operations within the general formula	29	3
Approximation	Comparison to the area of a square of length r or $2r$	6	1
Decomposition	Attempt to decompose the circle into equal parts and sum.	7	59
Total		**69**	**70**

Decomposition was not common prior to exposure. Figure 2 shows one of the most advanced examples from the pre-justifications. The drawing and the accompanying explanation showed some evidence of finding the area of the circle by first decomposing a circle into many tiny sections and then summing the area of these tiny sections.



I will cut a circle into a lot of pieces of triangles and get the area of one triangle and then add them together.

Figure 2. Justifying the area formula using decomposition pre-exposure.

Compared to the work submitted prior to the applet experience, the justifications submitted following the experience were more diverse and contained explicit reasoning about decomposition and limit. For example, Elizabeth made the following argument:

> *I made a circle and divided it into equal sections. Then, I arranged those parts in a formation that barely looks like a rectangle. However, after dividing the circle into very small sections, the shape looks more like a rectangle. The circle will transform into a perfect rectangle by dividing the circle into infinitely many times so we can't point out the lines of the sections.*

After exposure, most students chose to decompose the circle. Some were able to connect that action to the symbolic formula, still others showed a partial understanding of limit in this context, and a select few (like Elizabeth) provided a deductive argument based on limits and infinity. This diversity of sophistication led us to refine the category (Table 2).

At one end of the progression is *disconnected decomposition*. In these justifications, PSTs decomposed the circle, but made no attempt to link that imagery to the symbolic formula. Rather, the focus was on describing the "actions" on the screen. Tina wrote, "*unravel the circle so that it lays flat and the triangles are all sticking up, still split down the middle by color*." At the other end of the progression were ten PSTs who were able to articulate the concept of limit as Elizabeth did above.

Table 2: Progressive Reasoning for Strategies Incorporating Decomposition

Description	Pre	Post
Disconnected Decomposition: no attempt to link to the symbolic formula	4	3
Attempted Symbolism: made an unsuccessful attempt to link to the symbolic formula		7
Achieved Symbolism Connected the area formula symbolically with a decomposed circle.	1	33
Partial Limit Justification included the reasoning that more sectors yield a better approximation		6
Developed Limit Justifies the symbolic formula including reasoning about infinity.		10
Total	5	59

Discussion

Data from this study indicated that Or's applet is a tool that PSTs consider interesting and useful. Imagery was represented by many in sketches, screenshots, and narrated video segments. Even deeper, most utilized multiple images and the storyline of the applet was clearly present in the set of post-justifications. It was a tool for mathematical exploration and learning, but also became a tool for justification. Unfortunately, for most PSTs, this one experience was not enough to overcome the epistemic gap between approximation and deductive reasoning based on limits. However, 16 PSTs did write or talk about subdividing the circle into ever-smaller segments (Figure 1e) in their post-justification and of those, 10 approached a full justification. That suggests that more effort to draw PSTs attention to this part of the storyline is warranted.

There are two additional considerations for the PME-NA community. First, given the opportunity to explore differences between discrete and continuous data, *is this an important opportunity to build that connection?* Second, almost all participants showed conceptual growth, yet only few were able to walk across the epistemological bridge. *How important is it for elementary teachers to provide a full justification for this formula?* In these uncertain times, we feel more and more pressure to reduce the amount of content in preparation programs and discipline-specific professional development. We welcome the opportunity to talk about our future and conceptual priorities at the conference.

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TEACHING FOR DEEP CREATIVITY THROUGH QUALITATIVE GEOMETRY: THE CASE OF OPAL, AGE 5

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The study reported here is developing the curricular and pedagogical components of an approach to teaching qualitative change as it relates to deep creativity within elementary mathematics education. Through this work, the project aims to make three contributions to the fields of creativity, mathematics education, and the learning sciences: (1) new understandings about the development and multimodal expressions of children's qualitative mathematical thinking, (2) new insights about the benefits children experience from their engagement in activities that develop their understanding of qualitative difference as a foundational dimension of learning for radical change, and (3) evidence of the transformative role of deep creativity in elementary mathematics education. In this brief research report, we share preliminary findings of pilot research through a case study of a 5-year-old child named "Opal."

Keywords: Elementary School Education, Geometry and Spatial Reasoning, Design Experiments, Cognition.

Simply stated, *STEM needs creativity*. From equity to the economy to the environment, addressing major global issues demands both "deep creativity" and STEM knowledge and skills. Innovative solutions to address these large-scale problems can be derived through creative processes fundamental to scientific thinking (Cropley & Cropley, 2010). Regrettably, however, creativity is almost absent from mainstream approaches to STEM learning. If we do not resolve the disconnect between creativity, innovation, and the sciences, STEM graduates will be ill-equipped to tackle the most critical and persistent global issues. The *Stretchy Minds* project brings together researchers working at the intersection of mathematics education, creativity, and embodied and emergent design to resolve this conflict. Working in collaboration with elementary math teachers, this project is leveraging the unique affordances of qualitative mathematics to develop a theoretical model for cultivating deep creativity. The vision is that this model will be used to catalyze the enactment of new and much-needed approaches to building foundations for children's STEM creativity within elementary mathematics education. *Conceived with an eye toward this imagined future,* the purpose of this report is to share preliminary findings of pilot research through an exploratory case study of "Opal," a 5-year-old participant in the project.

Mathematics carries the intrinsic capacity to develop young learners' skills and abilities in creativity and innovation through their engagement with *qualitative* forms of mathematical thinking and reasoning. *Qualitative geometry* (Greenstein, 2014, 2018), in particular, has unique affordances for this endeavor because it involves the core concepts of deep creativity – including qualitative change and divergent thinking – *and* it is accessible to young learners (Greenstein, 2014; George, 2017; Piaget & Inhelder, 1956). Accordingly, we've proposed that qualitative geometry can offer a transformative space in which to introduce young children to the concepts of deep creativity in ways that are conceptually understandable, materially tangible, and aesthetically driven. Project findings will enable us to contribute to the expansion of the theoretical understanding and instructional practice of creativity in the space of elementary mathematics education, and in STEM education more broadly.

Our first research question focuses on the design of curricular tasks and the determination of pedagogical principles that engage and develop learners' embodied understandings of qualitative difference as it relates to deep creativity. The following two research questions are the focus of this proposal: *What ideas about qualitative difference are elicited and developed as a result of learners' systemic engagement in these curricular experiences? How are their understandings of qualitative difference enacted, expressed, and made visible through their engagement?*

Theoretical and Conceptual Framing

Creativity, Novelty, and Change. To conceptualize what we mean by *deep creativity*, we first define creativity not as an internal characteristic but as an active process for doing something *novel* and *appropriate* within a situated context (NACCCE, 1999; Sternberg & Lubart, 1999). Next, we propose that creativity is *deep* when it yields novel forms of change that have both quantitative and qualitative dimensions. These dimensions are defined in relation to the two types of change (Bergson, 1911; Deleuze, 1994; Deleuze, 1998):

- *Change-in-degree* (Δ_d) is a ***quantitative*** change, which is (only) incremental.
- *Change-in-kind* (Δ_k) is a ***qualitative*** change, which is wholly and genuinely novel.

At the Intersection of Embodied Creativity and Qualitative Geometry. Simultaneous to a renewed interest in embodiment and collaboration in creative work (e.g., Feldman & Benjamin, 2006), research suggests the promise of adopting an embodied and situated perspective on creativity – an *embodied creativity* as Malinin (2019) calls it. First, there is evidence that the features and qualities of the socio-material environment play a role in enhancing creative expertise (Malinin, 2016); insights into the roles they play can inform the pedagogical practices that lead to creativity. Second, an enactive approach to cognition (Thompson, 2007) has demonstrated that qualitatively novel ideas emerge via embodied actions, where doing *precedes* knowing. It's the doing that enacts meaning into being (Kerr & Frasca, 2021). Conceived from this perspective, this project is testing the conjecture that learning qualitative difference requires open-ended, exploratory, experimental, and emergent educational experiences.

Although mathematics is often viewed as the quantitative domain, it is also home to creative and qualitative processes. This is evident in the domain of topology, which is a non-metric, *qualitative* geometry. The "rubber sheet" conception of topology is useful for demonstrating its qualitative foundations. In general, any two curves are topologically equivalent if each can be bent and stretched, but neither glued nor broken, to produce the other. The property that determines their equivalence is the quality of connectedness that is invariant through the distortion of the rubber sheet. These elastic transformations produce curves that are equivalent precisely because they vary by (quantitative) *differences in degree*. A segment cannot be bent or stretched into a triangle because their connectedness is a (qualitative) *difference in kind*. We chose Speks magnetic toys as a medium for children's learning about qualitative difference, because one can imagine the topological concepts of connectedness and genus (i.e., the number of "holes" in a surface) embedded in them as loops formed by connecting their magnetic ends. The configurations of Speks toys in Figure 2 illustrate the two forms of change using "number of loops" as the criteria for equivalence (i.e., sameness in kind), and "size of loops" as the criteria for difference in degree. Shapes A, B, and C are different in kind, since they each contain a different number of loops. Shapes B and D are different in degree in that the loop in D is larger than the loop in B, and size is a matter of degree.

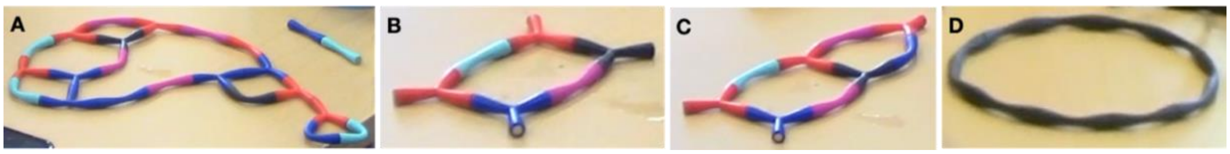

Figure 2: Configurations of Speks toys express differences in kind and degree.

Methods

This project is employing multi-phase design-based research (Brown, 1992; Collins, 1992; DBRC, 2003) to produce responsive curricular experiences that engage and develop learners' thinking about qualitative difference as it relates to deep creativity. Informing the design of those experiences are small-group teaching experiments (Cobb, 2000; Steffe & Thompson, 2000) that yield models of children's thinking (Thompson, 1982), and microethnographic methods (Nemirovsky et al., 2012) that characterize children's multimodal expressions (Edwards, 2009; Goodwin, 2014) of their thinking by tracing their moment-to-moment bodily and situated mathematical activity. Analyses of these expressions entail a grounded, bottom-up approach of constant comparisons (Glaser & Strauss, 1967) that generates inferential interpretations of the children's elicited understandings of qualitative difference.

Participants range in age from 5 to 10 years old. The age-related criterion is informed by research which finds that the development of children's spatial thinking begins as early as 3.5 years of age (Piaget & Inhelder, 1956). A related rationale is found in researchers' calls for deliberate creativity learning to begin in early childhood (Resnick, 2007; Yardley, 2011). Collected data includes video recordings of the children's small-group participation in design sprints; their drawings, magnetic toy structures, and other artifacts; and analytic memos (Strauss & Corbin, 1998) and field notes (Van Maanen, 1988) about the children's goal-oriented actions. Video recordings allow for in-depth analysis of the dynamic interplay between the children's whole-body movement, their material interactions, and their verbal explanations, conversations, and collaborations, as they relate to processes of deep creativity. A review of research that employed qualitative methods to analyze observational data of children's embodied expressions confirms the viability of this analytic approach (McCluskey et al., 2023).

Preliminary Findings

Our pilot study work is exploring whether we can support the development of children's understandings of deep creativity through game-based (Nguyen, 2020) learning activities we designed based on the following conjecture: Conceptual knowledge of difference-in-kind and difference-in-degree can emerge for the child in instrumented (Vérillon & Rabardel, 1995) fields of promoted sensorimotor interactions (Abrahamson & Sánchez-García, 2016; Abrahamson, & Trninic, 2014; Reed & Bril, 1996) with the cultural artifacts (Vygotsky, 1978) into which these target concepts are embedded (e.g., playing the game of *Sprouts* (see NRICH, n.d.) with Speks toys provided phenomenal grounding for the proto-conceptual learning of genus and connectedness). This conjecture about mathematical learning organizes the concepts of deep creativity as follows: In qualitative geometry, there is a property that defines the qualitative difference (or equivalence) of two shapes. That property can be considered a rule that determines whether two shapes are different or equivalent. By changing that rule, new shapes emerge as different or equivalent. These different shapes exhibit a truly *novel* distinction.

What new worlds arise when children come to understand that qualitative difference is a foundational dimension of learning for creative change? This is the question that framed the summer camp that provided the context for the second pilot study, which was held over 3 days with 8 children (3F, 5M) ages 5 to 10. Thirteen teacher-collaborators joined in to support and promote the children's agentive and exploratory play. Here we share some preliminary findings through an exploratory case study (Yin, 2014) of "Opal" (F), who was 5 years old at the time.

Opal was the youngest child at the camp; she was also the least verbal. A focal moment in Opal's activity elucidates her emergent thinking. At one point, we incorrectly inferred from her struggle to learn how to use the Speks toys to play Sprouts that the issue may be developmental. When we revised our approach to try and give her access to it, we learned from her responses that it was the scaffolding that we provided to Opal that was the source of the problem. Our approach was confusing Opal's already emergent ways of knowing about differences in kind and degree. For reasons that have to do with the rules of Sprouts, we had been promoting the use of "3-piece" Speks toys to form loops, and "2-piece" toys to enlarge or reduce the size of those loops. At the point at hand, Opal had already realized that a 3-piece could accomplish both tasks! At various points throughout her play, she used these 3-pieces to form loops and then make distinctions between shapes in terms of differences in kind based on *the number of loops* they possess. She also used those same 3-pieces to enlarge or reduce *the size of loops* and then make distinctions between shapes in terms of differences in degree. We had very little access to Opal's cognition through verbal expressions, but the enactive perspective (Thompson, 2007) revealed them via instrumental and embodied actions with Speks toys that provided evidence of her emergent cognitive structure (Varela et al., 1991) of qualitative difference.

A related conflict arose shortly thereafter when we sought to promote the use of "same" to name shapes that are alike in kind (i.e., topologically equivalent). In contrast, Opal enacted a more elaborate conception of same-ness, saying that two topologically equivalent shapes are actually *not* the same if they have "the same number" of loops, yet those loops and other features of a shape vary by degree. Thus, according to Opal's conception, two shapes are the same if and only if they are alike in kind *and also* alike in degree.

Conclusion

This case of Opal's experiences suggests the promise of this empirical exploration into developing the curricular and pedagogical components of an instructional approach to teaching qualitative change as it relates to deep creativity within elementary mathematics education. As this work continues, this project aims to make three contributions to the fields of creativity, mathematics education, and the learning sciences: (1) new understandings about the development and multimodal expressions of children's qualitative mathematical thinking, (2) new insights about the benefits children experience from their engagement in activities that develop their understanding of qualitative difference as a foundational dimension of learning for radical change, and (3) evidence of the transformative role of deep creativity in elementary mathematics education. In this brief research report, we share preliminary findings of pilot research through a case study of a 5-year-old child named "Opal."

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KINDERGARTEN THINKING: CONNECTIONS BETWEEN FORMAL MATHEMATICS AND STUDENT'S FUNDS OF KNOWLEDGE IN 3D SHAPES

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Keywords: Elementary Education; Geometry and Spatial Reasoning; Equity, Inclusion, and Diversity; Classroom Discourse

Students come with a wealth of knowledge in elementary mathematical spaces. Moll and colleagues (1992) refer to this informal cultural knowledge as students' Funds of Knowledge (FoK). Each culture brings their own unique perspective, background knowledge, and experiences (Banks, 1993; Pradhan, 2020). However, there exists the expectation that students be fluent in dominant mathematics (Gutiérrez, 2017). In the elementary classroom, dominant mathematics is displayed through strict processes, structures, and algorithms, leaving little room for instructional variation and multiple ways of knowing (Atwater et al., 2013). This limited view of mathematics continues to perpetuate the inequities in the US education system (Ladson-Billings, 2006).

Scholars document students' development of geometry concepts when students have agency around their learning (Civil, 2002; Natalija et al., 2019; Ng & Sinclair, 2015; Ng & Ye, 2022). Nevertheless, there is little to no research observing and documenting what connections students make from their FoK to the formal mathematical content. Because Clements and colleagues (1999) show primary students continued struggle to identify shapes, this study investigates students' connection between the formal mathematics and their FoK in the elementary classroom setting, specifically in the three-dimensional (3D) shapes unit.

3D shapes are important to teach because they strengthen students' ability to identify and organize visual information (Tsamir et al., 2015). Two overarching questions guide this study: How do primary elementary students incorporate their informal mathematical home-culture knowledge into their geometric units? How do students make connections between their informal mathematical knowledge and the formal mathematical terminology in a geometry unit in 2D and 3D shapes? In this study, students participated in their usual 3D shapes unit. I audio and video recorded their daily lessons consisting of a fifteen-minute, whole group mini-lesson followed by practice opportunities through table work. During the table work, I recorded a small group of six students (one table) and conducted five-minute, one-on-one interviews with the same students. The interviews focused on what shapes students drew and where they might see those shapes at home. Additionally, I collected the written work from the entire class.

To analyze the data, I use Bloome and colleagues' (2005) discourse analysis. The data will be transcribed and evaluated for common themes found between interviews, small group work, and whole class discussions. Through this analysis, I anticipate identifying key ways students talk about geometric shapes informally. I will identify the home contexts that are particularly relevant for students. I anticipate that students will use their FoK to distinguish between 3D shapes. I also expect students to use their FoK to describe the attributes of shapes.

By documenting the common connections students make between formal mathematics in 3D shapes and students' FoK, educators will be able to help students more authentically engage 3D shapes in ways beyond the current dominant mathematics in the formal academics. The students will be able to see themselves, their culture and community in their learning.

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TEACHERS' USE OF INVARIANCE FOR GEOMETRIC REASONING IN A DYNAMIC GEOMETRY ENVIRONMENT

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In this study, we considered how two teachers' use of invariant properties of draggable geometric objects are related to their geometric reasoning. We used a Discernment of Invariance theory to look at these teachers' levels of invariant properties as a way to make sense of their reasoning. Our analysis suggests that these teachers' ability to discern invariant properties at an advanced level was a key aspect for making meaningful conjectures, justifying, and explaining them. However, such an ability seems necessary, but not sufficient to explain and justify a geometric phenomenon. We found that unpacking more level-1 invariant properties and making more invariants level-2 connections between them can offer a richer exploration. We also found that an invariant property can be described as a feature of change. Implications for professional development and teacher education are discussed.

Keywords: Geometry and Spatial Thinking; Reasoning in Dynamic Geometry Environment; Variance and Invariance; Technology

Purpose and Background

Research shows that teachers' ability to discern invariant properties is a necessary skill in developing rich and deep understanding of draggable geometric objects (e.g., Leung 2015; Nagar, Hegedus, & Orrill, 2022b; Sinclair, 2018; Sinclair, Pimm, & Skelin, 2012). Despite its importance, invariance is often not as readily apparent among teachers; and when teachers considered invariant properties, they associated them mainly with shape, measurement, location, and calculation (Nagar, Hegedus, & Orrill, 2022a). Leung, Baccaglini-Frank, & Mariotti (2013) developed a Discernment of Invariance theory distinguishing between two levels: *Invariant properties level-1* are invariant properties that a person might perceived, while *invariants level-2 connections* are possible logical connections between level-1 invariant properties. Most research is focused on examining how a person might discern invariance and what type of invariant properties are discerned. More research is needed to unpack possible connections between invariant properties as part of reasoning. Thus, in this study, we investigated the following question: *How do two teachers use invariant properties at two different levels (invariant properties level-1 and invariant level-2 connections) as part of their geometric reasoning?*

While Leung et al. (2013) focused on students' awareness of invariant properties and whether they succeeded or failed to express awareness of level -2 invariant properties, in this study, we focused on what invariant properties level-1 teachers might use as part of their reasoning; what connections they might make; the direction and frequency of such connections; and whether there are other types of invariant properties that a person might discuss compared to previous results (e.g., Nagar et al., 2022a).

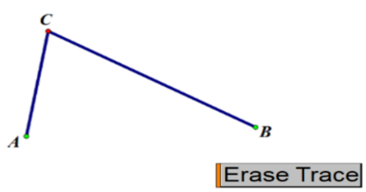
Theoretical Framework

Invariant properties are certain geometrical properties that remain unaltered when a transformation on the object is applied (Baccaglini-Frank, Mariotti, & Antonini, 2009; Hadas, Hershkowitz, & Schwarz, 2000; Laborde, 2005; Yerushalmy, Chazan, & Gordon, 1993). We consider a variable object as a draggable object under transformation, and the property that is 'an invariant satisfied' (i.e., remains unaltered) as an invariant property (Nagar et al., 2022b).

We drew on Leung et al.'s (2013) Discernment of Invariance theory to distinguish between two levels of discernment of invariant properties in a DGE. Discernment of invariant properties level-1 denotes to the awareness of invariant properties of a dynamic figure perceived under different dragging strategies; and discernment of invariant level-2 connections refers to the awareness of different types of control on level-1 invariant properties to imply logical relationship between them. We used a tablet based DGE, so the user could place one or more fingers to enact movement (by dragging an object). The use of dragging allows the user to "feel" the variation of the construction (Baccaglini-Frank & Mariotti, 2010) and to discern invariant properties (Nagar et al., 2022b; Sinclair et al., 2012), which can be help in unpacking the situation at hand.

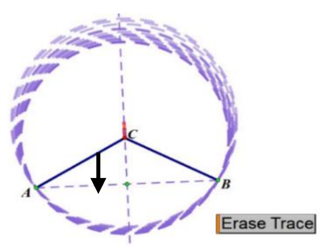
Data Sources and Method

(a) Activity 1



Participants could manipulate a draggable point (point C) connected to two fixed points A and B. Based on idea presented by Leung (2003). Point C could be dragged anywhere on the screen and leaves a colored trace.

(b) Possible Generalization



Generalizing that as long as point C (the center of the desired circle ("dashed circle") is being dragged along the perpendicular bisector of segment AB, it is possible to maintain a circle that changes its size.

Construction: Drawing three points, setting A and B to be fixed points, and constructing segments AC and BC.

Figure 1: Activity Used in the Study

The work reported here is part of a larger study where the primary data came from a set of two 45-minutes video recorded task-based interviews with six high school mathematics teachers: Four females - Amanda, Lisa, Diana and Susan; and two males - Andy and Mark (pseudonyms). Each interview focused on a set of four activities that were designed using the Sketchpad® Explorer (Jackiw, 1991). Figure 1 presents one of these activities. All interviews were videotaped and transcribed verbatim. We focus on Activity 1 Part 2 where participants were asked to think about ways to drag point C so that a circle passing through A and B with C as its center can exist; and then, if possible, to maintain such a circle.

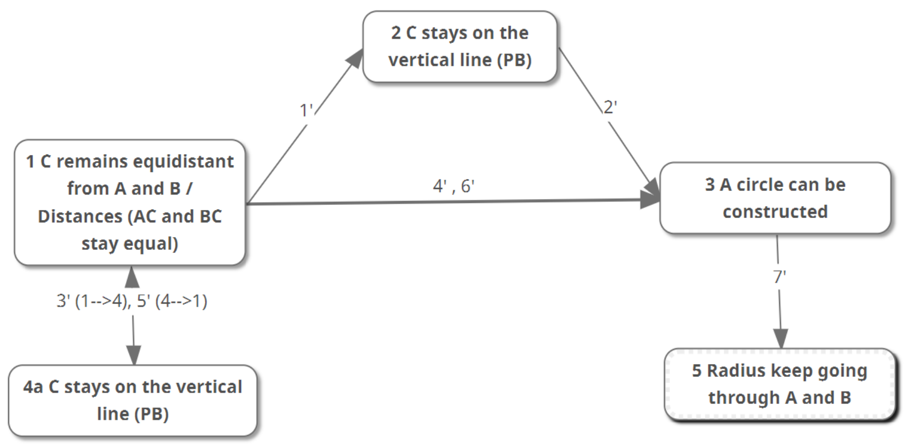
The analysis included two cycles of coding. In the first, we used *descriptive coding* (Miles, Huberman, & Saldaña, 2014) and created four spreadsheets to look at what participants said (verbally) and did (through actions). In the second cycle, we coded characteristics related to the two levels of invariants (invariant properties level-1and invariant level-2 connections). We also tracked the justifications and explanations provided by participants (if any), as well the direction and relative frequency of these connections. We then created *connection maps* (see below) to focus on both the invariant properties level -1 (represented by nodes) and invariant level-2 connections (represented by arrows). Connections map visually shows the logic relationships (level-2 connections) between the level-1 invariant properties. Lastly, we used a *pattern coding* method (Miles et al., 2014) to look for major themes and patterns.

Results

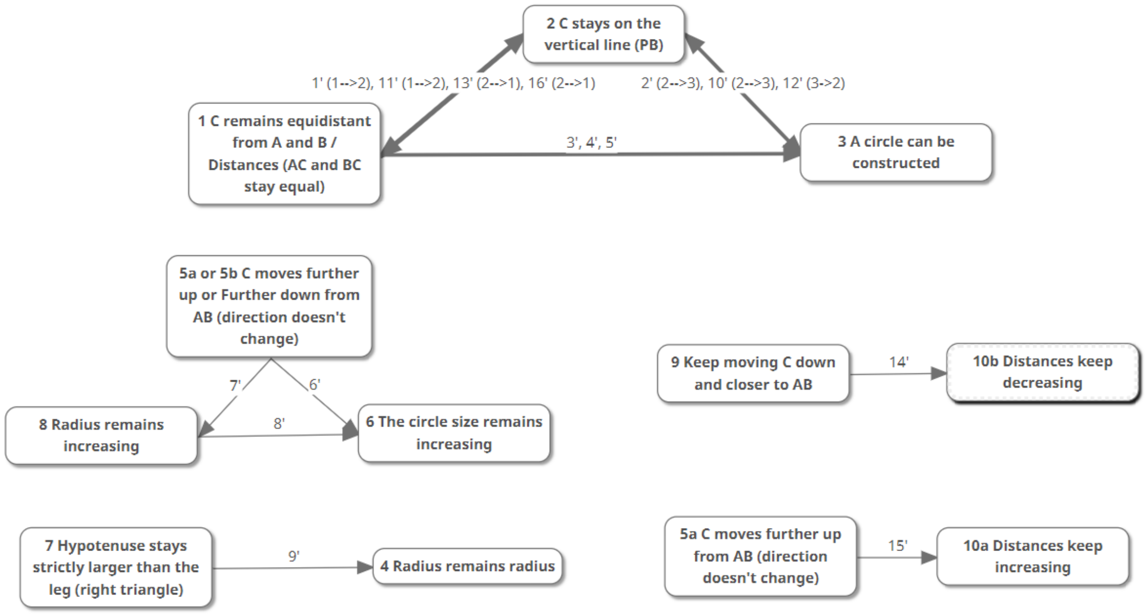
Invariant Properties Connections Maps

Table 3: Invariant Properties Connections Maps

Andy's use of invariant properties connections map



Mark’s use of invariant properties connections map



Note: A node represents the invariant property discerned (Level-1) and describes it.

The numbering inside the nodes is used to order the properties according to the participant's order of discernment.

An arrow between two nodes represents a connection between two invariant properties and its direction (Level-2).

A connection is a conditional/conjecture of the form "if ... then..." statement.

The prime numbering (e.g., 1', 2', etc.) indicates the order of the connections participant made. When a bidirectional arrow appears then the parenthesis indicates the direction of that connection – for example in Andy’s map the connection 3’(1→4) means that the participant made a statement (in his 3rd connection) saying that if the distances AC and BC stay equal (Node 1), then C remains on the vertical line (PB-perpendicular bisector) to AB (Node 4).

The thickness of an arrow indicates the relative frequency of the connection. The frequency can also be calculated by counting the prime numbers appearing on a single arrow – that is the number of times a connection was made.

Comparing between Andy’s and Mark’s Invariant Properties Connections Maps

In looking at both connections maps, it seems that both Andy and Mark were able to make important connections between key level-1 invariant properties to unpack the situation and to generalize that the perpendicular bisector of AB is a path that allows for the desired circles to be constructed. In addition, they both drew on the definition of a circle to make these connections. However, while Andy discerned only five invariant properties, Mark discussed twelve invariant properties. Mark offered advanced reasoning combining several big ideas in geometry such as: circle, right triangles, hypotenuse, perpendicular bisector, etc., and he also discussed several interesting invariant properties related to a feature of change.

While Andy made seven invariant level-2 connections, Mark made sixteen such connections and it seems that his reasoning was more coherent (using the same ideas more frequently). Mark was also able to generalize and formulate more connections between level-1 invariant properties and to unpack the situation in more depth. Lastly, although both participants were able to discern invariant properties at an advanced level (in both levels of invariant properties), still we saw that Andy did not justify all his conjectures even when was asked directly to try. Also, Mark did not provide justifications for his last two invariant level-2 connections (14' and 15').

Discussion and Conclusions

Our analysis shows that identifying different level-1 invariant properties and making invariant level-2 connections between them were key aspects in unpacking a dynamic geometric situation. Similar to Leung et al. (2013) who "were able to gain insight into explorations within DGE, analyzing in fine detail how discernment unfolds" (p. 458), we saw that both levels of invariant properties seem to be the basis for having the skills to make conjectures, state theorems, justify them, and make links to generalization. Thus, it is suggested that teachers might benefit from not only discerning multiple invariant properties, but also making and justifying possible connections between them. It also seems that if a person is able to unpack more level-1 invariant properties and make more connections between them (level-2), the exploration can be richer and deeper. We conclude that future research should focus on how to design opportunities in dynamic environments to support teachers in developing such skills.

We also found that some invariant level-2 connections were not justified. Maybe some connections are difficult to explain (e.g., Mark's connections 14' and 5'). Thus, it seems that the ability to identify level-1 invariant properties and make connections (level-2) is necessary, but not sufficient to explain and justify a geometric phenomenon. More research is needed to examine what challenges teachers might express in working with both levels of invariants, and how it is possible to overcome such challenges.

Lastly, Mark drew on an interesting idea - *an invariant feature of change* (e.g., describing a distance as changing in length but having a feature that remains the same – always increasing). He used this not only to explain a phenomenon about the size of the circle as increasing or decreasing, but also to experience the object has having different formations (continuous variation). Doing so, he offered a richer exploration and unpacked more sophisticated ideas. This adds to previous results (Nagar et al., 2022a) where we analyzed more than 150 descriptions and found four categories of invariant properties (Shape, Measurement, Location, and Calculation). So, in this paper, instead of just looking at something that stays constant like a circle (Shape) or distances (Measurement), it is possible to have a feature of change as invariant property. This might be an important aspect of geometric reasoning. Thus, the role of such invariant property (feature of change) in reasoning is worth further examination, as well as whether such an invariant property can be easily accessible to teachers or not.

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STUDENTS' MEANINGS FOR COORDINATE SYSTEMS: CONTINUOUS AND ORDERED-DISCRETE REFERENCE FRAMES

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Graphical representations are commonly used in everyday life and are important in STEM fields. Interpreting graphs entails understanding the underlying structures of graphs, including coordinate systems and reference frames. In this report, we characterize one student's constructions of coordinate systems. These constructions indicate two distinct types of reference frames not currently distinguished in the literature: (a) continuous and (b) ordered-discrete. Using data from a 10-session teaching experiment, we discuss the interplay of a student's perception of tasks, the reference frames she reasoned with, and differences in those reference frames. We consider how the interplay of the aforementioned items may have influenced the quantities she considered as well as the coordinate systems she constructed. We conclude with suggestions for research and teaching that support students' productive graphing activity.

Keywords: Geometry and Spatial Reasoning, Mathematical Representations, Cognition, Middle School Education

Graphs are a powerful way to visualize, explore, and communicate relationships between quantities. In STEM contexts, graphs can be used to mathematize spatial situations or represent relationships between covarying quantities (Paoletti et al., 2020; Glazer, 2011). In our view, students' meanings for graphs should depend on their meanings for coordinate systems, especially if their meanings for graphs are to be productive (Lee et al., 2020). Recent research has focused on differences in the underlying coordinate systems that students construct and how reasoning within these coordinate systems explains their graphing activity (Paoletti et al., 2018, 2022; Parr, 2023). In this report, we offer another contribution to this literature by characterizing two novel types of reference frames that underlie coordinate systems and by describing how students may use these reference frames to reason about quantities represented in coordinate systems. We begin with a theoretical background that defines two different types of coordinate systems and establishes a distinction between types of reference frames. We describe one student's use of both types of references frames within each coordinate system. We conclude with implications for teachers, curriculum designers, and researchers.

Theoretical Background: Two Types of Coordinate Systems and Reference Frames

Researchers (Lee, 2017; Lee et al., 2019, 2020) have distinguished between two kinds of coordinate systems (CSs): spatial and quantitative. Each type of CS is built by coordinating one or more reference frames. *Reference frames* (RFs), which are constructed to gauge relative extents of attributes in phenomena, consist of some orienting reference objects, directionality, and some anticipation of a measurement process that could be carried out (Lee et al., 2020; Joshua et al., 2015). When students consider quantities (Thompson, 2011), there must be at least one RF involved. We provide several examples of abstract quantities, using parentheticals to provide specific examples of situational quantities with explicit RFs: Distance (e.g., number of miles east a person is from school), time (e.g., number of minutes after passing a rest stop), and temperature (e.g., degrees Fahrenheit above 0).

A *spatial CS* involves the mental coordination of one or more RFs and a selection of units of measure which are imposed onto a physical space of interest. In this case, RFs are used to gauge the relative locations of objects within that space. In a spatial CS, locations in the space may be tagged with coordinates guided by these RFs and obtained through carrying out the anticipated measurement. For example, a student might organize the map in Figure 1a by constructing a spatial CS consisting of two RFs that imply the consideration of distinct distances from a reference object (like the star icon). The spatial CS could then be used to describe the X's (or any object's) location in terms of unique pairs of distances from the star icon.

A *quantitative CS* involves the mental coordination of one or more quantities which are activated upon assimilation of a situation, disembedded from it, and inserted into a new representational space through the coordination of their RFs. For example, a person may coordinate the relationship between the time and temperature throughout a day and represent this relationship via a graph in a quantitative CS. Within both spatial and quantitative CSs, locations within the CS are imbued with quantitative extents, which necessarily involve RFs.

In this paper, we add a fourth dimension to thinking within RFs: continuity. In addition to reference object, directionality, and some anticipated measurement process, we have found in our work with students that the notion of directionality and some anticipated measurement process could be established either discretely or continuously. Hence, we distinguish between two kinds of RFs that students indicated when reasoning in both types of CSs: *ordered-discrete RFs* and *continuous RFs*. A *continuous RF* involves understanding a continuum of an attribute's extents relative to the reference object and guides measuring activities that would lead to measurements as continuous quantities. An *ordered-discrete RF* involves segmenting an attribute's extents according to distinct, bounded regions that are arranged in some (implicit or explicit) sequence. Ordinal or directional language can be an indication of an established ordered-discrete RF and guides measuring activities that would lead to measurements in discrete units. For example, an individual who has established an ordered-discrete RF within a designed region might describe sub-regions in ordinal terms (e.g., second row or last circle from the center) or directional terms (e.g., left side or near the middle).

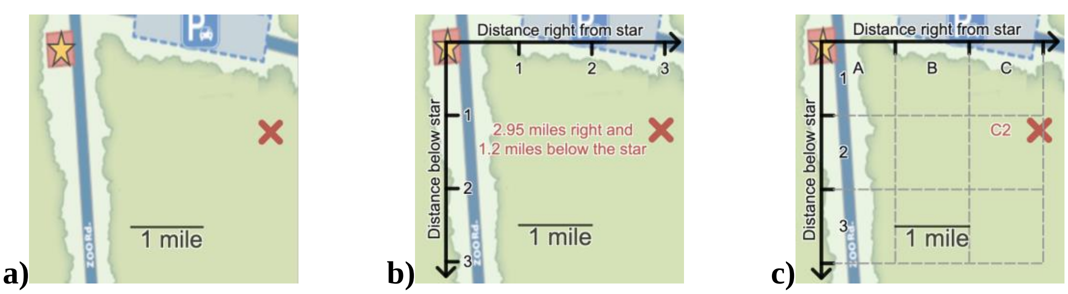

Figure 1: A map indicating: a) no CS, b) a spatial CS constituted by coordinating two continuous RFs c) a spatial CS constituted by a coordination of two ordered-discrete RFs.

We note an individual may understand an attribute as continuous and still construct an ordered-discrete RF; such a construction is dependent on the student's conceived context, goals, or current quantitative constraints in their reasoning. For example, Figure 1b shows how a spatial CS could be constituted by coordinating two continuous RFs. Figure 1c shows how a spatial CS could be constituted by coordinating two ordered-discrete RFs. In this report, we address the research question: *How does a student's construction of continuous and ordered-discrete RFs impact her reasoning in spatial and quantitative CSs?*

Methods

To address our RQ, we report on data from a teaching experiment (Steffe & Thompson, 2000) with three sixth-grade students: Nina (who self-identified as Latina), Tara (who self-identified as a White female), and Jacobi (who self-identified as an African American male). We focus this report on Nina's activity because she provided the strongest indications of the RFs of interest. The teaching experiment took place in a middle school whose population consisted of over 75% students of color. Participants were recruited based on teacher recommendation and student availability. Nina attended 10 teaching experiment sessions each lasting 35–40 minutes (Table 1). We video- and audio-recorded each session to capture utterances and gestures. Student activity on the Desmos platform was screen recorded, and we digitized all written work.

Table 1: Small Group Teaching Experiment Sequence

Session	Students Present	Task	Intended Student Goal
0	Nina	Pre-Interview	Various
1	Nina, Tara, Jacobi	X Marks the Spot – Guess Where	Construct and/or interpret spatial RFs and CSs to describe and/or identify locations in space
2	Nina, Tara	X Marks the Spot – Anywhere	
3		X Marks the Spot – Classmates' Descriptions	
4	Nina, Jacobi	X Marks the Spot – Anywhere	
5	Nina, Tara, Jacobi	North Pole Task	
6 7	Nina, Tara, Jacobi	Zoo Task	Interpret points in a quantitative CS
8	Nina, Tara, Jacobi	Kodiak Task	Interpret graphs in a quantitative CS
9	Nina, Tara		
10	Nina, Tara	Post-Interview (Growing Fruit)	Various

Tasks

We describe Nina's activity across several tasks from Sessions 3, 4, and 10. In the *X Marks the Spot-Anywhere* and *-Classmates' Description* tasks (Sessions 2–4), students took on the roles of Describer and Guesser in Desmos. The Describer was prompted to mark an X on the map and then generate a description of the X's location. The Guesser used that description to mark an X on their own version of the map. Students had access to a set of digital overlays (e.g., vertical lines, horizontal lines, concentric circles anchored at the star icon) that could be activated to potentially support students' location descriptions. For example, in Figure 2, the 'Horizontal' and 'Vertical' overlays are activated. In the *Anywhere* variation of the task, a pair of students take turns as Describer and Guesser for each other. In the *Classmates' Descriptions* variation, the students worked together as Guessers, with hypothetical classmates as Describer. The hypothetical classmates' descriptions were researcher-authored and sequenced to progress from (what we then considered) less precise to more precise descriptions in both polar-like and Cartesian-like CSs. We had yet to distinguish between continuous and ordered-discrete RFs when we authored these descriptions. However, in retrospect, the descriptions that we considered less precise used language indicative of ordered-discrete RFs while the descriptions that we considered more precise used language indicative of continuous RFs. In Session 10 Nina and Tara completed a post-interview together wherein they attempted tasks individually, and the teacher-researcher (TR) facilitated discussion across their responses. The fourth task of the post-interview, *Growing Fruit,* asked students to describe a situation that would be reflected by a given graph representing the relationship between a hypothetical fruit's weight and calorie content, both of which changed over time (Figure 3a).

Analysis

Consistent with teaching experiment methodology, we analyzed the data via conceptual analysis, which entails "building models of what students actually know at some specific time and what they comprehend in specific situations" (Thompson, 2008, p. 45). We watched all videos and identified moments that offered insight into the CSs and RFs Nina constructed as she addressed each task. We then created models characterizing whether Nina was constructing quantitative or spatial CSs. As we described Nina's reasoning in each type of CS, we characterized continuous and ordered-discrete RFs as an important distinction in her reasoning; we had not considered this distinction prior to conducting this analysis.

Results

Nina used two distinct types of RFs, *ordered-discrete* and *continuous* to construct and interpret both spatial and quantitative CSs. Further, the RF Nina constructed influenced her reasoning in each CS. Because Nina constructed both types of RFs within both types of CSs, we present these types of reasoning in a two-by-two matrix and detail four examples from the teaching experiment that demonstrate each combination (Table 2).

Table 2: Task and activity in which Nina constructed a CS using each type of RF

	Ordered-discrete RFs	Continuous RFs
Spatial CS	Describing locations in *X-marks the Spot-Anywhere*	Interpreting locations in *X-marks the Spot Anywhere*
Quantitative CS	Interpreting a given graph in the *Growing Fruit task*	Interpreting a modified graph in the *Growing Fruit task*

Continuous Reference Frames in Spatial Coordinate Systems

In Session 3, Nina interpreted continuous RFs in a spatial CS when she followed a hypothetical classmate's description to mark an X in the *X Marks the Spot – Classmates'*

Descriptions task. Nina's interpretation of the following description shows her ability to construct continuous RFs in a spatial CS:

> Click the Star (1) and Circles (3) options. Imagine the star is like a clock with the line going straight up being 12 o'clock and the line going straight down being 6 o'clock. The X is 1.25 miles from the star [icon] halfway between 10 and 11 o'clock.

This description is intended to introduce a pseudo-polar, spatial CS in which the continuous RFs are the radial distance (explicitly in 'miles') from the star icon and angle measure (implicitly in 'hours') from the top vertical line. After reading the description, Tara moved the cursor to an approximately correct location. Nina grabbed a measuring device (a wax-covered string bent at a length equivalent to the '1 mile' key on the map), which she used to confirm Tara's approximation. Specifically, Nina placed one end of her measuring device at the center of the star overlay and oriented the other end halfway between the lines representing 10 and 11 o'clock, near where Tara had placed the cursor. Nina reasoned that if the string piece was one mile, then the X must be slightly beyond it. Hence, Nina reasoned about distance from the center as a continuous quantity (i.e., 1.25 miles is slightly more than 1 mile) while also attending to the clock description as a continuous quantity (i.e., a location halfway between 10 and 11 o'clock). Thus, Nina used continuous RFs to generate an exact location in a spatial CS.

Ordered-Discrete Reference Frames in Spatial Coordinate Systems

In Session 4, Nina primarily used ordered-discrete RFs. For example, in her third turn as Describer in the *Anywhere* variation of the task, Nina established a spatial CS using two ordered-discrete RFs to describe a region in which her X was located. Nina marked an X as in Figure 2a and provided the description "Use horizontal and vertical lines. The lines make squares so count from the left, go all the way to the bottom, and count 6. Then go up 2, the x is in the right corner." To Nina, the combination of the vertical and horizontal overlays created distinct 'squares' (discrete regions) that Jacobi could count (ordering language) to identify which region contained the X (Figure 2a). Reflecting the non-continuous nature of the RFs Nina was constructing, the relative size of these 'squares' was not relevant from her perspective; her description did not distinguish the partial boxes in the bottom-most row and left-most column from the other boxes in the grid. Hence, we infer she was reasoning about discrete, ordered, regions from the bottom left corner. Hence, Nina established a spatial CS to describe a region by coordinating two discrete ordered RFs (The number of boxes to the right starting from the left side of the map and the number of boxes vertically up from the bottom of the map).

a) b)

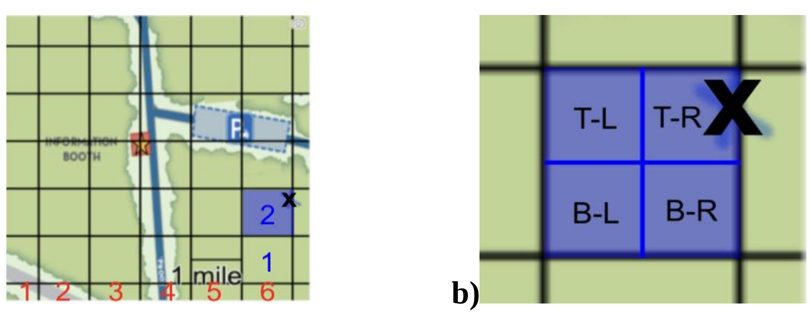

Figure 2. Representation of Nina's ordered-discrete reasoning in space in a) the first part and in b) the second part of the description

Nina’s use of ordered-discrete RFs influenced her activity in the spatial CS as it led her to using more than one set of ordered-discrete RFs as she described increasingly narrow regions in which points were located. That is, we infer that Nina’s addition of “x is in the right corner” was a second ordered-discrete RF she constructed within the first box she described. (We note Nina did not specify between top or bottom right corner, but we conjecture she meant top-right based on the X’s placement.) Our inference is based on her use of “right corner” as a location rather than a reference object (i.e., “in the right corner” as opposed to “1 cm from the right corner”). One possible way she could have done this is by mentally subdividing the ‘square’ into (at least) four discrete, ordered quadrants (i.e., top-left, top-right, bottom-left, bottom-right; Figure 2b). Thus, we infer that Nina could have coordinated two ordered-discrete RFs (left/right and top/bottom from the midpoint of the box) to describe a narrower region within a particular region of a spatial CS.

Reference Frames in Quantitative Coordinate Systems

In Session 10, Nina addressed the *Growing Fruit* task (Figure 3a). A normative explanation would include a description that at first the fruit gains weight while its calories remain the same and then the fruit’s weight and calories increase simultaneously. Based on Nina’s activity in Sessions 7-11, we anticipated Nina would use continuous RFs to interpret the given graph and produce a normative explanation. However, we infer that Nina initially reasoned about the quantities using an ordered-discrete RF and shifted to using a continuous RF when the TR added numbers to the horizontal axis. We provide evidence for each claim in the next two sections.

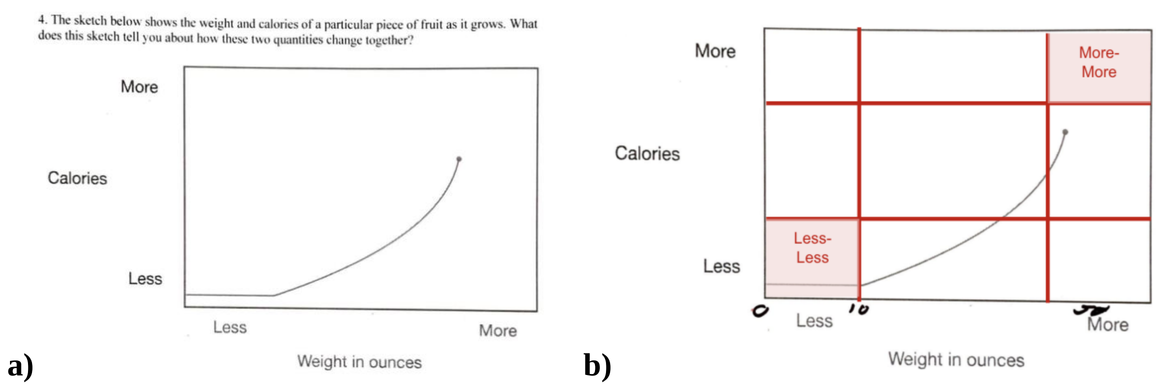

Figure 3: a) Growing Fruit Task as presented and b) a potential depiction of how ordered-discrete RFs could be coordinated to reason about the horizontal segment of the graph.

Ordered-discrete reference frames in quantitative coordinate systems. When initially interpreting the horizontal segment (and possibly the entire graph), Nina employed an ordered-discrete reference frame. Describing a situation that created this graph, Nina explained:

N: The fruit starts off, like, without any calories [*points to horizontal segment*] and doesn’t weigh a lot. And then while it grows [*traces curve*] it gains calories and … weighs more.
TR: Gains calories and weighs more?
N: Yeah.

TR: [*referring to the curved part of the graph*] So that's sort of what you [Tara] were saying, too. So, I think you're both in agreement. Now let me ask you [Nina] this question. If we start say, here [*gestures to the vertical intercept*] and just paying attention to this part [*tracing the horizontal portion of the graph*]. What's changing?
N: Nothing.

Considering Nina's argument, Tara disagreed with it. Tara traced the horizontal part of the graph saying, "Well as you go right the weight is getting bigger because it's getting closer to 'more' weight, I guess. But the calories would stay the exact same right here." Nina explicitly disagreed with Tara's argument stating, "I don't think here it's getting bigger [*traces horizontal segment*]. 'cause it's like [*pointing to "Less" markers on each axis*]...[3 second pause] For me it's like not getting bigger cause it's like still at less."

We interpret Nina as reasoning with ordered-discrete RFs as she interpreted the weight and calories of the fruit for the horizontal segment. In particular, she argued the horizontal segment was representing the quantities as both being in a static state of 'small' because the segment was close to the 'Less' label on each axis. Like her activity in *X Marks the Spot - Anywhere,* Nina was reasoning about ordered-discrete RFs on each axis by creating regions based on the 'Less' labels along each axis. We show one potential illustration of the resulting regions Nina may have been reasoning about in Figure 3b.

We note Nina's initial description of the curved segment ("while it grows it gains calories and … weighs more") could be indicative of reasoning with either ordered-discrete RFs or continuous RFs. If Nina understood that the curved graph spanned the (Less, Less) region and the (More, Medium) region, then she might have argued that the weight and calories both increased by some unknown amount as each moved into a higher-ordered region. If, however, Nina understood there to be a continuum of values beyond the (Less, Less) region, then she could have been using a continuous RF to reason about this part of the graph. As the TR was not aware of the distinction between the two types of RFs in the moment, he did not explore this possibility further. However, he did conjecture the 'Less' and 'More' labels on the axes, which were novel relative to quantitative CS used in previous sessions, may have been the catalyst for her reasoning about the straight segment. Hence, he opted to add numbers to the horizontal axis (0, 10, 50; seen in Figure 3b) to see if this change would lead Nina to a different interpretation.

Continuous Reference Frames in Quantitative Coordinate Systems. When the TR added the numbers to the horizontal axis, Nina immediately engaged in reasoning about the horizontal segment using a continuous RF in a quantitative CS and generated a normative interpretation of this part of the graph:

TR: But say if there were numbers here. Say this was like 0, 10, and like 50 [*writes in numbers on horizonal axis as shown in Figure 3b*]
N: Then it would get bigger
TR: Then you think it would-
N: It would weigh more
TR: You think it would weigh more?
N: Yeah
TR: And what about the calories? Would that be changing?

N: No [*shakes head*]

TR: No? Okay so it's sort of like this distinction between sort of like 'less' like we're in this less state-

N: Yeah

TR: -versus if there were numbers, you'd say they were changing? [*Nina nods head*]

When the TR added numbers to the horizontal axis, Nina immediately interpreted the horizontal segment as showing the weight increasing ("It would weigh more") as the calories remained constant. Thus, we infer Nina understood the horizontal axis as a continuous RF representing the weight of a hypothetical fruit. Furthermore, she agreed with the TR that the distinction between viewing the horizontal segment as representing a state of 'less-ness' versus viewing it as a record of change was based on the addition of the numbers. Hence, we infer the addition of numbers to the axes changed Nina's interpretation of the graph (and of the situation) as she shifted from using an ordered-discrete RF to using a continuous RF. Further, she exhibited reasoning compatible with a continuous RF on the next task in the post-interview, which asked her to construct a graph to represent the weight and calories of a novel fruit.

Discussion, Implications, Limitations, and Concluding Remarks

Addressing our RQ, we have shown how Nina used ordered-discrete and continuous RFs to reason within both spatial and quantitative CSs. Within a spatial CS, Nina's use of different RFs led to different strategies to mark or describe a location. With continuous RFs, Nina could identify an exact location, but she reasoned about increasingly narrow regions when using ordered-discrete RFs. In a quantitative CS, Nina's use of RFs impacted her interpretation of a situation represented graphically. When engaging with ordered-discrete RFs, Nina treated a segment of the graph as a single object, representing a static condition (i.e., Less-Less). However, with a minor alteration to the task, Nina considered the weight RF as continuous, thereby interpreting the segment as representing a record of change.

We note that Nina's construction of different RFs was influenced by her interpretation of and/or goals in the task. Although she was capable of reasoning with continuous RFs in both spatial and quantitative CSs, she opted to use ordered-discrete RFs when they satisfied the demands of a given task as she perceived it. We have observed other students who, like Nina, are capable of reasoning about continuous RFs in a spatial CS but opt to use ordered-discrete RFs to satisfy their perceived demands of a given task.

Implications for Curriculum and Instruction

We consider it likely that a continuous RF supersedes an ordered-discrete RF. Our hypothesis is that individuals who have constructed a continuous RF in a context would necessarily be able to construct an ordered-discrete RF in the same context, whereas an individual who constructs an ordered-discrete RF may not yet be able to construct a continuous RF in that context.

However, we emphasize that one type of RF is not inherently preferred; rather, their utility is determined by an activity's (or student's) context and goals. In spatial CSs, regions can be described using continuous RFs (e.g., Webb & Abels, 2011), but there may be instances in which ordered-discrete RFs are sufficient or even more appropriate. Although continuous RFs are more commonly used when constructing quantitative CSs, there are situations in which ordered-discrete RFs are useful. For instance, Webb and Abels (2011) describe using combination charts to describe the relationship between three quantities, such as cost of a number of pencils (represented along a horizontal axis), cost of a certain quantity of erasers (represented on the vertical axis), and total cost of n-pencils and m-erasers (represented in the cell (n, m)). Such a combination chart is an example of a quantitative coordinate system made up of two ordered-discrete RFs, in which number of pencils and number of erasers are discrete quantities.

It is important to be aware of the distinctions between these types of RFs, as their conflation can lead to unintended graphical interpretations. For instance, Figure 1c depicts a spatial coordinate system, but it is ambiguous whether each RF should be treated as continuous or ordered-discrete. On one hand, the vertical numeric labels suggest that students could describe the X's position using a continuum, but the use of letters as labels on the horizontal axis limits the ability to refer to non-discrete positions. Further, the positioning of the labels between tick marks rather than on tick marks may promote the creation of regions rather than a continuum. Depending on how an activity using a similar map is enacted, students may not conceive a distinction between the two types of RFs. Teachers and curriculum designers should be deliberate in crafting tasks and graphs such that students are prompted to engage with both types of RFs and explore the affordances and limitations of each in a variety of spatial and quantitative contexts.

Limitations and Concluding Remarks

This report is limited in that we only analyzed the activity of one student in a particular set of tasks. Future researchers may be interested in exploring how a wider range of students spontaneously construct and utilize both continuous and ordered-discrete RFs in quantitative and spatial CSs. Such research can support the field's understanding about how students reason about graphs and how such reasoning can be supported towards more normative graphing meanings. Understanding the ways in which students interpret and construct the fundamental components of graphs, such as RFs, is crucial to supporting students' developing meanings for graphs, which are ubiquitous in STEM contexts.

Acknowledgments

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HOW PICTURE BOOKS PRESENT AREA AND PERIMETER

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We conducted an analysis of children's literature centered on area and perimeter, using an adaptation of a framework for assessing the characteristics of picture books for learning mathematics. This paper focuses on the presence and absence of specific elementary and middle school Common Core State Standards related to area and perimeter within twelve picture books. We also consider whether the mathematics content is explained within each picture book. Findings reveal there are available picture books for teaching area and perimeter throughout elementary and middle school grades, except for sixth grade. We also found that most of the books presented the mathematics with explicit explanations, instead of implicitly. These findings could aid in teacher's selection and use of picture books when teaching mathematics.

Keywords: Elementary School Education, Geometry and Spatial Reasoning

Introduction

As educational practices evolve in these changing times, it is critical to envision the future of mathematics teaching. With traditional approaches being reevaluated, innovative teaching methods are becoming essential to meet the changing needs of students. Using picture books in mathematics instruction can meet those needs in the following ways.

Picture books enhance student engagement by immersing them in captivating stories, where interactions with characters and mathematical content make learning exciting (Skoumpourdi & Mpakopoulou, 2011). These books provide relatable contexts that allow students to apply their knowledge and see the relevance of mathematics in their daily lives (McAndrew et al., 2017). Lastly, picture books offer a promising way of teaching novel terms and associated properties of geometry specifically, by using both pictures and words. Further classroom discussion of these terms within the context of the books encourages students to make sense of and communicate with new mathematical language (Capraro & Capraro, 2006; McAndrew et al., 2017; Skoumpourdi & Mpakopoulou, 2011).

Despite these advantages, not all picture books related to mathematics are equally useful for teaching (Skoumpourdi & Mpakopoulou, 2011; Nurnberger-Haag et al., 2021). For example, how the mathematical content is presented influences the way a teacher incorporates the book in their teaching. If the mathematical content is only implied in the story, and not explained, the teacher would need to bring out the mathematics and help students build their understanding (Austin, 1998). However, if the mathematical content is explicit, there is danger it is being used to "tell" the mathematics, which can hinder cognitive engagement among students (Lobato et al., 2005). Thus, picture books' effectiveness in teaching mathematics must be assessed individually. However, there is limited research evaluating picture books for their effectiveness in teaching mathematics (Nurnberger-Haag et al., 2021), and most focus on the topic of shape (e.g., McAndrew et al., 2017; Skoumpourdi & Mpakopoulou, 2011). Our goal is to aid in broadening this field, by focusing on assessing picture books covering the concepts of area and perimeter, chosen due to students' common struggles in understanding these topics (Milinia & Amir, 2022). Using an adaptation of van den Heuvel-Panhuzen and Elia's (2012) f *Framework of Learning-Supportive Characteristics of Picturebooks for Learning Mathematics*, we intend to address the overarching question: How does the presentation of area and perimeter in picture books relate to their usability for teaching, based on the framework's learning-supportive characteristics? In this preliminary study, we focus on two questions: (1) Which elementary and middle school Common Core State Standards about area and perimeter are represented in picture books?; and (2) How are these concepts presented—implicitly or explicitly?

Theoretical Framework

In this study, we draw upon van den Heuvel-Panhuzen and Elia's (2012) *Framework of Learning-Supportive Characteristics of Picturebooks for Learning Mathematics* as a foundation. This framework identifies characteristics of picture books that contribute to the learning of mathematics by kindergartners. It consists of two main sections: (1) *Supply of mathematical content* identifies the mathematical content within a picture book; and (2) *Presentation of mathematical content* indicates how the mathematics is presented (e.g., explicitly or implicitly) and the characteristics that could prompt specific behaviors from children.

A. Supply of Mathematical Content		
A.1. Mathematical processes and dispositions		
A.2. Mathematical content domains The picturebook deals with CCSS.MATH.CONTENT… ***2nd grade*** **G.A.2:** Partition a rectangle into rows and columns of same-size squares and count to find the total number of them. ***3rd grade*** **G.A.2:** Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. **MD.C.5:** Recognize area as an attribute of plane figures and understand concepts of area measurement. - **A:** A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area. - **B:** A plane figure which can be covered without gaps or overlaps by *n* unit squares is said to have an area of *n* square units. **MD.C.6:** Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units). **MD.C.7:** Relate area to the operations of multiplication and addition.	- **A:** Find the area of a rectangle with whole-number side lengths by tiling it, and show that the area is the same as would be found by multiplying the side lengths. - **B:** Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning. - **C:** Use tiling to show in a concrete case that the area of a rectangle with whole-number side lengths *a* and *b* + *c* is the sum of *a* × *b* and *a* × *c*. Use area models to represent the distributive property in mathematical reasoning. - **D:** Recognize area as additive. Find areas of rectilinear figures by decomposing them into non-overlapping rectangles and adding the areas of the non-overlapping parts, applying this technique to solve real world problems. **MD.D.8:** Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.	***4th grade*** **MD.A.3:** Apply the area and perimeter formulas for rectangles in real world and mathematical problems. ***6th grade*** **G.A.1:** Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems. **G.A.4:** Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems. ***7th grade*** **G.B.4:** Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle. **G.B.6:** Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.
B. Presentation of Mathematical Content		
B.1. Way of presenting The mathematical content… - is addressed *explicitly* (something mathematical is happening that is explained) or *implicitly* (something mathematical is happening that is not explained) - is *integrated* in the story (either explicitly or implicitly) or *isolated* from the story (e.g., there is a picture of somebody wearing a dress with a nice geometrical pattern, but the story does not mention the dress)		
B.2. Quality of presentation		

Figure 1: Mathematical Content Domains and Way of Presenting (Adapted from van den

Heuvel-Panhuzen & Elia, 2012)

We adapted van den Heuvel-Panhuzen and Elia's (2012) framework to apply to picture books focusing on area and perimeter across various grade-levels. The primary change was replacing A.2—*Mathematical content domain*—with the Common Core State Standards of Mathematics (CCSSM) related to area, perimeter, circumference, and surface area for elementary and middle school grades. Notably, kindergarten, first, fifth, and eighth grade have no standards on these topics. We defined each characteristic of the framework, except for the CCSSM, was defined using our interpretations of van den Heuvel-Panhuzen and Elia's (2012) explanations. Although other adaptations were made, they fall beyond the scope of this paper.

In this report, we will focus on characteristics A.2—*Mathematical content domains*—and how they were presented, implicitly or explicitly, within B.1—*Way of presenting*—as shown in Figure 1. Content presented *explicitly* means the mathematics was explained in the words or images, while an *implicit* presentation includes no explanations of the mathematics.

Methods

The picture books we analyzed were chosen based on their connection to area, perimeter, circumference, and surface area. These were found by searching Google and the Western Michigan University library database for the following keywords: children's book or picture book and area or perimeter or circumference. Once we identified a potential book, we read it and kept it if it was related to area and/or perimeter, resulting in twelve books (Figure 2).

Adler, D. A. (2012). *Perimeter, area, and volume: A monster book of dimensions.*	Neuschwander, C. (1997). *Sir Cumference and the first round table*.
Brucke, C. Y. (2009). *Wrappers wanted: A mathematical adventure in surface area*.	Neuschwander, C. (1998). *Sir Cumference and the dragon of Pi.*
Burns, M. (1997). *Spaghetti and meatballs for all!*	Neuschwander, C. (1999). *Sir Cumference and the Isle of Immeter.*
Gabriel, N. (2006). *Sam's sneakers squares.*	Pilegard, V. W. (2004). *The warlord's kites*.
Murphy, S. J. (2001). *Racing around.*	Pollack, P., & Belviso, M. (2002). *Chickens on the move*.
Murphy, S. J. (2002). *Bigger, better, best!*	Reisberg, J., & Hohn, D. (2006). *Zachary Zormer: Shape transformer: A math adventure*.

Figure 2: Picture Books Focusing on Area and Perimeter

We began by independently analyzing each picture book using the framework in Figure 2. We individually highlighted any characteristic in the framework present in the book and included significant notes about its presence or absence. We then met to ensure consistency, discussing any differences, and clarifying definitions until agreement was reached.

Findings

Figure 3 identifies the CCSSM about area and perimeter addressed in each picture book.

Most CCSSM standards in second, third, fourth, and seventh grade were addressed by multiple books; yet, no picture book focused on the third grade standards, **G.A.2** or **MD.C.7.C**, which refer to partitioning shapes to represent fractions and modeling the distributive property with area, respectively (see Figure 1). However, these standards teach other concepts through area, rather than area itself, which may explain their absence from the books in our study. We also found that no books in our collection covered the sixth grade standards, **G.A.1** or **G.A.4** (see Figure 1). While one book, *Sir Cumference and the First Round Table* cut apart shapes and moved portions around, it did not find area by decomposing shapes as in **G.A.1**. Similarly, *Wrappers Wanted* described finding surface area by adding the area of each face of a box, but the idea of nets was not utilized as in **G.A.4**. Thus, none of the sixth grade standards were fully covered within any of the picture books analyzed.

When considering the *way of presenting*, *Spaghetti and Meatballs for All!* and *Chicken's on the Move* presented mathematics implicitly, as opposed to the other ten books which did so explicitly. For example, in *Spaghetti and Meatballs for All!*, the characters consider how many chairs fit around different configurations of tables. This implies the relationship between area and perimeter without explaining the mathematical concepts or using mathematical language.

Grade	**2nd**	**3rd**									**4th**	**6th**		**7th**	
CCSSM	G.A.2	G.A.2	MD.C.5 A	MD.C.5 B	MD.C.6	MD.C.7 A	MD.C.7 B	MD.C.7 C	MD.C.7 D	MD.D.8	MD.A.3	G.A.1	G.A.4	G.B.4	G.B.6
Adler	X		X	X	X	X	X			X	X			X	
Brucke				X			X				X				X
Burns										X					
Gabriel	X		X	X	X	X	X								
Murphy 01										X	X				
Murphy 02	X				X	X	X		X		X				
Neuschwander 97									X	X					
Neuschwander 98														X	
Neuschwander 99			X	X	X	X				X	X			X	
Pilegard			X	X	X	X	X				X				
Pollack & Belviso										X	X				
Reisberg & Hohn				X			X			X					

Figure 3: CCSSM for Area & Perimeter Addressed in Picture Books

Discussion and Conclusion

Our findings reveal there are picture books available for teaching a range of CCSSM related to area and perimeter throughout elementary and middle school grades. However, a gap exists in the representation of the sixth grade standards as seen by the absence of coverage for **G.A.1** and **G.A.4**, which describe decomposing shapes to find area and using nets to find surface area. Both standards involve manipulating and visualizing spatial elements, ultimately enhancing students' ability to understand and solve complex spatial problems (Davis et al., 2015). Thus, picture books focusing on these standards would provide much needed resources for teachers.

Ten of the twelve picture books explicitly incorporated mathematics content. Teachers may find it convenient to include these books in lesson plans as they require minimal interpretation on the teacher's part. However, it is crucial to use picture books as part of an integrated approach that enhances students' understanding of mathematical concepts (Austin, Thompson, & Beckman, 2005). Presenting mathematics implicitly requires students to actively engage with the material and draw connections between the story or illustrations and the mathematical concepts, fostering a deeper understanding of the mathematical content (Austin, 1998). Therefore, increasing the availability of picture books that implicitly present mathematics related to area and perimeter would benefit both elementary and middle school students and teachers.

By assessing these twelve picture books based on the CCSSM for area and perimeter, teachers can identify which ones may be beneficial for teaching their intended standard(s). We recognize we may have missed relevant books, which could be a limitation of our study. For example, books that present mathematics implicitly may not use the terms area and perimeter, so they may not have come up in our search. Our framework directs teachers' attention to specific learning-supportive characteristics (e.g., *way of presenting*) that may influence how picture books are used in their classroom. Looking forward, we anticipate this framework serving as a guide for categorizing the usability of picture books for teaching mathematical content, aiding educators in making informed decisions tailored to their classrooms' needs.

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A THIRD-GRADER'S DISTINCTION BETWEEN QUALITATIVE AND QUANTITATIVE TIME

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This paper explores how nine-year-old Loren described the duration of common experiences through qualitative perceptions and quantitative measurement. After organizing four "everyday" activities from shortest to longest duration (sleeping at night, watching a movie, eating lunch, brushing teeth), Loren justified their ordering through contrasting the paradox of how each experience felt against a quantified length of each duration. These distinctions demonstrate the complexity of temporal and durational reasoning and evidence potential conceptions that elementary-age children may have when reasoning about time—a consideration that mathematics teachers and researchers have often overlooked.

Keywords: measurement, elementary school education, problem-solving

When reflecting on the duration of common activities, elementary-age children draw on numerous aspects of their lived experiences (Smith, 2021). Many of these attributes are quantifiable in and of themselves, such as knowing that you have a long way to walk (distance) so it will take a while or building a Lego with 500 pieces (500 being a larger quantity) taking longer than a mini-figure (with only three pieces to construct). However, some aspects of experience are not measurable in the same, mathematical sense, such as having the long walk be on a hot (or cold) day which will make it *feel* like it is taking longer or having a knowledgeable sibling help build the Lego so it will not take as long (but a less-practiced sibling may cause it to take even longer). These less well-defined attributes of experience are not considered when formal time teaching and learning are established in early elementary school (see Common Core State Standards Initiative, n.d.); yet, they are commonly, informally prevalent in durational experiences throughout an individual's life, and thus, should be explored. The present case exemplifies one such child's attempt to articulate this mathematical paradox.

Theoretical Framework

Young children develop an intuitive sense of duration based on their perceptions of succession—the sequence in which events occur, and duration—the intervals of and between events (Piaget, 1969). These perceptions develop through conclusions made as they reflect on their lived experiences. Children who reason intuitively, for example, may consider duration as a result of their efforts, believing that *because* more work was done, the duration was necessarily longer (Piaget, 1969; Smith, 2021). A child reasoning in such a manner might conclude that *because* they drew lots of lines, the duration was long. As they grow and through experience, children's conceptions of time develop from intuitive—based purely on observations—to operational—based on the relationships abstracted from their observations (Long & Kamii, 2001; Piaget, 1969). When reasoning operationally, children recognize the reciprocal relationship of succession and duration (Piaget, 1969). When the previous child, for instance, can reason that they were able to draw a lot of lines because they took more time to complete it *and* that taking more time allowed them to draw more lines, they would demonstrate an operationalization of durational reasoning. This reasoning is necessary for durational measurement (Kamii & Russell, 2010, 2012; Piaget, 1969).

Flaherty and Meer (1994) posited a general theory of lived time, where the retention and recollection of durational events are defined by an individual's engagement, attention, emotional connection, and habituality of the event. In other words, how an individual might measure the duration of their experiences is impacted by how they processed the experience in the first place. Events that are common or less engaging might be perceived as taking more (or less) time than the actual, measured duration. This processing can lead to uneven durational perceptions, such as time feeling like it is "flying by" (temporal compression) or "dragging on" (protracted duration; Flaherty & Meer, 1994), which can impact durational measurement and reasoning (Smith, 2021).

Qualitative durational reasoning corresponds with Piaget's gross quantification (1952), where asymmetrical relations engender the difference between two quantities. Such comparison allows for the distinction of more (longer), less (shorter), or the same. *Quantitative durational reasoning* utilizes numbers as the units of measure (Russell, 2008). This occurs when an individual can distinguish space measurement from time measurement (i.e., can account for the duration of an event disparate from their perception of the event). This durational measure is no more or less cognitively demanding than qualitative reasoning because both require the individual to reflect on, and abstract, the durational relationships they experience (Piaget, 1969). Rather, these are two distinct ways that individuals might reflect on their durational conceptions.

Methodology

The case presented (Yin, 2003) came from a larger study that explored how elementary-aged children (age 4-11) organize the duration of common experiences (Smith, 2020, 2021). Each participant was asked to organize four activities: brushing teeth, eating lunch, watching a movie, sleeping at night, from the shortest to longest duration. Then, the interviewer inquired into the child's reasoning. This case illustrated markedly different reasoning from other participants.

Participant and Data Collection

The data presented come from an interview with nine-year-old Loren[1]. Loren was a third grader from a large suburban city in the Midwestern United States. The interview was video recorded to capture Loren's words and actions. During the interview, Loren was given four cards with images and words of the four activities and asked to put the activities in order from what took the least amount of time to the most amount of time, then discuss their reasoning.

Data Analysis

With the distinct nature of Loren's responses, I used a phenomenological lens during analysis (Creswell, 2013; Hycner, 1999) to understand as much as possible about Loren's reasoning. Following Wolcott's (1994) three-part analytic framework, I worked to create "data out of experience" (p. 13), by first incorporating all of Loren's words, actions, wait times, and inflections into a *descriptive* transcript. From my *analysis*, I distinguished Loren's quantitative durational reasoning from their qualitative perceptions of the durations. Reflecting on Loren's

[1] Gender neutral pseudonyms were used as gender identity was not the central to the research focus.

data as a whole, I returned to existing theories to better *interpret* how Loren conceived of duration as a measurable attribute of their lived experience.

Findings: Description and Analysis

When initially asked to organize the four activities, Loren placed the "sleeping at night" card on the far-left side of the table, indicating the shortest duration. They then placed "brushing teeth" at the far-right side, indicating the longest duration. This process was similar to how individuals bound the duration of their experiences, finding the upper and lower durational limits, then constructing the unit within (Smith, 2021). Interestingly, Loren's placement of these cards (longest/shortest), was backward/opposite of all other participants interviewed. Loren then placed the "watching a movie" card after sleeping at night, indicating that to them, watching a movie took longer than sleeping at night. Finally, they placed "eating lunch" between watching and movie and brushing teeth.

Sleeping at Night

Loren began describing how they considered the duration of each activity with what they conceived of as the shortest duration—sleeping at night. Loren explained, "When I fall asleep and I know that's like 12 hours, I wake up [swiped hand through air] and it feels like it's 30 seconds." This description was one of Loren's clearest examples of their qualitative reasoning distinct from quantitative reasoning. Qualitatively, and paradoxically, Loren explained that they *knew* the duration of their experience as one quantity (12 hours) despite that it *felt* like a different, much shorter quantity (30 seconds). Quantitatively, Loren attributed the duration of their sleep through two different standardized units of measure: 12 *hours* versus 30 *seconds*. I inferred that their motioning of swiping their hand through the air was an iconic gesture (McNeill, 1992) of the comparison between these two units, symbolizing their acknowledgement that 30 seconds is a quantifiably different durational amount than 12 hours. Remembering that Loren placed this activity as the shortest of the four, it seems that their qualitative durational reasoning guided their overall consideration of the durational organization.

Watching a Movie

After justifying their placement for sleeping at night, Loren immediately went on to explain, "I feel like movies are short, but when you look at the time, it's already like, if you started watching at 3:30 it's already 5:54 and I just feel like watching a movie is fun, and something that I enjoy." Similar to sleeping at night, Loren seemed to openly distinguish between their qualitative perceptions of the duration and the quantitative measurement—even providing a somewhat reasonable approximation of the duration of a typical movie (around 2.5 hours). Multiple times, Loren reflected specifically on how the duration felt, noting that it felt "short" and was "fun"/enjoyable. These perceptions were countered by Loren's hypothetical description of looking at a clock to note the start and end times of the movie (quantitative measurement).

Eating Lunch

Loren placed the eating lunch event last when organizing the events. Unlike the previous two experiences, Loren's reflection on the duration of eating lunch was less defined both quantitatively and qualitatively. Loren stated, "Eating my lunch, I usually don't eat that much, um, I have a salad and eggs, but I chose eating my lunch here [pointed to placement] because I feel like it's just, um, [hesitated three seconds] uh, I don't know." Initially, it seemed that Loren reflected on the quantity of food as related to the measure of the duration. This reflection on gross quantity as a proxy for durational length has been evidenced by other children (Smith, 2021), though never with explicit relation to qualitative durational reasoning. Loren did go on to state that they *felt* like it belonged in the placement it was but was unable or unwilling to elaborate on why this experience felt the duration it did, either qualitatively or quantitatively.

Brushing Teeth

Brushing teeth was the second event placed during Loren's initial organization and bound the "upper limit" (longest duration) of Loren's established durational timeline. Loren explained, "Brushing my teeth, I feel like it's two minutes, but when I go downstairs my dad will say, what are you doing, I told you to brush your teeth, you've only been up there for 30 seconds [laughed]." This explanation echoed Loren's previous descriptions in the use of measurable, quantitative durations based on established time units (two minutes, 30 seconds). However, Loren's qualitative explanation that brushing their teeth *felt* like two minutes but was, in fact, much less than that (30 seconds), yet was situated as the longest duration of the four events, highlights the impact that qualitative reasoning had on Loren's overall durational reasoning. I see Loren's last response, a laugh, as a critical piece of evidence of this durational paradox. Loren seemed aware that the quantitative duration of this event did not align with their qualitative experience of the event. Yet, even with this awareness, Loren maintained the order of the events as they were originally situated, from shortest (sleeping at night) to longest (brushing their teeth).

Discussion: Interpretation

The data presented highlight Loren's diverse durational reasoning. Quantitatively, Loren demonstrated understandings of various standard durational units—seconds versus minutes versus hours—and provided verbal evidence and non-verbal cues of an awareness of the relationship between these units (that seconds are a smaller duration than hours, etc.). Additionally, Loren quantitatively bound events (starting a movie at 3:30 and ending it at 5:54), providing evidence of unitizing duration (Piaget, 1969; Smith, 2021). Qualitatively, Loren explicitly stated that each of the four events "felt" a specific way. Some of these "feelings" provided context for their numerical quantification (12 hours of sleep feels like 30 seconds) while others provided evidence of Loren's appreciation for the paradoxical nature of duration and temporal events ("time always goes slow on something you don't enjoy").

Loren's ability to distinguish between the qualitative nature of the duration of their experiences and the quantified measurement of these durations aligned with various established durational and temporal conceptions. Specifically, Loren's descriptions provided evidence of four temporal and durational conceptions: 1) how engagement/enjoyment can impact perceived durations (Flaherty & Meer, 1994; Smith, 2021); 2) the correlation between gross quantity and durational measurement (Piaget, 1969; Smith, 2021); 3) how an individual's actions can impact the duration of their experiences (Piaget, 1969; Smith, 2021); and 4) how parents can impact children's perceptions of durations (Piaget, 1969).

Implications

When considering Loren's durational understandings, it is important to consider implications this may have both in formal schooling and in life. Clearly, Loren has a much richer bank of conceptions to their durational reasoning than established state and national standards consider. As they are currently situated, standards for time teaching and learning focus solely on analog and digital clock reading and hypothetical elapsed time calculations (CCSSI, n.d.). There are no math standards nor expectations for durational reasoning nor is there any consideration of how qualitative perceptions of time might impact or coordinate with quantified, standardized time (hours, minutes, seconds). Instead, time teaching and learning is siloed to elementary classroom clock reading, yet time-related concepts span the K-12 curriculum (e.g., time as an independent variable, Thompson, 2012, or scientific uses of time, Tasar, 2010). Shifting out of the elementary classroom, temporal and durational reasoning is an essential "life skill" in our Western society. Time management and scheduling are critical skills that rely heavily on durational reasoning, both quantitatively ("clock time"; Earnest, 2018) and qualitatively ("event time"; Earnest, 2018). To support students like Loren, we need to appreciate the complexity of time and consider how children's perceptions of this abstract, invisible quantity (Earnest, 2019) might impact their durational measurement.

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GEOMETRY REPRESENTATIONS ON STANDARDIZED TEST

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Keywords: Geometry and Spatial Reasoning, Mathematical Representations, Assessment

Teachers, students, and designers of exams must make choices in how they represent a mathematical object so that it helps them attain their goal. To be able to make choices in the use of geometry representations, teachers, students, and test designers need to know which representations are available, how to use them, and what are some benefits of using one over another. In this study, I analyze the geometry representation on state, national, and international exams at the secondary level in order to see what is deemed necessary as students leave high school and prepare for college and/or a career.

Besides the above, there are a few more reasons why studying representations is valuable. First, we all want students to solve problems with a high degree of accuracy and efficiency, and representations help us manage mathematical information. Students who coordinate different mathematical representations well solve mathematical problem better than those who do not (Gagatsis & Shiakalli, 2004). Second, to solve problems students need to use aids or tools like representations to reduce the complexity of a problem. We use representations as a kind of external memory, where we can offload some of the information from our working memory (Scaife & Rogers, 1996). Third, in the digital age, there are also practical reasons to study geometry representations. We are beginning to develop digital textbooks, online environments, etc., where the choice of representations is important (Presmeg et al., 2016).

Methods

I have examined standardized test given at the end of secondary school. Some of the exams are designed and administered at the state level like New York's Regents. Some are more regional/national like the PAARC and Smarter Balanced. Some exams like the SAT are national and designed as college readiness exams. Finally, some are international like PISA.
The unit of analysis is one question/problem from the exams. I searched for a minimum of 10 questions from each test, which were downloaded from the agencies that administered and/or designed the exams. If there were fewer than 10 questions, I looked at a related website that had a recent exam. If an exam had more than 10 question, I used no more than 20, selecting those that most resembled Euclidean geometry.

Results

From the preliminary result, many exams have typical problems with diagrams, but some avoided using diagram. There were few problems using real-world physical language on many of the exams, but the PISA math assessment contained multiple examples of this type with very complex diagram. On some exams the meta-language was quite complex, e.g., refute a claim is an abstract level higher than actually solving a problem. Calculator buttons were used to enter an answer such as a ration instead of selecting an answer as in multiple choice questions or writing the answer.

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HIGH SCHOOL STUDENTS' FIGURATIVE AND OPERATIVE THOUGHT WHEN REASONING ABOUT DISTANCES

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In this report, we share analysis of 55 students' interview responses to a "suspension of sense-making" story problem involving how far two people live from one another. The prompt occasioned different ways of reasoning. Some students reasoned about lines; others about triangles; and still others about circles. Students' differing responses indicate across-student variation in spatial organization and the quantitative roles of distance in those spatial organizations. We describe students' mental operations of reasoning about points as varying and static and relate it to research on students' meanings for a circle as representing all points an equal distance away from a center point. Overall, this study provides a quantitative reasoning lens to better understand how students make sense of a well-studied problem.

Keywords: Geometry & Spatial Reasoning, Mathematical Representations, Cognition

Over the past several decades, researchers have described the importance of students' quantitative reasoning (Thompson & Carlson, 2017). Research has also shown that school mathematics can constrain what students might consider when responding to certain problems (Palm, 2008). In this study, we report on high school students' responses to a question about the distance between two places (Figure 1). One productive response involves students constructing circles. An individual's spontaneous consideration of a circular path when tasked with identifying locations that are a specified distance from a fixed point is an indication of what Hardison et al. (2017) referred to as an operative conception of a circle; such a conception involves a smooth radial operativity if the individual anticipates a rotating segment of constant radial length around one fixed endpoint while the other endpoint traces out a circular path. However, there exist responses contraindicating students considering circles with this problem, and thus, we sought to answer the question: "What ways of reasoning do students indicate when considering distances between objects? What are the figurative and operative aspects of these ways of reasoning, and how do they relate to the realistic nature of students' responses?"

Bruce and Alice go to the same school. Bruce lives at a distance of 17 kilometers from the school and Alice lives at 8 kilometers. How far do Bruce and Alice live from each other?

Figure 1: *The School Problem* (adapted from Treffers & de Moor, 1990)

In this report, we first review existing literature around *The School Problem*, the word problem used in this study (Figure 1), highlighting reported findings on students' reasoning. We then consider this problem and prior findings in relation to the Piagetian constructs of figurative and operative thought (Piaget, 1970; Thompson et al. 2024); we specifically build upon Hardison et al.'s (2017) definitions and examples of figurative and operative conceptions of circles. We then report on the ways of reasoning indicated by 55 high school students who answered this problem. We focus on distinguishing figurative and operative aspects of the imagery indicated by students' responses and compare this analysis to previous findings on the realisticness of responses to this problem.

Literature Review

In this literature review, we report on findings on the *School Problem* and situate those findings within the literature on quantitative and covariational reasoning. We conclude the literature review with a focus on figurative and operative thought.

Problem Solving

The context of the study reported on here developed from a larger interest in understanding students' realistic mathematical sense-making (Reusser & Stebler, 1997). Researchers have reported on students' suspension of sense-making (Schoenfeld, 1991) when solving word problems (e.g., Palm, 2008), noting that students rely more on procedural mathematics than conceptual applications to solve these word problems. Researchers have also reported that students' suspension of sense-making increases over time as they develop more experience in problem solving and modeling in school (Mellone et al., 2017).

Verschaffel and colleagues explored this suspension of sense-making through word problems designed to have different responses depending on the extent to which students are attending to the realistic nature of the problem. One of these problems was used in this study (Figure 1) (from Verschaffel et al. (1997), as worded in the English-translated Treffers & de Moor (1990)). Verschaffel et al. classified it as a "problematic" item because the underlying mathematical modelling assumptions are problematic from a realistic point of view. According to the authors, there are two non-realistic answers and one realistic answer to this problem. The two non-realistic answers they provide are 9 km (i.e., 17−8) and 25 km (i.e., 17+8). The realistic answer is "you cannot know how far Saskia [*Alice in our study*] and Bruno [*Bruce in our study*] live from one another" (p. 344). Verschaffel et al. (1994) conducted a study with 75 students (aged 10–11), and only 3% of students provided a realistic answer to this problem. In Verschaffel et al. (1997), out of 332 pre-service teachers from three different institutes for elementary school teacher training in Flanders, 48% gave realistic reactions. This 48% includes responses with realistic answers and other responses researchers indicated as including "activation of real-world knowledge" (p. 345) (e.g., gave an indication that more than two answers were possible). In both studies, few details were provided into specifics of the "non-realistic" and "realistic" reasoning the participants engaged to determine their solutions. In this study, we use the ideas of quantitative reasoning and figurative and operative thought to understand students' ways of reasoning when solving this problem.

Distances and Quantitative and Covariational Reasoning

The *School Problem* can be interpreted in terms of quantities and their relationships. Quantities are measurable attributes an individual conceives in a situation, and, over several decades, researchers employing principles of quantitative reasoning have provided insights into the quantities students construct and quantitative operations students enact (Thompson & Carlson, 2017). This subfield has offered explanatory models for students' construction and interpretation of various representations (e.g., graphs, tables, geometric shapes).

For example, Carlson et al. (2002) elaborated differences in students' reasoning about the relationship between volume and height of water in a bottle as it is filled. Some students engaged in directional covariational reasoning by stating that the water's height increases as the volume of water added to the bottle increases. Here, students consider height and volume as quantities and reason about them changing in tandem via quantitative operations (e.g., addition, subtraction of quantities or states of quantities).

Previous research on the *School Problem* has not taken up this focus on quantitative reasoning. In this study, we do, and we specifically identify three quantities are of interest (from our perspectives): the distance Bruce lives from the school, the distance Alice lives from the school, and the distance Bruce and Alice live from each other.

Figurative and Operative Thought

Beyond quantities and quantitative reasoning, the Piagetian constructs of figurative and operative thought are relevant for the present study; these are two complementary aspects of thought. Piaget (1970) defined the figurative aspect of thought as "an imitation of states taken as momentary and static" (p. 14). In Piaget's view, figurative thought serves important cognitive functions involving, for example, perception and mental imagery. In contrast, the operative aspect of thought "deals not with states but with transformations from one state to another…it includes actions themselves, which transform objects of states" (p. 14). In Piaget's work, operative thought necessarily entails reversibility and is at the level of interiorization. Reversibility can be considered as follows: if an initial state is subjected to an operation, *A,* then an inverse operation, *B*, can be subsequently enacted to revert to the initial state. To say that operative thought is at the level of interiorization means that the operations can be carried out in the mind without being enacted physically. We emphasize that the figurative-operative distinction characterizes complementary aspects of thought in that the latter cannot exist without the former; operations require some (figurative) state on which to operate. However, Piaget (1970) clearly stated his interest in understanding operative aspects of thought in particular remarking, "...to my way of thinking the essential aspect of thought is its operative and not its figurative aspect" (p. 15).

For example, in Lee et al., (2018), the authors describe a student, Lydia, who conceived a slope of a given graph as "rising" three, which perturbed her when the researcher rotated the given graph counterclockwise 90 degrees. She struggled to conclude if the slope value should stay positive or become negative, given that the graph was no longer "rising." Moore et al. (2019) described this reasoning as figurative because her meanings for increasing and decreasing values were constrained by orientation.

In Moore et al. (2019), emergent shape thinking is provided as an example of operative thought. Emergent shape thinking involves covariational reasoning with quantities (e.g., reasoning with volume and height in the bottle problem) and transforming figurative entailments constructed from that reasoning (e.g., identifying amounts of change in volume for a change in height) into a quantitatively equivalent representation (e.g., a coordinate system).

Hardison et al. (2017) introduced the idea of figurative and operative circle concepts and reported on reasoning associated with an individual reasoning with figurative circle concepts. In the present study, we identify figurative and operative aspects of student thinking as indicated by their responses to the *School Problem* (Figure 1), specifically attending to the construct of an operative circle construct. The next section details this construct.

Conceptual Framework

In 2017, Hardison et al. considered figurative and operative aspects of thought in the context of one prospective teacher's thinking about circles. Specifically, they offered definitions for *figurative circle concept* and an *operative circle concept* (Figure 2). A *figurative circle concept* "consists of an individual's ability to recognize or re-present a circle as a static form." A figurative circle concept functions as a mental template for producing or assimilating a circle. An individual can bring forth an image of a circle in visualized imagination or recognize circle models (e.g., a hula hoop) as being circular in shape. In contrast, an *operative circle concept* entails an individual's mental image of a segment of fixed length rotating around a fixed endpoint while the other endpoint traces out a circular path. We emphasize that this smooth radial operativity is one of several operative possibilities for circles. In their 2017 work, Hardison et al. did not consider other forms of operativity for circles, and we will not either in this report because they were not identified within our given dataset.

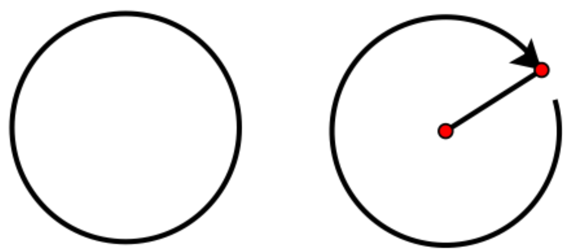

Figure 2: (left) Figurative circle concept and (right) smooth radial operative circle concept

We extend the work of Hardison et al. (2017) by noting that the smooth radial variety of an operative circle concept entails the following two aspects: Constructible and Quantitative (Figure 3). The constructible aspect of an operative circle concept involves an individual's anticipation that a circular arc will necessarily result when one considers a set of locations that are a fixed distance away from a fixed point (Figure 3 top). In contrast, the quantitative aspect of an operative circle concept involves an individual's anticipation that, when starting with a path known to be circular, all points along the path are necessarily the same distance from a particular (center) point (Figure 3 bottom). The distinction between these two aspects of an operative circle concept lies in what is taken as (a) the initial state and (b) the resultant state. In the constructible aspect, the initial state involves a fixed distance from a fixed point, and the resultant state is a (subset of) a circular path. In the quantitative aspect, the initial and resultant states are reversed. Common to both constructible and quantitative aspects is a mental rotational motion of a radial segment that enables transitioning between initial and resultant states in each case.

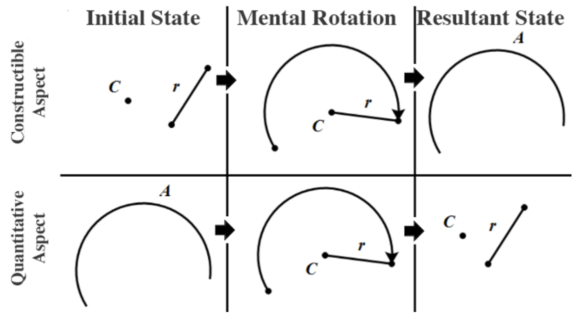

Figure 3: Reversibility of an Operative Circle Concept: Constructible (top) and Quantitative (bottom) Aspects

To demonstrate these concepts, Hardison et al. (2017) reported on a 12-session, semester-long teaching experiment (Steffe & Thompson, 2000) with a prospective secondary mathematics teacher (Lydia) at a large institution in the southeastern U.S. Lydia was working on the two tasks in Figure 4: *Going Around Gainesville* and *Where Did They Go?* For *Going Around Gainesville*, students were tasked with creating a graph relating the car's total distance travelled and its distance from Gainesville during the trip (Figure 4 top left). For *Where Did They Go?*, students were tasked with constructing a path representing the same relationship between distances from two points (A and B) in the given graph.

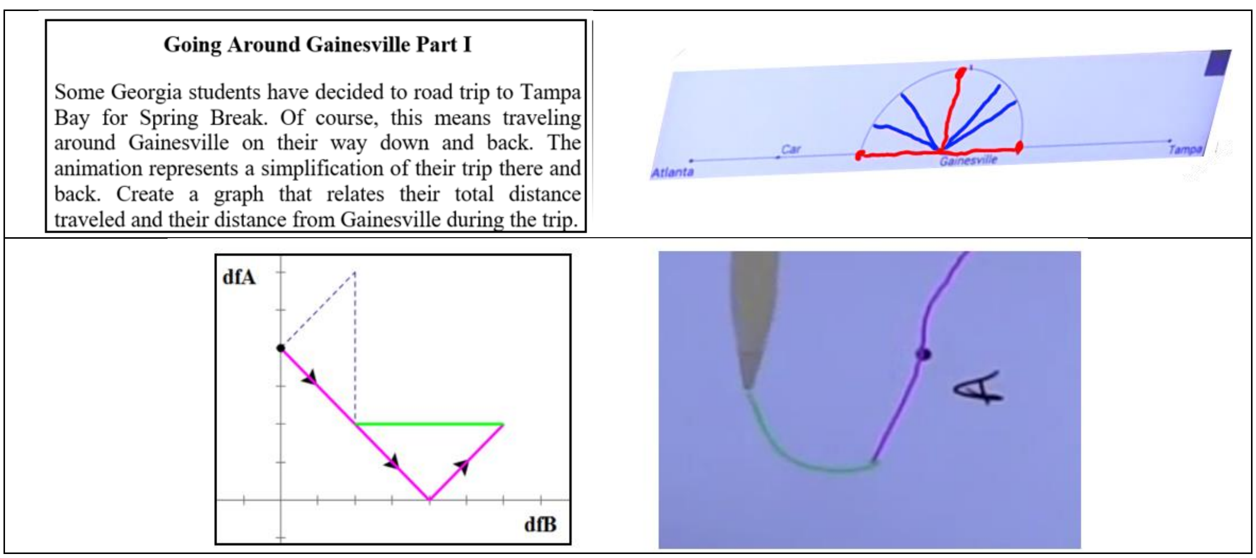

Figure 4: *Going Around Gainesville* Task and Lydia's work (top) and *Where Did They Go Task?* and Lydia's diagram (bottom)

In *Going Around Gainesville*, Lydia recognized the (semi-)circular portion of the road and drew red and blue line segments emanating from Gainesville and extending to the semi-circular portion (Figure 4, top right). She identified the distances represented by the red segments as "the same…because that's the radius length." Thus, Lydia recognized the static form of the a (semi-

)circle in this situation which is evidence of at least a figurative circle concept. The same was not true for the blue lines where she thought "at some points in the circle we're going to be closer to Gainesville". Thus, the quantitative aspect of an operative circle concept is contraindicated since Lydia's recognition of a semicircular path did not entail her immediate anticipation of equal radial distances from the center (i.e., Gainesville).

In *Where Did They Go?*, after drawing portions of a road map (Figure 4, bottom right) to appropriately account for the pink portions of the provided graph (Figure 4, bottom left), Lydia considered drawing a curved green path, which she called a "loop" in the road, to account for the green horizontal segment in the graph. She rejected her "loop" because she recognized from the graph that the distance from City A was "not supposed to change" and, according to Lydia, the loop had "changing distances." Thus, although Lydia recognized the need for a path that was a fixed distance from City A, this did not immediately evoke for Lydia a circular arc. In summary, Lydia's initial reasoning on this task contraindicated an operative circle concept precisely because Lydia's observable activities were contrary to the constructive aspect of operativity.

We expand on Hardison et al.'s (2017) work with Lydia in two ways. First, we identify examples of students who indicated operative circle concepts. Second, we discuss how other students' reasoning and diagrams seemed to contraindicate an operative circle concept.

Methods and Analysis

The study included 55 high school students from three settings: 17 enrolled in a 90-day English-speaking, island-based semester-program with a place-based curriculum, and 38 in a regular high school setting (16 with placed-based tasks). Each student participated in a 30-minute clinical interview, which included responding to three story problems, the third of which was the *School Problem* (Figure 1). The individual interviews occurred via video-conferencing software, and there was an audio recording and an image recorded of students' final written solution. Students were encouraged to think aloud. Video records were transcribed using natural language processing software (NLP) and then refined by a research assistant.

Analysis began with two rounds of coding: the first replicated Verschaffel et al.'s (1994) coding scheme, categorizing student work according to realistic reasoning (No answer, Expected Answer, Technical Error (e.g., subtraction miscalculation), Realistic Answer), and the second involved open coding to identify differences in students' approaches and strategies. In this second round, approaches were distinguished in terms of: (i) how many configurations of the locations of the houses and school the student considered (ii) how many configurations the locations of the houses/school the student thought were possible, (iii) whether students identified the minimum or maximum distances possible, and (iv) the various diagrams the students used. This second round of analysis inspired further consideration regarding how students' configurations foregrounded figurative and operative thought, given the variety of student responses and diagrams. Thus, a third round of analysis attended to students' figurative and operative thought—with students' solutions grouped by type of diagram and ways students within in each diagram type category described their images of the situation (i.e., more than a mental picture [see Thompson, 1996]).

Results

In this section, we discuss the responses from the 55 high school students. Recall that for the *School Problem* that an awareness of multiple solutions indicates realistic reactions. From the first round of analysis using the Verschaffel framework, we identified that 31 (~56%) of the students provided a realistic reaction. However, in the subsequent analysis of student responses, we identified that these realistic responses varied in the extent of potential alternatives. For example, 12 (~22%) students identified the maximum and minimum distances Bruce and Alice could live from each other by rearranging the order of the locations on a straight line, but this indicated awareness of only two potential Bruce-Alice distances. Two others indicated different configurations were possible, but not necessarily all values between 9 and 25; that is, the number of configurations of buildings was more than two, but the number of possible Bruce-Alice distances was still finite. Only five students clearly identified a range of values (two others indicated a range but did not provide numerical bounds). The remaining realistic responses indicated a multitude of configurations, but not necessarily a continuous range of Bruce-Alice distances. A summary of these considerations, including non-realistic responses (i.e., one configuration considered; no others), is in Table 1 (left). This distinction is relevant to figurative/operative thought because an activated operative circle concept would necessarily result in the consideration of two circles with each delineating the possible locations for each house, thus representing all possible configurations simultaneously.

Table 1 (right) summarizes the results from the second round of analysis, and the remainder of the results are from the third round of analysis which highlights the figurative and operative nature of select students' reasoning as indicated by the representations they drew. It is important to note that two students interpreted the distances as along paths and not as the crow flies and seven (different) students did not physically draw diagrams when working on the problem, so their results are excluded from Table 1 (right). It is also important to note that several students drew more than one diagram (e.g., triangle and circle), and so their work is counted more than once in Table 1 (right) unless they drew multiple cases of the same diagram (e.g., straight line segment switching whether the school is on the left or the middle of the segment).

Table 1: (left) Diagram Configurations Considered and (right) Student Diagrams

Consideration	# Students
One configuration considered; no indication others are possible	24
One configuration considered; indication others are possible	6
Multiple configurations considered; no indication others are possible	13
Multiple configurations considered; indication others are possible	5
Range of possible configurations	7

Diagram/Configuration Count	# Students
Straight Line Segment	34
Multiple Line Segment Cases	3
Triangle	9
Circles	3

Operative Circle Conception

Seven (~13%) students identified a range of possible configurations for the locations of Alice's and Bruce's houses respective to the school. Three of these students drew diagrams with circles (one of whom also drew a triangle). Two others drew multiple line segment cases (one of whom also drew a triangle) and the remaining two did not draw diagrams. For those who drew circles, the school was the center for two circles, each of which had a radius length corresponding to the given distances Bruce and Alice were from the school. Student A stated, "We don't actually know how far Bruce and Alice live from each other". He then noted a range of potential solutions: "At the least, they live nine kilometers away. At the most. I think it's, um, yeah, at the most it's 25 kilometers away." Student B wrote that Alice and Bruce "could be anywhere", providing several options around the perimeter of the circles. Student C, similarly, noted that the distance would be "at least nine". These three student responses demonstrate an operative circle construction. Like Lydia from Hardison et al. (2017), these students began with an initial state involving a fixed distance from a fixed point (i.e., the school) and a goal to construct a continuous path maintaining that distance from the fixed point. But unlike Lydia who did not demonstrate an operative circle concept, these students produced representations suggestive of a mental radial rotation resulting in circular paths (Figure 4).

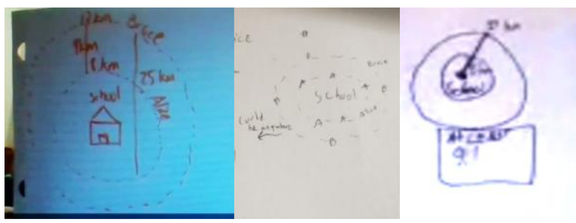

Figure 4: Student diagrams of circles for Student A, Student B, and Student C

Figurative/Operative Results Amongst Various Diagrams and Configurations

Most students created diagrams that were not circles. For instance, 24 students (44%) considered a single configuration of the buildings. Of these, 16 students considered a single straight-line configuration of the homes and school, answering either the sum or difference of the given distances (see Figure 5a). Four constructed triangle diagrams (two of which were right triangles), and the remaining four did not draw diagrams. Given that an operative circle concept necessarily involves considering multiple locations, we infer the 44% of students who considered only a single configuration likely did not leverage an operative circle concept in their solution strategy. If so, these students did not assimilate the *School Problem* to an operative circle concept or were yet to construct such a concept.

Instead, these students indicated relying on operations on static locations. For example, one student described the problem as "your classic triangle problem," which is, due to the static locations involved, evidence against an activated operative circle. These students considered the three buildings as three vertices of a triangle and solved for the unknown distance. For example, in Figure 5d, Student D used the Pythagorean Theorem to calculate the hypotenuse as 17.

However, not all students who only drew straight line diagrams or triangles only considered a single configuration. Eleven of the 34 students who only drew straight line segments considered multiple configurations of them. Three of the six students who only drew triangles considered multiple configurations of them. The remaining three students who drew triangle diagrams also drew other diagrams, indicating they considered multiple configurations. These students, along with one of the three students who drew multiple line segments, are examples of the 24 total students (43%) who were aware that multiple configurations of the buildings were possible, but who did not provide a range of values as their solution to the problem. Thus, an awareness of multiple configurations is not only insufficient for the construction of a circle but also in providing a range of potential distances as the solution to this problem.

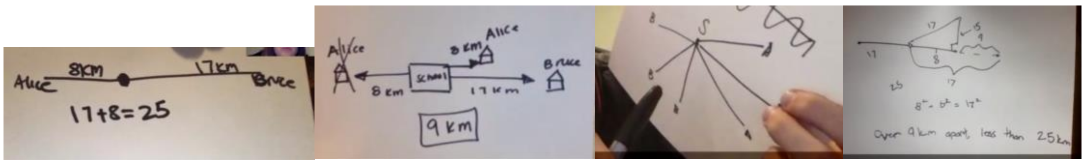

Figure 5: Student diagrams of a (a) line segment, one configuration (b) line segments, multiple configurations, (c) multiple line segments and (d) triangle

Conclusions

Verschaffel and colleagues' research on the suspension of sense-making on the *School Problem* has provided initial insights into the realisticness with which students solve the problem. This realisticness involves considering more than one configuration of the locations of two houses, each a fixed distance from the school. In this report, we have connected that consideration to operative and figurative circle concepts. In particular, we have expanded on Hardison et al.'s (2017) initial characterization of figurative/operative circle concepts by proposing the constructive and quantitative aspects involved. These aspects provided novel insights into students' quantitative reasoning on the *School Problem*. For instance, although 56% of students provided what Verschaffel would classify as realistic responses, only 12% of students identified a range of response, and 3 of the 7 of them did so with indications of enacting operative circle concepts. We also noted that 44% of students only considered a single configuration of buildings, which given our definition of an operative circle concept as including a variation, entails contraindications they were engaging in operative thought. These results imply a need to understand the relationship between students' realistic responses and the nature of their quantitative reasoning, particularly with regards to figurative and operative thought.

Acknowledgments

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TRANSFORMATIONAL REASONING WITH ADINKRA SYMBOLS: A LESSON PLAY ON SYMMETRIES

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We investigate prospective geometry teachers' (PSTs) transformational reasoning when analyzing symbols and construct a lesson play based on the results. The primary symbols used are Adinkra, prominent icons in Ghanaian culture. We find that the PSTs exhibited formal and informal reasoning about transformational relationships about the symbols, exercising precision in describing some features but not others. This paper contributes a lesson play which may help teacher educators apply similar activities focusing on non-Western cultural symbols and support student development of transformational reasoning.

Keywords: Geometry and Spatial Reasoning, Teacher Educators, Culturally Relevant Pedagogy

Introduction

Transformational reasoning can refer to (at least) two related but distinct forms of reasoning. First, there is reasoning about geometric transformations. One might evoke this type of transformational reasoning to justify that the opposite angles of a parallelogram are congruent, via an argument about rotating the parallelogram 180 degrees about its center and finding the coincidence of the preimage and image. Transformational proofs in geometry use the existence and properties of transformations such as rotations, reflections, translations, and dilations in deductive reasoning (GeT: A Pencil (2022; St. Goar & Lai (2022). The Common Core State Standards (2010) advocate for students to provide justifications involving transformations for defining congruence and similarity.

Secondly, transformational reasoning can also refer broadly to dynamic mental imagery (both with or without perceptual referents). Building on Piaget's extensive works (e.g., 1970), Simon (1996) defined this second form of transformational reasoning as the mental or physical enactment of an operation or a set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or continuum of states are generated. (p. 201)

In this report, we describe an activity for prospective secondary geometry teachers (PSTs) which utilized both forms of transformational reasoning about *Adinkra*, symbols rooted in Ghanaian culture (Babbitt et al., 2015). We created a *lesson play* (Zazkis et al., 2009) to extend the activity to deepen connections to secondary geometry teaching.

Theoretical Background

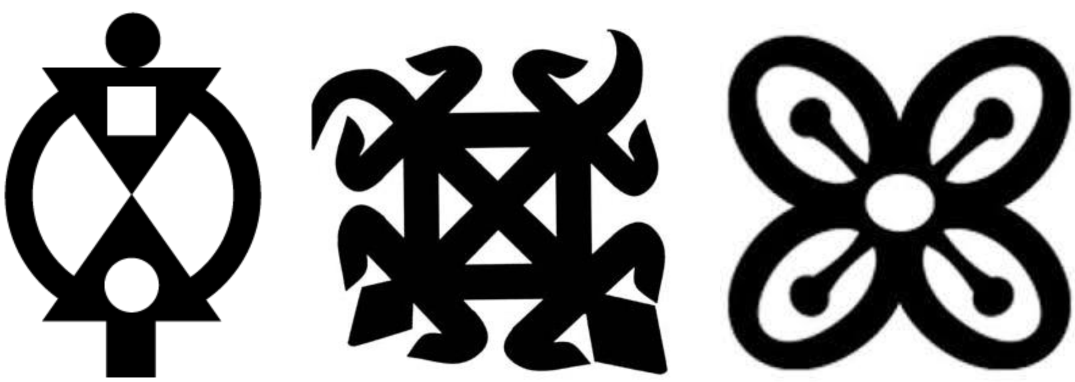
The research in this proposal fits broadly within the umbrella of design research in that it networks multiple theoretical perspectives to inform the design and refinement of instructional materials (Swan, 2014). This includes applying a radical constructivist lens (Glasersfeld, 1995) for modeling students' transformational reasoning as the product of their mental actions as well as a socio-cultural lens for understanding students' learning about the meanings of Adinkra as both a window into another culture and a mirror into their own (Gutiérrez, 2008). This report uses activities from a lesson study (Cerbin & Kopp, 2006) to investigate the transformational reasoning of PSTs, which also included objectives for learning about Adinkra. The creation of the lesson play itself can be considered the product of duoethnographic research, with its focus on simulating and fostering salient aspects of our observations of PSTs interacting in classrooms over the course of iterations of instructional design (Zazkis & Koichu, 2015).

Methods

The methods we used in the lesson study were qualitative, as we analyzed video-recordings of classroom observations as well as (primarily text-based) written artifacts. Our coding approaches fit within the grounded theory tradition (Corbin & Strauss, 1990), as we used the constant comparative method and axial coding to identify and code themes in the data as part of a larger analysis team (Boyce et al., 2023).

Participants and Activities

The participants in this study are 15 PSTs from a college geometry course. The PSTs finished a unit on transformations and symmetries prior to the lesson on Adinkra. We analyzed the students' responses from two lesson activities. The first activity involved students' observations during class about the three Adinkra symbols Boa Me Na Me Mmoa Wo, Funtummiereku Denyenmireku, and Bese Saka (see http://www.adinkra.org and Figure 1). They provided individual, written observations about the symbols' symmetries and features in small groups via Geogebra classroom. In the second activity, which followed the class as a homework assignment, students were asked to create their own symbol, and then identify transformations within another student's symbol. IRB approval was granted for analyzing artifacts from students' work. Pseudonyms are used in the analysis section of this report to protect student privacy.



Figures 1A, 1B, and 1C. Adinkra Symbols Boa Me Na Me Mmoa Wo, Funtummiereku Denkyemiereku, and Bese Saka

Analysis of Data

We analyzed each of the PSTs' written responses to investigate the qualities of their transformational reasoning. After compiling responses across both activities, we categorized sections of their responses based on which transformations they described, and then we analyzed their transformational reasoning by categories: *reflection, rotation, translation, and dilation*. After analyzing the responses in these categories, we took note of a central theme: the degree of precision in students' transformational reasoning.

Creating the Lesson Play

Zazkis et al. (2009) defines *lesson play* as "imagined interaction related to a particular student's difficulty" (pg. 43). These interactions usually take the form of a pre-written script before teaching instruction. The key difference between a lesson plan and lesson play to Zazkis and colleagues is the spotlight remains on students' mathematical topics which emerge from interaction rather than guided notes for teaching instruction. In research, lesson plays can be used to investigate a teacher's interaction with a student or group of students (e.g., Dooley & Grimes, 2023), and sometimes lesson play script writing can be delegated as a task for a group of prospective student teachers (e.g., Zazkis & Zazkis, 2016). We construct our lesson play to be used for these purposes as well by using a virtual duoethnography approach (Zazkis & Koichu, 2015). This type of lesson play creation involves characters which typify the writers' experiences with students learning the particular topic. After finishing the analysis, two student characters were drafted by the first author to reflect the themes in our participants' transformational reasoning. The second author then reviewed the lesson play to suggest revisions to reflect his teaching experiences.

Analysis of Student Reasoning About Transformations

Rotations and Reflections

The most frequent type of symmetries PSTs identified were rotational and reflection symmetries. Some students connected the presence of reflectional symmetry with 180 degree rotational symmetry when referring to Boa Me Na Me Mmoa Wo (Figure 1A):

> Saul: There is "near" reflective symmetry horizontally and vertically, the exception being the negative/positive space of the square and circle. In this way there is also rotational symmetry (180 deg)

It appears that some students may have conflated reflections and rotations. Although it is possible in some cases–such as Kip below–students were using the term "reflect (around a point)" to refer to a 180-degree rotation.

> Kip: two triangles that are reflecting around the vertex that is touching…

Dilations and Translations

Some students also reasoned about dilations and translations. Of the five students who described dilations, two seemed to only consider the scale factor as relevant, neglecting or omitting identification of the center of dilation. This was the only time a student described a dilation with a scale factor less than one; the rest of the students who stated dilation relationships started with the smallest shape and described the larger, similar ones as the scaled up images.

As for translational relationships, six students identified them in symbols where a regular shape appeared to them to be "copied and pasted" in different locations. One PST even identified a star that seemed to be "created" from copied and translated triangles.

Role of Precision

A common theme across PSTs' writing about transformations and symmetry patterns regards the structure of precision in their articulations. We posit that PSTs may view their transformational reasoning *within* the symbol and their transformational reasoning *of* the symbol (within the plane) as having different standards for precision. The figures they were transforming were not explicitly embedded within a coordinate system, and most students did not add labels for points or lines to identify with transformations. PSTs instead mostly described reflectional symmetry via sketching lines of reflection or using descriptors such as horizontal or vertical. For rotations and dilations, PSTs did not usually identify a center.

A main struggle emerged with identifying symmetries in symbols like Bese Saka (Fig. 1C). By this we mean those symbols also had four congruent subshapes attached to a common center shape. At least a couple PSTs identified diagonal lines of symmetry or extra rotational symmetries (one incorrectly said this is demonstrated via a 45-degree rotation rather than a 90-degree one). However, even then, students ignored the details of the center shape. Bese Saka has an oval which would disrupt this symmetry; the student-created symbols "like" Bese Saka involved hexagons in the center, which do not have 90-degree rotational symmetry.

Lesson Play Construction

We wrote a lesson play following the analysis to assist teachers in navigating dialogue with students about the concept of transformations. This lesson play draws inspiration from our interactions with the PSTs in a college geometry course; however, it is tailored to include a broader audience of teachers introducing the concept of transformations for the first time with students.

Hypothetical characters for the lesson play, Art and Bo, were based on PSTs who reflect familiarity with communicating transformational reasoning in different ways. Specifically, Art communicates his transformational reasoning with drawings, gestures, and verbalized visual imagery. Bo communicates her transformational reasoning with specific terms and values (e.g., "center of rotation"). It seemed common for students to have a general concept of transformational relationships and symmetries but either omit necessary mathematical details explaining their reasoning or fail to visualize the complete image of a transformation.

In our presentation, we will present and get feedback about this script.

Discussion and Next Steps

Our analysis of the Adinkra activities gave insight to how this particular group of PSTs in our lesson study reasoned in their descriptions of symbols' properties and meanings. After analyzing the data of PSTs engagement with the two activities, we described ways in which they reasoned both formally and informally about transformations. We then wrote a lesson play to capture some identified PSTs' strengths and struggles and provide a resource for instructors.

The analysis results provide two key research takeaways we will explain in further detail in the presentation. The first is findings consistent with St. Goar and Lai's (2022) study with PSTs: even college students in mathematics or education tracks did not fully understand that transformations map every point on the plane. It was common for students to imagine transformations affecting only parts of a figure and not others to argue the existence of symmetries. The second takeaway is evidence of students exercising transformational reasoning through analyzing figures, both existing symbols like Adinkra and student-created ones. A few

PSTs even imagined how a figure could have been 'created' through copying and transforming parts of a figure.

The lesson play is another contribution. Teacher educators can have PSTs in a workshop or course create or edit an existing lesson play script around a specific math topic. This activity can serve as a conceptual analysis (Thompson, 2008) of a math topic; that is, a means of detailed thinking from a student's perspective on how to make sense of a concept mentally. Creating a lesson play might also unveil a teacher's implicit biases: what characters do they create? What details about them matter? Who are the authoritative voices? Researchers can test the generalizability of this script by seeing how well the student characters in it (Art and Bo, in our case) represent the thinking of other high schoolers or PSTs when working with transformations.

We presented our lesson play to another group of PSTs in the same undergraduate geometry course. We gave these PSTs the same symmetry task with the same Adinkra symbol, Bese Saka, as provided in the lesson play and determine if they encounter the same hurdles as characters Art and Bo. We also gave the PSTs a chance to engage with the lesson script after and modify it themselves. For the presentation, we will report our findings from this activity in the presentation along with the original lesson play script.

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STUDENTS’ MISCONCEPTIONS IN DEFINING AND REPRESENTING TRIANGLES

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Keywords: Geometry and Spatial Reasoning, Cognition

We are interested in the following questions: 1) What misconceptions do students have about defining and representing triangles? 2) How do students resolve these misconceptions over time?

A mix of middle-school and high-school students participated in our study, including 10 seventh graders, 8 eighth graders, 5 tenth graders, and 5 eleventh graders. All participants attended the same school in a mountainous area of the California Central Valley. In each interview, participants were asked to do the following tasks: (1) define a triangle and (2) split up a rectangle, hexagon, and a star into triangles.

The students’ definitions of a triangle were scored (out of 3 points) based on if they mentioned: (1 point) three sides, (1 point) three vertices/three angles/closed, and (1 point) planar/polygon. We deemed a students’ attempt of splitting up a shape as successful if: (1) the shape was completely split up into triangles, (2) the edges of the triangles were straight, and (3) the triangles were closed.

Our results suggest that the high school students perform better on both tasks, but the differences are less significant than we originally expected. Due to the small sample size and the limited details the students gave as reasoning, we cannot draw any substantial conclusions at this time.

Instead, we see this study as a jumping off point to further explore findings that piqued our interest. For example, six out of the 28 participants did not split the inner pentagon of the star into triangles. Three out of the six participants mentioned above successfully split up the hexagon into triangles. Since a pentagon and hexagon are structurally similar shapes, we are curious about this inconsistency and want to further explore whether it is due to the students’ limited geometrical understandings or merely due to inattention.

During several interviews, we noticed that the participants interpreted our “simple” tasks as an insult of their intelligence. We believe we may be able to resolve this issue by asking participants to complete a longer and more challenging test with our original tasks included in the assessment. This way participants will feel mentally stimulated, and we will be able to better assess their geometric knowledge.

We are also interested in if there is a statistically significant correlation between the teachers’ geometric knowledge and their students’ geometric knowledge. Hence, in our future research, we have plans to assess the teachers the same way we do the students.

Based on conversations with other educators and working with students one-on-one, it seems that geometry often falls between the cracks in K-12 classrooms, which we believe is a great disservice to students. Furthermore, we believe that even small gaps in a student’s geometric understanding have a lasting impact on their mathematical performance. So, we hope our research sheds light on how fundamental geometric concepts, such as the definition of a triangle, relate to more involved geometric tasks, such as splitting shapes into triangles.

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Chapter 5: Mathematical Knowledge for Teaching

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IMPROVING PRESERVICE SECONDARY TEACHERS' KNOWLEDGE OF SOLVING ALGEBRAIC EQUATIONS

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Keywords: Preservice Teachers, Secondary Teachers' Knowledge, Algebraic Equations

Algebraic equation solving is a fundamental skill within the secondary education mathematics curriculum, playing a pivotal role in shaping students' mathematical reasoning and problem-solving abilities as well as serving as a foundation where more complex mathematical concepts can be built (Mamba et al., 2017; Salani & Jojo, 2023). Mastering algebraic concepts is essential for academic success and equipping students with critical thinking skills applicable to various real-world scenarios (Mamba et al. 2017; Sandoval et al., 2023). The competence of preservice secondary teachers in imparting this knowledge is crucial, influencing the quality of instruction and the development of students' mathematical proficiency (Kennedy & Ebuwa, 2022, Kleickmann et al., 2013, Sam et al., 2023, McCrory et al., 2012). Van Dooren et al. (2002) focused on underscoring the relevance of preservice teachers' proficiency in discerning and guiding students through problem-solving processes, particularly in the context of algebraic reasoning. Therefore, secondary mathematics preservice teachers are required to possess mathematical content knowledge which will help them to teach effectively in the classroom (Daniel, 2015). This study aims to explore how preservice teachers' pedagogical content knowledge relates to their ability to effectively teach and create meaningful learning opportunities for secondary school students in solving algebraic equations.

The Mathematical Knowledge for Teaching (MKT) framework, developed by Ball et al. (2008), expands on Shulman's (1986) concept of Pedagogical Content Knowledge (Nolan et al., 2015). MKT can be referred to as mathematical knowledge required by teachers to teach mathematics in the classroom (Nolan et al., 2015). The MKT framework addresses PCK and Subject Matter Knowledge (SMK), which are two important domains of teacher knowledge described by Schulman (1986). The MKT framework emphasizes the significance of teachers' adaptive expertise in problem-solving, encouraging flexibility in selecting appropriate strategies and recognizing the diverse ways students may approach algebraic tasks (Van Dooren et al., 2003).

This study comprehensively reviews the existing literature on preservice secondary teachers' knowledge of solving algebraic equations. The PRISMA principles were followed in this review (Liberati et al., 2009). ERIC and Google Scholar are selected as online databases to find relevant articles. There are inclusion and exclusion criteria along with the justification. The ERIC and Google Scholar databases were searched using the following keywords with connectors "AND" "OR" words: ("pedagogical content knowledge" or PCK) AND ("teacher candidates" or "preservice teachers" or "student teachers" or "pre-service teachers") AND ("secondary school" or "high school" or" secondary education") AND "solving algebraic equations" AND ("math education" or "mathematics education") OR algebra while in Google Scholar the same search terms are used without quotation and parenthesis. The search terms produced a total number of

184 articles (ERIC 157 and Google Scholar 27). To narrow down these numbers using the inclusion and exclusion criteria produced a dataset of 15 research publications, which served as the basis for the current contribution. The preliminary findings align with these datasets, highlighting that preservice teachers' pedagogical content knowledge is crucial for effectively teaching algebraic equations and promoting meaningful learning opportunities for secondary school students.

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PRACTICING MIDDLE GRADES TEACHERS UNDERSTANDING OF REFERENT UNIT

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Referent unit is foundational knowledge for rational numbers and quantity. This exploratory study examined how sixteen practicing teachers reasoned about a mathematical situation during an interview requiring the use of referent unit. Half of the teachers correctly reasoned with referent unit and a quarter made a referent unit error. This error may have been connected to a previous interview question as opposed to indicating a misunderstanding about referent unit. However, attending to details, particularly language, is important. Considerations for professional development and preservice education are discussed.

Keywords: Teacher Knowledge, Rational Numbers & Proportional Reasoning, Rational Numbers

The importance of referent unit for both students and teachers for understanding fractions has been established (e.g. Hackenberg, 2007, Izsák, 2008; Steffe & Olive, 2010). Focusing on teacher understanding of referent unit; Izsák, Tillema, and Tunc-Pekkan (2008) showed data from Ms. Reese's lessons on addition and subtraction of fractions on number lines and a student's interpretation of these lessons. These researchers suggested teachers need to focus on particular details such as language. Ms. Reese tended to refer to fractions as "amounts," and this added to a misunderstanding of referent unit for at least one of her students. Armstrong and Bezuk (1995) documented challenges middle school teachers faced as they explored multiplication and division of fractions in a PD setting. One challenge was identifying the referent unit for each number in a fraction number sentence. Orrill, Izsak, Jacobson and de Araujo (2010) focused on teachers using drawn representations to solve fraction problems, and one of their findings indicates that the teacher often "resisted thinking in terms of nested levels of units" (p. 339). For example, one teacher struggled to correctly explain why $2/3 \div 1/4 = 8/3$ using drawings. In her representation, she drew a rectangle and divided it into thirds. After shading two-thirds, she discarded one-third, stating that the problem only concerned 2/3. She further divided the remaining thirds into four pieces and claimed that these eight pieces represented the numerator of the answer, while the thirds served as the denominator. However, her interpretation of fourths in the answer was incorrect as she "eliminated crucial units that the answer referred" (p. 339) from her representation. These studies highlight the importance of referent unit to teachers' own understanding as well as their support of their students' understanding.

In this study, our research question was how do practicing teachers reason about a question requiring the use of referent unit?

Theoretical Framework

In this exploratory study investigating practicing teachers understanding of referent unit, we define referent unit as the unit of measure (Olive & Çağlayan, 2008) or how a quantity is measured (Epstein, Orrill & Brown, 2023). The knowledge of referent unit is foundational knowledge, and we view the knowledge of teachers through the lens of the Knowledge Quartet (Rowland & Turner, 2007; Orrill, Brown, Thapa, & Nti-Asante, 2022). As Rowland et al. (2005) state: the Knowledge Quartet "is about raising awareness; it is not about being judgmental. Whilst we see certain kinds of knowledge to be desirable for elementary mathematics teaching, we are convinced of the futility of asserting what a beginning teacher, or a more experienced one for that matter, ought to know." (p. 257). Thus, the intent of this study is to become aware of how some practicing teachers found the answer to a question involving referent units.

Methods

As part of a larger study, we interviewed 16 practicing teachers from two different states on the east coast of the United States. Eight of the teachers were currently working in public schools and eight were currently working in a private, religiously affiliated school. Two identified as male; the average years of teaching math was a little over 15. Three participants identified as Hispanic and 14 identified as White/Caucasian (with 1 indicating both). Half the teachers had facilitated professional development themselves with two of them being related to mathematics education. Half also had a master's degree with two in mathematics education. One teacher indicated they had non-math related accommodations at some point in their educational process.

The interviews were conducted via Zoom using a Google jamboard to share items and see any markings the participant made while solving the mathematical problem shared with them. Eleven situations were shared with two to six questions asked for a total of 36 questions asked. The interviewer asked clarifying questions about their solution. The interviews lasted about an hour. All interviews were recorded and transcribed. The transcript was used to determine if a teacher's response was correct and to note how the teacher solved the problem. One participant experienced technological issues so was unable to write on the jamboard during the interview.

For this exploratory study, we considered one of the subitems in the interview requiring the teachers to find the size of part of a rectangular area (see Figure 1). This was the fifth question asked regarding the Garden situation. All of the garden questions were designed to address a teacher's understanding of referent unit. This particular item was selected because teachers were familiar with the context and were applying their knowledge of referent unit to find the answer.

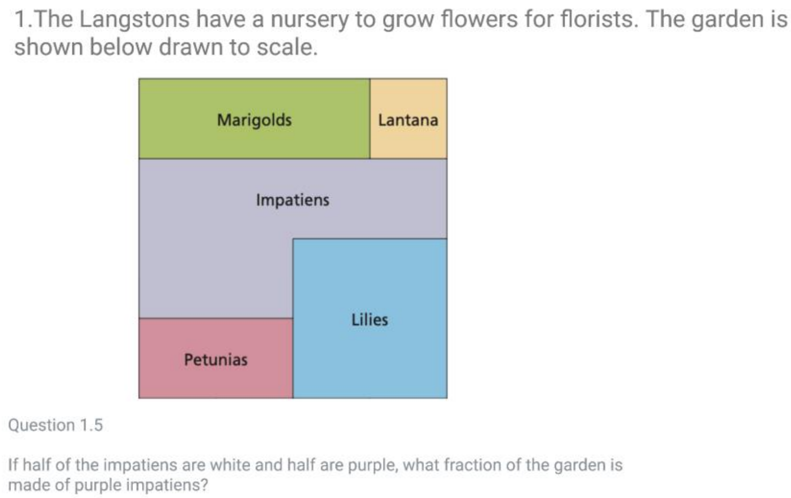

Figure 1: Math problem from interview

Three researchers coded the response for correct reasoning with 100% agreement. Then one researcher went back through transcripts and images from jamboards to detail how the teachers approached the question. These descriptions were shared with the other two researchers to see if there were any disagreements and to discuss the analysis.

Results

Of the 16 teachers interviewed, eight correctly reasoned three sixteenths of the garden was purple impatiens. In their explanation of how they found their answers, half the teachers used sixteenths and the other half used eighths.

For the sixteenths, one teacher explained their reasoning this way (representative of how the other 3 solved the problem): "so I like counted in basically in their 16 little lantanas, right? So if I break it all up, it's four by four. So it's 16". This teacher described how they broke up the entire garden into sixteen pieces (and sketched that on the jamboard as well). The teacher went on to explain the impatiens were 6/16 of the garden so the purple impatiens would be 3/16. One teacher who reasoned this way simplified 6/16 to 3/8 but then interestingly decided taking half of 6/16 was easier.

For the eighths, the teachers found all the impatiens to be 3/8 of the garden. For example, one teacher said: "I think that the lilies is a quarter, and that's the same. And then this is half of a quarter, which is an eight. So I've got two eighths plus one eighth. So I'm saying that impatiens make up three eighths of the garden". Once they knew the whole area of the impatiens, the teachers found half of 3/8 with three teachers writing a numeric representation of either ½ * 3/8 or 3/8 * ½.

For the remaining eight teachers who did not reason correctly, one teacher looked at the five colors of the garden and concluded 1/5. Another teacher estimated their answer, while another teacher just generically described the steps they would take which may or may not have been correct reasoning (thus not coded as correctly reasoned). Of the remaining five teachers, one teacher made a calculation error, one made both a referent unit and calculation error and three made a referent unit error. For the calculation errors, if the teacher did not realize the unreasonableness of their answer than we did not code as correctly reasoned even if they correctly described the area of the impatiens and needing to find half. The referent unit errors were based on not attending to the whole in the question (the whole garden). One teacher used the area of impatiens as the referent unit, and another identified the area of impatiens as ¾. Two used the area of the lily as the referent unit, based on the previous interview question that asked if the Lilies section is one acre of land, what fraction of an acre are the Impatiens? One of these teachers also miscalculated.

Discussion

Given the small sample size, this exploratory study highlights a few ways in which teachers may apply their understanding of a referent unit to a specific garden problem. Half the teachers correctly reasoned both about referent unit and arithmetically. Another quarter of the teachers made a referent unit error. This referent unit error may have been connected to the previous question in the interview and may not reflect a misunderstanding. However, if a teacher does not attend to the details around language, this may have a negative impact on their students, similar to the findings of Izsák et al. (2008).

Looking at the erroneous thinking, there is lots of potential to support teachers to correctly reason by modifying the question to emphasize the whole garden, asking if an answer is reasonable, or asking for more detail in the situation. The intent of the interview was to see what knowledge teachers already had about referent unit prior to a professional development experience that would focus on referent unit. Future research should examine how these teachers continue to use referent unit reasoning in the professional development experience.

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BROADENING & CONNECTING MATHEMATICS: EXPLORING HOW ELEMENTARY TEACHERS CONCEPTUALIZE THE BIG IDEAS

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The new California Mathematics Framework provides a roadmap for organizing mathematical knowledge around broader, integrated concepts ("big ideas"), with equity as a focus (CDE, 2023). This study involved collaborating with a local school district to select and train eight elementary teacher leader fellows in professional development (PD) focused on big ideas instruction and examining their conceptualization of the big ideas in mathematics after PD. Analysis of interviews with the eight teachers revealed they employed real-world analogies, education-specific schemas, and distinct content examples to understand and explain the big ideas. These methods allowed them to perceive the big ideas as central and interconnected within their mathematics instruction, encompassing a range of concepts wider in scope than traditional standards. We discuss implications for teacher learning around mathematics content.

Keywords: Mathematical Knowledge for Teaching, Professional Development, Elementary School Education

Purpose of Study

Critical to envisioning a just future for mathematics education is addressing the persistent inequities in students' access to meaningful opportunities to learn. While numerous initiatives focus on supporting teachers through the adoption of inclusive and dialogic pedagogies (Smith & Stein, 2018; Joseph et al., 2019) and/or culturally relevant curricula (Gutstein, 2007), an underlying concern remains: Could the very nature of the mathematical canon contribute to the perpetuation of these inequities? Our study investigates mathematical knowledge, emphasizing overarching, unified ideas over fragmented, smaller concepts, as this shift facilitates the enactment of equitable pedagogy (Boaler & Staples, 2008; Cabana et al., 2014).

The evolving landscape of mathematics education, notably marked by the adoption of the National Council of Teachers of Mathematics (NCTM) standards in the 1980s and the Common Core State Standards in the 2010s, has heralded calls for a coherent teaching approach. Despite these aspirations, the predominance of state standards emphasizing a disaggregated list of skills and procedures has persisted, influencing both assessment and instruction. The new California Mathematics Framework delineates a visionary shift towards organizing mathematical knowledge and practices around foundational "big ideas," with equity as a focus (California Department of Education [CDE], 2023). Our study directly engages with this transformative moment, responding to the critical need for empowering teachers and administrators to navigate and implement this innovative framework. Researchers partnered with a local school district to support elementary teachers in engaging with a big ideas approach in their mathematics teaching, thereby advancing equitable mathematics instruction for all students (CDE, 2023). This paper explores these teachers' emergent understandings of the big ideas. In particular, we ask: After a professional development (PD) focused on big ideas instruction, how do elementary teachers conceptualize the big ideas of mathematics for their grade level?

Theoretical Perspectives

Our study is informed by two related yet distinct theoretical perspectives on mathematics content. First, we discuss what we mean by the term "big ideas" as it relates to mathematics teaching and learning. Second, given that our focus here is on teachers' conceptions of big ideas, we draw on Ball and colleagues' (2008) notion of "mathematical knowledge for teaching" to theorize teachers' interactions with and perceptions of the big ideas.

Big Ideas in Mathematics

Too often, students view mathematics as an abstract, arbitrary, and disconnected set of rules. In the big ideas approach, however, students engage with the essential concepts in mathematics, which are composed of smaller, connected ideas. We adopt Charles's (2005) definition of a big idea as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole" (p. 10). This focus on the "coherent whole" enables teachers and students to grapple with the "essential mathematical meaning" of the idea (Charles, 2005, p. 10). Although the big ideas approach overlaps with similarly minded pedagogies (e.g., conceptual teaching, reform-oriented instruction), it explicitly relates to mathematics content and curriculum. As such, teaching to the big ideas necessitates a focus on a smaller set of essential mathematics concepts that coherently link numerous ideas.

The National Research Council (1999) recommended this approach over two decades ago: "Superficial coverage of all topics in a subject area must be replaced with in-depth coverage of fewer topics that allows key concepts in the discipline to be understood" (Bransford et al., 2000, p. 20). Students who work "in-depth" forge connections between mathematical ideas. This approach builds on research that has shown that teachers who organize content around big ideas and teach with an equity focus bring about higher and more equitable achievement (Boaler & Staples, 2008; Cabana et al., 2014).

Mathematical Knowledge for Teaching

To implement this approach, teachers need to be able to construct their own meanings of the big ideas, growing both their subject matter knowledge and their pedagogical content knowledge (Koellner et al., 2007). In their work on mathematical knowledge for teaching, Ball and colleagues (2008) specify three types of subject matter knowledge that mathematics teachers need: common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK). CCK refers to mathematics knowledge and skills that are used in non-teaching settings. In contrast, SCK refers to mathematical knowledge and skills unique to teaching, involving the "unpacking of mathematics" (Ball et al., 2008, p. 400). Further, HCK is teachers' awareness of how mathematics concepts are "related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403).

We contend that when teachers engage with and make sense of the big ideas of mathematics, they are simultaneously building all three types of subject matter knowledge (CCK, SCK, and HCK). As they grapple with the big ideas on math tasks as learners themselves, they build CCK, while as they analyze their curriculum to unpack what comprises each big idea and where the big ideas live, they build SCK and HCK, respectively. Accordingly, we view teachers' emergent conceptions of the big ideas of mathematics as amalgamations of all three types of subject matter knowledge. In examining how teachers make sense of the big ideas, we take up Ball and colleagues' (2008) call for supporting and empowering teachers to understand how mathematical knowledge is generated and structured.

Methods

This qualitative study uses a multiple-case design with eight participants (Yin, 2016). The case study methodology enabled us to provide an in-depth and nuanced exploration of teacher conceptualization of big ideas for a small number of participants.

Setting and Participants

In spring 2021, research team members met with administrators from a local school district in Northern California to understand their needs and discuss possible ways of supporting them during the pandemic, given prior work together. One-third of the elementary schools in this district are classified as Title I, and 37.4 percent of students received Free and Reduced-Price Meals (FARM) in the 2019-2020 school year.

The administrators and research team members co-constructed a teacher leadership fellowship to leverage the research team's expertise to develop teacher leaders who could support their colleagues with math instruction. In the 2021-2022 school year, the research team and administrators worked together to recruit site-based teams of two to three elementary teachers who applied to become "mathematics teacher leader fellows." The teachers admitted to the fellowship also consented to participate in this study. Our purposeful sample of teachers included sufficient variability in teaching experience and grade levels taught (Table 1).

Table 1: Participant Characteristics

Participant*	School*	Grades	Years Teaching
Amber	Tabitha Elementary	4/5 Combo	24
Sam	Pinewood Elementary	3	9
Stephanie	Pinewood Elementary	3	22
Denise	Jackson Elementary	5/6 Combo	6
Erin	Jackson Elementary	3/4 Combo	5
Nicole	Golden Sierra Elementary	2	13
Elizabeth	Golden Sierra Elementary	2	15
Heather	Golden Sierra Elementary	2	20

**All names are pseudonyms.*

Professional Development Context

The teachers participated in five professional learning days (PD) in the spring and summer of 2022. In these sessions, teachers worked in small groups on mathematics tasks and participated in mathematics discussions of these tasks and the big ideas embedded within them. Additionally, they examined the big ideas in their grade level and collaborated with grade-level colleagues to construct unit and year-long plans for the 2022-2023 school year.

Data Collection

The data collected for this paper are part of a larger data corpus. Here, we focus on two semi-structured Zoom interviews with eight teacher participants (n = 16 interviews). These interviews took place after the PD during the school year: in Oct/Nov 2022 ("Int2") and in Apr/May 2023 ("Int3"). In both interviews, interviewers asked teachers role-playing questions (Patton, 2002) to ascertain their conceptualization of the big ideas, e.g., "How would you explain 'big ideas in math' to a colleague who was not familiar with the term?"

Data Analysis

Our data analysis centered on teachers' responses to questions about the big ideas during the mid-year and end-year interviews. A team of five researchers engaged in a collaborative process of codebook development, refinement, and application. The analytic process included multiple rounds of coding, memo-writing, and discussions about emerging themes (Saldaña, 2009).

Codebook Development. The codebook was created through inductive analysis. Our team initially coded a subset of transcripts, then engaged in discussions to compare coding approaches and resolved disagreements regarding the definitions and criteria for inclusion or exclusion (Campbell et al., 2013). We then developed a second iteration of our codebook that expanded it from three codes to six, all under the domain "Conception of Big Ideas," with a definition and example quotes from the data (Creswell, 2013). We applied this codebook to the interview excerpts related to big ideas. Using Dedoose, the researchers eliminated one undersaturated code and determined the final codebook to be applied to all 16 interviews (Table 2).

Table 2: Code Descriptions

Code	Description
Analogy	Draws parallels or constructs similes and metaphors that liken big ideas to universally comprehensible or visually representational objects, such as umbrellas, webs, and circles.
Schema	Refers to a structure that helps organize and interpret big ideas based on common attributes, experiences, or concepts – particularly using educational terms, e.g., essential questions, enduring understandings, guiding questions, key points, units, frameworks, etc.
Open Tasks	Refers to big ideas as open tasks utilized in instruction.
Active Pedagogies	Refers to big ideas as student-centered instruction and other active learning pedagogies, e.g., changing environment and questioning; describing how students engage with the content and each other in the classroom, how they enact big ideas in the classroom through particular practices.
Specific Content	Statements in which teachers refer to a specific idea in their grade-level mathematics content, e.g., equivalence.

Thematic & Content Analysis. To examine the data (n = 72 excerpts) for emerging themes, we exported the Dedoose quotes grouped by code to Google Sheets. This extraction allowed us to re-code the interview excerpts with pattern coding (Saldaña, 2013). Through this process, we continued to memo on emerging themes, descriptive notes, and representative quotes. Once our final themes were decided (Table 3), we conducted content analysis (Patton, 2002) to provide a quasi-quantification of the number of teachers aligned with each theme.

Results

Teachers drew on various conceptual tools and content examples when asked how they would explain the big ideas of math to a colleague unfamiliar with the term. Table 3 shows the three most common themes in teachers' responses. To differentiate between analogy and schema, we draw upon Parsons and Davies (2022), who articulate that reasoning by analogy is a comparison of two concepts at the same concrete level of abstraction, whereas mapping a more general schema may help learn the abstract sense of an idea. Half of the teachers used real-world analogies to illustrate how they conceptualize the big ideas in mathematics, while all teachers drew on education-specific schema to articulate their conceptualization. Additionally, all teachers shared examples of specific grade-level content to explain the big ideas further. In this section, we delve into these three findings, then close by how these come together in describing teachers' emergent conceptualization of big ideas in mathematics.

Table 3: Themes and Representative Quotes from Interviews

Themes	Representative Quotes
Analogy used to conceptualize big ideas. (4 / 8 teachers)	
Concretized big ideas by using analogies to connect to something visual.	"I see the **visual of the web** in my mind and how all of these big ideas really connect to each other to form the standards that my students will learn in third grade" (Erin_Int2)
Illustrated the centrality and connectedness of big ideas using analogies.	"And so to me, a big idea is this **umbrella** of one idea that has a bunch of ideas within it. All of these little things that we go through that all tie back to that and intertwine with each other." (Amber_Int2)
Schema used to conceptualize big ideas. (8 / 8 teachers)	
Built on prior knowledge through a professional lens.	"In the old days, when you looked at state testing, you wanted to know what your **power standards** were. I kind of likened the bigger circles to the power standards." (Elizabeth_Int3)

Situated big ideas as broader in grain size.	"My current understanding is a lot of super smart people looked at the standards and distilled it down to some **monster conceptual** topics that I feel vertically align throughout the grade levels, are foundational and integrated." (Elizabeth_Int2)
Specific mathematics content referenced in relation to big ideas. (8 / 8 teachers)	
Referred to specific math topics as big ideas.	"For third grade, big ideas are multiplication, division, fractions, time down to the minute, area and perimeter." (Sam_Int2)
Discussed connections between topics.	"I would explain it as an understanding of how to process math, like learning about patterns in place value or understanding what 100 is, is kind of a general descriptor of what you might think or connect to." (Heather_Int2)

Analogy Used to Conceptualize Big Ideas

Four of the eight teachers utilized analogies (e.g., umbrella, web, tree, brainstem) to help them concretize this abstract concept into a familiar visualization, many of which specifically highlighted the centrality of big ideas and their connectedness. One participant explained:

> And when we look at some of these main ideas, these big things that all kind of go together, I think of it as an umbrella. Everything is underneath, and it's, there's so many things that are connected with each other. A big web, I mean, that's just like a big web, and you can make the connections between different concepts and different reasonings and activities. And to me, the big ideas is the idea that everything's connected. So, it's not just these small little bits of things and then you go, and you're done. (Amber_Int3)

In this description of the big ideas, Amber emphasized two everyday visuals: an "umbrella" and a "big web," while also attending to the specificity those two analogies brought to her conceptualization – connectedness and centrality. In this excerpt, she stated some iteration of "connect" three times. To her, this analogy was also based on the resemblance of everything being "underneath" the umbrella, eliciting a visual focal point for the big ideas to reside under.

Schema Used to Conceptualize Big Ideas

Expanding beyond a visual analogy, all eight teachers used a schema with teacher-facing language to discuss how they conceptualized a big idea (Table 3). Participants built on prior knowledge through a professional lens, referencing backward planning, units, frameworks, and red checks. As participants drew on other educational terminology, they conceptualized big ideas as broader in grain size than other units of content. For example, one participant noted: "If you take your long plan for math, and you have your standards, you have all these standards, your standards are then grouped" (Sam_Int2). Here, Sam not only drew on her prior work with standards to think about big ideas but emphasized that big ideas are bigger than standards; they are groups of standards.

Specific Mathematics Content Referenced in Relation to Big Ideas

In addition to drawing on the conceptual tools of analogy and schema, all teachers referenced mathematical content to explain the big ideas. Teachers identified specific topics, such as “fractions” (Erin_Int3), as big ideas at their grade level. Further, some teachers demonstrated an appreciation for the depth of these topics. A notable example includes Amber’s discussion of fraction multiplication:

> If one of the big ideas was understanding fraction multiplication, visually, that's not one thing. That's not one activity, and you're done. There's so much underneath that, so much within that that it just comes off of it. It's not one thing. A big idea is just not one great idea. It's a bunch of things that make up this whole big topic with all these different appendices. (Amber_Int2)

Amber’s nuanced understanding of depth mirrors the analogies (e.g., umbrella) frequently employed by teachers to describe their grasp of big ideas. Our analysis also showed that teachers' discourse demonstrated a more generalized understanding of mathematical content. For example, conventional topics such as addition and subtraction were conceptualized in broader terms like “groupings of things” or “composing and decomposing numbers” (Heather_ Int3). Rather than emphasizing content as standards, they discussed broader concepts.

Another emergent pattern pertains to the interconnectedness of specific pieces of content. For instance, Amber highlighted how the big ideas encapsulated a holistic view that interlinked “multiplication, factors, and number sense” (Int3), illustrating the content's connected nature within her teaching practice. Similarly, Heather discussed how the big ideas allowed her to draw connections between addition and subtraction properties, along with number sense, to support her students in utilizing number lines and hundreds charts—underscoring how big ideas facilitate a deeper understanding and teaching of mathematical concepts, enabling teachers to link specific classroom strategies with the foundational concepts their students are exploring.

Teachers’ Emergent Conceptualizations of Big Ideas

Taken together, our findings uncovered three key takeaways about teachers' emergent understandings of big ideas: the centrality and connectedness of big ideas using analogies, the big ideas as broader in grain size, and the connections between topics. The centrality and connectedness of big ideas, illuminated through the use of analogies such as webs and umbrellas, enabled teachers to grasp how individual concepts are not isolated but rather part of a larger, interconnected framework. This visualization process helped teachers conceptualize the curriculum not as a series of discrete lessons but as a cohesive narrative where each concept contributes to the understanding of a central, larger idea. Addressing the big ideas as broader in grain size, teachers acknowledged that these encompass larger conceptual domains compared to the more finely partitioned content typically encountered in standard curricula. Lastly, the connections between topics as part of the big ideas were further emphasized. Teachers recognized that effectively teaching to the big ideas involves drawing explicit connections between seemingly disparate topics, thereby unveiling the cohesive structure of mathematical knowledge. These articulations of the big ideas demonstrate that teacher participants employed a variety of conceptual tools to both comprehend and conceptualize the big ideas.

Discussion

Our results show that teacher participants drew on various real-world and education-specific conceptual tools to understand and explain the big ideas. Further, teachers utilized examples of specific grade-level content to support them in describing the big ideas. Through these tools, teachers saw the big ideas as central to their content, composed of multiple connected ideas, and broader in grain size than the standards with which they were used to working. Our findings highlight several main points, with implications for related research and professional learning.

Tools for Conceptualization

First, we found that conceptualization tools–schemas and analogies–were pivotal in helping teachers make sense of big ideas. Analogies help by linking new concepts to familiar ones, facilitating the application of known knowledge to new ideas (Parsons & Davies, 2022). This cognitive strategy enhances learners' ability to abstract, generalize, and transfer knowledge across different contexts, which is crucial in education. Our study supports existing literature indicating that individuals connect new information to prior knowledge when learning new concepts (Bransford et al., 2000). For instance, participants used the analogy of a web, relying on their understanding of a spider web or a graphic organizer, to grasp the interconnected nature of essential mathematics concepts. Here, our data highlighted the effectiveness of using analogies to connect new abstract concepts with familiar understandings.

Schemas, on the other hand, act as the cognitive structures that organize and interpret information, playing a crucial role in the way new knowledge is integrated into existing cognitive frameworks (Gick & Holyoak, 1983). As "mediators" of knowledge transfer, schemas enable individuals to categorize and store new information efficiently, facilitating easier retrieval and application in future learning situations (Gick & Holyoak, 1983, p. 2). Our findings showed that teachers used existing schemas to make sense of big ideas within their professional knowledge. Teachers assimilated new pedagogical approaches by mapping them onto familiar educational constructs such as standards, units, and frameworks. Therefore, providing teachers adequate time and support is crucial to linking big ideas with prior knowledge and tangible concepts during their learning journey.

Second, teachers drew on specific content examples as another tool to support their learning and uptake of some of the big ideas at their grade levels. Interviews showed that teachers offered deep, conceptual descriptions of topics such as addition and subtraction, pointing to their “unpacking of mathematics” (Ball et al., 2008, p. 400). This approach transforms abstract mathematical concepts into accessible knowledge for students, requiring a dynamic interplay between common content knowledge (CCK) and specialized content knowledge (SCK) as teachers master and teach the subject matter (Ball et al., 2008). Likewise, incorporating horizon content knowledge (HCK) into this discourse reveals a forward-thinking aspect of teachers' engagement with big ideas. By acknowledging the interconnectedness of mathematical concepts across different grade levels and curricular structures, teachers are aware of the "span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). This awareness aligns with the complex nature of mathematical content knowledge (MKT), which includes multiple forms of knowledge about mathematics, students, curriculum, and pedagogy. This study demonstrates the utility of MKT in theorizing teachers' interactions with and perceptions of big ideas.

Implications for Policy and Practice

The new California Mathematics Framework advocates for moving away from lists of discrete and disconnected skills and procedures (CDE, 2023). This shift is challenging for teachers, administrators, and professional developers because state standards have driven assessment and instruction for decades. Additionally, viewing mathematics as a list of procedures aligns with dominant discourses that mathematics is a fixed body of knowledge to be practiced, further complicating the shift (Louie, 2017).

As teachers in our study conceptualized the big ideas in mathematics education—a central component of the new framework—they highlighted elements of centrality, connectedness, and broad grain size. Teachers recognized that effectively teaching to the big ideas involves drawing explicit connections between seemingly disparate topics, thereby unveiling the cohesive structure of mathematical knowledge. This approach challenges the traditional compartmentalization of mathematical topics, advocating instead for a curriculum that mirrors the inherent connectedness of mathematical concepts. Viewing mathematics content as a set of fewer, connected, and conceptual big ideas has the potential to make mathematics activity more expansive and inclusive (Louie, 2017; CDE, 2023). Importantly, this approach supports teachers in developing mathematical knowledge for teaching (MKT), particularly horizon content knowledge (HCK), as it emphasizes connections across grade levels and units (Ball et al., 2008).

Our findings indicate areas for growth in how teachers develop a deep, conceptual, and connected understanding of the big ideas at their grade levels, particularly at the elementary level. Traditionally, teacher training has emphasized common content knowledge (CCK) more heavily than specialized content knowledge (SCK). Our results suggest the benefits of engaging teachers in grade-level teams to develop their SCK related to the big ideas and to build HCK through connections to other grades. This underscores the necessity of ongoing conceptual work to aid teachers and administrators in navigating this pedagogical shift. Future research should explore how teachers and school administrators adapt their practices to incorporate big ideas, including and beyond the new California Mathematics Framework. This inquiry could reveal insights into the systemic adoption of the framework and its impact on mathematics education, contributing to a more equitable and effective system for all learners.

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FACTORING QUADRATICS: HOW SECONDARY TEACHERS VIEW AND FOSTER PROCEDURAL FLUENCY

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While researchers have characterized what procedural fluency of numerical computation entails at the elementary level as well as identified instructional methods that support its development, secondary teachers do not have access to the same resources. To support these teachers, we must first understand how they perceive and attempt to foster procedural fluency. To do so, we interviewed eight experienced secondary teachers about how they teach factoring of quadratics and their associated goals. Results indicate that teachers lack an in-depth understanding of procedural fluency associated with factoring. While some teachers hope students develop more conceptual understanding, they ultimately teach rote algorithms that lack meaningful insight into the process. We outline what methods teachers use to teach factoring and provide a framework highlighting three different ways teachers approach the teaching of procedural knowledge.

Keywords: Algebra and Algebraic Thinking, High School Education, Mathematical Knowledge for Teaching

The relationship between the role of conceptual and procedural understanding in mathematics education has long been explored and argued (Rittle-Johnson, et al., 2015). This debate was at the center of the heated math wars in the 1990s (Schoenfeld, 2004) and continues today as states like California rewrite their mathematics curriculum. Researchers, often critical of the elevated attention to procedures in classrooms, have advocated for an emphasis on conceptual understanding, making this the focus of the reform movement. However, as Kilpatrick et al. (2001) explained, pitting one against the other creates a false dichotomy. Procedural and conceptual understanding are not separate, competing forms of knowledge, but are interwoven competencies that support each other. As such, rather than characterizing one as more or less valuable than the other, a more productive approach is to describe ways in which different manifestations of each are robust or not (Star, 2005).

Nevertheless, researchers continue to emphasize conceptual understanding, paying less attention to procedural fluency (Bay-Williams, 2020). The one exception has been at the elementary level where a significant effort has been made to explicitly characterize what procedural fluency of numerical computation entails as well as identify instructional methods that support its development (Bay-Williams & San Giovanni, 2021). This has led to a robust understanding of the types of strategies and reasoning involved in this critical mathematical understanding, previously only known by vague terms such as “number sense.” Such details have begun to appear in our standards and curriculum documents (CCSS, 2010), providing guidance for teachers to shift away from teaching rote algorithms and now target robust procedural knowledge. In addition, math educators have collectively developed various activities that support its development as well as numerous tools (rekenreck, number lines, area models, etc) that promote such ways of thinking. Such findings mean that what was once viewed as an innate quality in only a few students can now be developed in all students (Bay-Williams & San Giovanni, 2021).

Notably, in light of these advances, instruction in the US still tends to predominantly emphasize rote procedures. While we know that at the elementary level teachers' lack of mathematical knowledge is a contributing factor (Ma, 2010), there are limited studies that explore how teachers understand and attempt to foster procedural fluency at the secondary level. There have been multiple theoretical pieces characterizing the relationship between procedural and conceptual knowledge as well as empirical studies exploring how different instructional foci support student learning, but we do not have a strong understanding of what procedural fluency looks like for different secondary topics or what teachers' perspectives of procedural fluency is. Establishing such views is critical, enabling the field to develop targeted ways to better support teachers in engaging in more conceptually connected instruction of different procedures. The aim of this study was to fill this void. To do so we focused on the topic, quadratic factoring, to explore the question, *What understandings and perceptions of procedural fluency shape secondary teachers' instruction of factoring quadratics*?

Conceptions of Procedural Knowledge

Both procedural skills and conceptual understanding have long been valued components of mathematics education, but have historically been viewed as separate entities. Notably, when Hiebert and Lefevre (1986) first introduced and defined the terms procedural and conceptual knowledge, while they explored relationships between the two and acknowledged that it is hard to imagine students developing one without the other, they still positioned them as different types of reasoning, defining procedural knowledge as a familiarity with syntactic manipulation and conceptual knowledge as a deep, connected understanding. Fundamentally, they characterized all learning with meaning as conceptual, aligning with Skemp's (1978) notion of robust relational understanding and all procedural knowledge as rote, equivalent to the more superficial instrumental understanding. Critical of such an association, Star (2005) argued that knowledge type and quality should be treated as independent dimensions, challenging the field to conceptualize deep procedural knowledge and make it an instructional goal.

Today, scholars agree that robust procedural fluency should be an instructional goal, but there is debate over how best to foster it. NCTM (2014) advocates for instruction that extensively develops conceptual knowledge before procedural knowledge. However, Rittle-Johnson, et al. (2015), after a comprehensive analysis of multiple empirical studies, found no evidence for fostering one type of knowledge prior to the other. Instead, they concluded that the two serve to support each other and should be developed simultaneously.

More recently, in an effort to formalize this work, Fan and Bokhove (2014) offered a framework characterizing instructional foci that foster procedural fluency at three different levels (see Figure 1). In the first level, the focus is on rote practice with the goal of developing an instrumental understanding to consistently and correctly carry out the steps. The second level is characterized by a relational understanding of why the algorithm works and how it can be modified to accommodate different situations. The final level inherently involves familiarity with different algorithms as it consists of the ability to judge and compare the efficiency of different algorithms. Level 3 also includes the ability to construct or generalize new algorithms.

In addition to highlighting the type of student thinking associated with each level, Fan and Bokhove (2014) also highlight how the nature of the algorithms taught shape the level of thinking. While certain algorithms make the foundational ideas driving the algorithm quite accessible, others mask these underlying concepts making it difficult or impossible to unpack them. Drawing on Kilpatrick et al. (2001), they propose five different features (certainty, reliability, transparency, efficiency, and generalizability) to analyze the quality of algorithms themselves, which they see as aligning with the different levels of their proposed framework. Specifically, they argue that algorithms that only focus on the certainty that given steps are fixed and unambiguous correspond to level 1. Procedures that provide transparency for why they work as well as the reliability to consistently obtain correct answers characterize level 2 thinking. Finally, algorithms that allow for efficiency and generalizability are associated with the top level.

Level of Thought	Fluency with Algorithms	Features
Level 1: Knowledge and Skills	• Reproducing steps of a procedure	Certainty
Level 2: Understanding and Comprehension	• Describing why a procedure works • Applying procedure in a complex situation	Reliability Transparency
Level 3: Evaluation and Construction	• Comparing different algorithms • Judging efficiency of an algorithm • Constructing new algorithms (strategies) • Generalizing	Efficiency Generalizability

Note. Recreation of image created by Bay-Williams (2020) based on Fan & Bokhove (2014) framework including algorithm features.

Figure 1: Procedural Fluency Framework with Corresponding Bloom's Taxonomy

While this framework provides a theoretical lens to categorize procedural knowledge in general, it lacks details about what these different categories entail for different content areas. As noted, one exception where such details have been developed has been at the elementary level, where Bay-Williams and colleagues (2021) have developed an in-depth characterization of fluency with numerical operations. In addition, they articulate how fluency goes beyond simply speed and accuracy, identifying different strategies which illustrate flexibility as well as provide a broader view of efficiency that incorporates appropriateness. While such work offers a powerful resource to support teachers at the elementary level, we have a limited understanding of procedural fluency at the secondary level. One example is Durkin et al. (2017), who worked with algebra teachers to explore if instruction that focused on comparing different solution methods of symbolic algebraic equations would support students in developing procedural knowledge. They found students developed more robust procedural knowledge in classrooms where teachers used the curricular materials more, but also found that most teachers did not engage students in these activities and struggled, in particular, to facilitate conversations around different methods. Such results hold promise, but indicate that more work is needed exploring how teachers understand and attempt to foster procedural fluency.

Factoring

Factoring provides a productive context to explore teachers' perceptions of procedural fluency as there are multiple different and well-known ways to approach this topic. However, to analyze these different approaches, we first define the conceptual understandings associated with factoring. We see two interrelated concepts that are critical to understanding factoring. First, students must recognize that the factoring process serves to transform algebraic expressions from an additive to a multiplicative structure. They must understand that the two forms are equivalent expressions, but that the different structures elevate different characteristics. However, in most cases this process cannot be determined directly. Instead, it requires carrying out the distributive property and observing patterns to be able to reverse the process. Such pattern recognition characterizes the second conceptual component. While often taught as rote rules, the ability to effectively identify and generalize useful forms involves the mathematical practice *Looking for and Using Structure* (CCSS, 2010). As teachers know, fostering this practice is challenging, as the ability to strategically decompose algebraic expressions relies on a combination of goals and student understanding and thus cannot be taught as a rule.

Features of Quadratic Factoring Algorithms

With these underlying conceptual ideas in mind, we apply the Fan and Bokhove framework to the different quadratic factoring methods that have emerged. Considering which features each algorithm possesses, highlights the levels of thinking they afford.

Slide, divide, bottoms up-level 1. A well-known algorithm that focuses on certainty, without transparency is Slide, Divide, Bottoms Up, also known as Slip Slide (Steckroth, 2015). However, while it provides a sequence of easy-to-follow steps to factor all nonmonic quadratics, most are not only unjustified, but mathematically incorrect. So illogical are the different steps that the only way to know how to carry out the algorithm is by following the steps outlined in the name (see Figure 2). Since the underlying rationale is hidden, this algorithm allows for only a level 1 level of thinking. Moreover, it encourages students to treat algebraic expressions as a collection of disconnected symbols that can be moved and manipulated as isolated characters, serving to undermine the development of any structural sense.

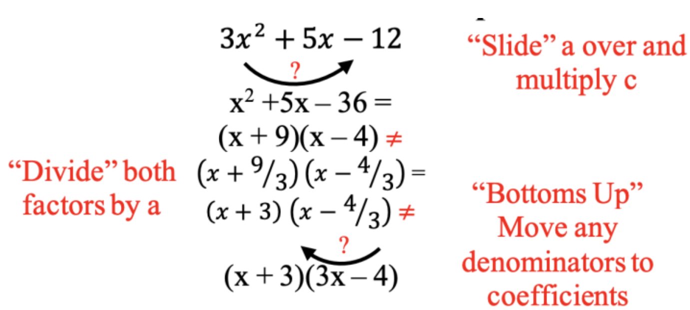

Figure 2: Outline of steps involved in Slide, Divide, Bottoms-up

AC method-level 1+. Another systematic approach that provides a consistent and reliable algorithm for all cases is the AC Method. This method relies on creating a 4-term expression, which can be factored using grouping, by splitting up the linear term into two factors of the product AC that add together to form b (see Figure 3). However, while each step follows correctly from the previous and the grouping provides a clear connection between factoring and distributing, there is no transparent reason for why the factors of AC should lead to 4 terms that can be reliably grouped. Moreover, to avoid challenges associated with grouping, teachers often use a 2x2 Box. Each of the 4 terms are placed in a separate box and the GCD from each row and column is factored out. In contrast to grouping, this avoids conceptualizing the binomial (x + 3) as a single algebraic entity, a key component of structural reasoning (Musgrave, et al., 2023) and again encourages students to treat each symbol as an isolated object, rather than a meaningful expression. As such, this method has attributes of level 2, offering some insight into how the multiplicative structure is formed, but still lacks full transparency, leaving it at the lower level 1.

AC = -36

9(-4) = -36

Grouping

$$3x^2 + 5x - 12 =$$
$$3x^2 + 9x - 4x - 12 =$$
$$3x(x + 3) - 4(x + 3) =$$
$$(3x - 4)(x + 3)$$

Box

	x	3
3x	$3x^2$	9x
-4	-4x	-12

Figure 3: Symbolic and Box approaches to the AC Method of Factoring

Guess and check-level 2. As noted above, conceptually understanding factoring requires seeing the connection to the distributive property which inherently involves pattern recognition to reverse the process. Such a method is often referred to as guess and check. While the pattern for monic quadratics is reliably the same, this is not the case for quadratics when the leading coefficient is not 1. Such a method can definitely be characterized as transparent and at times efficient, However, when applied to numbers with multiple factors, this method loses efficiency.

Scaling-level 3. One algorithm that is not only consistent and transparent, but also generalizable is a scaling method (see Figure 4) offered by Cuoco (2009) as part of the CME project. This algorithm uses substitution to transform all quadratics into a monic quadratic. Such a method provides a systemic approach that allows students to use the general pattern of looking for two factors of the constant term that add to the linear term.

$$3x^2 + 5x - 12$$
$$\frac{1}{3}(9x^2 + 15x - 36) = \frac{1}{3}((3x)^2 + 5(3x) - 36)$$
$$\frac{1}{3}(u^2 + 5u - 36)$$
$$\frac{1}{3}(u + 9)(u - 4)$$
$$\frac{1}{3}(3x + 9)(3x - 4)$$
$$(x + 3)(3x - 4)$$

Multiply expression by leading coefficient to scale up by a factor of a.

Create a monic quadratic by substituting u = ax

Divide expression by a

Figure 4: Scaling method which transforms quadratics to a monic form

Methods

To explore how teachers view and foster procedural fluency associated with quadratic factoring, we conducted and videotaped one-hour individual semi-structured interviews (Ginsburg, 1997) with eight high school teachers (1 male, 7 female; all white). Overall, the teachers possessed a wealth of experience and expertise, with four to 34 years in the classroom and an average of 17 years teaching. All but one had a graduate degree, with the majority studying specifically mathematics education, and had experience teaching multiple grades and classes. To ensure a diversity of perspectives, teachers were selected from seven different schools across two different states. The schools represented a wide range of student populations in terms of their socioeconomics (30%-88% poverty rates) as well as prior achievement (27%-65% meeting minimal proficiency levels on state algebra test). In particular, two schools were among the top performing in the state and two were among the lowest in the state.

Interview questions focused on identifying what methods the teachers have taught and currently teach in different classes as well as different questions aimed at eliciting their instructional motivation for choosing these. In addition, we asked the teachers about their instructional goals associated with factoring, their familiarity and understanding of different factoring methods, and ways they differentiated instruction including the use of different tools (i.e. algebra tiles). We began our analysis by identifying statements that referenced the methods they teach, the rationale behind those methods, and their understanding of the conceptual value of factoring. This led to several codes marking the overall conceptual understandings of factoring teachers possessed which we refined through multiple iterations of analysis. Ultimately, we found three categories that were representative of how teachers understand procedural fluency and their teaching of factoring.

Results

We organize our results around two major findings. First, we outline the actual methods that teachers instruct in class (see Table 1), separated by the tracked level assigned to students (see Prins & Hawthorne (2024), for more information about how tracking shaped teachers' instruction of factoring). Because some used more than one method, totals add to more than the number of participants. We then outline the different goals that teachers communicated about factoring. Together these methods and associated goals provide insight into the approach these teachers use to foster procedural fluency.

Table 1: Factoring Methods Used by Teachers for Different Levels

	Level 1			Level 2	Level 3	
		AC Method				
	Slide Divide	Grouping	Box Method	Guess & Check	Scaling	Calculator
High Track	1	2	0	6	0	0
Low Track	2	2	5	1	0	1

As the above table illustrates, the majority of the teachers use the Guess and Check method in their honors classes, while teaching the Box-AC method for their lower tracked students. Thus, teachers implement more conceptually grounded methods with their higher tracked students, but teach rote algorithms with limited to no transparency to their lower tracked students. Notably, when asked why they teach guess and check in honors, only one teacher made reference to the conceptual value of such a method. All others, even when probed multiple times, explained that they taught it for efficiency reasons, as they saw it as the best way to quickly arrive at the answer. Conversely, the teachers collectively taught the lower tracked students methods void of meaning, explaining that they saw the consistent steps of the Box-AC and Slide-Divide methods as the easiest to follow. Looking through these results through the Fan & Bokhove (2014) framework, such methods ensured that only students deemed advanced had access to more robust procedural fluency, but to support efficiency, not conceptual understanding. Ultimately, these teachers picked these methods not to foster procedural fluency grounded in flexibility or sensemaking, but rather out of a desire for speed in the higher tracks and a deficit perspective towards students in the lower tracks.

Probing deeper into the motivations behind such choices, we found that the majority of teachers saw factoring as simply a tool used to produce an answer, rather than a conceptual topic. As a result, they teach their students rote algorithms which allow students to arrive at a factored form, but without having to struggle with the deeper mathematics that are at play. Moreover, when asked, the teachers were unable to provide justification for these different methods and in most cases were unaware. However, results provide nuance to the teachers' understanding of factoring which led to three distinct categories that characterize the different ways in which teachers viewed factoring: conceptual value, general value, and performance value (see Table 2).

Table 2: Characterization of Instructional Value Associated with Procedures

Categories	Teachers	Characterization
Conceptual value	1	Articulated conceptual value in factoring and taught methods that aim to foster such understanding
General value	3	Alluded to vague notions of conceptual value (logical and reasoning, graphical and symbolic, multiplying and dividing), and focused on speed and answers
Performance value	4	Saw no conceptual value in factoring and taught methods that solely got an answer

Conceptual value

The first category, conceptual value, consisted of a single teacher who was able to speak to the conceptual value of factoring and as a consequence taught methods that aimed to foster such understanding. Throughout the interview, her responses indicated that she wanted students to understand the basis of different factoring forms and intentionally pushed back against methods that did not allow students to see the conceptual foundation associated with factoring. When asked about these other methods, she said, "kids don't understand the concept of what factoring is… they get caught up in learning this trick and memorizing it verses teaching them that factoring is just the reverse of multiplication." It is important to note that because of these beliefs, she teaches the guess and check and grouping methods to all students, regardless of track, and was the only teacher to do so. When asked why she preferred teaching guess and check, she emphasized that this method supports students in "understand[ing] the concept" and "it sticks with them longer." While her description of the underlying concepts was not necessarily robust, her appreciation of such understanding was significant enough to strive to support all students to engage in the thoughtful understanding of factoring.

General value

The second category, general value, was composed of three teachers who, when prompted, alluded to vague notions of a conceptual value in factoring. These notions included logic and reasoning skills, understanding graphical and symbolic relationships, or multiplying and dividing. As one teacher explained, "Understanding that factors are things you multiply to get this, and now we can do this with expressions the same way we do it with numbers." In particular, they predominantly focused on applications when describing the value of factoring (as opposed to thinking), citing that factoring is needed for finding zeros, and simplifying rational expressions. While these teachers were able to identify instructional goals that went beyond simply getting answers, these were vague without clear methods to foster such goals. Consequently, without more explicit understanding of the value of factoring, their instruction was characterized by a focus on speed and answers. All teachers in this category opted for methods they saw as producing answers quickly for their honors classes, and taught their lower tracked either the Box-AC method or Slide-Divide, believing these would be the easiest to produce correct answers. A quote from one teacher summed up how teachers' vague notion of conceptual understanding shaped their teaching of procedural fluency, "It is definitely a good thing to use sound methods, I don't know if that means that it is more advantageous to another." Despite her seeing value in using more conceptual methods, she ultimately opts for an algorithm that is less mathematical and easier for students.

Performance value

The last category was representative of 4 teachers who did not see any conceptual understanding in factoring and therefore teach factoring methods they believe will most easily produce correct answers. When asked about what they see as the value of factoring, these teachers cited needing to know how to factor for future math classes, application problems and solving for zeros. Representative of this category was one teacher who said, "The whole point [of factoring] is can you see some function and find out features of that function... most of the things we want to know is what can we solve this function for, which relates to the x-intercepts, so factoring is just simply a way to do that." Due to a lack of more in-depth goals, teachers in this category focused on getting to the answer most accurately and with the least amount of struggle for students. For the lower track, these teachers used the Box-AC method or Slide-Divide, with the exception of one teacher who had eliminated factoring from her instruction to exclusively teach the use of the quadratic formula as this had proven to be the easiest for students to get to an answer. While these methods do not foster a robust procedural knowledge, their instruction makes sense given their understanding and perspective. If teachers do not see any value in the process of factoring, then there is limited justification for using more challenging methods that lead to more student pushback and possibly lower accuracy.

Discussion

We found that teachers lack an in-depth understanding of procedural fluency associated with factoring, with half of the teachers teaching mathematically superficial and incomprehensible methods that serve to only produce answers. In addition, another group of teachers wanted students to develop a more robust procedural fluency, but were unable to articulate what this would entail. Without clear goals, these teachers eventually focused on speed and ease as well. However, there was one teacher using more conceptually grounded methods to instill a more meaningful understanding in all her students.

Notably, all of the teachers we interviewed had tried a variety of different methods, suggesting they had autonomy for their instruction of factoring. Furthermore, none of them spoke of stresses associated with testing, curricular demands, or department or administrative pressure. While this might be because most of the participants were highly experienced and taught in states without an end of the year exam past Algebra I, such independence distinguishes secondary teachers from their K-8 colleagues. Instead, what shaped their instruction was a narrow view of procedural fluency of factoring with a sole focus on answers combined with beliefs about their students. These teachers had each invested significant amounts of time learning multiple factoring algorithm along with different scaffolds to help students easily and reliably carry them out. So focused on simplifying the process that issues of productive struggle, mathematical practices, or possible messages about mathematics were overlooked. Even questions about why such algorithms worked were not considered, as teachers were not only unable to explain various steps of these algorithms, but had clearly never thought about this.

As noted earlier, the field has limited insight into how secondary teachers perceive and teach procedures. These results fill that void, highlighting that teachers need guidance in what procedural fluency entails and how to foster it. Moreover, they need support in understanding how different algorithms (and tools) aid or hinder procedural fluency.

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EXPLORING TEACHERS' SPECIALIZED KNOWLEDGE USING TECHNOLOGY AND MATHEMATICAL MODELING: INSIGHTS FROM MEXICO AND CANADA

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The study of teachers' knowledge and competence in integrating mathematical modeling has gained recent attention, particularly in German research (Greefrath et al., 2022). Such knowledge and competence have been explored by academics with the purpose of, on the one hand, designing teacher training strategies and, on the other hand, measuring the impact of these strategies. The present study explored eight educators' testimonies and how they implement modeling in their classrooms to address the research question: *What pedagogical approaches do practitioner teachers use to design and apply mathematical modeling tasks that integrate virtual simulations?*

Methodology. This study conducted eight semi-structured interviews. Data sources encompassed video recordings, transcripts, activity handouts, and researchers' notes. The constant comparison method (Merriam & Tisdell, 2015) was employed for data analysis, involving open coding to identify main themes across all activities and interviews. Additionally, we utilized Kaiser and Sriraman's (2006) classification of modeling perspectives to categorize the pedagogical approaches described by the teachers. Our focus on simulations serves as a compelling example of the purposes of the whole project. The themes we will discuss are: (1) simulation promotes a change of beliefs about chance, and (2) simulations through embodied modeling generate empathy.

Conclusions. Overall, our findings underscore the importance of teachers' specialized knowledge in designing effective simulation-based activities that cater to their students' diverse needs and contexts. Teachers can create transformative learning experiences that promote critical thinking, problem-solving, and socio-cultural awareness by recognizing and leveraging the power of simulations in mathematics classrooms. A limitation of this study corresponds to the number of participants and tasks included. However, we identified differences with the perspective on teacher knowledge described by Gerber et al. (2023), which are based on work done mainly by European authors, primarily German. In this sense, our results add to the efforts of Cordero et al. (2022) to seek alternative perspectives to the dominant demands of Europe.

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ELEMENTARY PRESERVICE TEACHERS' MATHEMATICAL MEANINGS FOR TEACHING FRACTIONS

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Introduction

Students' stages of units coordination are closely related to constructing schemes for fractional knowledge (Hackenberg, 2013). While students learn fractions in their elementary school, there are many middle school students who have difficulty coordinating multiple levels of units (Zwanch & Wilkins, 2021). Furthermore, some Pre-Service Teachers (PSTs) have not reached to stage 3 of units coordination (Jacobson & Izsák, 2014; Son & Lee, 2016). Although there are many studies about students' stages of units coordination (e.g., Hackenberg, 2007; Hackenberg & Lee, 2015; Hackenberg & Sevinc, 2024; Hackenberg & Tillema, 2009; Steffe, 1992, 2003), little is known about teachers' stages of units coordination (e.g. Izsák et al., 2012).

Teachers' mathematical meanings for teaching fractions may give more context to teachers' understanding and their scheme related to teaching. While there are studies about teachers' mathematical knowledge for teaching fractions (Copur-Gencturk, 2021; Izsák et al., 2012; Olanoff et al., 2014; Veloo & Puteh, 2017), little is known about teachers' mathematical meaning for teaching fractions (Thompson, 2013). This study examines how PSTs' capacity for units coordination is related to how they construct fraction schemes and create mathematical meanings for teaching fractions with two research questions: (1) Which cognitive processes might PSTs go through? What modifications to their fractional reasoning schemes might be noticed? and (2) How are PSTs' unit coordination stages related to the development of their Mathematical Meanings for Teaching Fractions?

Methodology

I intend to initiate the data collection by administering a written task to eight elementary PSTs, aiming to select four (out of the 8) candidates for the subsequent 10-episode teaching experiment and post-test. The data collection will involve two distinct phases. Initially, I will enlist 8 PSTs for a written task focusing on units coordination and fractions for the pre-test by using instrument by Norton et al (2015). Subsequently, I aim to conduct 10 episodes to delve into fractional knowledge. The initial five episodes will concentrate on their first-order knowledge, engaging participants in fraction-related problems accompanied by 45-minute discussions. The latter five episodes will center on the PSTs' second-order knowledge, involving their interpretation of elementary students' problem-solving videos related to fractions. Following the 10 episodes, the 4 selected PSTs will undertake a post-test similar to the pre-test, again performing written tasks on units coordination and fractions while recording videos.

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INTENTIONALITY OF DIAGNOSTIC INTERVIEW DESIGNS FOR MATH AND PEDAGOGICAL KNOWLEDGE

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This theoretical analysis examines the diverse designs and intentions behind mathematics teacher educators (MTEs)' diagnostic interview assignments for elementary preservice teachers (PSTs). It underscores the necessity for PSTs to develop integrated specialized content knowledge and pedagogical skills. The study establishes a framework to assess alignment between MTEs' design goals and actual practices. Through cross-institutional collaboration, common phases of diagnostic interviews were identified, focusing on evaluating PSTs' abilities to analyze student thinking and apply pedagogical strategies. The analysis emphasizes the importance of intentional design tailored to PSTs' needs, prompting educators to review interview design and rubrics for effective course planning.

Keywords: Instructional Activities and Practices; Mathematical Knowledge for Teaching; Preservice Teacher Education; Elementary School Education

Effective mathematics teachers evaluate and reflect on both the mathematics content and teaching methods they use in the classroom to support the learning of a diverse student population (Association of Mathematics Teacher Educators [AMTE], 2017). To learn how to enact that reflective practice, well-prepared elementary preservice teachers need opportunities to build a cohesive understanding of specialized mathematics content knowledge (SCK) and how to apply that knowledge through effective pedagogical practices (Ball et al., 2008; Li & Howe, 2021; National Council of Teachers of Mathematics [NCTM], 2014). However, preservice teachers (PSTs) in the U.S. experience a wide variance in the number of opportunities to learn how to teach mathematics effectively in their teacher training due to differences in program and course structures (Bertolone-Smith et al., 2023; Cochran-Smith et al., 2015). Mathematics teacher educators (MTEs) with limited time to prepare their PSTs, must design methods courses in innovative ways and scaffold instruction so that their PSTs build an integrated knowledge base of mathematics content and pedagogy (Harr et al., 2014; Saclarides et al., 2022).

Diagnostic interviews are one type of assignment often used by MTEs in math methods courses to create integrated opportunities for PSTs to develop various types of knowledge and practices necessary for their future career, while assessing PSTs' competencies for teaching.

However, MTEs' designs and goals for diagnostic interviews are diverse. The purpose of this theoretical analysis is to understand the diversity and intentionality of MTEs' designs and goals for diagnostic interviews and create a framework for them to reflect on and self-evaluate the alignment of these two elements.

Theoretical Framework

There are two core elements to teaching mathematics that all PSTs must know: 1) Specialized Math Content Knowledge (SCK); and 2) Pedagogical Content Knowledge (PCK). These two elements are the foundation of the theoretical framework for the theoretical inquiry presented in this analysis.

Mathematics teachers should elicit students' mathematical thinking and use it as evidence to inform their instruction (Forzani, 2014; NCTM, 2014). To enact this practice effectively, teachers need a strong foundation in domains of SCK (Ball et al., 2008). These domains include being able to notice and understand students' mathematical thinking; evaluate the accurate, flexibility, and fluency of students' mathematical strategies; as well as the differences between conceptual understanding of a topic and procedural strategies to apply that conceptual understanding (Bahr & de Garcia, 2010; Ball et al., 2008; Jacobs et al., 2010). Teachers also need a strong foundation in the domains of PCK. These domains include knowledge of the content, curriculum, students, and teaching practices (Ball et al., 2008). Pedagogical practices for teacher noticing focus on interpreting students' knowledge of the math content, where that knowledge falls within the curriculum and using evidence to adjust instruction as needed to support student learning (Jacobs et al., 2010; NCTM, 2014). Additional pedagogical curriculum teaching practices that support mathematics learning important to our theoretical framework include establishing goals, purposeful task design for reasoning and problem solving, using representations, posing purposeful questions, building understanding and using evidence of student thinking for curriculum planning (Litster et al., 2020; NCTM, 2014).

Analysis of the Issues

This theoretical analysis is one component of a cross-institutional study that examines how PSTs in elementary mathematical content and methods preparation courses develop and use SCK and PCK for teaching. During 2023, representatives from eleven universities met and discussed activities in their programs that integrate specialized math content and pedagogical methods. One common activity across multiple institutions was the use of a diagnostic interview given to K-12 students in a clinical setting. However, the interview at each university was unique in design. Through an iterative qualitative process, the group identified and refined five common phases of the interview process that may showcase evidence of PSTs' SCK and PCK from the theoretical framework (see Table 1). Then, each researcher reviewed their own design using a self-evaluation model that identified the tasks they require in each phase of the interview, whether the design allows the teacher educator to strongly evaluate, partially evaluate or not evaluate PSTs SCK and PCK from Table 1 for that phase, as well as any factors that contributed to the intentionality of that design. This analysis will first present potential evidence of SCK and PCK at each phase from our analysis in Table 1 and then present findings from the diverse MTE interview designs and intentionality behind those designs.

It is important to note that the potential tasks and knowledge that can be evaluated in this table is a compilation of information from all university interviews; there was not a single interview that encompassed every aspect of the table. The knowledge that can be evaluated is based on the design of the individual interview and the purpose behind the interview design

Table 1: Diagnostic Clinical Interview Phases and Knowledge Types

Interview Phase	Specialized Math Knowledge	Pedagogical Knowledge
Preparing Interview -choosing tasks -solving tasks/answer key -narrating background -establishing trajectory -adapting question	-open/closed quality of math tasks -accuracy/strategies for solving the task/problem - mathematical decomposition of the learning goal	-trajectory/ standard alignment -appropriate for student/goal -quality of task - common misconceptions -reasoning/problem solving - purposeful question design
Implementing the Interview -asking questions -adapting questions -requiring representations -taking notes/ transcript	-adaptability for follow-up questions -cognitive demand -accuracy of questions and responses to students -teacher noticing: attend to math	-adaptability -pose questions -productive struggle -representations -teacher noticing: attending to student thinking
Evaluating Student Thinking -concepts/procedures -math accuracy/fluency -flexible strategy/represent -strengths vs areas need	-common vs specialized "acceptable" strategies. -noticing math strengths/weaknesses -difference between procedures and concepts	-use evidence of student thinking -teacher noticing: interpreting student thinking
Informing Instruction -identify objectives -develop individual plan -develop class lesson plan	-Math in new lesson/individualized plan is accurate (math aligns with next step)	-appropriate math progression -teacher noticing: deciding how to respond based on student thinking -differentiation
Self-Reflection -reflecting (Multiple Stages)	-reflection on personal math understanding	-reflection for personal professional growth

In order to more closely evaluate the purpose of the interview design and how that design influences what PST knowledge can and cannot be measured, we used a collective self-study. Each researcher reviewed their own design using a self-evaluation model that identified the tasks they require in each phase of the interview and whether the design allows the teacher educator to strongly evaluate, partially evaluate or not evaluate PSTs' knowledge from Table 1 for that phase. Researchers justified any rationale to support their design choice such as point in the program, content vs. methods course, or access to children. Finally, they identified any formal evidence collected to evaluate PST knowledge.

Results showed that the needs identified by the teacher educator for their PSTs influenced the design. Results also showed areas of self-evaluations indicating a strong evaluation of PSTs' knowledge or skills were directly tied to lines in the assignment grading rubrics. A few examples organized by interview phase are shared below.

Time, type of course, and PSTs' mathematical and pedagogical abilities played a part in preparation design. For example, one researcher noted in their self-evaluation that due to time constraints they did not feel PSTs in their math methods course were ready to design their own questions in the preparation phase; as a result, they provided appropriate questions to choose from. This led to no strong areas of math or pedagogical knowledge evaluation, and there were no items on the rubric relating to preparation of the interview. In a second example, a researcher noted in their self-evaluation that one goal of the interview in their content course was to consider a variety of student strategies to solve math problems; as a result, they provided specific questions that allowed for a variety of strategies. Although nothing in the preparation phase was evaluated on the rubric, a major component of the rubric was "rich descriptions of how the student solved the problem" which ties directly into the purpose of the preparation design. In a final example, a researcher noted that their partner schools were concerned about PSTs' ability to align math content to the standards and solve the mathematics themselves; as a result, they required students to design three questions that align to a standard and solve the problems. The rubric for this design has points for alignment to the standard as well as accurate solution strategy and representation.

Two key aspects of implementation design focused on teacher noticing of student thinking and adaptability based on student responses. All self-evaluations indicated a strong ability to evaluate teacher noticing and ability to evaluate accuracy of student responses or strategies. Although many of the interviews required a copy of notes or a transcript, rubric items to evaluate these two areas of implementation were done concurrently with phase 3 (evaluation of student thinking). Adaptability was another common area with high evaluative potential (partially or strongly). A few interviews with strong potential for adaptability required PSTs to preplan potential misconceptions and follow-up questions (scaffolds or higher-order) based on student responses to help facilitate adaptations and measure adaptability.

Across all self-evaluations, results showed the evaluation phase is the keystone of the interview design and purpose. Researchers indicated the ability of their design to help PSTs notice and interpret student strength in accuracy, fluency or flexibility of mathematics. Additionally, PSTs were required to use evidence from their interview to support their interpretations across all interview designs. This did look slightly different across rubrics for grade alignment. For example: Rubric 1: "The profile contains rich descriptions of how the student solved the problem, with ample relevant and revealing evidence of their thinking. Key mathematical details from the student's thinking are attended to and interpreted in a way that creates reasonable models of the student's understanding." Rubric 2: "Describe the student's specific math strengths and weaknesses relating to accuracy, fluency, and flexibility. Support with evidence from notes."

The key aspects for design when planning next steps relied on when and where the interview was taking place as well as the similarity in interview topics. For example, one researcher who had all the PSTs do the same interview with students of the same grade level, had groups of PSTs co-create a lesson plan based on evidence from the interviews. The purpose was on using information about multiple students to plan for instruction. In another interview where PSTs' interviews covered a variety of math topics and grade levels, the next steps focused on the individual child's needs and scaffolds.

As this analysis has shown there is diversity in the intentionality of interview designs across institutions. There are many factors that can influence design such as timing, PSTs' knowledge, type of math preparation course, access to children, or other needs of the community.

Implications for Practice

In conclusion, this study supports our collective development and refinement of elementary math methods course assignments that seek to integrate content and pedagogy. Working across institutions will allow us to develop adaptable assignments that can be used in diverse contexts, and will support our collective efforts to improve elementary math teacher preparation.

As shown in this theoretical analysis, there is not one correct way of designing a diagnostic clinical interview. However, the design should be intentional for preparing, implementing, and evaluating the interview to meet the needs of PSTs at that institution. As societal shifts change the needs of PSTs entering the teaching profession, this analysis presents one way MTEs can review and analyze the intentionality of an interview design or rubric in their mathematics preparation courses or analyze a public interview design they may want to adopt for use to ensure it meets the needs of their unique population.

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EXPLORING PRESERVICE TEACHERS' UNDERSTANDING OF FRACTIONS FROM A COMMOGNITIVE LENS

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Fraction is one of the most challenging mathematical topics to learn and teach. Teachers often lack a profound understanding of fractions, which may lead to students working with fractions procedurally. In this paper, by drawing upon the commognitive framework, we offer an in-depth examination of two preservice elementary teachers' (PSTs') fraction understanding. We show how two PSTs who solve the same comparing fraction task, follow similar steps, and get the same correct answer, participate differently in the discourse about fractions. This different participation was identified by the extent to which the PSTs individualized the standard routine of comparing fractions, which we conceptualized in this paper. We discuss the affordances of the commognitive discursive lens on PSTs' understanding of fractions and highlight the study's contribution to teacher educators and teacher preparation programs.

Keywords: Preservice Teacher Education, Teacher Educators, Classroom Discourse, Learning Theory, Rational Numbers.

Fractions are widely recognized as one of the most challenging mathematical topics for both learning and teaching across K-12 education (Liu & Jacobson, 2022; OECD, 2014; Siegle, 2017). Despite the importance of teachers supporting students in developing a deep understanding of fractions and providing opportunities to explore underlying concepts, studies reveal that many teachers, especially prospective teachers (PSTs), often lack this foundational knowledge (Olanoff et al., 2014). While PSTs can often correctly solve fraction problems using established procedures, they frequently overlook the conceptual meanings and mathematical connections behind these procedures. This gap significantly limits their ability to effectively teach fractions in a way that fosters true understanding among all students in the future.

To address this challenge and enhance PSTs' proficiency in teaching fractions for understanding, the first author conducted a research project aimed at improving PSTs' teaching knowledge toward teaching fractions for understanding (Liu, 2021). This current study serves as a follow-up, focusing on a subset of data from the larger project and conducting a detailed analysis using a commognitive theoretical lens. Through this focused approach, we aim to uncover fresh insights into how PSTs conceptualize and understand fractions.

Specifically, our study examines how two PSTs individualize a standard routine for comparing fractions, known as the common denominator strategy. Our research questions guide this exploration: How do these PSTs individualize the standard routine for comparing fractions? Additionally, what are the differences in their de-ritualization processes?

Theoretical background

A Commognitive Theoretical Perspective

From a commognitive theoretical perspective, mathematical thinking is viewed as a discourse, a form of communication (Sfard, 2008). Learning within this framework is seen as a transformation in discourse—specifically in how learners talk about and interact with mathematical objects. According to commognition, any discourse can be characterized by four key elements: the use of specialized terms, visual tools, endorsed narratives, and discursive routines—patterns of actions performed in discourse (Steiner, 2018).

The evolution of learners' routines can be understood as a process of discourse development, where learners personalize established routines (Lavie et al., 2019). Initially, they *repeat* shared routines in society. Over time, they *individualize* the shared routines– “into independent, agentive implementers of those routines"(Lavie et al., 2019, p. 156). This gradual shift from ritual to explorative engagement is termed de-ritualization (Lavie et al., 2019).

Lavie et al. (2019) identified six key desirable characteristics of routine: flexibility, bondedness, applicability, performer's agency, objectification, and substantiation. They distinguish between ritual and explorative actions for each characteristic to describe learners' participation in discourse. This study specifically investigates how PSTs demonstrate bondedness, substantiation, and objectification within their mathematical discourse.

Bondedness refers to the interconnectedness of steps where the output of one step feeds into the next (explorative). When a performer precisely follows an expert's procedure but lacks awareness of these step relations, it results in disjointed steps (ritual).

Substantiation involves justifying or explaining a routine through reasoning. Ritual performances often focus on detailing the actual actions taken (object-level) for substantiation. In contrast, explorative performers base their justification on the meta-rules accepted by the community (meta-level).

Objectification encompasses reification, alienation, and saming. Reification occurs when participants shift from discussing processes to nouns (e.g., from "multiply 3 and 8 to get 24" to "24 is the common denominator"). Alienation involves excluding human agents from the discourse (e.g., from "I multiply 3 by 4 to get 12" to "12 is the product of 3 and 4"). Saming refers to treating seemingly different concepts as identical (e.g., "2/3 could be rewritten as 16/24"). More explorative objectification is indicated by increased reification, alienation, and saming in discourse.

A Standard Common Denominator Routine

In this study, we define and standardize a commonly used strategy for comparing fractions, known as the common denominator strategy, referred to as the CD routine. The CD routine addresses a mathematical challenge: the inability to directly compare or order fractions with unlike denominators and unlike numerators (Fractions_udun) in their symbolic form because a fraction's value depends on both its numerator and denominator. Thus, by applying the CD routine, we can standardize one variable (the denominator) by converting Fractions_udun to their equivalent forms with a common denominator (Fractions_cd) and then order the Fractions_cd by comparing their numerators.

This CD routine represents the culmination of historical development and expert consensus in fraction comparison strategies, typically codified in textbooks, curriculum standards, and educational guidelines. It comprises three essential subroutines: finding a common denominator, converting fractions to their equivalents using this common denominator, and listing them in order (referred to as finding, converting, and listing).

The first subroutine is finding a common denominator among the given fractions' denominators. A common denominator is a common multiple of these denominators, and while numerous numbers can serve as a common denominator, the least common multiple is often chosen for efficiency. Methods such as factor trees and listing multiples are commonly used to determine this common denominator.

Once the common denominator is established, the following subroutine is to convert each fraction to an equivalent form using this common denominator. This conversion follows a meta-rule: multiplying both the numerator and denominator by a nonzero number preserves the fraction's original proportion. This principle stems from the mathematical property that multiplying any number by one leaves its value unchanged.

By executing these subroutines—finding the common denominator and converting fractions—we transition from comparing Fractions_udun to comparing Fractions_cd. The third subroutine involves listing fractions in order, where the order of Fractions_cd is established by sorting their numerators. This is based on the understanding that a fraction a/b can be interpreted as the product of its numerator a and the reciprocal of its denominator (a * 1/b).

By the above conceptualizations, we could then examine how PSTs individualize the CD routine for comparing fractions, suggesting a better understanding of how they participate in the fraction discourse.

Methods

The setting, participants, and data source

This study draws data from a larger project aimed at enhancing PSTs' proficiency in fractions through an intervention. The focus of this study is on two participants: Hannah, a junior who completed required mathematics content courses on numbers and operations, as well as geometry; and Oprah, a senior who completed the same content courses along with a mathematics method course. Hannah and Oprah were chosen based on their positions within the teacher preparation program—Oprah being two years senior than Hannah. This educational difference potentially provided Oprah with more opportunities to develop a more explorative discourse, which in turn could increase variations in how they personalized and adapted the CD routine for fraction comparison. This selection is aligned with the purpose of this study.

As part of the intervention, PSTs were assigned the task of ordering fractions (2/3, 3/4, and 3/8) using various strategies, documenting their thought processes through written and verbal explanations (similar to a think-aloud activity) both before and after the intervention (pre- and post-test). The PSTs' responses, including their use of the CD routine, were recorded on video and transcribed for detailed analysis. This study specifically aims to explore how PSTs individualize the CD routine differently. Therefore, our focus is solely on analyzing the initial data from Hannah and Oprah: their respective pre-test excerpts.

Data analysis

We started the data analysis by consolidating each participant's written responses and verbal transcripts from the pre-test excerpts into cohesive documents. When verbal communication was ambiguous, we referred to video recordings to capture non-verbal cues. Each excerpt was systematically divided into lines based on the PSTs' speech patterns and narratives. Through regular meetings and discussions, we identified the specific subroutines within the CD routine that each PST utilized—namely, finding the common denominator, converting fractions, and listing them in order. To answer the research question, our analysis focused on pinpointing the characteristics of these subroutines, particularly examining bondedness, substantiation, and objectification. Subsequently, employing a comparative perspective, we assessed to what degree the characteristics of their routines leaned towards ritual or explorative practices. Since we view learning as a progression from ritual to explorative, our approach focuses on comparatively situating each PST's discourse along this continuum rather than categorically labeling any PST's discourse as strictly ritual or explorative.

Findings

As depicted in Figure 1, both PSTs compared 2/3, 3/4, and 3/8 using the three subroutines of the CD routine outlined earlier. Initially, they identified the common denominator 24 for the fractions 2/3, 3/4, and 3/8 (subroutine 1). Subsequently, they converted each fraction to its equivalent form (e.g., 2/3 to 16/24) (subroutine 2) and then established the order of the fractions (e.g., stating 2/3 as the second largest) (subroutine 3).

Our analysis revealed that while both PSTs followed these subroutines and achieved consistent outcomes, their discourse surrounding the process differed, indicating variations in their individualization processes and suggesting different levels of engagement in the discourse on fractions. In the following sections, we provide a detailed description of how each PST performed the three subroutines. Additionally, we delve into how they personalized each subroutine by examining variations in three key characteristics: bondedness, substantiation, and objectification.

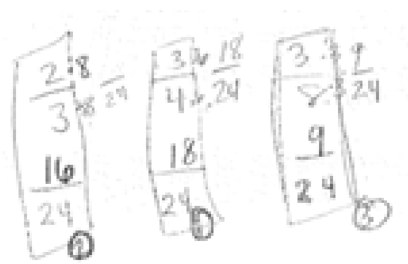
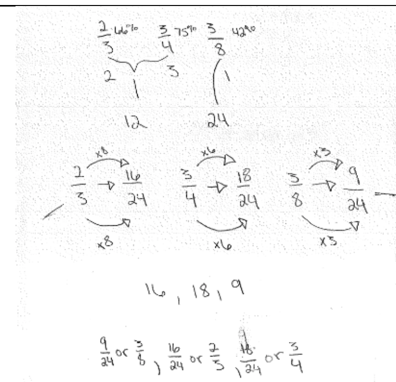
	Oprah	**Hannah**
Written response		
Finding	“And a multiple of 3, 4, and 8 is 24.”	"So, instead, I'm going to use 24"
Converting	e.g., “So, a new fraction is 16/24.”	“So, 2 times 8 becomes 16, and 24.”
Listing	"Our second largest is that (circled 2/3) because then we have 16 over twenty-fourths."	“Then the next lowest, the middle one, is going to be the 16/24.”

Figure 1. PSTs’ Written Responses and Indication for The Performed Routines

Subroutine 1: Finding the common denominator 24

Oprah: Oprah initiated the task by clearly connecting the subroutine of finding the common denominator with the broader goal of ordering fractions. She expressed, "You can solve it (the given task) by getting common denominators for all of them," indicating a direct alignment with the macro task. Oprah further detailed her approach by considering 24 as a common multiple of 3, 4, and 8. She justified her choice of 24 by explaining that it is divisible by each of these denominators, noting that they are factors of 24 ("a multiple of 3, 4, and 8 is 24, 24 is divisible by all of these numbers.").

Hannah: Hannah's initiation to the task was more procedural-focused. She stated her intent to "find a common denominator" without explicitly linking it to the larger task of comparing fractions. She then elaborated on the steps she took to arrive at 24, mentioning specific calculations ("between 3 and 4…is 12 to me… I also have to include 8… So, instead, I'm going to use 24").

The Characteristics of PSTs' Individualization of Subroutine 1

The descriptions above show that both PSTs identified the common denominator 24, yet they differed in their approach to bondedness, substantiation, and objectification. Regarding bondedness, which pertains to the awareness of connections between steps, Oprah integrated subroutine 1 of finding the CD with the overarching goal of comparing fractions. In contrast, Hannah focused solely on the procedural steps for finding the CD, highlighting a less integrated approach compared to Oprah's more explorative bondedness.

In terms of substantiation, which concerns how the procedure was justified, Oprah supported why 24 was chosen as the common denominator using meta-rules endorsed by the mathematics community. These included the concept of a common denominator as a multiple of denominators and its divisibility by each individual denominator. In contrast, Hannah detailed the calculations to explain why 24 was selected as the CD, focusing more on the process than the conceptual rationale behind it. Thus, Oprah's approach demonstrated more explorative substantiation compared to Hannah's.

Regarding objectification, Hannah's discourse involves more references to herself performing the procedures to find the common denominator. She uses phrases like "I'm going to use..." or "I'm going to do..." which place greater emphasis on her actions and decision-making processes rather than on the properties of the resulting object (24). In contrast, Oprah's discourse treated 24 as an abstract concept with inherent properties, discussing its divisibility without explicitly tying it to the procedural steps undertaken. This distinction suggests that Oprah's discourse was more objectified, focusing on the abstract properties of 24 itself, whereas Hannah's discourse was more human-centered and process-oriented, reflecting a more ritualistic approach.

Subroutine 2: Converting to equivalent fractions

Similar to subroutine 1 of finding the common denominator, subroutine 2 also concluded with both PSTs achieving the same outcome – three fractions with a common denominator of 24 that are equivalent to the three given fractions, respectively (16/24, 18/24, 9/24. See the written responses row in Figure 1). However, upon examining their discourse comprehensively, several differences in their discourse of this subroutine become evident.

Oprah: Oprah began by meta-discursively describing the actions required to maintain equivalence between two fractions. She stated: "What you have to do to the bottom, you have to do to the top." She applied this meta-rule to convert 2/3 to 16/24 and justified this approach by aiming "to make it proportionate." Oprah consistently followed this procedure of multiplying both the numerator and denominator by the same number for the other two fractions, resulting in three equivalent fractions.

Hannah: Hannah initiated this subroutine with a clear arithmetic objective – "to make all these [three fractions] out of 24." She then detailed the operational procedures while working with the fractions. Starting with 2/3, she first aimed "to get to 24 from 3." Hannah explained that achieving 24 from 3 required multiplying 3 by 8. After identifying this factor of 8, she proceeded to multiply both the numerator and the denominator of 2/3 by 8, resulting in 16/24 ("So I'm going to do the same thing on the top. So, 2 times 8 becomes 16."). Hannah followed a similar process for 3/4 and 3/8, obtaining 18/24 and 9/24, respectively.

The Characteristics of PSTs' Individualization of Subroutine 2

Similar to the first subroutine, Oprah's approach in subroutine 2 demonstrates both bondedness and substantiation. When revealing the equivalent fractions, Oprah seamlessly integrated the output of each procedural step into the next, such as increasing the denominator eight times, followed by increasing the numerator eight times to convert 2/3 into 16/24. Such kinds of interconnectedness reflect an explorative bondedness. Moreover, Oprah substantiated her actions by first establishing meta-rules ("what you have to do to the bottom, you have to do to the top") and then illustrating this rule through specific examples, like converting 2/3 to 16/24. She articulated this as creating "a six more relationship," emphasizing the systematic approach of multiplying both the numerator and denominator by the same number to maintain proportionality, which adds to the explorative substantiation in her discourse.

In contrast, Hannah's approach to subroutine 2 is characterized more by ritual substantiation. While she also applied the meta-rule of "doing the same thing on the top and the bottom," Hannah primarily focused on detailing the procedural steps without justifying them with broader meta-rules or conceptual explanations. For instance, she methodically described steps like setting 24 as the new denominator, dividing this new denominator by the old denominator (24 ÷ 3 = 8), multiplying this outcome (8) by the old numerator (2), and obtaining the new numerator (16). However, Hannah did not delve into the rationale behind these steps or the mathematical principles supporting her actions, illustrating a ritualistic approach to substantiation.

Regarding bondedness, Hannah connected her steps using transitional phrases like "So," indicating her awareness of the sequential relationship between actions. However, her discourse was characterized by a loose structure where individual steps appeared somewhat disconnected, lacking the cohesive flow observed in Oprah's approach. This mixture of goal-oriented phrases and rigidly performed steps suggests a relatively more ritual bondedness in Hannah's routine.

In terms of objectification, both Oprah's and Hannah's discourse employed human actors and emphasized the process (such as "You have to …" and "you would" in Oprah, and "I am going to …" and "You have to…" in Hannah), indicating ritual objectification. However, Oprah's discourse is a slightly more objectified exploratively regarding how she communicated about the number used to multiply. Oprah named it as "times of relationship" (e.g., eight times more relationship), while Hannah called it as a number itself.

In summary, while both PSTs achieved the same outcomes in subroutine 2, their discursive approaches differed significantly in terms of bondedness and substantiation and slightly different in terms of objectification. Oprah's discourse exhibited explorative bondedness and substantiation by integrating procedural steps with meta-rules and conceptual reasoning, whereas Hannah's discourse leaned towards ritualistic substantiation and bondedness, emphasizing detailed processes without the same depth of conceptual justification.

Subroutine 3: Listing the fractions in order As a result of the preceding routines, both PSTs obtained the same converted fractions: 16/24, 18/24, and 9/24. These fractions served as the starting point for the third subroutine.

Oprah: Oprah structured her approach by ranking the fractions based on the quantity of "twenty-fourths" each possessed. She asserted that 18/24 was the largest because it represented "eighteen of the twenty-fourths," identifying it as equivalent to three-fourths. She then identified 2/3 and 16/24 as the second largest by noting they both represented "sixteen over twenty-fourths." Finally, she designated 3/8 and 9/24 as the smallest due to having "the least amount of twenty-fourths."

Hannah: Hannah's method began with a focus on comparing the numerators directly. She determined that 3/8 was the smallest fraction by noting, "So we have 16, 18, and 9... So, the smallest one is going to be 9 out of 24, or the original fraction we were given is three-eighths." Hannah repeated this process with the other two fractions and got the correct fraction order.

The Characteristics of PSTs' Individualization of Subroutine 3

Both Oprah and Hannah demonstrated strong bondedness in their discourse by explaining how they determined the order of the fractions based on the results of subroutine 2, which provided fractions with a common denominator. Hannah emphasized the process by stating, "We can just compare the numerators," while Oprah provided a more detailed explanation. Oprah substantiated her process by describing how each fraction's numerator related to the total twenty-fourths, such as 18/24 having "eighteen of the twenty-fourths" and 9/24 having "nine of the twenty-fourths," thereby facilitating their comparison based on their numerators. This detailed substantiation made Oprah's discourse more explicit than Hannah's, which primarily focused on the method of comparing numerators, reflecting a higher level of explorative bondedness and substantiation in Oprah’s discourse.

In terms of objectification, both Oprah's and Hannah's discourse employed human actors and emphasized the process (such as "I have …" and "our second largest" in Oprah, and "so we have…" and "We were given…" in Hannah), indicating ritual objectification. However, Oprah's discourse demonstrated a relatively more objectified approach in how she communicated about fractions compared to Hannah. For instance, Hannah primarily referred to the numerator as an object ("we can just compare the numerators"), whereas Oprah discussed the fractions themselves as objects ("amount of twenty-fourth") and contextualized the numerator within the fractions (e.g., "our second largest is that [circled 16/24] because then we have sixteen over twenty-fourths". Meanwhile, while both Oprah and Hannah samed equivalent fractions, Oprah’s saming was much more straightforward (Oprah: "eighteen of the twenty-fourth…which was three-fourths" vs. Hannah: "nine out of twenty-four. Or the original fraction we were given is three-eights).

In summary, while both PSTs successfully completed the third subroutine with consistent outcomes, their discursive approaches varied in terms of bondedness, substantiation, and objectification. Oprah's discourse demonstrated deeper substantiation, stronger bondedness, and more objectified language compared to Hannah's, which was more process-oriented and human-centered.

Summary and Discussion

Based on the examination of each PST's individualization of the CD routine, with particular attention to routine characteristics, we present two different prototypes for individualizing the CD routine, which also reflect varied participation in the discourse on fractions. These cases are intriguing because despite following the standard routine, both PSTs exhibit notable similarities and differences.

Both Hannah and Oprah correctly identified the common denominator of 24, converted the fractions accordingly, and ordered them, adhering to the three subroutines typical in such comparisons. As Lavie et al. (2019, p. 156) observe, "Since the source of an individual's routines is in what other, more experienced performers are doing, we all end up acting in similar, compatible ways." Hence, it's expected that Hannah and Oprah follow the routine similarly.

However, upon closer examination of their discourse characteristics, we observed that Hannah and Oprah individualized the CD routine of comparing fractions in distinct ways, indicating varying degrees of de-ritualization from ritual to explorative characteristics. Generally speaking, Oprah's engagement in the discourse exhibits more explorative elements compared to Hannah's, particularly evident in how they approached substantiation, objectification, and bondedness. This study demonstrates that beyond mere correctness, there exists ample opportunity for further exploration and improvement (Liu & Zhuang, 2013). This study also expands upon our prior understanding of PSTs' fractional knowledge (Olanoff et al., 2014) and offers valuable insights into improving PSTs' proficiency with fractions.

The commognitive theory, which offers a unique perspective on learners' discourse as a reflection of their mathematical understanding, provided insights into the PSTs' potential learning processes and their individualization of standard routines. In the realm of teacher education, this analysis and its insights into how PSTs engage differently in the discourse about fractions can assist teacher educators in identifying specific areas for improvement. For instance, there could be a focus on supporting Hannah in justifying her procedures using meta-rules.

Our findings underscore the value of analyzing PSTs' discourse through a commognitive lens, offering a practical framework for examining how they individualize common mathematical routines. This approach provides an operationalized tool for understanding their learning and growth in mathematical pedagogy.

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MANIFESTATIONS OF MATH ANXIETY IN MIDDLE AND HIGH SCHOOL MATH TEACHERS

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Math anxiety can affect students and teachers in many ways, including student performance and instructional practices. Math anxiety has been well documented in elementary teachers, but little research has occurred on math anxiety at the middle and secondary levels. This study analyzed the ways in which math anxiety can manifest in middle and high school math teachers. Using phenomenological analysis methods on interviews with 11 teachers, findings from this study revealed that middle and high school math teachers with math anxiety see their anxiety manifest in three different ways: (1) in specific mathematical subjects; (2) when doing math in front of others; and (3) as a fear of new or unknown mathematics. These results demonstrate

Keywords: Affect, emotion, beliefs, and attitudes; teacher beliefs

Math anxiety, which researchers have long defined as feelings of fear or discomfort when dealing with mathematics (Ashcraft, 2002; Maloney & Beilock, 2012), can greatly impact student learning. This is true whether it is the student who is anxious (e.g., Sorvo et al., 2017) or the teacher (Beilock et al., 2010). Studies have shown that elementary teachers commonly have math anxiety (Hembree, 1990) and that it can impact their teaching practices (Hadley & Dorward, 2011). Research has also shown a negative correlation between math anxiety and attitudes about mathematics in elementary teachers (Çatlıoğlu et al., 2014; Jackson, 2015). However, little is known about math anxiety in middle and high school mathematics teachers.

A recent doctoral study (Mannix, 2022) demonstrated the presence of math anxiety in middle and high school math teachers. This study further uncovered a curious phenomenon in that secondary math teachers (which, for the purposes of this study, those teaching middle or high school) do not seem to experience math anxiety in the same ways as students or elementary teachers. Specifically, SMTs do not seem to feel math anxiety as a fear of mathematics itself. In fact, many of the participants from Mannix's (2022) study described pleasant, even joyful, feelings toward mathematics. This curious finding begs the question, in what ways does math anxiety manifest in middle and high school math teachers? Note, this brief report will examine this question as it manifests generally, not specifically in their teaching practices.

Methods

This study is part of a larger research project on the experiences of secondary math teachers with math anxiety. The larger study consisted of a survey that used convenience sampling to recruit participants with experience teaching at the middle and high school level, defined as grades five through 12. In this survey, participants responded to questions about their teaching experiences and their anxieties about math and teaching math. The current study focused on a set of interviews with 11 middle and high school math teachers who identified themselves as having some amount of math anxiety. Some information about each of the participants can be found in

Table 1. Transcripts from the interviews were analyzed using descriptive phenomenology, as outlined by Giorgi (Vagle, 2018).

Table 1: Participant Information

Pseudonym	Years Teaching	Degrees Held	Teaching Experience
Autumn	34	BA Math Ed; MS Math	High School
Brittany	7	BA Psychology; MA Math Ed	Elementary and Middle School
Charlie	17	BA English; MA Education	Elementary and Middle School
Daisy	33	BS Education; MA Education	Middle School
Georgia	16	BA Elementary Ed (Math Add-on); MA Education (Math Add-on)	Middle and High School
Hannah	2	BS Math; MS Math	High School and College
Michelle	11	BA Math Ed; MA Math; PhD Math Ed	High School and College
Oliver	5	BS Math	High School
Rachel	7	AA English; BA Math Ed; MS Math; MA Education	High School
Sarah	30	BA Math Ed; MA Education	High School
Whitney	4	BS Math; MA Math Ed	High School

Findings

The teachers in this study shared many interesting experiences and observations about their math anxieties. Notably, it seems that their math anxiety manifests in three different ways. For some, their math anxiety manifests within or around a particular subject, like geometry or statistics. For others, their anxiety manifests in front of other people. Finally, several of the teachers explained that their math anxiety manifests when confronted with new or unknown mathematics.

Math Anxiety in Particular Subjects

Each of the teachers in this study cited an anxiety with a particular branch of mathematics or a specific math course. For many, those anxieties stemmed from negative experiences in math classes at one level or another. For example, Georgia enjoyed math until she got to geometry, which she said was "frustrating in a bad way and made math lose its excitement" in her eyes. Similarly, Charlie shared an experience that would make anyone turn away from mathematics. He explained that, when he struggled in his high school AP calculus course, his teacher told him that he "may have hit his [mathematical] limit." This insidious comment made Charlie feel as though he had gone as high as he could with mathematics and that he wouldn't be able to understand the material in any further math courses. While he wouldn't consider it a traumatic experience, he does feel that it altered his trajectory and stunted his mathematical journey, causing him to refuse to take math courses in his undergraduate career.

Several of the teachers cited an anxiety with statistics. Specifically, Autumn said she doesn't like statistics because "it is often about making decisions that don't always make sense and may not seem logical," though she admitted that she didn't take a statistics course until graduate school, which might explain her anxieties around and biases against the subject. Rachel, on the other hand, did not care for the overlap between statistics and probability and found the multiple types of probability to be very confusing, causing her to feel anxious around any sort of statistical topic.

Brittany and Hannah shared an aversion to higher-level mathematics. Brittany explained that she was most anxious with courses like linear algebra and calculus because she feels she "lacks a foundation in those areas." Hannah, on the other hand, pursued a PhD in mathematics, but she had to leave the program after failing one of her qualifying examinations. "I had all of these dreams revolving around math, and I felt like they all shattered. So my relationship with math is definitely different." Hannah went on to say that she enjoys teaching math at the high school level, but she doesn't think she'll ever revisit the doctoral level mathematics that she had been doing.

One teacher, Whitney, said she felt most anxious not in a particular course but with a particular type of problem. For Whitney, multiple choice math problems cause her the most anxiety, particularly those with an answer choice of "answer not here." These problems, she says, cause her to second guess herself, so she prefers problems that are open ended as they allow her to share her thinking.

Math Anixety in Front of Others

For six of the participants in this study, their math anxiety affected them greatly when they were tasked with doing mathematics in front of people. Charlie and Georgia both felt anxious in front of their peers who taught higher-level math classes than they do or who have been teaching the courses for much longer than they have. Charlie shared that he "feels like he doesn't know what he is doing" when in these situtations, while Georgia said she knows she's smart enough to do and teach the math that her colleagues can do and teach, she just "doesn't have the knowledge right now," in the moment like they do, which makes her feel anxious and lesser than her peers.

Similarly, Daisy said she feels most anxious when being observed or when she knows someone might look over her work.

Oliver said he felt anxious when students would ask him for help with advanced math problems, like those they faced in math competitions. These problems are generally geared toward problem solving and tend to be critical-thinking heavy, and this teacher explained, "If I feel in the moment that I can't help with that question, I'm a little bit anxious…I don't feel as confident on those [types of problems]." Similarly, Whitney said she used to get anxious when students would ask about why certain topics or structures worked the way they do, but she said this has gone away with time and practice. For Michelle, that anxiety has not gone away. She said she can feel anxious when standing at the board in front of students, "even if I'm prepared or over prepared, I get that anxious feeling… I think that comes from the fact that I really do strive to let the students create discourse and conversation." She went on to explain that it was letting go of the control in a mathematical space, where there are so many different solution strategies, that made her anxious. "It's not just black and white, like everybody thinks." This ambiguity, mixed with having to make judgment calls on whether or not a specific strategy will always work, still cause her to feel anxious when in front of students.

Math Anxiety with New or Unknown Mathematics

Two of the teachers, Georgia and Autumn, emphasized moments when their math anxiety manifests as a fear of the unknown. Material they are unfamiliar with (as a teacher or student), in topics ranging from geometry to statistics to calculus and beyond, can cause them to feel anxious. Specifically, Georgia said she was most anxious when "performing math [she] doesn't know yet," and Autumn explained that she was most anxious when she had to tackle a problem with which she was unfamiliar or one that required her to combine skills and content areas in new ways. Similarly, any topic that requires her to memorize things can cause her anxiety as she struggles with memorization. "I like math because I am a problem solver. I'm not a good memorizer." It's the problem solving aspect of mathematics that she likes and is largely why she decided to be a math teacher. In general, though, the teachers said they feel least anxious with math they have taught previously, since they have had time to get used to it and figure out the ins and outs of the material.

Conclusion

While there has been a great deal of research in math anxiety throughout the years, only recently have researchers turned their attentions to math anxiety in middle and high school math teachers. This study examined how math anxiety manifests in 11 teachers of middle and high school mathematics and found that there were three primary manifestations: (1) in particular subjects; (2) in front of others; and (3) in new or unknown mathematics. These findings indicate a very real presence of math anxiety in secondary teachers and suggest that additional research is needed to further explore this phenomenon. Are there other ways in which math anxiety can manifest for secondary teachers? What effects, if any, does math anxiety have on math teachers' instructional strategies or pedagogical beliefs and attitudes? What impact does a secondary teachers' math anxiety have on their students' learning? Additionally, what supports would help these teachers overcome their math anxiety? These are all questions that should be addressed by future research.

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COMPARING PRE-SERVICE ELEMENTARY TEACHER'S BELIEFS ABOUT OPERATIONS WITH THEIR FACILITY FOR SOLVING WORD PROBLEMS

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Keywords: Teacher Beliefs; Teacher Knowledge

Research has shown that typical undergraduates have difficulty solving compare word problems (Heagarty, Mayer, & Green, 1992; Haegarty, Mayer, & Green, 1995) in which a central term such as "more" is indicated but the problem is most easily solved with its inverse operation, such as subtraction. This study examined 1) if pre-service teachers also make these errors, 2) if there are differences in these errors before and after receiving instruction and 3) if these errors are related to beliefs preservice teachers hold about operations (addition, subtraction, multiplication and division) and 4) if these beliefs are influenced by their preservice training.

Method

Ten participants (one male and nine female) were given pre and post instruction think aloud problem-solving interviews and 22 item surveys asking how much they agreed or disagreed with true and false statements about operations such as "if you the see the word 'more' then that means you should always use addition." The identical survey items were repeated pre and post while the interview problems were each given a different cover story to make them seem less familiar but retain the same mathematical relations. The interviews were retrospective think aloud interviews in which participants reasoned with problems ranging from start unknown, change unknown, result unknown or compare word problems and explained their reasoning upon completion (Van den Haak, de Jong, & Schellens, 2003).

Results

Results indicate that participants were successful both pre (93.75%) and post (85.25%) instruction solving the range of word problems but scored slightly lower on the post after seeming to become more complacent when a compare problem came directly after a typical result unknown item.

Despite being able to solve these problems, survey data indicated that pre and post data demonstrated distributions of faulty beliefs about operations overall and keywords more specifically at the posttest. Qualitative comparison of the pre-and post-survey data suggests that pre-service teachers still hold patterns of misconceptions about operations that are influenced by instruction to shift but remain part of their mathematical knowledge. In many cases increased support for true statements would be offset by greater support for false ones and vice versa.

Discussion/Conclusion

Results indicate that although pre-service teachers can successfully manipulate word problems, they frequently still have misconceptions in their understandings for the underlying operations. These misconceptions may not simply increase or decrease but vary in their support for believing true or false statements about the operations making them complex and difficult to address.

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AN EXAMINATION OF IN-SERVICE TEACHERS' RESPONSES TO A FRACTION DIVISION PROBLEM

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The purpose of this study was to investigate 236 U.S. in-service elementary and middle grades teachers' responses to a measurement fraction division problem including their strategies and types of errors. Using an inductive content analysis approach, one main result was that relatively high number of teachers (218 out of 236 teachers; 92.4%) responded to the problem correctly. A second main result was that a greater percentage of teachers had the quantitative meanings of measurement division, indicating flexibility with referent units, in comparison to those teachers and preservice teachers in past research. Regarding informal strategies, teachers relied on the common denominator and decimal strategies; however, strategies such as repeated subtraction, unit rate, or dividing numerators and denominators were not present. We discuss implications for improving teachers' knowledge of fraction division.

Keywords: Mathematical Representations; Rational Numbers; Teacher Knowledge

Part of teaching students mathematical concepts entails the teacher knowing and understanding the concepts themselves (National Council of Teachers of Mathematics [NCTM], 2000). However, U.S. teachers have performed poorly on content-based assessments in comparison to their international counterparts from top performing countries (e.g., Center for Research in Mathematics & Science Education, 2010). Further, the concept of fractions has been identified as one of the "most cognitively challenging" topics in school mathematics for both teachers and their students (Lamon, 2007, p. 629).

According to recent curriculum standards such as the Common Core State Standards for Mathematics (CCSS-M; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), teachers and their students need to have both conceptual and procedural understandings of fractions. This includes knowing how to compute with fractions and explaining why computations work when fractions are situated within problem contexts. Much of the existing research on teachers' reasoning about fractions, however, has reported that in-service and pre-service teachers struggled with reasoning about fractions despite their facility with fraction computation based on memorized rules and procedures (Jansen & Hohensee, 2016; Ma, 1999). Additionally, in an extensive review of past research on preservice teachers' understanding of fractions by Olanoff and colleagues (2014), most preservice teachers were found to have a hard time reasoning about fractions in terms of quantities, indicating preservice teachers' limited conceptual understandings about fractions.

For improving teachers' conceptual understanding of fractions, studies show that understanding and reasoning about referent units is critical (Izsák, 2008; Izsák et al., 2019; Lee, 2017; Lee, Brown, & Orrill, 2011; Philipp & Hawthorne, 2015). As an example, Izsák et al. (2019) administered a novel fractions survey to a national sample of 990 U.S. in-service middle grades teachers and found that teachers' proficiency with referent units were linked to their performance with the remaining components of reasoning about fractions. On the other hand, Izsák and colleagues (2019) found that only 30% of the teachers in the national sample demonstrated proficiency in referent units. One key characteristic of referent units is flexibility with referent units, defined as "a teacher's ability to keep track of the unit to which a fraction refers … and to shift their relative understanding of the quantities as the referent unit changes" (Lee et al., 2011, p. 204). In the present study, we investigated U.S. teachers' flexibility with referent units in a fraction division situation, to better understand their strategies and types of errors.

Conceptual Framework

Starting in the 1980s, researchers attempted to identify what teachers need to know for examining mathematical knowledge of teachers. Based on this line of research, teachers need to have both content knowledge (CK) and knowledge unique to teaching, called as pedagogical content knowledge (PCK; Shulman, 1986). PCK includes understanding students' particular difficulties and facilities on a topic, identifying their potential misconceptions, and being familiar with particular representations that are helpful for students to learn specific topics. For example, PCK about fraction division implies knowing why the invert-and-multiply algorithm works, being aware of students' potential misconceptions about this algorithm (e.g., inverting the dividend instead of the divisor), and selecting specific visual models to uncover and support students' understanding (e.g., double number lines, area models). Ball and colleagues (2008) articulated and expanded the body of knowledge that Shulman (1986) introduced as *mathematical knowledge for teaching* (MKT).

By drawing from both frameworks, based on what teachers need to know for teaching mathematics to their students, we refer to this construct as *specialized content knowledge* that includes teachers' understanding of mathematics unique to teaching. In the context of the present study, the key characteristic of *specialized content knowledge* is flexibility with referent units. Because conceptual understanding is a main strand for proficiency in mathematics and the ability to represent problem situations is an important indicator of such understanding (Kilpatrick et al., 2001), using visual models such as number lines, length and area models are strongly recommended in fraction instruction to support students' learning (CCSS-M, 2010). On the other hand, most teachers use drawings for the purpose of illustrating final answers, instead of supporting early understanding (Izsák, 2008).

Referent units are units when numbers are embedded in problem situations, and they are necessary for conceptual understanding of fractions (Philipp & Hawthorne, 2015). For the problem "Julia's cat eats 1/3 cups of cat food each day. If she has 1/2 cups of cat food, for how many days can she feed her cat?", it is possible to obtain the answer by using the invert-and-multiply algorithm. On the other hand, a teacher who has conceptual understanding of fractions is expected to keep track of the referent unit and think accordingly as the referent unit changes (i.e., flexibility with referent units). In terms of the measurement meaning of division, the teacher may interpret this situation as how many groups of 1/3 are in 1/2. While the referent units for 1/2 and 1/3 are the same one whole, the referent unit for the quotient is 1/2 of the whole. Lee et al. (2011) documented the need for teachers to have flexibility with referent units to interpret drawings appropriately in a way that makes sense to them and their students. Thus, we assume that teachers' showing flexibility with referent units would support their students' conceptual understanding of fractions, especially in fraction multiplication and division problem situations.

Literature Review

Existing research on understanding of fraction multiplication and division have reported that teachers (e.g., Copur-Gencturk, 2021; Izsák, 2008; Izsák et al., 2019; Webel & DeLeeuw, 2016) and preservice teachers (e.g., Baek et al., 2017; Tobias, 2013) struggled with identifying the units appropriately. For example, Baek et al. (2017) examined 85 preservice elementary and middle grades teachers' understanding of referent units through their valid and invalid pictorial strategies to the Paycheck problem shown as follows:

> Emily receives her paycheck for the month. She spends 1/6 of it on food. She then spends 3/5 of what remains on her house payment. She spends 1/3 of what is then left for her other bills. Finally, she spends 1/4 of the remaining money for entertainment. This activity leaves her with $150, that she puts into savings. What was her original take-home pay? (p. 4)

Based on preservice teachers' responses to this problem, Baek and colleagues reported that most had trouble keeping track of the referent units in each step of the Paycheck problem. As an example for fraction division situations, Jansen and Hohensee (2016) examined how 17 preservice teachers' conceptions of partitive division with fractions were connected and flexible when solving a partitive division problem. As a result of interviews conducted with these preservice teachers, Jansen and Hohensee found that they demonstrated inflexible conceptions of partitive division with fractions. In a recent study, Copur-Gencturk (2021) examined 303 U.S. in-service elementary mathematics teachers' responses to a fraction division problem and found that only 14% of the teachers used the referent units correctly.

In addition to those difficulties, several studies have documented that in-service and preservice teachers had trouble showing flexibility with referent units in fraction multiplication and division situations (e.g., Baek et al., 2017; Copur-Gencturk & Olmez, 2022; Izsák, 2008; Lee, 2017; Lee et al., 2011; Webel & DeLeeuw, 2016). In one such study, Son and Lee (2016) analyzed 60 preservice primary and middle grades teachers' written responses to a fraction multiplication problem and found that while 40% were able to identify and draw the problem correctly, only 30% of preservice teachers treated the underlying concept of fractions as "finding portions of portions", an indication of flexibility with referent units, and others applied a standard algorithm without considering referent units. In another study, Lee and colleagues (2011) interviewed 12 in-service middle grades teachers and analyzed their responses to eight multiple-choice items that required using drawings. Lee and colleagues reported that teachers relied on referent units correctly in only 25% of the problems, indicating teachers' difficulty of making the referent units explicit. Specifically, in one item, Lee and colleagues asked teachers to identify the number line that shows 1/5 x 1/4 (Figure 1). While Figure 1a demonstrates 1/4 – 1/5 = 1/20 with each number's referring to the same referent unit, which was one whole, Figure 1b is the correct number line representing 1/5 x 1/4, where the referent unit for 1/5 is the 1/4 of the whole and the referent unit for 1/4 is the whole. Lee and colleagues reported that only four of the 12 teachers were able to identify the referent units correctly in this problem.

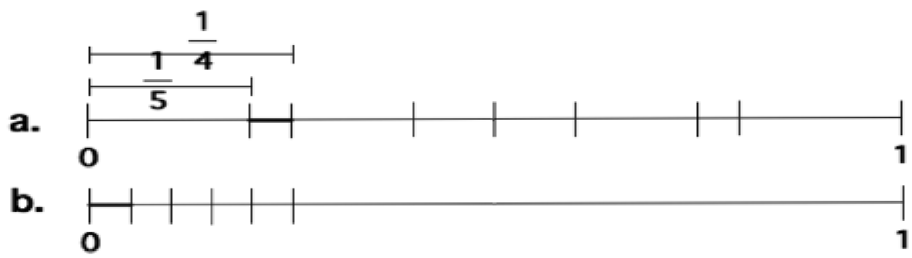

Figure 1: a) Number line for 1/4 – 1/5; b) Number line for 1/5 x 1/4 = 1/20 (Lee et al. 2011, p. 209)

In a recent study, Lee (2017) examined 111 U.S. preservice elementary teachers' written solutions to the stick problem using a length model in Figure 2. Lee found that while 52 preservice teachers (47%) gave a correct response to this problem, only 13 of those 52 preservice teachers (11.7% of the sample) reasoned through drawings, indicating flexibility with referent units. Eight preservice teachers relied on informal strategies of common denominator (or making equivalent fractions) and decimal strategies (Son & Crespo, 2009), whereas the remaining 31 preservice teachers gave a correct response using the invert and multiply strategy.

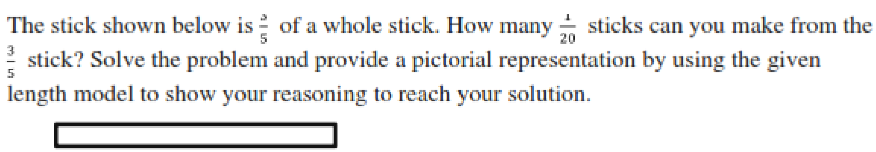
The stick shown below is $\frac{3}{5}$ of a whole stick. How many $\frac{1}{20}$ sticks can you make from the $\frac{3}{5}$ stick? Solve the problem and provide a pictorial representation by using the given length model to show your reasoning to reach your solution.

Figure 2: Stick problem (Lee, 2017, p. 335)

By building off the work of Lee (2017), the present study investigated in-service teachers' flexibility with referent units in the Stick problem and has two major contributions to the field. First, most studies that examined in-service and preservice teachers' flexibility with referent units were based on fraction multiplication situations rather than fraction division situations (e.g., Baek et al., 2017; Webel & DeLeeuw, 2016). Second, most studies examining in-service teachers' flexibility with referent units were conducted with either a small number of teachers (e.g., Izsák, 2008; Lee et al., 2011) or used an area model (e.g., Copur-Gencturk & Olmez, 2022). Given that Lee's (2017) study focused on preservice teachers who were taking the same methods course at the same institution, our study relied on 236 U.S. in-service elementary and middle grades teachers in the U.S with a wide range of backgrounds, giving us the opportunity to contribute a broader description of in-service teachers in terms of their strategies and types of errors.

The research questions guiding our study were as follows:

1. To what extent do teachers solve a measurement fraction division problem correctly?
2. What strategies and types of errors do teachers demonstrate in solving a measurement fraction division problem?

Methods

Participants

The sample of this study consisted of 236 in-service elementary and middle grades mathematics teachers in Grades 1–8 (i.e., children ages 6 to 14) from 48 states in the U.S. We collected data during Spring and Summer 2020 using an educational research company that provided contact information of teachers and partner professional development organizations. Teachers received an invitation email with an online screening survey that started with general questions about the teacher's educational background and teachers who were currently teaching mathematics in grades 1-8 were eligible to participate. In addition to providing information about their educational background, teachers answered the Stick problem by uploading their responses including drawings and descriptions. Participants were 80.5% female and 72% were White. The participants also had an average of 9.2 years of mathematics teaching experience, with a standard deviation of 8.04 years.

Fraction Division Problem and Coding of Teachers' Responses

We used the Stick problem to capture teachers' flexibility with referent units (see Figure 2). The stick problem was a measurement fraction division problem asking teachers to come up with a solution and a drawing that represents how they reached the solution. We coded teachers' responses to the Stick problem independently by following the categories in Lee's (2017) study and our agreement was over 90%. We discussed any disagreements and resolved them.

Specifically, based on Lee's (2017) coding scheme, we classified teachers' correct responses into three categories: (1) *computing based on algorithms without quantitative reasoning*, (2) *reasoning through computations not tied explicitly to quantities*, and (3) *reasoning with quantities*. For the category of *computing based on algorithms without quantitative reasoning*, teachers' solution was computationally correct, but did not include any reasoning about quantities. Teachers in this category used either a formal strategy (invert and multiply strategy) or cross-multiplication but indicated no flexibility with referent units. They also might have not produced any drawing, their drawings might have been incorrect, or their drawings might have only shown the final answer. For the category of *reasoning through computations not tied explicitly to quantities*, teachers' solution was based on computations not tied explicitly to quantities. Teachers in this category used an informal strategy (common denominator strategy; repeated subtraction strategy; decimal strategy; unit rate strategy; and strategy of dividing numerators and denominators; Son & Crespo, 2009), but indicated no flexibility with referent units. They also might have not produced any drawing, their drawings might have been incorrect, or their drawings might have only shown the final answer. For the category of *reasoning with quantities*, teachers' solution was based on reasoning with quantities. As opposed to previous categories, teachers in this category focused on drawings before using formal or informal strategies, identified referent units, and demonstrated flexibility with referent units. Their drawings also indicated appropriate use of mental operations such as partitioning, iterating, and disembedding. Specifically, teachers obtained three parts through partitioning, added two more fifths to make a whole stick (i.e., the referent unit), then obtained 20 pieces from the whole stick through partitioning and counted the number of twentieths corresponding to the 3/5 stick.

Regarding incorrect responses, based on Isiksal and Cakiroglu's (2011) coding scheme, we classified their incorrect responses into five categories: (1) *algorithmically based mistakes*, (2) *intuitively based mistakes*, (3) *mistakes based on formal knowledge of fraction operations*, (4) *misunderstanding of the symbolism of a fraction*, and (5) *misunderstanding of the problems*. In a fraction division operation, *algorithmically based mistakes* included misapplication of basic operational rules such as inverting the first term instead of the second term (e.g., $\frac{7}{8} \div \frac{1}{3} = \frac{8}{7} \bullet \frac{1}{3}$). *Intuitively based mistakes* consisted of inappropriate application of the properties of whole number operations to fractions such as thinking that one could get a smaller number when dividing 7/8 by 1/3 because dividing a whole number by another whole number results in a smaller number. *Mistakes based on formal knowledge of fraction operations* were based on lack of knowledge about the properties of fraction operations such as thinking that a given problem was a fraction multiplication problem ($\frac{7}{8} \bullet \frac{1}{3}$) instead of a fraction division problem ($\frac{7}{8} \div \frac{1}{3}$). *Misunderstanding the symbolism of a fraction* was based on limited understanding of the notation of fractions such as considering 7 as denominator and 8 as numerator for 7/8. Lastly, *misunderstanding the problem* results from limited understanding of the problem due to lack of mathematical knowledge or language.

Data Analysis

We used an inductive content analysis approach for analyzing teacher responses to the Stick problem (Grbich, 2007). First, we created an Excel spreadsheet of the raw data of teachers' responses, including images and descriptions. Then, we made initial coding of a subsample of the data to ensure reliability in our analysis and finalized coding schemes with examples. We then identified each response as correct or incorrect and coded all data based on the finalized coding schemes. This process was done for all data individually and we met to discuss discrepancies in coding. Our coding scheme allowed us to examine how teachers solved the problem and we report percentages of teachers who demonstrated flexibility with referent units and/or used certain strategies (formal and informal) for the correct solutions, along with certain errors for the incorrect solutions.

Results

Based on teachers' responses to the Stick problem, while relatively high number of teachers (218 out of 236 teachers; 92.4%) provided correct solutions, 18 teachers (7.6%) provided incorrect solutions. Regarding the correct solutions of the 218 teachers, 93 teachers (39.4%) demonstrated reasoning with quantities through drawing (i.e., flexibility with referent units); 106 teachers (44.9%) showed reasoning through computations, although they were not tied explicitly to quantities; and 19 teachers (8.1%) showed reasoning based only on computations.

The 93 teachers who showed reasoning with quantities (i.e., flexibility with referent units) focused on the drawing before computing any formal or informal strategy, obtained three parts through partitioning, added two more parts to make a whole stick, obtained 20 pieces through partitioning each part into 4 pieces (totally 20 pieces), and counted the number of pieces that align with the 3/5 stick (Figure 3).

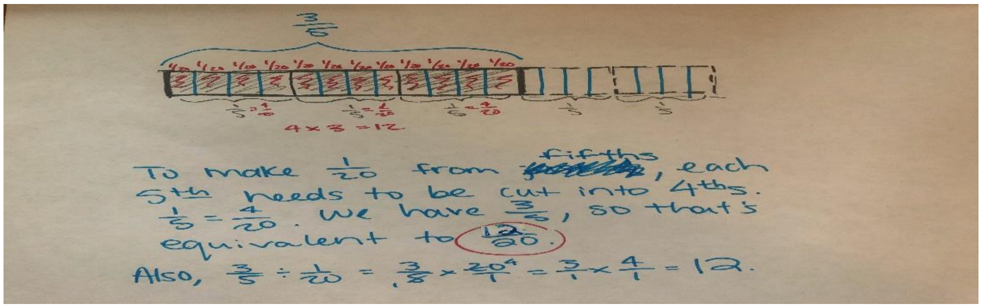

Figure 3: Example of teacher solution based on reasoning with quantities

The 106 teachers who showed reasoning through computations not tied explicitly to quantities used informal strategies for solving the Stick problem (Figure 4). In particular, while almost all of those teachers (103 teachers) used the common denominator strategy, by making equivalent fractions between 3/5 and 1/20, the remaining three teachers relied on decimal strategy by performing the division operation as a result of converting the given fractions into decimals. On the other hand, the strategies of repeated subtraction, unit rate, and dividing numerators and denominators were not present.

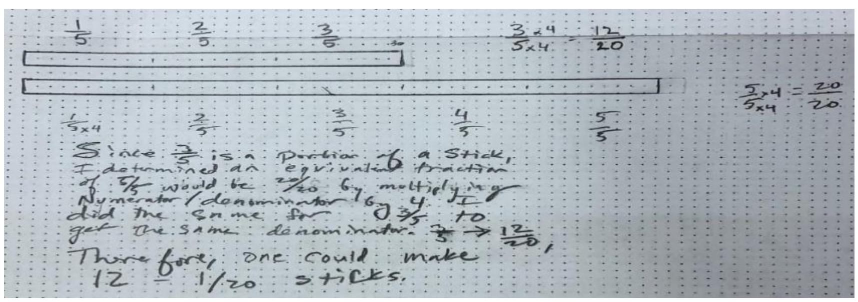

Figure 4: Example of teacher solution based on the common denominator strategy

The 19 teachers who showed reasoning based only on computations, 18 of them relied on the invert and multiply strategy and converted the fraction division expression into multiplication and only one teacher used the cross-multiplication by multiplying the numbers across each side of the equation (Figure 5).

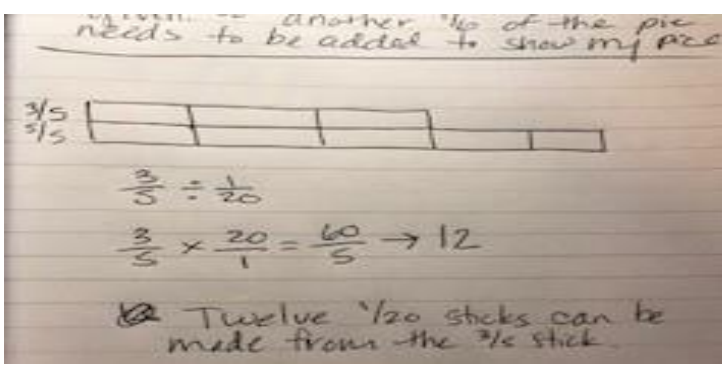

Figure 5: Example of teacher solution based on the invert and multiply strategy

Regarding the incorrect solutions of the 18 teachers, 11 teachers showed *misunderstanding of the problems*, indicating limited understanding of the problem due to lack of mathematical knowledge or language. These teachers interpreted 1/5 of the given stick as “20 pieces” and stated that the answer is “60 1/20 pieces” or partitioned 1/5 of the given stick into 5 pieces instead of 4 pieces. Three teachers showed *mistakes based on formal knowledge of fraction operations*, indicating a lack of knowledge about the properties of fraction operations. These teachers appeared to consider the Stick problem as a fraction multiplication problem by multiplying 3/5 by 1/20 instead of fraction division problem. One teacher showed *misunderstanding the symbolism of a fraction* by multiplying the numerator and denominator with different numbers to make an equivalent fraction. Three teachers made incomplete drawings and wrote that they have no idea about how to solve the problem. Lastly, none of the teachers made *algorithmically based mistakes* and *intuitively based mistakes*.

Discussion

The purpose of the present study was to examine U.S. in-service elementary and middle grades teachers’ responses to a measurement fraction division problem using a length model including their strategies and types of errors. One main result was that a relatively high number of teachers responded to the problem correctly. While only 47% of the preservice teachers (52 out of 111 preservice teachers) provided correct solutions for the Stick problem in Lee’s (2017) study, the percentage of the in-service teachers with correct solutions was 92.4% (218 out of 236 teachers) in our study. A second main result of the present study was that a greater percentage of teachers had the quantitative meanings of measurement division, indicating flexibility with referent units, in comparison to those teachers and preservice teachers in past research (e.g., Authors, 2022; Baek et al., 2017; Copur-Gencturk, 2021; Lee, 2017; Lee et al., 2011; Son & Lee, 2016). While only 11.7% of the preservice teachers (13 out of 111 preservice teachers) in Lee’s (2017) study showed flexibility with referent units through reasoning with quantities, the percentage of the in-service teachers who demonstrated flexibility with referent units was 39.4% (93 out of 236 teachers).

Furthermore, our results reveal that in-service teachers in our study outperformed preservice teachers in Lee’s (2017) study in terms of the use of strategies and types of errors. In particular, while 44.9% of the in-service teachers (106 out of 236 teachers) in our study showed reasoning through computations although they were not tied explicitly to quantities, the percentage of the preservice teachers who reasoned through computations not tied explicitly to quantities in Lee’s (2017) study was only 7.2% (8 out of 111 preservice teachers). Those in-service teachers relied on common denominator and decimal strategies as informal strategies, but not on repeated subtraction, unit rate, and dividing numerators and denominators. This result suggests that like preservice teachers, in-service teachers in the U.S. may have insufficient experience of providing alternative strategies for solving fraction division problems. Thus, professional development programs should emphasize teachers using a variety of informal strategies in classroom instruction.

In terms of types of errors, only 7.6% of the in-service teachers (18 out of 236 teachers) in our study responded to the problem incorrectly. Most of those teachers’ errors were based on mistakes resulting from misunderstanding of the problem based on Isiksal and Cakiroglu's (2011) framework. As opposed to our study, 53% of the preservice teachers (59 out of 111 preservice teachers) in Lee's (2017) study responded to the problem incorrectly and most of their errors were related to mistakes resulting from *misunderstanding of the problem*, *mistakes based on formal knowledge of fraction operations*, and *algorithmically based mistakes*.

Despite a better positive picture in terms of a greater percentage of in-service teachers who demonstrated flexibility with referent units compared to prior studies, teachers should be supported in teacher education and professional development programs to improve their flexibility with referent units. These supports include awareness of different visual models, such as length, area, and set, emphasis on making drawings in classroom settings, and encouragement for identifying the referent units in the drawings. Future studies should focus on examining in-service teachers' flexibility with referent units in partitive fraction division or fraction multiplication problem situations.

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DYNAMIC REPRESENTATION AS A TOOL FOR TEACHERS' CONNECTION MAKING

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In this paper, we present Epistemic Network Analysis of 32 teachers knowledge resources used to solve two different tasks. Using Drijver's (2018) framework about ways technology can be used for learning, we argue that using technology supports teachers to activate more knowledge resources and to use them in connected ways. We propose that this may offer insight into the design of professional development aimed at supporting teachers in the development of connected knowledge.

Keywords: Technology, Teacher knowledge, Rational Numbers and Proportional Reasoning

Purpose

In this paper, we address two research questions:

- What knowledge resources do teachers use to solve two proportional reasoning tasks?
- Are those knowledge resources connected in ways that seems to support Skills Practice, Conceptual Understanding, or some combination of both?

We argue that technology provides a different experience than static tasks for teachers as they engage with mathematics, thus allows a different kind of thinking. Further, we suggest that using technology to think about mathematics opens opportunities for teachers to develop more connected understandings of the mathematics they teach. We end with a discussion of the implications of this work.

Perspectives

We draw from two perspectives to make our case for the value of technology in supporting teacher conceptual understanding. First, we use a Knowledge in Pieces (KiP) lens to contemplate cognition and how it can be conceived of in teacher learning. Then, we draw from Drijvers' work on technology in mathematics education to situate the ways in which we use technology.

Knowledge in Pieces

Knowledge in pieces is a theory of conceptual change that considers knowledge to be comprised of fine-grained resources (diSessa, 1988, 2018; diSessa, et al., 2016). In KiP, learning can be viewed as creating new knowledge resources, refining existing knowledge resources, and/or creating connections between and among knowledge resources. For this paper, we are particularly interested in the connections participating teachers were making between their knowledge resources.

To make knowledge resources visible and to focus on interactions between knowledge resources, we rely on Epistemic Network Analysis (ENA; Shaffer et al., 2009). ENA is a quantitative ethnography (Shaffer, 2017) method that, in our case, allows knowledge resources to act as nodes and for lines between nodes to indicate the relative frequency of the co-occurrence of the knowledge resources. A co-occurrence, for us, is any instance in which more than one knowledge resource is used to solve a single task. We consider co-occurrences to be proxies for connections between knowledge resources.

Technology in Mathematics Education

In their conceptualization of the uses of technology in mathematics education, Drijvers and his colleagues (e.g., Drijvers, 2015, 2018) have developed a heuristic for technology use in mathematics education. They posit that the ways in which we think about technology in mathematics education should be driven by the function of the technology rather than the form of the technology. Thus, they conceive of technology as either helping us Do math or Learn math. And, in the case of technology that helps us Learn math, there is technology that helps with Skills Practice and technology that helps with Conceptual Development. It is this kind of technology that is the focus of this paper. By using KiP, we are able to think about knowledge as a series of fine-grained understandings that can be grouped in myriad ways. By using the Drijvers framework, we have a language for characterizing the ways in which the knowledge resources are interacting.

Methods

The data reported here comes from a pair of interviews conducted with a convenience sample 32 middle grades teachers from four states. The first interview was a think aloud protocol in which the participants responded to a set of tasks using a LiveScribe pen to capture and coordinate their voices and inscriptions. The second interview was a face-to-face interview with similar mathematics tasks, however some of the tasks used dynamic sketches on an iPad. The live interviews were recorded with two video cameras: one focused on the participants' faces and one on anything they wrote or at which they pointed.

For this analysis, we focused on two tasks: the Santa Task (Figure 2) and the Bears Task (Figure 3). For the Santa Task, which was based on a task from de Bock, Van Dooren, Janssens, & Verschaffel, 2002), we provided scaffolded prompts to support teachers in thinking about the situation. These included:

- Ms. Yarbrough's class had two favorite answers. About 40% of her class chose 18 ml and about 40% chose 54 ml. What might the students who were wrong been thinking about? Is that something you see commonly with your own students?
- One of Ms. Yarbrough's students drew rectangles around the images. Do you think this is a helpful strategy for a student? Why or why not?

For the Bears Task, we asked the teachers:

- Describe what is happening as you drag the slider. How is the image changing? When we started, the two figures were similar. Where you have ended, are they still similar?
- How would you characterize the growth as you move the slider? Is there anything in the relationship of the bears to each other that stays the same? How would you describe how many times larger the new bear is than the original? How would you describe the scale factor of the new bear to the original?

- What is the ratio of the bear's belly to the frame? Does that ratio stay the same as the bear change size?

These tasks were selected for this analysis because they engaged the teachers with similar mathematics, though Santa was done on paper and Bears was an interactive app.

Data were coded using an emergent coding scheme (Weiland et al, 2021) in which each code was a knowledge resource. We used ENA for the data analysis (Shaffer et al., 2009), which meant that each utterance was coded using a binary system (present/not present). In this case, an utterance was the answer to a single task. Once coding was done, the ENA webtool created the maps of participants' use of knowledge resources. For these maps (Figure 2a & b), the nodes indicate the knowledge resources being used by the participants. The size of the node is relative to its frequency in the dataset. The lines connecting knowledge resources indicate that those pairs of knowledge resources occurred together within an utterance. The thickness of the line indicates the relative frequency of the co-occurrence.

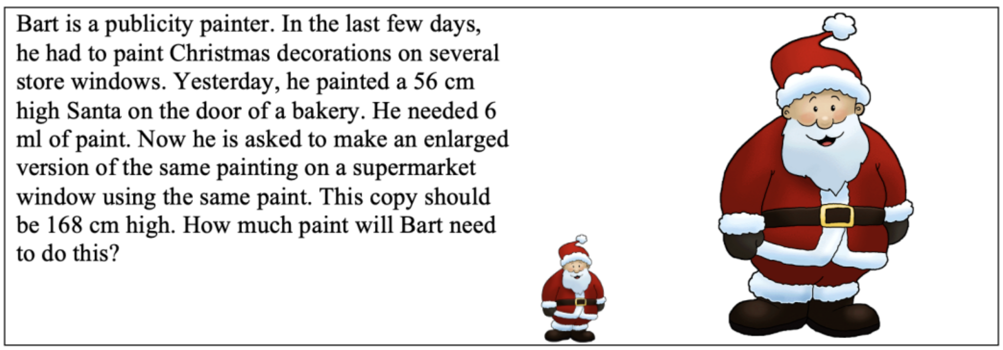


Figure 1. The Santa Task (based on de Bock, Van Dooren, Janssens, & Verschaffel, 2002)

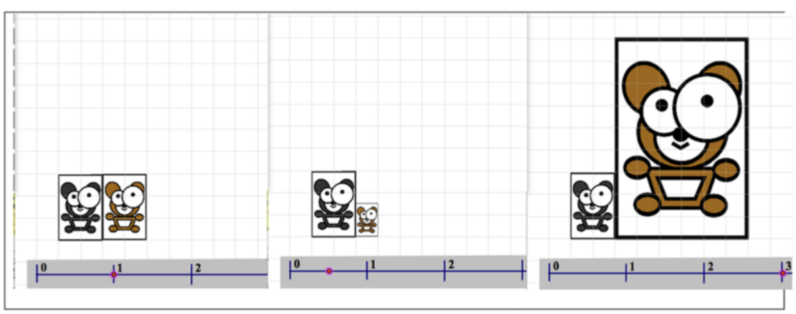
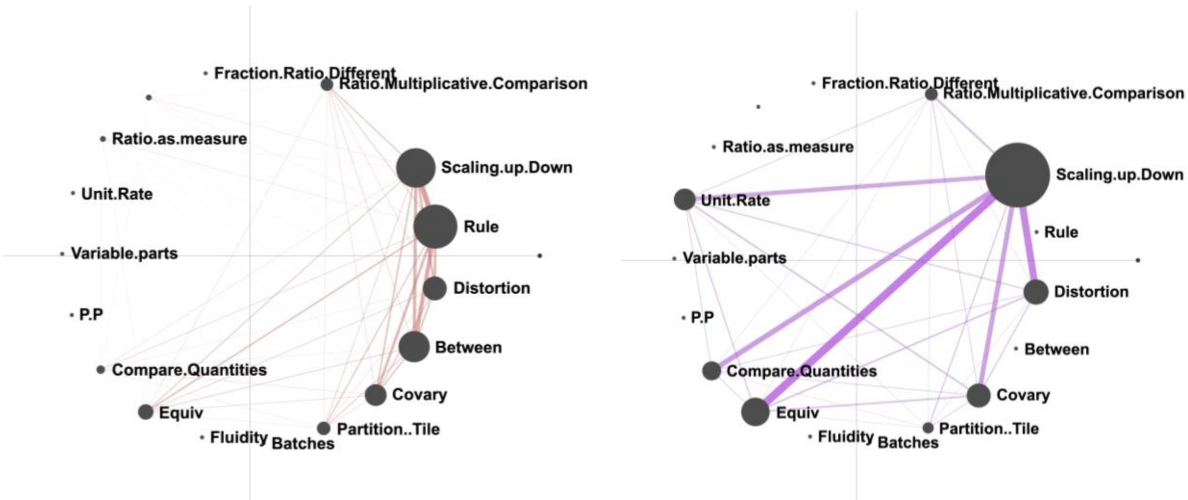


Figure 2. The Bears Task

Results

As shown in the ENA graphs (Figure 3a & b), the teachers relied heavily on Rules (mostly cross multiplication) and Scaling Up and Down to solve the Santa Task. Because they jumped to these two procedures, most of the teachers missed that the relationship between height and amount of paint is not a proportional one. The proportional relationship is between the area of the big Santa and the area of the small Santa (for a full qualitative analysis of the answers given, see

Brown et al., 2020). There were few connections made between knowledge resources, meaning that teachers often went straight to an algorithm and did not invoke other proportional knowledge resources. Most interestingly, most of the resources they did use were focused on answer finding, which is more related to Skills Practice than Conceptual Understanding (Drijvers, 2018). We argue that the only structures they attended to in connected ways were Covariation (e.g., the numerator and denominator change together) and the Between Measure Space relationship (e.g., attending to the relationship of one quantity, such as height, to the other, such as width), both of which were used less than the two skills.



(a) (b)

Figure 3. ENA results for the static Santa Task (a) and dynamic Bears Task (b)

In contrast, on the Bears Task, these same teachers did not rely on Rules at all. Further, there is much more interaction between knowledge resources., overall. While we still see a reliance on procedures with most of these interactions, we do see more variety in approaches. This suggests that there is something about the dynamic representation that both cues more knowledge resources and promotes more interaction between those resources.

Discussion and Conclusions

Given that learning in the KiP framework can happen through development of new resources, refinement of existing resources, or making new connections between resources, we posit that using technology-based tasks is a way to engage teachers in learning. Further, we suggest that doing this may support the development of connections between knowledge resources, which is potentially useful for supporting teacher learning. Consistent with Drijvers (2018) framework, the technology-based task seemed to offer more opportunity for Conceptual Understanding than did the static task. To this end, we propose that developing professional development that is designed to take advantage of technology in ways that supports the development of conceptual understanding may be fruitful for support teachers in better connecting their already-existing mathematics knowledge resources.

The implications of this are pragmatic. Too often, professional development is focused on building new knowledge without regard to the knowledge teachers already have. Because they are adult learners, often with degrees beyond a bachelors, teachers need professional development that caters to them. As shown in Figure 3, the teachers in our sample all had knowledge resources important for understanding proportional relationships. However, they needed the Bears task to activate some of them and to activate more than one in an utterance.

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SECONDARY PRE-SERVICE TEACHERS' MATHEMATICAL PROBLEM SOLVING KNOWLEDGE FOR TEACHING

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Keywords: Geometry and Spatial Reasoning, Mathematical Knowledge for Teaching, Preservice Teacher Education, Problem-Solving

There are urgent calls for teaching methods which foster students' problem solving in mathematics (NCTM, 2000; NGA, 2010). To do this, teachers and pre-service teachers (PTs) need the knowledge and skills to successfully teach for problem solving. This base of knowledge and skills is called *Mathematical Problem Solving Knowledge for Teaching* (MPSKT); that is, the knowledge needed to help students develop mathematical problem solving proficiency (Ball et al., 2008; Chapman, 2015). MPSKT consists of six subcomponents: knowledge of problem solving, problems, problem posing, students as problem solvers, and instructional practices, as well as affective factors and beliefs (Chapman, 2015). While some research has investigated the MPSKT of teachers and PTs (Chapman, 2016; 2017; Clivaz et al., 2023; Owens, 2023; Owens & Nolan, 2021a; 2021b; Piñiero et al., 2021), very little of this work has holistically examined the MPSKT of secondary PTs. This is an unfortunate limitation as secondary PTs are vital to our goal of fostering problem solving skills in all students.

Purpose, Methods, and Analysis

This poster presents details of a qualitative pilot study on secondary PTs' problem solving proficiency, MPSKT, and the relationships between them. PTs (n=4), who had recently completed a geometry methods course which focused on how to teach secondary geometry topics with student-centered approaches, took part in two interviews. The first was a task-based interview that assessed problem solving proficiency using think aloud protocols (Cowan, 2019). The second was a semi-structured interview examining each of the six subcomponents of MPSKT. Analysis focused on describing the problem solving proficiency and MPSKT of each PT as well as identifying any relationships that emerged between MPSKT and problem solving proficiency. A future study will build on this work with a larger sample.

Results and Implications

PTs' overall MPSKT was grounded in their personal experiences as students but lacked skills we would expect of experienced teachers. For example, when discussing their knowledge of beliefs and affective factors, PTs focused primarily on either the emotional needs of students or beliefs about the nature of mathematics, but never on both (as we would expect from experienced teachers). PTs typically held very little knowledge of problem posing. In addition, a complex interplay between PTs' problem solving proficiency and MPSKT emerged. PTs with weaker problem solving proficiency tended to demonstrate weaker MPSKT. However, higher problem solving proficiency did not necessarily predict comprehensive MPSKT in all six subcomponents, suggesting teacher educators may need to deliberately teach MPSKT. As we envision the future of mathematics education, we will need to better understand secondary PTs' MPSKT and our role, as teacher educators, in helping them further develop their knowledge and skills to ensure all students develop rich problem solving proficiency.

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EVALUATING MATHEMATICAL KNOWLEDGE FOR TEACHING FRACTIONS: DEVELOPMENT AND VALIDATION OF THE MKT-FRACTIONS MEASURE

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Keywords: Assessment, mathematical knowledge for teaching, Rational Numbers

In this poster, we discuss the development and validity argument for the Mathematical Knowledge for Teaching (MKT) of fractions measure (MKT-Fractions). Ball et al. (2005; 2008) defined MKT as including specialized content knowledge (SCK) and pedagogical content knowledge (PCK) for teaching mathematics. PCK includes knowledge of content and students (KCS) and knowledge of content and students (KCT). Whereas SCK involves understanding the different valid, and invalid, approaches to engaging in certain procedures or approaches, PCK involve understanding children's mathematical reasoning (KCS) and appropriate pedagogy to scaffold students' reasoning (KCT) (Ball et al., 2006; 2008).

We surveyed 103 undergraduate students from two Midwestern U.S. universities to evaluate an updated version of the MKT-Fractions measure (University A=49.5%; University B=50.5%). Prior versions of the measure included only KCS items, while this updated measure includes 18 multiple-choice items assessing: KCS (n=8), SCK (n=5), and KCT (n=5). Prior versions of the MKT-Fractions measure have collected validity evidence on test content, response processes, reliability and internal structure (Zolfaghari et al., 2021; 2022). In this paper, we used Rasch modeling to collect data on reliability and internal structure.

Rasch analysis began with a Principal Component Analysis (PCA), which provided support for unidimensionality across all KCS, KCT, and SCK items (λ=2.28, disattenuated correlation = 1.00). Item fit, another indicator of unidimensionality, suggested good model fit with an average mean square for item infit ($MNSQ$ =.99, Z=.00) and outfit ($MNSQ$=.98, Z=.00). The samples' model fit was also supported with person infit ($MNSQ$= 1.00, Z=.00) and outfit ($MNSQ$ = .98, Z=.00). Item and person reliability provided additional validity evidence. Item reliability was .96 with an item separation index of 5.09, indicating the measure's consistency in distinguishing between item difficulties. Person reliability was 0.64, with a person separation index of 1.33. Pragmatically, the statistics suggest participants can be categorized into two different levels (Boone et al., 2014). Analyses of prior versions of the MKT measure suggest the lower person reliability is primarily due to our sample being limited to undergraduate students, as in-service teachers typically score higher on PCK (Zolfaghari et al., 2021, 2024).

Examination of validity evidence for internal structure and reliability indicates that the MKT-Fractions measure is sufficient for evaluating teachers' mathematical knowledge for fractions. The results from this study show that the majority of PSTs demonstrated above-average knowledge in teaching fractions, as evidenced by the MKT-Fractions scores (M = .41, SD = .99), where 0.00 represents the average level of knowledge. This result is not surprising, given all of the participants had completed at least one mathematics methods coursework and grades 3-6 field experience. Both the mean scores and lower person reliability suggest a need for a more diverse sample, including experienced in-service teachers and more novice preservice teachers.

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Chapter 6: Mathematical Processes and Practices

MATHEMATICAL PROOF IN STANDARDS AND PRACTICES: A CONTRADICTION

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Mathematics educators universally agree that mathematical proofs stand as the core of mathematics, highlighting a significant consensus on their importance. Despite this agreement, research consistently shows that both students and teachers encounter substantial difficulties in teaching and learning proofs. These challenges underscore a critical gap in educational practices and comprehension. This study wants to bring the attention about the importance of studying the proof in standards and practices which directly affects teacher preparation programs.

Keywords: Reasoning and Proof, Standards, Teaching Practices.

There has been extensive research on mathematical proofs in school mathematics. Students are not only expected to learn but also to construct various types of proofs, including proof by contradiction, proof by induction, pictorial proofs, paragraph proofs, flowchart proofs, and two-column proofs. Despite the emphasis on its role and importance in school mathematics, research shows that no matter what grade they are in, students encounter difficulties when involved in proof-related activities (Weber, 2001; Knuth et al., 2002). This problem is universal and not limited to school students; even college-level mathematics major students struggle when working on proving mathematical statements (Zazkis, Weber & Mejia-Ramos, 2014). However, this doesn't mean students are incapable of building proofs. According to Lester (1975), students in all grades can understand and even construct proofs despite facing obstacles and difficulties.

According to Usiskin (1987), a root problem is how students are introduced to proof in school mathematics. It seems teachers are not adequately prepared, lacking exposure to proving tasks, which hinders their ability to teach proofs effectively and engage students in proof-related activities. A significant issue is the absence of explicit experiences to enhance content and pedagogical teaching knowledge (Abbaspour, 2022). It was for this reason that the researchers of this study tried to examine the standards and teaching practices to investigate if the expectations the mathematics education community has of students are consistent with the standards and practices defined for teachers to be prepared to help students learn mathematical proofs. For this purpose, this study analyzes the role of proof in NCTM's Principles and Standards for School Mathematics (2000) and Common Core State Standards for Mathematics (NCTM, 2010), Principles to Actions (NCTM, 2014), Catalyzing Change in High School Mathematics (NCTM, 2018), Standards for the Preparation of Secondary Mathematics Teachers (NCTM, 2020), and AMTE's Standards for Preparing Teachers of Mathematics (2017).

Standards and Teaching Practices

The Common Core State Standards for Mathematics mention proof primarily in high school geometry, without significant expansion. "Principles to Action" focus on reasoning and problem-solving but lack explicit proof instruction, missing guidance for teachers. "Catalyzing Change in

High School Mathematics" broadens proof application across disciplines, detailing the creation and validation of proofs. The "Standards for the Preparation of Secondary Mathematics Teachers" highlight the importance of reasoning and proof construction but lack detailed preparation standards, suggesting Linear Algebra for proof engagement. The "Standards for Preparing Teachers of Mathematics" emphasize mathematical arguments and reasoning, offering general course guidance but insufficient preparation standards for effective proof teaching.

Conclusion

It can be seen that the Standards go in depth with details when it comes to expectations for students regarding learning proofs. However, when reviewing the standards and preparations for teachers to teach proofs efficiently, there is a huge gap, which seems to be one of the underlying problems behind students' difficulties and teachers' obstacles in teaching and learning proofs. This study aims to highlight the importance of studying proof standards and practices, which directly affect teacher preparation programs.

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ANALYSIS OF A GRADE 4 STUDENT'S MATHEMATICAL REASONING IN A MATHEMATICAL LOGIC TASK

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Recent studies underscore the importance of teaching logic in elementary education, stressing its crucial link to reasoning. This study aims to analyze students' mathematical activity during logic tasks, employing Toulmin's (2003) theoretical framework to explore how students adapt to the specific rules of a logic game. The results provide diverse problem-solving strategies used by students, alongside the challenges they faced while reasoning. The discussion encompasses the challenges and the potential of such tasks to develop mathematical reasoning skills and the necessity of nurturing this skill in educational practices.

Keywords: Reasoning and proof, elementary school education, problem-solving

Introduction

In this project, we experiment a sequence of mathematical logic tasks, progressively integrating language and mathematical concepts, with two groups of 4th graders in primary school to explore how logic tasks could foster the development of mathematical reasoning. This angle is interesting given that mathematical reasoning lies at the heart of the mathematical activity (Mason et al., 2010). Moreover, its significance is reflected in various curriculums and institutional documentation (e.g., Common Core State Standards Initiative, 2024), highlighting its pivotal role in shaping educational practices. Yet recent and increasingly convergent research underscores the importance of teaching logic in elementary school: "logical thinking is not a natural talent [...], but a skill that can be trained, like say muscles are trained at the gym" (Adkhamjonovna, 2022, p. 915). The aim of this study is to describe and analyze the mathematical reasoning of elementary school students as they solve a logic game task.

Theoretical framework

The concept of mathematical reasoning is often used intuitively, without definition or characterization (Yackel & Hanna, 2003). Given the context of logic games established in this research, Toulmin's (2003) model helps us to describe the potential of mathematical reasoning for this type of task. Indeed, this model sheds light on the argumentative, logical and persuasive aspects of mathematical reasoning, offering a deeper understanding of the cognitive process involved. It emphasizes the importance of context and therefore can be used to understand how students adapt their reasoning to the game specific rules. The model enables the highlighting of reasoning steps and their analysis, facilitating the study of what Knipping (2003) calls chain of reasoning. Toulmin's model emphasizes three main elements: data, warrant, and claim. Starting from data and an often implicit warrant, it is possible to formulate a claim. Substantiating a claim requires offering evidence which constitutes the data. It is also necessary to convince oneself, that is to accept as valid or plausible the connection between the data and the claim. This is the role of the warrant, it supports the inference from the data to the claim (Toulmin 2003).

Three other elements may appear in a reasoning step: the backing, the qualifier and the rebuttal. The backing supports the warrant and consists of elements that have a certain consensus in a given community, upon which one can rely. The qualifier refers to the epistemic value of the claim, determining its truth value or plausibility. One way to decide about the status of this epistemic value is to use a rebuttal. A rebuttal is a statement that, if proven true, leads to the rebuttal of the claim, admitting it to be false.

Method

To achieve our goal, we selected one logic task and undertook a descriptive case study (Yin, 2018). The logic task examined in this paper was part of a larger project conducted over an entire year in two 4th grade classes (9- and 10-year-olds). Researchers visited the classrooms once or twice a month to introduce new types of tasks, which were then piloted by the teachers in the following weeks. The researchers interviewed six students to study their mathematical activity in greater depth. All activities were filmed. We chose this task firstly because it was used during the initial interview. Secondly, the task is a strictly visual mathematical logic game[2], minimizing the influence of reading skills and prior mathematical knowledge. Thirdly, the order in which the clues are presented does not strictly lead to a deductive chain, allowing for different types of claims and various reasoning steps.

The objective of this logic puzzle is to place all nine geometric pieces within a 3x3 grid, following the provided visual clues (See Figure 1). Each clue offers information related to the placement of pieces. While several strategies are possible, we opted to encourage students to interpret the clues sequentially, noting their resolutions as they progress (by temporarily positioning pieces in the grid).

Figure 1: Logic puzzle used in this research (Lyons and Sabinin, 2015)

A qualitative analysis helps us examine the meaning of the material collected through Toulmin's (2003) framework. Firstly, we looked at the task in terms of its potential to foster students' mathematical reasoning. Second, we construct an analysis grid for the task, highlighting the elements of potential reasoning steps in this type of task for each of the puzzle's clues. Third, we studied the nature of the task according to two criteria: explicit/implicit and positive/negative information. We determined a list of possible inferred claims that can come from the data and warrant; and analyzed their qualifier (or epistemic value). Secondly, we studied the mathematical activity of the six students who solved this task using the same framework.

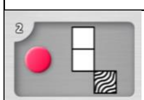
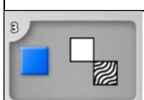
[2] https://apps.defimath.ca/gym-logique/

Results

Task Analysis Results

Based on Toulmin's (2003) model, we can say that it is the combination of multiple clues that provides enough information to generate the data and warrants necessary to infer all the claims to solve the puzzle. Furthermore, the mathematical reasoning that takes place relies on some shared ground (backing), i.e. the common knowledge (the name of the geometric piece, the color, the position) and to the "graphic" elements established by the task itself (the grid is 3x3, the waving pattern indicating the place of a piece (e.g., Clue 1), the X indicating the impossibility of placing a piece).To carry out our analysis, we processed the indices one after the other, since it was the strategy that has been valued in class. We present in table 1 a summary of the analysis of the first four clues.

Table 1: Analysis of the first four clues

Clues	Data	Inferred claims (qualifier)	Chains of reasoning
	*Any form* Blue Top left Corner	Blue circle is top left (P) Blue square is top left (P) Blue triangle is top left (P) One blue piece is top left (T)	
	Circle Red *Bottom line, center or right corner*	Red circle is center of bottom line (P) Red circle is bottom right corner (P) Red circle is bottom center or right (T)	
	Square Blue *Center or Bottom line, 2nd or 3rd column*	Blue square is center (P) Blue square is center of bottom line (P) Blue square is center 3rd column (P) Blue square is bottom right corner (P) Blue square is in one of the four right bottom places (T)	With Clue 1: Blue square is top left (F)
	Triangle *Any color* NOT 1st nor 2nd column. *3rd column*	Red triangle is top right corner (P) Red triangle is center 3rd column (P) Red triangle is bottom right corner (P) Blue triangle is top right corner (P) Etc.	With Clue 1: Blue triangle is top left (F) Blue circle is top left (T) With Clue 2: Red circle is bottom right corner (F) With Clue 3: Blue square is in the right column (F) Blue square is center (T)

		There is no circle nor square in the last column (T)	

Legend: italic (multiple possibilities), blue (implicit data), red (negative data), P (plausible), F (false), T (true)

The first thing we can notice from table 1 is that Clues 1 and 2 are independent and do not provide enough information to place a piece. We must process all clues up to the fourth one before being able to place a first piece with certainty, i.e. to infer a true positive statement about one singular piece. However, this true statement was first inferred as plausible while processing Clue 1. To validate that claim, Clue 4 must be combined with Clue 1 and Clue 3. Clue 4 also refutes inferred plausible claims from Clues 1, 2 and 3. Those refutations will be useful later to eliminate possibilities and solve the puzzle. Moreover, Clue 4 is the first one with negative information. For all those reasons, we can hypothesize that Clue 4 could be an obstacle to the resolution of this puzzle.

Analysis of students' mathematical activity

We analyzed the mathematical activity of the students by comparing their traces and discourses for each clue. Two strategies were observed for Clue 1. Four students placed all blue pieces in the top-left square, resulting in three plausible claims. Two students opted to place only the blue circle, inferring the claim "the blue circle goes in the top-left corner" and considering it true. Presumably, they initially scanned all the clues before processing them one by one. For Clue 2, all students positioned the red circle between the two bottom-right squares (See Figure 2). For Clue 3, all students place the blue square at the intersection of the four bottom-right squares (See Figure 2). Regarding Clue 4, students place a pile of triangles either at the top of the right-hand column or next to it (See Figure 2), except for E6, who stated, "we don't know yet." E6 seemed unable to infer that the triangles must be in the last column from the data and warrant for this clue.

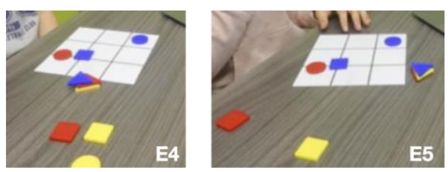


Figure 2: Traces of students E4 and E5 after processing Clues 1 to 4.

Our previous analysis of the task suggests that Clue 4 could lead to a chain of reasoning; it changes the epistemic value of the claims made in Clues 1, 2 and 3 from probable to true. We now know the place of the red circle and, consequently, the place of the blue square. However, none of the students relocate these pieces at first, thereby failing to alter the epistemic value of the previously inferred claims. This analysis confirms that Clue 4 certainly represents a turning point in this puzzle. Contrary to Clue 1, they cannot manage the consequences of this clue by relying solely on their traces. Indeed, for Clue 4, they seem to remain in a state of uncertainty. They are unable to make a definitive placement for the red circle and the blue square. Certainly, this chain of reasoning represents complex reasoning. It requires the student to manage several elements that can change status (e.g., the claim becomes a refutation for a previous reasoning step).

Discussion and Conclusion

This project aims to describe the mathematical reasoning of primary school students through their engagement in logic tasks. In both analysis (the task and the students' mathematical reasoning), Clue 4 appeared as an obstacle. Firstly, Clue 4 provokes a particularly complex chain of reasoning. The claims inferred from this clue can be used to validate or refute previously established claims with plausible status. This is reminiscent of Knipping's (2008) proposal around local and global arguments. She suggests that, starting from local arguments, it is possible to restructure the whole reasoning. That's what is at stake here. The processing of Clue 4 is local but enables a restructuring of previous reasoning steps. This type of reasoning chain is complex and represents advanced reasoning. The student participants seem not to be able to change the epistemic value of previously established claims, keeping their already produced traces. The change in epistemic value is based on an argument that must be accepted as true by the student. They must be convinced. It is the argumentative aspect, among others, that students must be able to use when constructing proofs, for example. However, even in a highly controlled environment, students have difficulty relying on these established facts as true. This environment is controlled, e.i. their choices are restricted by the task. A very narrow set of claims is possible, and the backing is rather clear and circumscribed. This is different from an open-ended task, where students rely on a broader set of mathematical facts to formulate a conjecture with no alternative if it turns out to be false. Our analysis also highlights the challenge of managing these traces. Students must decide what information to collect, how to collect it, and how to use these traces to guide their reasoning.

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THE MODELS OF TEACHING AND LEARNING MATHEMATICAL PROVING AND MATHEMATICAL PROOFS: A MULTIDISCIPLINARY PERSPECTIVE IN PK-20

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The models of teaching and learning mathematical proving and mathematical proofs are narrowly focused on the mathematics discipline. Scholars in mathematics education theorized the formal-embodied-symbolic triad as a basis of understanding teaching and learning mathematical proving and mathematical proofs. This paper, in contrast, adopts a multidisciplinary perspective of the teaching and learning of mathematical proving and mathematical proofs. By proposing a novel model of teaching and learning, this paper aims to improve the extant state of the teaching and learning mathematical proving and mathematical proofs in a PK-20 context.

Keywords: Proof; Reasoning; Modeling

Mathematical proving and mathematical proofs are the heart of mathematics; yet it is seldom the focus of students. Students in contrast view mathematics as discrete branches of mathematical knowledge that one usually picks up starting from Algebra 1 classes to Advanced Placement examinations (e.g., Calculus BC). From the teaching and learning perspective, the scholarly work on teaching and learning mathematical proving and mathematical proofs are bifurcated into teaching (by teachers) and learning (by students) perspectives. This thinking and way of doing things is outdated with the leveraging of generative artificial intelligence in learning mathematical proving and mathematical proofs.

Objectives

By adopting a practitioner-scholar perspective, this paper puts forth a novel theoretical framework of the teaching and learning of mathematical proving and mathematical proofs that incorporates insights from computer science, economics, epidemiology, mathematics, physics, psychology, and statistics. This model incorporates a diversity of approaches and emphasized the applications of mathematical proving and mathematical proofs beyond the discipline of mathematics.

Theoretical Framework

Discussion on the teaching and learning of mathematical proving and mathematical proofs tend to originate from the mathematics discipline (Stylianides et al., 2024; Tall et al., 2012). This paper goes beyond this narrow focus and examines mathematical proving and mathematical proofs from a multi-disciplinary perspective: computer science, economics, engineering, epidemiology, mathematics, physics, political science, psychology, and statistics (e.g., Brauer et al., 2019; Franklin, 1983; Gersting, 2007).

Coming from the perspective of mathematical thinking, Tall et al. (2012) adopts a learner-centered approach and posits that mathematical proving and mathematical proofs should be examined from three categories: formal, embodied and symbolic. Scholars in non-mathematics discipline tend to adopt an applied perspective by focusing on using mathematical proving and mathematical proofs to explain a real-life phenomenon. The theoretical framework is multi-level and its focus starts from fundamental forces from neuron level to humans, classrooms, schools, families, and to the large geographical areas (e.g., regions); thereby overcoming narrow dualism of teaching (by teachers) and learning (by students).

Methods

This paper adopts a review of the theories of the teaching and learning of mathematical proving and mathematical proofs in computer science, economics, engineering, epidemiology, mathematics, physics, psychology, and statistics by reviewing the top five journals of each field from 2010 to 2024. The first step is a title search, and the keywords include terms such as "proofs" and "mathematical proving," The second step is to review the abstracts and the third step is to read the shortlisted articles in-depth and to tease out the models of teaching and learning mathematical proving and mathematical proofs.

This paper adopts a practitioner-scholar perspective, and it will review the five most commonly use mathematics textbook in computer science, economics, engineering, epidemiology, mathematics, physics, political science, psychology, and statistics (e.g., Pipes & Harvill, 2014). It will also review mathematical proving and mathematical proofs courses in PK-20 sectors in California, Florida, Massachusetts, New York, and Texas.

Results

The preliminary results show that there are fundamental models that can explain real-life phenomena that students can relate to. These models can be a good starting point for courses that would incorporate theory-based and evidence-based methods of teaching and learning mathematical proving and mathematical proofs. The model focuses on fundamental forces from the neuron level to large geographical areas (e.g., regions); thereby overcoming narrow dualism of teaching (by teachers) and learning (by students). In the age of generative artificial intelligence, my model centers on self-learning and continual learning of mathematical proving and mathematical proofs under highly trained mentors.

Discussion

This paper goes beyond the conventional way of teaching and learning mathematical proving and mathematical proofs. By examining how non-mathematics disciplines use mathematical proving and mathematical proofs to examine real-world phenomena, this paper's model provides an alternative way of teaching and learning mathematical proving and mathematical proofs in PK20.

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SCAFFOLDING MOVES THAT ELICIT MODELING COMPETENCIES

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Situated within efforts to understand the complex interplay among learners, teachers, and tasks in mathematical modeling education, we examine how contingent scaffolding moves influence the modeling process. Using mixed methods, we coordinated qualitative frameworks for scaffolding and modeling competencies through their application to task-based cognitive interviews with undergraduate STEM majors. A mixed logistic regression model with participant random effect analyzed the temporally-linked frequencies of codes. The model sustains claims about the compatibility of the frameworks and predicts moves eliciting competencies.

Keywords: modeling, advanced mathematical thinking, cognition, mathematical representations

In any didactic situation, there is a triadic interaction among the learner, the teacher, and the task environment (Brousseau, 1997; Koichu & Harel, 2007). Understanding how the teacher influences the interaction between learner and task environment is a major research objective in mathematics education. In learning environments that focus on developing mathematical modeling skills, learners are assumed to enter with real-world knowledge (and therefore assumptions) that may not afford the intended mathematics (Cai et al., 2014). A number of studies have shown that educators may respond to the learners' work in ways that amount to consistent negative feedback or diminish learner autonomy in decision-making while modeling (Verschaffel et al., 2020). Additionally, support which may, on the surface, seem adaptive to error is not always contingent to a learner's in the moment needs (Wischgoll et al., 2015). For these reasons, educators have sought means for scaffolding learners' modeling processes that maintain cognitive demand, endorse and extend their autonomous ways of reasoning, and do not inadvertently teach the idea that there is a "school math" entirely distinct from "real math" (see Nunes et al., 1985; Watson, 2008). Our study is situated within the broader agenda to understand which kinds of scaffolding moves are effective in supporting modelers as they learn to construct and validate meaningful models of real-world scenarios. In particular, the aim of this study was to investigate the influence of facilitators' micro interventions on undergraduate STEM majors' modeling processes.

Literature Review

Mathematical modeling is a cognitive process.

Cognitive perspectives on mathematical modeling conceive it as a process of transforming a question about the real world into a mathematically well-posed problem (Kaiser, 2017). For example, one way the question *How rapidly will a disease spread through a community?* can be answered is by using the equation $\frac{dS}{dt} = \tau SH$ as a model of the scenario. Using an equation as a model means the modeler constructs quantitative meanings for the variables $\boldsymbol{S}$ and $\boldsymbol{H}$ which represent the number of sick people and healthy people at time $\boldsymbol{t}$, respectively. This sub-process is known as *mathematizing*. Another important sub-process of modeling is *validating*. This is done by adopting (implied or explicit) assumptions about how the world works and evaluating the adequacy of the resulting representation against those assumptions. Assuming that having the disease does not confer immunity to it is consistent with the model in the example. In general, modelers decide which real-world conditions and assumptions are important (or not) to incorporate into their model as mathematical properties, parameters, and relationships (Schwarzkopf, 2007; Zbiek & Conner, 2006). This sub-process is known as *simplifying & structuring*. Mathematical modeling cycles (MMCs) provide a descriptive framework that organizes the cognitive sub-processes as a set of phases connecting stages of model construction (Blum & Leiß, 2007). Table 1 shows Blum and Leiß (2007)'s cycle for the stages a modeler passes through and the sub-processes that connect those stages.

Table 4 Modeling competencies from (Blum & Leiß, 2007)

Sub-Processes	Definition	Connects Stages
Understanding	Forming an initial idea about what the problem is asking for	real world → situation model
Simplifying & Structuring	Identify (un)important real-world entities and relationships	situation model → real model
Mathematizing	Represent idealized version of the real-world problem using mathematical conventions	real model → mathematical model
Working Mathematically	Analyze or solve mathematical problem	mathematical model → mathematical results
Interpreting	Re-contextualize mathematical results	mathematical results → real results
Validating	Verify results against constraints	real results → real situation

Many studies have investigated the sub-processes and their manifestations across grade levels and content areas (Cevikbas et al., 2021), the characteristics of tasks that evoke them (Bock et al., 2015; Maaß, 2010), and the challenges learners face in carrying them out (Galbraith & Stillman, 2006; Klock & Siller, 2020). Importantly, many studies have found modeling does not proceed linearly through the sub-processes (Ärlebäck & Bergsten, 2010; Borromeo Ferri, 2007; Czocher, 2016, 2018). Despite the low predictive power of MMC's, they remain powerful descriptive models of desirable learner engagement with modeling tasks. There are robust analytic frameworks of observational indicators for which sub-process the modeler is engaged with that are applicable across content areas and grade bands (Czocher, 2016; Maaß, 2006). Within *working mathematically*, for example, learners are seen to exhibit procedural and conceptual mathematics knowledge whereas during *simplifying* and *validating*, learners are seen to articulate and justify assumptions they make and may not draw overtly on mathematical knowledge at all. Because carrying out the sub-processes successfully is critical to constructing a viable mathematical model, they are styled as *modeling competencies* (Maaß, 2006). Modeling competencies are learning objectives in their own right and a major goal of research in modeling is understanding how a teacher, who is using modeling problems to teach mathematics (or to teach modeling), can scaffold and thereby promote learners' modeling competencies.

Scaffolding in mathematical modeling ought to be contingent.

Scaffolding a learner as they develop and validate a model has two goals. The local goal is helping the modeler arrive at a viable model for a particular problem. The global goal is promoting competencies that can be used in other problems. Both are challenging because facilitators need to focus on learners' current knowledge and understanding as it is expressed within a given sub-process of the MMC (Blum & Borromeo Ferri, 2009; Doerr, 2006; Schukajlow et al., 2015; Stender & Kaiser, 2015; Wischgoll et al., 2015). The high-level idea is that because the nature of a learner's engagement in modeling changes across modeling competencies, there are likely to be differing (and specific) moves a facilitator can make that would support each sub-process. Providing hints towards a normatively correct mathematical representation when the learner is mulling over which variables are important to include robs her of the modeling experience and does little to cultivate competencies. Investigating this conjecture calls for a view of scaffolding suitable for studying learners' productions and their relations to facilitator moves at a within-task grain size, rather than broader views that take into account classroom-level organization or cross-lesson supports (Anghileri, 2006). For these reasons, the active trend in modeling research is to adopt a Vygotskian view of scaffolding as an interactive process between a teacher and a learner that gives support to the learner as she works on a task she might not otherwise be able to accomplish (van de Pol et al., 2010, p. 274).

Building on the scaffolding means and intentions framework (van de Pol et al., 2010; van de Pol et al., 2015), Stender and Kaiser (2015) assumed that scaffolding the modeling process may be productive under three conditions: the learner has disengaged (and therefore requires motivation to re-engage in the problem), the learner asks a question, or the learner has been working unproductively for an appreciable time and does not realize it. The latter case presents the most challenging aspect of designing and evaluating scaffolding moves. Effective in-the-moment scaffolding is *contingent*, meaning that the proffered support increases facilitator control when the learner is struggling and decreases control when the learner is succeeding. In some studies, contingency is conceived as being along three-point ordinal scale (van de Pol et al., 2015). Çakmak Gürel (2023) examined the interplay between teachers' participation structures and their scaffolding methods and found that the level of support could vary according to modeling competency. These findings did not directly relate level of support to modeling competency, instead showing that scaffolding method is mediated by the teachers' preferred form of engagement in the classroom. Additionally, modeling tasks can be quite open and learners' engagement in the modeling process is idiosyncratic, based in part on their highly individual previous knowledge and experiences (Borromeo Ferri, 2006; Stillman, 2000). Thus, contingency for scaffolding modeling processes means adapting support to be responsive to the particularities of a learner's constructed knowledge and how it manifests during the modeling process, not only attending to the accuracy of learners' intermediate productions – requiring the facilitator to engage in diagnostic activities before intervening (Kaiser & Stender, 2013).

To address the research need for analyzing contingent support, Stender (2016) developed a framework to capture contingent interventions in learners' modeling processes that are responsive to a learner's current conceptual and (partially formed) mathematical models of the real world scenario, anticipate the specific cognitive needs of the learner, and are calibrated to provide minimal in-the-moment support such that the learner will retain control of their modeling process (excerpt in Table 2). We focus on the "B3 Codes", which classify the contingent moves. Stender and Kaiser (2015) found that requesting leaners to summarize the work they'd done thus far (code B3.1 Work Status) enabled them to continue working independently or aided the facilitator in diagnosing their work to proffer further supports, regardless of how far along the learners were in model development. Stender and Kaiser (2015) also found some expected associations between scaffolding moves and particular modeling competencies. Thus, some scaffolding moves could be competency-general while others may be capable of promoting specific competencies. Stender and Kaiser (2015) also cautioned that it was not always possible for them to determine success of an intervention because there wasn't sufficient information in the students' work. They focused on only on the few minutes before and after the facilitators' intervention into a few focal groups' work and on normative correctness of the students' models. In this study, we used task-based cognitive interviews to generate facilitator-learner interactions that could be analyzed for the extent that scaffolding moves promote modeling competencies. This maximized the amount of information available to the facilitator for informing which moves to attempt and to the analysis for examining the impact of the proffered support. We address the research question: *Which modeling competencies were more frequently elicited by which kinds of scaffolding moves?*

Table 5 Scaffolding moves (Stender & Kaiser, 2015), fitted with instances from this study.

Code Name	Description	Rule to use	Example
B3.1 Work status	Learner asked to describe current work status or what they are currently working on	Can be a direct question or implicit; Intended to orient facilitator to the learner's reasoning	Can you summarize what you have done here, so far? Can you share what you're thinking about?
B3.6 Prompt to include real-world aspects	Learner asked or encouraged to include a certain aspect	Learner asked to add an aspect to the model. Can be used to increase complexity or to draw attention to specific variable or quantity	Are there any factors that negatively influence the number of current infections?
B3.10 Request reason or explanation	Interviewer requests a reason, explanation, or justification	The reason can be about assumptions made, refer to algebra steps, or to the whole modeling process	Why did you choose multiplication here? What leads you to think that way?

Methods

We used explanatory sequential mixed methods (Creswell, 2014). We deductively coded task-based interviews according to the modeling competencies framework for participants' modeling processes and the contingent scaffolding framework for interviewer moves. The quantitative analysis used mixed logistic regression model with a participant random effect.

Data Collection

Twenty four undergraduate STEM majors at a large southern university were recruited from differential equations courses or courses listing it as a pre-requisite to participate in a set of 10 hour-long task based interviews conducted over zoom. Each participant worked on between four and eleven modeling tasks across the sessions. The modeling tasks had well-defined goals and ill-defined answers (Yeo, 2007) and were designed based on canonical problems from differential equations featuring feedback loops. We studied participants' model construction (*simplifying & structuring, mathematizing*), interpretations of models (*interpreting*), and justifications of model adequacy (*validating*) and ignored *understanding* and *working mathematically*. The tasks were given as written statements so the *understanding* competency is primarily indicated by "reading the problem statement" (Czocher, 2016), and occurs without contingent scaffolding. Additionally, many of the resulting differential equations models cannot be solved analytically, so we did not ask for their solutions (also, contingent support would be highly tailored to the mathematical content instead of participants' modeling needs).

The tasks were sequenced so that later tasks presented scenarios whose mathematical structures subsumed the structures of earlier tasks. The first 4 tasks included embedded scaffolding (sub-tasks) oriented towards learners' quantitative reasoning to aid them in constructing or transferring quantitative structures to the task scenario (Moore et al., 2022; Thompson, 2011). The remaining tasks did not include embedded scaffolding and featured only contingent scaffolding provided by the interviewer. Not every participant saw all tasks, depending on how "far" they got through the trajectory, which was based on their capacity to work autonomously on the tasks. A lead interviewer and a witness from the research team were present during each interview (Steffe & Thompson, 2000). The interviewer intervened in the participants' modeling process if the participant requested help, if it seemed that the participant got stuck, or to generate and test conjectures about the participants' ways of reasoning about the mathematical or real-world aspects of the task. The probing questions were designed to focus on aspects of quantities and quantitative reasoning, but overall interviewer turns were formulated so they could map to the scaffolding moves framework. The alignment between interviewer moves and the scaffolding framework was achieved through several rounds of pilot interviews and subsequent analysis, not reported here. In this paper, we consider only the tasks without embedded scaffolding to isolate the influence of contingent scaffolding. The dataset for this study comprised 51 hour-long modeling sessions.

Table 6 Summary of interview tasks

Task Name	Intended Canonical Model	n	Median Task Time
Tropical Fish	Contaminated tank	18	1:10:11
Tuberculosis	Two compartment disease transmission	17	0:47:07
Ebola	Three compartment disease transmission	11	0:53:03
Bobcats & Rabbits	Two-species predator prey	2	1:04:44
Diffusion	Fick's first law (one dimension)	2	0:20:18
Kidneys	Dialysis across a one-dimensional membrane	1	0:42:26

Data Analysis

Qualitative analysis proceeded with deductive coding procedures based on pre-defined, published codes for engagement in modeling processes (MMC codes) and contingent scaffolding moves (B3 codes). Participant engagement and interviewer moves were coded separately. To code for participant engagement, we viewed the videos in MaxQDA and assigned an MMC code if an indicator for that code could be observed in the participant's speech or writing. The MMC codes were not mutually exclusive; a participant's actions at a given time could indicate both *interpreting* and *validating*, for example. Start and stop times for the codes were determined independently of the start and stop times for other codes. Because much of the modeling process takes place in the mind of the modeler, when a modeler "really starts" to make assumptions is not accessible information. Thus, we assigned a timestamp to the earliest moment there was verbal or written evidence of an indicator for the code. To code interviewer moves, the recordings were segmented into durations of 30s and each segment was assigned each intervention code the segment evidenced. In this way, the scaffolding moves codebook produced time series corresponding to if the code is "on" or "off" during each 30s segment. Pilot studies (not reported here) adapted the scaffolding moves codebook to the research setting. We then mapped each instance of a scaffolding move to the MMC by identifying which stage of model construction the intervention referenced (situation model, real model, mathematical model, mathematical results, real results). For example, the move "Let's work on just the susceptible and infectious. And we'll pick back up the removed later" was coded as B3.23 Narrowing scope because the interviewer suggested the participant to ignore the removed population. Because the move referred to the distinct populations identified by the learner (susceptible, infectious, recovered), we inferred it to refer to the Real Model stage of the MMC. In this way, we obtained a description of the move and its modeling-stage referent.

Due to the complexity of the codebooks, total duration of the 51 sessions, and planned quantitative models, our primary concerns about reliability were the chance of missing codable segments and consistent application of the codebooks across participants and tasks. Thus, two analysts independently coded each event. To mitigate coder drift, six pairs of analysts were formed from four research team members and rotated. Pairs met regularly to reconcile codebook application and resolve disagreements based solely on code definitions. Since neither codebook was mutually exclusive, multiple codes could be added to the same data segment if warranted. Remaining disagreements were considered by the whole group and resolved by consensus.

To investigate the impact of the contingent scaffolding moves (B3 codes) on the modeling competencies (MMC phases), at each instance of a B3 and for each MMC phase, we determined if the competency was observed during the subsequent two-minute window. If the competency was observed at least once in the window, we said the B3 move was taken up by the participant. Combining the results across tasks and participants, we estimated the probability of uptake for each competency and set of B3 codes.

As seen in Table 4, the number of instances observed varied considerably by competency. As expected, *understanding* (233) and *working mathematically* (261) competencies were rarely elicited, and so were excluded from analysis. However, *validating* was observed nearly twice as often as *interpreting*. To account for variation, we estimated a base probability under the null or no effect model where the null assumption is that MMC codes are uniformly distributed across the sessions. Under the null model, we let X_k be the number of instances of MMC code k in a given two-minute window. Then X_k follows a binomial distribution with size equal to the total number of instances of code k and probability equal to 2 divided by the total combined time of the sessions. The base probability is $p_{b_k} = P(X_k > 0)$. We then normalized the probability of uptake, by computing an odds ratio:

$$\text{OR} = \frac{(p/(1-p)}{(p_{b_k}/(1-p_{b_k})}.$$

If the uptake probability equals the expected value under the null model, then $OR = 1$, and indicates the contingent scaffolding move is not associated with an increased uptake of competency k. On the other hand, $OR > 1$ indicates that the contingent scaffolding move promotes competency k.

Initial investigations indicated that the proportion of uptake depended on the modeling stages referred to. Hence, we used the analysis of B3 code instances in terms of model construction stage to collapse to four broad categories: Real (scaffolding move refers to situation and/or real model), Math (scaffolding move refers to mathematical model), Both (scaffolding move refers to both Real and Math), and Neither (scaffolding move refers to neither Math nor Real). Only 15 instances of intervention codes referring only to math result, 14 referring only to real result, and 9 referring to both were observed. We excluded the low counts. Instances of B3.10 Request Reason or Explanation could fall into distinct secondary categories, depending on the stage referred to by the specific move at that time in the interview.

Results and Interpretations

Figure 1 shows variation in the odds of uptake of each competency following interviewer moves referring to the stages of model construction. To fully characterize the differences observed in Figure 1, for each competency we fit a mixed logistic regression model with a participant random effect, $\boldsymbol{u_i}$, to account for dependence (Agresti, 2012):

$$\log\left(\frac{p}{1-p}\right) = \beta_0 + \beta_1 \text{Real} + \beta_2 \text{Math} + \beta_3 \text{Both} + u_i$$

Results of the models are shown in the Relationship with Stage Referred to column of Table 4. "$A < B$" indicates that the Odds of uptake for that competency is significantly less for stage A than B. "$A = B$" indicates no significant difference.

Figure 3 Odds ratio of uptake of modeling competency by stage of scaffolding move referred to.

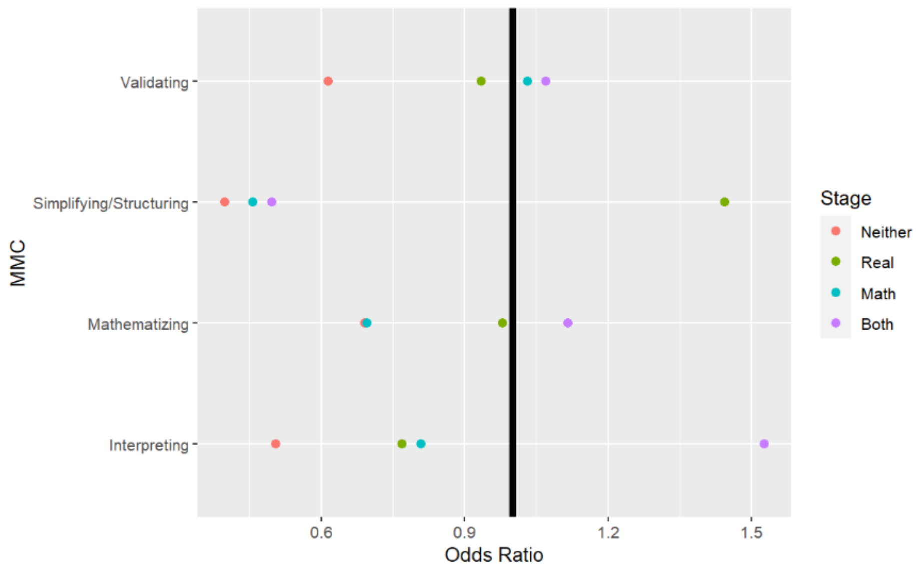

Table 7 Summary of competencies, relationship with stages referred to, and log odds for three of the contingent scaffolding moves from Table 2

Competency	Count	Relationship Stage Referred to	B3.1	B3.6	B3.10
Simplifying & Structuring	1218	Neither = Math = Both < Real	-0.289	0.027	-0.100
Mathematizing	929	Neither = Math < Real = Both	0.391	0.943	-.656
Interpreting	787	Neither< Real = Math < Both	0.208	0.408	0.192
Validating	1428	Neither< Real = Math = Both	0.552	-0.139	-0.101

As expected, contingent scaffolding moves referring to the Real stages were much more likely to elicit *simplifying/structuring* than those referring to other stages. Specifically, the odds are 1.4 times expectations under the null model. Scaffolding moves classified as Real and Both had the greatest odds of eliciting *mathematizing*, which is sensible because the mathematizing competency bridges thinking about real-world conditions and assumptions to reasoning about mathematical properties and parameters (Zbiek & Conner, 2006) it also, according to the MMC, ought to follow chronologically from thinking about real-world conditions and assumptions. Additionally, the odds of eliciting the *interpreting* competency are greatest when a move refers to Both (Math and Real) stages of model construction. This makes sense theoretically because *interpreting* competency, like *mathematizing*, bridges real-world and mathematical knowledge. Finally, a wide range of scaffolding moves can elicit *validating* – as long as the move refers to either Real or Math or Both -- corroborating claims in previous work that validating arises throughout the modeling process in response to multiple knowledge sources (Czocher, 2018; Ishibashi & Uegatani, 2022).

Due to space constraints, we discuss only three of the 48 B3 Codes. The log odds (Table 4) show increased rates of *mathematizing, interpreting,* and *validating* followed a request for the participant to summarize their current work (B3.1 Work Status), relative to the base rates of occurrence for these competencies, while it is not an effective move for eliciting *simplifying & structuring*. Because *mathematizing* and *interpreting* are likely to follow B3.6 Prompt to Include Aspects (of the real world), moves which primarily refer to Real stages of model construction, it seems that prompting learners to attend to their ideas about how the world works leads to competencies associated with increasing model complexity and scope. We are not surprised to see a positive association between B3.10 Request Reason or Explanation and *interpreting*, because asking a learner their rationale for a modeling decision would often necessitate interpreting situationally relevant meanings. However, its negative association with *validating*, which shares aspects of justifying and explaining (Czocher et al., 2018) was surprising. We had anticipated that B3.10 would increase elicitation of *validating*.

Finally, there were several scaffolding moves for which the odds of eliciting any of the four focal competencies were less than the base rate predicted by the null model. The move 3.11p Math Procedure was one such example. In contrast, the move B3.14 Suggestion for Action, Related to Content (focus on directing attention to variables and quantities) elicited all four focal competencies more than expected. Thus, it seems attending to the role of quantities and quantitative reasoning promotes modeling competencies.

Conclusions

We conclude that the logistic regression model adequately captures expected relationships between contingent scaffolding moves, their referents relative to the MMC, and elicitation of modeling competencies. We view it as an initial model capable of sustaining claims about (a) the compatibility of the analytic frameworks and (b) predicting which moves are capable of eliciting which modeling competencies. Importantly, the regression model quantifies variation and differences across competencies, scaffolding moves, and the likelihoods of their interactions. This is a promising advance for work seeking to understand the impact of modeling-forward learning environments on learners' modeling competencies. The approach retains the nuance of the critical aspects of contingent scaffolding, as articulated by the scaffolding moves framework, while offering a vision of the larger cross-participant and cross-task patterns. One limitation is that presently, it is unclear the extent to which the random effect model adequately accounts for the person-dependence of each competency. Future iterations would improve on this uncertainty. In the end, the holy grail is a model capable of informing facilitators which contingent scaffolding moves are most and least likely to promote which competencies so they may focus on developing powerful moves. Due to the large number of codes, we clumped them according to the stage they referred to. Future iterations can examine individual moves to understand which perform similarly with respect to competency elicitation and distill move types into strategies.

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STRATEGY COMPARISON AS A PRECURSOR TO STRATEGY ADOPTION IN THE CONTEXT OF INTEGER MULTIPLICATION

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Our study explored how fifth-grade students compare and adopt integer-multiplication strategies. They solved ten problems before instruction and answered seven reflective questions to reveal new strategies. Responses were analyzed across four dimensions: Focusing, Judging, Comparing, and Adopting. The findings show students' intuitive grasp of negative numbers and their willingness to engage with strategies. Teachers can support adoption and enhance understanding by attending to students' focus, judgment, and comparison processes.

Keywords: Algebra and Algebraic Thinking, Number Concepts and Operations, Elementary Education

Perspectives and Theoretical Framework

Comparing and discussing solution methods is recognized as an effective learning strategy in mathematics (Rittle-Johnson & Star, 2007; 2009; Silver et al., 2005). The National Council of Teachers of Mathematics Standards also endorses this practice (NCTM, 1989; 2000). However, concerns exist regarding potential overreliance on teacher guidance versus student-led thinking (Richland et al., 2007; Rittle-Johnson & Star, 2007). Effective implementation of comparing and discussing solution methods often involves presenting two examples with instructional support (Rittle-Johnson & Star, 2009). Our study builds on this research, focusing on integer multiplication and strategy adoption.

Previous work by Rittle-Johnson & Star (2007; 2009) explored how comparisons affect procedural knowledge, flexibility, and conceptual knowledge. They found that students who compared examples were more likely to transfer methods to novel tasks, enhancing conceptual knowledge and flexibility (Rittle-Johnson & Star, 2007; Rittle-Johnson, Star, & Durkin, 2020). Comparing correct and incorrect methods helped students identify important characteristics and differentiate usefulness in various situations (Rittle-Johnson & Star, 2007). Students who compared strategies focused more on methods for solving rather than final solutions, thus attending to the key similarities and differences between methods, which influenced subsequent judgment (Rittle-Johnson & Star, 2007).

Research in cognitive science suggests that exposure to novel transfer problems prompts adaptation (Paas & Van Merrienboer, 1994; Rittle-Johnson & Star, 2009). Analyzing examples with unfamiliar solution strategies and answering similar problems aids in studying the adoption process. Researchers must assess the extent to which learners change strategies to accommodate task differences (Lamb et al., 2023). Providing opportunities for students to explore multiple strategies may enhance learning. Identifying how students focus and judge examples during comparison tasks can offer insights into adopted strategies for future problems.

Our study aims to understand how students compare strategies and their subsequent use in tasks, particularly in the context of integer multiplication. Carpenter and Wessman-Enzinger

(2018) highlighted the inventive nature of students' pre-instructional strategies in early integer multiplication, making it a valuable domain for comparison and adoption research. We explore how focusing, judging, and comparing influence strategy adoption through two main research questions: How do students attend to the strategies and descriptions of others and to what extent is that attention influenced by similarities to their own reasoning? Additionally, to what level and in what ways do students adopt the strategies of other students in the domain of integer multiplication?

Methods

Seven fifth graders from two different schools were interviewed for our study. We share the results of our interview with one student, Nico, because of the variations in adoption he exhibited. Throughout the interview, Nico was asked to answer open number multiplication problems (e.g., $\square$ x -3 = -18) before being asked to explain his strategy (adapted from Bishop et al., 2018; Lamb et al., 2018). The unknown and the distribution of negatives varied across all problems (e.g., -4 x 3 = $\square$, 5 x $\square$ = -20, $\square$ x -3 = 9). Interspersed throughout the interview were opportunities for the students to reflect on sample strategies that we provided to them. We developed these strategies by adapting the strategies of students from our own pilots and from the strategies described by Carpenter and Wessman-Enzinger (2018). Each of these strategies included the name of a fictionalized student and a brief description of their work, written in first person natural language. For clarity, the interviewed students will continue to be referred to as the students and those whose strategies we introduced will henceforth be referred to as the Sample(s). After a student had completed two or three tasks, we presented the student with a Sample's strategy for answering the most recent question(s). Once a student had read a Sample's description aloud, they were asked to describe how the Sample was thinking and how it was similar to or different from their own reasoning. The student was then asked to rate the Sample's strategy on a modified scale (Rittle-Johnson, Star, & Durkin, 2012) with the following options: *Very good way; OK to do, but not a very good way; Not OK to do; Not sure if this is OK to do.* Student responses were analyzed on three a priori dimensions – Focusing, Judging, and Comparing (adapted from Rittle-Johnson & Star, 2007) and the induced adoption dimension (see Table 1 below). Researchers also took note of the multiplication strategies, following Carpenter & Wessman–Enzinger's (2018) framework, which were used to identify changes over the course of the interviews.

Table 1: Dimensions and Codes

Dimension	Code	Description. Students...
Focusing	On the Method	Focus on how a Sample reasons
	On the Answer	Focus mostly on the result of the Sample's work
Judging	Efficiency	Judge clarity or speed
	Confidence	Judge a Sample's certainty and/or confidence
	Accuracy	Judge correct or incorrect use of a rule

Comparing	Between Samples	Compare two Samples to one another
	To Own Work	Compare one or both Samples to their own work
Adopting	Authentic Adoption	Use a Sample's strategy
	Quasi-Adoption	Use a strategy that is modified from a Sample
	Lack of Evidence of Adoption	Use a strategy that is not from a Sample

Results

From Nico's written work, explanations, and use of provided materials, we found that he aligned closely with many of our dimensions and codes. Although all our codes are presented in Table 1, the following results and discussion only include the dimensions and codes that fit Nico's responses. Nico focused predominantly on Sample methods, not their answers, and judged those methods on matters of confidence and accuracy. For question nine (-2 x □ = 6), he had episodes of judging methods based on a Sample's confidence and accuracy. Sample Sofia had identified a counter movement technique that yielded not only the correct answer but also the same answer that Nico had found. However, Sample Sofia also conveyed a lack of confidence in her strategy when she shared, "Wait, is it six? Can you do that? But they're negatives. Hmmmm, I'm not positive." Nico thus chose to assess Sample Sofia's work as *OK to do, but not a very good way* because "she isn't sure, and you should always try to be sure." This response suggests that uncertainty was important to Nico in judging the worth of a strategy. The act of gauging clarity and confidence may be a result of classroom practices and teacher feedback. For this same problem, Nico judged accuracy when he noted that Sample Louis' methods were flawed because he had "treated negative numbers as subtraction, which would be okay if this was adding." Sample Louis had answered the problem by filling the box with 8 and justifying his choice by saying that 8 - 2 was 6. Throughout the interview, Nico noted that treating a negative as subtraction was inaccurate. This identification seemed to affect how he assessed Sample strategies.

The Sample responses to question five (□ x -3 = -18) sparked an episode in which Nico compared the Samples to each other and to his own work. His response that Sample Fatima and Sample Gabriel "thought the same way as me…[but] I think that Gabriel did a miscalculation" seemed to double down on his choice to focus and judge the method primarily over the answer. Comparing them to one another presented the idea that Gabriel's "strategy was good…but result wasn't." Nevertheless, Nico marked both methods as *very good way*, implying that the miscalculation did not affect the quality of the strategy. Earlier in the interview, Nico had used the same methods of repeated addition that were used by both Samples and comparing his work to theirs seemed to serve as a method of validation.

The path to adopting a strategy seems to relate to a chain of focus, judgment, and comparison. What Nico attended to and how he did so ultimately informed the degree to which he made sense of and replicated the strategy in novel tasks. When answering problems four (5 x □ = -20) and five, Nico explicitly named Sample Abbi as influential to his thought process. Sample Abbi used the language of groups in her descriptions, particularly when working with negatives which she described as groups of negatives. Nico suggested that Abbi would have read problem four as "five groups of negative four" and problem five would similarly be read as "how many groups of negative three would you need to get negative eighteen." Nico's work on questions one through three suggested that he was relying on multiplication facts and making attempts to induce a rule for how to work with negatives, a rule of which he was uncertain. In being exposed to Sample Abbi's work, Nico was able to identify a legitimate strategy that was based on existing techniques with which he was comfortable, thus adopting authentically.

Discussion and Implications

The results of this study illustrate some of the features found in the work of Rittle-Johnson & Star (2007; 2009) and the strategy adoption that may follow. We hypothesize that Nico attended to the reasoning of other students because he was looking to identify familiar patterns as well as a justification of his own intuition. His tendency to judge accuracy suggested that he was looking to uncover and understand methods that were seemingly familiar. At a pre-instructional stage, he was hesitant to completely discount or support a solution or a strategy, but he was comfortable judging the elements of a strategy. His judgment, however, was not just a matter of accuracy, but also a matter of the Sample's confidence, which affected his perception of their work and his willingness to adopt that strategy.

When comparing, the decision to weigh one element of a strategy more than another may have been due to comfort with procedural multiplication skills (such as repeated addition), fact memorization, or could have resulted from comfort with strategies used throughout a Sample's method. In line with the findings of Rittle-Johnson & Star (2009), Nico benefited from viewing two Samples at once as he was able to compare methods more deeply than if he had seen them one at a time. Attempting to uncover the mathematical thinking of the Samples seemed to suggest that Nico was prepared to go beyond the rules of integer multiplication, looking for potential traces of the underlying relationships between integers under multiplication.

The decision to adopt a strategy was the result of many considerations for Nico, such as whether he felt like he could reproduce the strategy in a similar setting. However, the decision to adopt, or not, could also have been based on the need for, or lack of, a strategy. Adoption is informed by all three of the a priori dimensions but is perhaps most influenced by the comparison stage. In some sense, adoption is built upon the constructivist concepts of accommodation and assimilation, in which decisions are made based on whether and how to accept new environmental stimuli into an existing conceptual understanding. In the comparison stage, students get an opportunity to identify how their work aligns with that of others. By identifying the varying degrees of similarities and differences, Nico had the potential to create a new understanding that combines the most salient and perceived accurate elements of each strategy. This process is iterative as it continues to occur with exposure to every new Sample as well as with the ideas that arise when a student attempts to reason through a problem themselves.

Nico held some intuition of negative numbers and was willing to attempt to reason through integer multiplication before and after exposure to Sample strategies. There are rich opportunities for growth if instructors better understand what students are coming into the classroom knowing and thinking. Attending to how a student focuses, judges, and compares their strategies to others may reveal what they are comfortable accepting as mathematical fact as well as the content knowledge and intuition they already possess. Furthermore, interpreting how those factors result in their level of adoption may inform how to enact teaching practices so that all students have the opportunity to authentically adopt rich strategies. Future work on this project might benefit from tracing the process of adoption across multiple mathematical topics in a classroom setting. We may also wish to explore other factors that contribute to flexibility and the willingness to adopt strategies, such as mathematics maturity and mathematics anxiety.

A thorough understanding of the intuitions and thought processes of students is vital to the future of mathematics education. Students come to the classroom with vast arrays of prior knowledge and experiences. The ways in which they employ those in the classroom is of particular interest when considering how they might try to incorporate new strategies into their own problem-solving schemes. We envision a future that highlights student thinking and the teaching methods that might nurture student learning.

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PLAYFUL MATH: WHEN STUDENT AUTHORING GENERATES NOVEL MATHEMATICS

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We propose the construct of playful math to support instances of student authoring. Authoring positions students as authors of mathematics in an environment in which students and teachers meet as epistemological equals. By emphasizing student agency and autonomy, playful math encourages students to explore self-selected goals as they design novel problems for one another or for their teachers. We introduce two types of novel mathematics that emerged from student authoring, Unfamiliar Problem and Catalyst, and share one example of each to envision a mathematics education future that celebrates student authoring.

Keywords: Algebra and Algebraic Thinking, Cognition, Problem-Based Learning.

Supporting students to develop and solve their own problems can enhance creativity, understanding, and positive attitudes towards mathematics (Kaur & Rosli, 2021; Kontorovich et al., 2012). Problem posing is seldom considered in relation to creating new mathematics, but some researchers have written about the experience of learning new ideas from their students' activity. For instance, Norton and Flanagan (2022) described how the ideas they developed about nested number sequences and logarithms as maps between multiplicative worlds were informed by their research on children's mathematics, and Ellis (2022) noted that her participants' mathematics "served as a source of novel mathematics for me as a researcher, as it could also do for teachers" (p. 24). We propose that student authoring of mathematics can create opportunities for both students and their instructors to experience new ideas, and that one way to foster authoring is through playful math. By *authoring of mathematics*, we refer to students producing something original (Cheng et al., 2022), using their mathematical voices to "enquire, interrogate, and reflect upon what is being learned" (Povey et al., 1999, p. 243). This use of authoring draws on Povey et al.'s (1999) notion of author/ity, in which teachers and students consider themselves to be members of a knowledge-making community where they "meet as epistemological equals" (p. 234). This perspective positions students as creators, not just doers, of mathematics.

Playful math describes the activities and features of an instructional environment that can facilitate mathematical play (Ellis et al., 2022). This can include task features, instructional moves, and engagement with artifacts. In playful math, students have agency to explore self-selected goals and to author novel problems. We present two examples of student authoring that introduced new mathematics both for the students and for us. We distinguish two types of new mathematics that can emerge from these contexts, Unfamiliar Problem and Catalyst.

Problem Posing and Mathematical Play

Problem-posing tasks are ones that "require teachers or students to generate new problems and questions based either on given situations or on mathematical expressions or diagrams" (Cai et al., 2020, p. 2). Problem posing can counteract the belief that there is one right way to do mathematics, as there is no one "right" question to ask (Palmér & van Bommel, 2020). It can also promote a sense of agency (Brown & Walter, 2004) and can improve motivation and creativity (Kontorovich et al., 2012). We have developed playful math environments as a vehicle for fostering student agency and author/ity. When students experience author/ity, they author problems that raise new mathematics not only for them, but also for us as teacher-researchers.

Defining and Designing for Mathematical Play

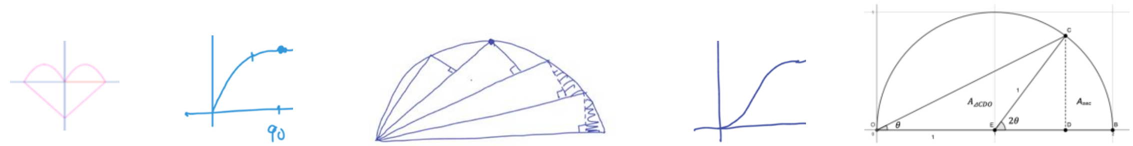
We define mathematical play to include (a) agency in exploration, (b) self-selection of goals, (c) immersion, and (d) enjoyment (Ellis et al., 2022). *Agency in exploration* means that students choose whether and how to participate (Huizinga, 1955) and how to accomplish their goals (Jasien & Horn, 2018). *Self-selection of goals* acknowledges that a learner's agency in determining goals (or sub-goals) is crucial for play (Dewey, 1916/1966). The final two traits are *immersion* and *enjoyment.* Mathematical play is imaginative and creative (Featherstone, 2000), and most accounts of students' mathematical play mention enjoyment (e.g., Sukstrienwong, 2018). Mathematical play can support experimentation, reflection, and persistence (Barab et al., 2010; Gresalfi et al., 2018), and it can provide a productive route for exploring and conjecturing (Mason, 2019; Williams-Pierce, 2019). Given these benefits, we set out to see if we could encourage mathematical play for secondary and undergraduate students.

We have established playful math five design principles to encourage (but not guarantee) mathematical play. They are (1) enable free exploration within constraints; (2) engender anticipation within the task; (3) provide a method for intrinsic feedback; (4) offer meaningful challenge while still being feasible; and (5) allow the student to act as both designer and player. As an example, we draw on an activity in which students investigate growing shapes, graphing a shape's area compared to its length as it sweeps left to right. To playify the task, we created the Guess My Shape game, in which students create secret shapes of their choice (design principles 1 and 5), construct graphs comparing length and area (principles 2, 4, and 5), and challenge each other to determine the shape based on the graph alone (or vice versa; principles 2, 3, and 4). Our principles are consistent with several features of problem-posing tasks, but the open nature of the Guess My Shape game offers greater agency than typical tasks to support author/ity and enable students to author problems reflecting their own mathematical interests.

Data Examples: Sector Areas and Vertical Line Segments

Unfamiliar Problem: Determining Areas in a Semicircle

In the following example, Phyllis and Ryan (secondary pre-service teachers) decided to create a heart shape (Figure 2a). They imagined a line segment on the x-axis that swept counterclockwise, rotating 360° to sweep out the shape. The task for the other students was to graph the area swept as a function of the angle swept by the line segment. Phyllis and Ryan reasoned that the initial part of the graph, from 0° to 90°, would increase at a decreasing rate (Figure 2b). However, when the other students encountered the challenge, they thought that the area would first increase at an increasing rate from 0° to the peak of the semi-circle, and then increase at a decreasing rate from the peak to 90°. To determine this, they created equiangular partitions and reasoned perceptually about the rates of change (Figure 2c), deciding that the area should be "bigger and then smaller" (Figure 2d).



(a) (b) (c) (d) (e)

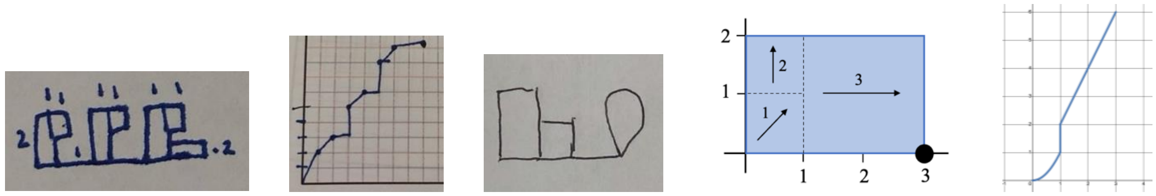
Figure 2: Heart shape (a), Phyllis and Ryan's graph (b), partitioning the semi-circle (c), Meredith and Toby's graph (d), and a semicircle with radius 1 (e)

When the groups compared their solutions, they resolved the discrepancy by redrawing a more precise version of Figure 2c with smaller partitions. They concluded that the area indeed increased at a decreasing rate throughout the semi-circle, but they acknowledged that this decision was based on a perceptual judgement. The students' disagreement led us to realize that we did not know how to directly compute the area of these equiangular portions. We wondered how to find the area between two non-radii chords without computing a double integral in polar coordinates. Thus, the students' authoring led to a novel Unfamiliar Problem for us. An Unfamiliar Problem is a problem addressing a new mathematical idea or challenge for the problem-solver, in this case, us as the teacher-researchers. Certainly, the mathematical ideas in the problem are not novel, but we experienced them as unfamiliar in that we were not aware of a solution method. We solved the problem by drawing a semicircle whose radius is 1 (Figure 2e). Denote by A the area covered by $\angle COB$, which can be decomposed as the sum of the area of triangle $\varDelta COE$, denoted by $A_{\varDelta COE}$, and the area of the sector corresponding to $\angle CEB$, denoted by A_{sec}. If we take OE as the base of $\varDelta COE$, the length of the height is $\overline{CD}$, which is $\sin(2\theta)$. So, we find $A_{\varDelta COE} = \frac{1}{2}(1)\sin(2\theta)$ and $A_{sec} = \frac{1}{2}r^2\alpha = \frac{1}{2}(1)^2(2\theta) = \theta$, hence $A = \frac{1}{2}\sin(2\theta) + \theta$. This function does indeed increase at a decreasing rate from 0 to $\frac{\pi}{2}$.

Unfamiliar problems can emerge when students have the freedom to explore directions of their own interest. They introduce genuine problem-solving experiences for one another and, in this case, also for us as teacher-researchers. Even though the mathematics was not novel from the perspective of the field, we found the problem to be interesting and worth exploring. Unfamiliar problems create problem-*solving* experiences, rather than problem-*posing* experiences.

Catalyst: Vertical Line Segments

The second example comes from teaching sessions with three middle-school students, Artemis, Apollo, and Francis, who had limited familiarity with graphing or linear functions. In this example, the students decided to create a Guess My Shape challenge for the teacher-researcher (TR), inventing a shape that they called "waves" (Figure 3a). The students graphed the first "wave" correctly, but beginning with the second "wave", they made an iconic translation of the vertical section of their shape directly into the graph, in which the vertical segment represented an increase of 2 square units with no change in the horizontal distance of the graph. They repeated this iconic translation for the final "wave."



(a) (b) (c) (d) (e)

Figure 4: The wave drawing (a), the associated area-length graph (b), TR1's graph (c), up-down square task (d), graph of up-down square task (e)

The vertical line segments created a perturbation for the TR, who wondered how to represent an increase in area without gaining any horizontal length. She proposed a solution in the moment by setting up a new convention of a "bubble", in which there is only one point on the line of horizontal sweeping that nevertheless generates an amount of area (in this case, 2 square units, Figure 3c). This task also resulted in establishing a new convention that any area generated in future tasks should be attached to the line of sweeping, to avoid the difference between the rectangular area in the first "wave" of Figure 3a with that seen in Figure 3c.

The bubble was a spontaneous response to a puzzling situation, but it led us to wonder whether we could create a swept shape that would produce a legitimate vertical segment for its area / length graph. In this case, the students' authoring resulted in an Unfamiliar Problem for themselves, as they tackled the challenge of creating the graph, but it also created a Catalyst for us, in that it provoked a new question: What if an area / length graph *could* have a vertical line segment? What shapes could produce such a segment? A Catalyst is a situation that challenges or reveals an ambiguity about an accepted (or implicit) convention or rule. It can thus engender problem-posing activity, such as the creation of novel sweeping shapes.

We continued to wonder about this question and reasoned that the x-axis quantity would need to stop growing as the area continued to grow. This led to the shape in Figure 3d. In this shape, the square first grows both in length and height, producing area at a constantly changing rate of change (a quadratic graph). Once the square reaches 1 square unit, it then grows up to produce an additional square unit, but without sweeping additional horizontal length, resulting in a vertical line segment. The rectangle then sweeps to the right, producing an additional 4 square units of area at a constant rate (producing a linear graph, Figure 3e). In the graph, the x-axis quantity is the horizontal distance traveled by the dot. We also realized that once the dot stopped moving horizontally, the area could grow up and down multiple times. In creating this problem, we reflected on the fact that the shape of the graph and the trace of the graph are different. This realization led to further problem posing, creating related tasks that incorporated both linear and quadratic growth in the vertical line segments, which can only be distinguished by considering the graph's trace.

Discussion

In both examples, the students experienced Unfamiliar Problems through authoring. However, the novel mathematics that we experienced as teacher-researchers differed. With the semicircle, we experienced an Unfamiliar Problem that required us to devise a solution method we had not previously encountered. The mathematical ideas were not new, but we were challenged to solve a novel (to us) problem. In contrast, the vertical line segment acted as a Catalyst to challenge us to imagine new mathematics. By asking “What if an area / length graph has a vertical line segment?”, we introduced new questions for ourselves, such as “Are there sweeping shapes that could produce such a graph, and if so, what would they look like?” This led to a novel set of problems inspired by up-down square, as well as a consideration of the ways in which two graphs can look identical even as their traces differ.

Student authoring can raise unique challenges for teachers, who may be faced with navigating unfamiliar ideas or puzzling situations while interacting with their students. We acknowledge that this can be difficult. However, we envision author/ity environments in which it is allowable for teachers and students to occasionally shift roles, in which teachers experiencing puzzlement or new learning can be normalized and celebrated, and in which we see our students' activity as sources of new learning for us, just as our instruction can be for them.

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INVESTIGATING CARTESIAN COORDINATE SYSTEM IN HIGH SCHOOL

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This large-scale study investigated high school students' reasoning of spatial and quantitative Cartesian coordinate system, meanings of a point and graphing. By integrating related frameworks, data from 229 students across different grade levels were analyzed through nine open-ended questions. Results revealed that students had critical difficulty in conceiving axes as frames of reference to represent quantities as horizontal and vertical directed distances from the origin in coordinating quantities. A significant number of students carried non-normative meanings for points in terms of multiplicative objects, and relatedly had difficulties in envisioning the graph as emergent trace of multiplicative objects. We discuss the implications of the results for learning, teaching, and curriculum development.

Keywords: Cartesian Coordinate System, Graphing, Multiplicative Object, High School Students

Coordinate systems are one of the most commonly used representational tools in learning and doing mathematics, and in science, technology, mathematics, and engineering (Paoletti et al., 2016; Roth et al., 1999). Cartesian coordinate system (CCS) enables representing attributes of two or more quantities on axes, where uniting their orthogonal projection results in forming a *multiplicative object* in mind. This allows students to conceive and represent the relationship between quantities' values and magnitudes (Thompson et al., 2017), which contributes to the development of covariational reasoning and reasoning about graphs (Moore et al., 2013). CCS also lays the foundations for function and rate of change ideas (Thompson et al., 2017) as well as to reason about ratios and proportional relationships, number systems, geometry, algebra, functions, vectors, and matrix quantities (CCSM, 2010). Therefore, forming a solid understanding of CCS and closely related concepts such as meanings for point and graphing is of great significance, particularly at the high school level.

Despite their significance in mathematics, the construction of coordinate systems receives little instructional time, and is taken for granted by researchers, teachers, and curriculum developers (Lee, 2017). Yet, constructing a coordinate system (Lee, 2020) and plotting a point at the quantitative level is non-trivial (Frank, 2016). Students from middle school to undergraduate level face several difficulties in constructing and interpreting graphs (Moore & Thompson, 2015). Despite demonstrating an understanding of quantitative relationships, preservice teachers struggled in clinical interviews when confronted with unconventional aspects of coordinate systems, such as conceptualizing y as the horizontal and x as the vertical axis. Researchers (e.g., Frank, 2017; Stevens & Moore, 2017; Thompson & Carlson, 2017) suggest that part of these challenges arise from an inability to conceive points as multiplicative objects.

While not exhaustive, the reported difficulties above highlight the need for research on students' meanings of CCS, especially on a large scale and at the high school level. It is in this regard that we investigated the following research questions: 1) How do high school students reason about Cartesian coordinate system and graphs within spatial and quantitative CCS? 2)

What are high school students’ meanings of a point in terms of multiplicative object, point on a graph, and outputs of a function?

Bringing together frameworks related to CCS, this study extends the literature by investigating high school students’ reasonings of CCS in a more comprehensive way through investigating their construction of spatial and quantitative CS, reasoning of a point in terms of multiplicative object, and graphing in spatial and quantitative CS, which are all components of robust and coherent understanding of CCS. Results shed light on high school students’ weaknesses and strengths, hence imparting implications for learning and teaching mathematics at various academic levels.

Conceptual Framework

We used the literature on CCS (e.g., Battista, 2007; Demir, 2012; Knuth, 2000; Lee et al., 2019; Moon, 2019) the framework on the use of multiplicative object (Tasova, 2021), and the framework for reasoning about graphing in spatial and quantitative Cartesian coordinate systems (Paoletti et al., 2018) in juxtaposition to each other, as these ideas relatedly and collectively might provide a more comprehensive picture of high school students’ reasoning about the CCS.

In this study, we define a coordinate system as “a mental system of coordinated measurements [of quantities] obtained through coordinating multiple frames of reference” and a *frame of reference* as “a mental structure through which an individual situates a quantity where the structure is constructed through the process of committing to a reference point, a unit measure, directionality of measure comparison” (Lee et al., 2019, p.82). A quantity is a measurable quality of an object that emerges when conceptualizing a situation by considering the measurable attribute of the object (Thompson, 1994).There are two types of coordinate systems based on their intended uses: spatial and quantitative CS (Lee et al., 2020).

A *spatial CCS* can be constructed by overlaying a reference point and two orthogonal lines passing through reference point onto physical or imagined space. Then quantities are produced by measuring attributes of the space using frames of reference and coordinating such measurements to represent attributes of the objects in the space or situation. For instance, a spatial CCS can be laid onto a Ferris Wheel such that the axle is located at the origin. Then the location of the car at a specific moment is described by Cartesian coordinates found by orthogonal distance from the origin. Constructing *quantitative CCS*, entails extracting quantities from the space a situation or phenomenon occurs and projecting them onto new space. In quantitative CCS, quantities are overlaid onto two orthogonal number lines and a point is formed by uniting their orthogonal projections forming a multiplicative object in the context of graphing (Tasova & Moore, 2020). For instance, in the Ferris Wheel example, height of the car from the ground and time can be extracted from the problem situation and their varying magnitudes are represented on number lines, e.g. time by the horizontal and distance from the ground by the vertical axis. By Cartesian product of distance and time {time x distance}, a two-dimensional CS is constructed which is different than the space containing the problem situation (Foerster, 2005).

Method

In this study phenomenography was used to identify and categorize student’s meanings of CCS as phenomenographic study mainly aims to identify various ways in which people experience, interpret, understand, perceive, or conceptualize a certain phenomenon (Orgill, 2012). In phenomenography, collecting data through open-ended questionnaire is advantageous when the number of participants is relatively high (Han & Ellis, 2019). The study took place in the Spring semester in 2021-22 academic year at a private high school in Istanbul which requires high performance at the national high school entrance exam, indicating high potential and capability in learning and doing mathematics. The sample consisted of 229 volunteer students from grade level 9 to 12 who had former experience with graphing, modeling, and forming various types of functions such as linear, quadratic, and exponential. Therefore, students' such lived experiences (i.e., the phenomenon) provided a rich information base for the study. Students worked on the hardcopies of the inventory in 70-80 min. within their two back-to-back classes. Students had access to computers, tablets, or phones to explore the simulations in some of the questions. Since the study aimed to explore students' reasoning skills with quantities, questions mostly did not include numerical values. Therefore, we provided them with scissors, wire, and papers to use as straight edge as optional tools to use in questions.

Inventory was developed in light of the existing research (e.g., David et al., 2019; Lee et al., 2020; Sencindiver, 2020; Tasova, 2021). After taking expert opinion from four mathematics educators who conducted research on students' understanding of CCS, opinions of three high school mathematics teachers teaching at the same school where the study took place were taken about the content and language of the inventory. Next, we conducted a pilot study with two high school students and revised the inventory again. This way, content's appropriateness of the and the inventory's language were ensured. The final version consists of nine open-ended questions with question 6 through 8 including sub-questions.

In this paper, in lieu of space we report on Q1, highlighting prominent findings on students' construction of CCS. In Q1, students' construction of CCS, more specifically, how students leverage their ways of spatial coordination to coordinate quantitatively was explored. In the problem, two ants (represented by points) move haphazardly in tubes that can be rotated and moved in a dynamic geometry environment. The question was "Can you describe mathematically the locations of the two ants with a single point that moves along with the ants?".

In the analysis, students' responses were categorized using coded analysis mainly based on the frameworks as shown in Table 1. In order to obtain reliable and consistent results, the authors analyzed, reviewed, and discussed each item in the inventory regularly.

Table 1: Table of Specification for the Analysis

Goal	Questions	Analysis
Students' meanings of a point in Cartesian coordinate system as a multiplicative object	Q2, Q3, Q4, Q5, Q7	Framework for representing a multiplicative object in the context of graphing (Tasova & Moore, 2020; Tasova, 2021)
Students' graphing in Cartesian coordinate system	Q6a, Q6b, Q8a (in spatial CS)	Framework for reasoning about graph in spatial Cartesian coordinate system (Paoletti, Lee & Hardison, 2018)
	Q8b, Q9 (in quantitative CS)	

In coded analysis of Q1, students' spatial and quantitative coordination was investigated according to their uses of frames of reference (Joshua et al. 2015). Students established either a spatial or quantitative CS; they gave no answer or incorrect answer.

Results

As depicted in Table 2, results showed that unfortunately 86% of the students couldn't generate a solution for Q1. Only 5% of the students constructed a quantitative CS by spatially orienting the tubes and using them as axes to represent distances with respect to the reference point origin. These students were aware of the quantities to describe location of a point mathematically and how to measure and represent them using a CCS (Figure 1a). On the other hand, 9% of the students formed a spatial CS by laying a CCS onto the figure and assigned coordinates of the points accordingly. These students failed to describe location of the two ants as one single point that moves along with the ants. Instead, they relied on conventional tasks such as finding mid-point and connecting two points by line (Figure 1b). They didn't seem to conceptualize axes as tools to represent the directed distances from the origin and coupling these quantities to represent the location of a point.

Table 2: Frequency and Percentages for Question 1

Responses for Question 1	
Quantitative CS	12 (5%)
Spatial CS	20 (9%)
No Answer	129 (56%)
Incorrect	68 (30%)
Total	229 (100%)

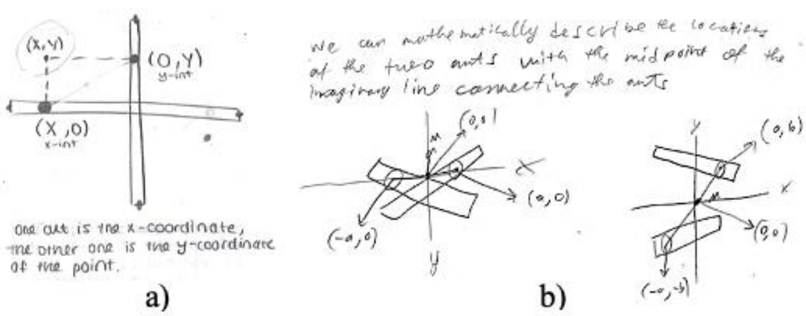

Figure1: a) Quantitative Cartesian CS b) Spatial Cartesian CS

Discussion and Conclusion

Results from Q1 support that constructing a coordinate system by organizing multiple frames of reference is a non-trivial task for high school students even when they have high capability in learning and doing mathematics (Drimalla et al., 2020; Lee et al., 2020). Results showed that students exhibited critical difficulty in conceiving axes as frame of references to represent the horizontal and vertical distance from the origin, thus failed to envision point as union of orthogonal projections of quantities represented on axes. Instead, they relied on procedural tasks that they were familiar from their mathematics classes such as finding mid-point or drawing graphs to depict distance traveled by time. As suggested by Karagöz Akar et al. (2022), results highlighted conceptualizing a point, for instance (*a*, *b*), as combination of directed lengths between origin and (*a*, 0) and origin and (0, *b*) rather than combination of just two labels. Future studies might investigate students' construction and understanding of coordinate systems further through design-based research studies. Finally, although this study provided a picture of 229 students' understanding of CCS, point and graphing in CCS, their meanings might be elaborated more in depth through clinical interviews.

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STRENGTHENING SMALL GROUP LEARNING ENVIRONMENTS

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Keywords: Classroom Discourse, Instructional Activities/Practices, Middle School Education

Background and conceptual framing. PEARL (Peers Engaged as Resources for Learning) research project studied an integrated understanding of effective small group learning environments, drawing on conceptual frameworks addressing demanding and groupworthy tasks (Lotan, 2003; Stein et al., 1996), productive mathematics discourse (Sztajn, Heck, & Malzahn, 2021), and peer cultures of effort and achievement (Hamm et al., 2012). Small group environments offer opportunity to address goals in *Principles to Actions* (NCTM, 2014) and *Common Core State Standards for Mathematics* (NGA, 2010).

Research Question. In what ways do resource lessons providing teacher and student supports and a teacher professional learning program enhance high school Math 1 students' experience of small group learning environments?

Research Design and Findings. The research team combines expertise in mathematics instruction for conceptual understanding and adolescent classroom peer processes. In this naturalistic study, three high school teachers taught four lessons on algebra and statistics topics intended for group work in their Math 1 courses during a baseline year. They engaged in two-day professional learning (PL) program in the following summer focused on high-functioning groupwork and were introduced to re-designed versions of the lessons that incorporated teacher and student supports for engaging with the task, in math discourse, and with peers as partners in learning. During the subsequent year, the teachers taught the re-designed lessons in comparable classes. Following each lesson (at baseline and post-PL), the students individually completed a survey to report on their experience working in small groups. The survey included scales for task, discourse, and peer interaction, as well as a single item on group collaborative focus. Surveys from 328 students in 46 groups in 4 lessons across the 3 classrooms were analyzed using 3-level (students in groups in classrooms) HLM to compare results from the baseline and post-PL. Students' self-report of their groupwork experiences indicated a significant, positive difference in their attention to demands of the task ($M = 4.32$, $SD = .79$ pre vs. $M = 4.51$, $SD = .69$ post; treatment coeff = .23 $p \leq .01$), and group focus ($M = 4.11$, $SD = 1.03$ pre vs. $M = 4.39$, $SD = .77$ post; treatment coeff = .27, $p \leq .01$;). Comparisons of student-rated group discourse and peer interaction were not significantly different pre- versus post-PL. Qualitative evidence from audiorecordings of these students engaged in groupwork will be presented, that provides corroborating illustrations of interaction sequences within groups that demonstrate greater engagement with task demands and stronger collaborative group focus post-PL using the enhanced resource lessons.

Conclusion. The project aims to build stronger theoretical and practical framing of small group learning environments to inform teachers' professional practice. Situated authentically in classrooms, the study attends to mathematics content, student learners, and the learning context. Initial findings reveal opportunities and challenges for teachers to produce and maintain balance within and across three dimensions of small groupwork to deliver on the promise of peer-to-peer learning.

Acknowledgments

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TOWARDS A DYNAMIC NARRATIVE: UNDERSTANDING A CALCULUS STUDENT'S DECISION MAKING ALONGSIDE HER EVOLVING MATH IDENTITY

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This paper juxtaposes Schoenfeld's (2010) decision making theory with the literature on math identity to investigate the ways in which a Calculus student's decision making and math identity co-evolved. Through a detailed analysis of the student's identity stories, goals, orientations, resources, and decisions over the course of a semester, this study illuminates the complex process through which she transitioned from pursuing deep understanding as an aspiring math major to dropping Calculus and admiring math only from afar. Notably, the student's evolving math identity stories were closely connected to her consequential decision making. The findings suggest a plausible mechanism for the co-evolution of students' math identity and decision making, extending Schoenfeld's theory across temporal dimensions.

Keywords: Calculus; Equity, Inclusion, and Diversity; Affect, Emotion, Beliefs, and Attitudes

High attrition rates in STEM fields significantly impact students, families, communities, and the nation (Lee & Ferrare, 2019). Calculus exacerbates this issue by eroding students' confidence (Bressoud et al., 2015) and disproportionately "filtering" students based on gender, race, ethnicity, socio-economic status, and first-generation college status (Seymour & Hunter, 2019). This study examines a Latina Calculus student's recalibration of her math identity alongside her decision making, linking the literature on identity with Schoenfeld's (2010) theory to expand both.

Scholars have varied views on math identity (Langer-Osuna & Esmonde, 2017). This study adopts a narrative approach to foreground historically marginalized students' narrativization of their math-related selfhoods. Gee (2000) defines identity as "being recognized as a 'certain kind of person' in a given context" (p. 99), while Sfard and Prusak (2005) view it as "reifying, endorsable, and significant" (p. 16) stories. Martin (2000) underscores the role of community and institutional forces, framing identity as beliefs about self and math. Prior research suggests that identity can change across time by analyzing students of different age groups (Nasir, 2002) or by focusing on micro-level positioning moves (Langer-Osuna, 2011). This study expands on these approaches by linking identity narratives with decision making mechanisms (explained below) and tracking their co-evolution over time.

Schoenfeld's (2010) theory models how individuals make decisions in the moment: Decisions are shaped by three factors: resources, goals, and orientations. Resources include all elements (e.g., knowledge) available to individuals during decision making, goals encompass both conscious and unconscious aims, and orientations cover "beliefs, dispositions, values, tastes, and preferences" (p. 29). Given the theory's focus on short time frames, there is a lack of studies examining how individuals' goals, orientations, and resources evolve and shape decision making over extended periods. This paper addresses this gap by offering an explicit mechanism for implementing Schoenfeld's theory across temporal dimensions.

Methods

From over 1,511 students enrolled in a first-semester Calculus course for STEM majors at a prestigious U.S. university in Fall 2023, 57 participated in the larger Calculus Learning Experience study. The course included large lectures and smaller discussion sections, with 80%-90% of students having taken Calculus in high school, yet 36.7% either dropped out or received failing grades (C-, D, F, NP). This paper focuses on Grace, a second-year Latina student who demonstrated significant changes in major selection, career path, and approach to learning math within a semester, reflective of the students who failed the course.

The primary data for this case study include video recordings of three 50-minute interviews, observations of lectures and discussions, field notes from 13 informal tutoring sessions, and Grace's written artifacts and survey responses. Interviews involved open-ended questions and problem solving tasks designed to elicit conceptual understanding and deep contemplation of Calculus concepts. Data analysis followed five phases: 1) conducting a holistic review of recordings, interactions, and notes to develop an overview of Grace's major decisions and math identity trajectory; 2) transcribing recordings to detail Grace's self-narratives and utterances during problem solving; 3) developing a conceptual framework grounded in Schoenfeld's theory and the referenced math identity studies; 4) chronologically organizing data to identify emerging themes; and 5) conducting a comprehensive review of all data to ensure accuracy and thoroughness in the analysis and revising the conceptual framework accordingly (see Figure 1).

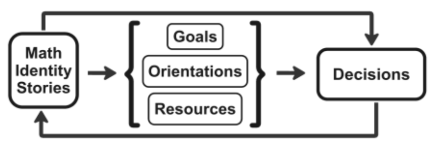

Figure 1: Conceptual framework connecting mathematical identity and decision making

Results

Weeks 1-4: An Aspiring Math Major Who Proactively Sought Deep Understanding

Math Identity Stories. Grace noted that she did not have a solid math foundation in high school but transformed from a state of confusion to "thoroughly enjoy[ing] math" through the Pre-Calculus course she took at the university (Interview, week 14). Although none of her family members pursued STEM-related fields (Field Note, week 2), Grace aspired to double major in math and philosophy (Field Notes, weeks 1 & 3). Her confidence was palpable. She stated, "My relationship with math, before I took this [Calculus] class, was really good. I think I understood, like, all the foundational aspects of math before going into Calculus" (Interview, week 14). She elaborated, "This [Calculus] class is pushing me more towards math because I really enjoy it… I came to understand why there's Pre-Calculus in the first place… I don't think there's a challenging part [about the course] because it all makes sense" (Interview, week 4).

Goals. Grace succinctly summarized her goal: "To understand Calculus!" (Survey, week 1).

Orientations. Grace perceived math as interconnected, logical, and coherent, believing it makes sense and prioritizing understanding in learning (Field Notes, weeks 1-4).

Resources. Grace leveraged her Pre-Calculus foundation, consulted textbooks, actively participated in discussions, derived meaningful insights from lectures, and regularly engaged in in-depth conversations about Calculus with the author (Field Note, week 2; Interview, week 4).

Decisions. Grace actively engaged in the course, sought deeper math understandings, and assisted peers with challenging concepts (Discussion Observations, weeks 2-4). She demonstrated a strong commitment to understanding by posing questions such as “What if…?”, “Why must…?”, “Why don’t you…?”, “Without…, how would you…?” (Field Notes, weeks 1-4).

Weeks 5-9: A Disheartened, Resilient Math Learner Who Adapted Learning Strategies

Math Identity Stories. Grace faced increasing challenges in the course, noting the accelerated pace and her need for more time to understand concepts (Interview, week 14). Her relationship with and confidence in math declined significantly after the first midterm (Figure 2).

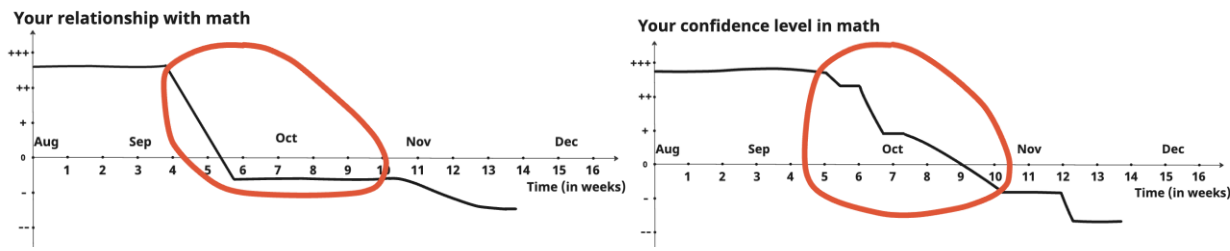

Figure 2: Grace’s drawing of her relationship with and confidence in math (Interview 3)

Despite growing frustrations, Grace persisted. During problem-solving, she encouraged herself with affirmations like “That’s hard. It’s okay, I got this.” or “I don’t wanna give up. OK, let me think.” ((Field Notes, weeks 5 & 7; Interview, week 9). When receiving low scores, she reassured herself, saying, “At least I’m learning,” or “It’s okay. The class is about learning, right?” (Field Note, week 6-9). She remained hopeful: “After the midterm, my relationship with math stabilized because I changed my learning strategies, and then I was like, okay, maybe I could do this” (Interview, week 14). She continued to find joy in math (Field Note, week 9).

Goals. Grace prioritized a deep understanding of Calculus (Interview, week 9; Field Notes, weeks 5, 7, 8) and planned to persist in the course by “toughing it out” (Interview, week 9).

Orientations. Grace recognized the substantial effort required for understanding Calculus concepts and derived satisfaction from achieving conceptual clarity (Interview, week 9). Her resilience was evident in her perseverance through challenges (Field Notes, weeks 5-9).

Resources. Grace supplemented the instructor’s lectures with Youtube tutorials, engaged in challenging problems during discussions, practiced additional textbook exercises, and engaged in in-depth mathematical conversations with the author (Interview 2; Field Note 6 & 8).

Decisions. Grace adjusted her learning strategies and sought deeper conceptual understanding in this phase (Field Notes, weeks 8 & 9; Interview, week 9). She reflected:

> After the first midterm, I realized I needed to change something. I started watching Youtube tutorials and re-watching lecture recordings… I tried to do the homework more before the quizzes because I learned that’s what you need to do in order to do well on the quiz. I also increased my study time and did additional problems from the textbook. (Interview, week 14)

Grace consistently posed insightful questions to deepen her understanding, such as “How would you see a horizontal asymptote without a graph?”, “Why does the derivative of the sine inverse function not have any trigonometric expressions in it?”, and “What if a non-continuous function has a zero? Will the theorem still apply?" (Field Notes, weeks 6, 7, 9).

Weeks 10-14: A Mathematics Admirer from Afar Who Dropped Calculus

Math Identity Stories. Grace's informal calculations of her potential course grade threatened her established identity as an "A+" student, instilling a fear of failure and a sense of alienation in the course (Field Notes, weeks 11 & 12). Grace's parents' advice to prioritize her mental health and withdraw from Calculus negatively impacted her confidence in math (Interview, week 14). Reflecting on her experience, Grace revealed that while law school had been a distant consideration, Calculus unexpectedly propelled her towards a career in law and humanities (Interview, week 14). Notably, Grace maintained a genuine appreciation of math after withdrawal:

> The course was challenging but it was definitely a growing experience… I just think I needed more time to process it in order to really do well on exams, and I didn't have that time. But yeah, math is an interesting field. It's delicate, complicated, and beautiful. I don't think I'll ever not like math… So math is like, my soulmate, I guess? (Interview 3, week 14)

Goals. Grace's primary objective was to "just pass the class" (Field Notes, weeks 11 & 12). Additionally, she emphasized the need to "focus on other things" (Interview, week 14).

Orientations. Grace continued to appreciate the beauty of math, believing that everyone, including herself, can excel through hard work (Field Notes, weeks 10-12; Interview, week 14).

Resources. In addition to the resources used in the previous phase, Grace sought emotional support from friends, input from family, and advice from the author (Interview, week 14).

Decisions. Grace reconsidered her career paths, asked fewer questions aiming for a deeper conceptual understanding, and dropped the course. There was a noticeable decrease in Grace's enthusiasm and instances of seeking deep understanding (Field Notes, weeks 11-13). Figure 3 summarizes the evolution of Grace's math identity and decision making over the semester.

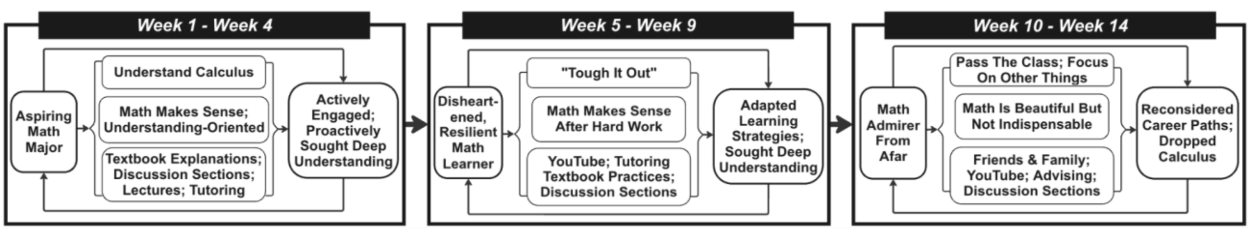

Figure 3: The evolution of Grace's mathematical identity and decision making

Discussion

This paper documents how Grace's math identity and decision making co-evolved over time. Notably, Grace's evolving math identity stories were closely connected to the decisions she made. Her active engagement in the course and proactive pursuit of deep understanding were congruent with her identity as a math major, holding potential benefits for her aspirations. As a resilient learner, Grace's adaptation of learning strategies and quest for deeper understanding were her natural responses to unsatisfactory feedback. The dissonance between her established identity as an "A+" student and the emerging narrative of failing in the course contributed to Grace's reconsideration of career pathways. Considering her new future self-image where math was no longer indispensable, it was understandable that Grace asked fewer questions aiming for a deeper conceptual understanding. Dropping Calculus allowed her more time and energy to focus on things that mattered to her new career trajectory while safeguarding her GPA.

Examining students' co-evolving identity narratives alongside their goals, orientations, resources, and decisions yields valuable insights into how they navigate learning challenges. These theoretical linkages are particularly crucial given the lack of connections between math identity literature and decision-making studies. Furthermore, since decision-making theories have yet to extend to temporal dimensions (A. Schoenfeld, personal communication, June 2, 2024), this study paves the way for more in-depth investigations into the evolution of decision making over time, as well as the identity-related factors influencing these changes.

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TRANSFORMATIONS OF SEMIOTIC REPRESENTATIONS IN MODELING: THE CASE OF LIV

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Students' use of representations yields insights into their learning of mathematical concepts. Thus, utilizing students' representations can reveal how they construct mathematical models for dynamic situations. In this brief research report, we investigate an undergraduate STEM major's use of multiple representations while modeling an exponential growth scenario. Through constructing second-order accounts of Liv's modeling activities, we demonstrate the ways in which she used multiple representations to arrive at a symbolic representation that predicts the growth of a yeast colony.

Keywords: Cognition, Representations, Mathematical Modeling

Mathematical modeling (hereafter, modeling) is the process of constructing mathematical structures to mathematically represent a situation. Scholars within the cognitive modeling perspective (see Kaiser, 2017) have extensively studied the process of modelers representing activities, with explicit attention on *mathematizing* (e.g., Suh et al., 2017; de Almeida, 2018). Yet, the difficulty in examining students' modeling process and helping them advance continues to be a significant challenge (Cevikbas et al., 2021). Particularly arduous is the task of guiding modelers towards constructing mathematical equations that establish relationships among conceived quantities symbolically (Jankvist & Niss, 2020)—a predictive component of a model. Fundamentally, modeling is the process of re-presenting a modeler's conception of the scenario through representations (Lesh et al., 2003). Therefore, the cognitive process involved in modeling can be investigated via the different representations a modeler uses (Duval, 2006). In particular, a modeler's representations and transformations of those representations over time are fertile ground for guiding modelers towards constructing expressions for situations. In this preliminary study, we report on one undergraduate's use of multiple representations to construct a symbolic representation for an exponential growth scenario.

Theoretical Framing and Background

Duval (2006) proposed looking at semiotic representations as a means for analyzing students' cognition as they engage in mathematical activities. This is because "mathematical processing always involves substituting some semiotic representations for another" (p. 107), and therefore the analysis of mathematical activities can be afforded through the transformations of representations. Duval distinguishes semiotic representations from mental representations. While mental representations are in the mind of the learner, semiotic representations are externalized mental representations that are observable by an outsider. In this study we consider semiotic representations as the objectifications (Radford, 2013) of mathematical thinking. Within this perspective, a sign can be verbal or non-verbal, and its role is to represent something else (Pierce 1998, Colapietro, 1993). Pierce (1998), defined *sign* as a triadic relationship among an object (signified), representamen (signifier) and interpretant (the effect it has on an individual whether it is the utterer or the interpretant).

Examples of non-verbal registers include graphs, symbols, equations, and tables. These are often referred to as the types of representations. Whether a representation is verbal or not is determined by the mode of production, such as written, spoken, or gestured (Goldin, 2020). A mathematical object can be represented via multiple semiotic representations. For example, an exponential growth can be represented via the symbolic form $y = e^x$, or through a graph that depicts how y changes with respect to x.

Duval (2006) posits two types of transformations of semiotic representations: treatment and conversion. While *treatments* are transformations of representations that happen within the same register, *conversions* are transformations of representations that entail converting to a different register but the same mathematical object (pp. 112-113). For example, a student who used an exponential function to model the growth of a continuously compounding bank account, may choose to work mathematically within the function to evaluate the amount of money in the account after 5 years. In contrast, the student may choose to create a table that coordinates the amount of money in the bank account with number of years, to represent her perception of the scenario. Despite the use of multiple representations, the underlying mathematical object—exponential function—remains the same. Duval (1999) calls the ability to use multiple representations to reason mathematically about the same object as *coordination*. Dreyfus (1991) posited that students' ability to use more than one representation, connecting and integrating representations can be taken as an indication of advancement in their learning process. More recently, Fonger (2019) defined representational fluency as "the ability to create, interpret, translate between, and connect multiple representations" (p.1). Scholars have investigated students' construction of mathematical ideas (Fonger, 2019; Selling, 2016) through the representations students produced and the evolution of those representations. However, research is still scarce on students' transformations of semiotic representations while constructing a mathematical model for a situation. In this report, we explore *how did an undergraduate STEM major used multiple semiotic representations to construct an exponential growth model*?

Methods

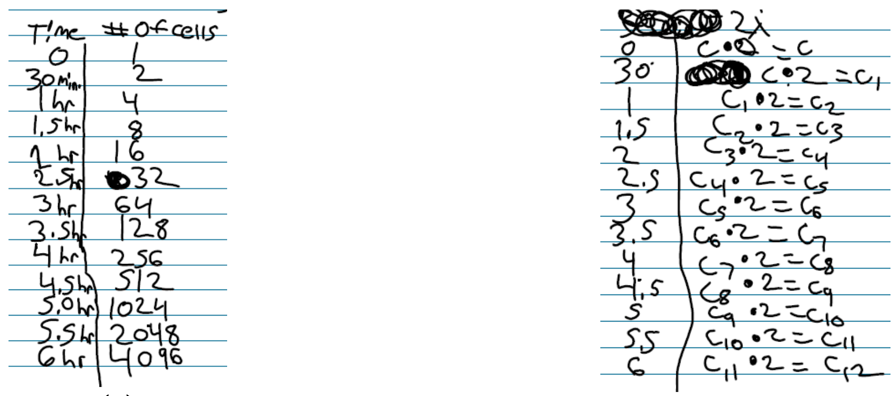
For this preliminary analysis we draw on one undergraduate STEM major's—Liv (pseudonym)—work on the Baker's Yeast task. The Baker's Yeast task was an exponential growth modeling task in which a colony of yeast cells reproduces every 30 minutes. Liv was tasked with constructing an expression for the number of cells present in the bowl at any given time. The data was collected through a 1-hour clinical interview (Goldin, 2012). Data was analyzed through constructing second-order accounts (Steffe & Thompson, 2000) of Liv's mathematizing activities. Explicit attention was paid to Liv's written semiotic representations and their transformations, while utilizing her spoken and gestured representations primarily as an explanatory source.

Findings

We present three phases of Liv's work which were pivotal in helping her construct a general symbolic representation of the number of cells present in the bowl.

Construction of Source Representation

Liv represented the scenario using a table where she coordinated the values of time and number of cells (Figure 1). Liv wrote $30\text{min} + 2 \cdot x$ to represent the number of cells after a particular time. She defined x to signify the number of cells. She decided that the equation did not require 30min, and scratched it out. Liv expressed frustration that she did not know "how to figure this out," referring to the construction of an expression that would give the number of cells in terms of time. Liv further expressed, "for me the easiest is to draw chart, but I don't know there is some faster way [referring to a formula]."



(a) (b)

Figure 1: Liv's Source Table (a) and Treatment of Table (b)

***Treatment* of Source representation to Determine a Symbolic Representation**

To help Liv transform her table into a symbolic representation, the interviewer asked Liv what the reproduction would look like if there were c cells present in the bowl initially. Liv constructed the chart in Figure 1(b), where she wrote the number of cells at time t in terms of the number of cells present at time $t-1$. On her calculator, Liv checked the values of 2^4 and 2^5 to see if they aligned with the number of cells at $t = 2$ hours and $t = 2.5$ hours as depicted in her table in Figure 1(a). In doing so, we interpret that Liv was implicitly considering $C_4 = C_3 \cdot 2 = C_2 \cdot 2 \cdot 2 = C_1 \cdot 2 \cdot 2 \cdot 2 = C \cdot 2 \cdot 2 \cdot 2 \cdot 2$. After confirming that the numbers do align, Liv first wrote C^{X+1} to represent the number of cells. Unable to explicitly state what C or X in her expression represented, Liv wanted to determine a way to include "number of half hours" in her expression. To accomplish this, Liv considered a specific case of $t = 10$ hours and posited that there would be 20 half hours. She then modified her symbolic representation to be 2^X, where X was the number of half hours. She then wrote that $1 \cdot 2^X$ would be the number of cells if there were 1 cell initially.

***Conversion* of Source Representation to Determine a Symbolic Representation**

To determine a more general expression, Liv considered a specific case: 5 cells were present initially. Liv wrote $5^{2 \cdot 20}$, where 20 in her expression signified the number of half hours and 2 signified the cells doubling. Upon request, Liv compared her model for the number of cells present when starting with 5 cells — $5^{2 \cdot 20}$—to her model when starting with 1 cell—2^X. She compared the values of 2^{20} and $1^{2 \cdot 20}$, and realized that they did not have equal outputs.

Upon suggestion by the interviewer, Liv drew diagrams for when there was initially one cell (Figure 2(a)) and 5 cells (Figure 2(b)) in the bowl. After drawing the diagram, Liv reasoned: "each individual cell [when starting with 5 cells] will follow the same pattern as if it were just starting out with just one cell." She further stated that in Figure 2(b), 5 copies of the process in Figure 2(a) were taking place. Through her reasonings, we interpret that Liv was comparing the number of cells present in the scenario as depicted in Figure 2(b) to the number of cells present in the scenario depicted in Figure 2(a), at each time step. She referred back to her table in Figure 1(a) and said that there would be $5 \cdot 4096 = 20{,}480$ cells at the end of 6 hours.

Consequently, she modified the number of cells present in the bowl at $t = 10$ hours as $5 \cdot 2^{20}$ (Figure 2(c)). Liv generalized this expression by nominalizing what each of the numbers in her expression signified: 5 signified the number of cells, 2 signified the process of cells doubling, and 20 signified the number of half hours (see Figure 2(c)). She checked her model by considering a known case of $t = 6$ hours and got $20{,}480$ yeast cells, where she simultaneously moved between her representations in figures 1(a) and 2(c). Liv produced 3 types of written semiotic representations (table, diagram, symbolic) in which the treatment and conversion of her source representation (Figure 1(a)) aided her in constructing a symbolic representation for the Baker's yeast scenario (see Figure 2(a)).

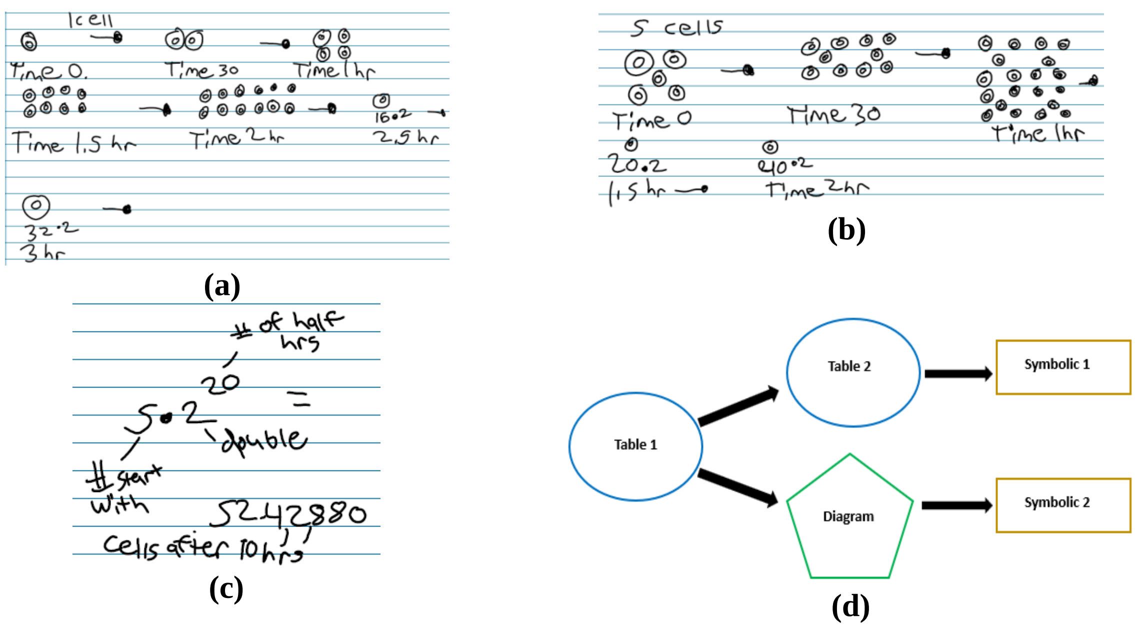


Figure 2: Liv's diagram for cells populating (a, b), symbolic representation (c), and network of types of representations (d).

Discussion

Students' reasonings, as they engage in modeling activities, are ongoing. However, students transforming from one representation to another can be taken as pivotal moments in their cognitive processes. This is because, during these shifts, students are either reorganizing their already existing conceptual structures or developing new ones. Therefore, these shifts can be seen as sensitive points to learning. Our findings suggest that students' representations can be leveraged to not only trace the evolution of their model through the inscriptions they make (Czocher & Hardison, 2019), but also in guiding them towards the modeling goal educators have for them. Students' *learning* of modeling and students' creations, interpretations, and coordination of multiple representations while *doing* modeling are interconnected (Dreyfus, 1991). Therefore, our study instigates a conversation within the modeling education community to answer questions such as how can a modeler's representations be leveraged to articulate what that modeler has learned during model construction? —a research problem yet to be solved. While our analysis attended to the transformations of representations, the focus of analysis was not on the mental operations that aided those transformations. Future research can network theories (Radford, 2008) of representations together with other compatible theories to capture the complexities and intricacies of model construction.

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THE USE OF TAPE DIAGRAMS: A SCOPING REVIEW

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Tape diagrams, which are also known as strip diagrams, bar models, bar diagrams, length models, model-drawing strategy, percentage bars, and model methods, may be a powerful tool for developing students' conceptual understanding and problem solving abilities in mathematics. Murata (2008) observes that students in Japan and Singapore outperformed U.S. students in the Trends in International Mathematics and Science Study (TIMSS) in 2003, and associates this finding with a systematic and consistent curricular emphasis on tape diagrams across grades, unlike the U.S.'s inconsistent usage. She and Timothy (2022) argue that mathematical word problems across grade levels can be solved using tape diagrams. Empirical studies have also shown that tape diagrams have the potential to improve students' problem-solving skills in the context of algebraic word problems involving both whole numbers and fractions (e.g., Ng et al., 2009), addition and subtraction problems involving whole numbers (e.g., Osman et al., 2018), algebra word problems involving whole numbers (e.g., Baysal et al., 2022), fraction and percentage problems (e.g., Dennis et al., 2016; Sharp et al., 2017).

In the above-mentioned studies, none of the studies focused on lower-achieving students. The participants in these studies were students with learning disabilities, average-ability students, above average-ability students, and randomly selected students. Of these studies, some studies randomly recruited their participants (e.g., de Koning et al., 2022). One study focused on students with mathematics difficulty (i.e., Morin et al., 2017). One study focused on higher-achieving students and average-achieving students (i.e., Ng et al., 2009). One study focused on higher-achieving students (i.e., Maglicco et al., 2016), and one study focused on average-achieving students (i.e., Mahoney, 2012). Two studies applied different scales of criteria. The synthesis of the participants in these studies revealed that there is no study focused on lower-achieving students. Lower-achieving students constitute a significant proportion of U.S. students and should be researched and offered support. Therefore, the focus on lower-achieving students might be one study area of using tape diagrams.

In the above-mentioned studies, few studies used tape diagrams as the sole intervention. Of these studies, eight studies used tape diagram drawing with step-by-step instructions on problem solving (e.g., Green, 2009; Preston, 2016; Dennis et al., 2016). Four studies did not have any information about the intervention (i.e., Putrawangsa et al., 2021; Osman et al., 2018; Van Galen et al., 2013; Madani et al., 2018). Two studies included some information about the intervention (i.e., Shah et al., 2021; Baysal et al., 2022). The synthesis of the intervention showed that tape diagrams might not be the sole intervention. Other types of interventions are always accompanied with tape diagrams. More studies can be conducted on students' understanding of tape diagrams and how students develop tape diagrams as an effective tool to solve problems.

Of these works, most of the studies focused on students' correct problem solving (e.g., Preston, 2006; Mahoney, 2012; Maglicco, 2016; Putrawangsa et al., 2012). Only three studies described students' obstacles while using tape diagrams to solve problems (i.e., Baysal et al., 2022; Green, 2009; Madani et al., 2018). By learning students' obstacles in using tape diagrams, more effective instructions can be developed to support students' learning. Further exploration on "what do students notice and wonder about while they are using tape diagrams and how are the things that they notice and wonder about related to the errors they make?" can be done.

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CONOCIMIENTOS MATEMÁTICOS UTILIZADOS POR ESTUDIANTES UNIVERSITARIOS PARA PREDECIR LA ESCASEZ DE AGUA DE UN LAGO

MATHEMATICAL KNOWLEDGE USED BY UNIVERSITY STUDENTS TO PREDICT THE SCARCITY OF WATER OF A LAKE

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En respuesta a las iniciativas globales que abogan por la integración de problemas reales en la educación matemática para estimular el desarrollo de conocimientos y habilidades matemáticas, así como reflexiones sobre los desafíos ambientales, este estudio se centró en analizar el conocimiento matemático que emerge cuando un grupo de estudiantes universitarios aborda una actividad (MEA) en el contexto de la escasez de agua. El marco teórico fue la Perspectiva de modelos y modelación. El análisis cualitativo permitió concluir que la MEA propició que los estudiantes profundizaran en el contexto e incorporaran sus experiencias personales en el proceso de modelación. Además, exhibieron el uso de sus conocimientos como razones, porcentajes, proporciones, variación, razón de cambio constante, y estimación para describir y predecir la escasez de agua en el lago de Chapala.

Palabras clave: Modelación, función, representaciones matemáticas

Introducción

De acuerdo con la UNESCO & MGIEP (2017) se requiere ampliar los esfuerzos educativos para introducir en la educación matemática problemáticas reales para apoyar el desarrollo de conocimiento, habilidades matemáticas y la reflexión de los estudiantes hacia los retos ambientales del mundo. En este artículo se reportan los resultados de una investigación que se enfocó en describir el conocimiento matemático relacionado con el concepto función que emerge cuando un grupo de estudiantes de nivel universitario resuelven una situación-problema cercana a la vida real. La situación-problema fue creada en el contexto de la escasez del agua, tópico importante en los Objetivos de Desarrollo Sostenible. La pregunta de investigación fue ¿qué conocimiento matemático relacionado con funciones exhiben los estudiantes universitarios al construir modelos para resolver una situación problema cercana a la vida real?

Marco Teórico

Este estudio se basó en la Perspectiva de Modelos y Modelación (MMP, por sus siglas en inglés), la cual plantea que el proceso de desarrollo de conocimiento matemático puede describirse como un proceso no lineal de desarrollo de modelos que se modifican de manera continua durante la interacción de un individuo o estudiante con sus compañeros o maestro para resolver una situación problemática (Ärlebäck & Doerr, 2015; Lesh, 2010).

Dentro de esta perspectiva, un modelo se define como un sistema formado por elementos, relaciones, reglas y operaciones que pueden utilizarse para dar sentido, explicar, predecir o describir otro sistema (Doerr & English, 2003; Lesh & Doerr, 2003). De acuerdo con Lesh y Doerr (2003) cuando los estudiantes construyen modelos, sus ideas se expresan a través de una variedad de medios de representación, en los cuales los significados matemáticos están distribuidos (Figura 1). La elaboración de modelos implica que los estudiantes sigan múltiples iteraciones donde expresan, prueban y validan el modelo (Lesh & Lehrer, 2003). Estas iteraciones implican "la comprensión de la situación problemática, el desarrollo de un modelo matemático como solución al problema planteado, la expresión del modelo a través de alguna forma de representación como tablas, gráficos y ecuaciones, la comprobación de la utilidad del modelo y la revisión/refinamiento del modelo si es necesario" (Sevinc, 2021, p. 80).

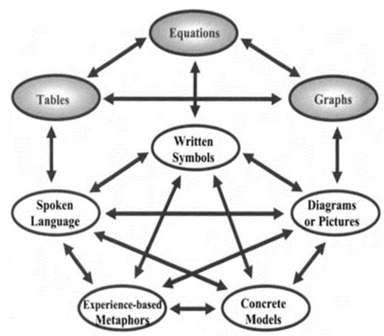

Figura 1: Medios de representación que pueden ser utilizados en un modelo. Esquema tomado de Lesh y Doerr (2003, p. 12)

La MMP propone que se utilicen Actividades Provocadoras de Modelos (MEAs, por sus siglas en inglés Model Eliciting Activities) para propiciar el aprendizaje de las matemáticas. Las MEAs son actividades diseñadas para conocer las concepciones e ideas matemáticas iniciales de los alumnos mientras desarrollan sus habilidades como solucionadores de situaciones problemáticas reales y significativas (Doerr, 2016). Resolver MEAs implica que los estudiantes puedan construir maneras útiles de interpretar la naturaleza de los datos, las metas y posibles trayectorias de solución de la situación problema; estas actividades demandan procesos de matematización que apoyen la toma de decisiones de un cliente a través la creación de modelos compartibles y reutilizables (Ärlebäck & Doerr, 2018; Sevinc & Lesh, 2018).

Metodología

La investigación fue de tipo cualitativo. Participaron 12 estudiantes que cursaban los últimos semestres de una licenciatura en matemáticas, quienes fueron organizados en cuatro equipos (A, B, C, D), cada uno conformado por tres integrantes. De acuerdo con el profesor del curso, esta fue la primera vez que los estudiantes participaban en la resolución de una MEA.

La MEA titulada "El día cero" fue construida con base en los seis principios de diseño descritos por Lesh et al. (2003). El contexto aborda uno de los problemas medioambientales relacionados con la escasez de agua, específicamente el decrecimiento del volumen de agua del

Lago de Chapala del estado de Jalisco. En la MEA se solicita a los estudiantes que ayuden a Sonia a describir la disminución de agua del Lago de Chapala y predecir el desabasto a la Zona Conurbada de Guadalajara (ZCG) con el paso de los años. En la nota periodística los estudiantes pueden identificar información como la siguiente: la población total de la ZCG según el Censo de Población y vivienda en 2010, la capacidad total del Lago de Chapala, así como el porcentaje de agua en el Lago hasta el 28 de febrero de 2023. La actividad se llevó a cabo en dos sesiones, cada una de aproximadamente 60 minutos. En la primera sesión los estudiantes escribieron la carta y la leyeron a todo el grupo. En la segunda, en una discusión plenaria, explicaron con detalle su carta. Para ello realizaron dibujos, tablas y gráficas en el pizarrón. La toma de datos fue a través de la recolección de las cartas escritas, audios de la sesión grupal, y fotos de las contribuciones en el pizarrón durante la sesión plenaria. Tres de los autores de este reporte apoyaron al investigador principal en la recolección de datos, transcribieron y codificaron la información en: a) conocimiento matemático asociado al concepto de función utilizado en la construcción de los modelos y b) uso de representaciones (Figura1). Finalmente, analizaron y discutieron en grupo los hallazgos y redactaron el reporte.

Resultados y Discusión

A continuación, se hace una descripción de los modelos construidos por los estudiantes y los conocimientos matemáticos subyacentes. Los modelos se denominaron de acuerdo con la idea principal utilizada en las cartas. Durante el proceso de creación de modelos, todos los equipos discutieron el fenómeno y reflexionaron sobre el cúmulo de variables que lo afectaban. Sin embargo, ante la solicitud de la MEA de escribir una carta para ayudar a Sonia, tomaron decisiones que les permitieron construir respuestas.

Modelo: el volumen del lago cambia

El equipo A sugirió medir el volumen de agua del lago de Chapala. Propuso atar un objeto pesado a una cuerda, sumergir el peso al lago y determinar la profundidad del lago. Con base en ello, sugirió calcular el volumen del lago y seguir este procedimiento cada año para predecir la disminución del nivel de agua. El equipo explicó que podrían partir de la hipótesis de un lago con forma semiesférica para hacer más fáciles los cálculos. Considerando las representaciones mencionadas por Lesh y Doerr (2003), el equipo A utilizó principalmente: lenguaje hablado, diagramas o dibujos y modelos concretos. El conocimiento matemático distribuido en estas representaciones fue: longitud, volumen y razón de cambio (volumen/año).

Modelos: el desabasto de agua puede ser en tres años

El equipo B sugirió calcular el gasto del agua del lago de Chapala mediante el uso de los datos incluidos en la nota periodística. Obtuvo el volumen porcentual del agua del lago y supuso un consumo constante de 8.27% cada tres meses. Este planteamiento de una razón de cambio constante de consumo de agua permitió al equipo crear su modelo y estimar cuándo podría ocurrir el día cero, sin necesidad de conocer el volumen preciso del agua del lago. Predijo que en tres años se acabaría el agua del lago bajo las condiciones iniciales planteadas.

Considerando las representaciones mencionadas por Lesh y Doerr (2003), el equipo B utilizó: lenguaje hablado y símbolos escritos. El conocimiento matemático distribuido en estas representaciones fue: porcentajes, razones, proporciones y estimación, todos ellos asociados a una función lineal ya que se partió de la idea inicial de un consumo constante de agua.

Modelo: muchas variables y poca información

El equipo C comentó que había poca información en el problema y para describir el fenómeno influían diversas variables como la cantidad de agua utilizada por los habitantes y las empresas, la influencia del crecimiento poblacional y de la cantidad de empresas, y el historial del uso del agua en el transcurso de los años. En la sesión plenaria el equipo construyó un histograma para explicar cómo podría comportarse la disminución de agua del lago de Chapala cada año. Además, argumentó que la fecha en la cual llegaría el día cero dependía del consumo por los habitantes y del consumo por las empresas, el cual es mayor.

El equipo consideró que debido a que no tenían muchos datos en el problema y la disminución del agua podía deberse a muchas variables, no podía construir y proponer un modelo matemático preciso. Requerían de más información por la gran cantidad de variables que influían en un fenómeno como este. Esta percepción de falta de información es común que emerja al resolver problemas abiertos (Vargas-Alejo et al., 2018a; Vargas-Alejo et al., 2018b). Algunos estudiantes enfocan su discusión en la diversa cantidad de variables que hay alrededor de un fenómeno y la dificultad de construir algún modelo ante esa situación.

Considerando las representaciones mencionadas por Lesh y Doerr (2003), el equipo C utilizó: lenguaje hablado y gráficas. Algunos de los objetos matemáticos distribuidos en estas representaciones fueron: razones y porcentajes.

Modelo: el desabasto del lago puede ser en 60 años

El equipo D hizo suposiciones con base en la información de la nota periodística y datos que investigaron en internet, por decisión propia, para describir la disminución de agua del Lago de Chapala y predecir el desabasto a la ZCG. Estimaron que había un decrecimiento constante que equivalía al 1% por año, desde el año 2010 hasta el 2023. Con este dato, propusieron un modelo lineal y predijeron que bajo estas condiciones iniciales planteadas podríamos tener agua por un periodo de 60 años más. El uso de porcentajes obtenidos en su búsqueda en internet y en la nota periodística, así como el planteamiento de la hipótesis de una razón de cambio constante para estimar el desabasto de agua, permitió al equipo construir su modelo y estimar cuándo podría ocurrir el "día cero" para el Lago de Chapala, que definió como el momento en el que quedara del 5 al 8% de agua en el lago.

Considerando las representaciones mencionadas por Lesh y Doerr (2003), el equipo D utilizó: lenguaje hablado, símbolos escritos y gráficas. El conocimiento matemático distribuido en estas representaciones fue: variables, variación, proporciones, razón de cambio (porcentaje de desabasto / año), función lineal y estimación.

Conclusiones

El conocimiento matemático exhibido por los estudiantes al construir sus modelos para describir la disminución de agua del Lago de Chapala y predecir el desabasto a la ZCG puede resumirse en lo siguiente. El equipo A usó longitud, volumen y razón de cambio (volumen / año), mientras que el equipo C utilizó razones y porcentajes para describir el comportamiento del fenómeno. Ninguno de los equipos A y B predijeron una fecha de desabasto. Los equipos B y D usaron razones, porcentajes, proporciones, variación y razón de cambio constante para estimar una solución. Los equipos consideraron la necesidad de describir el desabasto con base en alguna tasa de cambio, conocimiento asociado al concepto de función. Tal como señala Lesh y Doerr (2003) las experiencias personales, así como académicas influyeron en el proceso de modelación y, por lo tanto, en las reflexiones sobre la problemática.

La MEA, diseñada bajo los seis principios de Lesh et al. (2003), permitió que los estudiantes se enfrentaran a la construcción de un modelo, una experiencia novedosa para ellos. Los estudiantes matematizaron, es decir, tomaron decisiones para reducir variables y relacionarlas, estimaron una respuesta que funcionara bajo ciertas condiciones; finalmente, documentaron y auto evaluaron su modelo. La MEA posibilitó que los estudiantes conectaran conocimientos matemáticos con la situación planteada, desarrollaran y discutieran sus habilidades de modelación y reflexionaran sobre temas de sostenibilidad, respondiendo así a las necesidades planteadas por la UNESCO y MGIEP (2017) para ampliar los esfuerzos educativos frente a los retos ambientales del mundo que estamos viviendo.

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INFERENTIAL KNOWLEDGE: THE GLUE THAT HOLDS THE STEPS IN A PROCEDURE TOGETHER

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In mathematics education, knowledge is often divided into conceptual knowledge and procedural knowledge. These two knowledge types are sometimes seen as competing for teachers' attention and curricular focus. Similarly, there exists a perceived dichotomy between proof-based mathematics and procedure-based mathematics. In this context, learning procedures, which include computations and calculations, is frequently viewed as learning how to execute them to obtain answers. However, from our perspective, procedures should be understood as a sequence of inferences. Thus, we propose the construct of inferential knowledge *as an alternative to the traditional conceptual-procedural divide. We present* inferential decomposition *as a technique to deconstruct the knowledge required in understanding a procedure inferentially. We advocate for using inferential knowledge to integrate sensemaking and explanation within procedures.*

Keywords: Algebra and Algebraic Thinking, Reasoning and Proof, High School Education, Undergraduate Education

Deductive inference-making is paramount to mathematics. However, in some sub-disciplines of school mathematics, the core inferential basis is sometimes hidden behind routines and algorithms. We address this by introducing an *inferential knowledge* framework that serves a dual purpose of highlighting the inferences within procedures and also deconstructing the types of knowledge underlying an understanding of the validity of these inferences. Our inferential knowledge approach: 1) sheds light onto decades-long debates about procedural knowledge versus conceptual knowledge, 2) highlights a viable opportunity for incorporating sensemaking into learning already standardized procedures, and 3) provides potential avenues for smoothing the transition from calculation-centric mathematics (where inferences are often below the surface) to proof-centric mathematics (where inferences are more transparent).

Inferences Hidden in Procedures: Two Examples

To help illustrate the fundamental perspective driving this work, that inferences underlie procedures, we discuss two typical procedure-centric examples: The first is in the context of introductory calculus, specifically "implicit differentiation"[3].

Example 1: The Ladder Problem and Implicit Differentiation

Consider the procedure of implicit differentiation. Figure 1, below, displays an example of a fairly common introductory calculus problem whose standard solution utilizes such a procedure.

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[3] We refer to the described method for solving this problem as "implicit differentiation" for convenience, even though it may not technically qualify as such (Mirin & Zazkis, 2020).

A 3-meter ladder is sliding down a vertical wall. The top of the ladder is sliding down the wall at 0.1 m/s. Using y to represent the height of the ladder's top from the floor, and x to represent the distance of the ladder's bottom from the wall (both in meters), we have the equation $x^2+y^2=9$. Find the speed of the bottom of the ladder

Figure 1: *The Ladder Problem*

A student is likely to procedurally take $\frac{d}{dt}$ of both sides of

1. $x^2 + y^2 = 9$,

substitute in the given information, perform some manipulations, and then arrive at the answer. Mirin and Zazkis (2020) propose a way of conceptualizing this implicit differentiation procedure in a way that coheres with typical introductory calculus material: viewing the equation (1) as an equality of functions (of *t*), we use the fact that sameness of function implies sameness of derivative to infer that the derivative (with respect to t) of the function represented by the left hand side is the same as the derivative of the function represented by the right hand side. This then implies that:

2. $\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(9)$.

While there are more steps (inferences) to solve this particular problem, here we focus on the very first step in transitioning from equation (1) to equation (2). We note that understanding why a procedure is valid encompasses other mathematical understandings. In the context of the ladder problem, Mirin and Zazkis (2020) propose that understanding why implicit differentiation is valid is tantamount to viewing the procedure as an inference from function equality to derivative equality (from equation (1) to (2)), and the authors consider the conceptualizations involved in understanding such an inference. Understanding equation (1) as a statement of function equality acts as a warrant for writing (2), a statement of derivative equality, which in turn acts as a warrant for performing the differentiation procedure. While there might be other productive ways of understanding (1) and (2) besides function and derivative equality (e.g. using a calculus grounded in differentials, as in Ely, 2021), our central point here is that underlying this common procedure of differentiating both sides of an equation is an inference from equation (1) to (2). Our broad point is that deductive inferences, such as the one shown above, are omnipresent in procedures but are often left tacit outside proof centered contexts.

Example 2: The Number Line Problem and Solving Inequalities

We now consider another example to guide our discussion: a typical secondary school algebra problem, which we hereafter refer to as the number line problem.

Graph the following on a number line:

$2x-5x \leq 12$

Figure 2. The Number Line Problem

The typical procedure for solving this problem is to start with the initial inequality shown below:

3. $2x - 5x \leq 12$

Then the left-hand side is usually simplified by collecting like terms, resulting in the inequality:

4. $-3x \leq 12$.

The standard final step for solving the inequality is to divide each side by -3 and reverse the inequality sign, yielding:

5. $x \geq -4$.

Finally, the number line is shaded to the right of the point representing the value of -4, and a solid dot is drawn on the point representing -4.

Let's consider how we could conceptualize this procedure inferentially. Like with the Ladder Problem, these inferences are often left tacit. Because $2x - 5x = -3x$ for all values of x (and underlying understanding this equation entails other conceptualizations, which are not discussed here), we can infer that the values of x that satisfy (3) are precisely those values of x that satisfy (4). Therefore, solving (4) is tantamount to solving (3). Similarly, using the fact that $a \leq b$ is equivalent to $a/(-3) \geq b/(-3)$ and that $12/(-3) = -4$, we can conclude that the values of x that satisfy (4) are precisely those values of x that satisfy (5). Hence, we can conclude that the values of x that satisfy (3) are the same as those that satisfy (5), and then we can appropriately highlight all values on the number line that are greater than or equal to -4.

Our Framework: Inferential Knowledge and Inferential Decompositions

The two problems in the previous section can be solved by implementing known procedures with little or no attention to why those procedures work. However, as we demonstrated, each step in those procedures relies on an underlying inference. This observation is true in general of any legitimate mathematical procedure. We leverage this observation to define and delineate a type of knowledge that underlies understanding procedures inferentially. In other words, we define our novel construct, ***inferential knowledge,*** to characterize the ***knowledge involved in understanding a procedure as a chain of valid inferences.***

To contextualize and demonstrate the utility of our inferential knowledge construct, we first explain how *inferential knowledge* is novel in relation to mathematics education discourse on knowledge types. Then, we describe and illustrate what we call ***inferential decomposition***. An inferential decomposition is the process by which one makes explicit the often tacit inferences involved in performing a procedure and then characterizes the types of knowledge and understandings (Figure 3) associated with the procedure. We then make the case that performing an inferential decomposition is a valuable activity for both educators and students.

Situating Inferential Knowledge in the Literature

Much of the discourse around procedures within mathematics education has been situated within the procedural-conceptual dichotomy (e.g., Hiebert & Lefevre, 1986). In these works, procedures are commonly characterized as little more than a rote series of steps to be followed. For example, Hiebert and Lefevre (1986) describe procedural knowledge as skill-based, rote, and lacking meaning. In contrast, they characterize conceptual knowledge as meaningful and based on understanding. Star (2005) contributed nuance to this discussion by separating out knowledge *type* (procedural versus conceptual) from knowledge *quality* (shallow versus deep) and thus introduced the notion of *deep procedural knowledge.* Star's deep procedural knowledge, like Hatano's (2003) *adaptive expertise,* accounts for the fact that procedures can be implemented in skillful, teleological (goal-oriented), and non-rote ways. However, even this approach omits any clear indication or discussion of the inferential basis that lies at the core of why procedures work. In other words, the "deep" part of deep procedural knowledge appears to come not necessarily from understanding its inferential basis. Instead, it comes from flexibility and adaptability in implementing the procedure. This means there is little space within the procedural-conceptual literature for highlighting the inferential basis for procedures. In fact, a review of the literature (e.g., Baroody, 2003; Baroody et al., 2007; Hiebert & Lefevre, 1986; Ma, 1999; Peled & Zaslavsky, 2008; Rittle-Johnson et al., 2015; Star, 2005, 2007) on procedural and conceptual knowledge indicates no explicit reference to the notion of procedures being understood as valid inferences.

However, the literature does provide some hint into how some authors might categorize inferential knowledge in the procedural-conceptual divide. Hiebert and Lefevre's (1986) classification of procedural knowledge as *how-to* and conceptual knowledge as *why* seems to suggest that inferential knowledge goes in the conceptual knowledge classification – however, this classification appears not to be associated with procedures, nor is the inferential basis explicitly discussed. On the other hand, Star's (2005, 2007) characterization of procedural knowledge as being knowledge about procedures suggests that inferential knowledge associated with a particular procedure can be viewed as a type of procedural knowledge – indeed, this is the approach that Mirin and Zazkis (2020) take when discussing implicit differentiation. However, Star (2005, 2007) does not provide inferential examples when illustrating the construct of deep procedural knowledge. So, authors using the procedural-conceptual dichotomy do not seem to agree on how to classify inferential understanding of a procedure, nor have they specifically addressed the inferential basis for procedures.

Further, discussing the inferential knowledge associated with a particular procedure, such as implicit differentiation, will often entail discussing conceptualizations (e.g. of functions) that do not directly reference any procedures and are thus not necessarily *about* procedures. While the construct of adaptive expertise (Baroody, 2003) keeps procedural and conceptual knowledge intertwined, this construct appears to be more about transfer and flexible adaptation of procedures to varying contexts than it is about understanding the validity or legitimacy of procedures. One aspect of Baroody's adaptive expertise is knowing when to use a particular procedure, which is not wholly unrelated to knowing when to use the procedure. For example, part of knowing when to use implicit differentiation could relate to recognizing statements of function sameness. Yet, recognizing appropriate situations and understanding the inferential basis for *why* those situations are appropriate are not necessarily the same.

Although not explicitly addressed, we believe our approach, focusing on the inferential bases underlying procedures, echoes ideas hinted at by others in the mathematics education community. For example, Ma (1999) discusses ways that someone might be able to make sense of subtraction algorithms in terms of place value. Peled and Zaslavsky (2008) also seem to hint at the notion of inferential understanding by focusing on the meta-knowledge of procedures, such as in the context of the regrouping procedure when performing subtraction. Although not explicitly referred to as such, this type of understanding is closely related to understanding procedures inferentially. The procedure of regrouping when subtracting 53 by 25 could be understood, for example, by reasoning that a 1 in the tens place is the same as ten 1's in the one's place. In other words, someone could understand facts about place value as warrants for performing a grouping procedure. Another context where the notion of inferential reasoning within procedures is alluded to can be found in the National Council of Teachers of Mathematics (NCTM)'s *Standards and Procedures*. Specifically, in the document titled Procedural Fluency, the NCTM (2023) suggests that an "effective strategy" for solving for x in 4(x+2)=12 involves "using relational thinking to recognize that the quantity inside the parentheses equals 3, thus x equals 1." (p.1). While the NCTM seems to be advocating for this sort of reasoning as an efficient way of getting answers and as part of procedural fluency, we find it notable that they seem to be at least implicitly referring to inferences that one might make in an equation-solving context.

In summary, even if we find where our notion of understanding procedures inferentially falls within the discourse on knowledge types, we note that there is no consistent agreement amongst authors, nor has this past discourse explicitly addressed the inferential basis for procedures. Our approach does not ignore the procedural-conceptual distinctions, nor does it attempt to categorize every aspect of inferential knowledge as procedural or conceptual. Instead, by focusing on the inferential basis for a procedure, we are highlighting *where* the inferences lie within the procedure as a way to approach mathematical teaching, learning, and understanding.

Inferential Decompositions

We delineate three types of inferential knowledge associated with understanding a procedure inferentially: (I) Content-Specific Knowledge, (II) Deductive Knowledge, and (III) Inferential Orientation.

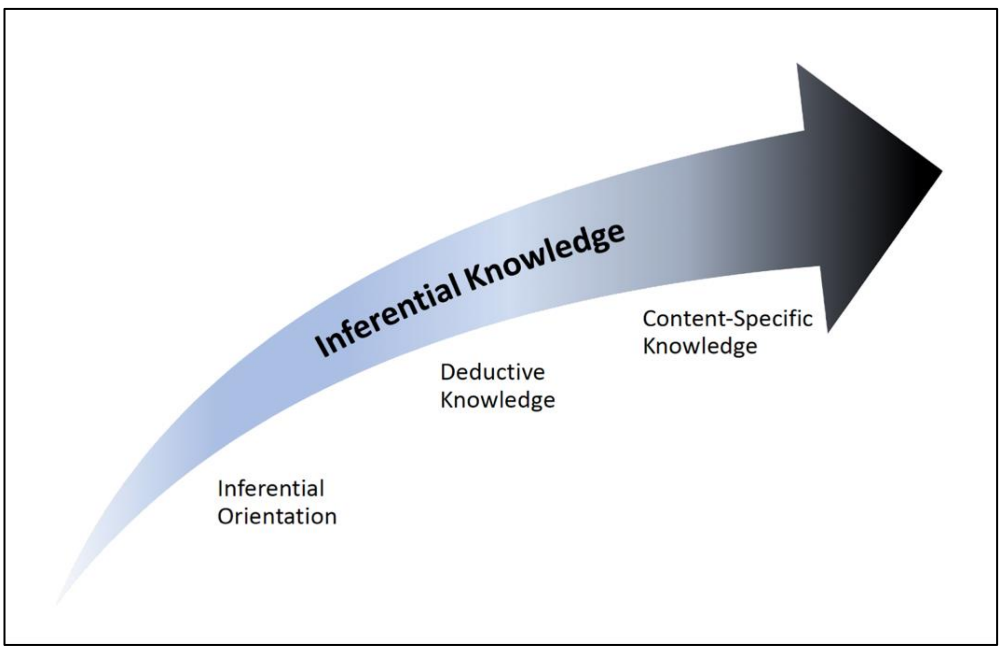

Figure 3: Inferential Knowledge

Using the ladder problem (Figure 1) and the number line problem (Figure 2), we outline *inferential decompositions.* An *inferential decomposition* is the process of identifying the knowledge and conceptualizations, and where they fall in the associated three categories (Figure 3), involved in understanding a procedure inferentially. This involves looking at how one conceptualizes the specific mathematics involved as well as how one understands the logic connecting those inferences and the role of inferences within procedures more generally. Figure 3 reflects how these different layers of knowledge relate to how "zoomed in" the type of knowledge is in relation to the procedure at hand, which we describe in more detail below. It is important to note that the descriptions below serve the purpose of illustrating our constructs of the three different types of inferential knowledge, which in turn helps illustrate the process of inferential decomposition. We are not claiming that these are complete inferential decompositions, nor are we claiming that these are the only ways of understanding the procedures described. Instead, we are illustrating how one might approach deconstructing the types of knowledge and understanding involved in conceptualizing a procedure inferentially.

We define ***Content-Specific Knowledge*** of a procedure as the conceptualizations that enable one to understand the individual inferences associated with each step of a procedure as valid. For example, Mirin and Zazkis' (2020) describe the *content-specific knowledge* involved in understanding implicit differentiation as a valid inference from function identity. This involves knowledge about functions, derivatives, the equals sign, and so on. In other words, this is what we think of as typical mathematical content knowledge and is often the focus of mathematics education research. In doing an *inferential decomposition,* we determine the specific content knowledge involved. By focusing on function and derivative knowledge in relation to implicit differentiation, Mirin and Zazkis have performed an aspect of an inferential decomposition. They then used this inferential decomposition to investigate the various obstacles that students may encounter on their way to developing such conceptualizations. We understand content-specific knowledge as being the most zoomed in of the knowledge types within an inferential decomposition in the sense that it largely focuses on the types of knowledge involved in conceptualizing an individual line, equation, or step in the procedure. For example, content-specific knowledge, in this case, involves how someone understands the equation (1) on its own as a statement of function equality and the equation (2) as a statement of derivative equality. This is the calculus content knowledge someone should have in order to view (2) as a valid inference from (1). In other words, content-specific knowledge is how someone understands the inscriptions and their mathematical referents (e.g. function, derivative) and need not entail how someone understands the relationships between these inscriptions.

We define ***Deductive Knowledge*** to be knowledge of the logical relationships between the different steps of a procedure. This is what is required for understanding the relationships between each line and is needed to make inferences and string them together to achieve a desired goal. In this sense, while content-specific knowledge can be understood as intra-line, deductive knowledge can be understood as inter-line. Returning to the ladder problem example, a series of line-by-line inferences are required to go from the initial statement of function equality, $x^2 + y^2 = 9$, to the final answer $\frac{dx}{dt} = \frac{0.1y}{x}$. Deductive knowledge can also appear in single-line inferences. Consider how someone could reason from line (1) (function equality) to line (2) (derivative equality). They could first have the content-specific knowledge about functions and derivatives that a derivative is determined by a function's graph or that the same function implies the same derivative. They could also have the content-specific understanding that line (1) expresses that two functions are the same. From these two facts, they would then need to conclude that the functions defined on the left side and the right side of line (1) do indeed share a derivative. This sort of reasoning is inter-line and is akin to a modus ponens argument (Same function → Same derivative. Same function. Therefore, same derivative). Similarly, in the number line problem, deductive knowledge is used for concluding that line (3) is equivalent to line (5) on the grounds that they are both equivalent to line (4).

Knowing how to identify and sequence inferences to reach a desired goal is non-trivial and requires a more global view. Our choice of the word *inferences* here is intentional. If the individual steps are chained together skillfully without attention to their inferential basis, then what is happening is inherently not deductive. In such a case, what is happening is akin to what Star (2005) calls "deep procedural knowledge". What we describe as *deductive knowledge* differs in terms of understanding the core activity involved. That is, deep procedural knowledge is goal-oriented, where the goal is to get a result/answer, while deductive knowledge is oriented toward deducing why that same procedure results in that particular answer.

Someone can have strong deductive knowledge *and* content-specific knowledge yet still not understand procedures inferentially. We define ***Inferential Orientation*** as the view *that* a given procedure has a logical structure that *can* be understood as a chain of inferences. This is the most zoomed-out category in that it does not concern specific lines in a procedure. It entails understanding that, at the end of a procedure, we have constructed a chain of inferences to get from the premises we stated in the problem to the deduction (the answer) required by the activity. An inferential orientation reflects a view regarding what procedures, taken as mathematical objects, are. A student can theoretically have a robust understanding of logical arguments with strong deductive reasoning skills and a mastery of content-specific knowledge, yet still not have an inferential orientation due to not viewing implementing mathematical algorithms as grounded in inferences.

The precise delineation for categorizing each particular piece of knowledge is beyond the scope of this paper. The focus of this paper is to shed light on the inferences required to choose, understand, and complete a valid procedure. From this perspective, inferences are the glue that hold together any valid mathematical procedure. If we view procedures as stemming from an inferential basis, then exploring this basis is an avenue for a deeper understanding of procedures. We argue that we should not dismiss procedures as a series of steps (that can potentially be implemented skillfully), but rather treat procedures and inferences as inextricably linked.

Treating procedures inferentially can help students evaluate and interpret their own work. Students are better positioned to interpret their answer in relation to the original problem, and thus also check for reasonableness of their answers, if they attend to the logical relationship between the initial problem being operated on and their result or answer. That is, presenting procedures as inferential reasoning can empower students to verify their answer by a using a series of inferences rather than troubleshooting by revisiting each of the calculations that led them to that answer. Consider, for example, the common mistake in inequality-solving in which students forget to flip the inequality symbol when dividing by a negative number, such as when transitioning from lines (4) to (5) in The Number Line Problem (El-Shara' & Al-Abed, 2010).

Taking an inferential approach enables a student to catch this common mistake by evaluating the original inequality at some number and comparing the truth-value of the resulting inequality at the same number. Relatedly, approaching procedures inferentially is also a useful way for helping students interpret atypical results from procedures. Multiple mathematics education researchers (e.g. Frost, 2015; Sfard & Linchevsky, 1994) observed that students struggled to interpret the results of their solution procedures when such procedures yielded atypical solutions. For example, Sfard and Linchevsky (1994) observe that students tended not to differentiate between dependent and inconsistent systems of linear equations since in both cases "the system disappears" (p. 298). Through the lens of Inferential Knowledge, students did not have the inferential knowledge to interpret their result in terms of the original problem.

Conclusion and Discussion

This work contributes to the literature in three significant ways: 1) It provides a new lens for making sense of a several decades-long debate on conceptual vs. procedural knowledge. 2) Inferential Decomposition is valuable because it parses where the inferences and by extension the concepts are within a given procedure. It also illuminates the varying layers at which that inferential structure occurs. Finally, 3) it provides a tool that could potentially smooth the transition from calculation-centered mathematics to early proof education.

Our construct of inferential knowledge provides a valuable lens for making sense of the procedural verses conceptual debate that's been a theme in mathematics education for several decades. From the perspective of inferential knowledge, procedures are not treated as inherently rote applications of pre-determined steps. Instead, they are treated as a sequence of steps glued together by inferences (content-specific knowledge), which are, then, linked together toward the goal of a logically deductive argument (deductive knowledge) and broadly situated within an inferential mathematical landscape (inferential orientation). Procedures are thus not inherently purely "procedural"; instead, they can become procedural when those implementing them lose sight of or are unaware of their inferential basis.

Inferential decomposition provides a valuable tool for identifying the inferences, and hence the conceptual bases, for procedures. An inferential decomposition performed by an educator can help highlight to both that educator and their students where the inferences and concepts are within a procedure. This may allow for a more conceptual approach to teaching a procedure that is grounded in reasoning and sense-making. Additionally, an inferential decomposition can be used to identify gaps in students' knowledge of the inferences involved in a procedure, which in turn can illuminate aspects of the student's conceptual knowledge. Finally, as a research tool, an inferential decomposition can highlight which inferences within a procedure are less or more common in comparison to other procedures. This may facilitate the creation of activities and interview questions which may lead to a deeper understanding of how students understand certain aspects of procedures inferentially.

The transition to proof-based mathematics both in high school, when students encounter Euclidean geometry, and in post-calculus undergraduate introduction to proof courses is notoriously difficult (Stylianides et al., 2017). We believe that inferential decomposition can be used to help highlight the core inferential structure present in procedure-based mathematics, making the transition to proof less abrupt. Thus, introducing inferences within procedure-centric mathematics may help promote much-needed curricular coherence particularly with regard to this transition (Stigler & Hiebert, 1999; Thompson, 2008). We are not diminishing the differences between calculation and proof, nor are we suggesting that every calculation or procedure be written up as a formal proof. Proofs often require stringing together inferences in novel ways, or in collegiate proof contexts, generating novel inferences. However, both proof and procedure can be viewed as constituting strings of inferences, and highlighting this similarity has the potential to increase curricular coherence and in doing so improve both proof-centered mathematics and calculation-centered mathematics.

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HOW IS MATHEMATICAL MODELING USED IN ENGINEERING AND MATH EDUCATION?

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Keywords: Mathematical Modeling, Engineering Education, Mathematics Education

Purposes of the study

The purpose of this study was to explore the use of mathematical modeling in engineering and mathematics. I interviewed an engineering professor, a math professor, and a high school math teacher to better understand how they use modeling. As my guide for this study, I used the following research question: What are the pedagogical approaches to utilizing mathematical modeling problems in teaching among an engineering professor, mathematics professor, and high school mathematics teacher? How do these approaches compare across different educational levels and disciplines?

Theoretical Framework

This research was guided by an iterative modeling framework. This involved taking a real-world problem and transferring it into a theoretical context for analysis using a model. The insights gained were then applied back to the real world. This approach is similar to the interdisciplinary math modeling (IMM) framework proposed by Doğan et al. in 2019 and Stacy's mathematisation cycle, which consists of a four-step iterative process: real world, mathematical problem, mathematical solution, and real solution. Additionally, Galbraith et al. (2013) begin their iterative framework for mathematical modeling with a real-world "messy" problem.

Methods

In this qualitative exploratory study, I triangulated the modeling investigation by interviewing an engineering professor, a mathematics professor, and a high school math teacher. Each participant had a minimum of 10 years experience in their field. Through a semi-structured interview process, I asked them to define mathematical modeling, identify how it is used in their classroom, if and how engineering is represented in their classroom, and what they think students need to understand about engineering. The interviews were analyzed using an inductive open-coding process (Saldaña & Omasta, 2022).

Summary

Results of this study show that participants in different disciplines have similar applications of mathematical modeling, challenges with students, and challenges with creating good mathematical modeling. Unsurprisingly, the use or application of mathematical modeling differed between the engineers and mathematics instructors. However, all three participants used mathematical modeling to generate questions about the curricula (Blomhøj, 2019); identified that designing mathematical models takes time and experience (Blomhøj, 2019; Diefes-Dux et al., 2004); and modeling provides students opportunities to persevere (Firouzian et al., 2012; Kashefi

et al., 2012). The study demonstrates the need for more intentional planning when using mathematical modeling to represent engineering. Collaborating with diverse groups of people can boost confidence in teachers and students when using mathematical modeling for coursework.

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UNDERSTANDING THE CONCEPTUAL REASONING OF STUDENTS WITH DISABILITIES THROUGH MATHEMATICAL SELF-CORRECTIONS

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Keywords: High School Education; Middle School Education; Reasoning and Proof; Students with Disabilities.

A research divide between mathematics education and special education has led to significant gaps in research on the conceptual understanding of students with disabilities (Lambert & Tan, 2017). This study addresses this research gap through an exploration of disabled students' self-corrections of mathematical thinking as an indicator of conceptual reasoning.

Theoretical Framework

Disability Studies scholars consider disability to be a socially constructed disadvantage that can be altered through changes in the environment and in society (i.e. social model) rather than deficits within an individual (i.e. medical model), implying that all students, disabled or not, have the capacity to engage in problem-solving and conceptual reasoning (Lambert & Tan, 2017). Conceptual understanding is vital to students' learning of mathematics (NCTM, 2000) and students with disabilities deserve the opportunity to develop their conceptual understanding.

Methods

Two 9th grade participants, Adam (Moebius Syndrome, high-functioning autism, ADHD, and anxiety) and Cohen (math and reading specific learning disabilities), were individually interviewed in three video-recorded interviews. Each interview comprised of two problem-based, open-ended tasks on rate of change, a topic integral to students' conceptual understanding.

Findings

I used a general inductive approach (Thomas, 2006) to analyze the students' self-corrections. First, I coded moments of self-correction, where the students verbally or physically adjusted their work. Then, I created subcodes for what seemed to prompt self-corrections. Findings show that 66.67% of Adam and Cohen's self-corrections led to improved or fully corrected mathematical thinking. They were prompted to self-correct their work in three different ways: self-detections (61.9%), routine questions (28.6%), and facilitating questions (9.5%). A self-detection prompt occurred when the participant adjusted their mathematical thinking with no feedback or input from the researcher. Routine questions were asked for each task regardless of the correctness of the student's work. Facilitating questions were asked with the purpose of moving the student forward in their thinking after a mistake was made by the student.

Scientific or Scholarly Significance

The participants' ability to self-correct suggests: first, students with disabilities can engage in mathematical reasoning without constant correction and handholding; and second, the mistakes they make can be poor indicators of whether or not they are capable of engaging in deep mathematical thinking. Their ability to correct their thinking was likely a result of time with the task and reflection opportunities created by routine and facilitating questions. Future research should extend this existence proof to include more participants with a wide variety of disabilities.

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"IT DOES SHOW IT BOTH WAYS, THOUGH": EMMA'S REASONING THROUGH GRAPHING CONVENTIONS

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As graph literacy continues to be necessary to communicate in STEM fields, conventions around such graphs have developed for students to work and reason with. We describe a fifth grader's, Emma's, thinking through non-conventional graphical representations of a linear relationship. We argue that Emma relied on mathematical reasoning when faced with conflict in conventions and was able to make sense of unconventional graphs by using quantitative strategies. Although Emma acknowledged her known conventions of graphing, she was not bound by these conventions but rather leaned on her reasoning about quantities and flexible use of reference frames. We use Emma's activity to argue possible implications for research and teaching regarding graphing conventions.

Keywords: Mathematical Representations, Cognition, Middle School Education

Graphical representations are commonly used in STEM fields, and relatedly, the ability to read and write graphical representations is important for students to progress in STEM coursework and careers (Costa, 2020). These graphical representations commonly draw on conventions. For example, many graphical representations are constructed upon the Cartesian plane, with two perpendicular axes (i.e., x and y axes) with the intersection of the axes at (0, 0), named "the origin". Because such conventions are used widely and often, it is important that students know these conventions and use them to communicate ideas with others. However, despite their effectiveness for communication, too much emphasis on conventions can become a hurdle for students. Researchers have shown that students' meanings for graphs are often constrained to a 'a set of rituals' (e.g., Mamolo & Zazkis, 2012; Thompson, 1992). For example, researchers have noted an over-reliance on the vertical line test to determine if a graph represents a function even in cases where this procedure does not apply (Breidenbach et al., 1992; Even, 1993; Montiel et al., 2008; Moore, Silverman, et al. 2019; Oehrtman et al., 2008). Student adherence to conventions used for the Cartesian plane has similarly provoked struggles while creating/interpreting a polar coordinate system (Sayre & Wittman, 2008; Moore et al., 2014). Further, some researchers have shown that some conventions commonly used in math classes are not consistent with how STEM fields use graphical representations in practice. For example, Paoletti et al. (2022) showed that the origin is typically not (0, 0) in graphs used in several STEM fields. Collectively, these studies show that too much attention to conventions might take away students' focus from more important reasoning that could support their graph literacy.

Although the aforementioned studies provide insight into the complexities students can experience when it comes to graphing conventions coming in conflict with their graph reasoning, we note that these studies involved older students, who have had many years of experience with graphing conventions. In our work, we have been working with Grade 5–8 students who are yet to or are in the early stages of learning about graphs in school. We aim to document how students at this earlier stage are capable of reconciling conflict between learned graphing conventions to view them as conventions rather than as required rules, in conjunction with their budding quantitative strategies and thinking within frames of reference (hereafter referred to as "reference frames" (RFs)). In this paper, we describe a fifth grader's, Emma's, thinking about graphical representations of what we deemed to be a linear relationship. We describe how Emma's attention to graphing conventions, quantitative strategies, and thinking within RFs interplayed throughout her engagement with a graphing task. We argue that Emma relied on her quantitative reasoning when faced with conflicts with learned graphing conventions to make sense of unconventional graphs. We close with a discussion on the implications of Emma's work for future research and teaching regarding students' developing meanings for graph conventions.

Theoretical Underpinnings and Relevant Literature

In this section, we discuss the theoretical underpinnings that guided our task design and data analysis. We also review literature relevant to our specific focus on students' interpretation of $y = 2x$ graphs in both conventional and unconventional forms.

Conventions

Thompson (1992) differentiated students' understanding of conventions as conventions (*conventions qua conventions*) versus students' understanding of conventions (to teachers and researchers) as rules that must be followed (*ritual use of conventions*). We used Thompson's distinction between conventions qua conventions and ritual use of conventions to characterize Emma's attention to graphing conventions in our analysis. That is, we attended to whether Emma viewed certain features of graphs presented to her as mere conventions that could be changed or as rules that need to be followed when constructing or interpreting graphs.

Moore and colleagues examined students' interpretations of simple graphs, like $y = 3x$, constructed in nonconventional variations of the Cartesian plane (Moore & Thompson, 2015; Moore, 2016; Moore, Stevens et al., 2019; Moore, Silverman et al., 2019). Graphing tasks like this were used to develop models of students' graphing activity, with specific attention to what aspects of the graphs were prioritized in students' focus. In doing so, the researchers were also able to examine students' meanings for conventions interplaying with their reasoning about quantitative relationships. Moore, Stevens et al. (2019) provided numerous examples of pre-service teachers (PSTs) whose graphing activity was constrained to maintaining conventions as rules. In many cases, the PSTs' reliance on conventions took precedence over their quantitative meanings for the situation, leading them to claim that mathematically accurate graphs (from the researchers' perspective) were wrong due to the graphs differing from their expected conventions in some way. For example, only 31% of PSTs from the study deemed an accurate graph of $y = 3x$ with x and y represented on the vertical and horizontal axis, respectively, to be an accurate representation of the relationship defined by $y = 3x$. Inspired by this line of work, we designed the "Variations of $y = 2x$" task to vary conventional features of the canonical $y = 2x$ graph and asked students to check whether the graph accurately depicted the relationship between x and y. Variations included changing the axes and/or the orientation of axes like in Moore and colleagues' work. Other features, such as the location of the origin and the scale of each axis, were also varied (see the Methods section for more details).

Quantitative Reasoning and Reference Frames

We conjectured students could rely on their quantitative reasoning to develop meanings for graphing conventions as conventions. We adopt Steffe, Thompson, and colleagues' (e.g., Smith & Thompson, 2008; Steffe, 1991) description of quantitative reasoning, which characterizes *quantities* as conceptual entities individuals construct to interpret their experiential worlds (von Glasersfeld, 1995). Quantitative reasoning, then, entails an individual conceiving of and reasoning about the relationships between quantities (Smith & Thompson, 2008; Thompson, 2011). Engaging with algebraic situations should entail quantitative reasoning (Smith & Thompson, 2008; Steffe & Izsák, 2002). With respect to "$y = 2x$" in our work, a student reasoning quantitatively may quantify a relationship between y and x as multiplicative (i.e., the y-value is always twice the x-value).

In the context of quantitative reasoning, Joshua et al. (2015) defined a RF as "a set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 2). Joshua et al. identified three related mental actions—committing to a unit of measure, committing to a reference point, and committing to a directionality of measure comparison (p. 32). Further, Joshua et al. defined a coordinate system as the product of the mental activity involved in conceptualizing and coordinating multiple RFs, which allows individuals "to represent the measures of different quantities simultaneously when those measures stem from potentially different frames of reference" (ibid., p. 35).

Similarly, but more broadly, we use RFs to refer to mental structures used to gauge the relative extent of various attributes in the phenomenon (Levinson, 2003; Lee, 2017; Joshua et al., 2015). Thinking within RFs entails attending to and establishing reference objects, directionality, and having an idea of what and how to measure the quantities being depicted (Joshua et al., 2015; Lee et al., 2019). For example, to create or interpret the graphical representations like those in Figure 1, an individual will need to establish x and y in terms of where they start, in which direction they move/change, and how each quantity is measured (e.g., unit of measure). Relatedly, coordinate systems refer to the geometric coordination of the RFs (e.g., axes). A coordinate system allows an individual to systematically express and coordinate RFs; a graph refers to a collection of points depicted upon the underlying coordinate system. Considering such a collection of points, an individual can hold in mind both quantities' (potentially varying) magnitudes simultaneously (Thompson et al., 2017). The nature of graphs and hence, ways of thinking about a graph, fundamentally depends on the RFs and coordinate systems upon which the graphs are created and how individuals make sense of the quantities depicted.

Lee et al. (2019) documented shifts in how a PST constructed and reasoned within RFs when engaging in graphing activities with non-canonical coordinate systems. Specifically, Lee et al. attended to the PST's reference points and directionality of measure comparison, which shifted from relying on perceptual features of graphs to focusing on coordinated actions such as quantitative relationships. The researchers hypothesized that the PST's shift was supported by perturbations from the unconventional graphs. Building on this work, in our work with Emma, we attended to her RFs, specifically, her attention to some reference point(s) and directionality of measure comparison.

Guided by these ideas, our research question is, “When faced with unconventional graphical representations of $y = 2x$, what reasoning does one fifth grader employ between her conventions, quantitative meanings, and reference frames?”

Methods

In this paper, we present data from a larger project that uses clinical interviews (Ginsburg, 1997; Clement, 2000; Goldin, 2000) to examine students’ current ways of graph thinking. The project goal is to examine middle school students’ graphing activities that could inform theory and practice.

Participant and Data Collection

The participants were recruited locally via social media and ranged from fifth to eighth grade. Four students met with the researchers on a university campus in the southern United States to participate in a sequence of four hour-long individual clinical interviews. Interviews had an interviewer (IR) and witness-researcher (WR) present; they were video-recorded with a focus on student work and any interactions and gestures between the student and IR. We digitized student work through scanning and screen-recordings. The participant we focus on in this paper, Emma, was a fifth grader. Specific to the task, Emma self-reported that in school, she had not seen graphs like the ones from the task. However, Emma did describe exposure in school to using coordinate grids to plot points, where the origin would be placed at (0, 0). Although she had experience with “conventional” coordinate systems in school, these conventions had not necessarily been emphasized yet in relation to linear graphs such as $y = 2x$. We note that Emma reported studying additional mathematics outside of school, and she demonstrated familiarity with linear graphs throughout her interviews.

This paper focuses on data from one task in Emma’s third interview, “Variations of $y = 2x$”, implemented through the online, interactive teaching and learning platform, Desmos. We designed the task while considering the work discussed above with unconventional coordinate systems and graphs. Our task contained four slides, where each slide contained a graph of the line $y = 2x$ with differing orientations of axes, scaling, or origin changes (Figure 1). Specifically, Graphs A and B (Figure 1a and b) showed the x- and y-axis with differing scales, Graph C (Figure 1c) had positive x-values oriented to the left and positive y-axis values oriented downwards, and Graph D (Figure 1d) showed the axes intersecting at (-2, 0). When opening each slide, we asked Emma if the graph represented the relationship between x and y in the equation $y = 2x$ by selecting “Yes,” “No,” or “I don’t know”. Because our goal was to investigate how the student might make sense of the quantitative relationship and not their ability to read an equation, if the student had a difficult time interpreting the equation $y = 2x$, we explained to the student that the equation meant the y value is always twice the value of x.

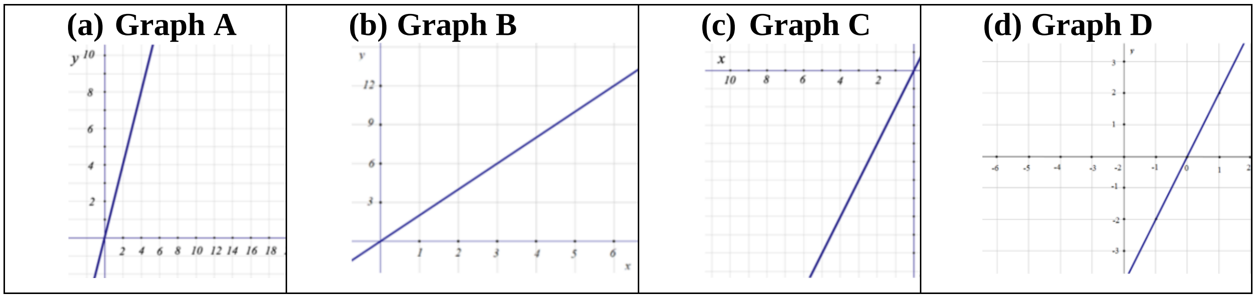

Figure 1: "Variations of *y = 2x*" Graphs

Data Analysis

In our analysis, we created a thick description of Emma's activity with the task (Geertz, 1973). We used this description to build a model of Emma's current meanings through conceptual analysis (von Glasersfeld & Steffe, 1991; Thompson, 2008). As we attempted to build this model, we characterized Emma's quantitative reasoning, attention to conventions, and thinking within RFs. Specifically, we examined Emma's activity for her quantitative reasoning, potential habitual use of conventions, relevant RFs Emma used, and shifts between habitual use of conventions and using conventions qua conventions. During this process, we re-examined previous parts of the description to support our working model, identify possible shifts in Emma's reasoning over the episodes, or negate our original interpretations.

Results

Although Emma expressed her known conventions around graphs, she was able to rely on her quantitative meanings for the relationship $y = 2x$, in conjunction with the use of flexible RFs, to determine if a(n unconventional) graph accurately depicted the relationship. Notably, her flexible use of RFs included interpreting shifts in directionality (i.e., representing positive x-values to the left), unconventional units (i.e., tick marks not representing 1 unit), and different reference points (i.e., unconventional intersection of axes). In all four graphs, Emma consistently used quantitative reasoning and RFs to resolve conflicts that arose when aspects of a graph did not match the conventions she assumed needed to be maintained.

Conventions, Quantitative Reasoning, and RFs Aligned: Graph A

In Graph A (Figure 1a), Emma's meanings for conventions, RFs, and quantitative reasoning aligned. After some conversation about how $y = 2x$ may be represented in a graph, the IR asked Emma what she thought about the relationship as meaning y is always twice x. Emma first implicitly considered if the graph represented a rule in which x was two more then y by checking if the point (0, 2) was on the graph before realizing she should consider if y-values were double x-values. She then moved her cursor to (0, 0) and over horizontally to $x = 2$, claiming, "If x is that [two], y is that [*moving her cursor up vertically to intersect the graph and then horizontally over to $y = 4$ on the y axis*]." With the cursor on (2, 4) on the graphed line, Emma argued that this point was correct based on four being "two times x". Emma decided to answer "yes" to the prompt and provided more explanation to back up her claim, such as (4, 8) being another point on the graph reflecting her quantitative meaning for $y = 2x$ of y being "two times x".

Across her activity, we infer Emma used the x- and y-axis each as a RF. She identified 0-points for each axis, worked with an implicit direction, and understood each tick to represent the appropriate number of units. For example, for Emma, $x = 2$ meant starting at 0 and moving two units right via 1 tick mark jump. Finally, we note Emma relied on a quantitative meaning for the relationship (y is "two times x") to determine if the graph reflected the relationship. Emma continued to use this *quantitative meaning* in the rest of the graphs of the task. In some cases when Emma became perturbed as she addressed a novel graph, the IR referred back to her quantitative meaning to help remind Emma of the connection of the equation to the relationship.

Conventions Superseded by Quantitative Reasoning and RFs: Graph B and C

The unconventional nature of Graphs B and C (Figure 1b, 1c) created perturbations for Emma as she attempted to interpret novel coordinate systems. However, Emma leveraged her quantitative meanings along with flexible reasoning about RFs to interpret both graphs as accurate representations of the relationship $y = 2x$.

In each case, as Emma tried to apply her quantitative meaning, the unconventional nature of the graph created a complexity. When initially addressing each graph, Emma decided that the graphs did not reflect the relationship. In Graph B, this happened as Emma was looking for $x = 2$ and $y = 4$ to touch on the graph; as she moved up from $x = 2$ to the graph, she said, "It doesn't [represent the relationship]. Four would be right there [*motioning over the graphed line above x = 2 between the y-values tick values of 3 and 6*]." We conjecture the point not being at the intersection of two gridlines created a complexity for Emma. As the y-axis was scaled by increments of 3, 4 was not represented on the scale or by a gridline; we infer this broke from the convention Emma (implicitly) used in the prior graph that each tick mark along the y-axis represented a change of 1. Emma rejected Graph C even faster, calling it "wrong" due to its unconventional nature, saying, "From what I see, those [*referring to the x and y values on the left and down of the intersection of the axes*] have to be negative numbers because that is the… I think that's the third quadrant…" In each case, we inferred that conventions around coordinate systems, implicitly in Graph B and explicitly in Graph C, influenced Emma's initial decisions for if the graphs represented the relationship.

However, Emma reconsidered each graph as she returned to her quantitative meanings and adapted her RFs when asked to explain her original decision. In Graph B, Emma reorganized her RFs such that the unit of measure of each tick mark represented matched those depicted. After her last comment above regarding Graph B and her conflict with the point (2, 4), she tilted her head and wondered aloud, "actually, it does [represent the relationship]." She then decided to check that $x = 3$ corresponded to $y = 6$ in the given graph, confirming that the graph represented the quantitative relationship. Emma then returned to checking $x = 2$. She placed the cursor directly above the x-axis and defined the distance between the x-axis and her cursor as a unit length, "the top of the circle [cursor] would be one". She then iterated that length by moving the cursor up three more times, intersecting the graph at the y-value of 4. We infer Emma had re-established her RFs constituting each axis to attend to the non-normative scaling of the y-axis as compared to Graph A. Using this reorganization in conjunction with her quantitative reasoning of checking the pairs of points, she determined that Graph B accurately depicted the relationship.

Emma similarly switched decisions with Graph C by reasoning flexibly with RFs and maintaining a focus on the quantitative relationship. In particular, after verbally identifying the unconventional axes different from Emma's expected signs for "Quadrant 3," the IR asked Emma what her answer to the prompt would be. After a 6-second pause, Emma responded "it does [represent the relationship]." Emma then gestured to the 2 on the *x*-axis down to the graphed line and horizontally over to the 4 on the *y*-axis explaining, "It shows it because the two and the four touch right there, on the line." Emma smiled and decided confidently to answer "Yes." She further demonstrated how when *x* equaled 4, *y* equaled 8 on the line as "another way I can prove it." Although Emma's initial reaction was to reject Graph C, she re-considered her decision after reorganizing her RFs, attending to the changed direction of the values on the *x*- and *y*-axis. This reorganization of her RFs allowed her to use her quantitative meaning for the relationship to confirm Graph C did, in fact, represent the relationship. Emma's work with Graphs B and C evidence her understanding of convention qua conventions, where she leaned on her re-organization and use of RFs and quantitative reasoning to overcome an initial hesitation towards the representation that was depicted differently than she seemed to expect.

Conventions in Conflict with Quantitative Reasoning and RFs: Graph D

The unconventional location of the intersection of the axes in Graph D created a greater complexity for Emma as she considered if the graph correctly represented the relationship. However, as before, she eventually was able to reorganize her RFs and leverage her quantitative meanings to interpret the graph as correct. When first viewing Graph D, Emma expressed concern with the intersection of the axes:

> Why do they have…Um. I think that this line [*traces y-axis on the screen*] has to be moved over more… the *y* line, has to be moved over more [*gestures to the zero on the x-axis*] to the zero because, um, I, well, maybe it doesn't… uh, it does. Um, it has to be, the origin is always (0, 0).

Emma's reaction seems indicative of a ritual use of conventions regarding the intersection of the axes ("always (0,0)"). In fact, her tone changed as she emphasized the origin "*has to be*" (0, 0). However, there was also a note of suspicion that "maybe it doesn't" have to be at zero. Immediately after making this comment, Emma critically investigated between the current origin (Figure 2a) and her desired origin, (0, 0) (Figure 2b). She then discussed (0,0)'s placement, "Hmm. That does… That shows zero, too. That's showing… Hmm, actually… Actually, that shows (0, 0). But I don't think, was it… I don't understand this. How are the, why is the *y* line like that?" We infer that Emma realized the point (0, 0) was on the given graph, which she understood was consistent with the given relationship $y = 2x$. However, the unconventional placement of the *y*-axis persisted in creating confusion as Emma again declared that the graph would not represent the relationship.

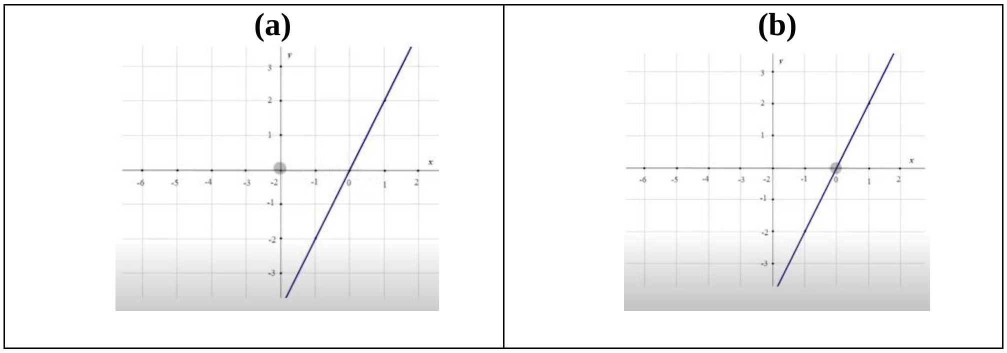

Figure 2: Graph D’s (a) Depicted Origin, and (b) Emma’s Desired Origin at (0, 0)

Emma continued to consider whether the intersection of the axes at (0, 0) was a rule that must be followed in relation to her inferences regarding RFs and her quantitative meaning. As Emma considered Graph D, she motioned along each axis to show that the graph represented should depict $x = 2$ corresponded to $y = 4$. However, we infer that in the moment, Emma still considered the graph to be incorrect. She opted to check another point, moving her cursor along the x-axis to 1, then moving up and horizontal to the y-value of 2. As she did this, she paused and looked closer, “Wait… but it does! It shows it… Hmm, it does.” She then moved on to the point (0, 0) and reasoned that doubling zero should achieve that point, laughing to herself, seemingly with surprise. The IR then asked her where zero should be on the graph, and Emma repeated her original reasoning, “If I could have the zero anywhere, I would have the zero right here [*places cursor on the intersection that currently had* $x = -2$ *in Figure 2a*].” We infer Emma wanted the intersection of the axes to be (0, 0), not (-2, 0). She stared at the screen again for about five seconds and calmly decided, "It does show it both ways, though… because, I can do it with the one and the two [*gestures up to the graph from* $x = 1$]… Oh, one and a half would be about there [*puts mouse between the 1 and 2 on the x-axis*]… One and a half, three [*motioning from the x-axis to the graph at* $y=3$]!” She then pulled herself back and smiled, concluding, “I think it’s actually yes”. We infer that Emma’s initial reaction to the graph involved the intersection of the axes at (0, 0) to be a rule rather than a choice. However, as she focused on the RFs represented by each axis (rather than the intersection point), she reconsidered the graph in terms of her quantitative meaning, concluding the graph accurately reflected the relationship. Although she still expressed preference towards the intersection of the axes to be at (0, 0), Emma treated this as a conventional choice (convention qua convention) rather than a rule that must be followed (ritual use of convention).

Although Emma initially rejected each of Graphs B, C, and D due to something unconventional about each, she eventually reorganized her RFs to consider if the graph reflected the underlying quantitative relationship. Reflecting a conscious awareness of the unconventional nature of such graphs, Emma referred to unconventional aspects of the graphs such as the axes as possible “mistakes” or “there to confuse [me].” But, consistent with understanding graphical conventions qua conventions, Emma understood each graph as reflecting the quantitative relationship defined by $y = 2x$.

Discussion

Addressing our research question, we showed interactions between Emma's meanings for conventions, quantitative reasoning, and use of RFs as she explored representations of $y = 2x$. Emma's strategies were powerful in leading her to reconcile unconventional graphs by focusing on the quantitative relationship and re-organizing her RFs. Emma's flexibility was illustrated through her reasoning through unconventional axes directions, scaling, and origin as she continued to rely on y being twice x and thus checking if appropriate points met on the graph. Emma's activity exemplifies the merit in students grappling with conventions on their own before directly being adopted throughout their schooling; we conjecture such discussions that allow students to consider quantitative meanings for algebraic equations and RFs may be fruitful in supporting students understanding graphing conventions as conventions. Further, such unconventional graphs can also be fruitful for supporting students in moving beyond a ritual use of conventions, such as realizing the intersection of the axes did not have to be "always (0, 0)".

Connecting back to the literature, several researchers have conjectured that students' meanings for algebra and graphs as a set of rituals may stem from a lack of opportunities to construct and reason about relationships between quantities (Moore, Silverman et al., 2014; Moore, Silverman et al., 2019; Paoletti, 2020; Paoletti et al., 2018; Thompson & Thompson, 1995). We note how Emma, as a fifth grader, was focused on a quantitative relationship in her activity, which allowed her to exhibit more flexible reasoning than the PSTs reported on addressing similar tasks (Moore, Silverman et al., 2019; Moore, Stevens et al., 2019). We conjecture Emma's flexibility relative to the PSTs may be due to her having significantly less school experiences adhering to conventions. That is, we conjecture conventions become rules for students when they are always used without explicit conversations or opportunities to consider other choices. In reality, students need this flexibility when faced with unconventional representations found to be applied in real-life contexts (e.g., as in STEM fields), especially as fields continue to evolve unpredictably over time along with possible new developments for representing quantities and needs for students reasoning within those developments arising.

Based on Emma's interactions with the given representations moving beyond conventions to determine the quantitative relationship depicted, we conjecture providing students with such unconventional coordinate systems early in their learning about graphs could support them in developing meanings for conventions qua conventions. However, our sample consisting of one student in one session limits our ability to evidence such conjecture. We call for future research to explore this possibly. Such research can support teachers and researchers in understanding and supporting flexible meanings for graphs that support students across STEM fields and real-world contexts.

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NUMBER SENSE AND GROUPITIZING: LOOKING UNDER THE HOOD OF ELEMENTARY MATH ACHIEVEMENT

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Educators and education researchers have looked for many years at methods to improve mathematics achievement for all students and to address gaps in achievement. In a study conducted in the early 1990s, Gray and Tall (1994) showed that math achievement in elementary students correlated with flexible use of numbers, also known as number sense. This study sought to replicate their findings in the context of a US elementary school and incorporate a new cognitive development in understanding students' mathematical thinking, "groupitizing." The results from 76 students in grades 2-5 confirm the earlier finding with a strong association between achievement and number sense. Groupitizing was also found to associate positively with math achievement and strategic flexibility in the use of numbers. These findings have important implications for early childhood mathematics instruction.

Keywords: Number Concepts and Operations

For many years, the average US student has remained in the middle of the pack in international assessments of mathematics achievement (Mullis et al., 2020; OECD, 2023). This middling "average" statistic covers over the more educationally relevant story that US student performance reveals staggering differences across various groupings of students, driven largely by differences in access to high quality educational opportunities and fiscal supports for schools (Reardon et al., 2019). In the 1990's two researchers in the United Kingdom, Eddie Gray and David Tall, provided a keen insight into how high and low achievers think through the most basic aspects of mathematical operations. In their interviews of groups of young children rated 'high' versus 'low' in math achievement, striking contrasts emerged in the step-by-step *thinking process* the two groups employed when showing how they reason through elementary problems. Remarkably, these two groups in the very same grade appeared to use fundamentally different strategies in basic arithmetic problems, even for problems that both groups solved accurately. More recently, cognitive scientists have established profound links between a child's global math achievement and more elemental *cognitive developments in number sense*. 'Number sense', according to Dehaene, is short-hand for our rapid intuitive ability to approximate, manipulate, and understand numerical quantities (Dehaene, 2011).

This study seeks to determine how well Gray and Tall's insights into United Kingdom students, circa the 1980s-1990s, holds true in a US elementary school decades later. Our more contemporary take on their question includes new measures of number sense and strategic flexibility, including recent advances in the cognitive development of “groupitizing”— a child’s emerging ability to conceptualize and enumerate larger sets of objects via flexible combinations (i.e. grouping) of smaller number group concepts (i.e. subitizing).

Research questions:

- To what extent does a student’s overall math achievement reflect their use of arithmetic strategies or step-by-step thinking processes?
- Does cognitive development of groupitizing help explain the emerging influence of “number sense” in the context of arithmetic strategy use?

Perspectives/Theoretical Framework

Number Flexibility

Achievement scores in elementary mathematics have been strongly linked to children’s ability to approach simple arithmetic problems in flexible ways (Gray, 1991; Gray & Tall, 1994). Gray (1991) distinguished different approaches to solving addition and subtraction problems. For example, when given two numbers to add, a child may: (a) *count-all* by applying their counting skills, starting at one, to each number in succession to arrive at the sum (e.g., when finding the sum of 4 and 7, a child will count to 4 and then keep counting 7 more units to get to the sum of 11); (b) *count-on* by starting with the cardinal value of one of the numbers and using the other number as a way to know how many times to increment the count by one (e.g., when finding the sum of 7 and 4, a child might begin with 7 as a whole, then count-up 4 more places in the number sequence to get to 11); (c) *derived fact*: mentally break up one of the operands to enable the convenient use of a memorized fact (e.g., when finding the sum of 7 and 4, a child might decompose 7 into 6 and 1 so they can apply their “6+4=10” knowledge, then increment by 1). Since this strategy depends upon a child’s larger emerging construct of *“number sense”* (McIntosh et al., 1992), for the rest of this paper we will equate use of this strategy with the development of number sense. Finally, a much larger shift in arithmetic processing occurs as older students engage in (d) direct retrieval of declarative memories of *known arithmetic facts*.

In the current study we focus our analysis on strategy (c) described above as critical, since the emergence of number sense strategies has been associated with greater mathematics success, not only in terms of students’ efficiency with simple arithmetic but also their subsequent success in more complex mathematics (Hornung et al., 2014; Jordan et al., 2010). Gray and Tall (1994) found that higher achieving students tend to display more flexible strategies (including the selection of more appropriate procedures), consistent with number sense, whereas lower achieving students rely on the less flexible procedural methods of counting.

Groupitizing and Subitizing

Groupitizing, a novel theoretical construct introduced by McCandliss et al. (2010), refers to rapidly enumerating arrays that are spatially grouped into subitizable subgroups. Groupitizing is typically assessed by presenting 4 to 9 dots and requiring an exact numerical answer, so that even very young children can accurately respond with minimal instructions (i.e., “How many dots all together?”). In the grouped condition the dots are grouped by spatial proximity into obvious subgroups of 1-4 dots. In the ungrouped condition no grouping cues are present. A child’s groupitizing ability is calculated by comparing the speed and accuracy of enumeration between grouped and ungrouped dot arrays. When grouping cues allow a child to enumerate the whole set faster, it is taken as evidence that they are accessing and combining the cardinal values of the subsets, rather than counting the dots in a successive fashion,

Extensive research reveals a grade-level effect in groupitizing abilities. Starkey and McCandliss (2014) studied 378 typically developing children from kindergarten through third-grade and found no evidence for groupitizing in kindergarten, an emerging effect in first grade, and a robust effect in second and third grade. This effect progressively develops across additional years of schooling, through at least 9th grade, as demonstrated by Guillaume et al. (2023), who investigated 1,208 children ranging from beginning of 3rd through the end of 8th grade. Their study demonstrated a remarkable growth trajectory in groupitizing that shows significant within school year growth within every school-year. In addition, groupitizing also underlies complex cognitive processes that are essential for understanding children's overall math abilities. Research indicates that groupitizing not only strongly correlates with neuro-cognitive measures of math fluency, but also outperformed every other measure investigated in its ability to account for unique variance in state mandated standardized test scores for mathematics, even after taking into account socioeconomic, domain-general, and domain-specific factors (Starkey & McCandliss, 2014; Guillaume et al., 2023). These findings collectively emphasize that groupitizing is a continuous construct that grows stronger year-over-year across K-9 schooling and it is fundamental for capturing key features of number cognition and math achievement beyond what is evident in symbolic numerical tasks.

Methods

Participants

Participants in this study were elementary students from a school in the California Bay Area in grades 2-5 and were nominated by teachers as their 'highest' and 'lowest' achieving math students. Teachers were not told what factors to use when selecting 'high' and 'low' achieving students, although most teachers reported using district test scores on a computer based standardized test as their metric for choosing students. Ultimately 76 students were included in the final data.

Data Collection

The research team conducted individual cognitive interviews with students. These interviews consisted of students completing a dot enumeration task on an ipad to measure their groupitizing ability and then they were asked six short arithmetic questions. For the dot enumeration task, dots were randomly set to appear as either grouped or ungrouped arrays. In ungrouped arrays the dots were roughly evenly spaced but appeared random on the screen while in the grouped arrays, the distance between groups was at least three times larger than the distance of dots within any group. For the grouped arrays, the number of subgroups and the maximum number of dots in any subgroup was varied. In total the task asked students to count 90 arrays in approximately three minutes - see Guillaume et al. (2023) for further details. Math problems were printed on a sheet of paper and handed to students so that they could write with a pencil or pen, if desired. They were also provided with small double-sided counters for optional use. Problems consisted of addition and subtraction of single digit numbers, addition and subtraction of a single- and double-digit number, and addition and subtraction of two double digit numbers.

Data Analysis

Using the framework from Gray (1991), students' strategies were categorized as *known fact*, *derived fact* (which we describe as number sense), *count on*/*count back*/*count up* or *count all*/*take away*. Two researchers initially coded independently, and then came together to discuss and reconcile all disagreements. Groupitizing ability is calculated based on the number of items children solve correctly per minute (Guillaume et al., 2023). We classified students into two groups based on their performance: individuals whose performance on grouped arrays improved by more than 10% compared to ungrouped arrays are classified as "high groupitizing"; those who showed minimal improvement on grouped arrays relative to ungrouped arrays are categorized as "low groupitizing". After all student strategies were coded, the data was compiled together with student characteristics such as grade level, whether the student was rated as 'high' or 'low' by the teacher, students' results from the groupitizing task, and students' results and strategy from each math problem. Since many students used different strategies on different questions, each instance of a strategy was treated separately.

Results

The results of this study confirm the conclusions from Gray and Tall (1994) - showing a clear relationship between achievement and number sense in arithmetic. Students, rated by their teachers as 'high' achievers in mathematics, used strategies reflecting number sense on over half the problems (mean=57.1%, 95%CI [49.1, 65.0]) - nearly four times the frequency of students rated as 'low' achievers (mean=15.3%, 95%CI [7.4, 23.2]). Students with high performance on the groupitizing measure also demonstrated higher rates of number flexibility. Specifically, the high groupitizing students also used number sense strategies over half of the time (mean=54.5%, 95%CI [45.5, 63.5]), which was roughly double the frequency with which the low groupitizing students did so (mean=26.7%, 95%CI [16.6, 36.7]). The alignment of these results further supports a relationship between the emergence of groupitizing, students' use of number sense strategies in the context of solving arithmetic problems, and ultimately achievement in the mathematics classroom. Additionally, students who were rated by their teachers as the lowest achievers were significantly less likely to exhibit high groupitizing profiles, and conversely, students rated as high achievers by their teachers were more likely to be identified with high groupitizing ability in the cognitive test. A chi-square test confirmed the significance of this result (X^2(1, n=74) = 15.074, p<0.000).

Discussion

These results replicate the prior study of Gray and Tall (1994) which is now over 30 years old. Students who were rated highly by their teachers in mathematics were significantly more likely to use number sense when solving simple arithmetic problems. As noted in the earlier study, the students rated 'low' in math were in fact often doing much harder work as they attempted to count numbers of increasing magnitude which made the possibility of error much higher and required extra time. The groupitizing effect also replicates Guillaume and colleagues' (2023) findings that increases in groupitizing abilities are tightly linked to increases in estimates of math achievement. This work extends the research on groupitizing by identifying associations between students' groupitizing ability (high versus low) and number sense. It suggests that these skills may share the same underlying cognitive mechanisms.

The results suggest two major implications for teaching and learning elementary mathematics. First, to improve fluency in simple arithmetic, teaching students to visualize how subgroups of numbers (or sets of objects) constitute larger numbers (or sets of objects) may be a powerful and highly inclusive scaffolding device for eventually escaping counting strategies and beginning to employ and master number sense strategies. While verbal memorization of declarative number facts can be helpful in some ways, it seems preferable that students learn how to work flexibly with numbers so that they can make use of a smaller set of learned facts and apply them in many novel situations.

Second, the relationship between groupitizing and number sense suggests that teaching groupitizing and a flexible approach to number, rather than a focus on memorization of facts, will help improve students' number sense. Classrooms across the US typically focus on the memorization of math facts, and the use of worksheets to practice them (Boaler, 2019). This study found that the ways students visually see the composition of numbers, and their use of flexible strategies of decomposition was associated with higher achievement.

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COORDINATING PRACTICES: IN-SERVICE SECONDARY TEACHERS' USE OF 5-PRACTICES TO SUPPORT MATHEMATICS DISCUSSIONS

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We present three cases of how in-service secondary teachers took up the 5-Practices instructional framework to support mathematics discussions in their classrooms. The data comes from the teachers first implementation of researcher designed and teacher modified mini-units in their classrooms that were part of a 3-year design experiment. The cases illuminate common yet subtly different affordances and challenges that the in-service teachers experienced in using the 5-Practices. They also highlight how the research team adjusted the professional support they provided to the teachers after the first implementation of the mini-units. One outcome of this adjustment was the development of a multi-tiered framework that relates teaching moves, practices, instruction, and instructional routines. The findings and framework contribute to the body of research on how in-service teachers can learn to support mathematics discussions.

Keywords: Core practices, teacher learning, mathematics discussion, design experiments

Mathematics educators have characterized rich mathematical discussions as ones that are centered on students' mathematical reasoning and are aimed at accomplishing specific instructional goals (Jacobs & Spangler, 2017). As such, mathematical discussions have been found to be a vehicle through which students have opportunities to learn substantive mathematics (Boaler & Staples, 2008). Opportunities for learning arise as students clarify (Goos, 2004), refine (Richland et al., 2019), expand (Webb et al., 2014), generalize (Land et al., 2014), and justify (Brodie, 2010) their mathematical reasoning through sharing it with their peers and by engaging with their peers' reasoning. Facilitating productive mathematics discussions relies on complex teaching capacities including making quick decisions, eliciting students' thinking, being responsive to students' contributions, and managing cognitive demand. Given these complexities, researchers have found that teachers often need support to effectively learn to facilitate productive discussions (Boston & Smith, 2009) where this support often occurs through decompositions, representations, and approximations of practice (e.g., Staples & Truxaw, 2010).

We use this paper to report on a cross-case comparison of three experienced secondary teachers who were learning to adapt and incorporate the *5 Practices* (5Ps) (Smith, Steele & Sherin, 2020) into their teaching. The teachers were part of a 3-year design experiment whose aim was to study how secondary in-service teachers facilitated students' *mathematical generalizing*. We introduced the teachers to the 5Ps framework for two reasons: (a) we considered ourselves more likely to see high quality and varied instances of how teachers' supported student generalization if the teachers were also supporting student discussion; and (b) we wanted teachers to have a common instructional framework to support cross-classroom conversations about instruction. The data we report on in this paper is from the first time the teachers implemented researcher-designed and teacher modified mini-units in their classroom. We respond to the following research question: What affordances and challenges did the teachers experience in their re-composition of the 5Ps over the course of implementing their mini-unit?

Literature Review

Many early studies that investigated how teachers facilitated productive mathematical discussions occurred in classrooms with teachers who had considerable content and pedagogical expertise: often the teacher was a researcher (e.g., Ball, 1993, Heaton, 2000; Lampert, 1990). These studies yielded substantial information about teaching practices that support mathematics discussions. Subsequently, mathematics educators transformed this information into practitioner friendly frameworks that could be used to support both pre- and in-service teachers (PSTs and ISTs, respectively) to support mathematical discussions (e.g., Chappin, et al., 2003/2013; Stein, et al., 2008). Researchers have used these frameworks with PSTs in mathematics methods courses (e.g., Ghousseini & Herbst, 2016; Tyminski, et al., 2014) and with ISTs in professional development settings (e.g., Reinsburrow, et al., 2022) to study the PSTs and ISTs learning of discussion-based practices. However, Ghousseini (2015) and Pang (2016) both identified a dearth of research that examines how PSTs or ISTs learn to use discussion-based frameworks *in actual classrooms* (as opposed to in methods courses or professional development settings).

Researchers (e.g., Bağdat & Yanik, 2023; Heyd-Metzuyanim et al., 2019; Kooloos, et al., 2023; Martins et al., 2023) have begun to respond to this lack of research. Within this work, they have reported that practices that occur prior to teaching (e.g., setting goals, identifying tasks, and anticipating) are easier for teachers to learn than those that occur during teaching (e.g., monitoring, selecting, sequencing, and connecting) (Pang). We use this study to contribute to this growing body of research by identifying how teachers adapt and incorporate discussion-oriented practices, specifically the 5Ps, into classroom instruction. We were particularly interested in how the teachers transitioned from using decompositions, representations, and approximations of practice in professional development to implementation of these practices in live instruction.

Analytic Framework

Jacobs and Spangler (2017) define teaching moves as “actions that teachers take that observers can see or hear, such as asking a question, providing a representation, or modifying a task” (p. 778). They differentiate teaching moves from goals, which they define as “the intentions teachers have….(which) typically must be inferred by researchers because they are not usually stated explicitly” (p. 778). Jacobs and Spangler acknowledge that moves and goals take place at different grain sizes. However, they do not introduce a language to differentiate among the different grain sizes. To capture these differences in grain size, we introduce a four-tiered nested framework that preserves Jacobs’s and Spangler’s distinction between observable actions and goals a teacher has for these actions. Moving from smallest to largest grain size, we use the term *teaching move* to mean actions that occur during moments of interaction with students. *Teaching practices* (Smith & Stein, 2017) occur over a longer timeframe within a lesson where multiple teaching moves are embedded in each teaching practice. Teaching practices, then, function together to comprise *instruction* where we use the term instruction to refer to teaching that occurs in a single lesson. We use the term *instructional routines* for patterns in a teacher's instruction that can be discerned over multiple days of instruction. At each grain size, it is possible to characterize both a teacher's observable actions and their goals.

In outlining the 5Ps, Smith, et al. (2020) do not make the same distinctions we do in grain size, but it is possible to interpret the 5Ps relative to these distinctions, particularly the distinction between teaching moves and practices. In Table 1 we define Smith, et al.'s practices that occur while teaching, Practices 2-5—monitoring, selecting, sequencing, and connecting—and give examples of observable actions and goals that are at the practice and move grain size. In Table 1, we omit Practice 0, setting goals/selecting tasks, and Practice 1, anticipating, because they occur prior to teaching, and our primary interest in this paper is what happened while teaching.

Table 1: Monitoring, Selecting, Sequencing, and Connecting Practices (Smith et al., 2020)

5Ps Defined	Actions Related to Teaching Practice	Goal(s) of the Teaching Practice	Actions Related to Teaching Moves	Goals of the Teacher Moves
Monitoring: Attending to student thinking while students work on a problem	• Circulates among groups of students, revisiting groups when appropriate • Asks assessing and advancing questions	• Track student thinking • Assess student thinking • Advance student thinking (p. 86)	• Asks an assessing or an advancing question • Uses a talk move	• Understand student thinking • Move a student toward a learning goal
Selecting & Sequencing: Choosing what student work to discuss and organizing that work in a specific order	• Records the range of strategies from which to choose (selecting) • States the order of student presentations (sequencing)	• Ensure student work to be shared connects to all learning goals • Establish a coherent storyline for the work presented (p. 122)	• Asks a student (privately) if they will share their work • Calls on a specific student to share their work first	• Ensure a student is willing to present • Indicate to an individual student when to share
Connecting: Using student work to make connections to learning goals or connections among	• Asks questions to all presenters to highlight connections to learning goals • Records student observations about similarities	• Connects student work to the full range of learning goals for the lesson • Connects the *set* of selected	• Asks a question to highlight how a particular piece of student work connects to a learning goal(s) • Has students turn-and-talk about	• Makes connections between an individual presenter's work and learning goals

different pieces of student work	and differences across all student work	student work to each other (p. 172)	how a new piece of work is related to their own	• Makes connections between two specific pieces of student work

The four-tiered framework was an *outcome* of the first year of our design work. We introduce it here because it supports what follows. We see the distinctions in grain size as important to framing the claims a researcher is making about a teacher's teaching. That is, it is different to make claims about a teacher experiencing success at the smallest grain size, teaching moves, than it is to make claims about a teacher experiencing the same success at the largest grain size, instructional routines. It is different because the grain size of an action impacts the ease with which a teacher will successfully integrate that action into their instruction. For example, it is easier for a teacher to introduce smaller grain size teaching moves like wait time, re-voicing, asking an assessing question, or inviting students to participate than it is to adopt larger grain size instructional routines like consistently using the 5Ps over the course of a mini-unit. Our aim in this study is to describe instructional routines that we could discern over multiple lessons taught by the ISTs; in particular, we investigate the way that the ISTs fit together practices, from the 5Ps, in instruction, and identify patterns (i.e., instructional routines) related to how they did so. Although important, we have a smaller focus on specific teaching moves and practices.

Methods and Methodology

Design experiment research involves researchers designing an intervention, testing that intervention, and then refining the intervention during subsequent iterations (Cobb et al., 2003). As part of this process, researchers identify conjectures they have that guide the design of the intervention, where one result of a study involves documenting how they modified their conjectures for future iterations of the intervention. We focus on one conjecture we made related to our work with the teachers relative to the 5Ps. *Conjecture:* Decompositions, representations, and approximations of *individual teaching practices* and *moves* within practice was sufficient support for ISTs to recompose these practices in instruction in ways that would support them to develop reliable new instructional routines.

During the summer, prior to implementing the mini-units in their classrooms, all three teachers participated in eleven, 3-hour professional collaboration sessions. The first and third authors designed these sessions to focus on four themes: (a) the mathematical content of the mini-units; (b) student reasoning related to the mathematical content, which included video cases (Burch, et al., 2021); (c) instructional planning for the mini-units using the 5Ps as a guide for this planning (Smith, Steel, Sherin, 2020); and (d) a teacher-appropriate framework for supporting generalization in the classroom (Driscoll, 1999). The mini-units were initially developed by the third and first author (Burch & Tillema, unpublished) and were subsequently modified by the teachers as part of the process of planning for the implementation. Each mini-unit lasted 3-7 days, depending on the length of each teachers' class period (i.e., 45-, 50-, and 70-minutes). All lessons in the mini-unit were videotaped using three cameras: one captured the whole classroom, one captured the teachers' interactions with students, and one focused on a small group of students as they participated in the lessons. The three participating teachers—Celine, Felix, and Hazel—had 30, 21, and 6 years of teaching experience, respectively. We use Table 2 to briefly characterize classroom structures and opportunities for discussion in each teachers' classroom. The observations are based on field notes taken during five visits to each teacher's classroom and an interview about their instruction, which occurred in the year prior to teaching the mini-units. None of the teachers used student driven discussion as the primary organizational tool for instruction in their classrooms, but all incorporated, to varying degrees, elements of discussion.

Table 2: Brief Characterization of Teacher' Instruction Prior to the Project

Celine	Celine relied largely on problems from traditional textbooks in which more open-ended application problems occurred after problems involving a particular skill. Celine's normal instructional routine was to have students begin class in small groups to discuss the previous evening's homework. She, then, introduced new content through a teacher led problem solving session, which she considered a guided-discovery approach. Celine's guided discovery consisted of her publicly solving problems where she directed the solution of the problem, but students were expected to contribute key pieces to the solution of a problem. Celine had carefully identified the key pieces students were expected to contribute based on what they had already worked on. Celine, then, offered students time to solve several similar problems in small groups where she had an array of mechanisms in place to support student to student interactions. She assigned 3-5 homework problems at the end of class to work on outside of class.
Felix	Felix relied largely on problems from traditional textbooks. He began class with two students each presenting a homework problem where students explained their solution to the class. During this time, other students in the class asked the presenters questions. Felix, then, used the student presentations to highlight key ideas that he anticipated other students might have struggled to understand. After student presentations, Felix gave a lecture on the topic for the day, with some students taking notes and others listening to the information. Felix, then, assigned 8-10 homework problems and gave students time to work.
Hazel	Hazel often started her class with an open-ended problem that students worked on individually, in small groups, or as a whole class. She used the problem to hook students and highlight key ideas she intended for them to work on in class that day. Once the class discussed this problem, they typically worked either individually or in small groups on more common textbook problems. The outcome of this work was for students to present their solutions to other students either in small groups or whole class. Hazel, then, assigned students 3-5 homework problems at the end of class to work on outside of class.

For analysis, we mixed the whole classroom video and teacher video into a single video file. One mathematics education faculty member and six graduate students coded 45-minute segments of Celine's mini-unit using the 5Ps framework. After coding a 45-minute segment, the team met to discuss and refine codes where this part of analysis focused on what observable actions corresponded to codes for each practice. The research team continued this process until coding for Celine's entire mini-unit was complete; at this point code definitions were relatively stable. Next, the team coded one lesson from Felix and one from Hazel to make additional minor adjustments in the code book based on differences across the three teachers' instruction. The result of this process was a code book with stable code definition for 10 codes: launch; monitoring; assessing questions; advancing questions; selecting: evidence of teacher choosing student's work; selecting: evidence of what student work was presented; sequencing; connecting to learning goals; connecting student work; and student work time. The research team, then, broke into two subgroups, and used the, now stable, code book to code the remaining lessons from Felix and Hazel's mini-units. This coding occurred similarly to what is described above. To respond to our research question, we used descriptive statistics to capture the percentage of time each teacher spent on each practice, which supported our qualitative interpretations of how the teachers recomposed the 5Ps in instruction and what patterns emerged as instructional routines.

Results

In Table 3, we identify the percentage of time and coding frequency for each code. We use this information to highlight salient features in each teacher's re-composition of the practices in instruction to characterize their instructional routines. It is important to note that none of the teachers wanted to use a monitoring chart (cf. Smith et al., 2020) the first time they implemented their mini-units. Their concerns were rooted in perceived trade-offs between a monitoring chart's helpfulness to organize their thinking and distraction from staying present with their students.

Table 3: Descriptive Statistics Related to Each Practice[4]

	Felix			**Celine**			**Hazel**		
# of Lessons	3			7			5		
Total Time	3:45:59			5:44:16			3:53:15		
	Total Time (hr:min:sec)	**Coverage (%)**	**Code Fre-quency**	**Total Time (hr:min:sec)**	**Coverage (%)**	**Code Fre-quency**	**Total Time (hr:min:sec)**	**Coverage (%)**	**Code Frequency**
Launch	57:00	25.22%	13	57:26	16.69%	33	47:59	20.57%	17
Monitoring	1:08:45	30.43%	16	1:57:57	34.27%	21	1:56:23	49.90%	20
Assessing Questions	7:02	3.11%	11	10:52	3.16%	20	34:31	14.80%	75
Advancing Questions	6:49	3.02%	8	22:03	6.41%	43	47:13	20.25%	59

[4] Percentages exceed 100% because of overlapping codes (e.g., monitoring and assessing questions).

Selecting (Choosing)	11:38	5.15%	11	9:37	2.79%	38	3:44	1.60%	16
Selecting (Presenting)	31:28	13.92 %	11	34:47	10.11 %	39	12:51	5.51%	19
Sequencing	25:33	11.31 %	2	49:48	14.47 %	9	16:06	6.91%	7
Connecting Learning goal	57:31	25.46 %	8	1:41:10	29.39 %	46	35:32	15.23 %	21
Connecting Student work	31:11	13.81 %	9	3:01	0.88%	3	10:11	4.37%	10
Student work time	6:54	3.06%	4	22:12	6.45%	10	0	0.00%	0

Felix spent 30.43% of the total instructional time monitoring students, while they worked on problems. Of that time, he spent the lowest percentage of time asking assessing or advancing questions (i.e., 10.18% and 9.80%, respectively, for a total of 19.98% of his monitoring time[5]). Instead, Felix tended to listen to and observe student-to-student conversations while they worked in small groups. For 15.35% of the time he was monitoring, we double-coded selecting (choosing)—a code we used when there was observable evidence that a teacher was choosing a particular student's work to share later in the lesson. We infer, then, that, while he was listening to and observing student conversation during monitoring, he was also focused on determining what work he would *select* to have students share with the class. Moreover, Felix did not have any sequencing codes that occurred while he was monitoring. This indicates there was no observable evidence that he was considering how to sequence student work while monitoring. We infer from this combination of codes, and our qualitative observations, that Felix was challenged to coordinate asking students assessing and advancing questions (relatively low percentage of his monitoring time) while also aiming to determine what student work to select (relatively high percentage of his monitoring time) and sequence that work (none of his monitoring time). Our inference is that he was heavily focused on what work he would select while he was monitoring over, for example, assessing differences in student thinking.

Among the teachers, Felix had the highest percentage of time coded for selecting (presenting) (13.92%) and the lowest number of code instances (11 coded instances). We used this code when students presented their own work to the class. This combination of codes meant that Felix had *fewer, but substantially longer periods of time* during which students presented their work to the whole class than the other teachers. We attribute the length of student presentations to two factors: Felix spent minimal time asking assessing and advancing questions, which meant that sometimes significant mathematics surfaced for the first time during these presentations; and he, among the teachers, allowed for the most open ended whole class discussion of student ideas.

Felix managed the sequencing practice by having multiple students put their work on the white board at the same time. Doing so meant that he had very few instances of the code

[5] These percentages differ from those in the table because they represent the percent of time he was asking each kind of question relative to the total time he was monitoring rather than relative to the total time of instruction.

sequencing, in part, because all student work was displayed simultaneously with students often comparing multiple pieces of student work to each other at the same time. One affordance of this instructional decision was that Felix had ample opportunity to engage in the practice of connecting student work (13.81% of his total instructional time). This code was double coded with selecting (presenting) for 63.38% of the total selecting (presenting) time. Thus, Felix often actively asked questions of students as they presented their work—questions that focused on making connections to other students' work. We also coded a substantial portion of Felix's total instructional time as connecting to learning goals (25.46%); however, this code was only double coded for 4.6% of the total time coded selecting (presenting). This indicated that Felix often made connections to learning goals after students presented their work either by asking further questions of them or by making his own explicit statements of connection.

Celine spent 34.27% of her instructional time monitoring student work. She spent a relatively low percentage of her *monitoring time* asking assessing questions (9.18%) with a larger percentage of that time spent asking advancing questions (18.64%) for a total of 27.82% of her monitoring time. Celine, like Felix, spent much of her time monitoring by listening intently to small group conversations and observing the work that students produced during this time. One reason Celine had a lower percentage of time coded for assessing questions than advancing questions was she often used her assessing listening as a basis to ask advancing questions.

Celine's data also indicates that she experienced a challenge in coordinating asking assessing and advancing questions with engaging in initial phases of selecting and sequencing while monitoring. However, this challenge expressed itself differently in her re-composition of the practices in instruction than it did in Felix's instruction. That is, in contrast to Felix, only 1.6% of her total time monitoring was double coded with selecting (choosing). Her monitoring, then, included little observable evidence that she was considering what student work she would select.

Celine had a lower percentage of total instructional time coded for selecting (presenting) (10.11%) as compared to Felix (13.92%), but a high number of instances of the selecting (presenting) code (39 coded instances). This set of code combinations indicates that she had *frequent but short times* during which *multiple* students had the opportunity to present their work, and they presented sequentially. She did have a substantial percentage of her total instructional time coded as connecting to learning goals (29.39%). However, the code connecting to learning goals was double coded only 1.4% of the total time the selecting (presenting) code was used. This indicates that Celine tended to have students present their work, and then once they had presented it, she made connections to learning goals by either asking the class additional questions or by making her own explicit statements of connection to the learning goals. This sequential code structure indicates that Celine often prepared questions to ask the class as students presented their work but did not integrate this questioning into student presentations. Another consequence of Celine's students presenting their work one-by-one was that, in many instances, the record of student work was gone after it was presented. As such, it was challenging for Celine to make connections across student work. This is supported by only 0.88% of her total instructional time being coded as connecting student work; moreover, this code was never double coded with the selecting (presenting) code.

Hazel spent 49.90% of her total instructional time engaged in the practice of monitoring, a substantially higher percentage than either Felix or Celine. Hazel also spent 70.28% of the time she was coded as monitoring asking either assessing or advancing questions (29.66% and 40.62%, respectively)—also substantially higher than Felix or Celine. Among the three teachers, Hazel's monitoring practice provided the most concrete, observable evidence that she was using her time monitoring to make sense of details about her students' thinking and, therefore, would be well-positioned to both choose student work to share with the class and to consider a sequence for this work. However, only 0.01% of her monitoring time was double coded with selecting (choosing), meaning there was little observable evidence she was engaged in the initial processes of selecting student work while monitoring and no evidence she was considering sequencing it. This code structure again indicates that Hazel was challenged to coordinate monitoring with the initial phases of selecting and sequencing student work.

We infer a challenge for Hazel was balancing the time she spent monitoring with foreseeing the amount of time she would need to effectively engage her class in whole-class discussion about their work. Of the three teachers, she had the lowest percentage of her instructional time coded as selecting (presenting) (5.51%), as sequencing student work (6.91%), and connecting to learning goals (15.23%). These percentages support our observation that she often did not have sufficient time at the end of her lesson to connect to learning goals even though she did engage in this practice at the end of each lesson. The combination of percent of instructional time with frequency for the selecting (presenting) code indicates that Hazel was in between Celine's frequent, short student presentations, and Felix's less frequent, longer student presentations.

The percentage of instructional time coded selecting (presenting) that was double coded as connecting to learning goals was 59.33%. This indicated that, when students were presenting their work, Hazel was often actively questioning them in ways that supported connecting to learning goals, a phenomenon we attribute to her careful use of assessing and advancing questions while monitoring. Overall, 4.37% of Hazel's total instructional time was coded as connecting student work—again between Felix (Felix had a percentage about 3 times higher) and Celine (Celine had a percentage about 1/5 as much). Hazel, like Celine, frequently had multiple students present their work one-by-one often without having a way to simultaneously display multiple pieces of student work. This organization for presenting student work meant that, while students were presenting, Hazel focused on connecting to learning goals rather than connecting student work, which did not occur as a double code with selecting (presenting).

Discussion

One possible way to read the data is that each of the three teachers, in one way or another, was relatively far away from a high-level implementation of the 5Ps. We caution against this interpretation; we were specifically interested in documenting the affordances and challenges that experienced ISTs faced when coordinating the practices together in *instruction* for the first time, and how to support them in the emergence of new *instructional routines*. With this observation, we return to the *conjecture* that guided our design: Decompositions, representations, and approximations of *individual* practices and *moves* within practice was sufficient support for in-service teachers to recompose these practices in instruction in ways that would support them to develop reliable new instructional routines. During the professional collaboration sessions, the ISTs were all able to engage with the practices individually, demonstrating, for example, what we considered to be high level approximations of each individual practice. However, in their classroom teaching, a substantial challenge they faced was how to coordinate the practices with each other in instruction to produce new reliable instructional routines. Moreover, that was how they judged their success; they judged their success, at least initially, in terms of how the practices fit together within a single lesson, and how multiple days of instruction produced new reliable instructional routines—criteria that were neither explicit for them, nor to us, until they implemented the mini-units. Given these observations, we refine our conjecture: the ISTs needed support, in decompositions, representations, and approximations, that focused more explicitly on coordinating practices with each other to produce reliable instruction and instructional routines.

The three cases offer insight into what this support could look like. That is, one common challenge across all three teachers was to coordinate monitoring with the early phases of selecting and sequencing. This challenge may have been due to the teachers' choice not to use a monitoring chart, however, we do not attribute this challenge only to this decision. Moreover, there were subtle differences in their experience of this challenge, and thus differences in the support that could address it. Felix, for example, while monitoring, was consumed with what work to select, and as such asked a relatively low number of assessing and advancing questions. His listening, while monitoring, was often focused on what work he would choose rather than using an assessing, and then advancing question cycle to help him determine what students were thinking and then to move that thinking forward. On the other hand, Hazel focused extensively on asking assessing and advancing questions, while monitoring. Doing so gave her the most detailed information about her students' thinking and thus prepared her to ask students questions to make explicit connections to learning goals. However, there were very few observable teaching actions focused on preparing to select student work, which was related to her inefficiency in transitioning from monitoring to selecting and sequencing student work. These different challenges call for differences in support each teacher needed.

We close by identifying one contribution of this study. Research reports on teachers' use of the 5Ps often focus on characterizations of an individual practice (e.g., Dunning, 2022; Reinsburrow, et al., 2022; Tyminski, et al., 2014) even in reports where multiple of the practices are considered (e.g., Bağdat & Yanik, 2023). These accounts offer important details about a specific practice, including the distinct goals and uses that teachers have for the practice. We think another important point of focus is on how teachers learn to coordinate the practices together (e.g., Pang, 2016), and how this coordination evolves over time into new instructional routines. Our assertion is supported by Felix, Celine, and Hazel's initial judgements of the success of a given lesson (i.e., their instruction) as based—*not* on their use of individual practices, but rather—on how the practices fit together for them within the lesson and across the mini-unit. Given that experienced ISTs' instruction is rooted in established instructional routines, working with them to experience what they deem to be successful instruction is important as they determine for themselves whether to adopt new practices that alter established routines.

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MODELING STUDENTS' STRATEGIES WHEN CREATING A GRAPH: A FOCUS ON REFERENCE FRAMES AND COORDINATE SYSTEMS

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We examined students' thinking of graphs around a graphing task from 14 individual interviews, in terms of three layers—frames of reference, coordinate systems, and graphs—and explored their productive and intuitive strategies. As a result, we present a framework that offers a characterization of students' graphing activities. We then discuss implications of the framework.

Keywords: Mathematical Representations, Cognition, Middle School Education.

Graph literacy is important for students to progress in STEM coursework and careers (Paoletti et al., 2020; Costa, 2020) and for making sense of, and responding to, information in the real world (Yore et al., 2007). Sherin (2000) argued researchers should move beyond identifying students' difficulties to explore students' natural inclinations when developing graphical representations and how these inclinations can be leveraged to support graph literacy. In line with researchers who have focused on asset-based accounts of students' strategies, the work we report in this paper was guided by the question, 'What cognitive strategies and intuitive insights do middle school students invent or draw upon when representing quantities in a graphical representation?' To address this question, we present a framework we developed and refined through analyzing interviews with 14 middle school students on the Family Frenzy graphing task. We close by discussing the broader implications of the presented framework.

Some Relevant Literature and Brief Theoretical Underpinnings

Researchers have identified many difficulties students encounter with graphs. Of relevance to this report, researchers identified that students often treat graphs as literal representations of a situation (Bell & Janvier, 1981; Clement, 1989; Lai et al., 2016; Oehrtman et al., 2008). For example, Clement (1989) described students interpreting a speed-height graph of a bike rider as representing a hill the bike rider traveled over. To explore ways students may reason as they construct graphs, we modified Swan's (1985) "Bus Stop Queue" task (Figure 1a), which requested students to interpret a scatterplot by matching each person in the picture to their appropriate point. Note that height and age were labeled along the horizontal and vertical axis, respectively; from this we inferred one goal of the task was to perturb students who interpreted graphs as literal pictures, i.e., interpreted the height of a point as the height of a person. We modified the task by switching the axes labels (Figure 1b) and asking students to create their own graph, as our goal was to examine students' generative activities and intuitions they can build on.

Our work builds on previous work that examined students' generative activities (diSessa et al., 1991; Sarama et al., 2003; Sherin, 2000). Sherin (2000) described students' intuitive representations when tasked to create a picture to describe a motorist's motion over time. Students' depictions often contained pictorial features (i.e., using symbols such as lines to represent more or less of a quantity) that could lead to ideas akin to conventional graphs. However, as Sherin stated, he did not "attempt to be more specific about how this collection is constituted in detail (for example, in terms of knowledge structures)" (p. 413). In this paper, we account for cognitive strategies students draw upon to identify knowledge structures (i.e., thinking patterns that might be involved in students' graph literacy).

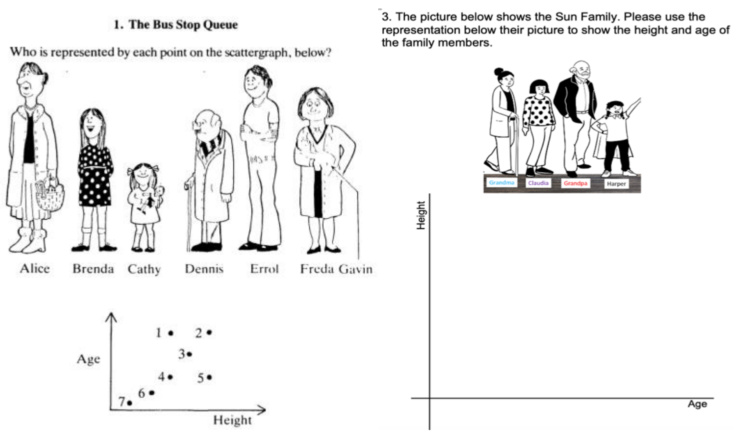

Figure 1: (a) Bus Queue task from Swan (1985); (b) The Family Frenzy task

Frames of Reference, Coordinate System, and Graph

Graphical representations involve spatial depictions of quantities (Thompson, 2011) and are a way to mathematize phenomena. A graphical representation consists of three layers: frames of reference, a coordinate system, and a graph (a collection of points). Frames of reference refer to mental structures used to gauge the relative extents of various attributes in the phenomenon (Levinson, 2003; Lee, 2017; Joshua et al., 2015). Thinking within frames of reference entails attending to and establishing reference points, directionality, and having an idea of what attributes to consider and how to measure them (Joshua et al., 2015; Lee et al., 2020). The nature of graphs and hence, ways of thinking about a graph fundamentally depends on the frames of reference and coordinate systems upon which they are created.

Methods

The data presented here comes from 14 clinical interviews (Ginsburg, 1997) across two projects, both aimed to examine middle school students' (5^{th} to 8^{th} grades) graphing meanings. We collected video recordings, screen recordings, and digital copies of students' written work. The projects recruited students from various mathematical and socio-economic backgrounds. In this paper, we present data from the Family Frenzy task (Figure 1b) which was used in these clinical interviews. We initially examined students' thinking in Family Frenzy and sorted them related to frames of reference, coordinate systems, and graphs (three layers) using the Analytical Framework for Making Sense of Students' Graphical Representations (Lee, 2024). Next, using open and axial techniques (Corbin & Strauss, 1996), we created descriptions of themes within each layer; from these descriptions, we further abstracted and classified the students' strategies, and we present those results in Table 1. We note that the resulting codes are meant to be a holistic characterization of the students' strategies for each attempt they made at the task. Each graphing attempt received a set three of codes where one code was from each category (graphing activity, reference frame activity, coordinate system activity). Results

Students demonstrated a variety of intuitive approaches, which is organized in Table 1. In the table, *representational objects* refers to the (often geometric) objects students physically inscribed on the paper, which included stacked dots, stick people, and bubbles (regions). To distinguish students' inscriptions from the pre-made, two-line segments labeled as Age and Height (what the researchers intended as axes), we call the totality of the two-line segments and the space they span as the *graph space*. We take both the graph space and students' representational objects to constitute their representation of the Sun Family's height and age. We next present one student's strategies to exemplify a subset of these strategies.

Table 1: Summary of Students' Representation Strategies

<table>
<tr><th></th><th>Graphing Activity</th><th>Reference Frame Activity</th></tr>
<tr><td>Height</td><td>• Spatial Transfer: Uses fingers or other physical materials to transfer the height of members in the picture to the graph space and marks the height using representational objects.
• Non-physical Transfer: Estimates relative heights of each member, without using any observable physical action or object to transfer length and indicates such heights in the graph space using representational objects.</td><td>• Pictorial Ordering: Represents height in the order of the members standing in the picture (e.g., Grandma, Claudia, Grandpa, Harper) in the graph space.
• Quantitative Ordering: Represents height in ascending or descending order of heights of the members (can be different order than in picture; e.g., Harper, Claudia, Grandma, Grandpa).</td></tr>
<tr><td>Age</td><td>• Indexing: Estimates relative ages of members based on picture and writes the age of members near the representational object used for height in the graph space. Ages' representations are add-ons to those used for height.
• Non-indexing: Estimates relative ages of members based on picture and indicates such ages using representational objects in the graph space. Ages' representations are independent of (though could be related to) those used for height.</td><td>• Pictorial Ordering: Represents age in the order of the members standing in the picture (e.g., Grandma, Claudia, Grandpa, Harper) in the graph space.
• Indexed Ordering: Represents age in the same order of height in the graph space because age is indexed onto height's representational objects.
• Quantitative Ordering: Represents age in ascending or descending order of ages of the members (can be in different order than in the picture).</td></tr>
<tr><td>Height and Age Together (Coordinate System Activity)</td><td colspan="2">• One, implied axis as an ordered number line: One of the axes in the graph space is acting as an ordered number line while the other is not; 1-D coordination.
• Two, separate, implied axes as number lines: Both axes in the graph space are acting as an ordered number line for each quantity but the two number lines are used individually; two 1-D coordinations.
• Two, overlapping, implied axes as number lines: One axis in the graph space acts as an ordered number line for both quantities; both quantities are represented on a single axis: stacked 1-D coordination.
• Two, coordinated, implied axes as number lines: Each axis in the graph space is acting as an ordered number line for a quantity; both quantities are represented in the two-dimensional space produced by the product of the two axes: 2-D Cartesian coordination</td></tr>
</table>

Thomas' Representation and His Strategies

Six students used a *spatial transfer* strategy when graphing the family's height. Transferring was evidenced by measuring the height in the picture in some manner (e.g., using a ruler, using the span of two fingers) and then marking this measurement directly in the graph space, resulting in a literal copy of the cartoon's height. Figure 2 shows Thomas enacting spatial transfer (and his final representation). Thomas partitioned the Height axis into what he called centimeters. He then used his fingers to measure Grandma's height and then maintained this gap to represent her height on the vertical axis (Figure 2 left and middle). He used this strategy for all the family members, which yielded a set of stacked names on the y-axis (Figure 2 right). Further, this strategy yielded a *quantitative ordering for heights* in that the heights of family members were ordered from shortest to tallest in his representation.

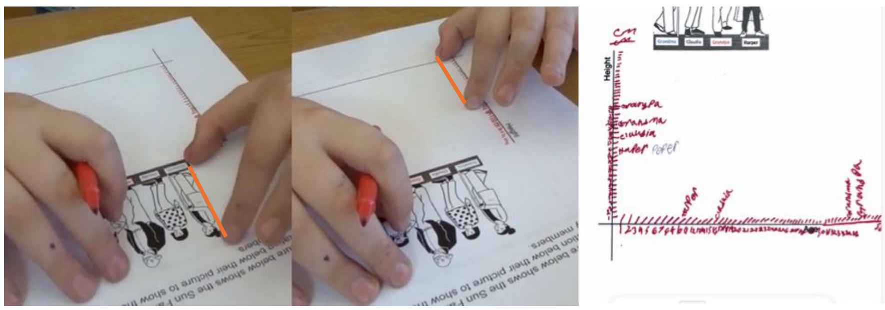

Figure 2: Thomas' Strategy and Final Representation

Thomas used a *non-indexing strategy for age* as he inferred ages based on the picture and represented them along the horizontal axis in the graph space. Specifically, he placed 60 tick marks on the Age axis, and plotted the family members from youngest (Harper) to oldest (Grandpa) along the axis. Thomas ordered the ages in ascending order (see Figure 2 right), and we inferred this order was independent of his representations of height, yielding a *quantitative ordering for age*. Thomas' graphing was indicative of using *two, separate, implied axes as number lines*. Based on how he partitioned each axis into unit-heights and unit-ages and plotted family members' height and age on each axis, we inferred he treated each axis as a number line. Note, Thomas plotted each family member twice, once along each axis. When the interviewer asked if he could find a way to mark each family member only once, Thomas maintained that age and height could not be represented together with a single point. Thus, we inferred his graph space remained as two, separate, implied axes as number lines.

Discussion

We presented a framework characterizing a variety of strategies students used when creating graphical representations given a pictorial scenario. Our framework attends to students' graphing activities of each quantity, height and age before potentially being coordinated together. The framework provides more nuanced "knowledge structures" (Sherin, 2000, p. 413) that students draw on when constructing graphs than previously described, attending to their graphing activities in relation to their reference frame and coordinate system activities. These activities refer to mental actions we inferred from observing students' physical graphing actions. We do not intend our framework to be exhaustive, but instead a starting point for future research that can contribute additional strategies to the framework. We believe the students' strategies in the framework can be leveraged to support students in achieving more conventional graphing meanings. For example, we can build from students' creations of 1-dimensional graphs as conceptual starting points to motivate the potential construction of a 2-dimensional coordinate system from their 1-dimensional graphs. While most research has described students' literal translations as hindering, we view it as a tool that could be productively used and subsequently modified to lead to more productive graphing meanings. We will be further examining these constructions as we continue in our research.

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Chapter 7: Number Concepts and Proportional Reasoning

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ELICITING EARLY MULTIPLICATIVE IDEAS: USING DIFFERENT TYPES OF ARRAYS AS QUICK IMAGES

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In this paper, we present part of a larger study investigating the development of multiplicative ideas in young children before their introduction to multiplication. Arrays were displayed as Quick Images (briefly displayed for 5-10 seconds), including those with hidden and missing amounts, over nine days across 3 weeks. Second-grade students (n=20) were asked to determine the total amount, followed by whole-class sharing and discussions. Using a constant comparative method, we analyzed students' problem-solving strategies reported here. Results indicated an initial reliance on subitizing and counting strategies, with neglect of row/column size in determining the total amount. Students exhibited a shift in attending to rows/columns as composite units, indicating early multiplicative ideas, particularly when making sense of arrays with partially missing amounts. Implications for teaching and research are also discussed.

Keywords: Number Concepts and Operations, Mathematical Representations

Introduction

Multiplicative reasoning holds crucial significance in mathematics, particularly for proficiency in advanced concepts such as algebraic reasoning, ratios, proportions, and measurement, extending beyond the primary grades. Students gradually cultivate multiplicative thinking (Vergnaud, 1994), and making sense of multiplication as part of this thinking requires understanding the iteration or replication of a composite unit rather than a mere enumeration of individual elements (Killion & Steffe, 1989; Steffe, 1992). Specifically, recognizing multiplication as distinct from addition involves acknowledging its binary nature, perceiving the factors as two distinct inputs—the number of groups and the quantity in each group—rather than a singular entity. Scholars contend that students' grasp of the "equal grouping" structure is fundamental to their ability to think multiplicatively (Killion & Steffe, 2002; Sullivan & Mousley, 2001).

Many studies (e.g., Barmby et al., 2009) provide evidence that array models can support students' understanding of multiplication and have the potential to pave a way for multiplicative ideas. In particular, arrays encompass both an "equal grouping" structure and a binary spatial structure with rows and columns as composite units, and thus can offer a visual model "to fully appreciate the two-dimensionality of the multiplicative process" (Young-Loveridge, 2005, p. 39). Unfortunately, the prevailing literature also provides evidence that many children do not perceive the row-by-column spatial structure of an array and resort to inefficient strategies such as counting the objects one-by-one (Battista et al., 1998) and do not consider rows and columns as composites when asked to find the total amount (Barmby et al., 2009). In fact, several researchers have argued that students do not automatically attend to spatial structure but construct it according to how they perceive a situation, shape, or object (Battista & Clements, 1996; Mulligan et al., 2005). For example, when considering a 2-by-3 dot array, students may recognize the total amount as 6 due to subitizing it without needing to consider coordinating the

2 units of 3 each.

Furthermore, tasks in U.S. third-grade textbooks, which is when multiplication is formally introduced, primarily involve direct counting of objects in arrays to determine the total amount (Kosko & Singh, 2018), potentially further constraining the utility of the array representation in fostering multiplicative concepts. Arrays are introduced to students in second grade with the size of the arrays presented under 5-by-5, with a focus on repeated addition. However, research indicates that students tend to count one-by-one when presented with smaller arrays, again ignoring the row-by-column structure the representation affords. Given the research examining the potential of arrays in supporting students' development of multiplicative ideas, our interest was in exploring how arrays could be employed to prompt strategies beyond simple counting to notice the rows or columns as composites. Additionally, in light of some evidence that Quick Images can prompt students to observe the row and column structure of the array when determining the total amount (e.g., Jacob & Mulligan, 2014), and arrays with hidden and missing amounts can reveal how students make sense of the arrays, we leveraged Quick Images as an instructional routine to examine the emergence of multiplicative ideas in young children. Specifically, our research question was:

- How do Quick Images using arrays, including those with missing or hidden amounts, impact students' strategies, particularly, their emergence of multiplicative ideas?

While Quick Images are incorporated in primary education, there is limited focus on using this routine to reveal students' array conceptions and elicit early multiplicative ideas. This paper presents the strategies students employed in transitioning from subitizing and counting to considering rows or columns as composites, along with operating on them, when presented with array images and arrays containing missing objects.

Theoretical Background

We considered learning from an emergent perspective wherein it evolves from participation in social practices, classroom norms, and interactions around the mathematical content employed in problem-solving scenarios. As noted by Cobb and Yackel (1996), children's learning is shaped by their ability to articulate their thoughts to others, while also being influenced by discussions with peers in the classroom. Thus, classroom interactions served as the backdrop where meaning was co-constructed by engaging in social practices and the mathematical content as facilitated by the teacher through prompts and discussion of student strategies.

To identify early multiplicative ideas, we draw from Steffe's (1992) research on unit construction based in children's numerical progression, involving singletons (units of 1), composite units (units larger than 1), and the types of units they construct within their number sequences – rooted in counting activity. Steffe characterizes early multiplicative actions as extensions of children's natural counting behavior, emphasizing that construction of a composite unit marks an essential milestone in early multiplicative reasoning. Specifically, children initially count objects one-by-one (referred to as being at Initial Number Sequence stage). The first milestone is achieved when they move from counting one-by-one to counting repeatedly by the same group size (termed as Tacitly Nested Number Sequence). According to Steffe, within the context of array structures, a student who simply counts by same group size does not necessarily demonstrate multiplicative thinking. To think multiplicatively, a student must also recognize the array as a composite formed of individual units and understand the role of individual units within a row (or column) as an iterative unit (Steffe, 1992). For example, they should be able to count three singles as one unit (the composite or row) and then count the composite unit. Additionally, because arrays are spatial structures, for this study, we considered unit organization with spatial structuring (Mulligan & Mitchelmore, 2009; Mulligan et al., 2005) to identify multiplicative ideas.

Methodology and Methods

This study was conducted with 20 second grade students in a small public elementary school in the Midwestern United States. Students were seen within their whole classroom and were engaged in Quick Image routines for 25-30 minutes 3 times a week for 3 weeks, for a total of nine days, outside of their regular math instructional time. We started with dot images displayed to elicit equal grouping structures on the first day (see Bajwa et al., 2023 for sample images). In each session, 3-4 Quick Image tasks were administered one at a time and displayed for approximately 5-10 seconds. Consequently, students had a limited timeframe to observe the image and were asked to determine the total count of dots/squares/objects within the presented image. After presenting each image as a Quick Image, students were asked to share their solution in whole classroom discussions that focused on students' solutions and strategies for determining the total amount. Thus, we focused on analyzing their collective activity.

Data Collection and Analysis

The gathered data included video recordings of all 9 days of whole class sessions, subsequently transcribed, as well student's written responses on selected tasks that required them to draw a representation prior to discussing their solutions, and notations written on the whiteboard during classroom discussions. Additionally, the dataset included research team field notes and pre-and post-planning meeting notes.

To analyze students' collective activity as part of their classroom discussion, we analyzed the data using a constant comparative method (Glaser & Strauss, 2017). Both authors independently coded transcripts from each of the sessions to identify student strategies used for determining the total amount and then met to compare codes for each session to identify employed strategies, which included instances of counting, adding, and early multiplicative strategies. These codes were then cross-examined with other students to determine if additional ideas beyond those initially identified had emerged.

Results and Findings

Student Strategies Involving Subitizing and Additive Strategies with Complete Arrays under 5-by-5

On Days 1 and 2, students demonstrated additive strategies including repeated addition, chunking, and subitizing to determine the total amount. Specifically, on Day 1, students utilized both subitizing and repeated addition when presented with a 4-by-3 dot array. The teacher introduced the term "rows" for the first time during this session as depicted in the excerpt below. This exchange occurred after the teacher presented the dot card with the array and instructed the class to discuss their observations with their partner. Following the pair-share activity, during which the teacher listened in on students sharing from a distance, the teacher summarized what they had heard several pairs discuss and selected a volunteer to represent a common strategy that was heard. Below is an excerpt from one of the students, Foti. All names are pseudonyms.

Foti: I saw 6.
Teacher: Where did you see 6?
Foti: I saw 6 at the top (referring to the top 2 rows of 3).
Teacher: Okay. Did you see them all the way across, 6 at the top?
Foti: No, I saw three, and then under that 3…
Teacher: Okay, so you saw 2 rows. Then you saw other two rows? (Circling the two rows on the image)

We approached this strategy with caution, recognizing it may entail a form of subitizing, specifically conceptual subitizing, which entails identifying a whole set of objects by quickly (perceptually) subitizing and combining smaller amounts (Sarama & Clements, 2009). MacDonald and Wilkins (2019) note that conceptual subitizing is fundamentally linked to students' construction and coordination of units. By composing two sets (3 and 3 is 6), Foti was likely developing a "units of units" understanding. Subsequent prompts did not clearly establish whether students perceived the 6 first and the 3 as parts of 6 or considered 3 as a row (viewing a row as a composite unit) to reach 6.

Apart from subitizing the number in a row, certain students employed additive strategies. These strategies involved breaking the array into smaller chunks and subitizing groups to determine the total (see Figure 1).

Neha: So, this was 7, I knew that, so there was 3 here, right, and 3 here (top two rows of 3). That was 6. I added these 2, that was 8. Then…there was 4 here. I added 4, and that was 12.

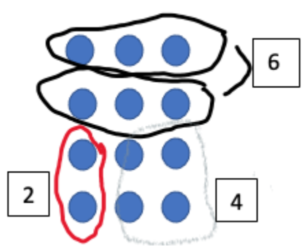

Figure 1. Using Chunking to Find the Total

Neha's strategy included breaking the array into different sized chunks by considering the top two rows and breaking the bottom two rows into a 2 and a 4 and adding those numbers together to find the total amount without considering a row or column as a composite unit. They initially state 7 but pointed to what they saw when figuring out the total, to get to the right answer of 6 dots with the top two rows.

In addition to subitizing and chunking, we also found some students began to use repeated addition or skip counting to figure out the total amount by Day 3. However, we found that not all who skip counted used the size of a row or column. Some students skip counted without considering the size of a row or column. For example, when presented with a 4-by-4 square array on Day 3, we had some students who skip counted by 4s. However, Odelia figured out that the image had 16 squares by skip counting by 2s.

Teacher: You counted by 2s? How did you count by 2s, can you show me where you counted by 2s? (shows Odelia the card again).
Odelia: 2, 4, 6, 8, 10, 12, 14, 16 (moves finger while counting sets of 2 squares, left to right, top to bottom).
Teacher: Okay. So, you went along the rows and skip counted by 2s, like that.

On these initial days, it first seemed that students counted based on the numbers in a row or column. However, as they elaborated their strategies, it became apparent that students weren't consistently utilizing or identifying the amount in a single row or column. This became evident when students, such as Odelia, noted that while they could determine the total in the image, they weren't necessarily focusing automatically on the quantity in one row or column but instead on a number they could readily recognize or subitize – in this case 2. Notably, when using this strategy, students did not engage with three levels of units. In this task, the students were shown the full array (image) instead of being told that there are 16 dots in the full array. Therefore, the students did not need to double-count to note that two repeated 8 times equals 16 as they could refer to the visual of the image to know when to stop. Despite these inconsistencies between what initially appeared and what students were actually focusing on, we found that most students were abandoning counting one-by-one to find the total utilizing other counting strategies, such as, repeated addition, chunking, and skip counting to figure out the total amount in an array by Day 4.

Student Strategies Involving a Row or a Column as Composites in Incomplete Arrays

Students were presented with incomplete arrays with missing parts (or composite arrays) on Day 7. We first presented students with a 5-by-5 composite array (see Figure 2). We were interested in getting students to utilize rows or columns as composites and so when this image was presented the teacher asked students to find the total and utilize what they know about arrays. The use of composite arrays with missing elements indeed draws students' attention to considering the composite of rows or columns in different ways as depicted below.

Arianna: I counted 3 all the way down. So it was 3, 6, 9, 12, 15, 18, and then there was one 3 on the side, so I counted that, which was 21. Then there were 2 left, which was…
Teacher: Like that? 3, 6, 9, 12, 15, 18? [circling 3 squares, from left to right, in each row].
Arianna: Yeah, and then one on the side. That way [finger pointed to the board, circling vertically]…Yeah, then there was a 2 going down. And then I added, so, it was 21 plus 2 got me 23.

Arianna utilized composites of the initial row, made of 3 squares and used skip counting to determine the total. On the other hand, another student (Keena, see below) also utilized arrays but by mentally moving the squares to form a complete array using the given squares.

Keena: See that two right over here [pointing to the column of 2 on the card]? I actually just moved this 2, like, all the way into this space [pointing to the space at the top of the column with 3 squares].

Teacher: So, you moved this up here?
Keena: Yeah, and then I know that there was four here [pointing across card, indicating the number of columns] and there was five here [pointing up and down on card, indicating the number of rows], and it was 20.

Keena used the template of a complete array to visually create a full array by moving the squares mentally. This then led them to notice that using all the given squares a 5-by-4 array could be formed, resulting in the answer of 20. While we cannot say which particular strategy they utilized to get to 20 (e.g., repeated addition or multiplication), this strategy and others, representative of Arianna indicated that students were getting comfortable with creating and utilizing rows and columns as composite to find the total.

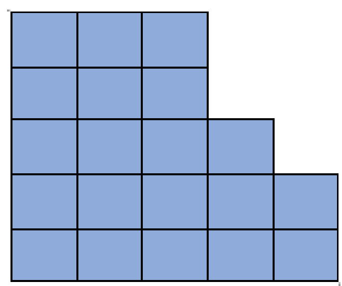

Figure 2. Example of an Incomplete Array

Student Strategies Involving a Row or a Column as Composites in Arrays under 5-by-5 with Missing Objects

We found that when presented with arrays smaller than 5-by-5 with missing elements (See Figure 3, left), students initially resorted to either one-by-one counting or attempted to memorize the number of squares, often facing difficulties in accurately tracking the total count, partly due to the limited time for which the images were presented. These difficulties prompted students to explore alternative strategies when presented with larger arrays. On subsequent days, to get students to consider or invent alternative strategies, we introduced a real-life scenario involving a tray with some seedlings missing (see Figure 3, right). After reminding the students that the image would be displayed for 5 seconds, we posed the question to the students, “How many seedlings can fit into the container?”

Figure 3: Examples of Arrays with Missing Objects

When sharing their strategies, we found that most students regarded the row or column size as a composite unit. Among those few who explored alternative approaches, they drew from their experiences with complete arrays as a reference, disregarding or intentionally ignoring the absence of seedlings to help them determine the total amount. Overall, student strategies varied, from determining the total by forming equal groups unrelated to the row (or column) size to treating the row or column size as a composite, iterating it, and mentioning multiplication as depicted in the excerpt below.

Sage:	20. I counted the columns.
Teacher:	You counted the columns. So, how did you count the columns?
Sage:	[while drawing her finger downward] 4, 4, 4, 4, and 4
Owen:	20. I used multiplication
Teacher:	Okay, you said you used multiplication? Okay, how?
Owen:	So, I counted all of them down, and that was 4, and I counted across and that was 5. So, I did the 4 times 5 and counted by 5. So, I counted 5, 10, 15, 20.

By the last day, we observed that the majority of students demonstrated flexibility in making sense of the total amount in an array despite certain objects missing. In the seedling task, for instance, numerous students determined the total by identifying the size of a row or column as a unit and iterating it to find the total number of seedlings. Notably, there was a shift in language usage as students explained their process of iterating rows and columns. Among those identifying rows or columns, some began using the term "times" to indicate the number of times they added (iterated) the row or column. For instance, one student when figuring out the total in another seedling tray (3-by-9) with some seedling missing, mentioned, “I counted by 3s, so I counted by '3s nine times.'” Intriguingly, a specific student (Owen) even used the term "multiplication." In summary, the introduction to arrays larger than 5-by-5 with missing amounts prompted various responses, with many students engaging in early multiplicative ideas.

Student Strategies Utilizing the Rows-by-Columns in Arrays Over 5-by-5 with Hidden Objects

We found that when presented with a task involving larger arrays with hidden amounts, students began to utilize the row-by-column structure of the array inherent in their strategies. We presented students with an image with windows, an example of a real-world 6-by-10 array. Initially, we presented the windows without the hidden amount. The teacher reminded the students that the image will be showed for a short time. As anticipated, students employed various strategies to determine the total number of windows—many focused on identifying both row and column sizes. After concluding the discussion on the presence of 60 windows, we presented the next card to students using hidden windows as a "challenge card." In this card, we used the same image and obscured windows in the middle of the top three rows (see Figure 4).

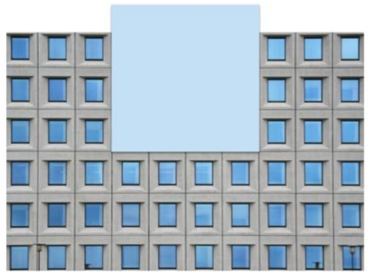

Figure 4: Windows Task - Array with Hidden Amount

Since the students had already established the total number of windows in the picture, they were asked to determine the quantity that was hidden. We found that while students used a variety of strategies, certain students concentrated solely on the hidden section, deducing the number of hidden rows/columns and the size of each column/row.

Omar: 12. Because, I see that there are 1, 2, 3, 4 here and 1, 2, 3 there, so 4 columns of 3.
Teacher: Did you hear what Omar had to say about that? That there were 4 columns, he said, of 3 that were hidden. Austin, did you do something different?
Austin: Yeah, I just pretended to move that down and counted by 3s down.
Teacher: Okay, so you, you imagined that this was actually moved down here and counted this part here? Okay. Alright.

The use of hidden amounts in a larger array resulted in eliciting a variety of strategies. Some students identified the how many rows and columns might be hidden to determine the hidden amount; representative of what Omar did. This indicated that these students may have been considering the hidden amount as an array itself, but we say this with caution as we did not ask students to determine if this was indeed the case. Some identified the number of rows and amount in each column (like Austin) and repeatedly added those.

Discussion and Implications

Quick Images with arrays provided a valuable avenue for students to delve into early multiplicative concepts prior to their formal introduction to multiplication. Initially, smaller arrays of 5-by-5 or less were presented, facilitating subitizing and counting to determine the total. As expected, some students counted the objects one-by-one, some formed various-sized groups, not necessarily tied to the size of a single row or column but instead based on groups they could subitize or effortlessly skip count. As the instructional days progressed and students encountered arrays with missing objects as well as larger arrays exceeding 5-by-5, there was a noticeable shift in most students' approach. They began considering the quantity within a single row and/or column, iterating or coordinating the two to calculate the total but inconsistently. Specifically, by restricting the display time of images and providing multiple opportunities for students to construct and count composites, students began to leverage the spatial structure of arrays. Most found the total by combining both the spatial arrangement (arrays as rows of objects) and numeric composites (row size as a unit). The introduction of arrays with missing or hidden elements further encouraged students to consider the structure of the array as comprised of equal-sized rows and columns, and iterate them to find the total amount in the presented image.

Our results indicated that the presence of arrays larger than 5-by-5 may impact students' development of early multiplicative ideas. More research is needed to determine if the use of composite arrays and/or arrays with hidden amounts fostered that understanding or if this was due to the array itself being larger, because it could not be counted quickly. Additionally, more research is needed examining how the amount of time students have to view an image, especially when the array is larger, impacts the strategies they use to find the total amount. Our results seem promising that a combination of these infused during instruction may contribute to students' development of multiplicative ideas prior to formally being introduced to multiplication.

Conclusion

This paper presented results from a larger study examining second grade students' understanding of multiplicative ideas who have not yet been formally taught multiplication. Initially, students were found to count objects one-by-one, subitize, or use groupings to find the total amount in an array and could do so without attending to a row of a column as a unit. When presented with larger arrays, in particular composite arrays and those with hidden/missing pieces, we found that students began to attend to a row or column as a unit which could then be iterated to find the total amount in the array. Our findings have yielded encouraging results illustrating that young students can consider and utilize multiplicative ideas prior to a formal introduction to multiplication as they engage in making sense of images presented as Quick Images.

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CHILDREN'S USE OF MANIPULATIVES WHILE INVENTING NEGATIVE INTEGER MULTIPLICATION STRATEGIES

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Negative integers are a contextually difficult subject for children to comprehend due to their abstract nature. 35 grade 5 children, from two qualitative studies, invented strategies to solve open-number sentences involving the multiplication of negative integers prior to formal instruction. The strategies that children used were categorized in order of success as (1) direct modeling, (2) repeated addition and subtraction, (3) recalled fact, (4) procedure, (5) counting, and (6) analogy. Upon examining children's invented strategies, the results indicated that manipulative use (i.e. unifix cubes, two-sided chips, and number lines) could inform educators as they provide negative integer operation instruction. Implications for teaching integers and future research are also discussed.

Keywords: Number Concepts and Operations, Elementary School Education, Cognition

The mathematics education field recognizes the importance of examining children's thinking across multiple number domains (Carpenter et al., 2015; Empson & Levi, 2011). Although there is a rich understanding of the ways that children think about whole number operations (Carpenter et al., 2015) as well as negative integer addition and subtraction (Bofferding & Hoffman, 2019; Wessman-Enzinger, 2019a; Whitacre et al., 2017), we lack descriptions of how children think about and invent uses of multiplication with negative integers. Making sense of how children invent strategies to reason with negative integer multiplication is important because it empowers children, educators, and researchers by highlighting the sophisticated mathematics children already know. Because invented integer operation models often differ significantly from traditional models (Wessman-Enzinger, 2019a), examining children's use of operations with negative integers provides vital insight into the knowledge that children already possess prior to formal instruction.

Children invent sophisticated ways of reasoning about integers (Bishop et al., 2014; Bofferding, 2014). However, the transition from additive structures to multiplicative structures can be challenging (Carpenter et al., 2015). As children approach addition and subtraction of integers they often use strategies such as moving forward for addition and backward for subtraction. When multiplying, children are taught to move forward on a number line, therefore when introduced to negative integer multiplication, moving backwards on a number line may seem like an illogical solution. Negative integers are difficult for children to model and create contexts for due to their abstract nature.

Baek (1998) demonstrates that children understand multiplication with greater success when they are permitted to invent their own strategies. These invented strategies can also be used to teach children as they begin to reason with negative integers. As children solve problems involving the multiplication and division of integers, they often invent strategies that make use of counting, repeated addition, or direct modeling (by grouping collections of countable objects) (Carpenter et al., 2015). These strategies provide insight into the ways that students may solve multiplication tasks involving negative integers.

Theoretical Perspective

Cognitively Guided Instruction (CGI) is a framework developed to help educators understand how children's mathematical ideas evolve. Integer addition and subtraction literature indicates that children use several strategies that are derived from the CGI framework. Some of these strategies include, creating analogies (Bishop et al., 2016; Bofferding, 2011; Wessman-Enzinger 2017; Whitacre et al., 2017), using procedures (Bishop et al., 2014), and drawing upon recalled facts. This report will draw upon both single-digit (Carpenter et al., 2015) and multi-digit strategies (Baek, 1998) for multiplication of whole numbers, as well as the CGI strategies for integer addition and subtraction.

Additionally, this report will draw upon the framework, seen in Figure 1, created by Carpenter and Wessman-Enzinger (2018) to analyze children's thinking as they invent ways to multiply using negative integers. This previous study identified the following six categories of strategies that children invented while interacting with negative integer multiplication: (1) *direct modeling*, (2) *repeated addition and subtraction*, (3) *recalled fact*, (4) *procedure*, (5) *counting,* and (6) *analogy*.

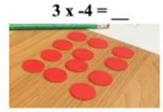
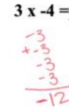
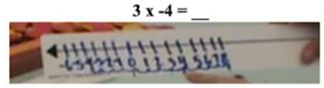
Invented Strategy	Description	Children's Representations
Direct Modeling	Children use physical tools or draw pictures in order to solve the integer multiplication sentences (e.g. unifix cubes, chips)	**3 x -4 = __** 
Repeated Addition and Subtraction	Children describe multiplication as adding or subtracting an integer repeatedly	**3 x -4 =__** 
Recalled Fact	Children extend their factual knowledge about whole number multiplication quickly to integer multiplication without explanation	**3 x 5 = __** "I just knew it was 15 because I have been doing multiplication for a while and I know that answer"
Procedure	Children invent a rule or algorithm to find the solution (often applying the rule across multiple tasks)	**-2 x 3 = __** "I thought, I will minus out this (the negative sign) and then I will times these two (2 and 3) and after I will add on the minus, so it becomes minus…or I mean negative 6"
Counting	Children count or skip count in sequential orders up or down (i.e. using a number line)	**3 x -4 = __** 
Analogy	Children connect previous knowledge about whole numbers to integers in order to justify new claims when solving integer multiplication number sentences (i.e., 2 × 4 = 8, so -2 × -4 = -8)	**-2 x -4 = __** "Well, because it wouldn't really make as much sense for a negative multiplied by a negative to equal a positive. It's like, um, I'm not sure how to... it just wouldn't make as much sense. Because if a positive multiplied by a positive would equal a positive, then I would assume that it would be the same for a negative. And, it would be a negative times a negative would equal a negative"

Figure 1: Sample Children's Invented Strategies for Integer Multiplication

Methods

The data presented in this report was gathered from two qualitative studies. The first study, conducted by Carpenter and Wessman-Enzinger (2018) examined how children invent strategies for solving negative integer multiplication prior to formal instruction. The participants consisted of 23 grade 5 students attending a rural school in the Pacific Northwest, United States. The children each participated in one clinical interview (Clement, 2000) and were asked to solve four open-number sentences, all with varying degrees of negative integer inclusion. They were asked to solve each number sentence and describe their strategy, before creating a story to represent their solution path. This study resulted in the production of an inaugural framework that described the ways that children invent strategies to solve negative integer multiplication.

The second study was conducted by Carpenter (2024) to validate the findings of the first study. The participants of the second study consisted of 12 grade 5 students attending a laboratory school in the Midwest, United States. In congruence with the first study, all children participated in one clinical interview. However, as seen in figure 2, half of the participants (6 children) were given the four open-number sentences in the order presented in sequence 1, identical to the order presented in the first study by Carpenter and Wessman-Enzinger (2018). The other half of the participants (6 children) were given the open-number sentences in the order shown in sequence 2. The sequence that each child received was chosen at random and was blind to the researcher. Following their strategy invention for each number sentence, children were asked to create a story to represent their solution path.

Sequence 1	Sequence 2
3 x 5 = __	3 x 5 = __
-2 x 3 = __	3 x -4 = __
3 x -4 = __	-2 x 3 = __
-2 x -4 = __	-2 x -4 = __

Figure 2: Open-Number Sentence Order

The results of the second study confirmed the validity of the framework previously created by Carpenter and Wessman-Enzinger (2018), despite the tasks being presented in a different order and to a different participant pool. While the results of both studies highlighted the sophisticated reasoning present within the strategies that children invented; the present report will provide distinct recommendations for the use of specific strategies and manipulatives that educators should include within their instruction to help students successfully develop an understanding of negative integers and their operations. This report will answer the following research question:

1. What types of strategies do children invent while operating with negative integer multiplication?

Interpreting Negative Integers

Negative integers are a difficult area of study for children to conceptualize. Since negative integers are difficult to represent physically, many children struggle to develop strategies that can represent them in various operations. Conceptualizing negative integers requires children to acknowledge the existence of numbers less than zero (Bofferding, 2014). While there are a few models that can be used to represent the addition and subtraction of negative integers, such as moving on a number line, solving tasks involving multiplication and division can become much more difficult. The children in this report were not yet taught any de-contextualized rules or algorithms about integer multiplication, which is a common practice among classroom teachers when they begin negative integer instruction.

Without prior formal instruction about negative integers and how to operate with them, these children developed insightful strategies that have the possibility to help future educators understand the best methods to introduce children to negative integer multiplication. These invented strategies do not always produce mathematically correct results; however, they are often rooted in logic (Carpenter and Wessman-Enzinger, 2018; Wessman-Enzinger, 2020) and provide useful information about misconceptions that children may have as they make sense of such an abstract concept. Understanding the connections between the strategies that children use and the misconceptions that arise as a result of an invented strategy can provide educators with useful information about the tools they should be presenting to children as they interact with negative integer multiplication for the first time. The successes and challenges that children face as negative integers are introduced into their operations for the first time can be seen in Table 1.

Table 1: Children's Correct and Incorrect Solutions

Task	Solutions		Percentage of Correct Solutions
	Correct	Incorrect	
3 x 5 = __	35	0	100%
-2 x 3 = __	14	21	66%
3 x -4 = __	14	21	66%
-4 x -2 = __	2	33	6%

When presented with an open-number sentence containing two positive integers, all children (100%) were able to successfully invent strategies to solve for the correct answer. This indicates a robust understanding of single-digit whole number multiplication. However, as the open-number tasks involve more negative integers, success in creating a viable strategy for solving drastically declines. Two-thirds of children (66%) correctly solved multiplication number sentence tasks containing only one negative integer. Interestingly, this finding was identical for both tasks containing only one negative integer, despite children receiving the tasks in two distinct ordering sequences. Finally, only two children (6%) were able to invent a strategy to correctly solve a multiplication number sentence containing two negative integers. The decline in children's success while solving each open-number sentence involving negative integers is partially caused by a lack of understanding of how to interpret a negative integer (Fuadiah et al., 2019). As seen in Figure 3, the children demonstrated a wide variety of negative integer interpretations while solving each task.

Negative Integer Interpretations	Children's Responses
Negative integers are equivalent to zero	**-2 x 3 = __** **Addie:** "Negative 2 is I'm guessing its zero and zero times this is zero"
Negative integers are equivalent to nothing	**-2 x 3 = __** **Mia:** "There are three groups with nothing."
Negative sign is completely ignored	**-2 x 3 = __** **Benny:** "I got my answer by counting by 2 3 times, 2, 4, 6 so I counted by 3 and got 6"
Negative sign is equivalent to the subtraction symbol	**-2 x 3 = __** **Cash:** "2 times 3 is 6 and then took away 2 for the negative… I just knew there was a negative 2 so I took away 2"
Two negatives multiplied together will equal a negative.	**-4 x -2 = __** **Jasper:** "I'm pretty sure that a negative multiplied by a negative will just stay as a negative and not go into a positive at all… Well because um it wouldn't really make as much sense for a negative multiplied by a negative to equal a positive. It's like um I'm not sure how to… it just wouldn't make as much sense. Because if a positive multiplied by a positive would equal a positive then I would assume that it would be the same for a negative. And it would be a negative times a negative would equal a negative."
Multiply both integers as positive integers and then add on the negative sign to the product	**3 x -4 = __** **Warren:** "I just crossed out the ... the negative symbol. Multiply 3 times 4 and then... got 12 and just brought the negative symbol back."
Multiply both integers as positive integers and then subtract the negative integer(s) from the product	**-2 x 3 = __** **Jasper:** "You do the multiplication and then you take that and then you make the negative a positive and do… subtract. And the answer for your subtraction problem would be your overall answer."

Figure 3: Children's Negative Integer Interpretations

The broad range of negative integer understanding amongst the children in both studies is far from abnormal. There are many common misconceptions that arise as the children invent strategies for operating with negative integers for the first time. In congruence with previous literature, some children believe a negative integer is equivalent to zero and others ignore the negative sign altogether (Bofferding, 2014). Some children approach the negative sign as equivalent to the subtraction symbol (Stephen & Akyuz, 2012), while others simply assign a negative value to the product after they multiply both integers as positives (Wessman-Enzinger, 2020). However, a less common approach to operating with negative integers is also present within the data. Several children invented a procedure to multiply the integers as if they were both positive and then proceeded to subtract the negative integer present in the original number sentence from the product. The difference between the product and negative integer was then declared as the solution. This understanding builds upon the interpretation that the negative sign and subtraction symbol are equivalent. Children who use this strategy are drawing upon their understanding of whole number multiplication and subtraction, in a logical attempt to make sense of a new symbol (the negative sign) that they have not seen before.

Modeling Negative Integers

Using the framework developed by Carpenter and Wessman-Enzinger (2018), the student-invented strategies are sorted into six categories: (1) *direct modeling*, (2) *repeated addition and subtraction*, (3) *recalled fact*, (4) *procedure*, (5) *counting,* and (6) *analogy*. These strategies are used with varying degrees of frequency and success, as seen in Table 2.

Table 2: Children's Use of Invented Strategies

Invented Strategy	Open-Number Sentences								All Responses Combined		
	3 x 5 = __		-2 x 3 = __		3 x -4 = __		-4 x -2 = __				
	Correct	Total	Correct	Total	Correct	Total	Correct	Total	Correct	Total	Success Percentage
Direct Modeling	9	9	5	8	2	9	0	4	16	30	53%
Repeated Addition & Subtraction	8	8	1	3	0	1	0	0	9	12	75%
Recalled Fact	15	15	0	0	0	1	0	0	15	16	94%
Procedure	0	0	7	16	12	21	0	15	19	52	37%
Counting	3	3	0	4	0	3	0	9	3	19	16%
Analogy	0	0	0	0	0	0	0	4	0	4	0%

As children solve tasks using their invented strategies, they have the most accuracy with the implementation of (1) *recalled fact* (94% correct), (2) *repeated addition and subtraction* (75% correct), and (3) *direct modeling* (53% correct). Despite being used more than all three aforementioned strategies; *procedure* (37% correct) and *counting* (16%) prove to be the least successful strategies other than *analogy* (0% correct) which is not used accurately in any interviews.

Recalled fact is the most accurately used strategy by the children to invent a solution. They also rely heavily on *direct modeling* and *repeated addition and subtraction*. *Direct modeling* demonstrates a child's attempt to physically represent the negative integers within the task, while *repeated addition and subtraction* is used to justify the role of multiplication within the task. Both are important to understand in order to invent a viable solution path for negative integer multiplication. Children's strategies are characterized as a *procedure* when they invent a rule for solving and continue to apply it throughout the other tasks. Although children also produce solutions by inventing *counting* strategies, these solution paths, often involving the use of a number line, are much less successful than the other invented strategies.

Although many children demonstrate proficiency with the use of *recalled fact* to produce correct solution paths, it will not be a focus of this report. *Recalled fact* is used almost exclusively with open-number sentences containing two positive integers and is often the result of children memorizing de-contextualized rules which are demonstrated to them by others. Since the children in both studies were exploring negative integer operations prior to formal instruction, it makes sense that they would not know, and therefore not use, any memorized rules for solving tasks involving negative integers. Due to its lack of use among any of the open-number sentences containing negative integers, it will not be highlighted as a recommended strategy in this report. Despite being the most used invented strategy, developing a *procedure* did not produce a high level of solution accuracy. Common procedures included appending a negative sign to the solution, interpreting negative integers as equivalent to zero, and assuming exclusive negativity. An uncommon procedure also presented itself in both studies, however, due to their complexity, these findings will be discussed in future proceedings.

Children developed several different strategies to make sense of negative integer multiplication, however, this report will recommend the use of (1) *direct modeling* (i.e. unifix cubes and two-sided chips) and (2) *counting* (i.e. number line) to current and future educators as they approach negative integer operations with their students. Both strategies use physical manipulatives or contexts to model negative integer multiplication and although they were ultimately coded as *direct modeling* or *counting* because of the manipulatives used, both strategies often also relied on *repeated addition* to complete their strategy. Thus, the recommended use of *repeated addition* is not included as it is embedded within the use of all contextual manipulatives discussed in this report. Understanding how often children use and are successful with these invented strategies, seen in Table 3, informs the recommendation of this report.

Table 3: Children's Use of Physical Manipulatives and Accuracy

Task	Manipulatives Used By Children to Invent Strategies					
	Unifix Cubes		Two- Sided Chips		Number Line	
	Correct Use	Total Use	Correct Use	Total Use	Correct Use	Total Use
3 x 5 = __	6	6	2	2	0	0
-2 x 3 = __	4	5	1	3	0	4
3 x -4 = __	2	4	1	3	0	3
-4 x -2 = __	0	3	0	0	0	9
Cumulative Totals	12	18	4	8	0	16
Percentage of Correct Use	66%		50%		0%	

Children's Use of Manipulatives

Twenty-one children (60% of participants) invented strategies using manipulatives as they attempted to model negative integers physically. Manipulatives are necessary for instruction of negative integers as they provide children with operational context, rather than the rote memorization of rules. Additional guidance for the use of contextual manipulatives such as (1) unifix cubes, (2) two-sided chips, and (3) number lines will be demonstrated through the invented solution paths of three children.

Unifix Cubes

While they are closely related in many aspects, unifix cubes differ from two-sided chips because they lack the binary restriction of just two sides or two colors. Unifix cubes present as 3D objects that come in color groupings of ten and are often used to physically represent whole number operations. Despite being used in almost equal comparison to the number line, this manipulative had a 66% success rate among children who used it to invent their strategies. Children using unifix cubes to model negative integer multiplication apply their whole number multiplicative reasoning of creating groups to construct "groups of" negative integers that represent their solution, as seen in Figure 5.

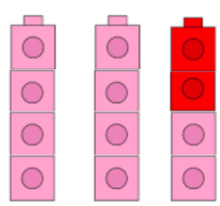


Figure 4: Koda's Use of Unifix Cubes to Solve 3 x -4 = __

Koda invented his strategy by organizing the unifix cubes into three groups of four cubes. He did not assign a negative value to a specific color. Instead, he stated that all cubes were representative of a negative (regardless of color) and used both red and pink cubes in his representation. unifix cubes are used by children to solve each open-number sentence, regardless of the number of negative integers present, making it a versatile tool for negative integer exploration. They differ from two-sided chips in their ability to link together. Because of their ability to connect, children do not use them regularly to create arrays, but invent representations using groups instead.

Two-Sided Chips

Two-sided chips are uniquely qualified manipulatives for representing negative integer operations because of their ability to represent both a negative and positive value using the same object. The children who invent strategies successfully with the two-sided chips regularly assign a negative value to one side of the chip and a positive value to the other. In this study, most children who use two-sided chips do so by inventing strategies that create arrays of negative integers, as seen in Figure 4.

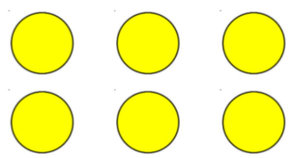


Figure 5: Zach's Use of Two-Sided Chips to Solve -2 x 3 = __

Zach invented a model using an array that assigns a negative value to the yellow side of the two-sided chips. He explained his strategy by saying "I got 2, 4, 6, so my answer was negative 6… I multiplied negative 2 by 3." Two-sided chips encourage children, like Zach, to discover that values less than zero can be modeled with physical objects by assigning a negative value to that object. Surprisingly, this tool is only used by children in three of the four tasks given. Although this manipulative is used less than all others available, children who use two-sided chips in their invented solutions demonstrate the second highest levels of success in solving the tasks. Given the success rate of children who use the two-sided chips, educators should consider these manipulatives as paramount for implementation within their instruction of negative integer multiplication.

Number Line

Although number lines are a useful tool for the instruction of whole number operations, this instrument does not serve as a benefit to children who use it to invent strategies for negative integer multiplication prior to formal instruction. This manipulative is used more than any other by children in the study, however, it also produces a 0% success rate among its users. Children face a variety of obstacles while using the number line, but many instinctively gravitate towards it initially due to their familiarity with it in primary mathematics and whole number operations.

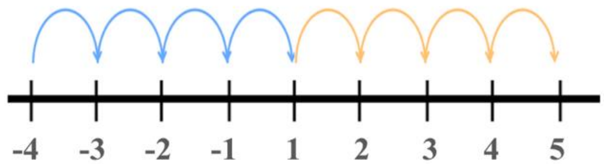


Figure 6: Arlo's Use of the Number Line to Solve -4 x -2 = __

Arlo started their number line at -4 and skip-counted two groups of four until determining a solution of 5. Arlo decided to start skip-counting at -4, because it was the first integer presented in the open-number sentence. This invented strategy presents some successes, but also highlights several challenges that accompany the use of a number line to represent negative integer operations. Arlo's directional movement and total number of "jumps" indicate promising conceptual foundations but fall short of the correct solution due to Arlo's chosen starting point and omission of zero on the number line. Although number lines are regularly used to teach whole number operations, the starting points and directional movements required for negative integer operations are not intuitive for children. Additionally, minor computational errors, such as omitting zero from the number line can cause children to arrive at an incorrect solution, even if they performed all other steps accurately. For these reasons, number lines should be used with caution while providing instruction on negative integer operations.

Discussion and Conclusion

Although the literature examining the ways that children operate with negative integers using addition and subtraction is growing (Bofferding & Hoffman, 2019; Wessman-Enzinger, 2019a), there is a continuous need for research on children's thinking about negative integer multiplication and division. The ways that children interpret the negative symbol has direct implications for how they approach negative integer operations. Future research should examine the ways that children approach the negative symbol that are unique to negative integer multiplication (i.e. multiplying both integers as positive and subtracting the negative integer from the product).

Implications

Although children in both studies were asked to pose stories for negative integer multiplication number sentences, it was not discussed in this report. Storytelling has proven to be an excellent tool to encourage contextual understanding for abstract mathematical concepts. Future research should examine whether there is a connection between the stories children create and the strategies that they invent while solving negative integer multiplication tasks.

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FEW TEXTBOOK ILLUSTRATIONS OF MULTIPLICATION SHOW STRUCTURE

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Keywords: Mathematical Representations, Number Concepts and Operations.

Researchers have argued that pictorial illustrations of multiplication should contain visual information that highlights the various meanings of multiplication (e.g., equal groups, rectangular array, cartesian product) and its structural elements (e.g., relations between units; Jitendra et al., 2022; Kosko, 2020). Not all illustrations that accompany mathematical tasks are equally effective for learning, however (Berends & van Lieshout, 2009; van Lieshout & Xenidou-Dervou, 2018). First, illustrations designed to serve different purposes (e.g., to represent concepts or to decorate) vary in their effectiveness (Carney & Levin, 2002). Second, the perceptual features in illustrations can be relevant to the target concept or not (Belenky & Schalk, 2014), thereby impacting student learning in different ways (e.g., Kaminski & Sloutsky, 2013; McNeil et al., 2009). The objective of the current study was to analyze the illustrations used in multiplication problems in elementary textbooks. The findings will provide insight into the ways in which the conceptual structure of multiplication is conveyed through illustrations and more generally, how multiplication is taught in elementary classrooms (Porter, 2002).

Third- and fourth-grade textbooks from two publishers (TAM TAM, Deshaies et al., 2022; Lincourt et al., 2022; and Matcha, Borduas et al., 2019a, 2019b) adopted in a large metropolitan area in Eastern Canada were selected for analysis. The textbooks contained 171 exercises, each of which contained either one task or a subset of tasks of the same type. We first coded the exercises as either including a *decorative* illustration or not, defined as one that was not related to the targeted concept but intended to capture student interest (Lenzner et al., 2013). Second, the tasks (n = 573) were coded as: (1) *symbolic* (i.e., words or symbols only) or (2) *illustration*. Illustrations were further coded as either: (a) *representational*, showing the meaning and structure of multiplication (e.g., three bunches of 5 bananas); (b) *irrelevant*, a symbolic problem embedded in an illustration unrelated to the meaning of multiplication (e.g., 3 x 5 = __ placed in an image of a bowling pin); or (c) *organizational*, a structural framework for solving a given problem (e.g., an empty number line to show three groups of 5).

Over half (59%) of the exercises included a decorative image in the margin, revealing a relatively large number of illustrations that were not related to multiplication concepts. The task analysis revealed that of all multiplication tasks, only 25% incorporated a pictorial illustration, revealing a preponderance of symbolic problem presentations. Just over 55% of the illustrations were representational, meaning that only 14% of all tasks contained illustrations showing the structure of multiplication. Of all illustrations, 26% were not related to multiplication and used as a pictorial backdrop for a symbolic problem. Given the existing empirical evidence on pictorial illustrations supporting the understanding of multiplication (e.g., Jitendra et al., 2022), the small percentage of representational illustrations currently used in at least some popular Canadian textbooks is noteworthy. Future research should examine how the distributional frequencies of the different types of multiplication tasks with illustrations may influence student learning.

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DEVELOPMENT OF SPATIAL REASONING AND COUNTING SKILLS THROUGH SCAFFOLDING IN A PROGRAMMING GAME

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This study investigates the development of spatial reasoning and counting skills in third-grade students through instructional scaffolding within a programming game. We focus on two pairs of students who engaged in the game Coding Awbie, using number and movement blocks to navigate a grid. Over six 30-minute sessions, qualitative analysis revealed challenges in counting within rows and columns, alongside effective use and potential misdirection caused by game scaffolds. Peers offered suggestions that partially alleviated scaffold-related difficulties. The study's findings illuminate the role and challenges of programming games as instructional tools in enhancing children's spatial cognition and numerical fluency.

Keywords: Number Concepts and Operations, Geometry and Spatial Reasoning, Computing and Coding, Problem Solving

There has been growing recognition of the importance of spatial reasoning and counting skills in children's mathematical development (Baroody & Wilkins, 1999; Battista et al., 1998; Owens, 2015; Siegler & Ramani, 2009). One avenue of exploration for supporting these skills lies in using instructional supports and scaffolding within programming games. Programming games offer a unique environment where children can engage in problem-solving activities that require spatial manipulation and numerical reasoning (Jiau et al., 2009; Ma et al., 2011). These games often embed scaffolds, supportive structures, or cues to assist learners in navigating challenges (Kim & Hannafin, 2011). However, the extent to which such scaffolds effectively facilitate the development of spatial reasoning and counting skills remains an area of inquiry.

Spatial Reasoning and Counting

Spatial development encompasses a child's ability to understand and navigate space (Clements & Sarama, 2009). These abilities are fundamental in mathematical problem-solving, providing the cognitive basis for concepts such as geometry, measurement, and spatial relationships (Mulligan, 2015). Spatial reasoning involves identifying and manipulating objects' spatial characteristics and relationships (Lowrie et al., 2016; Mulligan, 2015). Kocabas et al. (2022) noted that children commonly employ spatial connections to determine an object's location relative to reference points, utilizing mathematical cues like distance. Strategies for counting or adding distances can be either informal or formal, developing as children participate in mathematical tasks. A common problem seen in young learners in both programming and mathematics is the challenge of double counting, where pupils tend to count an object or space more than once (Kocabas et al., 2019; Fuson, 2012; Shumway et al., 2021). In the game used for this study, students have made double counting errors when the game character changed directions, and they counted a corner square twice (Kocabas et al., 2019). Although students have issues with double-counting, there is less detailed evidence on how they overcome their double-counting difficulties.

Theoretical Framework

According to Vygotsky (1978), children's cognitive development is intertwined with their social interactions and cultural context. In the context of programming games like Coding Awbie™, Vygotsky's theory suggests that collaborative peer interactions play a crucial role in how students interpret symbols within the game and respond to scaffolds provided by the game environment (Schunk, 2012). Additionally, Vygotsky's concept of the Zone of Proximal Development (ZPD) is highly relevant to understanding the role of scaffolding in facilitating learning (Vygotsky, 1978). Within the ZPD, students can complete tasks with the support of a more knowledgeable peer or instructor. The scaffolds provided in games, such as modeling, probing, and hints, serve to support students as they navigate the complexities of spatial reasoning and counting within the game environment (Wood et al., 1976; van de Pol et al., 2010).

This study adds to the literature by examining how instructional support and scaffolding within a programming game, Coding Awbie, play a role in third-grade students' spatial reasoning and counting skills. Drawing upon qualitative analysis of six sessions, this research investigates the challenges encountered by students and the efficacy of scaffolds in supporting their learning processes (Van de Pol et al., 2010; Wood et al., 1976). Therefore, we investigate the following research questions: (1) How do third graders utilize instructional scaffolding in developing spatial reasoning abilities? and (2) How do their use of scaffolds (highlighted path) in the game influence their counting and use of programming blocks?

Method

We recruited first and third-graders from a midwestern public elementary school in the US where 45% of students qualified for free and reduced lunch, and 11% were classified as English Language Learners. A total of 55 students, 28 first graders, and 27 third graders, participated. For this paper, we focus on two pairs of third graders: Emma and Quincy, Sophie and Marcus. In the game Coding Awbie, the objective is to move Awbie to gather strawberries while avoiding falling into the water, thereby advancing to the next level. Students use physical programming blocks to control Awbie's actions (walking, jumping, grabbing), direction (up, down, left, right), and number of movements (ranging from 1 to 5). When students place the blocks, with the initial command placed at the top, the screen highlights where the character would go based on the code, an important in-game scaffold. During their six game sessions, pairs played until the end of their 20-minute session. Upon completing a level, students advance to the subsequent one. If students did not complete a level during a session, they restarted it the following session.

Through analysis of video recordings and transcripts of peer conversations during sessions of playing Coding Awbie™, we examined how students used instructional supports and scaffolds within the game to coordinate movements, use numbers, and count spaces. We coded instances where students employed single blocks versus multiple blocks to denote movement, used addition to cover distances, and relied on movement blocks, hints, and highlighted paths. Furthermore, we observed whether students counted spaces manually or relied solely on the highlighted path feature. We analyzed cases where students' use of coding blocks interacted with peer dynamics and instructional supports. We documented changes in students' strategies and interactions, marking shifts in their use of numbers and counting methods, and delved into the spatial characteristics of the game during these transitional moments.

Findings

Using Highlighted Path, Board, and Explicit Hints

The game had hints that helped students to navigate the space with varying success. For instance, Sophie and Marcus thought the highlighted path coming out of trees meant their code would work, even though it had Awbie hitting trees, which did not work (see Figure 1, left). However, when they came to a board with a question mark that revealed an explicit hint, they matched their code to move Awbie accordingly (see Figure 1, right).

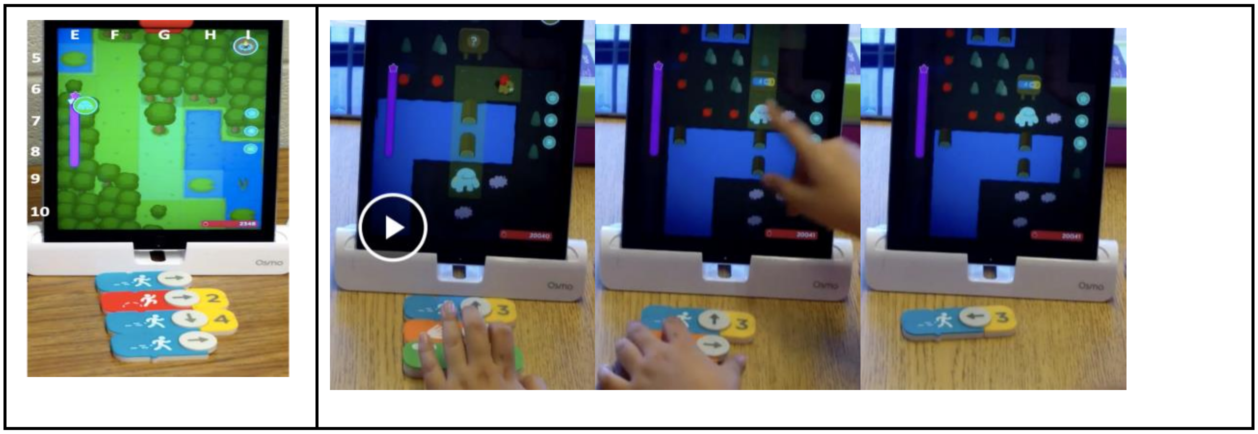

Figure 1: Highlighted Path into the Trees and Using Board Hints

In Emma and Quincy's second program, they traced the screen and used the highlighted path as a guide to check each block as they placed them (see Figure 2, left). Later, they received a hint from the game to guide Awbie off of a lilypad. However, instead of repeating the hint for the next, similar span, they omitted the *walk down 1* instruction and once more stopped on a lilypad.

Using Partners

There were times when partners helped each other. For example, Sophie struggled to understand the rows and columns despite correctly predicting a jump up two spaces from Q16 to Q12 (see Figure 2, left). She mistakenly thought their next move should be a *walk right one* space to Q11, but Marcus placed a *walk up one* instead. Despite Sophie's attempts to use jump blocks, Marcus intervened when she next tried to place *jump right two*, pointing out that it would lead Awbie into the water (on U11). Sophie tried placing two *jumps right one* blocks, but still faced the same challenges, prompting her to let Marcus take over. Quincy also helped Emma change her use of numbers so that Awbie would not jump into the water (see Figure 2, right).

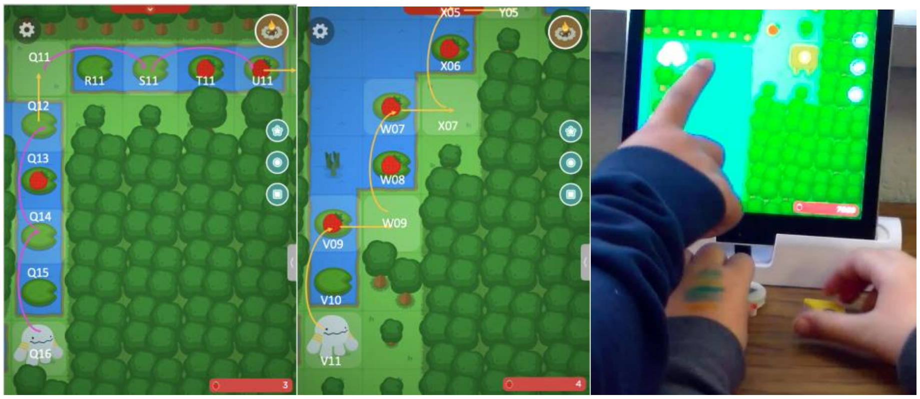

Figure 2: Partner Help and Use of Numbers

Use of numbers and counting

Sophie and Marcus's code to navigate Figure 2 (left) used patterns of numbers to make Awbie's movements through jumps very smooth (*jump up 2, walk up 1, jump right 2, walk right 1)*. As they encountered a new path layout, they strategically adapted their approach. They omitted the numbers, changed the walk direction (see Figure 2, middle) and retained the movement sequence (*jump up, walk right, jump up, walk right, jump up, walk right*).

Emma and Quincy struggled to coordinate their counting to match a lengthy, straight section of the path. Initially, they moved Awbie right twice, successfully reaching a strawberry, but Awbie stopped on a lilypad, causing him to fall into the water and return to his original spot. They adjusted and instructed Awbie to walk right five and three times. Once again, they found themselves on a lilypad with a strawberry, and Awbie fell into the water again. This sequence of events suggests that the duo's challenge might lie in their focus on reaching the next strawberry.

Discussion and Implication

In this study, we investigated the role of instructional support and scaffolding in the development of spatial reasoning and counting skills among third-grade students, focusing on the context of the programming game Coding Awbie^{TM}. Drawing upon Vygotsky's Sociocultural Theory and the concept of the ZPD, we explored how game scaffolds influenced students' spatial reasoning abilities. Our findings revealed that instructional scaffolding was vital in supporting students' spatial reasoning abilities. Through collaborative peer interactions and providing scaffolds such as modeling, probing, hints, and direction, students could navigate the complexities of spatial manipulation more effectively. These scaffolds served to bridge the gap between student's current level of understanding and the desired learning outcomes, facilitating their progression within their ZPD. The use of scaffolds, particularly the highlighted path feature in the game, had a significant impact on students' counting strategies and utilization of programming blocks. The highlighted path scaffold provided students with visual clues and guidance, helping them to overcome challenges related to double counting and spatial structuring. As a result, students demonstrated increased proficiency in spatial reasoning and counting skills throughout the study. The findings of this study have important implications for educational practice and research. Our results underscore the importance of incorporating instructional supports and scaffolding mechanisms into educational games and learning environments.

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THE MERGING OF VIVIAN'S FRACTION WORLDS

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Undergraduate developmental mathematics students' difficulties with fractions, a known gatekeeping topic for this demographic, are well documented. Yet, research on fraction understandings for this population is scarce. In this paper, I synthesize relevant literature regarding undergraduate developmental fraction understandings and related K–12 fraction literature. I then report preliminary findings from a teaching experiment that highlights part of the fraction journey of Vivian, an undergraduate developmental mathematics student, as she constructs an understanding of unit fractions. Vivian merges two separate, but equally valid, fraction worlds as she constructs a unit fraction understanding.

Keywords: Rational Numbers, Undergraduate Education

Many students experience difficulties when working with fractions, a gatekeeping topic for college-level mathematics courses (Mesa et al., 2014; Ngo, 2019). In this preliminary report, I share part of Vivian's journey as an undergraduate developmental mathematics student at a large research institution in the southern United States, as she merged the two fraction worlds she experienced, both of which she deemed as equally valid ways to view fractions. Vivian came to question her own thinking and merged these fraction worlds. This preliminary report comes from a larger dissertation study. I draw from data from a teaching experiment (TE) (Steffe & Thompson, 2000) with an initial task-based clinical interview (CI) (Ginsberg, 1997).

Literature Review and Theoretical Framework

Research on the fraction understandings of undergraduate developmental mathematics students is scarce (Alexander, 2013; Mesa et al., 2014). Studies involving the mathematical thinking of undergraduate developmental mathematics students builds on K–12 research since undergraduate developmental mathematics students study K–12 topics while being college students (Alexander, 2013). Fraction knowledge has been shown to function as a gatekeeper for algebra readiness, though further need for research has been suggested (Booth & Newton, 2012; Ngo, 2019; Siegler et al., 2012). In my study, I draw on the fraction literature of Lamon's (2020) fraction understandings and Steffe & Olive's (2010) fraction schemes.

For the purposes of my study, I define fractions as numbers written in the form of *a/b,* where a and b are not necessarily integers, such that b is not equal to zero (Lamon, 2020; Empson & Levi, 2011). Vivian's fraction story includes the way she understands unit fractions, or the unit of measure when a whole is partitioned into unit fractions. For example, $1/n$ is a unit fraction, where n is a natural number, for a whole that is partitioned into n segments (Hackenberg et al., 2016; Lamon, 2020; Steffe & Olive, 2010). Unit fractions can then be used as building blocks as they are iterated to create composite units (Steffe & Olive, 2010).

Methods

In this paper, I present data from a larger dissertation study. The overarching goal of the dissertation is to get a picture of the way the participant is currently understanding fractions using Lamon (2020) and Steffe & Olive (2010) and then to attempt to create an environment to foster the strengthening of the measurement understanding of fractions (Lamon, 2020). In this section, I will briefly explain TEs, CIs, and the methods used for this study.

Teaching Experiments and Clinical Interviews

A TE is a type of qualitative research that allows the teacher-researcher (TR) a way to have firsthand experience of their participant's mathematical learning and reasoning (Cobb & Steffe, 1983; Steffe & Thompson, 2000). TEs are composed of episodes using tasks designed to test hypotheses the TR has formed about their participant's thinking (Steffe & Thompson, 2000). In a TE, the TR seeks to create an environment to foster a change in thinking (Steffe & Thompson, 2000). But the TR does not assume they are the cause of thinking changes (Thompson, 1979). Thus, the construction of knowledge is the result of the participant's work. The witness's (WR) main job is to observe the interaction between the TR and the participant and may also suggest follow-up questions the TR may not think of (Steffe & Thompson, 2000; Steffe & Ulrich, 2020). The CI is a semi-structured interview that focuses on describing the participant's current thinking and reasoning (Ginsberg, 1997; Steffe & Thompson, 2000). Data for TEs comes from video recordings, the participant's written work, and any field notes taken (Cobb & Steffe, 1983; Steffe & Thompson, 2000; Steffe & Ulrich, 2020). Both ongoing analysis and retrospective analysis are utilized (Steffe & Thompson, 2000).

Participants, Data Collection, and Analysis for This Study

Vivian was recruited from a randomly selected undergraduate developmental mathematics class at a large research university in the southern United States. She participated in a CI that included 12 fraction tasks, divided among two 45-minute sessions. She continued into TE phase of the study, participating in six hour-long teaching episodes. Data was collected during the spring and fall semesters of 2023 in the form of video recordings, Vivian's written work, and field notes taken by the TR and WR. We filmed sessions with two cameras. The primary camera captured Vivian's written work. The secondary camera recorded the interactions between Vivian and the TR. Screen recordings were captured when Fraction Bars software was in use. Ongoing analysis took place during and after each session and retrospective analysis is still underway (Steffe & Thompson, 2000).

Results

I realized that Vivian viewed fractions as living in two different worlds during episodes 3 and 4 of the TE. Once viewed as equally valid, Vivian's and the teacher's worlds merged in episode 5. Note: 1/5 often refers to Vivian's thinking, not necessarily the normative meaning of a fifth.

Vivian's Two Worlds

Vivian demonstrated a discrete view of fractions during her CI. This was evident while sharing her understanding of 2/3, which she described 2/3 as being the majority of something (see Figure 1(a)). Vivian also drew discrete representations of 1/2 and 1/3 when comparing the two fractions during a number line task (see Figure 1(b)).

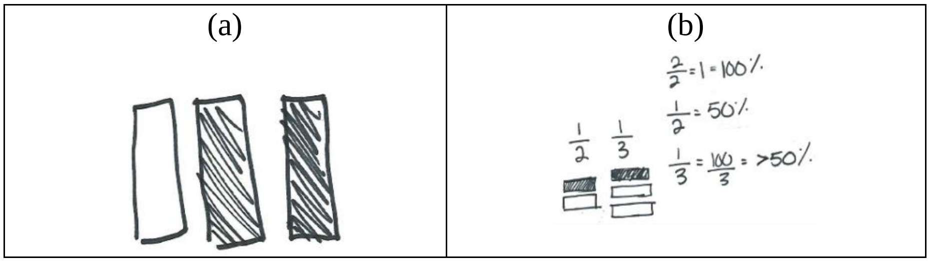

Figure 1: Vivian's representations of 2/3, 1/2, and 1/3 during her clinical interview.

The third teaching episode brought more insight into Vivian's view of unit fractions. Vivian was asked to cut 1/4 from each of four different sizes of rectangles. When explaining how she made 1/4, Vivian said, "It's folded and then folded again so you have four significant parts. I cut one off so that's one of four pieces." Vivian used the phrasings 1/4, "one fourth," and "one of four pieces" interchangeably during this session. Next Vivian was tasked with showing 1/5 using Fraction Bars software. The program had a split function to equipartition units automatically. However, Vivian chose to manually split the unit into five unequal pieces. Vivian pulled out two of her splits and said that each of these pieces was 1/5 (see Figure 2). She used the context of a farmer's field to explain that while the pieces were different sizes and that "one buyer needs to go look somewhere else, but these are both still one fifth." Vivian added, "Comparing this [top] slice of land to this [second from bottom] slice of land, this buyer is getting a lot more land for the same fraction." In short, for Vivian, the size of the piece did not seem to matter.

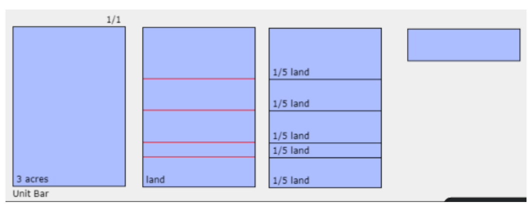

Figure 2: Vivian splits the land into fifths.

However, it wasn't until the fourth teaching episode that I became aware of Vivian's two fraction worlds. When revisiting our task to cut off 1/4 of a paper rectangle, Vivian showed two different ways of finding the 1/4 (see Figure 3). After creating different fourths, Vivian shared that the paper shown in Figure 3(a) is "more precise" than the one in Figure 3(b), saying, "one is bigger than the other so they can't both be 1/4 on a test." Vivian described a teacher's fraction world where it would be incorrect to call both 1/4. Vivian also shared that there were other situations where her folded "fourth" would still be one fourth since there were four pieces using paper folding (see Figure 3(b)). In Vivian's eyes, one section in each figure was equivalently 1/4, even though she recognized that a teacher would view them differently. Vivian later created a farmer's field, where each section was "one fourth but a different size" (see Figure 3(c)). She then defined 1/4 as "an obvious portion because each section is one of four pieces," adding a caveat that the meaning of 1/4 would change if you add in measurements. Vivian focused on the number of pieces in the whole to determine the unit fraction, but not the visual size of the part.

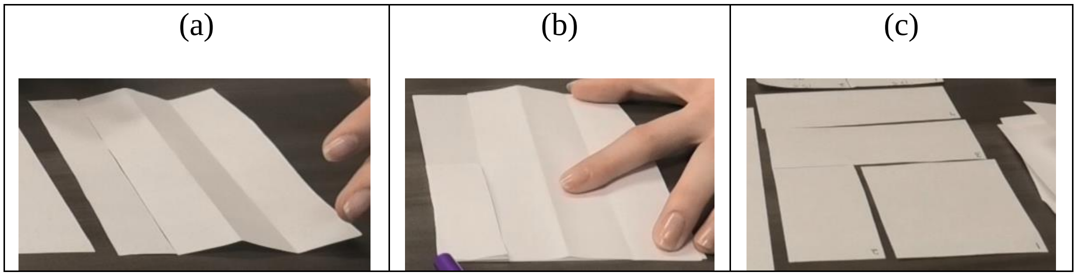

Figure 3: Vivian shows various ways to make a fourth.

Reconciling Two Worlds into One

Near the end of episode 4, Vivian was asked again about whether sections 2 and 3 of Figure 3(c) were still 1/4. Vivian responded, “Yes but no,” since they were each one of four pieces but had different areas. We picked up this line of thinking in episode 5 using Fraction Bars. Taking a cue from Vivian, the first task of the day used the context of a farmer fairly dividing a field for their children to use as they see fit. This scenario was used for three and then five children. Using program functions, Vivian automatically split the field into thirds but manually split the field into fifths (see Figure 4(a)). The WR asked Vivian how many copies she would need to make of one of her fifths to show that a split was really 1/5. Vivian chose one of her fifths and stated that she would need at least five copies to check if it was 1/5 (see Figure 4(b)). Vivian also acknowledged that the computer could have done a better job than she did in splitting the whole. However, she still viewed each split as 1/5 regardless of the method used in the process. Vivian questioned whether her perspective is valid, saying,

> If we’re just gonna take one of the one-fifths, which it happens to be smaller than the other one-fifths, then I don’t think on its own it would be able to create perfectly the pink again if we were to copy it a bunch of times. But does that mean that it’s not one fifth anymore?

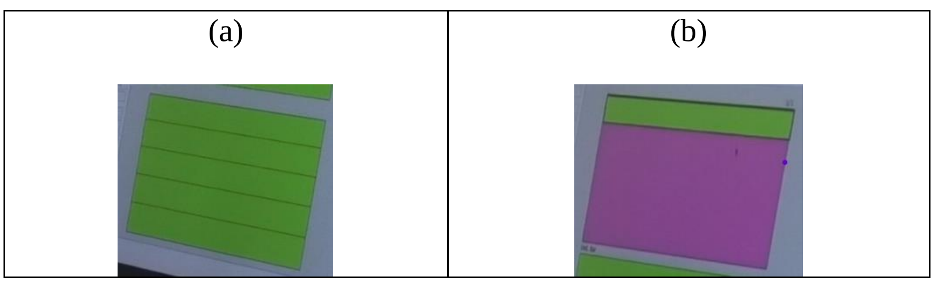

Figure 4: Vivian (a) splits a "field" into fifths and (b) checks if a split is really 1/5.

When asked what she thought, Vivian responded that she did not know, adding, “This whole time I’ve been thinking that this is still a piece of the whole. But if the piece by itself, copied, can’t make up the whole, is it the correct fraction?” We changed the language in our sessions to distinguish between pieces of the whole and fractions. The session continued with creating wholes and splitting them to find various unit fractions. This unit fraction was copied to create the requested fraction of the whole. When checking her work for 5/7, Vivian demonstrated that she needed five copies of 1/7, which could come from any 1/7 of the unit. She also recognized that it took seven copies of the same size pieces of 1/7 to get to the whole.

Discussion

Vivian had a system of working with fractions from the beginning of our time meeting together. This system, which reminded me of Erlwanger's (1973) Benny, included understanding unit fractions as one of *n* pieces. During the fourth episode, Vivian spontaneously started labeling one of her farmer's fields with sided lengths, areas, and perimeters as she showed different ways to find 1/4. It was during this task that Vivian may have begun to realize that the areas need to be the same for unit fractions to be equivalent. This is suggested in her "Yes but no" response to deciding if two different pieces out of 4 were both 1/4.

During episode 4, Vivian stated "The whole is easier to understand when the pieces are the same." However, it wasn't until Vivian was asked if it was important for all the pieces to have the same size to be a fraction that she expressed her question about if her fifth was really 1/5. Vivian's recognizing that 5/7 is five copies of 1/7, using copies from any 1/7, provides evidence that she had established a unit fraction understanding consistent with Seffe & Olive (2010).

Acknowledgments

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THE NATURE AND DEVELOPMENT OF STUDENTS' DIGITAL EXPLANATIONS OF THEIR PROPORTIONAL REASONING OVER TIME

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Keywords: Technology, Middle School Education, Rational Numbers and Proportional Reasoning

Engaging students in mathematical practices such as collaborative problem-solving, justification, and explanation has long been accepted as beneficial for student learning (Staples, 2007; Stein, Engle, Smith, & Hughes, 2007) As the presence of technology in classrooms continues to grow, students are asked to explain their mathematical ideas through typing and digital tools that can be easily shared and saved (Engelbrecht, Llinares, & Borba, 2020; Thomas & Palmer, 2014). The increased prevalence of student explanations in digital form can offer new opportunities for teachers and students (e.g. the ability to edit/revise explanations, increased access to student work for teachers), but can also provide increased insight into student thinking over time for researchers and teacher educators. Large databases of students' typed explanations have been used to develop machine learning-algorithms to provide feedback to students on isolated items in science education research (Shin & Shim, 2021), yet we know less about student mathematical explanations and how they develop over time and across contexts. Further, students' use of proportional reasoning to make sense of our world does not receive enough attention across different mathematical contexts.

In our analysis of students' digital explanations, we draw from larger design-based research to develop a digital collaborative platform for students and teachers, which allowed students to collaborate with peers and type explanations across multiple connected units of a problem-based curriculum. This research is guided by the following research questions: 1) *What are the characteristics of middle-grades students' digital mathematical explanations of proportional reasoning?* and 2) *How do middle grades students' digital explanations develop across mathematical contexts?* The primary data source is a large database of student written responses to mathematical problems across three curricular units focused on proportional reasoning, created from log files of student actions in the digital platform. We use learning analytics methods (e.g. text-mining) to identify key words of student written text describing proportional reasoning, as well as how characteristics of student explanations (e.g., length, mathematical focus) might change across number and operations, geometry, and algebra/functions contexts. Secondary data include classroom observations and teacher interview recordings to identify larger trends and themes in the students' digital explanations across units. Findings will provide insights into the nature and development of student proportional reasoning, and how teachers can support deep and flexible ways of knowing. Our analysis and findings provide insight into how students communicate their mathematical thinking and understanding, a critical perspective in technology design and the development of machine-learning algorithms for feedback/assessment.

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A Visual Approach to Developing Fluency with Multiplication Facts

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The purpose of this study was to determine if visual imagery in the form of Quick Looks could promote the development of multiplication fact reasoning strategies. It was conducted from August to March during one school year and involved three experimental third-grade classrooms who received instruction using visual images. Corresponding control classrooms from the same schools received instruction from the district-adopted curriculum. Using a mixed-methods design, students from experimental and control classrooms were compared with respect to progress towards fact fluency and overall multiplicative understanding. Analysis revealed that the experimental group made statistically greater gains, suggesting that image-based instruction may encourage greater progress towards multiplication fact fluency.

Keywords: number concepts and operations, elementary school education, curriculum

The learning of basic facts, or sums and products of numbers 0-10 and their related differences and quotients, has always been a high priority for elementary school teachers. These early skills form a necessary and significant component of a student's procedural fluency, or "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately" (National Research Council, 2001, p. 5). Although fluency with basic facts functions as a gateway for future mathematical success, "many educators find that children, even in the upper grades, continue to draw tally marks and count by ones as their dominant solution approach in solving problems" (National Council of Teachers of Mathematics [NCTM], 2020, p. 82).

In attempts to address this issue, numerous research studies (e.g., Brendefur et al., 2015; Cook & Dossey, 1992; Thornton, 1978) have established that multiplication facts instruction that is based on students developing reasoning strategies as opposed to rote memorization of isolated facts produced significantly higher levels of fluency. In their study, Brendefur et al. (2015) utilized a cognitive framework focused on Bruner's Three Modes of Representation (1966). Students first explored multiplication strategies with hands-on activities (enactive), then through pictures (iconic), and finally, with number sentences and words alone (symbolic). The idea of using representation to scaffold strategy development also aligns with recent research on the importance of visual imagery. For example, Park and Brannon (2013) found that individuals who were given frequent opportunities to compare approximate sums and differences of dot patterns (i.e., using approximation to determine if the sum of the dots on two arrays is greater or less than the dots on a third array) showed statistically significant growth in their symbolic computational abilities as compared to control groups. They note the results "strongly corroborate the proposition that nonsymbolic-arithmetic ability and symbolic-math ability share cognitive foundations" (p. 2017). Thus activating both visual and symbolic thinking simultaneously during instruction may have promise for enhanced fluency development.

Our study attempted to build upon these findings to explore the use of visual imagery as a means of motivating multiplication fact strategies with third grade students. More specifically, the study utilized dot patterns in equal group or array formations that were shown for 2-3 seconds followed by students sharing how many dots they saw and how they saw it. This "Quick Look" routine utilized carefully designed and sometimes sequenced images as shown in Figure 1 so that key multiplication fact strategies were likely to naturally emerge.

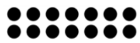
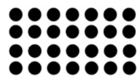
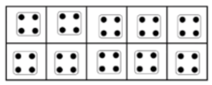
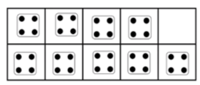
Strategy	Image 1	Image 2	Symbolic Representation of the Strategy
Doubling			4 x 7 = ? If I already know 2 × 7 = 14, I double to get 4 x 7 = 28.
Subtracting a Group			9 x 4 = ? If I know 10 x 4 = 40, then I subtract one group of 4 to get 40 – 4 = 36.

Figure 1: Quick Look Activities: Sequencing Images to Encourage Strategy Development

The purpose of this study was to examine the results of using an enhanced and deliberate collection of lessons with visual imagery in a Quick Look format. This involved a comparison between classrooms who received the Quick Look treatment and those who utilized the existing curriculum. The following research questions involved these ideas and were used to guide the study: 1) Is there a difference in fluency obtained by students who engage in visual imagery activities as compared to students who do not receive this instruction? and 2) Is there a difference in multiplicative understanding obtained by students who engage in visual imagery activities as compared to students who do not receive this instruction?

Theoretical Frameworks

This study drew from various areas of research, including intuitive multiplicative concepts, fact fluency development, and visual mathematics, and thus was grounded in several theoretical frameworks. Various researchers (e.g., Clark & Kamii, 1996; Mulligan & Mitchelmore, 1997; Wright et al., 2012) have developed similar trajectories for classifying intuitive models of multiplication. One limitation of these trajectories is that their levels often differentiated various counting-based methods, such as unitary counting, rhythmic counting, and skip counting, but combined reasoning strategies and recall into a single level. A more nuanced framework was required for the purposes of this study, leading to the development of the Multiplicative Understanding Levels (MUL) shown in Table 1.

Table 1: Sample Codes for Multiplicative Understanding Level (MUL)

Level	Description	Code
Numerical Composite	Counting by 1s using unitary or rhythmic counting	UC or RC
Abstract Composite Unit	Skip counting or repeated addition	SC or RA

Emergent Composite Unit of Composite Units	Recognizes the opportunity to apply a strategy but unable to successfully execute	ECU
Facile Composite Unit of Composite Units	Successfully implements the strategy	FCU

In particular, the distinction between the Emergent Composite Unit of Composite Units (ECU) and Facile Composite Unit of Composite Unit (FCU) levels was not incorporated in any of the previous studies on which this framework was based, but was used in our study to capture the subtle differences in how students make sense of and develop proficiency with fact strategies. For example, if asked to provide a strategy to solve 6 x 8, a student at ECU might just identify 5 x 8 as a viable helper fact but then not know how to proceed, where as a student at FCU could complete the strategy: 5 x 8 + 8 = 48.

While the MUL framework was utilized to capture multiplicative understanding, a three-phase framework adapted from Baroody et al. (2003, 2006, 2009) was used to provide a measure of overall progress towards fact mastery. With respect to multiplication, Phase 1 incorporates any methods that rely on unitary counting, rhythmic counting, or skip counting. In contrast, Phase 2 involves reasoning strategies based on known facts. Phase 3, the Mastery Phase, is reached when children can quickly generate an answer for a fact either through recall or highly efficient strategy application. These two frameworks were useful for measuring student understanding and fluency growth, but Bruner's Modes of Representation (1966) drove instructional design. In particular, our study focused on using discussion of Quick Look images to bridge iconic and symbolic thought as shown in Figure 1. An integration of these theoretical perspectives led to a grounded supposition that attaining a complete concept of multiplication, as well as attaining basic fact mastery, requires students to be able to operate in the symbolic mode with ease. Thus, frequently recording student thinking in words and symbols as they explained how they made sense of the images was an important part of the intervention, facilitating the eventual fading of the iconic as students transitioned to mainly symbolic work (Fyfe & Nathan, 2019).

Methodology

The study took place in three rural/suburban schools in Southwest Michigan, USA, from August to March of one school year. Each school contained one experimental classroom and one or two control classrooms with a total of 26 experimental and 25 control students. The intervention consisted of six experimental lessons taught by Kling that each utilized discussion around 4-6 purposefully sequenced Quick Look images, followed by a written partner activity. These lessons replaced existing lessons with the same objectives from the regular curriculum used in the control classrooms, which primarily used number stories to motivate multiplication fact strategies. There were no other differences in instruction between the control and experimental groups aside from expected natural variation due to different classroom teachers.

Data collection utilized five semi-structured interviews conducted throughout the school year, including a pre-assessment interview at the beginning of the year and interviews following Lessons 1, 3, 5, and 6. The pre-assessment interview included addition and subtraction facts and served as a baseline measure to establish comparability of groups. Interviews 2-5 used a semi-structured protocol consisting of seven bare numeral multiplication facts and one open response task. For the bare numeral tasks, students were asked to explain how they figured each one out, regardless of the speed or accuracy with which they answered. This allowed for both the phase and strategy to be recorded. The recorded phase (1, 2, 3) was combined with a 1 if correct for a total fluency score of up to 4 points per bare numeral fact, 28 points per interview. Mean scores were compared to determine statistically significant differences. The open response tasks prompted students to interpret or apply particular strategies and were coded qualitatively based on MUL. For example, Interview 4 contained the following open response task: *How could you use doubling to help you to figure out 8 × 4? Can you solve it a different way now?*

Results

The first research question explored differences in levels of fluency obtained by experimental and control students. Mean scores for the seven bare numeral items on Interviews 2-5 were compared. Steady gains were made by the experimental group while the control group fluctuated, culminating in the experimental group demonstrating a statistically higher fluency level ($p = 0.047$, $\alpha = 0.05$) on the final interview conducted in March. Furthermore, 77.5% of the tasks were answered by the experimental group using either a strategy or recall (Phase 2 or 3), compared to only 63.4% by the control group, for whom the second most common approach across the tasks was unitary counting (Phase 1). Proportion testing on these results found the experimental group scored statistically higher at the $\alpha = 0.05$ level ($p = .002$). Thus, it is possible that the intervention was more successful in encouraging movement away from counting to the development of reasoning strategies and higher levels of fluency.

The second research question explored differences in multiplicative understanding and thus called for a qualitative approach. Interview open response items were coded for MUL using the coding scheme shown in Table 1, with a particular focus on emergent and facile composite unit of composite units thinking (ECU and FCU) as both were indications of movement towards strategy acquisition. An examination of Table 2 shows that the control classrooms generally had a sizable portion still in ECU, indicating incomplete strategy development. However, when examining FCU, results favored the experimental group as they had statistically higher proportions of FCU on Interviews 2, 3, and 5 (p values shown for FCU only).

Table 2: Summary of MUL Codes: Interview Open Response Tasks

		ECU	FCU	p value
Interview 2	Experimental	3.8%	38.5%	0.036
	Control	12.0%	16.0%	
Interview 3	Experimental	26.9%	50.0%	0.013
	Control	40.0%	20.0%	
Interview 4	Experimental	11.5%	73.1%	0.060
	Control	20.0%	52.0%	
Interview 5	Experimental	0%	73.1%	0.004
	Control	20.0%	36.0%	

Discussion and Conclusion

The purpose of this study was to determine if the intentional design and use of visual imagery in the form of Quick Looks could promote multiplication fact strategy acquisition and fluency development. As summarized above, there were several indicators that the intervention was in fact successful in doing so. This adds to the research on the importance of visual mathematics for learning (Boaler et al., 2016). Visual imagery has promise perhaps because it is accessible to a wide variety of students; even students who are still in the counting phase can make sense of the images and the class sharing of strategies may help provide them with efficient alternatives to adopt in the future. Furthermore, the deliberate connections made between the images (iconic) and symbolic expressions used to represent student thinking during class discussion of the images may have helped promote a deeper understanding of the multiplication reasoning strategies developed in the lessons as well as more flexibility in working simultaneously within each mode of representation. Visual imagery, in the form of Quick Looks, may provide a much-needed pathway for all students to achieve the lasting and meaningful fact fluency necessary for future mathematical success.

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ANALYSIS OF STUDENT'S REASONING ON FRACTIONS FROM A VIDEO GAME WITH A COORDINATION CLASS

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Given that previous studies about video games and students' math learning focus on the quantitative aspect of students' learning (e.g., post-test scores), this research applies a coordination class from the Knowledge in Pieces framework in order to analyze fine pieces that emerge in students' reasoning around fractions in the context of a video game. This study analyzed one student's reasoning around fractions emerging during the interview about his gameplay. The analysis shows that rich inferences around fractions get activated in different contexts and how unrecognizably the inferences the student employs in each context shift and mix up with other inferences. In order to support students to expand their intuitive understanding of fractions into a more systematic understanding, the rich inferences from students' intuitive reasoning need to be incorporated into instructions on fractions.

Keywords: Rational numbers, Cognition, Elementary school education

Previous studies have demonstrated positive impacts of video games on students' mathematics learning by showing improved post-test scores (Braithwaite, & Siegler, 2020; Denham, 2015; Hulse et al., 2019; Ke, 2008a, 2008b; Kim & Ke, 2017; Litster & Moyer-Packenham, 2020; Liu et al., 2017; Moyer-Packenham et al., 2020; Vogel et al., 2006). It is surprising, however, that few studies analyze any qualitative changes in students' mathematical reasoning emerging during gameplay. Hence, this study proposes to design a math video game around fractions and to conduct a qualitative analysis on students' reasoning about fractions emerging from gameplay with a coordination class theory.

Literature Review

Researchers analyzed rational numbers into several sub-constructs (e.g., Kieren (1980) : part-whole, measure, quotient, ratio and operator, Behr et al. (1992) : part-whole, quotient, and operator) and the subconstructs have been applied in previous studies (Cramer et al., 2019; Lopez-Martin et al., 2022; Moyo & Machaba, 2021; Witherspoon, 2019; Wood et al., 2013). The various facets of rational numbers can be approached with the Knowledge in Pieces (KiP) framework (diSessa, 2002). Knowledge of rational numbers can be framed as a complex knowledge system that consists of various knowledge pieces of different types. Among the possible knowledge types in the system of rational numbers, this study approaches the subconstructs of rational numbers as a coordination class (diSessa & Sherin, 1998; diSessa & Wagner, 2005; Levrini & diSessa, 2008).

Coordination class theory

diSessa and Sherin (1998) introduce a coordination class as one type of knowledge pieces as "systematically connected ways of getting information from the world (p.1171)". There are two sub-classes in a coordination class. The one sub-class of a coordination class is called **extractions**. One example shared by diSessa (2004) is that we notice that an object is bigger or smaller than another object and that the information of size difference relates to mass. The other sub-class of a coordination class is **inferential net**, which is “the set of all possible inferences that lead to determining the relevant information (diSessa, 2004, p.141)”. A specific collection of parts of a full coordination class activated in a specific context is **concept projection**.

For example, let’s say there is a circle that is divided into four equal parts and two of the four parts are shaded. One student might attend to that the four parts are all equal size and only two parts are shaded, which are extractions. The student would coordinate the two extractions with inferences such as ‘the one circle is a whole’ and ‘two shaded parts of the four parts can be represented with fractions.’ These are the inferences that the student projected at the moment from the inferential net of a coordination class of fractions. The student would get to a determining information that the two parts represent $\frac{2}{4}$. The two extractions and the two inferences are one concept projection of a coordination class of fractions by the student in this context with a circle representation. All processes from extractions to information determining about the context from an inferential net are called as **readout** (for more details, see diSessa et al., 2016).

As students accumulate experiences where they apply a coordination class in various contexts, the **span** of the coordination class gets extended. As students construct more expert-like knowledge system by extending the span of a coordination class, they will come to conclude the same valid interpretation of a coordination class across contexts, which is **alignment**. (for more details, see Levrini & diSessa, 2008; diSessa & Wagner, 2005).

Reapproach fractions as part-whole with a coordination class theory

Behr et al. (1992) introduce two interpretations of the part-whole construct: parts of a whole and a composite part of a whole in both continuous quantity and discrete quantity. Behr et al. (1983) analyze children’s reasoning related to the two interpretations with a circle representation and categorize students’ reasoning around fractions into three levels based on whether students can or cannot label each part of the circle with both names. It is not enough, however, to categorize at which level students’ understanding on fractions as part-whole is in terms of why some students can label a given representation with two labels but other cannot, and in what ways students come to be able to label the representation with both labels in what contexts. It is worth analyzing the information students attend to and the inferences they coordinate with when interpreting fractions as part-whole in different contexts beyond just categorizing students’ understanding about fractions as a part-whole into a few levels.

Video games in elementary math education

In this research, math video games refer to games whose designs and goals focus on supporting students to have a designed experience (Squire, 2006) around mathematical concepts introduced in school mathematics (Braithwaite, & Siegler, 2020; Denham, 2015; Hulse et al., 2019; Kim & Ke, 2017; Litster & Moyer-Packenham, 2020; Liu et al., 2017; Moyer-Packenham et al., 2020; Vogel et al., 2006). Even though many studies show that video games are effective for students’ math learning quantitatively (Karki et al., 2022; Kiili et al., 2018; Tsai & Yen, 2016; Zhang et al., 2019), it has not been uncovered enough in terms of what happens in students’ reasoning behind their post-test scores. Given that there is a lack of a qualitative analysis around students’ reasoning around fractions in the context of video math games, this study conducted a qualitative analysis applying a coordination class theory within the Knowledge in pieces framework to focus on small knowledge pieces and coordination that emerge from the process of students' reasoning around fractions in a video math game context.

Methods

This method section is based on the Knowledge Analysis (KA) methodological framework suggested by diSessa et al. (2016).

Empirical set-up

1. Based on the definition of a video-based game as a designed experience (Squire, 2006) delivered on digital devices (e.g., tablets, computers or consoles, etc.) with openness to various ways to achieve game goals, the researcher designed and developed one math video game, Bridge the Cloud, with a Unity 3D game engine. (For more information about the game, please try the game here using a Chrome browser). In the game, players need to make clouds at different locations in order to block stones falling from the sky. To make clouds at position players want, players need to use different numbers for denominator and numerator (Figure 1).

Figure 1: Screenshots of the game, Bridge the Cloud

2. In each 1-hour game session, a student played the game on a given android tablet device for the first 30 minutes while the researcher sat next to the student. The gameplay was recorded using the screen recorder function of the tablet device. For the next 30 minutes, the researcher interviewed the student and asked questions about the student's gameplay. The entire game session was recorded with two camcorders.

3. One 4th-grade student, Jason (pseudonym) was recruited for this study at one public charter school in the southern US and total 10 game sessions (one game session per week) were held in one classroom at the school.

Capture and Reduction

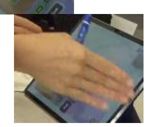
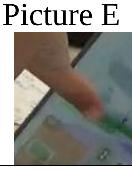
The focus of this study is to observe extractions and inferences, if any, that emerge in students' reasoning around fractions. To serve the purpose, an interview format was chosen because an interview allows researchers to ask additional questions to students about their reasoning. All verbal expressions, written expressions, and gestures were included in the analysis. All the interview sessions were transcribed to capture verbal expressions. Written expressions and gestures that were made along with verbal expressions were documented in transcripts. Verbal and written expressions and gestures were coded as a chunk around a specific reasoning of the student. In the next phase of coding, specific information that the student extracted and inferences with which the student coordinated the information were analyzed.

Results

Unrecognized shifts among inferences related to one whole and one-unit length

When Jason shared his thinking about how the green numbers (numerator) and the blue numbers (denominator) work in the game, the concept of a whole emerged in his reasoning. However, what he referred to as a whole changes quickly and even unconsciously, and a concept of a unit is entangled in his projected concept of a whole. The excerpt in Table 1 shows his reasoning about a whole.

Table 1: Interview excerpt from the 2^{nd} interview session on Sep 15^{th}, 2023

J : Jason & R :Researcher 2^{nd} interview 06:33- 11:28 on Sep 15^{th}	
R: What do you think 3 green and 2 blue are doing here (Picture **A**)? J: Hmm…What do you mean? R: Now, in this box, there is 2 in the blue box and 3 in the green box and the cloud is here (3/2), what do you think this blue 2 and green 3 are doing here? J: hmm. I think ,, what they are supposed to do? Like green makes it longer and blue makes it shorter?	Picture A 
R: I mean blue makes it shorter and green makes it longer right? But why the cloud is here (3/2) not here (pointing at near 1 – left red circle in Picture B) or there (pointing at near 2 – right red circle in Picture **B**) ? J: Because I put 3, and then put 2 in the blue and then that subtracts half the line, I guess? Yeah, half of the line which makes it to going that B: Half of the line... what line? J: This is the whole line (pointing at the one green stick)	Picture B  Picture C Picture D 
B: Wait, you mean this line (one green stick – Picture **C**) or this whole line (the whole number line – Picture **D**)?	
J: This line (one green stick from 0 to 1- Picture **E**), that's the whole line, and then if you're subtracting 2 from that line	Picture E 

This shows that the inferences Jason projects in each contextuality shift unrecognizably. When firstly asked what green number 3 and blue number 2 were doing in the given game context, Jason appeared to extract the information that the length of the stick changes with the input of green 3 and blue 2 and he concluded that the cloud was made at the current position because it is half of *the line*. However, when explicitly asked what the line he referred to right after he said half of the line, he suddenly pointed at the line from 0 to 1. At this moment, he appeared to be cued with another inference that a whole means 1 on the number line.

Fractions as both a whole quantity and a selected quantity present

When Jason explained about one-third shown up in the game, he kept explaining a one-third as one selected part among three equal parts that are present at the same time in one whole. So, when he explained one-third of the stick in the game, he brought up that the two other parts were subtracted. This shows that when Jason projected a coordination class of fractions as a part-whole, an inference got activated that a fraction, such as 1/3, represents cardinality contexts involving both a whole evenly divided into three parts and one selected part among the three parts. Wood et al. (2013) showed that students' understanding of fractions includes an operation of removal in addition to equal dividing but did not address why students come to include the subtraction operation in their reasoning. This analysis suggests one possible explanation; if students utilize an inference that fractions represent cardinality contexts where both a whole partitioned into *n* parts and *m* selected parts of *n* parts are present, this inference would lead them to focus on explaining that the *n-m* parts are gone so that they can make sense of why only *m* parts are left.

Discussion and Conclusion

The analysis results highlight that when a student interprets fractions in the specific video game context, how momentarily the inferences one employs in each context shifts unrecognizably. By approaching fractions as a coordination class, small, but rich, inferences that students use or get cued by contextuality can be uncovered. Based on the inferences from students' reasoning, we can figure out what instructional supports students need in order to expand their intuitive understanding of fractions into a more systematic understanding.

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NUANCED RELATIONSHIPS BETWEEN WHOLE NUMBER AND FRACTION UNDERSTANDINGS: DALTON AND ANGELA'S CONCEPTUAL RESOURCES

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The research base remains unfinished when considering how students with specific learning differences develop an understanding of part-whole number knowledge, then leverage it for fraction knowledge. From a conceptual analysis of two students' engagement with six task-based interviews, we provide insight into distinct nuances in each student's partitioning and iterating with whole number and fraction development. Findings indicate students had very different strategies and approaches to their part-whole number strategies, allowing the teacher-researchers very different conceptual resources to consider when leveraging their strengths throughout the study. Implications of these findings suggest an asset-based lens provides all students' equitable opportunities for mathematics learning.

Keywords: Students with Disabilities, Number Concepts and Operations, Rational Numbers.

The research base remains unfinished for students with learning differences, specifically students with learning disabilities and other health impairments, develop an understanding of part-whole number knowledge. Moreover, it is not yet known how students with learning differences leverage their part-whole knowledge for fraction knowledge, and what features of their diverse experiential and cognitive backgrounds (e.g., working memory or processing differences) might interact with their development (e.g., Hunt et al., 2016; Hunt & Tzur, 2017). Current research efforts document elements of students' diverse cognitive backgrounds thought to interplay with students' mathematical learning from an early point in their lives (Compton et al., 2012). These factors are then used to explain learning differences as variations in certain norms that predict performance over time (Vukovic, 2012). This approach to designing intervention has gained much knowledge over the years, yet there is still a great deal of opportunity for improvement. Here we examine how two third grade students with learning differences construct their whole number understandings in relation to their fraction understandings. We expect to provide insight into the conceptual resources students with learning differences enact as they leverage their whole number knowledge for their fraction knowledge development.

Theoretical Framework

The authors of this study frame this work by examining learning patterns in children's part-whole number and fraction understandings, using theoretical account of units construction and coordination (e.g., Hackenberg & Sevinc, 2024; Steffe, 2024; Steffe & Cobb, 1988; Steffe & Olive, 2010). We define *a unit* as a perceptual chunk, sometimes conceptualized as a discrete "one" or a continuous length (Hackenberg & Sevinc, 2024). Steffe and Cobb (1988) described children's construction of pre-arithmetic units through their development of number sequences; they explain counting activity as a means of developing sequential schemes for additive operations with number. Once children internalize their number sequence, they are capable of partitioning number sequences and using counting-on instead of recounting all (e.g., three more than six is seven, eight, nine). This is also described as development of *numerical composites*, meaning students can solve tasks requiring one numerical unit grounded in their number sequence (Steffe, 1994). Once children construct and count with abstract units, they are described as having *one-level of units interiorized*. One-level of units interiorized allows students access to *two-level units with one unit developed in activity* because students can actively distribute the elements of one unit across the elements of a second unit (described as part-whole number understandings, see below for details). Part-whole number understandings would require students to have *two-levels of units interiorized*, freeing up their working memory to attend to pre-arithmetic units in activity. Once a student constructs two-levels of units interiorized they have an assimilatory structure of unitary items embedded in composite units (Steffe, 2024).

Part-whole number understandings is a pervasive issue affecting many elementary mathematics students with learning differences (Landerl et al., 2004; Vukovic, 2012). We define part-whole number understandings as the development of students' understanding of numbers as part of a sequence (i.e., 1, 2, 3, 4, 5) and part of a unit (i.e., 2 and 3 compose 5) in such a way where a number becomes a coordination of counting and grouping operations (Piaget, 1968/1970). When a student solves the problem "How many threes in twelve?" one strategy may be to verbally count, "one, two, THREE (puts up one finger), four, five, SIX (puts up a second finger) …4 times" (Ulrich, 2016, p. 2). This "double counting" technique, involving counting each set of three as 1 group, is an application of internalized part-whole operations, as the parts (one, two, THREE and 4 sets of 3) are operated upon within the whole (12). If students require a sequential recounting of the 12 counters to find the total of 3 sets of 4, we describe that as a less sophisticated part-whole operation.

Fraction understandings are also framed, in part, with units coordination theories providing insight into students' earliest activity when constructing part-whole fraction and fractions as a form of measure understandings (e.g., Hunt et al., 2016; Steffe & Olive, 2010; Wilkins & Norton, 2017). Steffe and Olive delineate students' fraction development as a process of units coordination to characterize varying degrees of sophistication in students' scheme development. Steffe and Olive's work provided insight into students' operations when transitioning from fractions conceived in a part-whole manner versus measurement fractions. Hunt and colleagues provided insight about fraction development among students with learning differences and learning disabilities; they explained why early operations (e.g., partitioning) can range so widely between individual students. We focus next on the wide-ranging scheme development and look at implications for transitioning from whole number schemes to fraction schemes.

Students with learning differences often develop less sophisticated double counting strategies when understanding part-whole number operations (Landerl, Bevan, & Butterworth, 2004; Vukovic, 2012). Nevertheless, these are meaningful and comprehensive counting strategies. In schooling experiences, students with learning differences often need additional support to elicit their conceptual knowledge of part-whole operations from earlier grades; such knowledge is critical as they work to establish part-whole number understandings for numerical computation in upper grades (Butterworth, 2011). The research literature has described the development of number understanding and computational skills among students with learning differences in very different ways than for their more successful counterparts (Lambert & Tan, 2017). In contrast to these patterns of deficit comparison in the literature, we adopt an asset-based lens by identifying and elaborating upon the meaningful activity and scheme development that can allow for a productive bridge from whole number concepts and operations to fraction concepts and operations.

Methodology

Part of a larger pilot study, this paper describes qualitative data from two students, Dalton and Angela, each receiving specialized services for their mathematical learning needs. Dalton and Angela were enrolled in a small rural school in the midwestern portion of the United States. We engaged each student at their participating school in six task-based sessions: three whole number and three fraction task-based interviews. The first and second author worked with the students individually and, at one point, jointly with both students. Excerpts in this manuscript indicate the student by their pseudonym, "Dalton" or "Angela", and the Researcher as "R". We adapted Wright, Ellemor-Collins, and Tabor's (2012) number tasks (interviews one, two, and three) and Hunt and colleagues (2016, 2017) fraction tasks (interviews four, five, and six). After each interview, we transcribed data, conceptually analyzed it, and adapted the subsequent tasks in response to what we understood about the students' scheme development and their interpretation of each task. Conceptual analysis included reviewing the transcripts and identifying the conceptual resources we could attribute to each student based on their verbal interactions, their use of figural representations and their gestures as they engaged in tasks.

Findings

Whole Number Understandings

Dalton. Throughout interview one, Dalton heavily relied on physical activity, finger patterns, and self-created number lines. When asked "how many," he often recounted all items (starting with the number one). When the teacher-researcher covered objects, he sometimes needed to feel through the cloth to count them. Early on, in the first whole number task, Dalton relied on his fingers to stand in for the perceptual items. For instance, Dalton was given the task, *I've covered thirteen counters here. If I took seven of those counters away, how many would I have left?* Dalton responds with the solution "six" and says he first "counted by ones to see what the answer was on my fingers for ones. This is seven (flashing seven fingers), and I figured out that this is seven, and then I knew that was seven (showing how he counted to construct a figurative unit for seven), so I counted fourteen, thirteen, umm ... thirteen, twelve, eleven, ten, nine, eight, seven, six." This explanation suggests that he creates the figurative unit for seven through his counting. By creating this figurative unit, he could focus on reversing his number sequence starting at 13 and counted down to six, exhausting the figurative unit for seven.

We also gave Dalton the task*, So, there's twenty-two counters here. Now if I took five of these counters away, how many would I have left?*. After counting silently and looking at his fingers, he announces, "Seventeen!". When the second author asks, *how did you solve this task?*, Dalton explains:

Dalton: I knew the finger thing for five.
R: Okay.

Dalton: And then I just count ... I did this (shows two hands balled up with zero fingers showing). And I counted to five (flashes five fingers on his left hand) on this (referring to his five fingers on his left hand).
R: Okay.
Dalton:Just to be sure, I counted on this.

R: Can you show me what you were doing?
Dalton: So, I did…
R: Do it again, I guess…
Dalton: … twenty-one, twenty, nineteen, eighteen, seventeen (shows five fingers putting one finger down with each number word).

In this excerpt, Dalton did not need to create a figurative unit for five, as he knew its pattern. We believe this freed up his ability to focus on his reversed number sequence in activity. We gave Dalton, in both strategies, tasks with a known number, a number he could create with his fingers, affording him access to known mathematics and/or created finger patterns.

In the second interview, we gave Dalton tasks asking him to construct figurative material for the subtrahend or for the difference. The design of these tasks pressed him to depend on given perceptual manipulatives. For instance, when asked, *without a number line, tell me how many ones on a number line would you need to jump back from thirty to reach twenty-two?*, he counted backwards by ten and then forwards (e.g., thirty, twenty, twenty-one, twenty-two), answering by stating "twenty-two". To explain his solution to these tasks, he creates a number line, perturbing him to reflect on his counting activity with the number sequence represented on his number line. Strategies like this suggest that he could anticipate the starting number (the minuend) and keep track of the ending number (the difference), in activity. However, he was unable to coordinate this with the reversal of his number sequence. In response to the task, *without using a number line, what is 12 less than 30*?, Dalton stated:

Dalton: Twelve less than thirty. Thirty, twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine, thirty, thirty-one (counts up with fingers to twelve – keeping track of ten fingers and two more). Thirty-one.
R: Thirty-one. So, if you're going on the number line from thirty …
Dalton: Oh, that's up.
R: … and that's twelve ... Oh, okay. So, what can we do to solve this?
Dalton: Hmm ... (holds up one finger and uses his second finger to tap this finger repeatedly while he thinks).
R: So, I'm gonna read it to you again. What is twelve less than thirty? What do you think?
Dalton: Thirty minus twelve. That's basically what it is.
R: Okay.
Dalton: Thirty, twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine ... (counting up by fingers and pauses at twenty-nine).
R: What do you think?

Dalton: How do... how do I keep going from minusing to plus?

Again, Dalton struggles to coordinate the reversal of his number sequence with the two other units (minuend and subtrahend) in this excerpt relying on his known multiples for decades and his forward number sequence. Because he ends his count with a number higher than his starting number of 30, his counting scheme seems to be perturbed when he approaches this.

From Dalton's known finger pattern for five in interview one, we realize one of his conceptual resources may be his known unit for five. We do not yet know to what degree he is constructing and abstracting this known unit for five. For instance, it was unclear if he constructed a numerical composite of five from his numerical sequence (Steffe, 1994) or if he constructed a composite unit for five as a mental object with which to operate on.

Given this, we include the number five in interview three's task to more closely examine his activity and operations possibly associated with five. For instance, we first ask him, *how many fives does it take to create 30?*. Dalton first places counters a line of 30 and describes his grouping as "random," where he segments groups of varying sets of items before adjusting them (adding or taking away one or two) to represent groups of five. Following this, we ask Dalton, *how many counters are covered?* (showing him a line of five with 25 covered). Dalton experiences a perturbation, by first stating "25" as a solution and then changing his solution to "29", describing the group of five as "one." We infer Dalton conceptualizes five as a numerical composite, which we believe was numerical in relation to his numerical sequence (cf Steffe, 1994). As such, we posit he could not use this five as a mental object, pressing him to conceptualize the unit as a unit of "one," which allows him the capability to understand this unit as a component of his number sequence. These are characteristics of Dalton evidencing numerical composites.

Angela. At the beginning of the study, we note that Angela seems to have an internalized number sequence to count on from and some effective strategies with visual representations when solving the aforementioned backwards counting tasks. Additionally, we observe Angela relying on learned procedures or stating known/double facts without fully understanding their relevance to the solution (e.g., using an equation and "borrowing" next door in subtraction). These "tricks" seem to help alleviate her working memory constraints but prevent her from experiencing meaningful perturbations.

When asked *how many groups of five did it take to make 30?,* Angela forms rows of counters to make an array without simultaneously attending to the number of items in each row and the number of rows. Angela created groups of five and explain this was her strategy, "so, they are in a nice row, and they don't get mixed up." Angela then counts all counters and changes her groups by adding one counter to each group. She explains that she knew the groups were the same because "three and three make six," drawing from units of three in coordination with six. At this point, Dalton partitions his groups and adjusts them to represent six groups of five. We observe the following exchange when we ask them *where is the five?*.

Dalton: (Looks back at the first two groups and rearranges the counters and recounts) one, two, three, four, five. One, two, three, four, five.
R: Okay, so how many groups of five do you have [to Dalton]?

Dalton: One, two, three, four, five, six … six. (Places his hands on each group and counts the groups.)

R: Okay, six … six groups [to Dalton]? And then how many groups of five do you have [to Angela]?

Angela: (Counting groups.) One, two, three, four, five. Five.

Dalton: (Counting groups.) One, two, three, four, five, six.

R: Are these groups of five [to Angela]?

Angela: No.

R: So, let's look at Dalton's and let's look at yours [to Angela]. What's different about the way you each solved it?

Dalton: We didn't …

Angela: He did bigger groups. I did smaller groups.

R: What do you mean by bigger groups?

Angela: He got, … I don't know if I have thirty (counts counters in a group). (Counts all counters) one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty, twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine, thirty. I have thirty.

Angela and Dalton continue this exchange to check and verify their total number of counters, the number of counters in each group, and the number of groups of counters before Angela realizes that they "switch swapped!", meaning that she created five groups of six and Dalton created six groups of five. Throughout this third interview, she repeatedly generates items that need to align with one of the units (5 or 30, or 5 or 6) but cannot coordinate both. The characteristics of Angela's activity evidence two-levels of units with one unit in activity.

Fraction Understandings

Dalton. In session 4, we gave Dalton a piece of paper with a yellow rectangle on it to represent a french fry and the task *how do you think we could share a fry between the two of us?*. This leads him draw a line to "split it in half," suggesting that he anticipates using a partitioning strategy. Dalton justifies this by developing two shares stating, "Because that's umm ... the middle (used his finger to go up and down on the paper, over the line he drew – see Figure 1a). So, if, if you cut it right here (pointing to a different section of the fry), it wouldn't ... if I cut it right here, it wouldn't be equal." When the second author asks Dalton how he could *check if the shares were of equal size*, Dalton states that he could "measure it" and creates a ruler with a piece of blue paper (shorter than the yellow fry) by drawing small lines and marking them with numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 … while counting "two, three, four, five, six, seven, eight, nine, ten, eleven, twelve". As Dalton continues to justify his response, he draws lines at "the end" of the ruler (located by the 12 – see Figure 1b) creating four shares (see Figure 1c).

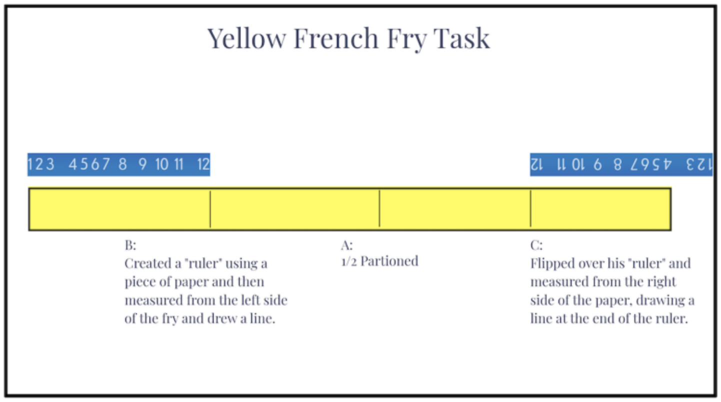

Figure 1: Dalton's "Measurement" When Determining if Two Units are "Equal" in Length

Dalton treats his "ruler" as a discrete object, laying his blue paper ruler on one end of the fry to determine where to mark a second line (see Figure 1b). Then Dalton flips over his "ruler" to align the "one" with the edge of the paper and the opposite end of the "ruler" marks a third line of the fry (see Figure 1c). We find it interesting that Dalton needs his "end" mark of this "ruler" to partition the fry, further evidencing that the entire length of the "ruler" might not have been used at all. We also wonder if Dalton is either iterating the end of his ruler or he is iterating the ruler as one pre-arithmetic unit. As such, by using an auxiliary item in this way, Dalton seemed to engage in some preliminary iteration and did not seem to coordinate the length of the fry or the length of his "ruler" in relation to the two shares.

This relates to Dalton's whole number activity, as he focused on either his iteration or partitioning operations when relying on perceptual or figurative material. When asked to iterate or partition a composite unit or a length unit in activity, his working memory disallows such coordination pressing him to develop early forms of operations (iteration or partitioning) in physical activity, which are separate from his unit development. This suggests Dalton unitizes physical material before focusing on his operations for this material, affording him meaningful whole number activity for his development of fraction knowledge.

Angela. In sessions four and five, we asked Angela to share a fry equally among three people. First, she partitions the fry in half and then adapts this partitioned line to create a third section. In subsequent attempts, Angela guesses and checks until she achieves three shares, always with left-over portions of the fry. She cuts off the end for the third piece to match the others and initially states that the leftover piece "was for the dogs." However, upon further probing on that attempt, she divides the fry piece into three additional shares, treating the remaining portion of the fry as an entire new fry.

Angela: Can I draw something on here (R shakes her head yes)? So, I can show you? It's not going to be that good (Angela draws a large fry on the white paper). So, this is the French fry. FR for French Fry (Angela writes those words on top of her drawing). You have to split it into 3 pieces. So, this is two (Angela draws a line down the middle of the drawn white rectangle).

R: So what part is this then (R, is pointing to the partitioned line of the drawing)?

Angela: This is two.
R: That's two fries?
Angela: Yeah, but if you do this and do this (Angela draws a line trying to split the 1/2 pieces down the middle). That would make 6.
R: Would it make six?
Angela: It would make four. One, two, three, four (Angela counted and tapped on the segmented pieces). I wish it made three. I just noticed it makes four.

At this point, Angela relies heavily upon halving partitioning but reorganizes this activity to intentionally develop a more sophisticated partitioning than a "guess and check." Following this activity, Angela tries to "measure" her original yellow paper fry to determine if her pieces were of equal size.

Angela: Okay. This is the size. Okay, so here's one. I'm going to do it this way (R helps Angela fold the paper). So, you can... Oh, wait, wait, wait, wait. So, this.
R: Hmmm, so what you're doing right now?
Angela: I'm measuring.
R: Oh, you're measuring.
Angela: Back like that. Okay (Refers to the fold she made with the blue paper).
R: I'll hold these down (R held the pens in place so they didn't move while Angela takes and measures each segmented piece of the yellow fry with the blue piece, repeated measure). Okay?
Angela: Stay like that (Refers to the pens and then measures each segmented piece of the fry).
R: Oh, uh oh.
Angela: But you know what you can do? Magic! Magical.
R: Magical. Do you want to fold that again?
Angela: Okay, it fits.
R: It fits? All of them?
Angela: No. See if.... (Keeps trying to measure with the same blue piece to check the size of all the segmented fry pieces). Okay, we just need this to be put right there.

This suggests to us that Angela "measures" her fry length by iterating to determine if her pieces were all the same length. By iterating she develops new means to coordinate her unit size with the length of the fry. Following this, Angela partitions the fry but does so by segmenting the fry into five shares with four partitioned lines. This activity is similar to the whole number task of five groups to create 30, whereby she could attend to the number of units and the total number but not to the group size. In the fraction task, she can attend to the length and to the size of the unit but not to the number of units. This also suggests that Angela is developing preliminary iterating of each unit but could only anticipate one length unit at a time and constructs the remaining units in activity. Given this, we believe Angela has two levels of units with one unit in activity for both her whole number and fraction development.

Discussion and Conclusion

Dalton and Angela each had very different conceptual resources for their whole number and fraction development. Dalton drew from a reliance on physical material but also was capable of counting-on, counting down, and working with numerical units from his numerical sequence (Steffe, 1994). Dalton drew from meaningful whole number understandings when conceptualizing five as a unit, sequencing his unitizing and operations. Dalton also evidenced early forms of iteration, providing him potential access to measurement fraction development in future schooling experiences. These conceptual resources allowed Dalton development with meaningful whole number and fraction activities. Angela was capable of meaningful counting-on, partitioning, unitizing, and iterating with the opportunity to coordinate two levels of units in activity. Angela's more advanced partitioning activity afforded her preliminary part-whole number reasoning and access to some successful fraction development. However, Dalton drew more often from iteration allowing him potentially more opportunities to develop measurement fractions than Angela might. We found that by leveraging both students' conceptual resources over the course of a small set of interviews, they were better positioned to solve more complex part-whole number and fraction problems. Questions remain about how particular nuances in conceptual resources of students with learning differences and their actions forming operations can be leveraged over time to promote more sophisticated mathematical development.

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THE INFLUENCE OF USING MANIPULATIVES AS MATH TOOLS OR TOYS IN PLAY CONTEXTS ON CHILDREN'S PART-WHOLE CONCEPTS

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Keywords: Instructional Activities and Practices; Early Childhood Education; Number Concepts and Operations.

Recent research has shown that the ways in which children use manipulatives impact their perceptions of the objects and their subsequent learning in mathematics. Osana et al. (2018) found that when children used manipulatives as representing quantities in a mathematics task, they perceived them as "math tools," but when they played with manipulatives in a non-mathematics environment, they perceived them as toys. Moreover, children's interpretations of manipulatives, as either representing quantities or toys, were resistant to change, even when the same objects were used in a subsequent mathematics lesson. In another study, Donovan and Alibali (2021) found that children's use of manipulatives as math tools was positively related to their subsequent learning about mathematical equivalence. From this research, it is tempting to conclude that children should be discouraged from playing with manipulatives, particularly when the same objects will be used to teach mathematics.

In both studies, the objects' use was experimentally manipulated by engaging the participants in either an instructional mathematics context or a play context, conflating context and the objects' use (math tools vs. toys). Thus, these data do not allow one to conclude that playing with manipulatives will result in perceptions that will hinder children's subsequent learning. Our objective in the current study was to conduct a conceptual replication of Osana et al. (2018) and Donovan and Alibali (2021) by comparing two ways of using manipulatives in teacher-directed play contexts. We hypothesized that children who used and perceived the manipulatives as math tools would outperform their peers who used and perceived them as toys.

We randomly assigned 64 first-grade students to one of two teacher-directed play conditions (math-play and toy-play) or a control condition, where children were not exposed to any manipulatives. In both manipulative conditions, children played a shopping game with beige plastic tiles. In the math-play condition, the manipulatives represented quantity, whereas in the toy-play condition, they were used to play with. We then assessed their perceptions of the objects and their part-whole understanding on two tasks using the same manipulatives from the intervention. The decomposition task (Manches et al., 2010) involved finding all possible two-way decompositions of six and nine. The evaluation task required children to evaluate a puppet's use of part-whole concepts in the strategies it used to solve eight additive word problems.

A significantly larger proportion of students in the math-play condition perceived the objects as math tools (96%) than in the toy-play (5%) and control conditions (0%), χ^2 (4, N = 64) = 48.89, $p < .001$. Contrary to our predictions, however, there were no condition or perception effects on the students' performance on either part-whole measure. Thus, when children in both conditions played with the manipulatives, the effect of using them as math tools or as toys on learning as found in previous research was no longer present. Future research should compare the effects of using manipulatives as representing quantities to using them as toys in both play and instructional contexts to gain a deeper understanding of the conditions under which children's use of concrete objects impacts their learning in mathematics.

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COUNTING: AN ESSENTIAL YET OVERLOOKED COMPONENT TO DEVELOPING INTEGER UNDERSTANDING

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Developing conceptual understanding with integers is challenging for middle school students (Fuadiah et al., 2019; Khalid & Embong, 2020; Schindler & Hußmann, 2013). Existing curricula traditionally support integer number sense through plotting, comparing, and ordering activities during introductory lessons (e.g., Lappan et al., 2009). Counting activities are notably missing, suggesting that educators assume middle school students can transfer whole number counting skills to integers without intentionally aimed activities. This poster focuses on a preliminary finding from a Design-Based Research (DBR) dissertation project that used an intervention game and contextual model, *Floats & Anchors* (Pettis & Glancy, 2014), as the throughline in an integer unit. The research question investigated: What characteristics of a comprehensive curricular unit best support students' understanding of integers and integer operations?

The *Floats & Anchors* unit consisted of 12-16 lessons implemented in three design cycle iterations with small groups during math intervention or special education mathematics classes. Each design cycle included an initial unit design, critical expert panel discussion(s), classroom implementation(s), teacher interview(s), a student survey, initial analysis, and re-design decisions. The unit was co-taught by a researcher and the classroom teacher(s) to students who were identified by their school as needing additional math support, with many students also receiving special education services. Altogether, 49 middle school students in sixth-, seventh-, and eighth-grade participated in 82 lessons. Data was collected from audio/video recordings of lessons, student work artifacts, transcripts from teacher interviews and expert panel meetings, student survey responses, and reflexive memos. The data was grouped by curricular characteristics (e.g., vertical number lines) and analyzed for evidence of student understanding.

One key preliminary finding emphasized counting as a distinct and essential component of integer understanding at the middle school level. Our study revealed that whole number counting skills did not transfer to integer counting without intentional activities. During the second iteration, students demonstrated a conceptual understanding of integer addition and subtraction by selecting appropriate count-on or count-back directional strategies. However, students were unable to count accurately in the desired direction despite demonstrating mastery with plotting, comparing, and ordering integers earlier in the unit. To address this, we routinely integrated a modified early childhood counting game where students take turns counting from 10 to -10. When the counting activity was first introduced, students consistently needed to reference the classroom number line, demonstrated sustained periods of thinking time (particularly between two and -2), and/or required support from a teacher before they were able to complete their turn. By the end of the third iteration, students demonstrated collective mastery of integer counting. Students could count up and down starting with values beyond 10 or -10, alternate directions with each new round, skip-count by sets of two, five, and ten in both directions, and demonstrate mastery individually with a teacher fluently and without the earlier supports. Moreover, integrating an intentional counting routine positively correlated with students' overall mastery of integers, as demonstrated on the summative assessment from the third iteration.

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Chapter 8:

Policy, Instructional Leadership, and Teacher Educators

RURAL MATHEMATICS EDUCATION LANDSCAPE: A STUDY ON FACTORS THAT FACILITATE OR IMPEDE AMIBTIOUS INSTRUCTION

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We provide an overview of a multi-stage project designed to generate a comprehensive understanding of the state of rural mathematics education and conditions impacting rural mathematics education in middle schools in the United States. We seek to explore: the curriculum resources and instructional practices that are prevalent in middle grades mathematics education in rural contexts; the factors that facilitate or impede efforts to develop local capacity to implement rigorous mathematics teaching and learning in rural contexts; and how state and federal education policies (e.g., assessment instruments, funding priorities, mandates) impact mathematics instruction in rural landscapes. The purpose of the poster is to describe the set-up for the study design and share intentions of the project, along with details related to data collection.

We will present the plan for this four stage study that includes: (a) a survey of rural middle grades mathematics educators (n=1000) to develop a picture and comprehensive understanding of the forms of curriculum resources, instruction, and professional learning experiences reported to be in use in a variety of rural contexts; (b) interviews with teachers, instructional leaders, and administrators (n=80) in selected, diverse rural districts about their perceptions of their district's/school's current mathematics curriculum and instruction, their goals related to their mathematics programs, the resources and opportunities for improving mathematics curriculum and instruction, as well as the challenges they face; (c) case studies of the implementation of a professional development program in roughly 10 districts, the purpose of which is to investigate factors that facilitate or impede efforts to implement rigorous forms of mathematics curriculum and instruction, and to gauge the resources that would be necessary to develop local capacity for a sustained implementation; and (d) establishing a collective dialogue with a range of stakeholders in education policy to gain greater insights into the rural mathematics education landscape and to develop recommendations for enacting rigorous mathematics instruction at geographic scale.

This study draws on research in mathematics education, including literature on curriculum characteristics, curriculum implementation (Jackson et al., 2017), and professional development in rural contexts (Irwin et al., 2010), as well as broader literature on rural education and rural economics (Schmitt-Wilson et al., 2018).

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MAPPING POLICIES FOR BECOMING A MATHEMATICS TEACHER

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In the US, mathematics teacher licensure is governed at the state level. Each state may have different policies about mandatory elements of teacher preparation. Within each state, educator preparation providers determine how to provide such learning opportunities. In the current era of accountability, the policy conversation around teacher preparation has focused on preparing highly effective teachers (e.g., Aydarova, 2023; Bales, 2006; Cochran-Smith et al., 2018; Tatto, 2021). However, little research has examined the specific laws or rules enacted at the state level. Professional organizations in mathematics education also strive to shape policy and practice around mathematics teacher preparation through reports (e.g., Conference Board of the Mathematical Sciences, 2012) and standards (e.g., Association of Mathematics Teacher Educators, 2017). Given the large number of stakeholders helping to shape mathematics teacher preparation and the potential for state-by-state variation in laws and rules, I ask the following research questions: (1) What laws and rules govern mathematics teacher preparation across the US? (2) How do those laws and rules differ by state or territory?

To answer these questions, I analyzed the laws and rules shaping teacher preparation across the US. Within each state, teacher preparation is governed by laws (written and passed by the state legislature, then signed by the governor) and rules (written by an executive branch agency, such as a Department of Education, which was empowered by the legislature to produce rules that help interpret the laws). Informed by prior research as well as my own work as a teacher educator, I anticipated laws and rules focused on mathematics content requirements and general pedagogical practices. I also anticipated laws and rules directed toward teacher candidates themselves as well as policies directed toward teacher preparation providers. Building on this framing, I engaged in an open coding process (Vollstedt & Rezat, 2019), starting from the specific words in each law or rule, and then looking across codes to generate themes.

Preliminary findings reveal different approaches to regulating mathematics teacher licensure across the US. Some states include detailed lists of content and pedagogical standards within state law or rule. Other states enshrine exams (Praxis or state-specific exams administered by Pearson) and accreditation (CAEP) into law and rule. Another variation across states was the extent to which laws and rules focused on teacher candidates or preparation providers. Indeed, in some states a requirement would be directed toward the candidate (e.g., a teacher must be able to elicit and interpret student thinking), while in other states the requirement would be directed toward the preparation provider (e.g., the program will ensure the teacher can elicit and interpret student thinking). Finally, states use different language to describe licensure and issue a wide range of license types. These variations contribute to a confusing licensure landscape. Despite these differences, there is also significant overlap in the focus of the laws and rules across states, particularly related to pedagogical practices.

Identifying the similarities and differences in laws and rules across the US is critical to understanding the mathematics teacher preparation landscape. Engagement with the laws and rules that shape teacher licensure will enable teacher educators to advocate for research-informed policy that supports preparing teachers to teach ambitious mathematics across the country.

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TRENDS IN MATH EDUCATION RESEARCH OVER THE LAST 10 YEARS

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In this paper, we use a systematic approach to explore trends in mathematics education research over the last ten years by sourcing articles from the two highest ranked mathematics education journals: Educational Studies in Mathematics (ESM) and Journal for Research in Mathematics Education (JRME). We examined the methods of research used in articles (quantitative, qualitative, mixed methods), the geographic region in which research was conducted, and the characteristics of research samples. The findings showed that methods of research were balanced in JRME, with a relatively strong tendency toward qualitative research in ESM. Furthermore, the findings revealed that there is little geographic diversity in published research in JRME, with more diversity in ESM. Finally, research sample sizes were larger in JRME than ESM, and both journals showed notable patterns in relation to sample characteristics.

Keywords: Research Methods; Policy

Over the past ten years, there have been technological advancements, ideological shifts, and critical events (e.g., COVID-19), all of which have the potential to change the nature of mathematics education research. It is critical to track changes in mathematics education research to discern how the field is evolving and discern the strengths and limitations of current research approaches. In this paper, we use a systematic approach to explore trends in mathematics education research by sourcing articles from the two highest ranked mathematics education journals: Educational Studies in Mathematics (ESM) and Journal for Research in Mathematics Education (JRME) (e.g., Niven & Otten, 2017; Williams & Leatham, 2017). These two journals are highly respected in the field and provide a reasonably reliable litmus test for trends in mathematics education research. Furthermore, looking at journals from both US and international organizations allows us to see a more complete picture of mathematics education research globally and compare trends between US based and international research. To that end, we ask the following research questions: (1) What methods of research are most prevalent in ESM and JRME over the past ten years and how have these methods changed? (2) Which geographic regions are most represented by research in ESM and JRME over the past ten years and how has geographic representation changed? (3) What are the characteristics of research samples in ESM and JRME over the past ten years and how have these characteristics changed?

Notably, these research questions are significant from the perspectives of the editors of ESM and JRME. For example, ESM's editors signified the importance of publishing research from underrepresented regions (Mesa & Wagner, 2019; Wagner & Prediger, 2023). Furthermore, JRME's editor provided a brief account of the methodologies used by researchers during PMENA 2023 (Herbst, 2023). Examining trends across both journals identifies areas in research that need greater emphasis. For example, are certain populations underrepresented? Is there a need for more qualitative/quantitative research in a particular global region? The different methodologies have different strengths and weaknesses and both make valuable and necessary contributions to research (Choy, 2014). This analysis will make researchers aware of needs in mathematics education that will guide future research and contribute to a more complete corpus of information. Thus, this study examines relevant trends of mathematics education research.

Method

To answer the research questions, we compiled all articles that were published in JRME and ESM from 2014-2023. Then, we coded the articles using a coding scheme that was relevant to the research questions. For Research Question 1, we coded all articles as: (1) empirical; (2) editorial; (3) book review; (4) other (e.g., systematic review, commentary, etc.). Then, we compiled all empirical articles and coded them as quantitative, qualitative, or mixed methods. While there are various methodological definitions, we relied on Gay et al. (2012) for the following delineations of research:

(a) "Quantitative research is the collection and analysis of numerical data to describe, explain, predict, or control phenomena of interest" (Gay et al., 2012, p. 7).
(b) "Qualitative research is the collection, analysis, and interpretation of comprehensive narrative and visual (i.e., nonnumerical) data to gain insights into a particular phenomenon of interest" (Gay et al., 2012, p. 7).
(c) "Mixed methods research combines quantitative and qualitative approaches by including both quantitative and qualitative data in a single study" (Gay et al., 2012, p. 481).

Finally, we calculated frequencies for each methodology and explored differences in methodologies across time.

To answer Research Question 2, we coded each article for the institutional affiliation of Author 1. The institutional affiliation of Author 1 acts a proxy for the geographic region in which the research was conducted. We coded the geographic region of each article according to Rosenburg's (2023) eight official world regions: (1) Asia; (2) Middle East, North Africa, and Greater Arabia; (3) Europe; (4) North America; (5) Central America and the Caribbean; (6) South America; (7) Sub-Saharan Africa; (8) Australia and Oceania. For JRME articles, we coded each article according their geographic region within the U.S. using the U.S. census bureau's regions: (1) Northeast; (2) Midwest; (3) South; (4) West. We calculated frequencies and percentages for geographic regions and explored differences over time.

To answer Research Question 3, we compiled all empirical articles and coded the sample according to its target population: (1) Grades Pre-K-2; (2) Grades 3-5; (3) Grades 6-8; (4) Grades 9-12; (5) Undergrad/Community college; (6) Graduate; (7) Preservice teachers; (8) Adult Learners; (9) Inservice Teachers; (10) Other. In studies where multiple populations were included, we coded each portion of the sample separately. In addition to sample population, we coded each study for sample size, and racial/gender characteristics. We calculated medians, frequencies, and proportions to compare sample characteristics in both journals independently and in conjunction with the other two research questions.

In what follows, we provide quantitative summaries for each research question, providing separate analyses for ESM and JRME.

Findings

Methods of Research

JRME published a total of 250 papers over the past decade. Of those papers, 50% of them were empirical papers, 16% were editorial, 10% were book reviews, and 24% were other (theoretical, meta-systematic reviews, textbook, and content analysis, commentary). ESM published a total of 729 papers. Of those, 63.5% were empirical papers, 4.7% were editorial, 7% were book reviews, and 24.8% were other.

For the empirical studies, we coded each article for the type of methodology used (quantitative, qualitative, mixed methods). Frequencies across time for each methodology are given in Table 1. Over the last decade, JRME maintained a balanced publication of both qualitative and quantitative studies, with the number of mixed studies being consistently comparable. In contrast, ESM published around three times as many qualitative studies as quantitative studies over the last ten years. Furthermore, ESM published considerably more mixed methods studies than quantitative studies.

Looking across time, the number of quantitative studies published in ESM shows an upward trend since 2016, with a decline last year (2023). In JRME, the number of quantitative studies has been about the same for the past 5 years. In relation to qualitative research, there are no discernible patterns for articles published in JRME. For ESM, the number of qualitative studies appeared to increase over time. From the frequency column, we can observe that the total number of empirical studies published in JRME each year has been fluctuating but averaging about 12 articles per year. Overall, the number of empirical articles published in ESM shows an upward trend, averaging about 46 articles per year.

Table 1: Methodology Across Time

	JRME				ESM			
Year	Quant	Qual	Mixed	Frequency	Quant	Qual	Mixed	Frequency
2014	8	5	3	16	8	25	7	40
2015	4	5	3	12	6	21	14	41
2016	1	8	2	11	5	22	12	39
2017	3	3	6	12	7	24	13	44
2018	6	2	3	11	6	20	14	40
2019	5	5	2	12	10	22	7	39
2020	3	5	2	10	9	18	22	49
2021	3	5	8	16	11	25	16	52
2022	5	4	3	12	20	29	14	63
2023	6	5	2	13	9	35	11	55
Total	44	47	34	125	91	241	130	462

Geographic Representation

Table 2 shows the geographic representation for articles in JRME over the last ten years (using the affiliation of Author 1). As illustrated in Table 2, most JRME articles were published by authors in North America (89%), and most of the authors were from the U.S. The journal publishes very few articles from other regions, with Europe ranking second at just 4% of the overall sample. There are no noticeable trends over time in relation to international geographic regions. Table 3 shows that, of articles published in the U.S., more articles are published from Southern regions than any other region, and Northeastern regions account for the smallest number of articles within the sample. It is important to note that many states are counted in the Southern region within the U.S. census bureau that many people would not identify as the "South." Therefore, these results should be interpreted with the U.S. census bureau boundaries in mind. There are no noticeable trends regarding changes in U.S. representation over time.

Table 4 shows the geographic representation for articles in ESM over the last ten years. As illustrated, ESM's geographic representation is somewhat more diverse, with Europe and North America accounting for the majority of articles (43% and 29% respectively). There are some noticeable trends over time in ESM's geographic representation. Namely, articles published from Asia and the Middle East, North Africa, and Greater Arabia regions have trended up over the last ten years.

Table 2: Geographic Representation in JRME over Time

Year	Asia	Middle East, North Africa, Greater Arabia	Europe	North America	Central America and the Caribbean	South America	Sub-Saharan Africa	Australia and Oceania
2014	1	0	3	23	0	0	0	0
2015	0	0	1	22	0	1	0	0
2016	0	0	1	23	0	0	0	3
2017	0	0	0	27	0	1	0	0
2018	0	0	1	30	0	1	0	0
2019	0	2	1	22	0	0	0	0
2020	0	1	1	19	0	0	0	0
2021	0	0	0	22	0	1	0	0
2022	1	0	1	20	0	1	0	0
2023	1	2	1	16	0	0	0	1
Total	3 (1%)	5 (2%)	10 (4%)	224 (89%)	0 (0%)	5 (2%)	0 (0%)	4 (2%)

Table 3: Geographic Representation in JRME over Time by USA Region

Year	Northeast	Midwest	South	West
2014	2	9	7	5
2015	3	4	11	3
2016	2	9	5	7
2017	4	4	12	7
2018	5	3	15	7
2019	2	1	14	5
2020	2	4	8	5
2021	6	8	6	2
2022	4	7	6	3
2023	1	5	7	2
Total	31 (14%)	54 (24%)	91 (41%)	46 (21%)

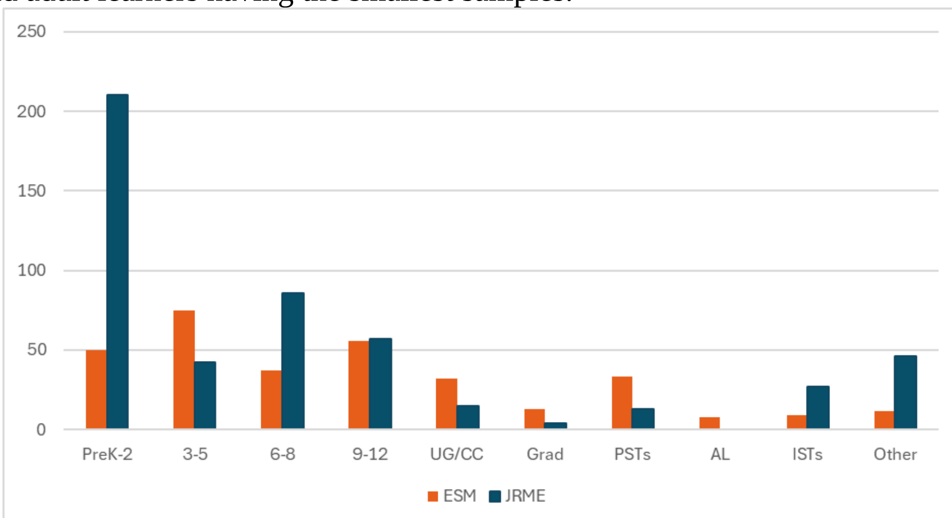
Table 4: Geographic Representation in ESM over Time

Year	Asia	Middle East, North Africa, Greater Arabia	Europe	North America	Central America and the Caribbean	South America	Sub-Saharan Africa	Australia and Oceania
2014	2	3	32	20	0	1	2	9
2015	3	8	25	23	0	0	1	5
2016	2	4	32	26	0	2	1	4
2017	4	5	24	24	0	2	1	5
2018	2	5	26	17	0	3	1	3
2019	3	7	43	13	0	2	3	3
2020	6	10	28	22	0	1	1	4
2021	8	8	34	21	0	4	3	7
2022	6	11	40	24	0	0	1	6
2023	13	10	31	20	0	2	1	5
Total	49 (7%)	71 (10%)	315 (43%)	210 (29%)	0 (0%)	17 (2%)	15 (2%)	51 (7%)

Characteristics of Research Samples

The sample sizes for the research studies examined in this analysis varied greatly. Quantitative sample sizes ranged from 1 to 132,747 while qualitative samples were understandably smaller, with a maximum sample size of 1,813. Mixed method empirical studies fell in between with the largest sample having 6,218 participants. The data were positively skewed with a small number of very large samples. Because of the outliers in the data, we describe sample sizes in terms of median values instead of averages. The median sample size in

JRME was 42 participants while the median for ESM was 25. Figure 1 shows the median sample size for each journal separated by the population studied. The graph shows that JRME had larger sample sizes for students in the preK-2 and 6-8 grade groups, as well as for inservice teachers and the "Other" category. ESM, however, had larger sample sizes for research studying grade 3-5 students, undergraduate students, and preservice teachers. ESM also included the only study focused on non-traditional adult learners. In addition to illustrating how the sample sizes vary according to population category, the figure also suggests that the largest samples in mathematics education were found in research conducted on K-12 students, with research on graduate students and adult learners having the smallest samples.



Note. Grad= Graduate Students UG/CC= Undergraduates/Community College, PST= Preservice Teachers, AL = Adult Learners, IST= Inservice Teachers

Figure 1: Median Sample Size by Population

Because quantitative studies generally have larger sample sizes than qualitative studies, we looked at sample size according to study methodology. Table 5 shows the median sample sizes for each journal while considering the study population and methodology. With the exception of the 9-12 grade category, the quantitative studies in JRME had larger samples than those in ESM. The table also shows the higher sample medians for ESM were due to larger samples in qualitative and mixed methods studies. The sample sizes were also examined to look for changes over time but no discernible patterns were found (see Figure 2). Generally, the sample sizes in JRME fluctuated greatly over the past ten years with lows in 2015 (median = 13) and 2016 (median = 6), and highs in 2014 (median = 96) and 2022 (median = 102). Median samples for ESM had much less variation but showed a slight increase over the past ten years from 18 in 2014 to 29.5 in 2023. Interestingly, both journals had their lowest sample median in 2016.

Table 5: Sample Medians by Study Population and Methodology

	Quantitative		Qualitative		Mixed	
	JRME	ESM	JRME	ESM	JRME	ESM
PK-2	**647.15**	136	**28.5**	10	**232**	129.5
3-5	**443.5**	213	1.5	**12**	72	71
6-8	**961**	350	**18**	7	**129**	102
9-12	198	**271**	10	8	**134.5**	82
UG	**522**	208	5	**13**	**450.5**	85
Grad	0	0	2	**5.5**	11	**24**
PSTs	**1044**	224	12	10	13	**73.5**
AL	0	0	0	1	0	**15**
IST	**93**	59.5	2	3	12	**29.5**
Other	**439.5**	278.5	8	7.5	**59**	36.5

Note. Grad= Graduate Students UG/CC= Undergraduates/Community College, PST= Preservice Teachers, AL = Adult Learners, IST= Inservice Teachers. Bold font indicates notably higher sample medians, but this was a subjective distinction made by the author.

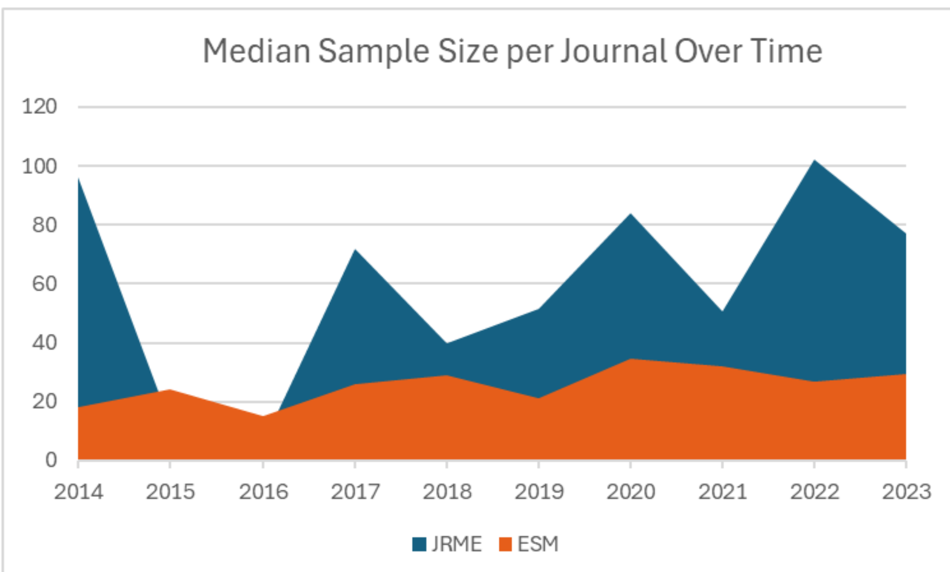


Figure 2: Median Sample Size Over Time

As a final inquiry into the samples that were the focus of study in JRME and ESM over the past ten years, we looked to see how gender and race were represented in the samples. In ESM, only about 50% or fewer of the studies reported gender representation, and student race was reported in less than 20% of the articles. In the articles that did report on gender, there was equal or greater representation for female subjects in 30-50% of the studies, which may be partially due to the majority of PSTs and ISTs identifying as female. Looking at studies on PSTs or ISTs that reported gender demographics, over 60% of the samples identified as female. Similar patterns appeared when looking at the representation of gender and race in JRME articles with a majority of articles not reporting the demographics of their sample. However, the proportion of articles failing to report has decreased since 2020, suggesting a shift in mathematics education research which indicates an awareness of the need to consider the needs of various student populations differently. This shift, however, only seems to be occurring in the United States, as the shift is not observed in the international articles of ESM.

Discussion

In this paper, we examined trends in mathematics education research by exploring methodologies, geographic representation, and sample characteristics for studies published in mathematics education's highest ranked journals (ESM and JRME). The findings highlight several implications for future research.

First, this study showed that methods of research (quantitative, qualitative, mixed methods) were balanced in JRME, with a relatively strong tendency toward qualitative research in ESM. There were few discernible trends over time except that ESM appeared to publish more quantitative studies from 2016 onward. Together, the data reveals methodological diversity in mathematics education research. From the perspective of potential authors, ESM may be more open to qualitative methods than JRME, though more detailed analysis is needed to certify this claim.

Second, this study revealed that many geographic regions are not well-represented by top journals in mathematics education. Mathematics teaching and learning is heavily influenced by culture. Therefore, it is important that the field draw upon research from diverse regions to create new knowledge and global perspectives on mathematics education. Notably, ESM has published more research from some underrepresented regions over time. JRME, on the other hand, seems primarily concerned with national interests. This is, perhaps, unsurprising since JRME was founded by a U.S.-based council. Yet, given the influence of ESM and JRME, the journals might consider placing more emphasis on scholarship from underrepresented regions.

Third, our analysis revealed a fairly consistent difference in sample sizes between studies in the two journals. The research reported in JRME tended to have larger median sample sizes than the research reported in ESM. Additionally, while the median sample sizes in JRME fluctuated significantly, the median sample size in ESM remained fairly consistent over time with a slight upward trend. Though sample size is not generally an important consideration in qualitative research, it is an important consideration in quantitative studies as it contributes to increased statistical power and generalizability. However, large samples can be difficult and costly to obtain, which may contribute to the lack of large quantitative studies from underrepresented regions.

Our analysis also revealed that race and gender are rarely reported for research samples, with the exception of research conducted in North America. Increased reporting in North America could be due to the emphasis on increasing equitable education in the U.S., as well as the diversity of cultures and races within the U.S. Still, 64% of studies in JRME did not report the race of their samples, and gender demographics were only reported for about 50% of the samples. These data may indicate that North American authors have different perspectives and priorities than other geographic regions in relation to reporting the characteristics of samples.

In conclusion, this study showed several important trends in mathematics education research published in ESM and JRME. These findings may support the mathematics education community in identifying areas of strength and improvement moving forward. Furthermore, these findings may support potential authors in choosing appropriate outlets for publishing their mathematics education research.

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OPERATIONALIZING RURALITY FOR MATHEMATICS EDUCATION: RURAL-CENTRIC APPROACH

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Rurality matters in mathematics education because geographic diversity affects students and teachers in their home, community, school, and district. Researchers have documented a number of factors which differ across rural, suburban, and urban settings, including student achievement, recruitment and retention of teachers, access to resources, and cultural funds of knowledge. However, mathematics educators have mostly focused on urban and suburban settings, often adopting a legacy urban-centric system developed by the U.S. government for applying 'city', 'suburb', 'town', or 'rural' labels to schools. We compare this categorical system with a new method for estimating the extent to which school districts are 'rural-serving'. Results suggest the new metrics offer more rural-centric information for mathematics educators.

Keywords: Research Methods, Professional Development, Rural Education

Introduction and Literature Review

Many educators, researchers, and policy makers appreciate the importance of context in public schooling. The U.S. government maintains demographic and financial data on more than 102,000 public schools (NCES, 2023), labeling 32% as located in a rural area. There is considerable variability in these data across rural areas (Showalter et al., 2023). Many rural schools are challenged by persistent inequities, such as higher levels of poverty and lower levels of educational achievement among historically excluded black and Hispanic students (Johnson et al., 2014; Lavellely, 2018; Lauzon et al., 2015; NCES, 2016). At the same time, students attending rural schools have higher graduation rates compared to their non-rural peers (NCES, 2023a), and, overall, rural children who experience poverty demonstrate higher achievement than their non-rural peers (Showalter et al., 20203). Given the unique challenges and circumstances rural schools face (e.g., limited resources, geographic isolation, technological challenges), leading policymakers and researchers highlight the importance of developing and testing innovative approaches to educational improvement that are specifically tailored to meet the needs of rural schools, teachers, and students (Parks, 2021).

How rural is defined greatly impacts who has access to these innovative approaches and the associated resources (Coburn et al., 2007). For example, our universities recently applied for and received funding for two mathematics education grants focused on supporting and researching outcomes in rural schools. How we define rural significantly impacts who is eligible to receive that support and who is included in rural mathematics education research outcomes. Though there are many ways geographers and others have defined "rural" (Woods, 2004), the historical focus has been on distinguishing rural areas from urbanized areas. Compared to cities, rural areas tend to be more sparsely populated, with more open spaces, differing cultural norms, and greater focus on ‘primary economic activities’ such as farming, ranching, hunting, forestry, mining, and oil and gas extraction (Montoya, 2023). There are many subtleties as local economic, housing, and demographic trends have shifted over time. The most recent *Why Rural Matters* report acknowledged issues with the urban-centric definition of rural adopted by the U.S. Census Bureau more than 100 years ago, highlighting that “…many children attend rural schools in districts that are not designated ‘rural’ by the U.S. Census Bureau.” (Showalter et al., 2023, p.2). We explore: How might mathematics educators might benefit from shifting from a reliance on the urban-centric list of locale types currently used by the U.S. government to a more direct measurement of the extent to which a school district is ‘rural-serving’?

Methods

Our primary data source was a large database of 19,714 public school districts maintained by the EDGE program in the National Center for Education Statistics (2023b). Records for the school districts included geographic boundaries and 12 “locale type” categories derived from the urban-centric definition of locales developed by the U.S. Census Bureau. In this system, school districts can be located within a “City”, distinguished by the population size of the principal city (Large = 250,000 or more, Midsize = 100,000-249,999, Small = less than 100,000). “Suburban” districts are located just outside one of those cities (similarly labeled Small/Midsize/Large by the size of the city). “Town” districts are located within “urbanized clusters” (at least 500 residents per square mile), with labels for their distance from the nearest city (Fringe = less than 10 miles, Distant = 10 to 35 miles, Remote = more than 35 miles). The remaining districts are “Rural”, with similar Fringe/Distant/Remote labels based on distance to the nearest Town or City. We also used “tract” population values from the 2020 U.S. Census. Tracts are relatively small, geographically-stable boundaries in use since 1920, typically corresponding to neighborhoods or natural boundaries, split or merged every 10 years to maintain populations between 1000 and 8000 people.

To obtain our new ‘rural-serving’ estimates for school districts, we used the statistical software system *R* to calculate the geographic overlap of census tracts with each school district. This allowed for calculating a weighted mean population density of each overlap. For each district, we computed the “% Rural-Servings” as the fraction of tracts served by the school district which would be considered non-urbanized by the U.S. Census (< 500 people per sq mi). We optimized the algorithm for computing the geospatial estimates for each district, then summarized the results, comparing the new estimates to the pre-existing locale type labels.

Results

Table 1 and Figure 1 illustrate similarities and differences between the existing Locale Type labels and the new % Rural-Serving measure for school districts. U.S. school districts are highly variable in the numbers of residents they serve (Mean = 55,323 residents, SD = 129,846). Nonetheless, City districts tend to serve greater numbers of residents than Suburban districts. Town and Rural districts tend to serve similar numbers of residents. In the aggregate, Table 1 suggests that Locale Types are broadly aligned to decreasing population density and increasing rural-serving distributions (with the only exception to the ordering being ‘Rural, Fringe’ school districts). However, as indicated in Figure 1, there is large variability in % Rural-Serving in some Locale Types, especially in 'City, Small' and Suburban districts.

Table 1. Summary of U.S. Public School Districts by Locale Type

Code	Locale Type	# Districts	Mean # Census Tracts	Mean Population	Mean Pop Density (people / sq mi)	Mean % Rural-Serving
11	City, Large	186	119	458238	6478	9%
12	City, Midsize	175	54	217581	4357	19%
13	City, Small	392	28	109702	2753	29%
21	Suburban, Large	2446	23	99160	3338	20%
22	Suburban, Midsize	327	19	77073	1369	47%
23	Suburban, Small	239	14	55625	1098	57%
31	Town, Fringe	516	11	43709	502	80%
32	Town, Distant	1109	11	38580	401	83%
33	Town, Remote	736	9	30675	324	85%
41	Rural, Fringe	1714	12	47412	426	81%
42	Rural, Distant	3028	7	26414	92	97%
43	Rural, Remote	2364	6	15734	19	100%
—	Not Applicable	150	12	44189	933	75%

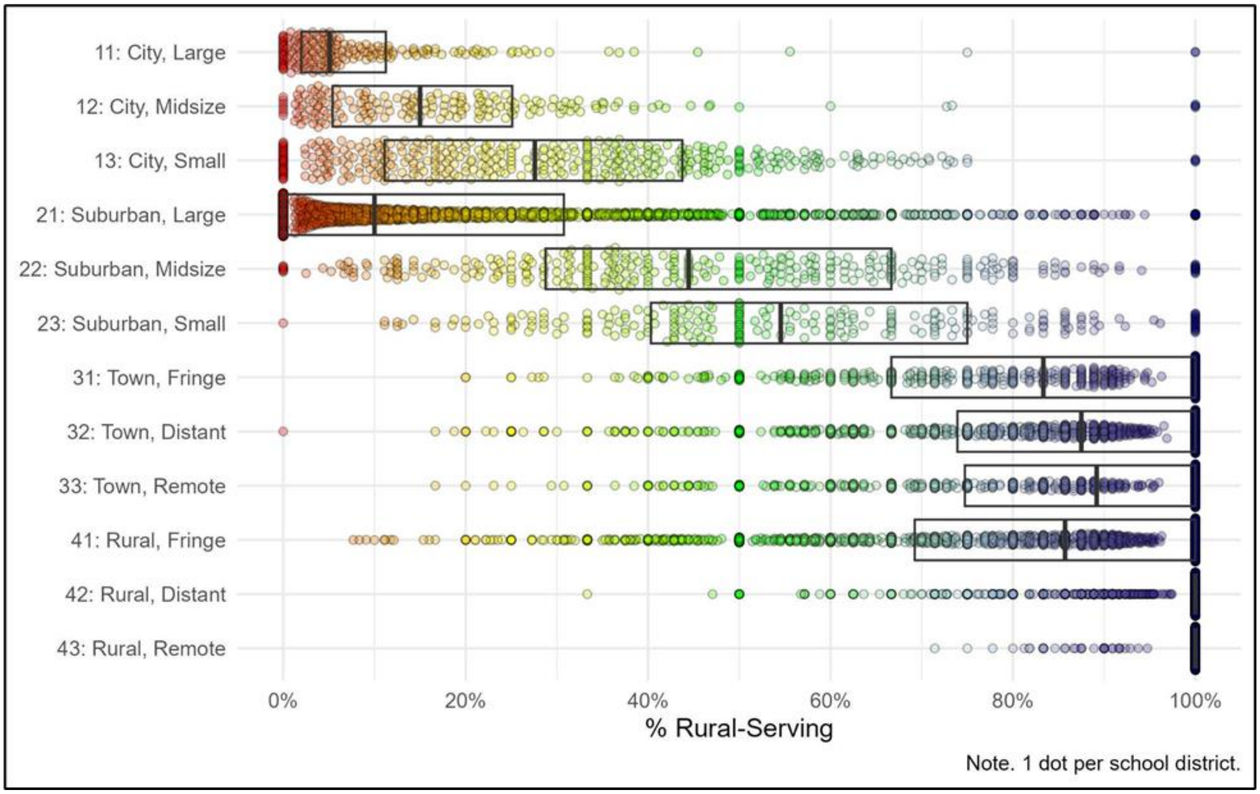

Figure 1. Distributions of % Rural-Serving by Locale Type among U.S. School Districts

Figure 2 compares U.S. maps of the existing Locale Type label and the new rural-centric measure or rurality. The maps highlight the increased precision afforded by % Rural-Serving.

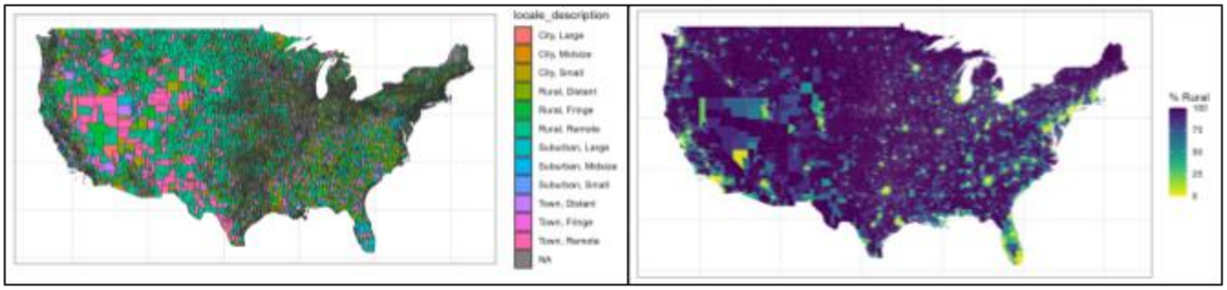

Figure 2. Maps comparing Locale Type (left) and % Rural-Serving (right).

Conclusions

We recommend math education researchers pay particular attention to the extent to which school districts serve rural populations. As indicated by our results, many school districts, especially those located near small cities or in suburban areas have relatively large proportions of rural areas within their districts. In particular, we recommend more mathematics education researchers, policymakers, and program developers adopt a *rural-centric approach* (using the % Rural-Serving or similar metrics) to reach more schools with rural student populations.

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INITIALIZING CROSS-INSTITUTIONAL COMMUNICATION AND PARTNERSHIPS IN MATHEMATICS TEACHER PREPARATION PROGRAMS

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Keywords: Policy, Preservice Teacher Education, Mathematical Knowledge for Teaching, Professional Development

Both the Conference Board of the Mathematical Sciences (CBMS) and the Association of Mathematics Teacher Educators (AMTE) provide standards and recommendations for the mathematical preparation of elementary teachers (AMTE, 2017; CBMS, 2012). Considering what professional organizations and state licensing requirements call for in the mathematical preparation of preservice elementary teachers (PSETs), teacher education programs aim to fulfill those requirements in a variety of ways, especially in the coverage, sequencing, rigor, and integration of content and pedagogy in mathematics content and methods courses (An et al., 2021). Individual states tend to have explicit mathematical achievement requirements for PSETs working toward initial teacher certification, but less specific course requirements for their training. With standards that need to be met, but no required course structures in place, access to an interactive platform being created in this study is expected to facilitate PSET preparation programs' decision making process on how to structure their programs, meet the needs of their PSETs and those PSETs' future students, and learn from other PSET preparation programs.

Research Questions and Method

Our study is guided by the following research questions: (a) How do teacher education programs account for the mathematical education of PSETs in the design of their programs?; (b) How do their program structures align with current standards for the mathematical preparation of teachers?; and (c) In a central resource for program design, what features would assist teacher education programs? There are two major components to our initiative: (a) survey administration to build knowledge about various structures of PSET preparation programs, and (b) platform development to foster communication and partnerships among PSET mathematics teacher preparation programs.

Results and Discussion

Our ultimate goals are to connect preservice mathematics teacher preparation programs of all grade levels (K-12) and to support cross-level communication and collaboration. For example, a secondary teacher preparation program and an elementary teacher preparation program could collaborate to build coherent mathematical knowledge for teaching curriculum so that future graduates would deliver coherent mathematical knowledge to their K-12 students. Starting by focusing on the preparation of PSETs will allow us to show and pilot the feasibility and utility of the proposed online platform. Although we will not have any results by the time of the presentation, we will present on our design and seek feedback on the initial phases of our project.

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ESSENTIAL DOCTORAL PROGRAM FEATURES IDENTIFIED BY MATHEMATICS EDUCATION UNIVERSITY FACULTY

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The purpose of this study was to learn from university faculty with a doctorate in mathematics education from across the world about how programs in mathematics education should prepare doctoral students for research and teaching in mathematics education. Online survey responses indicated that 99 mathematics education university faculty from 33 different countries stressed the importance of providing doctoral students with opportunities to examine and compare fundamental theories of learning mathematics; examine current and historical research in the field of mathematics education; and develop broad and deep knowledge of the big ideas ages 2-20 years (i.e., grades preK-14) mathematics.

Keywords: Teacher educators, teacher knowledge.

Over the past 20 years, there has been a significant increase in research into the development of and issues around mathematics teacher educators (MTEs). One subset of MTEs lacking a robust research base is the group of holders and pursuers of doctorates in mathematics education or didactics of mathematics, depending on your geography and background. Existing research on mathematics education doctorates, although limited, has highlighted the great variability in preparation and programs (e.g., mathematics knowledge preparation, research training) and focused on the potential to identify a common core of knowledge and experiences that would prepare graduates for diverse careers (Goos & Beswick, 2008; Kilpatrick & Spangler, 2015; Reys, 2002).

The research presented here is part of a larger project designed to identify features of doctoral programs in mathematics education that remain essential across institutions and countries and have the potential to become part of a core set of experiences, practices, and expertise for any mathematics education doctoral program, regardless of where it is located (Grevholm et al., 2008). In this paper, we report on ongoing international research to collect and examine data about the experiences, practices, and expertise of individuals with or working toward a doctorate in mathematics education. The following questions guide the research presented here: What features of doctoral programs, across countries, do individuals working as university faculty and possessing a doctorate in mathematics education identify as being essential?

Methods

A combination of purposive and convenience sampling was used to identify and contact (via email) potential participants for the larger study, which is composed of individuals with a doctoral degree in mathematics education (or didactics of mathematics) or currently working toward such a degree. Several proceedings from international and regional mathematics education conferences from the past five years (e.g., CERME 13, MERGA 45, NORMA 20, PME 46, PME-NA 45, The Mathematics Education for the Future Project, XVI CIAEM) were used to obtain the email addresses of potential study participants. Next, potential participants were emailed a letter introducing the study and inviting them to click on a link to a consent form and survey (the survey link is still active and available at https://tinyurl.com/DocMathEd). The survey was designed to identify the experiences, practices, and expertise of individuals holding or pursuing a doctorate in mathematics education. It was hoped that respondents would forward the email and link to their colleagues and/or doctoral students, which occurred in several instances.

Participation

Survey participants for this report comprised 99 mathematics education university faculty from 33 countries (e.g., Australia, Canada, Indonesia, Japan, Netherlands, Sweden, U.S.). Each participant self-identified as someone working as a university faculty member and possessing a doctorate in mathematics education (or didactics of mathematics). Forty-one participants (41.4%) received their doctorate in mathematics education from a Department of Education, 23 participants (23.2%) from a Department of Mathematics or Mathematical Sciences, 17 (17.2%) from a Department of Mathematics Education, and 18 participants (18.2%) received their degree in some other department (e.g., Department of STEM Education).

Data Collection

Participants were asked a series of survey questions regarding how important they believed specific features (see Table 1) were to a doctoral program in mathematics education.

Table 1: Doctoral Program Features

Doctoral Program Feature (F)	Doctoral Program Feature (F)
F1 - Analyze, design, and evaluate mathematics curricula	F8 - Develop broad and deep knowledge of the big ideas in preK–14 (e.g., ages 2-20 years) mathematics
F2 - Study the history of mathematics education	F9 - Examine how the big ideas in preK–14 (e.g., ages 2-20 years) mathematics develop in students
F3 - Examine historical, social, political, and economic factors that influence mathematics education	F10 - Utilize technology as a tool of inquiry in mathematics teaching and learning
F4 - Examine current and historical research in the field of mathematics education	F11 - Design learning experiences for students and teachers that utilize technology
F5- Examine and compare fundamental theories of learning mathematics	F12 - Supervise field experiences for prospective (pre-service, student) mathematics teachers
F6 - Examine the influence of curriculum frameworks, standards, and/or competencies on school mathematics programs	F13 - Examine issues of diversity, equity, and inclusion in mathematics learning and teaching
F7 - Examine and compare different forms and purposes of assessment	

Responses were limited to “Very Important,” “Moderately Important,” “Slightly Important,” “Not Important,” and “Not Necessary/Not Required.” All 99 participants responded to 13 of these Likert-type level of importance questions.

Analysis

The Likert-type level of importance questions was analyzed by weighing each possible anchor response as follows: “Very Important” = *4*, “Moderately Important” = *3*, “Slightly Important” = *2*, “Not Important” = *1*, and “Not Necessary/Not Required” = *0*. For example, the question focused on the importance of the doctoral program feature “Analyze, design, and evaluate mathematics curricula” received the following responses: “Very Important” was selected by 31 participants; “Moderately Important” was selected by 34 participants; “Slightly Important” by 22 participants; “Not Important” by two participants; and “Not Necessary/Not Required” by nine participants. Next, a Friedman Test was performed to determine if statistically significant differences existed between participants’ responses (i.e., level of importance) and the 13 program features. Furthermore, Dunn’s pairwise post hoc tests were used to determine which, if any, mean rank pairs of program features were significantly different.

Results

Results of the Friedman Test indicated a significant difference in the selected importance levels between the different program features, $\chi^2(12) = 222.57$, $p < .001$. The mean rank score of each program feature from largest to smallest is as follows: F5 (9.65), F4 (8.43), F8 (8.28), F9 (8.03), F13 (7.24), F3 (7.12), F6 (7.11), F7 (6.59), F1 (6.58), F10 (6.04), F11 (5.55), F2 (5.51), F12 (4.87).

The Dunn’s pairwise post hoc test adjusted by the Bonferroni correction for multiple tests indicated the mean ranks of the pairs indicated by * were significantly different (see Table 2).

Table 2: Dunn’s Pairwise Post Hoc Test Results

	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10	F11	F12	F13
F1	—	1.000	1.000	*	*	1.000	1.000	.097	.449	1.000	1.000	.089	1.000
F2		—	.170	*	*	0.180	1.000	*	*	1.000	1.000	1.000	.076
F3			—	.964	*	1.000	1.000	1.000	1.000	1.000	.214	*	1.000
F4				—	1.000	.918	*	1.000	1.000	*	*	*	1.000
F5					—	*	*	.680	.156	*	*	*	*
F6						—	1.000	1.000	1.000	1.000	.227	*	1.000

F7	—	.103	.474	1.000	1.000	.083	1.000
F8		—	1.000	*	*	*	1.000
F9			—	*	*	*	1.000
F10				—	1.000	1.000	1.000
F11					—	1.000	.097
F12						—	*
F13							—

Dunn’s pairwise post hoc test results indicated F5 was significantly different than nine other features, F12 was significantly different from seven other features, and F4 was significantly different from six other features. Therefore, F5, the feature with the largest mean rank score, significantly differed from 75% of the other features. Item F5 (Table 1) asked participants to indicate the importance (to a mathematics education doctoral student) of examining and comparing theories of learning mathematics. Similarly, F4, the feature with the second largest mean rank score, was significantly different than 50% of the other features, highlighting the importance (to mathematics education doctoral students) of examining current and historical mathematics education research during doctoral studies. Finally, F12, the feature with the smallest mean rank score, significantly differed from 58.3% of the other features, highlighting the relative lack of importance (to mathematics education doctoral students) of supervising field experiences for prospective (pre-service, student) mathematics teachers.

Discussion

As reported here, doctoral programs in mathematics education should, at a minimum, provide doctoral students with ample opportunities to examine and compare fundamental theories of learning mathematics; examine current and historical research in the field of mathematics education; develop broad and deep knowledge of the big ideas in ages 2-20 years mathematics (i.e., grades preK-14); and examine how the big ideas in ages 2-20 years mathematics develop in students. Less consequential are opportunities to supervise field experiences for prospective (pre-service, student) mathematics teachers, study the history of mathematics education, and design learning experiences for students and teachers that utilize technology. This report is part of a larger, international study with the intent to continue the discussion and promote actions toward more cohesive expectations, practices, and expertise for doctoral programs in mathematics education. Such discussions and actions have the potential to develop guidelines for robust mathematics education doctoral programs regardless of the program’s location.

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AN ARGUMENT FOR A SYSTEMIC APPROACH TO CHANGE IN THE FIELD OF MATHEMATICS EDUCATION

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Keywords: Systemic Change

The purpose of this poster is to begin a conversation about why systemic change needs to be brought into the field of mathematics education. This is a direct response to the conference's call to envision the future of mathematics education in uncertain times, and our reply is to open a dialogue by connecting systemic change-oriented literature from across fields to expand and adapt the conversations we are having as math educators in our present world of ever-increasing complexity. We additionally provide one possible argument for why mathematics education as a system is complex enough to warrant systemic approaches toward change.

First, it is important to establish a common understanding of the word "system" and how mathematics education interfaces with this idea. There is no single definition of a system; instead, multiple definitions co-exist in a manner that "reflects the multidimensionality of the concept" (Hieronymi, 2013, p. 585). Systems change likewise has many different conceptions but aims to answer "what change is needed, why is it needed, and what might be the unintended consequences" (Abercrombie et al., 2015, p. 9). Systems change discourse has not always upheld deep equity in the way true systemic change requires across individual, interpersonal, institutional, and systemic/societal domains (Petty & Leach, 2020). Gutiérrez (2017) similarly discusses the way that surface-level tinkering around equity in mathematics education cannot yield results in a fundamentally flawed system. Bringing the language of systemic change to mathematics education represents one potential opportunity to begin this necessary process of change.

How do we know that systems thinking is appropriate for the field of mathematics education? The Omidyar Group provides a guide that helps us to answer this question by considering four dimensions of challenge complexity: (1) the more unsure we are about the exact nature of the problem or solution; (2) the more there is a significant diversity of opinion or conflict between opinions and stakeholders; (3) the more "diverse and dynamic interconnections between the problem and the broader environment, which itself is unstable and dynamic (political, social, and economic)"; and (4) the more our goal is "to make sustained change at a broad scale," the more important it is to use systems thinking and design (p. 10). The higher the level of alignment with the above statements, the more appropriate systems level thinking is, argues the Omidyar Group. The multiple iterations of the Math Wars (Schoenfeld, 2004) and politicized backlash to "challenging the status quo" (Gutiérrez, 2017, p. 8) prove that mathematics education is no stranger to conflict due to differing opinions about the nature and purpose of mathematics instruction (Dimensions 1 and 2). Mathematics in education is also tied to broader educational structures, including in many cases standardized testing and accountability structures (Dimension 3). The poster presentation will center on linking research in mathematics education to these four elements of complex problems requiring systems thinking to advance our understanding of mathematics education's place within this discourse. Through this poster, we aim to create open dialogue about how some of the challenges we face in the field of mathematics education interface with existing literature on systems change.

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EXAMINING A CASE OF A MATHEMATICS COACH LEARNING SYSTEM

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We report on a case study examining one district's efforts to implement a district-wide system of support for mathematics coaches' learning. Through interviews with school-based mathematics coaches and district personnel, we identified four activities that coaches saw as beneficial in supporting them to enact one-on-one coaching cycles effectively: (1) traditional pull-out professional development sessions, (2) partnerships between pairs of school-based mathematics coaches at different schools, (3) school-based coaching days in which all mathematics coaches worked collectively on issues relevant to coaching practice, and (4) individualized support from the district-based K-12 mathematics specialist. We describe the four activities and what the coaches and a district mathematics specialist perceived to be their function(s) in supporting coaches' learning.

Keywords: Professional Development, Teacher Educators, Instructional Leadership.

One-on-one mathematics coaching can support improvements in mathematics teaching and learning when coaching *is done well* (Campbell & Malkus, 2011; Russell et al., 2020; Kraft & Hill, 2020). However, the work of mathematics coaching differs significantly from mathematics teaching (Kane & Saclarides, 2023; Kochmanski & Cobb, 2023a; Saclarides & Kane, 2023), and many mathematics coaches often transition to the role directly from the classroom with few opportunities to learn to coach effectively prior to working with teachers. Most novice mathematics coaches will therefore require support for their own learning if they are to coach in ways that can significantly enhance teachers' and students' learning (Kane & Saclarides, 2023; Saclarides & Kane, 2023).

Just as teachers can benefit significantly from systems of support that link together different types of teacher learning activities (Cobb et al., 2018), mathematics coaches are likely to benefit from similar systems of support that link together different types of coach learning activities. However, few studies have examined types of professional learning activities that could support mathematics coaches' development, beyond traditional pull-out professional development (Saclarides & Kane, 2023). Further, there is limited research examining how different types of coach professional learning experiences might cohere to form a system of support for coaches' learning. Schools and districts therefore face the daunting prospect of implementing school- or district-wide coaching initiatives without a clear research base that can inform their efforts to support coaches in learning to coach effectively.

In this paper, we address this gap in the coaching literature by reporting on one district's efforts to implement a system of support for mathematics coaches' learning. Specifically, we identify the types of coach learning activities in the coach learning system. We also define the function(s) those activities appeared to serve in supporting coaches' learning. The following research questions informed our investigation of the coach learning system: (1) What types of coach learning activities do mathematics coaches perceive to be beneficial for their learning? (2)

What do mathematics coaches and the district mathematics specialist perceive to be the function(s) of the coach learning activities in supporting coaches' learning?

Conceptual Framework: Toward a Coach Learning System

Teachers can benefit significantly from opportunities to engage in a coherent support system of supports that link together potentially productive teacher learning experiences (Cobb et al., 2018). For example, teachers are likely to benefit greatly when traditional professional development includes and is linked to classroom-based coaching (Rock, 2019). Given these potential benefits to teachers, it stands to reason that coaches are also likely to benefit from a system of support focused on their learning. Several recent studies have examined efforts to support coaches' learning in traditional, pull-out professional development settings (Stein et al., 2021; Swars Auslander et al., 2023), finding that coach professional development (PD) can support coaches in developing effective coaching practices (Stein et al., 2021). Yet, there may be other types of professional learning activities that could support mathematics coaches' development, beyond traditional pull-out PD. For example, coaches could benefit from engaging in collective coaching experiences, just as teachers benefit from collective teaching experiences like lesson study (Lewis et al., 2009) or mathematics labs (Kazemi et al., 2018). Understanding the nature and function of different types of coach learning activities beyond traditional pull-out PD is an important step in defining the elements of a productive coach learning system.

Methods

We employed a case study methodology (Yin, 2017) to answer our research questions.

Focal Case: Apple Valley Schools

We focused our case study on Apple Valley Schools (AVS; name is a pseudonym), which is a mid-sized school district located in the Southeastern United States. We selected AVS for this study because the district employs ten school-based mathematics coaches and a K-12 mathematics specialist, whose primary job function is to support the school-based coaches. At the time of the study, the district was also implementing several distinct coach learning activities. Specifically, AVS was partnering with faculty at a nearby regional university to design and implement pull-out coach PD. University faculty worked with the AVS K-12 mathematics specialist to design coach PD sessions focused on different aspects of coaching practice central to enacting one-on-one coaching cycles effectively. One-on-one coaching cycles are a common coaching routine for which there is significant evidence that they can support teachers' learning when facilitated effectively (Russell et al., 2020; Kochmanski & Cobb, 2023a). The K-12 mathematics specialist also established formal coaching partnerships between pairs of school-based coaches. Further, the specialists implemented school-based coaching days, which were opportunities for the school-based coaches to work together in a common school space.

Data and Data Analysis

We conducted semi-structured interviews with the ten school-based mathematics coaches and with the K-12 mathematics specialist. We used online video conference software to conduct the interviews. Each interview included questions focused specifically on the types of coach learning activities the district was implementing (e.g., pull-out PD, coaching partnerships). The interviews also included questions focused specifically on the purpose(s) of the activities in supporting coaches' learning. Additionally, we asked open-ended questions intended to elicit coaches' perspectives on other types of coach learning activities or learning structures beyond the three primary activities the district was implementing. For example, we asked coaches to respond to the question, “Imagine a district leader was about to begin designing a system of support for coaches. What advice would you give them?”

To answer our first research question, both authors consensus coded each of the ten coach interviews for the types of coach learning activities that the coaches saw as beneficial. We documented our codes in an excel spreadsheet. We also documented the coaches’ descriptions of each type of coach learning activity. To answer our second research question, we open coded the ten interviews with coaches and the interview with the mathematics specialist to characterize the coaches’ and the district mathematics specialist’s perspectives on the function(s) of the coach learning activities. We used open coding because the current coaching literature provides little to no guidance regarding the functions of such activities in supporting coaches’ learning.

Findings

The mathematics coaches in AVS identified four types of coach learning activities that they perceived to be beneficial for their learning: (1) traditional pull-out professional development led by an outside facilitator, (2) peer coaching partnerships, (3) the school-based coaching days, and (4) individualized, school-based support from the K-12 mathematics specialist.

Traditional Pull-out PD

All ten of the mathematics coaches perceived the pull-out coach PD to be beneficial to their learning. The ten coaches noted that it was beneficial for the PD to be led by a university faculty member, and thus a facilitator outside of their district. This meant that the participating coaches had the opportunity to engage with what one coach referred to as a “perspective that was different than the usual district” perspective. Eight of the ten coaches also found it beneficial when the PD sessions centered on case studies of one-on-one coaching. Coaches found the case studies to be especially beneficial when they originated from AVS coaches’ own work, as this meant they could spend more time discussing the nuances of coaching and less time trying to understand the context for the case.

Regarding the function of the PD, all coaches and the district mathematics specialist explained that the purpose of the coach PD was to introduce coaches to new and effective coaching practices. That said, several coaches also noted that the PD enabled the coaches to develop a common framework for and language to describe one-on-one coaching in the district. The coaches explained that this language was often formalized in the resources the PD facilitator shared in the PD sessions. Finally, several coaches also explained that the PD provided them with opportunities to think more deeply about mathematics teaching and learning, especially as the case studies often involved an analysis of mathematics lessons and students’ work.

Peer Coaching Partnerships

Eight of the ten coaches saw the district-initiated peer coaching partnerships as beneficial for their learning. The other two coaches saw potential for this type of partnership to benefit their work but outside circumstances (e.g., scheduling difficulties) limited their ability to realize that potential. All eight coaches who perceived the benefit of the partnerships visited their partner coach’s school. On these visits, the partnering coaches observed lessons together and then met to discuss potential next steps for working with individual teachers. Several coaching partnerships also discussed strategies for facilitating coaching conversations with teachers in the building. For example, one coach mentioned that her partner coach helped her rehearse for an upcoming coaching conversation that she thought might be particularly challenging.

Most of the coaches thought it was helpful to see instruction at different schools, as this helped them see common instructional challenges across the district. Five coaches and the district mathematics specialist saw the partnerships as functioning like a coaching professional learning community (PLC). As one coach put it, while she had a PLC as a teacher, she was on her own as a coach. The district-initiated partnership gave her someone else in a similar role who could serve as a collaborator and a "go-to person." Finally, three of the coaches noted that the coaching partnerships provided another set of eyes on the coaching process. This helped them identify their own coaching biases and look at teaching and coaching in new ways.

School-Based Coaching Days

All ten coaches saw the school-based coaching days as beneficial for their learning. According to the coaches, the school-based coaching days consisted of teams of 3-4 coaches visiting a classroom for a full lesson. The coaches then met afterward to discuss the data they collected during the observation and what they perceived to be next steps for supporting the teacher's learning. Finally, the whole group of ten coaches met to discuss what they noticed in the lessons they observed. The coaches saw the coaching days as opportunities to put into practice many of the ideas discussed in the pull-out PD sessions. As one coach put it, the school-based coaching days were a chance to "develop your craft as a coach." Coaches noted they had opportunities to get better at documenting students' thinking in the lesson and the teacher's actions during the lesson. They also had the chance to work with fellow coaches to connect instruction and students' thinking when identifying next steps for the teacher, which is an important coaching practice (Kochmanski & Cobb, 2023b).

Individualized Support from K-12 Mathematics Specialist

Three of the coaches described it as beneficial to receive individualized support from the district K-12 mathematics specialist. All three coaches were either new to the district or new to their school. The coaches described the individualized support as taking many different forms, from formal school visits in which the specialist helped the coach lead portions of a one-on-one coaching cycle to informal text exchanges where the coach asked the district mathematics specialist for advice. According to the coaches, the primary function of the individualized support was to provide the coaches with opportunities to bounce ideas off an experienced coach and to receive resources they may not have known about previously. The district mathematics specialist saw the individualized support as serving a different function. She noted that it provided her with an opportunity to help the coaches think about the culture of their individual schools, with the goal of making each school a productive place for improvement.

Discussion and Conclusions

This case study of a district-wide effort to support mathematics coaches' learning resulted in the identification of four types of coach learning activities that appear to be beneficial for coaches' learning. Describing these four types of coach learning activities and documenting their functions in supporting coaches' learning is an important step toward a research-based understanding of a productive coach learning system. Future research can build from this study by investigating the learning opportunities these types of activities present to coaches, what it takes to facilitate the coach learning activities effectively, and the conditions necessary for their effective implementation.

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MATHEMATICS TEACHER EDUCATORS DEVELOPMENT OF INSTRUCTIONAL ACTIVITIES TO IMPACT PRESERVICE TEACHER CURRICULAR REASONING

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We address mathematics teacher educators' (MTEs') instructional activity design practice to inform preservice teachers'(PTs') curricular reasoning. Findings from a self-study of three MTEs' instructional activity design in methods courses focus on opportunities for PTs' as curriculum reasoners. We argue MTEs' instructional activity designs are informed by ontological goals for teaching about curricular reasoning, such as curricular agency, and assumptions about curricular contexts.

Keywords: Preservice Teacher Education, Teacher Educators, Curriculum

Mathematics teacher educators (MTEs) develop instructional sequences to inform preservice teacher (PT) curricular reasoning (Tyminski et al., 2013). Though research includes principles for instructional design (Drake et al., 2014) and courses (Saclarides et al, 2022), MTE instructional design practices to support PTs' curricular reasoning are complex and require additional characterization (Ghousseini & Herbst, 2016). As relational (Kitchen, 2005) constructivist (Steffe & D'Ambrosio, 1995) MTEs, we use self-study to describe instructional design practice. Relationships with PTs (Kastberg et al, 2022) and evidence of their pedagogical concepts (Simon, 2008) inform our instructional decisions. We use evidence of PTs' learning to create opportunities for PTs to learn to teach from experience. We have described "layering instructional activities" (Kastberg et al., under review), though instructional activity design remained opaque. Theories of curricular reasoning address how contexts may constrain teachers' curricular decisions. MTE curricular decisions have few constraints (Tran & O'Connor, 2023) such that MTEs' domain of potential action (Brown & Coles, 2020) is broad. This research seeks to address: What informs MTE design of curricular reasoning activities?

Background and Literature

MTEs' instructional activity design process evolves as they internalize their ideas about PTs' learning of pedagogy, theories of mathematics education, and lived experiences as MTEs. In this paper we describe factors MTEs use as they design and enact activity sequences. Examples of MTEs' instructional activity design for PTs (e.g., Ghousseini & Herbst, 2016; Tyminski et al., 2013) illustrate two dimensions: MTEs' use of theories of teacher learning (e.g., Gutiérrez, 2018; Hammerness et al., 2005; Kazemi, 2018) and pedagogies of practice (Grossman et al., 2009). Using self-based methodologies (Borko et al., 2007) MTEs seek to make sense of and represent the intersectional nature of action and knowledge in practice (e.g., Kalinec-Craig et al., 2021). Thus, instructional design is guided by goals and histories of interactions with PTs (Coles, 2013). Jaworski (2021), Chapman (2021), and others address how MTEs' practices emerge and how practice informs MTEs' development. Such accounts characterize the particular (Hamilton & Pinnegar, 2014) providing exemplars of MTE learning and development producing knowledge as "a complex, integrated system and way of being/thinking and as knowing how to act in the teaching process" (Chapman, 2021, p. 415). Our interactions with PTs inform the design and modification of instructional activities in support of PTs as curriculum reasoners. We view MTE instructional activity design as situated and emerging through interactions with PTs.

Curricular reasoning is "thinking processes" teachers use with curricular materials (Breyfogle et al., 2010, p. 308). Curricular materials include print textbooks and online resources (Dingman et al., 2021). PTs must learn to read, analyze, and discuss curriculum (AMTE, 2017) to set goals and select tasks (NCTM, 2014). PTs' use of curriculum includes adhering, elaborating, creating (Nicol & Crespo, 2006), and adapting materials (Lloyd, 2008). This suggests MTEs must provide opportunities for PTs' curricular reasoning.

Mathematics methods includes opportunities for PTs to engage in curricular reasoning with attention to context, autonomy, and agency. Drake et al. (2014) showed reading curricular materials has subskills, with subskill development and recomposition (Sapkota & Max, 2023) during one methods course unlikely (Saclarides et al., 2022; Simon, 2008). Teacher curricular reasoning has variability even when agency is constrained. Tran and O'Conner (2023) identified constraints on teachers' curricular agency including centralized curriculum. Though curricular contexts may appear fixed and curricular autonomy constrained, teachers use curricular agency as they reason with and enact curriculum with learners. We describe our process of designing instructional activity sequences informed by our goal of supporting PTs' use of curricular agency in varied curricular contexts with conscious curricular reasoning and decision making.

Methodology and Methods

We are three white female MTEs, each with over 10 years' experience teaching mathematics methods at different institutions. Our institutional missions range from teaching-focused to research-intensive and our program foci span elementary to secondary teacher certification. We use self-study methodology, defined as self-initiated, improvement-aimed, and interactive using qualitative methods (LaBoskey, 2007) as means to study and improve our practice. Dialogue is a central process used in coming to know (Pinnegar & Hamilton, 2009). Weekly dialogues focused on our practice used "constructivist listening" (Weissglass, 2004) to enable the talker to represent her thinking and dialogue with "critical friends" (Schuck & Russell, 2005) regarding alternative perspectives on shared events.

We used four analytic methods: 1) 10 analytical dialogues (Guilfoyle et al., 2007) during fall 2023 to unpack the "details of the experience of teaching" and theorize about principles of practice from our experience (Brown & Coles, 2020, p. 99), 2) evidentiary maps of the "structure of events" (Jordan & Henderson, 1995, p. 57) in our fall 2023 methods courses involving events and artifacts from our instructional activity design practice, 3) cases of PTs as curricular reasoners to interpret the cumulative impact of instructional activities on PTs' curricular reasoning, and 4) descriptive coding (Saldana, 2016) of our recorded dialogues to search for confirming and disconfirming evidence. These four analytic methods created an evidentiary basis for findings common across three MTEs' design of instructional activities and related contexts.

Findings

This section describes two components influencing our instructional activity design for PTs' curricular reasoning: (1) goal of curricular agency, (2) imaginings of curricular contexts. Data from all three authors' design of instructional activities were used to derive the findings. Here, we represent findings addressing the question: What informs MTE design of curricular reasoning activities? using excerpts from transcripts of dialogic conversations focused on Alyson's experience. The following narrative highlights coming to know how our support for PTs' curricular reasoning was informed by our goal that PTs develop awareness of curricular agency.

My (Alyson) methods course is the second of two required mathematics methods courses in our secondary and middle grades preparation programs focusing on grades 8 – 12 mathematics content. Secondary PTs take this course the semester prior to student teaching and middle grades teachers take it earlier. Enrollment was low in Fall 2023; with three PTs in the course (Claire and Ashley – secondary, Crissy – middle grades; pseudonyms), I had freedom to experiment as I supported development of their ideas about curriculum. I used *Principles to Actions* (NCTM, 2014) as a text, with emphasis on the mathematics teaching practices, knowing the prerequisite course focused on student thinking. Two major tasks of this course are: 1) three teaching rehearsals and 2) two written multi-day lesson plans. No field experience is connected to this course, though PTs are in other field placements while taking this methods course.

I began by wanting to make use of textbooks matching the scope and sequence for courses in the local school system where most of the PTs would find jobs. I wondered how to prepare them to use textbooks as the *approved curriculum* due to state policy. I acknowledged this contextual constraint in a critical friend conversation:

> I'm very conflicted with this because I want to support the school system and that they're making decisions in the best way they possibly can [with scope and sequence documents] . . . But I also think a teacher teaching Precalculus ought to be able to teach circles as the first of the conic sections if they want to. (Conversation, 11/13/23)

I did not want to force PTs to use textbooks – though textbooks can be useful. My goal: PTs develop problem solving lessons with opportunities for learners to make sense of mathematics. Critical friend conversations focused on how I could learn about my PTs' thinking. I started my instructional sequence with a standards interpretation activity. PTs created posters mapping mathematics content across four courses. After this introduction to high school content, I needed evidence of PTs' ideas about curriculum. I was unsure of my goals for PTs' curricular reasoning.

> I'm really wondering if I should give them each a poster paper and say, "think about your lesson plan. Think about what was influencing your decisions about what you included - and what you chose not to include? And see what I get. (Critical Conversation, 10/30/23)

Knowing their thoughts about curriculum would help me clarify goals. I used reflective writing and a concept mapping activity to gather evidence of PTs' curricular ideas to create instructional activities built from these ideas. PTs' work was revealing. Claire focused on order of standards: what learners should know and connect to what is next. Claire questioned who decides the order for teaching concepts and if the sequence of concepts from the textbook could be changed. Crissy reasoned about concept order to simplify concepts for learners to understand. Ashley's curricular reasoning considered what learners *do* know (rather than *should* know) and selected resources based on learners' demonstrated knowledge. In a critical friend conversation Susan wondered about PT agency when Alyson brought up a PTs' question on order of concepts.

Susan: Do you think the PT is asking “Is there one right way to approach it?” Or were they saying, “Do I have the agency to change it when the textbook says this?”

Alyson: A little bit of both, and I think she wants to know why the authors of the textbook put it in that order - she wants to make an instructional decision with more information. (Conversation 11/13/23)

I designed additional layers of activities (Lischka et al., 2023) including textbook analysis. Textbooks and teachers’ editions as curricular resources can be overwhelming. Research findings provide insights about what PTs attend to – but would PTs in my course attend to elements identified in research, or have other ideas? I observed the PTs’ choices during the textbook analysis which led to the rehearsal of launching a lesson using provided curriculum materials.

PTs used textbook sections differently during rehearsals. Claire used a textbook activity but changed the context. Crissy used a textbook example but “simplified” the directions. Ashley used an activity not in the text but planned to use textbook examples later in the lesson. Each of the PTs reasoned about the textbook as a curricular resource, but I wondered about the impact of the rehearsal, or whether they brought curricular reasoning from other courses or experiences?

PTs created two lesson plans on assigned topics to demonstrate learning concepts in this course. PTs considered how they incorporated strategies from class into plans, and selected materials to support instructional goals, within constraints I provided. I struggled with feedback to help PTs find balance between developing conceptual understanding and procedural fluency, torn between telling them to use particular curriculum resources and wanting them to make their own curricular decisions to reach their instructional goals.

PTs final lesson plans had a better connection between conceptual development and procedural fluency. These PTs were provided with more textbook access, and they used the textbooks more than PTs from prior semesters. PTs seemed to be exploring their agency in curricular reasoning. PTs’ end of term reflections showed a desire to “make decisions, get them down and then evaluate after that” (Ashley, Final Self-Evaluation).

Discussion: Agency, Context, and Instructional Design Practice

Research provides evidence of how PTs read (Tyminski et al., 2013) and use curricular resources (Nicol & Crespo, 2006). The complexity of MTEs’ work using learning theories (e.g., Kazemi, 2018), core practices (Grossman & Dean, 2019) and pedagogies of practice (Grossman et al., 2009) to support PT development (Ghousseini & Herbst, 2016) suggests study of MTEs’ instructional activity design practice could provide findings connecting such practice to PTs’ development of curricular reasoning (Breyfogle et al., 2010). We used self-study methodology and qualitative methods to gather empirical evidence of MTEs’ ways of knowing in instructional activity design to impact PTs’ curricular reasoning. We found that MTEs’ instructional activity design was based on ontological goals for PTs’ curricular reasoning, namely supporting PTs’ awareness and use of curricular agency in curricular contexts with varied constraints.

Our findings represented by Alyson’s narrative illustrate ways of knowing involved in instructional activity design for the development of PT’s curricular reasoning, beyond time related tensions in MTE design described by Saclarides et al. (2022). Aligned with findings from explorations of MTEs’ instructional activity design practice for teachers (Ratnayake & Taranto, 2023), we found that MTEs use existing theories of learning and being in making instructional activity design decisions. Alyson’s commitment to working with PTs’ ideas as the basis for curricular reasoning instructional activities was informed by principles of relational practice (Kitchen, 2005) and constructivist teaching (Kastberg, 2014; Steffe & D'Ambrosio, 1995). Yet beneath these decisions was an ontological goal supporting the layers of instructional activity designed to support PTs' curricular reasoning; our goal was for PTs to become conscious of and use curricular agency in curricular contexts informed the series of instructional activities we developed. Assigning topics to the PTs constrained the curricular context as Alyson expected PTs to use curricular agency (Tran & O'Connor, 2023) to address the imposed constraints. Alyson's discussions with PTs focused on gaining evidence of how they were reasoning about standards and textbooks and using those resources to demonstrate agency within constraints. This study builds on prior research (e.g., Tyminski et al., 2013) and points to the need for more research on how MTEs design instructional activities to support PTs' curricular agency in increasingly constrained contexts.

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PROSPECTIVE MATHEMATICS TEACHER EDUCATOR LEARNING

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Mathematics teacher educators (MTEs) play a significant role in supporting preservice teachers (PTs) to develop the mathematical content and pedagogical knowledge they need for teaching. We examined what four novice prospective MTEs learn about supporting PT learning when they are part of a Community of Practice (CoP) in a mathematics content course taught through problem solving. We collected data through weekly reflections written by the prospective MTEs, researcher memos, interviews with each participant at three points in the semester, and notes from the CoP twice weekly discussions. We found that the prospective MTEs learned by observing their students' learning and by making sense of their roles in the course taught through problem solving. We also found that the CoP, weekly readings and having an intern role in the course were key to supporting the prospective MTEs' learning.

Keywords: Teacher Educators; Teacher Knowledge; Problem Solving; Mathematical Knowledge for Teaching

Mathematics teacher educators (MTEs) play a significant role in supporting prospective teachers (PTs) to develop the mathematical content and pedagogical knowledge they need for teaching. Research has shown, however, that most of the MTEs in the United States have little experience teaching students at the level of mathematics that they are preparing PTs to teach (e.g., elementary school), and that they receive little to no training or support either in their graduate programs or in their jobs (Masingila & Olanoff, 2022; Masingila et al., 2012).

Mathematics professional organizations (e.g., Conference Board of the Mathematics Sciences (CBMS) in the USA, Association of Mathematics Teacher Educators in the USA) have recommended that PTs develop deep and connected understandings of foundational mathematical ideas and be engaged in doing mathematics that "allows time to engage in reasoning, explaining, and making sense of the mathematics that prospective teachers will teach" (CBMS, 2012, p. 17) and "develop the habits of mind of a mathematical thinker and problem-solving" (p. 19). Masingila et al. (2018) proposed that one way to "foster deep mathematical knowledge development in PTs is to engage them in learning mathematics via problem solving" (p. 431). Since prospective MTEs have typically experienced traditional mathematics teaching and learning in their academic and teaching experiences, we argue that for MTEs to be prepared to support PTs in learning via problem solving, they need to experience, reflect on, discuss with others, and learn from teaching and learning through problem solving.

In this paper, our goal is to contribute to the mathematics education community's understanding of how novice MTEs may be prepared to support PTs in learning via problem solving. Specifically, our contributions are to provide evidence from our four participants of how this learning occurred through this experience. We examined how four novice prospective MTEs learn about teaching and learning mathematics through problem solving when they are part of a CoP comprised of instructors and interns (novice prospective MTEs) working together to support PTs' learning in developing mathematical understandings through problem solving.

We engaged prospective MTEs in an internship with particular design features situated within a Community of Practice (CoP) and the context of a mathematics content course for PTs taught through problem solving. The mathematics content of the course included the concepts of numeration, operations, number theory, and probability and statistics. The course met for 80 minutes twice a week with the 22-28 PTs working collaboratively on problem-solving tasks that the instructor introduced. The instructor and interns facilitated the small group problem solving and the instructor led the PTs in a wrap-up discussion bringing out the mathematical ideas from the tasks. For more information on this course and a second course with content including the concepts of geometry, measurement and rational numbers see Masingila et al. (2018).

Theoretical Framing

The bodies of literature that guided our work involve (a) Communities of Practice, (b) Mathematical Knowledge for Teaching and Mathematical Knowledge for Teaching Teachers, and (c) Learning through Problem Solving. Wenger, McDermott, and Snyder (2002) defined CoPs as "groups of people who share a concern, a set of problems, or a passion about a topic, and who deepen their knowledge and expertise in this area by interacting on an ongoing basis" (p. 7). Through a CoP, members develop and articulate new knowledge in response to questions and problems they have about their practice. A CoP offers a way for its members to engage in negotiating shared understandings, learning, meaning making, and identity. Wenger (1998) identified three dimensions of a CoP: (a) members interact with one another, and determine norms and relationships through *mutual engagement*, (b) members are held together by their understanding of a sense of *joint enterprise*, and (c) members seek to produce, over time, a *shared repertoire* of communal resources (e.g., language, routines, artifacts, stories). The members of our CoP were three instructors (one for each section of the mathematics content course), including a professor who was an experienced instructor of this course and served as the course supervisor, and six graduate students who served as interns. Two of the interns had previously been interns in the companion mathematics course with content of rational numbers, geometry, and measurement. The other four interns were new graduate students in mathematics education. Our CoP also included a graduate student who was part of the research team and had previously served as an intern. Our CoP provided us a space for working together to learn how to support PTs in their mathematical learning and grow as MTEs.

Based on Shulman's (1986) work, Ball and colleagues (Ball & Bass, 2002; Ball et al., 2008) introduced the term *mathematical knowledge for teaching* (MKT) and developed a framework for MKT that expanded on Shulman's descriptions of content knowledge and pedagogical content knowledge to include sub-categories of the mathematical knowledge that teachers need to know. More recently, researchers have examined the mathematical knowledge needed by MTEs to support PTs in developing MKT (Castro Superfine & Li, 2014; Olanoff et al., 2018; Zopf, 2010) – *mathematical knowledge for teaching teachers* (MKTT). Zopf defined MKTT as, "the mathematical knowledge used by mathematics teacher educators in the work of teaching mathematics to teachers" (p. 11), and claimed that, "the major purpose of the work of [mathematics] teacher education is beginning with people who already know some mathematics and developing that knowledge into mathematical knowledge for teaching" (p. 165). Our CoP supported us in developing our collective and individual MKTT and afforded us the opportunity to observe the growth of the novice interns' MKTT.

Problem solving as a concept and a practice has been around for as long as humans have tried to overcome challenges. Mathematics educators and professional associations have long advocated for engaging students in learning mathematics through problem solving. The Conference Board of the Mathematical Sciences (CBMS) (2001) argued that PTs can develop deep understandings of mathematical ideas "with classroom experiences in which *their* ideas for solving problems are elicited and taken seriously, their sound reasoning affirmed, and their missteps challenged in ways that help them make sense of their errors" (p. 17). The CBMS (2012) continued arguing for engaging PTs in problem solving with its recommendation that courses for mathematics teachers should "develop the habits of mind of a mathematical thinker and problem-solver, such as reasoning and explaining, modeling, seeing structure, and generalizing" (p. 19). Teaching through problem solving, however, is quite challenging with the need for the teacher to select and facilitate high-level tasks, scaffold student learning as appropriate, and orchestrate discussions about the mathematics arising from the students' problem-solving work. Masingila et al. (2011) argued that the teacher's responsibility is to "establish a mathematical community in the classroom where everyone's thinking is respected and in which reasoning and discussing mathematical ideas and meanings is the norm" (p. 14). We were interested to see how novice prospective MTEs make sense of the teaching and learning in a mathematics course taught through problem solving.

Research Methods

Our aim in this study was to investigate what novice prospective MTEs learn when serving as interns in a mathematics content course for preservice elementary teachers taught through problem solving. Our research design was a descriptive case study.

Context

The course content focused on whole numbers and operations, number theory, probability, and statistics. The CoP instructional team met formally twice a week, on Mondays prior to the two lessons for the week and on Thursdays after the second lesson, as well as conversations among the CoP members occurred informally before and after lessons.

This course was taught with an emphasis on PTs learning mathematics through collaborative problem solving. PTs worked together in small groups to solve problems with the goal of developing deeper understandings of the mathematics taught in elementary school and their own MKT. The role of the instructors of the course was to facilitate the PTs' problem solving and knowledge development and the role of the interns was to support the instructor and the students in their problem solving.

Data Collection

The participants for this study were the four new graduate students serving as interns. The data collected were (a) weekly reflections written by each novice intern as part of their intern work, (b) weekly or bi-weekly memos from the researchers who were working with the interns in teaching, (c) interviews with each intern individually at the beginning, middle and end of the semester, and (d) notes from the CoP discussions.

For the weekly reflections, the interns were given a published article or book chapter related to teaching and learning through problem solving to read each week. They were asked to reflect on what insights they gained from the article and what insights they gained about both student learning and their own learning through their experience in the two lessons that week. Given the challenge of teaching through problem solving, instructors first serve as interns to experience teaching and learning through problem solving so the instructors had read and reflected on the readings when they were interns. The course supervisor and instructor of record for the internship course (the first author) read and responded to the interns' reflections individually via email. Additionally, she would often mention points raised by the interns in their reflections and bring it to the CoP for discussion during the twice weekly meetings. In this way, the readings were shared texts for the CoP members and situated the discussions of teaching and learning problem solving within the mathematics education literature.

The researcher memos were written individually by three of the authors, each of whom was working as an instructor or experienced intern in one of the course sections, with their observations of the novice interns in the course. The interviews were conducted by the second author who was not serving as an instructor or intern. The purpose of the interviews was to understand each participants' experiences prior to, during, and after the internship.

Data Analysis

Our data analyses were both ongoing and retrospective. We used open coding to inductively develop codes from the data. Ongoing analysis occurred during the semester and was the basis for developing the questions for the midway and end-of-semester interview questions and the testing of emerging hypotheses. Retrospective analysis involved examining the larger corpus of data through a carefully structured review of all the data sources. We used qualitative data analysis methods as described by Creswell and Creswell (2018). Our approach involved preparing the data for analysis, reading through and coding the data, generating themes, and interpreting the meaning of those themes across the data set.

Results

Below we share our findings of prospective MTEs' learning in two categories: (a) MTEs learning by observing their students' learning, and (b) MTEs learning by making sense of their roles in a course taught through problem solving. Note that all names used are pseudonyms.

Observing the Learning of Preservice Teachers

We found that prospective MTEs learned through observing how the PTs in the course developed their mathematical understandings and confidence. We identified the themes related to these observations of preservice teacher learning: learning through collaboration, learning through problem solving, and connecting student learning with course readings.

Learning through collaboration. One intern, Alex, seemed to especially value the way that PTs in the course learned through collaboration. They noted this growth and attributed it to the collaborative nature of the course:

> At the beginning of the course, many students expressed their frustration at the tasks and had less perseverance in struggling with the math. They are now more readily engaged in the learning tasks and, while individual questions might pose a challenge, they are as a group more resilient and likely to continue trying a problem even when it is challenging. Part of this

> growth is owed to their development of good group dynamics, communication, and collaboration. (Reflection Week 6)

Here we see that Alex learned that PTs developed understanding of mathematics through collaboration.

Alex noted the importance of this collaboration early in the course experience. When asked what it would look like for PTs in the course to be successful, Alex said, "effective collaboration is huge, like the fact that [the PTs] work in groups is really important. Again, so that they carry that into their own teaching, sort of having this collaborative mindset" (Beginning Interview). Alex viewed success for these students as learning to be collaborative, not just among peers for their learning, but for developing a collaborative mindset in their future roles as teachers.

Another intern, Cameron, learned to value the collaborative approach of the course. When reflecting on how PTs collaborated, Cameron said, "Almost every student gets involved in classroom activities in their groups and it becomes easier to monitor students by groups and provide help when necessary. Students feel comfortable trying new ideas and discussing their problem-solving strategies among their peers and asking relevant questions when necessary" (Reflection Week 6). Cameron found that collaborative learning made teaching logistically effective and encouraged PTs to share their ideas and ask questions.

Learning through problem solving. Another way that prospective MTEs observed PTs' growth is in the way that PTs developed as problem solvers, both in their mathematical skills and in attitudes toward doing mathematics. For example, Michael shared during a post-class discussion that he observed PTs' developing knowledge as they connected new algorithmic ideas to prior concepts of the course (Researcher Memo Week 6). Similarly, Alex observed a notable change in students' problem-solving abilities by the end of the semester. Alex said, "it's cool to say, you know, very concretely, I could, through my observations, say like, they didn't have these skills at the beginning of the course. And now I think if we put a problem in front of them that they had never seen before, they would apply these strategies of problem solving. It's very cool" (Post Interview). These examples show how MTEs observed PTs develop mathematical knowledge and skills on specific days in the classroom and throughout the semester.

John observed PTs gain confidence in their problem-solving capabilities. In a team meeting near the end of the semester, John shared that at the beginning of the semester, most students would ask the instructor and interns, "Is this right?" when they are working on a task. But now, near the end of the semester, they were confident in their solutions. So much so that even when John said that he does not think they are correct, they replied that they know that their solution is correct (Researcher Memo Weeks 9-10). Cameron, too, observed a shift in students' attitude toward learning mathematics, which he attributed to their learning through problem solving. When asked what he thinks his students are learning, Cameron shared,

> I remember, during the first day of class, most of the students were asked how they think about mathematics. And most of the responses were, mathematics is difficult, boring, not interesting. But then now you could see students come to class, so eager to learn something new, to practice something different, to do problem solving and all that. And you will see the spirit of the class is very high. (Mid-Semester Interview)

Both John and Cameron observed how learning mathematics through problem solving influenced the attitudes of PTs toward their learning.

Connecting student learning with course readings. We found that the course readings were effective in providing shared language in the CoP as prospective MTEs observed student learning. In his second week reflection, John explicitly drew on the course reading to connect with his observations of student learning. John shared,

> Procedural knowledge refers to the step-by-step procedures executed in a specific sequence, while conceptual knowledge refers to the relationships between pieces of information, which enhances flexibility in accessing and using information (Carpenter, 1986). In performing Activity 2.3 and Activity 2.3 (Masingila et al., 2011), students applied procedural knowledge by laying down steps to write big numbers such as 476 in the Babylonian system, and conceptual knowledge by analyzing the relationship between different number systems. (Reflection Week 2)

The terms procedural knowledge and conceptual knowledge represent a common use of shared language within the CoP to make sense of student learning.

In addition to drawing on shared language, participants connected to the course readings through the ideas the readings presented as they observed student learning. Alex, for example, noted that "learning opportunities occur when students must create or discover a method to solve a problem and then justify the validity of their methods to classmates (idea from this week's reading), and I see both of these happening throughout the course so far" (Reflection Week 5). The language and ideas from course readings helped prospective MTEs to make sense of their observations of student learning and provided shared texts to draw on within the CoP.

MTEs Making Sense of Their Roles

In addition to learning through observations of their students' learning, the prospective MTEs in this study learned as they made sense of their roles in this course. We found that the participants learned about teaching and learning through problem solving in general. In addition to this, some participants developed a strategy of prompting PTs to think about their future careers as teachers as the interns began to recognize their roles as prospective MTEs.

Making sense of teaching and learning through problem solving. For three of the participants, their backgrounds in mathematics can generally be characterized as traditional or teacher-centered, and a course taught through problem solving was an opportunity to experience a new pedagogical style. For example, when debriefing with one of the researchers after class during the first week of the course, Michael shared that he had never seen a mathematics course taught through problem solving and that he was eager to learn more (Researcher Memo Week 1). By the end of course, Michael learned that teaching through problem solving involves centering student responses. Michael shared,

> And one of the few things that I discovered in a problem-solving class, we usually encourage students to give us their answers. Because one of the few things that we usually focus on, we're interested in seeing the way students think in different ways and obtaining different answers. And different answers, it is upon them to choose the method they're interested in. (Post Interview)

Having never experienced teaching and learning through problem solving before, Michael explained how eliciting student thinking is a key feature of this approach that he learned.

Cameron described his mathematics experience during his primary school as being teacher centered. Cameron shared, “hardly will the students start with the example themselves. The teacher gives the example, gives the procedure to solve by example, then allows students to solve different examples, the same procedure that they have solved” (Beginning Interview). However, Cameron suggested that this changed slightly in secondary school when teachers began giving “hints” and “guidelines,” which he saw as being closer to a problem-solving approach (Beginning Interview).

John described his prior mathematics experience as being one where teachers were “more focused on [students] passing the exams rather than understanding the concepts” (Beginning Interview). This evidences John’s traditional experience when he was a student of mathematics. Later in the semester, John reflected on his own growth in terms of a shift from a traditional approach to a problem-solving approach. John shared, “to highlight how much I've learned, it has helped me to redefine my role as a teacher of math by trying to center my teaching through problem solving, rather than the lecture method, where I'm the sole person giving up the knowledge” (Mid-Semester Interview). Together, these examples evidence how three of the participants who had not previously experienced teaching and learning through problem solving learned about this pedagogical style when making sense of their roles as MTEs.

Making sense of supporting PTs. Enacting a role as a MTE, for some, meant prompting PTs to begin to think about their future roles as educators. Alex elaborated on this prompting in the mid-semester interview when asked how they see themselves as supporting their students as future teachers. Alex said,

> I try to explain to them, like, the idea is, eventually you'll be engaging students in this type of thinking. So, you know, you need to be going through this yourselves. And experiencing this type of learning, so that hopefully, you'll see the value in, and you'll be able to teach this way someday. And I think more informally, like, during office hours, I've talked with some of the students about their goals. You know, like, by the end of their degree program, or, you know, what their sort of purpose in doing this program is. So, I guess more informally talking with them about, you know, where they want to be in education, what kind of roles they see themselves in schools. (Mid-Semester Interview)

In making sense of their role as an MTE, Alex supported PTs to consider their future roles as educators as they were learning in the course.

John also shared this idea in the way that he made sense of his role as an MTE. John said, “I think the purpose of encouraging my students to just see themselves as teachers in the classroom … they came as students taking math …” and now “they say I’m learning … as a teacher … I’ve really benefitted” (Post Interview). While the interns were not in a position to directly teach pedagogical methods to their students, Alex and John began prompting PTs to consider their future identities as educators. In this way, they made sense of their roles as MTEs as supporting PTs to maintain a long-term vision of their learning.

Discussion

Researchers have argued that one way to support PTs to deep mathematical knowledge is to engage them in learning mathematics through problem solving (Masingila et al., 2011; Schroeder & Lester, 1989). We propose that engaging prospective MTEs in supporting PTs to develop this mathematical knowledge through problem solving is also a problem-solving activity for the MTEs. The prospective MTEs are engaged in problem solving as they learn to support PTs' mathematical development in learning what type of questions to ask, how to support PTs in productive struggle, how to facilitate a class discussion to bring out mathematical ideas that arise through the PTs' problem solving, etc. Thus, the MTEs are also learning through problem solving and their learning is about supporting PTs' mathematical and MKT development while the MTEs themselves develop MKTT.

Van Zoest and colleagues (2006) in their study with prospective MTEs in mentored clinical experiences found that the novice MTEs learned the most through "observing, analyzing, and discussing classroom interactions" (p. 143). Our analysis aligned with this claim as we found that our code that occurred the most frequently in our coding of our data was what we called "observing student learning", in which our participants mentioned notable instances of individual PT or group learning that provided new insights for the intern about what the PTs were learning, their process of learning, their dispositions toward the problem solving work, and/or connections that the PTs were making among mathematical ideas. Our participants wrote about their observations of the PTs' learning in their weekly reflections, they talked with other members of the CoP about their observations and mentioned these in the interviews. Their observations were often brought up, analyzed and discussed during the CoP's twice weekly meetings. We found that observing and reflecting on the PTs' learning in the context of a mathematics course taught through problem solving was a key way that the prospective MTEs learned.

Van Zoest and colleagues (2006) recommended having key readings for the prospective MTEs to support their learning. We found that the weekly readings connected the interns to literature on developing mathematical understandings and teaching and learning through problem solving. The readings provided a grounding from which the interns could situate their observations of student learning as well as the pedagogical strategies that they observed and were trying to enact themselves. Since the readings were shared texts in the CoP, ideas from the readings became part of the shared language used by members of the CoP to discuss challenges and developments in students' understandings and in instructional strategies.

In the same way that "prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach" (CBMS, 2001, p. 7), MTEs need opportunities that will enable them to develop a deep understanding of the mathematics that they will teach to PTs and support them in understanding mathematical ideas deeply. As we envision the future for mathematics education in uncertain times, we argue that one site for prospective MTEs to gain this knowledge and pedagogical skills is through a CoP with experienced and novice MTEs working with PTs in which the novice MTEs have an active role in supporting the PTs. The prospective MTEs develop their MKTT in collaboration with the CoP as the PTs develop their MKT. The prospective MTEs learn how to facilitate (a) student thinking with prompting questions, (b) collaboration as a means of active engagement, (c) PTs' problem solving efforts, and to value multiple approaches to solving problems. While it is possible for prospective MTEs to reflect on their teaching alone by observing student learning and examining their own practice, without a CoP and being an active member in the mathematics content course taught through problem solving, there would have been no opportunity to reflect on the actions of others, to receive feedback on their observations and reflections and teaching, or to see other ways of approaching a teaching or learning challenge.

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EXPLORING HOW SCHOOL LEADERS CARE FOR TEACHERS OF MATHEMATICS AMID THE PANDEMIC

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Teachers' emotions and identities are intertwined (Zembylas, 2003), and have been disrupted by the isolation and stress of the COVID-19 pandemic (Jones & Kessler, 2020). Although some teachers have exhibited resilience during the pandemic, a majority have felt their well-being decreased (Gearhart et al., 2023). Workload stressors including school climate increased stress (Herman et al., 2021; Johnson & Coleman, 2023). Self-care strategies have done little to reduce these impacts (Walter & Fox, 2021). However, relationships with colleagues can be buffers (Blair et al., 2023; Johnson & Coleman, 2023; Stang-Rabrig et al., 2022), highlighting the potential positive impact of a caring coach or leader.

"Nel Noddings is closely identified with the promotion of the ethics of care, – the argument that caring should be a foundation for ethical decision-making" (Smith, 2020, para. 4). Ethical caring ala Noddings is relational at its core; the "carer" turns their attention and energy toward the "cared for." Noddings' (1984, 2013) ethical concept of care informed our analysis.

For this longitudinal study, we conducted 24 interviews with principals, instructional coaches, and teachers in a Title I, urban school district to explore how school leaders cared for math teachers amid the pandemic. Transcriptions were analyzed using Noddings' (1984, 2013) theory of care ethics. Intercoder reliability was maintained during coding (Cofie et al., 2022).

We identified three kinds of care that principals and coaches leveraged to show support for math teachers. Specifically, principals and coaches: (a) implemented new and recycled structures, (b) employed discursive practices, and (c) acted as buffers between district-level mandates and teachers. We also identified challenges that the principals and coaches faced while striving to enact care for teachers, which chiefly consisted of structural constraints, including a lack of time to physically be present with and support teachers. To illustrate, leadership teams at one elementary school implemented a "check in, check out" structure to respond to teachers' needs, as described by Principal Francisco: "We created a check in, check out. Like, if a teacher needed a restroom break, or they just needed a break, there was someone that would go and relieve them." To provide one further illustration, coaches and principals also exercised discursive moves akin to cheerleading. For example, Coach Theresa found opportunities to "give feedback and build that confidence that they were so in desperate need of." Last, coaches also served as buffers between teachers and district-level mandates to preserve teachers' sanity in the face of new initiatives or assessments. Coach Kristin shared: "I felt like I was a naggy voice for teachers…[school leaders had] expectations that I just knew weren't realistic."

The coaching and leadership practices here to support math teachers elucidate how leaders and teacher educators can serve as buffers and advocates for teachers. Further research is needed to elucidate how teachers' social and emotional well-being can be supported by school leaders.

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UNDERSTANDING THE DISCUSSION OF LEARNING STYLES IN MATHEMATICS EDUCATION LEADERSHIP: AN ANALYSIS OF NCTM PUBLICATIONS

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The prevalent belief in education is that individuals possess a distinct "learning style," whether visual, kinesthetic, or auditory, with a significant percentage of educators, up to 80-95%, subscribing to this notion (Nanceckivell et al., 2020). Despite its widespread acceptance, decades of research have failed to provide substantial evidence supporting the idea that tailoring instruction to match an individual's purported learning style enhances their learning outcomes (Pashler et al., 2008). The myth of learning styles has been “busted,” but remains prevalent in educational research (Authors, 2022). However, little is known about how widespread the discussion of learning styles has been within mathematics education. As the world’s largest mathematics education organization with publications since 1907, the National Council of Teachers of Mathematics (NCTM) is very influential in mathematics education (NTCM, n.d.). As Dickey (2020) states, “the legacy of the Council is found in its journals” (p. 82). Given the importance and impact of NCTM’s publications throughout the decades, they are an ideal place to understand how both mathematics education and learning styles have been discussed.

In this study using content analysis, we asked, *what are the characteristics of articles published by NCTM that mentioned “learning styles?”* We conducted searches for “learning style*” in all NCTM journal publications, specifically, The Mathematics Teacher (TMT, 1908-2018), The Arithmetic Teacher (TAT, 1954-1994), Teaching Children Mathematics (TCM, 1994-2019), Mathematics Teaching in the Middle School (MTMS, 1994-2019), Journal for Research in Mathematics Education (JRME, 1970-present), Mathematics Teacher Educator (MTE, 2012-present), and Mathematics Teacher: Learning and Teaching PK-12 (MTLT, 2020-present). We identified a total of 331 articles that met our criteria, with 92 articles in TAT (28.7%), 64 articles in TMT (19.3%), 69 articles in TCM (20.8%), 71 articles in MTMS (21.5%), 29 articles in JRME (8.8%), 1article in MTE (0.3%), and 2 articles in and MTLT (0.6%). Identified articles were published between 1962 and 2023, with 111 of the 331 (33.5%) articles published in the 1990’s. Two hundred and two (61%) articles were full articles about lessons, teaching practices, and related topics, 62 (18%) were book reviews, seven (2%) were reader letters, 29 (9%) were advertisements for professional development, 10 (3%) were author biographies, 14 (4%) were calls for manuscripts, and 7 (2%) were other types of articles. Findings highlight how the prevalence of mentions of learning styles spans decades in NCTMs’ journals, across various grades, populations, and positions of mathematics education leadership. As NCTM publications serve as the leading place for mathematics teachers and mathematics teacher leaders to learn and share ideas, the impact of these references may have been significant in endorsing the consideration of learning styles in mathematics teaching for decades.

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ALIGNMENT AND MISALIGNMENT IN TEACHER-COACH INTERACTIONS: APPLYING THE INSTRUCTIONAL TRIANGLE TO MATHEMATICS COACHING

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Coaching is increasingly seen as a critical component of teacher professional learning programs. However, questions persist about approaches to coaching in mathematics that might better foster teacher buy-in and support scalability. Our study adapted the instructional triangle (Cohen et al., 2003) to the context of math coaching to investigate teacher-coach interactions across nine different school groupings in the context of a school-mandated curriculum-embedded professional learning program. Our findings showed that coaches often described teacher mindsets as a barrier to their work, while teachers focused on their need for content expertise. We also addressed differences between school-based versus district-based coaching approaches, and generalist versus math-specific coaching approaches, suggesting implications for the design and use of math coaching as a lever for instructional reform.

Keywords: Professional Development, Mathematical Knowledge for Teaching, Curriculum, Research Methods

While coaching is broadly defined as a critical component of effective teacher professional learning (PL; Desimone & Pak, 2017; Darling-Hammond et al., 2017), mathematics coaching has not received the same attention in the research compared to coaching in other domains such as literacy (Kraft & Blazar, 2017). Indeed, the impact of mathematics coaches in supporting PL programs within schools is often hidden in research on teacher PL (Hjalmarson and Baker, 2020). This is in part because coaching is highly interactional and adaptive depending on the coach training, school context, and willingness of the teacher participants (Coburn & Russell, 2008; Russell et al., 2020).

This study adapted the instructional triangle (Cohen et al., 2003) to the context of mathematics coaching to unpack these interactions between teachers and coaches, and to indicate how the structure and approach to coaching relates to the perceived benefit of mathematics coaching, according to both teacher *and* coach participants. The ability of math teachers and coaches to mutually and productively adapt their shared work is a critical component in the scalability of coaching models (Russell et al., 2020). As such, this study used teacher and coach interviews in the context of a curriculum-embedded PL program to describe the shared – or divergent – perspectives of teacher-coach groupings. Given that participants' interpretations of policy ultimately inform their behavior in such systems (Desimone, 2002), our findings can be informative for those interested in using math coaching to help support instructional reform.

With this context in mind, we consider the following research questions:

1. In what ways do mathematics teacher and coach descriptions of their shared coaching work align or diverge?
2. What aspects of the coaching environment do teachers and/or coaches describe as beneficial or detrimental to improving teachers' instructional practices?

Theoretical Framework

We leveraged the instructional triangle (Cohen et al., 2003), especially iterations of the instructional triangle as it relates to the work of mathematics teacher educators or coaches (Nipper & Sztajn, 2008; Shaughnessy et al., 2016), to investigate the interactional nature of the teacher-coach experience. The instructional triangle describes teaching (and in our context, mathematics coaching) as "a collection of practices, including pedagogy, learning, instructional design, and managing organization" (Cohen et al., 2003, p. 124) that is inherently influenced by external environmental factors. Importantly, Cohen and colleagues stress that improving teacher learning outcomes depends not on the mere *inclusion* of educational resources, but rather the ways that individuals (e.g., teachers and coaches) are able to work with one another and *use these resources*. Our adaptation of this framework is shown below in Figure 1, and described in our review of the literature that follows.

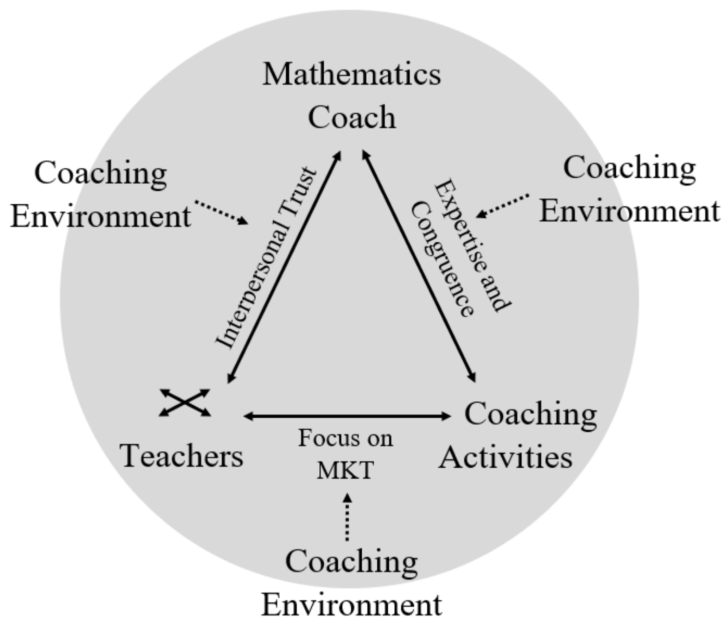

Figure 1: The mathematics coaching triangle

Mutual Adaptation: Alignment Between the Teacher and the Coach

Before examining the various interactions that are mapped onto our mathematics coaching triangle, it is important to note that this framework recognizes the importance of both teachers and coaches as actors within a system aiming for instructional reform. Successful educational reform projects have long been characterized as adhering to mutual adaptation (McLaughlin, 1976), a process “in which project goals and methods [are] modified to suit the needs and interests of participants and in which participants [change] to meet the requirements of the project” (p. 172). More recently, Russell et al. (2020) identified that such adaptations may support the scalability of mathematics coaching programs in particular, so long as coaches are able to adapt to teachers’ perceived buy-in to the coaching without violating the integrity of the coaching model. By considering how both teachers and coaches perceive of their shared work, our approach can indicate conditions where productive mutual adaptation has occurred, or conditions under which teacher and coach interactions have led to “lethal mutations” (Russell et al., 2020, p. 176) of the coaching model.

Teacher and Coach Interactions with Coaching Activities

Coaching activities in mathematics can vary considerably, from emphasizing an understanding of mathematics content itself to strengthening teachers’ understanding of their classroom interactions with students (Nipper & Sztajn, 2008). Activities that attend to both teachers’ mathematics subject matter and pedagogical content knowledge – which together is referred to as mathematical knowledge for teaching (MKT; Ball et al., 2008) – are of particular interest to our work. Teacher PL programs that explicitly focus on MKT have been found to improve teachers’ MKT for both pre-service teachers (Morris & Hiebert, 2017) and in-service teachers (Jacob et al., 2017), although findings connecting such programs to student achievement have been mixed. Importantly, Jacob and colleagues (2017) described decreasing district leadership support for the MKT-focused professional learning program over time as a probable reason for the program’s limited effects on student achievement. Such findings bolster the idea that the effectiveness of teachers’ interactions with their coaching activities are influenced (or constrained) by environmental factors of school and district policy and institutional support, discussed further below.

The nature and quality of a coaches’ own training also influences how they interact with teachers. When the training coaches receive around supporting teacher implementation of new curriculum materials includes active learning, this can enable the coaches to interact more deeply with teachers by attending to MKT (Coburn & Russell, 2008); however, if coaches’ training is of lower quality, their coaching may in turn be incongruent with the goals of district curriculum, and they may pass on this incongruence to teachers. Similarly, teachers whose coaches hold more expertise in attending to MKT are more likely to develop their own MKT expertise (Sun et al., 2014). Such research shows the importance of both the coaches’ own MKT and their congruence with the goals of instructional reform.

Teacher and Coach Interpersonal Interactions

In addition to activities related to MKT and curriculum, aspects of the teacher-coach interpersonal relationship can also influence teachers’ perceptions of their professional learning (Smith & Desimone, 2023) and influence the depth of their engagement with their learning (Coburn & Russell, 2008). Hence, it is perhaps no surprise that the relational aspect has long been considered an important element of effective coaching practices across multiple disciplines (Blazar, 2020; Desimone & Pak, 2017; Ippolito, 2010). Teachers’ social networks with one another (shown in Figure 1 above) also influence engagement in their learning (Coburn & Russell, 2008), though this is outside the scope of our study.

The Coaching Environment

As indicated in Figure 1, the above interactions between teachers and coaches do not occur in isolation; rather, in line with the original instructional triangle (Cohen et al., 2003), our framework presumes that external, environmental factors are impacting the interactions between teachers, coaches, and their shared activities. Two salient factors within the coaching environment that our present study considered were (1) whether the coach is placed full time at a school (i.e., "school-based"), which has been shown to correlate more highly with student achievement compared to when a coach is only part-time and/or has to spread their time across multiple schools (i.e., "district-based"; Harbour & Saclarides, 2020), and (2) whether the coaching role is designed as a generalist or math-specific position. Given the importance of attending to MKT noted above, the expertise and focus of the coach regarding the teaching of mathematics was conjectured to influence the nature of the coaching interactions.

Methods and Methodology

Our data are drawn from the Research on Curricular Alignment Partnerships (R-CAP) project. R-CAP was a multi-year study supported by the Bill & Melinda Gates Foundation, with the overarching aim of understanding how professional learning partnerships (PLPs) – or collaborative partnerships between school districts, PL providers, and curriculum developers – could foster ambitious instruction and culturally responsive (CR) instruction through curriculum-embedded PL (i.e., specific curriculum materials are integrated throughout the ongoing PL activities; Taylor et al., 2015), particularly for minoritized students.

Participants and Protocols

Our teachers (n = 18) and coaches (n = 9) were recruited from participants (n = 479) across six different school districts participating with a PLP as part of the R-CAP project, drawn to reflect teachers with a variety of perceptions about their coaching and school-mandated PL. Because of the participating PLP's focus on ambitious instruction and culturally responsive instruction, we selected teachers based either on their reported normative authority (or buy-in) to their professional learning (Desimone, 2002) or their reported frequency of CR instruction. This allowed for a sample of teacher participants with a variety of experiences with this school-mandated professional learning. This sampling approach is further addressed in Comstock et al., 2022.

Through this approach, we were able to draw from a diverse sampling of teachers from participating PLPs, including those with low normative authority (n = 4; pseudonyms Mrs. Menton, Mr. Brannon, Mrs. Cicero, and Mrs. Wendell), high normative authority (n = 3; pseudonyms Mrs. Leak, Mrs. Dratch, and Mrs. Seale), low frequency of CR instruction (n = 3; pseudonyms Mr. Mendes, Mr. Carrigan, and Mrs. Hembrow), medium frequency of CR instruction (n = 5; pseudonyms Mrs. Klein, Mrs. Muhr, Mr. Biscay, Mrs. Eccles, and Mrs. Gibbon), and high frequency of CR instruction (n = 3; pseudonyms Mrs. Giddings, Mrs. Sturman, and Mr. Hickson). With these teachers selected, we then recruited coaches from the same participating PLPs who had indicated working at these teachers' schools.

Our interviews were designed to elicit responses from both teachers and coaches about the nature of the coaching work, the relationship between teachers and coaches, and facilitators and barriers to implementation. For instance, teachers were asked "How would you describe the quality of your professional relationship with your coach?" while coaches were asked "How would you describe the quality of your professional relationship with your teachers?"

Analysis

We used an embedded case study design (Yin, 2018) to analyze the interview responses from our nine teacher-coach grouping (i.e., for each coach, we interviewed 1-3 teachers that they worked with). In alignment with the mathematics coaching triangle, we coded interview responses to identify descriptions of (1) interactions with different coaching activities identified by the teachers or coaches (2) interactions related to teacher-coach interpersonal trust, (3) references to coach expertise/congruence or the coaching focus on MKT, and (4) environmental factors outside of direct teacher-coach interactions that participants indicated had impacted the coaching work. We then also coded whether these descriptions were described as (1) beneficial or detrimental to teachers' learning and instruction, (2) as a mixture of beneficial and detrimental, or (3) neutral or unclear in relation to the benefit of the activity or aspect. To determine these categories, we looked for references to whether the benefit of an action interaction was described directly (e.g., "that was helpful" or "That doesn't work for my students"), or indirectly (e.g., "I think that my strength is just my emotional intelligence, being able to relate to my teachers and being able to be a trusted advisor for them" or "We just can't get through it all - it's just too much").

Each interview was double coded by members of the research team, identifying coaching activities, teacher-coach interactions, and environmental factors from the participant descriptions. After each teacher-coach grouping was coded by one member, another member would code the same grouping and note any discrepancies in the initial codings. The researcher members would then meet to resolve any disagreements. After this initial round of coding, research members wrote analytic memos (Saldaña, 2013) for each school-based teacher-coach grouping, describing the ways that the teacher(s) and coach aligned or diverged in their descriptions, and the relevant themes across the different participants in each school grouping.

Results

RQ1: Alignment and Divergence of Teacher and Coach Descriptions of Coaching Work

Our first research question focused on how teachers and coaches aligned or diverged in their descriptions of their shared coaching work. We found that teachers and coaches largely agreed about the focus of their coaching work. However, teachers held reservations about the perceived MKT of their coaches, while coaches held concerns about their teachers' mindsets around their learning and instruction. Table 1 and Table 2 show each activity, interaction, and environmental factor described by both teachers and coaches in each teacher-coach grouping, and their perceived benefit (or detriment).

Table 1: Coach-Teacher Groupings with Similar Perceptions

Teacher-Coach Grouping	Similar coaching activity (teacher and coach description)	Similar interactions & environmental factors (teacher & coach description)
Coach: Mrs. Line Teacher: Mrs. Cicero	Assessment (+)	Interpersonal Trust (+)
Coach: Mrs. Koenig Teacher: Mrs. Gibbon	Observe/Debrief (+)	Interpersonal trust (+) MKT focus (+) Coach expertise (+)
Coach: Mrs. Mattingly Teachers: Mr. Biscay, Mrs. Seale, Mrs. Hembrow, Mrs. Sturman	Observe/Debrief (+) Lesson design (+) Co-teaching (+)	Interpersonal trust (+) MKT focus (+) School Policy (#)
Coach: Mrs. Eccles Teachers: Mr. Hickson, Mrs. Tabor	Observe/Debrief (+)	+Interpersonal trust +Coach flexibility

Note. + = Described as beneficial. - = Described as detrimental. # = Described as both beneficial and detrimental. ? = Not described as beneficial or detrimental.

Table 2: Coach-Teacher Groupings with Differing Perceptions

Coach-Teacher Grouping	Similar coaching activity (description by coach, by teachers)	Similar interactions & environmental factors (description by coach, by teachers)
Coach: Mrs. Mallinson* Teachers: Mrs. Giddings, Mrs. Muhr, Mr. Carrigan	Assessment (+, ?)	Coach expertise (–, –)
Mrs. Lees (Coach)** Teachers: Mr. Mendes, Mrs. Klein	Observe/Debrief (+, #)	Interpersonal trust (#, #)
Mrs. Nowak (Coach)** Teacher: Mr. Brannon	Lesson design (+, #)	Interpersonal trust (+, +) MKT focus (+, -)
Mrs. Lyndon (Coach)* Teacher: Mrs. Wendell	Lesson design (+, #)	MKT focus (+, -) School policy (+, -)
Mr. Aras (Coach)** Teacher: Mrs. Fieser, Mrs. Menten, Mrs. Leak, Mrs. Dratch	No similar coaching activities described by both coach and teachers	Interpersonal trust (+, +)

Note. + = Described as beneficial. - = Described as detrimental. # = Described as both beneficial and detrimental. ? = Not described as beneficial or detrimental. * = Generalist (cross-content) coach. ** = District-based coach.

Broadly, we found that teachers and coaches described similar sorts of activities. However, while coaches in each grouping described their activities as supporting the goals of the curriculum-embedded coaching work, there was a split in how teachers perceived these activities. Just under half of the teacher-coach groupings were in agreement about the benefit of these activities, while in the rest of the groupings teachers described both beneficial and detrimental aspects of the activities.

For groupings that aligned regarding the benefit of the coaching activities, teachers and coaches also frequently aligned in describing positive aspects of interpersonal trust and (in two of the cases) the MKT focus of the activities. For groupings that diverged in their descriptions of the coaching work, some trends arose regarding how teachers and coaches described their interactions with one another and with the activities themselves. For instance, when discussing lesson design and planning, Coach Lyndon stated, “I can get in there and give you ideas based on what I'm seeing, based on how the curriculum should be presented” and that “being able to give them my first-hand experience, I think has helped them.” Yet, Mrs. Wendell, a teacher working with Coach Lyndon, described how Coach Lyndon “doesn't have a specialty in one particular area. She's a coach for the whole school.” Despite Coach Lyndon’s belief that she was facilitating lesson planning activities that were beneficial to teachers, Teacher Wendell’s perceptions of the usefulness of those activities appeared to be influenced by Coach Lyndon’s lack of teaching experience or specialization with mathematics teaching (i.e., a lack of MKT). Notably, every teacher-coach grouping where there was disagreement about the benefit of the coaching activities were also school sites with district-based or generalist coaches.

While concerns with coach expertise or the ways that coaching activities supported their MKT were chief among teachers, another trend arose distinctly among coaches. All but one coach (Mr. Aras) described the issue of teacher mindsets as an important aspect of their work. For instance, Coach Koenig described shifting teachers’ mindsets as part of the “biggest barrier” in her coaching and discussed how the teachers she worked with were “not really believing that we can do certain things.” Similarly, Coach Mallinson described how “some of our veteran teachers are just kind of reluctant to try new ways and new things.” The issue of teacher mindsets was mentioned by only one teacher, Mrs. Cicero, who also happened to have a stated goal of becoming a math coach herself. In sum, coaches described teacher mindsets about math instruction to be a major barrier to instructional change, while teachers described a need for specific content and teaching expertise as being the major barrier to their success.

RQ2: Perceived Beneficial or Detrimental Aspects of the Coaching Environment

As noted above, perceptions about the MKT focus or coaching expertise appeared to be a major divergence between the teacher and coach experience. For our second research question about what might be driving alignment or divergence of such experiences, we found that the design of the coaching role – specifically whether the coach was school-based and whether the role was math-specific – related to teachers’ perceived benefit of their coaching.

A common theme that emerged from both coaches and teachers was the importance of the coach in fostering teachers’ MKT which, in turn, was related to the coach’s own MKT. Teacher comments about the MKT focus of their coaching activities often also extended into descriptions of their coach’s perceived expertise in supporting MKT. Teachers with generalist coaches repeatedly described this aspect of the coaching to be inadequate and stifling to their desire to strengthen their own MKT. Indeed, of the four instances where participants discussed the perceived content expertise of their coaches in negative terms, three of these came from teachers with a generalist coach (Mr. Carrigan, Mrs. Muhr, and Mrs. Wendell), while the fourth came from a generalist coach themselves (Mrs. Mallison).

There were markedly different interviews given by teachers with school-based coaches compared to those with district-based coaches who supported multiple schools. Teachers working with district-based coaches expressed frustration at the lack of connection between their teaching context and their coaching interactions. For example, Mrs. Menten described how her coach (Mr. Aras) was "knowledgeable on the curriculum, but he's not knowledgeable on students with special needs." Similarly, Teacher Brannon described how his coaching activities (run by Coach Nowak) "[weren't] what I needed...I would have rather spent time with other seventh and eighth grade math teachers who were specifically [focusing on] the special ed population." These teachers specifically did not see the work with their district-based coach as aligned to the challenges that they felt around working with their students with special needs. On the other hand, teachers who worked with school-based coaches generally described the interactions with their coach as beneficial. Indeed, every school grouping that had teachers and coaches describe similar activities and in similar (positive) terms involved school-based coaches.

Limitations

This analysis should be interpreted in the context of several limitations. First, it was conducted in the context of a district-mandated curriculum-embedded PL program. While we believe that our findings likely extend beyond this particular context, future studies should consider the role of particular policies and reforms. Second, this study occurred in large, urban districts serving predominantly students from minoritized groups. The resources, visions, and challenges of these particular districts may influence our findings about coaches and teachers.

Discussion

Grounded in our adaptation of the instructional triangle to mathematics coaching, we explored teacher-coach alignment in descriptions of their interactions, and how key aspects of the coaching environment were perceived as beneficial or detrimental to teachers in improving their instruction. Our findings showed that teachers and coaches generally described engaging in similar types of activities. However, coaches tended to consider their activities to be beneficial – even when their teachers did not. Because productive mutual adaptation through content-focused inquiry is a promising feature of scalable coaching models (Russell et al., 2020), the teacher-coach groupings that varied in alignment offer clues about how teacher-coach interactions can support such opportunities for mutual adaptation.

A disconnect between focusing the coaching work on *teaching* versus focusing it on *teachers* seemed to be a driving cause of discontent between coaches and their teachers in certain school grouping. This sort of disconnect is not new (see Hiebert & Morris, 2012), but the lens of the mathematics coaching triangle may help to highlight the delicate balance of work required to achieve coaching that is well-received by coaches and teachers alike, and is thus positioned to support productive, mutual adaptation between coaches and their teachers. While almost all coaches described the challenge of shifting teacher mindsets as part of their work, coaches with teachers who found the coaching work beneficial were also those that seemed to foster positive interactions around interpersonal trust with their teachers and around MKT in their activities.

Given that teachers who found the coaching work detrimental rooted such problems in the lack of MKT focus with their coaching, this could indicate an opportunity for more aligned and beneficial coaching. Regardless of whether coaches feel that teachers hold mindset barriers toward instructional change, if they foster strong relationships with their teachers and focus on strengthening teachers' MKT, they may be able to achieve better buy-in from their teachers.

Finally, our findings indicated concerns with more generalist of district-based coaching approaches. Coaches who were hired to support multiple contents were not perceived by their teachers as knowledgeable enough to successfully support teachers' MKT. This indicated how district supported professional learning that carefully attends to aspects of MKT is not only a compelling model for impacting teacher practices and student learning outcomes (Jacob et al., 2017; Morris & Hiebert, 2017), but may also be a model that teachers themselves desire. School-based coaches were also perceived as more beneficial than district-based coaches. Previous studies have indicated that school-based coaches may be better positioned to foster stronger relationships with teacher may also be burdened with more administrative tasks by school leaders that take away time for coaching (Kane & Rosenquist, 2019). Our findings indicate that, at least from the perspective of teachers, school-based coaches were able to foster strong interpersonal relationships *and* facilitate beneficial learning activities. Therefore, while administrative tasks may have still arisen in these cases, they did not appear to prevent buy-in from the teachers regarding the benefit of the coaching work.

Because of the importance of these teacher-coach interactions and environmental factors in our findings, the coaching triangle may be a helpful lens for investigating the complexity of math coaching. By attending to interactions between teachers, coaches, and their shared activities, our hope is that future investigations on PL programs can give further voice to the experiences of both teachers and coaches in defining the effectiveness of such interventions.

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Boundary Spanning to Cultivate Family Connection: Perspectives and Recommendations of Academic Mothers in Mathematics Education

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Objectives and Purpose

Beswick (2021) describes the importance of mathematics teacher educator (MTE) voice generated from reflection on their learning and development as MTEs. Such a voice provides opportunities for colleagues to "enter the experience" (p. 422) of other MTEs. Drawing from *matricentric feminism* (O'Reilly, 2019; 2021) to frame our epistemological beliefs as mothers and motherscholars (Matias & Nishi, 2018), we understand that we bring our lived experiences as academic mothers into our work as MTEs. Further, a community of practice consisting of MTEs who are mothers can provide emotional and intellectual support that academic mothers desperately need to survive and thrive in academia (e.g., O' Brien Hallstein & O'Reilly, 2012). Our identification as academic mothers in mathematics education (AM-ME) positions us to identify a set of tacit knowledge that comes from the interconnectedness of these two roles. In this poster, we present findings from our research on *the knowledge we gain from our experiences as mothers (who are also MTEs and former teachers)* and *how we use this knowledge as MTEs to leverage pathways for Pre-service Teachers (PSTs) to disrupt the status quo when working with parents, caregivers, and the community in mathematics education*.

Methods

Using collaborative autoethnography (CAE) (Chang et al., 2012), we documented our lived experiences and the ways in which we leveraged our mother experiences in our MTE work. Our team engaged in narrative interviews (Jovchelovitch & Bauer, 2000) in which each member shared their experiences leading up to becoming both an MTE and mother, as well as the experiences when these two identities intersected. Following these interviews, we generated individual reflections around the ways that our mother roles had informed our MTE work in the areas of teaching, research, and service. These data points were used to construct narratives naming the challenges in our mother experiences that produced knowledge used to inform our practices as MTEs working with teacher candidates in methods coursework.

Findings and Discussion

Our experiences as mothers of school-aged children have heightened our awareness of the power and participation dynamics that exist between families and schools. These dynamics are largely rooted in home-school communication structures, such as conferences and communication home to families in relation to student progress, standards, assessments, or instructional practices; however, we note that they also connect to the use of family experience and expertise as a tool to support curricular design. Through our CAE, we unearthed a wealth of knowledge we developed over the years through parenting and interacting with our school-aged children and their teachers and utilized this knowledge to improve our instructional practice to better prepare mathematics teachers to build bridges with families. We hope to share these ongoing reflections and how they impact our work in preparing learning experiences for teacher candidates in methods coursework that center on cultivating reciprocal family relationships.

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Chapter 9:

Precalculus, Calculus, and Higher Mathematics

IMPLICIT DIFFERENTIATION IN A COLLEGE CALCULUS CLASSROOM: INTERACTIONS BETWEEN STUDENTS AND LEARNING ASSISTANTS

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Learning Assistants (LAs) are undergraduate students who, having successfully completed a certain course, return to classrooms in a different role – as near-peer tutors who help facilitate their peers' learning in that course. While students in courses with LAs have repeatedly evidenced more positive content-related learning outcomes compared to students in courses without LAs, little is known about how LAs facilitate these positive outcomes. This study explores LA-student interactions around the topic of implicit differentiation, in a university Calculus I course. Findings suggest that LAs help to demystify this topic for students by connecting to previous course material and focusing on computational correctness. These findings help to advance our understanding of the mechanisms of how LAs support student learning of implicit differentiation.

Keywords: Learning Assistants, Calculus, Classroom Discourse, Undergraduate Education

Introductory-level university courses, such as Calculus, aim to equip students with a robust foundation for future careers in science, technology, engineering, and mathematics (STEM) fields. However, Calculus courses were shown to have a high drop, withdrawal, failure rates, and student dissatisfaction with their learning experiences (Bryk & Treisman, 2010). Research suggests that the transformation of these courses (i.e., lowering failure rates and increasing student satisfaction) can be attained through *active learning* strategies such as studio course designs and group problem-solving (Freeman et al., 2014).

One means of facilitating these active learning approaches is through the incorporation of *near-peer tutors* (i.e., undergraduate students who have previously been successful in the course) into course instruction. While near-peer tutors can be incorporated into courses in several ways (Adreanoff, 2016; Otero et al., 2010), the *Learning Assistant* (LA) model of near-peer tutoring has been shown to be particularly effective in supporting active learning pedagogies (Knight et al., 2015). Under this model, near-peer tutors, called learning assistants (LAs) synchronously aid course instruction, typically by facilitating small group interaction around course content. LAs also practice in weekly meetings with course instructors to review the content they will be teaching and in a pedagogy course to learn about and reflect on teaching practice.

There is a growing body of research showing a variety of positive outcomes for students in LA-supported STEM courses, specifically, in Calculus. These include affective outcomes such as more positive attitudes toward mathematics (Castillo et al., 2022), and content-related outcomes such as lower drop, withdrawal, and failure rates (Alzen et al., 2018) and improved course grades (Bullock et al., 2015).

While these positive learning outcomes have been documented, there is little understanding of *how* LAs help facilitate them. Some attribute this to LAs' temporal proximity to the course content due to their recent completion of the course, and/or to their social proximity to student experiences (Alzen et al., 2018; Hernandez et al., 2021). These assumptions are based on research on LAs' classroom actions, such as providing students with feedback and increasing discussion time. However, these action descriptions are not subject-specific, leaving open a question of how LAs facilitate students' leaning of a particular course content.

This study seeks to examine how LAs facilitate student learning of specific mathematical content—implicit differentiation—by analyzing LA-Student interactions within a collegiate Calculus I course. Implicit differentiation, detailed in the following sections, was chosen as the content area since it has been identified as an under researched topic with which calculus students commonly struggle (Martin, 2000). The mathematical challenges characteristic of this topic offer opportunities for LAs to authentically aid student learning, which constitutes a rich research environment. This study explores the following research questions:

1. What mathematical aspects of implicit differentiation do students and LAs discuss during their interactions in Calculus I recitations?
2. What discursive moves accompany the mathematical aspects of implicit differentiation discussed by LAs and students during their interactions in Calculus I recitations?

Literature on Implicit Differentiation

Implicit differentiation is a technique for finding derivatives of equations where y cannot be explicitly expressed as a function of x (e.g., $xy + y^2 = 3yx^2$; graphs of such equations do not pass the vertical line test). Despite the importance of implicit differentiation in differential Calculus and its connections to other key topics such as chain rule and related rates, it received little research attention (Speer & Kung, 2016). Only recently have researchers begun exploring this topic, uncovering many challenges students have with applying implicit differentiation and understating its meaning. Mirin and Zazkis (2019) point to the inherent difficulty of recognizing implicit equations and making sense of applying differentiation to both sides of such an equation. Even when students recognize the need for implicit differentiation, they may struggle to apply prerequisite skills from algebra (e.g., simplifying exponential or radical expressions) and calculus (e.g. differentiation rules) to solve such problems (Borji & Martínez-Planell, 2020; Kandeel, 2021).

Researchers have also begun exploring ways to support student learning of this topic. Borji and Martínez-Planell (2020) studied how students' understanding of implicit curves and implicit differentiation changed following a series of interventions designed within the Action-Process-Object (APOS) theory. Another intervention, which combined the concepts of the chain rule, implicit differentiation, and related rates was designed and tested by Jeppson (2019). Both intervention studies developed highly detailed conjectures about of the learning stages students progress through when learning about implicit differentiation (called *genetic decompositions* in APOS theory, or *hypothetical learning trajectories* (Simon, 1995)). Buchbinder and Allen (in press) further adapted the elements of both learning stages trajectories— by Jeppson (2019) and Borji and Martínez-Planell (2020)—into a framework used in this study (as described below).

The literature points to the growing interest in the mathematics education community in describing and supporting student learning of implicit differentiation. This study aims to extend the description of students' difficulties with this topic by considering how students address and resolve these difficulties in an authentic classroom environment - to the best of our knowledge, to date, there have been no studies that examined this topic in the authentic classroom context.

Also, since LAs have a unique near-peer status with students, the conversations between LAs and students around implicit differentiation may shed new light on student learning of this topic.

Theoretical Framing

We utilize Vygotsky's (1978) sociocultural theory to study LAs and students' interactions around implicit differentiation. Under this theory, learning is a social process, mediated through language, and through interactions with individuals who have more knowledge than the learner. These More Knowledgeable Others (MKOs) facilitate the individual's learning by guiding them through the Zone of Proximal Development (ZPD) – a set of concepts, ideas, and skills outside of the individual's reach, which can only be attained with the help of others. This study frames LAs as MKOs who facilitate students' movement through the ZPD.

In addition to the general emphasis on language as mediating the learning process, we rely on the tools of Critical Discourse Analysis (CDA) to tie together deep linguistic analyses of LAs and students' discourse with the unique sociocultural features of collegiate mathematics classrooms in which the language is used. CDA has been successfully used in mathematics education research (McNeill et al., 2022) as well as in studies on near-peer tutoring (Butler & Buchbinder, 2023; DiMaio, 2020).

We rely on Gee's (2014) formulation of CDA, according to which language fundamentally reflects the lived experiences of the speakers within particular contexts. Linguistic structures of the spoken language, like grammar and words used, reveal individuals' ways of "saying (informing), doing (action), and being (identity)" (p. 2) in the world. We operationalized Gee's first two components of language through a two-fold qualitative coding scheme (detailed below). One part of the coding scheme captures the specific mathematics students and LAs discuss (saying), and the other part describes the actions of these speakers (doing). Gee's notion of "being" is also essential to this study as the information and actions only make sense in the context of the participants' identities as LAs and students in a Calculus I course.

Methods

Setting

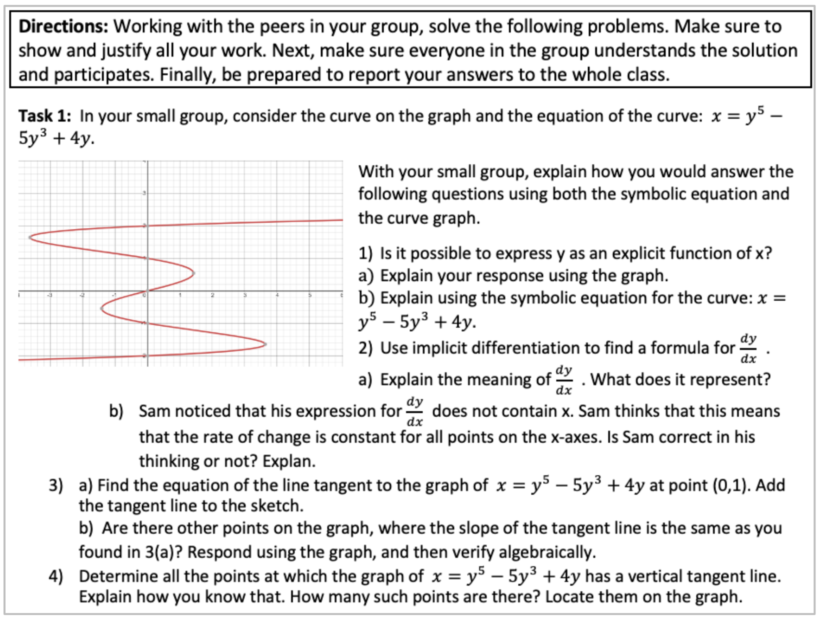
This study is a part of a larger NSF-funded project to transform introductory STEM courses at a large, research-intensive university in the northeast of the United States. In mathematics department, these transformation efforts focus on Calculus I recitations. This Calculus I course follows a lecture-recitation model, with students attending a large lecture (~160 students) taught by a faculty member three times a week and a smaller recitation session (~20 students) led by a graduate teaching assistant (GTA) twice a week. Transforming the recitations involved (a) the incorporation of researcher-designed, conceptually oriented activities focused on fostering metacognitive practices and representational fluency and (b) introducing LAs to facilitate small groups' discussions of problems in these activities.

Data Collection

This study focuses on the Implicit Differentiation activity (Figure 1), adapted from Boelkins et al. (2018). The activity aims to support recognition of the need for implicit differentiation and making connections between symbolic and graphical representations of implicit equations.

The data were collected over two semesters, in nine recitations taught by multiple GTAs, during the implementation of the Implicit Differentiation activity. The activity was enacted toward the end of the semester, so students were well familiar with LAs who had been working with them in recitations since the beginning of the course. In each recitation, one small group of three to four students (volunteers), was recorded using 360-degree table-top video cameras (Buchbinder, in press). Students in these small groups were mostly first- and second-year STEM majors, mostly white, and there was an approximately even split between male and female presenting students. LAs were mostly second- and third-year STEM majors, mostly female, and ranged in LA experience from one to three semesters.

From the total of approximately 7.5 hours of footage, we first identified video clips where the LA interacted with the recorded small group or one of its members. Some clips had to be excluded from the analysis due to low audio quality or if the content of the interaction was not mathematical (e.g., grades, office hours). From this process, a total of 25 minutes and 5 seconds of usable video data were identified for five LAs. semesters. The footage involved 22 separate LAs-student conversations containing 250 utterances. Each utterance was coded using the analytic framework described below.



Directions: Working with the peers in your group, solve the following problems. Make sure to show and justify all your work. Next, make sure everyone in the group understands the solution and participates. Finally, be prepared to report your answers to the whole class.

Task 1: In your small group, consider the curve on the graph and the equation of the curve: $x = y^5 - 5y^3 + 4y$.

With your small group, explain how you would answer the following questions using both the symbolic equation and the curve graph.

1) Is it possible to express y as an explicit function of x?
a) Explain your response using the graph.
b) Explain using the symbolic equation for the curve: $x = y^5 - 5y^3 + 4y$.
2) Use implicit differentiation to find a formula for $\frac{dy}{dx}$.
a) Explain the meaning of $\frac{dy}{dx}$. What does it represent?
b) Sam noticed that his expression for $\frac{dy}{dx}$ does not contain x. Sam thinks that this means that the rate of change is constant for all points on the x-axes. Is Sam correct in his thinking or not? Explan.
3) a) Find the equation of the line tangent to the graph of $x = y^5 - 5y^3 + 4y$ at point (0,1). Add the tangent line to the sketch.
b) Are there other points on the graph, where the slope of the tangent line is the same as you found in 3(a)? Respond using the graph, and then verify algebraically.
4) Determine all the points at which the graph of $x = y^5 - 5y^3 + 4y$ has a vertical tangent line. Explain how you know that. How many such points are there? Locate them on the graph.

Figure 1: Implicit Differentiation Activity

Analytic Framework

Transcripts of the selected clips were coded at the utterance level for mathematical content and actions of the speaker. The specific elements of implicit differentiation discussed were coded using the Implicit Differentiation Knowledge Components (ImDKC) framework developed by Buchbinder and Allen (in press). This framework classifies knowledge components of implicit differentiation according to three types of learning goals: *recognition* of implicit differentiation, *symbolic* manipulation of implicit formulas, and *graphic* representation of implicit curves. These categories are further broken into specific competencies and skills students must be able to perform to meet the broader goals. Further, *algebra* and *calculus* categories were added to capture LAs' and students' talk related prior knowledge (Kandeel, 2021).

To understand *how* LAs and students interact around the various goals of the ImDKC framework each LA and student utterance was coded for the type of discursive action (e.g., asking question, elaborating, explaining). Specifying the discursive actions is important for understanding how the interlocutors frame the implicit differentiation topics and how this framing evolves throughout the discussion (Gee, 2014). LAs' discursive actions were coded at the turn of talk level using the Action Taxonomy for Learning Assistants (ATLA) (Thompson et al., 2020) which catalogs 25 types of LAs' classroom actions (e.g., explain, check knowledge, clarify the goal of activity) across six broad categories: directed facilitation, guided facilitation, feedback, advice, course-related talk, and non-course related talk.

Students' discursive actions are also coded at the turn of talk level; however, due to the absence of a suitable coding scheme for college students' discursive actions, we developed one for this study. Taking ATLA categories as a starting point we used open coding and constant comparative method (Strauss & Corbin, 1998) to develop 11 categories of students' discursive actions such as "explaining work" or "asking for directions." The codes were grouped into three broader categories: asking questions, explaining work or thinking, and following LA directions.

Overall, each utterance or dialog turn was coded in two ways: with an element of the ImDKC framework to capture the mathematical theme, and with a discursive action of the speaker: LA or a student. Following this micro-analysis, we aggregated across all LAs and students for various coding categories to identify trends in the data and respond to the research questions: *what* aspects of implicit differentiation do students and LAs discuss, and *how* do they talk about them?

Results

In this paper, we report on the preliminary results of data analysis to provide an overview of the trends in LA-student discourse across all interactions about implicit differentiation. Table 1 shows the distribution of the ImDKC topics across interactions; Table 2 shows the distribution of discursive actions by LAs and students across interactions.

Table 1: Distribution of ID Topics Across Interactions (N =250 utterances; 22 interactions)

ImDKC Topic	Number and Percent of Utterances[(a)]	Number of interactions[(b)]
Symbolic N_S=65 (26% of all utterances)		
Implicit Equation (S_Eq): View y as an implicit function of x	5 (8%)	2
Chain Rule (S_Ch): Use of chain rule in implicit differentiation	3 (4%)	1
Differentiation (S_Diff): Compute dy/dx	14 (22%)	4
Evaluation (S_Eval): Compute dy/dx at a point	16 (25%)	5
Tangent Line (S_Tan): Compute equation of a tangent line at a point	3 (4%)	1
Procedure (S_Pro) Relate implicit diff. to other differentiation rules	24 (37%)	3
Graphic N_G=91 (36% of all utterances)		

Recognition (G_Rec): Implicit curves don't pass the vertical line test	9 (10%)	2
Tangent Slope (G_TS): Relate dy/dx at point as the slope of the tangent line at that point	32 (36%)	5
Vertical Tangent (G_VT): Implicit curves with vertical tangent lines	14 (15%)	2
Constant Rate (G_CR): Graphical meaning of constant rate of change	16 (18%)	2
Coordination (G_Cor): Coordination between graphic and symbolic representations.	20 (21%)	5
Background Concepts N_B=94 (38% of all utterances)		
Calculus (C) Topics previously learned in the calculus course	33 (35%)	5
Algebra (A) Algebraic operations when performing computations	61 (65%)	9

Notes: (a) The percentages of each code in the first data column are calculated out of the total number of codes in the related ImDKC category: Symbolic, Graphic or Background Concepts. (b) The total number of interactions exceeds 22, since there were multiple codes per interaction.

Table 2: Distribution of Student and LA Discursive Actions (N=250)

	Student Actions			LA Actions		
ImDKC Category	Asking questions	Explaining work or thinking	Following LA direction	Directing	Guiding	Providing feedback or advice
Symbolic	12	5	9	21	3	15
Graphic	15	7	13	28	12	16
Background	18	12	20	23	4	17

Symbolic Goals

Symbolic goals were mentioned in 65 (26%) of utterances across 22 interactions. Of modal categories of codes were *Symbolic Differentiation* (22%), *Symbolic Evaluation* (25%), and *Symbolic Procedures* (37%). *Symbolic Differentiation* (S_Diff) captures talk about finding dy/dx in implicit equations and *Symbolic Evaluation* (S_Eval) captures talk about evaluating this derivative at a specific point. Each of these topics was of particular concern for students since about half of the utterances for each of these codes were spoken by a student. Furthermore, in three out of four interactions involving symbolic differentiation, it was students who initiated the discussion. Similarly, students initiated three out of five interactions around symbolic evaluation.

LA-student conversations also focused on *Symbolic Procedures* (S_Pro) - 24 utterances (37% of symbolic codes). This topic was introduced by an LA in all three of the interactions in which it appeared, with the LA typically explaining procedures of implicit differentiation in connection to previously seen differentiation rules and with students following along the explanations. For example, one LA *directed* a student to "just take the normal derivative of it and then multiply by dy/dx" to complete a problem. The prevalence of this topic reflects the pragmatic role of the LA as a near-peer in classroom discourse with students; LAs explained how to do the computations involved in implicit differentiation in the context of these students' prior knowledge.

The pragmatic nature of these LA-student interactions is also evidenced in their discursive actions around symbolic codes (Table 2). The associated student action codes reveal that students were particularly concerned with correctly performing the algebraic steps of the procedures and arriving at the correct answers. Students primarily *asked questions* (12 of 26 student symbolic utterances), to check with the LA their computations (e.g., "Is this right? Like am I on the right track?") or to ask for guidance about the next steps (e.g., "How would you do, would that just be y?"). In response to these questions, LAs typically *directed* student work (21 of 38 LA symbolic utterances); they explained procedural steps that students should take (e.g. "So then you could just factor out the y prime and solve for it.") While one may hope that LAs would not default to explaining steps, LAs' direct address of students' concern for correctness reflects their effort at aiding students in correctly performing computations on which they will likely be assessed.

Graphic Goals

Graphic aspects of implicit differentiation appeared in 36% of utterances across all 22 LA-students' interactions. Of the graphic goals, *Graphic Tangent Slope* (G_TS) and *Graphic Coordination* (G_Cor) were the most prevalent, 36% and 21%, respectively. That is, when discussing graphical components of implicit differentiation, LAs and students mostly focused on lines tangent to implicit curves and relating graphical features of implicit curves to symbolic representations of these curves and their derivatives.

The category *Graphic Tangent Slope* (G_TS) appeared in five interactions; initiated by LA in four of them. LAs typically either explained how to interpret slopes of tangent lines (e.g., "remember that the derivative is the equation for the slope") or prompted students to use their prior knowledge of tangent lines in this new context (e.g., "So what is the meaning of dy /dx?"). In response to LAs' explanations and prompts, students occasionally explained their thinking (e.g., "You plug in the point and then see if it's equal"), but primarily provided simple factual answers (e.g., LA: "So if I take the derivative of a function, what is that?", Student: "Slope"). Overall, the talk on this topic was characterized by LAs prompting students to recall prior knowledge about tangent lines, and then explaining to students how to apply the facts they know within the new context of implicit differentiation. Since students did not often bring this topic up, it was LAs who supported students' making connections between previous course content and implicit differentiation, helping make these connections explicit.

Another topic initiated mostly by LAs was *Graphic Coordination* (G_Cor), with LAs introducing the topic in four of the five interactions in which it appeared. Typically, LA pushed students to connect their symbolic work to graphical properties. For example, one student asked an LA what value to plug in to find the derivative at a point. In response, the LA directed the student by pointing to the graph and asking "so what point were you looking at originally? (0,1)?" Instead of telling the student how to proceed, the LA probed the student's thinking about the problem by connecting the graph of the implicit curve with symbolic calculation of slope at a point. This shows the LAs aiding student's reasoning across representations and making sense of computations using graphical properties.

In relation to graphical topics, students primarily *asked questions* (15 of 35 graphic student actions) and *followed* LA instructions (13 of 35 graphic student actions). Questions revolved around interpreting graphs (e.g., How are you supposed to explain that they're [vertical tangent lines] vertical?"). In response to student questions, LAs *directed* work (28 of 57 LA graphical actions) and provided *feedback* (16 of 57 LA graphical actions), explaining concepts and correcting or confirming student work, similar to symbolic topics. LAs' discursive actions for graphical topics had a high number of *guiding* utterances (13) compared to the very low counts for symbolic topics (3) and background concepts (4).

Background Concepts: Algebra and Calculus Skills

LAs and students also discussed prior knowledge of calculus (e.g., calculus notation, tangent lines) and algebra skills (e.g., steps in solving an equation, exponent rules). These topics are needed to both understand implicit differentiation and do such computations. These *Background Concepts* were mentioned in 94 utterances (38%), of those, almost twice as many were related to algebra than calculus (61 and 33 utterances respectively) (Table 1). Algebra topics appeared in more interactions (nine out of 22) than calculus topics (five of the 22 interactions).

In relation to background skills, students *explained* their thinking and work on computations (e.g., "Okay I did the (0,-1), but didn't get the same", "5 minus 15, negative 6.") in only 12 out of 50 utterances in the background concepts category. Yet it was the highest compared to five and seven student explaining codes in symbolic and graphic categories. This suggests students may be more comfortable or confidant sharing their thoughts about topics with which they are already familiar. Students also *asked* LAs to check the validity of their work on background concepts (18 of 50 codes) saying things like "So this would be 15?" and "Am I doing this right?".

In response to student thoughts and questions on these topics, LAs typically *directed* student work (23 of 44 LA background concepts utterances), meaning that they explained procedures to students (e.g., "You can use the quadratic equation to solve for y squared"). LAs also provided students *feedback* (17 of 44 LA background concepts utterances) on their work (e.g., "Um, you're on the right track, but something's going wrong in these terms here"). Students appeared receptive to this explanation and feedback with 20 of the 50 student background concepts codes being *following LA* direction. Like with the symbolic goals, LAs were pragmatic in supporting students; their diligent oversight and confirmation (or correction) of students' basic calculus and algebra skills reassured students that they were successfully performing procedures on which they will be assessed.

Discussion

Our analysis reveals that classroom interactions between LAs and students are primarily pragmatic in nature. The topics of LA-Student conversations often revolve around computation, with symbolic and background topics together discussed in the majority of utterances. While this kind of talk could be dismissed as superficial, we interpret the focus on procedural correctness as a shared value between LAs and students as university student peers who know that success in Calculus 1 course hinges on displaying procedural fluency on exams. LAs also helped students consider graphical features of implicit equations by interpreting graphs in the context of computations, connecting to prior knowledge, and prompting students to explain their thinking.

Though some detail is lost by reporting this micro-analysis of classroom discourse in aggregate, this initial report allows us to make sense of the trends seen across various LAs' interactions with small groups as they discussed implicit differentiation. We are currently in the process of analyzing the sequencing of moves within each interaction to provide a sense of how conversations are reciprocally constructed by LAs and students in this setting.

The current study provides a novel insight into LA-student classroom interactions. We extend Thompson et al.'s (2020) taxonomy of LA's actions (ATLA) to student actions as well, and contextualize both types of actions within specific mathematical content. This study also exemplifies several ways in which LAs act as MKOs in a Calculus setting, including through pragmatic focus on computation and by explicitly connecting new ideas to prior knowledge. This emphasis on procedural knowledge could be one aspect of the mechanism contributing to success of students in LAs' supported classrooms, since procedural knowledge of mathematics has been shown to support development of conceptual knowledge (Rittle-Johnson et al., 2015).

Further, this study contributes to our understanding of how students learn implicit differentiation in an authentic classroom environment, compared to prior studies that used local interventions and teaching experiments (Borji & Martínez-Planell, 2020; Jeppson, 2019). LAs' emphasis on prior knowledge reframed implicit differentiation not as a new and potentially intimidating course topic, but as a familiar procedure with some modifications, thus potentially supporting students' emotional and cognitive processing of this topic. Continued analysis of these LA-student conversations at this detailed level will generate much needed insight into both students' real-time learning of implicit differentiation and the content-specific aspects of LA-supported learning.

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UNPACKING INFORMATIVE ASSESSMENTS TO YIELD STUDENTS' UNDERSTANDING OF PARTIAL DERIVATIVES USING MATH LITERACY ACTIVITIES

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Keywords: math literacy, partial derivatives, concept definition, concept image

Mathematical literacy is a critical component of teaching mathematics as its own language. In this study, I will use writing to learn mathematics (WTLM) activities to help students communicate their mathematical thinking and understanding. I will conduct an embedded single-case study with 43 undergraduate students enrolled in a hybrid calculus course, which focuses on integral and multivariable calculus topics. WTLM activities will provide students with an opportunity to explain their understanding either symbolically, procedurally, and/or conceptually. The students' understanding of partial derivatives will be analyzed using my posit framework, which is an integration of the frameworks Concept Definition and Concept Image, and Covariational Reasoning. In my study, I investigated students' initial and developing understanding of partial derivatives within the calculus curriculum. Using WTLM activities, I address my research question: What are students' concept definitions and evoked concept images when learning about partial derivatives in calculus? In this paper, I focus on the WTLM activities I implemented as my data collection sources.

One of my goals was to give students a larger voice in their thinking and understanding and a larger ownership in the learning process by teaching mathematics literacy as part of the learning process. Using writing to learn mathematics (WTLM) activities, students can communicate their mathematical thinking and understanding. The two WTLM activities I propose using are (1) "The Important Thing About…" Prompt [based on Brown's (1949)] Important Book, and (2) My Aspects of Mathematical Phenomena (AMP) – Chart [based on Frayer and colleagues (1969) with contributions of online blogger Musingsofamathteacher (2011)]. It is with these activities, that I can begin to unpack students' understanding of topics such as partial derivatives and address my research question: What are students' concept definitions and evoked concept images when learning about partial derivatives in calculus?

Summary

As I continue to analyze my data, I expect to learn about my students' concept definitions and concept images of partial derivatives. Afterward, I expect to see how these concept definitions and concept images of partial derivatives reflect their understanding. My work on this study will contribute to giving students (1) a larger voice in their thinking and understanding to their teachers through the usage of WTLM activities, (2) the skills to communicate their understanding inside and outside of the mathematics classroom, and (3) a tool to combat the difficulty that comes with abstract mathematical ideas.

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SIGNIFICACIÓN DEL CONCEPTO RAÍZ REAL DE UNA ECUACIÓN POLINÓMICA MEDIADA POR LA TECNOLOGÍA DIGITAL

MEANING OF THE CONCEPT OF REAL ROOT OF A POLYNOMIAL EQUATION USING DIGITAL TECHNOLOGY

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Presentamos una secuencia didáctica mediada por la tecnología digital para significar y calcular las raíces reales de una función real en un curso de cálculo diferencial para estudiantes de ingeniería. Para introducir las raíces diseñamos un Escenario Didáctico Virtual Interactivo, que simula un problema real, y usamos el sistema tutorial CalcVisual para apoyar el cálculo aproximado de las raíces. Implementamos la secuencia con una población de 45 estudiantes universitarios en México. Los datos se analizaron mediante los modelos emergentes de la Educación Matemática Realista. Mostramos el progreso en la actividad matemática de los estudiantes a través de cada uno de los niveles de actividad de los modelos emergentes quienes mostraron un avance significativo en la comprensión conceptual y cálculo de las raíces reales.

Precálculo, Cálculo, Tecnología, Experimentos de diseño.

Los polinomios son funciones fundamentales en la matemática, en particular en el cálculo, análisis matemático y álgebra lineal. Una de las propiedades más importantes de una función polinómica, son sus raíces reales y complejas, pero determinarlas no es tarea sencilla e incluso en algunos casos se llega a confundir la función polinómica con la ecuación que se deriva de ella (Dede y Soybas, 2011). Significar el concepto de raíz real de una función resulta determinante por tratarse de un concepto fundamental para la matemática e imprescindible para aplicaciones en procesos de optimización, cálculo diferencial e integral, álgebra lineal, cálculo multivariable y método Simplex de programación lineal, por mencionar algunos. La determinación de las raíces simples o múltiples es un problema complejo y vigente que tiene su origen desde los primeros vestigios de la humanidad, y que siempre ha estado asociado a problemas de variación, acumulación y optimización. Hasta nuestros días se mantienen problemas abiertos sobre el cálculo de raíces, sobre todo cuando son múltiples (Cuevas y Madrid, 2013), y cobra relevancia en el problema de cómo introducir desde el plano cognitivo este concepto en la enseñanza a nivel de precálculo y cálculo (Veuillez-Mainard, 2023). Al realizar una búsqueda sistemática de la literatura podemos constatar que existen pocos artículos que reporten las dificultades de enseñanza de raíces reales, minimizando la importancia y dificultad del concepto. Es conveniente recordar que la resolución de ecuaciones polinomiales generó el álgebra (Puig y Rojano, 2004).

Tradicionalmente el cálculo aproximado de raíces reales era un tema importante que tratar en cursos de análisis numérico donde se practicaban diversos métodos como: bisección, regula falsi y Newton-Raphson para aproximarse al valor de una raíz real. Sin embargo, al popularizarse el uso de herramientas digitales con la avalancha de diversos manipuladores simbólicos virtuales, el significado y proceso bajo el cual se desarrolló permanece oculto, dejando la incertidumbre de qué es una raíz real de una función real. Esto debido a que softwares como: Mathematica, Wolfram Alpha, Matlab, GeoGebra y Photomath, resuelven ecuaciones y encuentran sus raíces en cuestión de segundos. Este reto que la tecnología digital ha puesto en la enseñanza y aprendizaje de las matemáticas permanece sin respuesta, y ha creado el paradigma de ¿qué se debe de enseñar? Cuando los estudiantes utilizan cualquier dispositivo o software para calcular el valor de una raíz real ¿sabrán que la mayoría de las veces encuentran un valor aproximado? ¿qué cuando las raíces son múltiples y cercanas pueden confundirse por errores de aproximación? ¿qué significa gráfica y numéricamente una raíz? ¿en qué se puede utilizar el concepto de raíz real, más allá de calcular su valor? Estos significados, se extraviaron al perderse los métodos de aproximación de una raíz y difícilmente se recuperarán algún día. Nos preguntamos ¿cómo recuperar los significados del concepto raíz de una función real mediante actividades mediadas por la tecnología digital? Nuestra propuesta consiste en el desarrollo y creación de actividades didácticas que permitan recuperar los significados de las raíces reales aprovechando los recursos que la tecnología digital ofrece el día de hoy como la capacidad numérica, gráfica y simbólica.

Marco teórico

Cuando un profesor frente a un grupo de estudiantes explica y anota definiciones, fórmulas y ejercicios en el pizarrón, mientras los estudiantes lo observan, escuchan y anotan en sus libretas lo expuesto por él, a esta enseñanza se le denomina enseñanza tradicional, la cual se ha desarrollado durante varios años. Para evitar este tipo de enseñanza y promover una enseñanza participativa con el objetivo de dotar de un significado a los conceptos matemáticos, Cuevas y Pluvinage (2003) proponen una serie de principios – intranet conceptual, partir de un problema en un contexto real, un plan de acción, implementación de operaciones inversas, la articulación de diversos registros de representación, la validación de resultados y la aplicación del concepto en un contexto diferente al enseñado– para la enseñanza de un concepto matemático. Usamos estos principios para el diseño de las actividades.

Los modelos emergentes son una de las tres heurísticas de diseño instruccional de la Educación Matemática Realista (RME por sus siglas en inglés). Esta heurística describe como una serie de modelos puede apoyar el avance matemático de los estudiantes (Gravemeijer, 2020). La heurística de los modelos emergentes destaca la importancia de comenzar con problemas contextuales que ofrezcan oportunidades para desarrollar un razonamiento específico de la situación y con el potencial de crear problemas cuya solución hace necesario el uso de conceptos matemáticos más sofisticados (Gravemeijer y Doorman, 1999). La actividad matemática inicia con el uso o desarrollo de un modelo derivado del contexto y, con el tiempo, este modelo apoya la aparición de formas de conocimiento matemático formal (Doorman et al., 2012). Los estudiantes transitan por distintos niveles de actividad que van desde el uso de estrategias informales hasta el razonamiento matemático formal (Gravemeijer, 1999). Los cuatro niveles propuestos por la RME son:

1. Nivel situacional (actividad en el entorno de la tarea). En este nivel las interpretaciones y

las soluciones dependen de la comprensión de cómo actuar en el entorno (fuera del entorno escolar).

2. Nivel referencial. En este nivel los modelos-de se refieren a la actividad en el entorno descrito en las tareas. En consecuencia, los modelos que surgen se basan en la comprensión de los estudiantes del entorno real y forman parte de las explicaciones en las que los estudiantes describen cómo interpretaron y resolvieron las tareas centradas en los escenarios de partida.
3. Nivel general. Este nivel comienza a surgir cuando los estudiantes empiezan a razonar sobre las relaciones matemáticas implicadas. Por lo tanto, surge cuando el razonamiento de los estudiantes pierde dependencia de las imágenes específicas de la situación. En este sentido, los modelos-para sirven más como medio de razonamiento matemático que como forma de simbolizar la actividad matemática basada en entornos particulares.
4. Nivel formal. En este nivel se trabaja con los procedimientos y notaciones convencionales. Se alcanza cuando los estudiantes ya no necesitan el apoyo de modelos para la actividad matemática.

Usamos estos niveles para mostrar el progreso en el razonamiento de los estudiantes, sobre el concepto de raíz, al trabajar con las actividades propuestas.

Metodología

Este estudio se desarrolló con base en la Investigación Basada en el Diseño (IBD) por lo que esta investigación implica iteraciones de diseño, implementación y análisis mediante las siguientes fases: preparación y diseño, experimentos de enseñanza y análisis retrospectivo (Bakker, 2018).

Fase de preparación y diseño

Se diseñó una secuencia de cinco actividades para introducir de forma gradual el concepto de raíz real (ver figura 1).

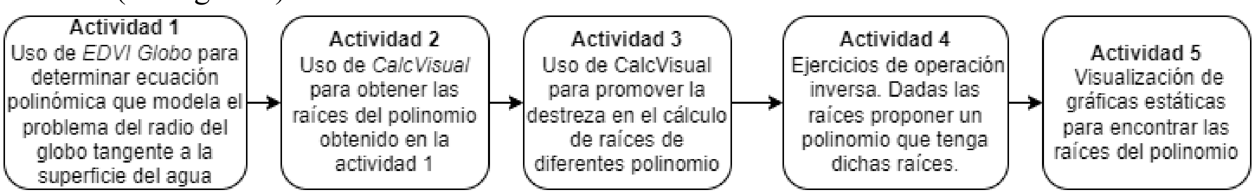

Figura 1: Secuencia didáctica

Como parte de la secuencia, se diseñó un Escenario Didáctico Virtual Interactivo (EDVI), al que denominamos EDVI "Globo" porque simula un recipiente cilíndrico de 10 cm de diámetro con un globo esférico atado al fondo. Este EDVI cuenta con botones para llenar y vaciar de agua el recipiente y botones para inflar y desinflar el globo a partir de los cuales, pueden observar cambios de forma dinámica en parámetros como: la altura inicial del agua con el globo desinflado, la altura del agua al inflar o desinflar el globo y el radio y diámetro del globo (ver figura 2a). En este artículo nos referimos a un EDVI como un manipulativo virtual que permite simular y visualizar diferentes representaciones semióticas de un problema real (Cuevas et al., 2023).

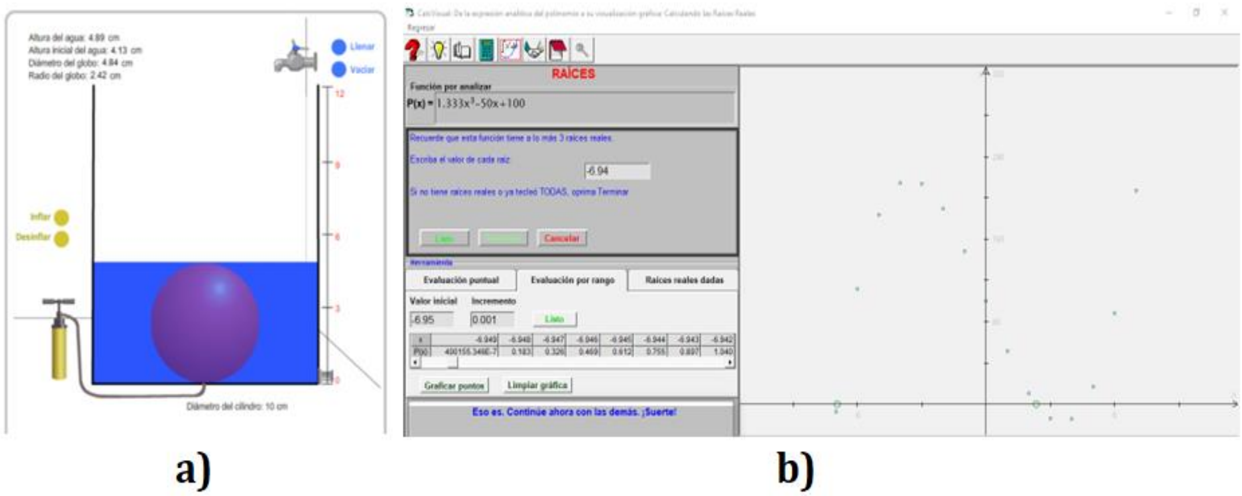

Figura 2. a) EDVI Globo; b) Sistema Tutorial Inteligente CalcVisual

Adicionalmente, se utilizó un Sistema Tutorial Inteligente denominado CalcVisual que apoyará a los estudiantes en el cálculo de las raíces (ver figura 2b). El CalcVisual es un software que no calcula las raíces como cualquier manipulador simbólico. Permite introducir el polinomio y mediante herramientas visualizar diferentes representaciones del concepto de raíz. Por ejemplo, su representación tabular y su gráfica sobre un plano cartesiano. Es importante señalar que, CalcVisual no trabaja con funciones racionales y radicales. Asimismo, se diseñaron Hojas de Exploración y Aprendizaje Guiado (HEAG) para cada actividad, las cuales guían al estudiante en la manipulación de las herramientas digitales y en la construcción del concepto matemático.

Fase de experimento de enseñanza

La intervención didáctica se desarrolló en una universidad pública mexicana con 45 estudiantes inscritos en un curso de "Matemáticas aplicadas a la informática". Las HEAG se enviaron de manera digital a cada estudiante y las actividades se desarrollaron en equipos de 6 integrantes. Después de resolver cada actividad, el profesor seleccionó unas HEAG al azar y realizó una discusión en clase para llegar a las respuestas correctas de forma consensuada. Las actividades fomentan tanto el aprendizaje individual como el colaborativo. Los datos se obtuvieron mediante las respuestas en las HEAG que los estudiantes enviaron al correo electrónico del profesor. Uno de los autores fue el encargado de impartir dicho curso. Los datos se analizaron de manera independiente por los investigadores. Se identificaron estrategias de solución y las respuestas de los equipos se clasificaron en los niveles de actividad (situacional, referencial, general y formal).

Resultados y análisis retrospectivo

En esta sección describimos como progresa el razonamiento de los estudiantes a través de cada uno de los niveles de actividad de los modelos emergentes al trabajar con las actividades propuestas. Debido a la limitación del documento, mostramos las respuestas de dos equipos (T1 y T2) seleccionados al azar.

Actividad Situacional

Como se ha mencionado anteriormente, la actividad situacional implica que los estudiantes trabajen en un entorno real para alcanzar objetivos matemáticos particulares. La actividad en el aula comenzó con la exploración del EDVI "Globo". Clasificamos esta actividad en el *nivel situacional* porque los estudiantes usaron las herramientas disponibles en el escenario para identificar variables, constantes y características cómo: una altura inicial del agua (h_0) con la que el diámetro del globo puede ser igual a la altura del agua (h_a). La tabla 1 muestra las respuestas de los equipos T1 y T2 a las preguntas de exploración.

Tabla 1. Preguntas y respuestas a las actividades de exploración

Pregunta	Respuesta T1	Respuesta T2
¿Qué elementos son variables al inflar el globo?	Radio del globo, volumen del contenido en el recipiente, volumen del globo.	Radio del globo, volumen del contenido en el recipiente, volumen del globo y altura inicial del agua.
¿Qué elementos son constantes al inflar el globo?	Radio del recipiente, volumen inicial de agua y altura inicial del agua.	Radio del recipiente, volumen inicial de agua.
¿Hasta qué valor puede crecer y disminuir el radio del globo (x)?	El radio puede crecer hasta 5cm y disminuir hasta 0cm	El radio puede crecer hasta 5cm y disminuir hasta 0cm.
¿Hasta que altura inicial (h_0) se puede llenar el recipiente de agua?	La altura inicial máxima es de 12 cm	La altura inicial máxima es de 5.47.
Escribe una altura inicial del agua h_0 con la que el diámetro del globo sea igual a la altura del agua h_a.	Si el recipiente se llena hasta una altura de 10cm, el diámetro se expandirá hasta los 10cm.	A una altura de 5.45

En general, los elementos constantes del EDVI son el radio del recipiente, el volumen inicial de agua y la altura inicial del agua. Sin embargo, nótese que el T2 indicó como variable la altura inicial del agua. Inferimos que dieron esta respuesta porque se trata de un parámetro que se puede modificar en el EDVI. Aunque, una vez establecido, al inflar y desinflar el globo este permanece constante. Observe también que, las respuestas a la pregunta 4 son diferentes. Ambas respuestas son correctas, ya que el T1 se enfocó en la h_0 con el globo desinflado, mientras que el T2 primero infló el globo hasta su valor máximo y posteriormente llenó el recipiente con agua. Finalmente, las respuestas a la pregunta 5 nos hacen inferir que los estudiantes confundieron la altura inicial del agua (h_0) con la altura del agua (h_a) aunque se puede observar que sí identificaron valores en los que el diámetro del globo es igual a la altura del agua.

Actividad Referencial

Tras la exploración del EDVI "Globo", la actividad en el aula continuó con el problema de identificar la ecuación polinómica que modela el volumen total del contenido del recipiente, cuando el radio del globo es tangente a la superficie del agua para identificar como solución la raíz de un polinomio. Clasificamos esta actividad como *referencial* porque los estudiantes comenzaron a establecer relaciones matemáticas en el contexto. Esta actividad se dosificó de modo que los estudiantes propusieran ecuaciones para determinar la altura del agua (h_a) en relación con el radio del globo (x), el volumen del globo (V_G), el volumen inicial del agua (V_A) y el volumen total del contenido en el recipiente (V_T). La tabla 2 muestra las ecuaciones propuestas por los equipos T1 y T2.

Tabla 2. Ecuaciones propuestas por los equipos T1 y T2 para modelar el problema del radio del globo tangente a la superficie del agua

T1	T2
$V_G = (\frac{4}{3})\pi x^3$ $V_A = \pi(R^2)\ h_0$ h_a=$2x$ $V_T = \pi(R^2)\ h_a$ $V_T(x) - V_A - V_G = 0$ Donde $V_T(x)$ es el volumen del cilindro con radio x y altura del agua h_a, V_A es el volumen inicial de agua y V_G es el volumen del globo.	$V_G = (\frac{4}{3})\pi x^3$ $V_A = \pi(R^2)\ h_0$ h_a=$h_0 - x$ $V_T = \pi R^2 h_a + V_G = \pi R^2(h_0 - x) + \frac{4}{3}\pi x^3$ $\pi R^2(h_0 - x) + \left(\frac{4}{3}\right)\pi x^3 - V_T = 0$

De las respuestas observamos que ambos equipos identificaron la relación entre el volumen total del contenido en el recipiente (V_T), el volumen sumergido del globo (V_G) y el volumen inicial del agua (V_A). Por ejemplo, los estudiantes del T1 mencionaron que "el volumen total del contenido en el recipiente es igual a la suma del volumen inicial del agua y el volumen sumergido del globo". Además, señalaron que esta relación se podía expresar mediante la siguiente ecuación "$V_T = V_i + V_g$". De forma similar, los estudiantes del T2 indicaron que "el volumen total del contenido del recipiente es igual a la suma del volumen inicial del agua con el volumen sumergido del globo". Sin embargo, ningún equipo llegó a la ecuación polinómica esperada $\frac{4}{3}x^3 - 2R^2x + R^2h_0 = 0$ donde, x es el radio del globo, R es el radio del recipiente y h_0 es la altura inicial del agua. Esta ecuación se desarrolló y explicó en la discusión grupal.

Actividad General

Después de identificar la ecuación polinómica $f(x) = \frac{4}{3}x^3 - 2R^2x + R^2h_0$ que modela el problema del radio del globo tangente a la superficie del agua, se propuso a los estudiantes que usaran CalcVisual para encontrar las raíces del polinomio, con $R = 5$ y $h_0 = 4$, $f(x) = \frac{4}{3}x^3 - 50x + 100$. Clasificamos esta actividad en el *nivel general* porque los estudiantes se enfocaron en representaciones gráficas y tabulares para encontrar las raíces, sin hacer referencia al contexto del globo. Por ejemplo, el T1 respondió "Nuestra función cuenta con 3 raíces, $-6.9493, 2.3430$ y 4.6063. Esto lo conocemos gracias a que al meter la función dentro de nuestro programa graficador, este genera 3 puntos exactos". Por su parte, el T2 mencionó "Este polinomio tiene 3 raíces, $x1 = -6.94, x2 = 2.34\ y\ x3 = 4.60$. En la gráfica podemos ver tres puntos, lo cual significa que cada uno de ellos es parte de una raíz". Nótese cómo en ambos casos mencionan la existencia de puntos en la gráfica del polinomio, los cuales asociaron con sus raíces. Al finalizar esta actividad, pedimos a los estudiantes que validaran sus resultados respondiendo la pregunta "¿Qué raíces tiene sentido para el problema del globo?". El T1 respondió que "las raíces positivas debido a que se no se puede tener un volumen negativo. Los valores de la segunda y tercera raíz el radio del globo es tangente a la superficie del agua". En cambio, el T2 señaló que "todas las raíces son correctas. Sin embargo, para la raíz $x3 = 4.6$ el radio del globo es tangente a la superficie del agua". Las respuestas a esta pregunta se pueden clasificar en el nivel referencial porque los estudiantes interpretaron las raíces encontradas en el problema del globo. El cambio de nivel de actividad va en acuerdo con la afirmación de Rasmussen y Blumenfeld (2007) acerca de que los niveles de actividad no imponen una jerarquía estricta. Sin embargo, es importante aclarar que esta pregunta se hizo con la intención de identificar el significado que los estudiantes estaban dando a las raíces reales en el contexto del EDVI Globo.

Actividad Formal

Después de que los estudiantes realizaron actividades con el uso de CalcVisual para encontrar las raíces de un polinomio, trabajaron con actividades de operación inversa como: "Escribe un polinomio que tenga al menos las siguientes raíces reales: $r1 = 2$ y $r2 = -5$ ¿será ese el único polinomio que tenga al menos esas raíces reales?". Las respuestas de los equipos T1 y T2 se resumen en la tabla 3.

Tabla 3. Respuestas de los equipos T1 y T2 a actividad de operación inversa

Escribe un polinomio que tenga al menos las siguientes raíces reales: $r1 = 2$ y $r2 = -5$	
Equipo T1 Respuesta: $x^2+3x-10$ 1) Se coloca el signo opuesto de las raíces dadas. 2) Multiplicamos los factores que obtuvimos. $(x-2)(x+5)= x^2+5x-2x-10$ 3) Expandimos el producto. $x^2 + 3x - 10$	Equipo T2 Respuesta: $(x-2)$ y $(x+5)$ $(x-2)(x+5)= x^2+5x-2x-10$ $x^2 + 3x - 10$
¿Será ese el único polinomio que tenga al menos esas raíces reales?	
No es el único polinomio que tiene al menos esas raíces porque podemos multiplicar el polinomio por cualquier otro factor lineal y obtendremos un polinomio que tenga las mismas raíces.	No, también puede ser $f(x)(x - 2) = (x^3 +3x-10)(x-2)=$ $x^3-5x^2-4x+20$

Como se puede observar en la tabla 3, los estudiantes escribieron primero los polinomios de forma factorizada y posteriormente realizaron la multiplicación de los factores para proponer un polinomio desarrollado. Clasificamos estas respuestas en el nivel formal porque los estudiantes trabajaron con procedimientos algebraicos desligados del EDVI Globo y el uso de CalcVisual.

La actividad formal también se observó en la actividad 5 que no involucraba el uso de herramientas digitales sino la visualización de gráficas estáticas como la de la figura 3. Esta actividad se diseñó como tarea final para identificar el aprendizaje que los estudiantes adquirieron sobre el concepto de raíz en un contexto distinto al del Globo.

Escala en el eje x es de 1 unidad por marca.

1. ¿Cuántas raíces reales tiene el polinomio de la gráfica?

 a) 0 b)2 c)3 d)No sé

Justifica tu respuesta

____

____

____

____

____

2. Escribe un valor aproximado para cada raíz:

____

____

____

Figura 3. Actividad de visualización de gráficas estáticas para identificar raíces

Los estudiantes iniciaron la actividad 5 en el nivel general porque para responder las preguntas de la figura 3, los estudiantes se enfocaron en identificar los puntos en los que la gráfica corta al eje x como muestran las respuestas de los equipos T1 y T2. Por ejemplo, en T1 mencionaron que “la gráfica está atravesando el cero en el eje de las x tres veces. Por lo tanto, la función de esta gráfica contiene 3 raíces: $-2, 0$ y 3”. De forma similar, el T2 contestó “debes contar el número de veces que la gráfica del polinomio cruza el eje x. Esta gráfica lo cruza exactamente tres veces. Entonces, el polinomio tiene al menos tres raíces: $-2, 0$ y $+3$”. Posteriormente trabajaron en el nivel formal porque representaron el polinomio mediante una factorización para llegar a su representación desarrollada.

Discusión y Conclusiones

Presentamos una secuencia de actividades que ayuda a los estudiantes a transitar de un razonamiento basado en un problema contextual a uno formal sobre el concepto de raíz. Si bien, Gravemeijer (2020) menciona que los niveles de actividad no necesariamente se observan de manera jerárquica, en nuestra investigación observamos que las actividades guiaron el razonamiento de los estudiantes sobre el concepto de raíz de forma secuencial. Es decir, la actividad 1 se trabajó en el nivel situacional y referencial, las actividades 2 y 3 en el nivel general y las actividades 4 y 5 en el nivel formal. Con lo anterior no queremos decir que los estudiantes no pueden regresar al contexto cuando trabajan en el nivel general y formal. Elegimos el contexto del Globo como un contexto con el que los estudiantes pueden significar el concepto de raíz, por ello, inferimos que se trata de una situación que perdura en la mente del estudiante.

Destacamos que, la actividad 1 con el uso del EDVI Globo fomentó el desarrollo del nivel situacional al identificar variables, parámetros y relaciones funcionales. Posteriormente, fomentó el tránsito al nivel referencial en la actividad de determinar la ecuación polinómica que modelaba el problema del radio del globo tangente a la superficie del agua. Este polinomio se usa en la actividad 2 para que los estudiantes identifiquen sus raíces, mediante el uso de CalcVisual, y les den un significado en el contexto del Globo. La actividad 3 fomentó el desarrollo del nivel general al trabajar con el CalcVisual mediante el tratamiento de funciones polinómicas fuera de cualquier contexto. Las actividades 4 y 5 fomentaron el tránsito al nivel formal al desarrollar un polinomio expresado mediante una factorización y localizar raíces en una gráfica. Los estudiantes ya no usan el EDVI Globo pero pueden usar el CalcVisual para ingresar el polinomio factorizado o desarrollado y visualizar representaciones gráficas, algebraicas y tabulares.

Una de las limitaciones de este documento es que se han presentado los resultados únicamente de dos equipos de estudiantes, aunque el análisis del nivel de actividad se ha realizado con los datos de los 45 participantes. De este análisis observamos que 30 estudiantes alcanzaron el nivel formal al trabajar con las actividades 4 y 5. Aquellos que no alcanzaron el nivel formal se debe a dificultades operativas y el uso incorrecto de procedimientos algebraicos.

Otra limitación del estudio tiene que ver con los problemas asociados a la instalación y uso de las herramientas digitales. A pesar de que se proporcionó a los estudiantes el CalcVisual para que lo instalaran en su computadora, algunos tuvieron problemas para instalarlo y no pudieron realizar las tareas en casa que necesitaban el uso de dicho software. Como solución, acudieron a otra herramienta como Wolfram, GeoGebra, etc para resolver las actividades. Aquí, el problema radica en que al usar herramientas como Wolfram los estudiantes obtienen las raíces sin saber cómo. Prueba de lo anterior es que en la retroalimentación con los estudiantes que trabajaron todas las actividades con el CalcVisual, manifestaron seguridad al resolver el examen del curso. En cambio, aquellos que utilizaron otros softwares, no sabían de dónde provenían los datos. En este sentido, el rol del profesor es importante porque debe intervenir para guiar a los estudiantes en el cumplimiento de los objetivos. En este estudio, durante la discusión en grupo, el profesor mostró a los estudiantes cómo usar el CalcVisual y discutió las desventajas de usar otro software.

Retomando nuestra pregunta de investigación ¿cómo recuperar los significados del concepto raíz de una función real mediante actividades mediadas por la tecnología digital? Sugerimos que el uso indiscriminado de la tecnología digital en la enseñanza de las matemáticas puede contribuir a la perdida de los significados y aplicaciones de los conceptos matemáticos. Por lo que recomendamos que su aplicación requiere de un cuidadoso diseño didáctico previo a su aplicación. Sugerimos iniciar la actividad en el aula con la simulación de un problema en contexto que permita a los estudiantes dar sentido al concepto matemático de interés. La simulación del problema del globo sumergido en un recipiente permitió al estudiante dotar de significado al concepto de raíz que por su propia naturaleza es abstracto. Usar un software no resolutivo como CalcVisual permite manipular diferentes representaciones de forma simultánea para establecer relaciones entre ellas, lo cual lleva a que los estudiantes adquieran significado del concepto de raíz en su representación tabular, gráfica y algebraica.

Destacamos la importancia de incluir actividades sin el uso de herramientas digitales para corroborar el aprendizaje del concepto de raíz. En este caso las herramientas digitales sirven como herramienta de verificación de resultados. Los estudiantes pueden identificar raíces de gráficas estáticas, obtener su polinomio y, posteriormente, ingresarlo en el software para comprobar los valores numéricos de las raíces. La aplicación de las HEAG es fundamental ya que guían la secuencia de actividades, contribuyendo a la comprensión del concepto de raíz de una función real.

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WHAT DOES CALCULATING LOOK LIKE IN CALCULUS? CALCULATING SITUATIONS FOR INTRODUCING DERIVATIVES

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In this study, I present four calculating instructional situations identified within 56 instructional tasks for introducing derivatives proposed by eight experienced college calculus instructors. During four 1–2-hour long interviews, the instructors proposed up to eight tasks for introducing derivatives physically, graphically, verbally, and symbolically, both at a point and as a function. The tasks were analyzed to identify calculus-specific instructional situations framing them. Here, I describe the four identified calculating situations, how they were realized in the tasks through calculus-task units (CTUs), and what mathematical works they expect of students.

Keywords: Undergraduate Education, Calculus, Instructional Activities and Practices, Curriculum

As basic units of instruction and "objects of students' activity" in mathematics classrooms (Ni et al., 2018; Sullivan et al., 2009, p. 859), instructional tasks are often seen as the conceptual bridge between teaching and learning (Christiansen & Walther, 1986; Stein & Lane, 1996). Empirical studies have revealed that the nature of tasks often changes during instruction, and that teachers' decisions about how students worked on the tasks has a significant influence on their learning outcomes. Nonetheless, the research on instructional tasks in first-semester college calculus is lacking. Given that much is at stake for students in these courses (e.g., progression to graduation or majoring in math-required fields), a better understanding of how the content is presented and what learning opportunities are created is imperative for improving students' learning outcomes. In this study, I aim to enhance our understanding of calculus teaching and uncovering how one specific content area—derivatives—is presented to students through instructional tasks. More specifically, I focus on the work of *calculating* in instructional tasks aimed at introducing derivatives.

Theoretical Framework

Using the Brousseau's (1997) notion of didactical contracts to delineate what is normative and what is not, Herbst (2006) introduced *instructional situations* as customary ways by which the teacher's and students' actions and interactions are framed into appropriate units of work regarding the knowledge at stake. Instructional situations regulate the "work on particular kind of tasks for particular objects of knowledge" (Herbst & Chazan, 2012, p. 605) as both distinct problem types used in a course of studies, as well as "the specific norms that regulate the teacher's and students' work on them" (Herbst et al., 2023, p. 402). Herbst et al. (2020) defined *task framing* from the perspective of the teacher in mathematics instruction as choosing a way of handling a problem, or choosing an instructional situation, about the knowledge at stake; once recognized by students, the task framing, that is the instructional situation, allows students to know what kind of mathematical work and interactions they should prototype. For instance, *graphing* is an example of an instructional situation in Algebra I. When students are asked to graph a line in the form of $y = mx + b$, by realizing that the instructional situation is graphing in Algebra I, they access the didactical contract of graphing. Teachers can select from the available set of instructional situations to frame their instructional tasks, consequently launching and organizing students' work (Herbst et al., 2018; 2020). Researchers have identified various customary instructional situations mainly in high school geometry and algebra, such as doing proofs; installing (defining) a new definition or a new concept; installing (introducing) a new theorem, property, or formula; solving equations with known methods; and solving word problems (Chazan & Lueke, 2009; Herbst et al., 2010).

Methods

The data for this study comes from a larger interview study with eight experienced U.S. calculus instructors. During four semi-structured 1–2-hour long interviews with each instructor, I prompted the instructors to propose up to eight tasks for introducing derivatives. Organized by Zandieh's (2000) framework, each interview was dedicated to one representation of the derivative (graphical, verbal, physical, symbolic) and its process-object layers (ratio, limit, function). The process-object layers are hierarchical, as each layer is found by taking the process of that layer over the previous layer as an object. For example, the limit layer is found by the *process of* finding the limit of the ratio as an *object*. The limit layer corresponds to when the denominator approaches zero; the function layer is presented as an array of numbers or set of ordered pairs of differences. The instructors were asked to propose tasks within each representation that would help transition students' conception from one layer to the next within that representation: from ratio to limit, and from limit to function.

The instructors collectively proposed 56 tasks, or what could be better described as 'a set of tasks.' Despite this, I refer to what they proposed in its entirety and to the unit of analysis as an instructional task. To pursue analysis, I first broke down the instructional tasks into their smallest calculus-specific problems, which I call *calculus-specific task units* or *CTUs*. By this, I mean that if I further divided a CTU into smaller tasks, the results would not be recognizable as calculus tasks; instead, they would be recognized as tasks from other content areas preceding calculus in a standard mathematics curriculum sequence. I then used thematic analysis (Saldaña, 2021) to reduce the 37 identified CTUs to 11 instructional situations categorized as four calculating, two exploring/conjecturing, two graphing, two installing, one proving, and one solving equation (see Gerami, 2024). In this study, I focus on calculating situations for introducing derivatives.

Findings

I identified 12 distinct calculating CTUs across all the tasks, which I then reduced to four instructional situations, as shown in Table 1:

Table 1. Calculating instructional situations and along with their CTUs

Instructional Situations	Calculus-Specific Task Units (CTUs)
1) Calculating/ Estimating a value using known formulas or definitions	**C1.** Calculating the average rate of change or average velocity or slope of secant lines or difference quotient given $f(x)$ and/or a table of values for various x_i and $f(x_i)$, or the equation of the curve of best fit based on various x_i and $f(x_i)$ **C6.** Calculating instantaneous velocity at x_0, using equation of

	continuity **C7.1.** Calculating instantaneous velocity or instantaneous growth or the slope of tangent or derivative values or $f'(x_0)$ or $f'(a)$ given $f(x)$, using a limit definition **C7.2.** Calculating $f'(x_0 +/-)$ using the limit definition **C8.** Calculating $f'(x_0)$ or slope of a tangent line at x_0 or a using the formula for $f'(a)$ or $f'(x)$ **C11.** Calculating instantaneous velocity or instantaneous growth or $f'(x)$ using a limit definition
2) Calculating/ Estimating a value by collecting information from graphical representations to use in known formulas or definitions	**C2.** Calculating slope of secant lines to the left and right side of x_0 using the graph of $f(t)$ **C5.1.** Calculating the slope of tangent line at x_0 using the graph of $f(x)$ on plain or grid background **C5.2.** Calculating slope of tangent line at a point x_0 by zooming in
3) Writing an algebraic statement using known formulas or definitions	**C3.** Finding the equation of a secant line **C9.** Finding the equation of a tangent line at x_0 or a **C12.** Finding instantaneous velocity $f'(x)$ using average velocities on consecutive equidistant intervals for a quadratic function
4) Estimating a value using number sense	**C4.** Estimating instantaneous velocity or slope of a tangent line or rate of change or $f'(x_0)$ or $f'(a)$ using a pattern of average velocities or slope of secant lines or average rates of change **C10.** Estimating y-values of f' as slopes of tangents of f

The first instructional situation—calculating/estimating a value using known formulas or definitions—can be seen in calculating CTUs in which information is provided to students, expecting them to put forward known formulas/definitions to calculate the unknown value in the problem. In C1, C6, C7, C8, and C11 students should use various definitions to calculate derivatives at different layers. At stake in these situations is students' knowledge and proficiency to use the formulas to find a numeric value. Given that students derive the final answer by reasoning, their work here is deductive. The second instructional situation—calculating/estimating a value by collecting information from graphical representations to use in known formulas or definitions—requires students to first interact with the graphs of mathematical objects (such as lines and functions) to collect related information about the variables in their known formulas/definitions and then use the information in a known formula to compute the value of a variable. In C2 and C5, students are expected to use the graph of secants, tangents, and functions to find/calculate slope values using $\frac{\Delta y}{\Delta x}$. As well as students' knowledge and ability to use the formulas, these situations make room for students' ability to interact with and use graphs of lines and functions to find information and values needed in their slope formulas. Therefore, students need to know how to extract x- and y-values of various points on given lines or curves. Similar to the first instructional situation, the work is deductive. I use ‘estimating’ in the first two instructional situations when the inputs are non-exact values (e.g., finding rough values for x- and y-intercepts of a function based on a graph) or the output is irrational and students round these values up or down for simplification. This is different than how I use estimating in the fourth instructional situation.

The third instructional situation—writing an algebraic statement using known formulas or definitions—requires students to utilize information (given or collected from graphical representations) and known formulas or definitions to write algebraic expressions/statements or algebraic representations of mathematical objects. Other than knowing the formulas, these situations also make room for the recognition of properties of lines in general, as well as properties of secant and tangent lines in relation to values of rate of change for a function in particular. Because the final solution is not a numeric value in these situations, students must recognize which variables they should substitute numbers for, which ones they should leave as variables, and what the desired algebraic expression looks like (e.g., in the form of $y = mx + b$). This work is also deductive.

The fourth instructional situation—estimating a value using number sense—engages students in a distinct form of calculating in which students use their number sense to estimate a value given a set of related values that were given or were calculated by students. I use *estimating* here in a distinct manner compared to the first and second instructional situations. Here, instructors expect students to use known procedures and/or formulas to collect/calculate multiple values close to the exact solution and decide, or *conjecture*, whether to stop collecting/calculating or keep going to get closer to the solution. Because conjecturing leads to proving or disproving, I did not consider this as a conjecturing situation but rather a special kind of calculating. In C4, after finding slope values of nearby secants around a tangent line, students must generate a ‘good enough’ value as an estimate for slope of the tangent based on the data about nearby secants. In C10, students can demonstrate their knowledge of the graph of a derivative function and recognize the relationship between the points on a derivative function’s graph and the points on the original function’s graph. Because the graphs include information about other points in the domain than what is necessary to solve the problem, students are supposed to choose their points wisely (local max/min, reflection points, x- and y-intercepts) and use their knowledge of the points and their derivative values. Because students can choose which sets of values to use to make their best prediction or educated guess, the reasoning work here is abductive. The expectations placed on students’ deductive and abductive reasoning in these calculating situations can be high, as the ultimate solutions remain unknown to them (Herbst et al., 2010)

I also observed two patterns across instructional tasks. First, calculating CTUs were employed as standalone situations, especially using the limit definition of the derivative at a point (C7) and as a function (C11). This suggests that by using only one CTU in their instructional tasks, instructors aimed to make these two calculating CTUs the primary focus of the instructional tasks rather than a means to achieve other CTUs or learning objectives. Second, I observed that when calculating situations were not the central focus of the instructional task, they functioned as ‘glue’ (or link) between CTUs to connect them in the instructional task. This highlights the integral role calculating situations play in linking other generic types of situations.

Conclusion

Here, I described four calculating instructional situations, how they were typically summoned and incorporated into tasks via CTUs, and the kinds of mathematical work they would engage students with. Although these calculating situations targeted introducing derivatives, they offer insights into the types of calculating situations that are used for teaching college calculus and the kinds of mathematical reasoning students are often expected to engage with, which are vital for improving its teaching via curating tasks that meet the needs of its diverse learners.

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"MY" MEANING FOR UNDERSTANDING THIS PROOF: COMPARING STUDENT PERSPECTIVES TO A MATHEMATICIAN-CENTERED MODEL OF COMPREHENSION

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The literature around proof comprehension focuses on the instructor perspective, we delineate from this by considering the student perspective. In this paper, we characterized how 5 undergraduate students consider what it means to understand a particular proof. With these characterizations, we compared the student participants to mathematicians (from the literature) and found close alignment with most ways a mathematician may identify what it means to understand a proof. Additionally, we found two distinct ways a particular participant identified what it means to understand a proof that may not be considered by mathematicians.

Keywords: Reasoning and Proof, Undergraduate Education, Proof Comprehension

In the realm of undergraduate proof comprehension research, scholars have initially focused on assessing or improving students' comprehension of proof through the development of comprehension tests (e.g., Mejia-Ramos et al., 2017), interventions (e.g., Hodds et al., 2014), or alternative written proof representations (e.g., Roy et al., 2017). Few have focused their studies on students (e.g., Dawkins & Zazkis, 2021; Weber, 2015). Considering the student perspective is important for researchers and instructors in developing productive interventions and activities. One way to consider the student perspective is to consider how it may align with the perspective of experts. In this paper, we set out to answer the following research question: In what ways do students' perceptions of what it means to understand a proof align (or misalign) with the literature's findings on mathematicians' perceptions of what it means to understand a proof?

Relevant Literature

Scholars have compared students with mathematicians and their beliefs about proofs in various ways – through comparing reading of mathematical text (Shepherd & van de Sande, 2014), eye tracking on proof reading tasks (Panse et al., 2018), or beliefs about the purpose of reading proofs (Weber & Mejia-Ramos, 2014). While others have investigated student perceptions on proof reading tasks (e.g., Krupnick et al., 2018; Lew & Mejia-Ramos, 2019; Weber, 2010), few have investigated student perceptions in proof comprehension (Weber, 2015).

Studies show students' reading habits do not differ from mathematicians when considering a particular validation or comprehension task (Panse et al., 2018), yet their actions may differ greatly as they are reading (Shepherd & van de Sande, 2014). In considering the student perspective, Weber (2015) found the students used various strategies such as, using examples to understand confusing claims and justifications, attempting to prove the theorem before reading the proof, and ensuring they understood all the terminology used within the theorem statement.

Theoretical Framing

Mejia-Ramos et al. (2012) developed a model that can be used to develop assessments of student proof comprehension. This model has seven types of questions one could ask students, three of which can be categorized as local questions of proof (L1: meaning of terms and statements, L2: logical status of statements and proof framework, L3: justification of claims) and the remaining four can be categorized as holistic questions of proof (H1: summarizing via high level ideas, H2: identifying the modular structure, H3: transferring the general ideas or methods to another context, H4: illustrating with examples). We consider these seven question types as a synthesis of the literature's findings on what it means to mathematicians to understand a proof. We do note that not all of these may be applicable to any given proof.

When researching the beliefs students can have on proof, it is important to acknowledge the possible ways in which students' argumentations are affected by the institutional meaning in which students are exposed (Recio and Godino, 2001). We argue students' beliefs about proof can be impacted by their schooling. In the context of understanding a proof, we believe students define what it means to understand a proof through their experiences in proof-based classes.

Methodology

The data presented is part of a larger dissertation study investigating the strategies mathematics majors employ as they read proofs with the task of understanding the given proof. Five students were recruited during the Fall 2023 semester after having passed an introduction to proof course (ITP) the prior Spring or Summer semester. Table 1 presents the demographic information of the students using their chosen pseudonyms.

Table 1: Participant information

Name	Race/Ethnicity	Gender	Grade Received in ITP
Adrian	White	Non-binary	B
Claire	White	Female	B
Marie	Latin/Hispanic	Female	A
Tifanni	Latin/Hispanic	Female	A
Zeus	Latin/Hispanic	Male	B

Each student met individually with the first author for a series of interviews. For this paper, we focus on the data from one task-based interview in which students were given a theorem statement and its proof and told to think-aloud as they tried to understand the proof to the best of their ability. The proof presented to students was taken from Chartrand et al. (2013) and showed f is bijective if and only if f^{-1} is a function. Once the students felt they understood the proof to the best of their ability, they were asked a series of questions. The focus of this analysis will be on the students' responses to the question, *What does it mean to you to understand this proof?*

Analysis

Student responses were transcribed and characterized into the different ways they discussed a meaning of understanding the proof (MUP) provided. MUPs were coded using Mejia-Ramos et al.'s (2012) assessment framework. When student responses did not align with any of the question types from the assessment framework, we coded a new instance of a MUP (N1 or N2).

Results

Table 2 shows the types of understanding discussed by the participants which did and did not align with the proof comprehension assessment model (Mejia-Ramos et al., 2012).

Each student had identified L1 (meaning of terms and statements) as a MUP. The participants noted this through indicating they had to know the concepts used in the proof. When asked what it means to understand a proof, Marie stated "one is to understand the concepts." All but Zeus had identified L2 (logical status of statements and proof framework) as a MUP. Students typically focused on how specific statements of the proof were proving or working towards proving the theorem statement. For example, Claire states "they're using f and then they talk about things in the inverse and that's how it and that kind of helps wrap it all together that f is one-to-one." Here, Claire is discussing how statements within the proof work together to show the function is one-to-one, a subgoal of the proof identified by Claire earlier in the interview.

Table 2: MUPs Identified by Participants in Relation to Mejia-Ramos et al., 2012

	Local			Holistic				New	
Student	L1	L2	L3	H1	H2	H3	H4	N1	N2
Adrian	X	X		X	X				
Claire	X	X	X	X	X		X		
Marie	X	X	X	X					
Tifanni	X	X	X		X			X	X
Zeus	X								

For three question types of the assessment model – L3 (justifications of claims), H1 (summarizing via high-level ideas), and H2 (identifying the modular structure) – three participants discussed these as being a MUP. In identifying that L3 is needed to understand this proof, participants typically focused their explanations as understanding each line or each step. For example, Claire explains "Well, I mean you do need to understand every line." H1 was identified by students either through the indication of needing to know the key ideas of the proof or in being able to explain the proof to someone else in simpler terms. Marie states that you would have to be able to "explain that from point A to B." Finally, for H2, participants usually noted the need to know the different parts or sections of the proof. Adrian explained "understand how this stuff, like lines 1 through 4 mean that f is one-to-one, that's one step."

Meanwhile, it was not always the case that the participants discussed a MUP that aligned with mathematicians. No participants had identified H3 (transferring the general ideas or methods to another context). Only one participant, Claire, identified H4 (illustrating with examples) as a MUP. Claire stated that understanding the proof would mean "if someone's like, well what about this function and then you could say 'OK well this is how you would show it's one-to-one or onto' […] you can apply it to a different situation and understand how it works."

One participant (Tifanni) also identified two MUPs that do not correspond directly to question types in the assessment model (N1 and N2). We classified N1 as theorem comprehension. In response to being asked what it means to understand the proof, Tifanni initially stated, "understanding what the theorem is saying first." We believe this statement counts as evidence towards L1, as the theorem statement uses key concepts such as inverse and bijection; however, we believe Tifanni is also considering the theorem statement itself as an object worthy of comprehension. We note that Mejia-Ramos et al. (2012) focused their assessment on the proofs and not necessarily on the theorem statements. As such, L1 focuses on key concepts and statements *within* the proof. Because of this distinction, we believe Tifanni's comment reveals a new MUP. We classify this dimension to be theorem comprehension, understanding what the theorem is saying – whether through the meanings of the key terms in the statement or in understanding what the theorem is saying more holistically.

Tifanni also identified N2, which we classified as disciplinary writing norms. We show two instances of Tifanni discussing disciplinary writing norms in different ways. First, in multiple arguments in the proof, the author uses subscripts to distinguish multiple elements (a_1 and a_2 in the domain or codomain) paired to one element (in the codomain or domain). Yet, for one argument, the author decides to use a prime (a and a') instead of subscripts. To an expert, this may just be chalked up as a notational choice that does not change the substance of the proof, for Tifanni, she questioned why the author made that decision (as if it could affect the substance of the proof). Tifanni showed concern with being able to explain to a class on why the author chose this notation. She worries that this impacts how well she understands the proof, as she would feel unable to explain why. For Tifanni, she believes the notation used for variables in the proof is important and the reasoning for this notation impacts how well someone understands the proof. The second instance where Tifanni discusses disciplinary writing norms is related to the structure and order of arguments. Tifanni notes that someone who understands the proof would know why the proof was structured, in terms of order. "Like for example they could have put this part first, but it well maybe it wouldn't make sense to say this thing first before proving that it was bijective which is one-to-one and onto. Like understanding why." In this quote, Tifanni is discussing the importance of the order in which the arguments are made within the proof. We interpreted both instances to be in reference to disciplinary writing norms, an aspect not discussed by Mejia-Ramos et al. (2012).

Discussion

The study presented extends Weber's (2015) findings by comparing student perspectives to mathematician perspectives. The findings report that student and mathematician meanings for understanding a given proof do seem to align in multiple ways. All students identified L1 and all but one student identified L2. At least three students identified L3, H1, and H2. Only one student identified H4 and no students identified H3. Additionally, Tifanni identified two MUPs that were not in the proof comprehension assessment model. The theorem comprehension MUP may be absent from the assessment model due to the model's focus on the proof and not necessarily on the theorem statement itself, but we suspect mathematicians would agree with the need to understand the theorem statement. Tifanni's second new MUP (disciplinary writing norms) could be due to the differing experience mathematicians have with proof writing in contrast to undergraduate students. Tifanni's ITP course was her first exposure to proof – suggesting Tifanni was still learning the proof writing norms at the time of the study. Questioning the notation indicated her need to increase her understanding of mathematical proof writing norms.

We note the sample size is a limitation of the study and that this report focuses only on the MUPs identified by students with regards to a single proof. Indeed, the participants identified H3 and H4 for other proofs in the larger dissertation study, which is outside the scope of this paper.

These misalignments amongst the student participants and mathematicians do indicate two things. First, students may benefit from being given more tasks that focus them on H3 and H4. Second, students are attending to normative styles of writing that mathematicians have become accustomed to. Scholars have identified norms and values the research mathematics community holds in proof and in writing proof (Lew & Mejía Ramos, 2020; Dawkins & Weber, 2017). Instructors hope to have students adopt these norms and values while taking proof-based courses.

Further research is needed in both mathematician perspectives and student perspectives. The larger dissertation study aims to expand on the knowledge of student perspectives in what it means to them to understand proof and the ways they attempt to understand given proofs.

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CÁLCULO PARA TODOS: UN ENFOQUE NOVEDOSO PARA ESTUDIAR DERIVADAS A TRAVÉS DE ACTIVIDADES DE LABORATORIO

CALCULUS FOR ALL: A NOVEL APPROACH TO STUDYING DERIVATIVES THROUGH LABORATORY ACTIVITIES

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Este estudio investigó el impacto que las actividades de laboratorio del proyecto Cálculo Para Todos tuvo en la comprensión de conceptos de cálculo en estudiantes de segundo año de bachillerato en México. Empleamos una metodología cualitativa para examinar las coordinaciones entre la modelización corporizada de los estudiantes y sus representaciones gráficas. Los datos incluyen un pre y un post test, con ítems de opción múltiple y preguntas abiertas. Los resultados revelaron un cambio positivo en el rendimiento de los estudiantes después de la intervención. Se identificaron múltiples transiciones de aprendizaje influenciadas por la fluidez representacional entre lo concreto y las representaciones gráficas. Este estudio contribuye al entendimiento de cómo los laboratorios pueden promover la comprensión profunda de conceptos de cálculo.

Palabras clave: Modelado, Tecnología, Cálculo, Representaciones Matemáticas, Pensamiento Estudiantil

En muchos países, incluido México, el cálculo se establece como una herramienta clave de la educación matemática en las etapas superiores de la escuela secundaria (grados 11 y 12) y en cursos a nivel universitario (Greefrath et al., 2021). Por un lado, el cálculo sirve como piedra angular para acceder a carreras STEM (ciencia, tecnología, ingeniería y matemáticas) y por otro, como una barrera que propicia el abandono de una trayectoria académica en estas áreas (Burdman et al., 2021) y muchas veces obstaculiza el avance de estudiantes con orientaciones hacia otras áreas como ciencias sociales o humanidades. En respuesta a este escenario, se han generado iniciativas que proponen cambios en la instrucción del cálculo; por ejemplo, la iniciativa *Macalester* (Burdman et al., 2021) utiliza una aproximación al cálculo desde la modelación—este curso reduce el foco en las técnicas de integración y derivación, y en su lugar enfatiza la programación con el software R. Además de destacar la modelación de sistemas dinámicos utilizando ecuaciones diferenciales.

El proyecto *Cálculo Para Todos* (CPT), en el que se basa la presente investigación, se propone como una alternativa factible en la que se entrelaza el trabajo colaborativo con las tecnologías digitales y con los contextos socioculturales de los estudiantes. CPT fue desarrollado a nivel bachillerato por una docente (segunda autora) en el Centro de Bachillerato Tecnológico Industrial y de Servicios (CBTIS) #164 en Ciudad Madero, Tamaulipas, México. El proyecto CPT se encuentra alineado con el nuevo plan nacional de estudios, de la Nueva Escuela Mexicana (NEM). El plan de estudios de la NEM se basa en el compromiso de aprovechar los recursos sociocognitivos de los estudiantes, prestar atención a sus formas de pensamiento emergentes y organizar secuencias de actividades de manera informada por la literatura internacional sobre *progresiones de aprendizaje* (Subsecretaría de Educación Media Superior [SEMS], s.f.).

El objetivo de esta investigación es indagar el impacto que las *actividades de laboratorio* de CPT tuvieron en los estudiantes, en lo referente a la apropiación de conceptos de cálculo al usarlos para modelar fenómenos de cambio desde sus experiencias compartidas. En particular nos preguntamos: *¿cómo los estudiantes coordinan la modelización corporizada y socialmente distribuida de fenómenos de cambio con la representación gráfica de funciones y sus derivadas?*

Antecedentes

Investigadores concuerdan (Lehrer y Lesh, 2003; Moore et al., 2018), que la fluidez representacional puede promover una comprensión más profunda de los conceptos matemáticos al conectar ideas abstractas con situaciones concretas y visualizar relaciones entre variables. Moore y colegas (2013), mencionan que un aspecto importante de la fluidez representacional es la habilidad de trasladar entre y dentro de modos de representación (y cf. Duval, 2017). Estos modos de representación incluyen el uso de modelos concretos refiriéndose a manipulativos físicos o virtuales. Estas investigaciones enfatizan que el uso de modelos concretos y representaciones ayudan a los estudiantes a dar significado a ideas matemáticas y sus aplicaciones. Brady et al. (2022) sugieren que los modelos concretos son parte de un proceso de modelización corporizada, en la que el cuerpo se sintoniza con el objeto físico o virtual (p. ej., un sensor ultrasónico) para generar diferentes tipos de representaciones. Por otro lado, la visión de Kaput y Roschelle (2002), se centra en una visión histórica sobre la evolución cultural de las representaciones, el potencial de los medios tecnológicos y los desafíos de satisfacer las necesidades sociales. Estos autores enfatizan el poder de las representaciones gráficas, dinámicas y ejecutables en el proceso de modelización, como objetos inclusivos y democratizantes, que permiten comprender el cambio y la variación de contextos dinámicos. Además, abogan por el uso de simulaciones computacionales para comprender conceptos del cálculo en contextos concretos. En resumen, desde la fluidez representacional hasta la modelización, el uso de laboratorios para el estudio del cálculo ofrece un entorno dinámico donde estudiantes encuentran oportunidades de construir conceptos abstractos con aplicaciones concretas, explorar representaciones gráficas, y experimentar con simulaciones computacionales.

Metodología

El laboratorio de cálculo consta de un estuche de prácticas basado en tecnología Arduino que, conectado a una computadora, brinda la posibilidad de realizar diferentes experimentos. A su vez, este estuche se divide en tres módulos, el primero orientado al estudio del movimiento unidimensional y la caída libre utilizando un sensor ultrasónico que permite estudiar funciones polinomiales y sus derivadas; las variables a operar son distancia, tiempo y velocidad. El segundo módulo permite analizar el voltaje generado por un mini panel solar con el que se puede analizar el comportamiento de una función senoidal y su derivada; en este caso las variables a operar son voltaje y tiempo. El tercer módulo permite analizar el voltaje producido por un mini aerogenerador (rehilete), cuyos datos tienen un comportamiento exponencial. El prototipo incluye todas las prácticas de laboratorio listas para ser utilizadas en clase con un enfoque transversal (más información en http://calculoparatodos.com/).

Estructura del test

Antes de iniciar con el curso se aplicó un pretest que constó de ocho preguntas de opción múltiple y dos preguntas abiertas en las que se debían elaborar la gráfica de posición y velocidad para un problema en contexto. Al finalizar el curso se aplicó un post test que incluía las mismas preguntas que el pretest. Las dos preguntas abiertas del test se enfocaron en el siguiente problema en contexto:

> "9. Lee cuidadosamente el siguiente enunciado y construye la gráfica de posición vs. tiempo correspondiente: Un robot inicia la toma de datos tocando el sensor (a una distancia 0). Empieza a moverse rápidamente alejándose del sensor, durante un intervalo de 1 segundo. Durante el siguiente intervalo de 2 segundos, disminuye su velocidad hasta que deja de moverse. Mantiene la pausa durante el siguiente segundo. Entonces, se empieza a mover hacia el sensor, durante el siguiente intervalo de 2 segundos, pero, no llega al sensor."

La pregunta 10 solicitó que se construyera la gráfica velocidad vs. tiempo para el mismo enunciado.

El curso en el que se aplicaron las prácticas de laboratorio y los test se llevó a cabo a lo largo del ciclo escolar 2022-2023 en el CBTIS #164 al norte de México con estudiantes de segundo año. En total se trabajó con 194 estudiantes, pero al final solo consideramos una muestra de 163 estudiantes ya que fueron los que resolvieron los dos test (pre y post).

Resultados

El análisis de las preguntas abiertas en los test se realizó mediante una codificación abierta por el método de comparación constante en el pre y post test (Charmaz, 2014; Strauss y Corbin, 1990). En esta etapa codificamos cada ítem en términos de cómo se relacionaba la representación gráfica y la representación verbal/lingüística. Es importante mencionar que nos encontramos en el proceso de analizar que su utilidad va más allá de simplemente identificar los *estados* individuales de los alumnos, ya que nos ha permitido reconocer la importancia de las *transiciones* observadas entre el pretest y el post test. Confiamos en que estas transiciones revelarán el progreso de las formas de pensar y de apropiación del lenguaje del cálculo de los estudiantes cuando describen fenómenos del mundo.

Posterior a la codificación abierta pudimos identificar treinta y cinco estudiantes que respondieron a las preguntas abiertas tanto en el pre como en el post test. Utilizando estos datos pudimos codificar transiciones entre el pre y post test de acuerdo con las modificaciones en la representación gráfica. Elaboramos un libro de catorce códigos que nos permitió catalogar estas transiciones. Éstas nos están ayudando a distinguir transiciones estables en el aprendizaje vivenciadas por los estudiantes antes y después del curso. Para este escrito, elegimos ejemplificar una de las transiciones que ilustran uno de los tipos de pensamiento que encontramos en nuestro análisis.

Transición de “disminuye su velocidad” como ir hacia abajo a la representación correcta. En la Figura 1, se muestran las gráficas de posición del pre y post test que el mismo estudiante elaboró siguiendo el texto del problema 9 (ver sección Estructura del test para el enunciado). En la gráfica del pretest el estudiante puso etiquetas que delimitan las frases en la representación lingüística (AB, BC, CD). Esto indica una estrategia de descomposición del problema que lo ayuda a su comprensión. Si seguimos las etiquetas en la gráfica el segmento de recta que va del origen a A es la interpretación del enunciado “Empieza a moverse rápidamente alejándose del sensor, durante un intervalo de un segundo.” El intervalo de tiempo también concuerda con el segmento dibujado. En esta lógica, el segmento AB corresponde al enunciado “Durante el siguiente intervalo de dos segundos, disminuye su velocidad hasta que deja de moverse.” En este momento la dificultad de representar “disminuye su velocidad” se hace evidente. El resto de los segmentos coinciden correctamente con la representación lingüística.

La gráfica elaborada en el post test muestra puntos sobre la gráfica que corresponden a la descomposición del enunciado, esta vez sin etiquetas. Si seguimos la gráfica en el segundo intervalo [1, 3], se ve una curva que indica que hay una disminución de la velocidad paulatina. Entonces tenemos evidencias de que este estudiante ha avanzado en dos sentidos. Primero, su representación del cambio de velocidad desde alto (positivo) a más bajo (todavía positivo) ahora concuerda bien con la gráfica esperada. Además, el uso de una curva suave refleja un entendimiento profundo del movimiento de los cuerpos reales en el mundo físico. Conjeturamos que las experiencias de capturar movimientos corporales con el sensor ultrasónico han favorecido la sensibilidad a estos matices. Dentro de la muestra de 163, 48 (aproximadamente el 35% del total) representaron “disminuye su velocidad” como en el ejemplo mostrado. Consideramos que estos estudiantes se encuentran en una etapa de transición como la que vivió el estudiante en el ejemplo.

Pre test | Post test

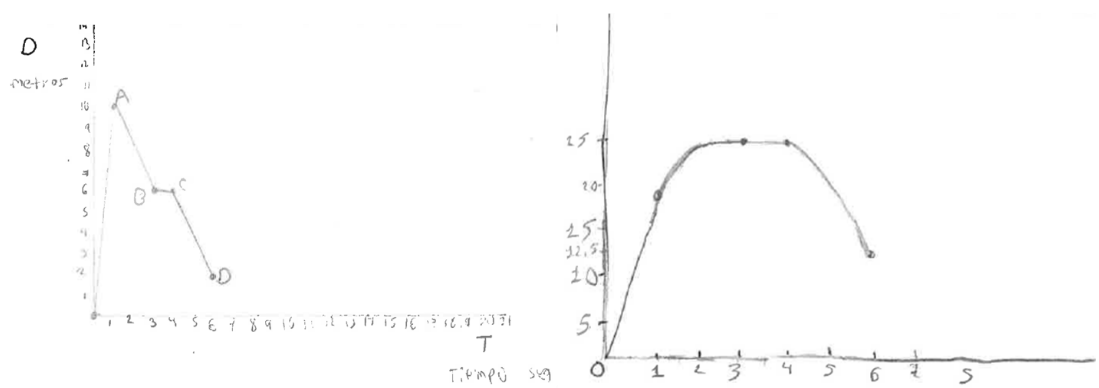

Figura 1. Pretest y post test mostrando Transición de *ir más lento como ir hacia abajo* a la representación correcta de la posición.

Discusión y conclusiones

Nuestro análisis cualitativo reveló la presencia de patrones de formas de pensamiento recurrentes en el grupo de estudiantes y que, en un estudiante fueron influenciadas por las actividades del laboratorio (p. ej., la actividad sobre el movimiento de un cuerpo). Este hallazgo sugiere la existencia de múltiples *transiciones de aprendizaje* que pueden aparecer en este terreno conceptual. Los resultados mostrados vislumbran que estas transiciones están orientadas por la fluidez representacional (Moore et al., 2013) que se da en la coordinación entre la modelización corporizada y la representación gráfica de funciones y sus derivadas (Kaput y Roschelle, 2002). Para fortalecer estos hallazgos, estamos en proceso de análisis de datos adicionales, incluyendo la revisión de grabaciones de video tomadas durante la aplicación de los laboratorios por la profesora, y los trabajos de un subconjunto de los mismos estudiantes, durante una siguiente clase de cálculo integral. Finalmente, consideramos que el proyecto *Cálculo Para Todos* es una propuesta innovadora que revierte la práctica tradicional de pasar de la enseñanza de los conceptos a su aplicación a una práctica en la que, partiendo de las acciones concretas, se significan y resignifican los conceptos matemáticos.

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WHAT IS PROOF? COMPARING PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' AND UNDERGRADUATE MATHEMATICS STUDENTS' PERCEPTIONS

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Key Words: Reasoning and Proof, Preservice Teacher Education, Undergraduate Education

Due to the shortages of secondary mathematics teachers across the United States, many of them receive little or no preparation for teaching mathematics (Sutcher et al., 2016). More specifically, they are encouraged to seek an alternative pathway to be a certified secondary mathematics teacher to meet their school district's hiring needs (Sutcher et al., 2016). With such hiring needs, undergraduate students with a minor in middle school mathematics are more likely to teach middle school students after graduation. Similarly, undergraduate mathematics students (UMs) will have opportunities to teach middle or high school mathematics after receiving their degree. Taken together, it is important to examine and compare pre-service secondary mathematics teachers' and undergraduate mathematics students' proof perceptions.

Three groups of PSMTs[1] and UMs[2] minor, from a Midwestern university in the United States volunteered to participate in this study. The first group consisted of 15 undergraduate students from the "College Geometry" course in Spring 2020. The second and the third group of participants were from the transition-to-proof course titled "Discrete Mathematics" in Fall 2020 and Fall 2021, respectively. We used each participant's description of what a mathematical proof is from the beginning of the semester to identify roles and characteristics of proof they perceived. More specifically, the undergraduate students were all asked to complete a written class activity titled "Getting to Know You" on the first day of a class that included a question of what a mathematical proof is. Thus, each participants' written work was the primary source of data for categorizing PSMTs' and UMs' descriptions of a mathematical proof.

Regardless the participants' mathematics backgrounds, the two most common roles of proof cited by them were verification and explanation. More mathematics teaching majors and mathematics majors mentioned the roles of systemization and communication than other participants did. Also, the majority of participants focused on the two characteristics of proof—set of accepted statements and modes of argumentation. If PSMTs could be provided with rich opportunities to engage in a variety of proof-related tasks at the undergraduate level, then they are more likely to view proof as a meaningful tool to teach and learn mathematics (Knuth, 2002).

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[1] PSMTs in this paper refer to a middle school teaching minor, a middle school teaching major, and a mathematics teaching major.
[2] UMs in this paper refer to an undergraduate student with a major or minor in mathematics.

Thus, it would be interesting to study how PSMTs' and UMs' roles of proof affect the ways in which they construct and evaluate proofs.

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QUALITY OF WRITTEN JUSTIFICATIONS AND EXPLANATIONS AS AN INDICATOR OF CONCEPTUAL UNDERSTANDING IN AP CALCULUS

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This study explored a correlation between the quality of students' written justifications and explanations on AP Calculus assessments and their achievement on related computational problems. The aim is to provide research that supports AP Calculus teacher's use of written justification and explanation as a teaching and learning tool. The study employs a convergent mixed-methods correlational design. Comparisons are made between the ratings students are given on their written justifications and explanations on the written portions of classroom assessments and their raw scores on computational problems. The two sets of data do not overlap. Written samples were examined using discourse analysis – a well-established qualitative method for analyzing student writing in mathematics (Austin & Howsen, 1979; O'Halloran, 2005; Pimm, 2004). Qualitative data was transformed into ordinal data using a rubric based on Toulmin's model for argumentation (Brockriede, 1960) which diagrams the basic components of an argument. The ordinal data was compared to quantitative data (on a ratio scale) using regression analysis to explore the possibility of a positive correlation. The study used a worldview of pragmatism, which, according to Creswell and Plano Clark (2018), focuses on the "importance of the question asked rather than the methods, and on the use of multiple methods of data collection to inform the problems under study" (p. 37).

Two research questions were explored: 1) How is mathematics conveyed by students in high school Calculus when justifying or explaining their responses? and 2) What is the relationship between the quality of AP Calculus students' written justifications and explanations and their conceptual understanding as demonstrated on computational parts of classroom assessments?

Participants include 28 AP Calculus AB students from the same suburban high school in Ohio. AP Calculus AB is a high school course that covers the content of a first-semester college calculus course. At the course's end, students take an exam for which colleges could award them credit. The assessment instrument for this study was a unit test given during the third quarter of the AP Calculus course.

Initial results of Spearman's rank correlation (2024) indicated a moderate positive association between justification scores and computational scores ($R_s[28] = 0.576, p = 0.005$). After looking at a scatter plot of the data, two data pairs that diverged from the line of best fit were removed. One of these pairs had a written justification score of 1 and a computational score of 3, indicating that this student was much more proficient at computing the answer than explaining its meaning. The other pair had a written justification score of 3 and a computational score of 1, indicating that this student could explain the answer without being able to perform a correct computation. These were the only two students of the original 28 who had these scores. When the Spearman correlation was performed with the remaining 26 data pairs, the results indicated a strong positive correlation ($R_s\ [26] = 0.707, p = 0.001$).

In a small number of cases, the data diverged from the general trend; because of this, I recommend a follow-up study in which interviews are performed with a subset of students from

the original population representing varying score ranges using an explanatory sequential mixed-methods design.

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EXPLORING THE USE OF HISTORICAL CONTEXTS IN TEACHING LOGARITHM

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This study investigated the effectiveness of integrating history-infused lessons on students' understanding and attitudes towards logarithms. The theoretical framework drew from sociocultural perspectives and embodied cognition, emphasizing the social and emotional dimensions of learning. Design-based research principles guided the iterative development of history-infused logarithm lessons. Data was collected through pre- and post-assessment tests, interviews, and classroom observations. The findings indicated a significant improvement in students' post-test scores, suggesting a reduction in their fear of logarithms. Additionally, interviews revealed a positive shift in students' perceptions of logarithms, from abstract and intimidating to practical and relatable.

Keywords: History-infused mathematics, logarithms, secondary mathematics education, student attitudes

The research literature emphasizes the importance of logarithms in both advanced mathematics and real-world contexts, including sound measurements (decibels), earthquake magnitudes (Richter scale), star brightness, and chemical properties (pH balance). However, many students struggle to grasp the conceptual underpinnings of logarithms and often resort to rote memorization of rules, as noted by various authors (Berezovski, T., 2008; Kuper & Carlson, 2020; Weber, 2016). The challenges faced by students include interpreting logarithms as the "inverse of exponents" and developing a coherent understanding of logarithmic notation, logarithm properties, and the application of logarithmic functions (Kuper & Carlson, 2020; Berezovski, T., 2008; Chua & Wood, 2005; Gol Tabaghi, 2007; Strom, 2006).

To address these issues, researchers and educators have suggested a variety of strategies. These include using concrete materials (Thompson, 1994), implementing authentic assessments such as project-based learning and computational thinking (Shin et al., 2021), engaging students with game-based learning (Barab et al., 2010), problem-based learning (Hmelo-Silver, 2004), and effective teaching methods (Larmer, 2018). The use of gestures alongside diagrams (Walsh & Hord, 2019), gestures combined with manipulatives (Beilstein, 2019), and incorporating the history of mathematics (Liu, 2003; Poh & Dindyal, 2016; Sampaio & Batista, 2018) have also been recommended. This study aims to bridge the gap in research regarding the teaching of logarithms by utilizing the history of mathematics in combination with gestures. The research questions guiding this study are as follows: (1) How do history-infused logarithm lessons aid in reducing students' fear of logarithms? (2) How do students' perceptions of logarithms change over the duration of the history-infused logarithm program?

Theoretical Framework

This study was grounded in sociocultural perspectives for student learning, a history-infused program, and embodied cognition for evaluating understanding. It focused on social constructivism, which emphasized the role of social interaction in cognitive development and suggested that learning occurred best in a social context. The study also incorporated the dynamic nature of assessment, including formative, diagnostic, prognostic, or summative assessment. Vygotsky's sociocultural perspective underscored the extensive impact of social learning, suggesting that learners did not engage with new knowledge in isolation. Key instructional aspects in Vygotsky's perspective were mediation, scaffolding, and creating a zone of proximal development (ZPD). Building on this, the instructional technique in the history-infused math lessons involved exploratory activities and mediational strategies, including scaffolding based on prior knowledge. This study also emphasized embodied cognition, focusing on how humans used their bodies to express thought processes, like gestures. Students' multimodal approaches, such as gestures, were coded and compiled based on McNeill's typological categories (1992), allowing for a holistic approach to teaching and learning logarithms

Methods

Participants, Settings, and Programs

The study took a holistic approach, incorporating both quantitative and qualitative data collection methods to assess the effectiveness of integrating historical insights into logarithm teaching. This approach was underpinned by design-based research (DBR) principles, enabling iterative refinement of teaching strategies based on observed student interactions and outcomes. The study took place at a private high school in Western New York, United States, with a student-teacher ratio of 12:1. The focus was on 14 students (10 girls and four boys) in Grades 11 and 12, all of whom had prior exposure to logarithms in their mathematics courses.

The curriculum, based on the Precalculus with Limits: A Graphing Approach by Ron Larson (High School Edition, 6th Edition), was adapted to incorporate historical insights into the discovery of logarithms. The goal of the first phase was to develop an initial design of the program, which consisted of three history-infused math lessons. These lessons were developed by integrating the history of logarithms, allowing students to explore historical perspectives on the discovery of logarithms by John Napier, and other mathematicians (e.g., Pythagoras), tailoring students' participation in peer collaboration through the lens of history of mathematics, and improving learner engagement in the instructional process in the form of mini projects on history-infused mathematics. Students were asked to use log tables and watch a video of the process. Students also explored the history of mathematics related to logarithms and did a presentation. In the second phase, the program underwent iterative design to test and refine it. This second iteration, to be conducted in the spring of 2023, involved designing three history-infused modules, each consisting of three lessons with scenario-based problems (e.g., Scenario-based Log) and historical approaches (e.g., Using Log Tables). The historical accounts of the discovery of logarithms will be introduced from existing sources, highlighting how logarithm computations were performed before calculators.

Data Collection and Analysis

Data collection involved pre- and post-assessment tests, interviews, observations, and analysis of classroom artifacts. Pretests assessed students' baseline knowledge, while post-tests measured their understanding after the history-infused lessons. Interviews and observations offered qualitative insights into student engagement, attitudes, and understanding. The study was conducted over two weeks, covering initial assessment, implementation of the history-infused program, and subsequent assessment and interviews. Data analysis was conducted using a mixed-method approach, combining quantitative and qualitative methods. For quantitative analysis, descriptive statistics were computed to compare pretest and posttest scores, and the Wilcoxon Signed-Rank Test was used for inferential analysis to assess the effectiveness of the history-infused lessons. Error analysis of participants' written responses was also conducted. For qualitative analysis, interviews were transcribed using the ELAN annotation tool, and thematic analysis was conducted. Open-coding was done using ATLAS.ti software to categorize data from surveys and interviews. Additionally, gestural analysis was conducted, categorizing gestures based on McNeill's framework, and disagreements between coders were resolved through consensus. The general inductive approach was employed to analyze qualitative data, systematically organizing and summarizing textual data.

Summary of Findings

Analysis of pre- and post-test showed that students' average scores in the posttest (M = 76.2, SD = 17.4) were significantly higher than their average scores in the pretest (M = 50.2, SD = 21.1).

How History-Infused Logarithm Lessons Alleviate Students' Fear of Logarithms:

The analysis of pre- and post-test scores shows that the history-infused logarithm program led to a statistically significant improvement in students' understanding of logarithms. This finding is particularly noteworthy considering the pre-existing fear and apprehension that many students typically harbor towards this complex mathematical concept. Interviews with students provided deeper insights into the impact of history-infused lessons on students' emotional engagement and attitudes towards logarithms. A majority of students expressed that the historical context provided in the lessons made logarithms seem more accessible, relatable, and less intimidating. Many students indicated that understanding the origins and evolution of logarithms gave them a sense of connection to the subject, and a better appreciation for its practical significance. Student L7 articulated this sentiment, saying, "It helped me understand it better because I can be more appreciative of the mathematicians back in the day and it gets me more interested in math, so I will be motivated to learn more about the concepts knowing the philosophers that contributed to it." Incorporating historical narratives and activities into the logarithm curriculum served as a cognitive scaffold for students, allowing them to contextualize complex mathematical concepts within a narrative framework. Students appreciated the opportunity to engage with mathematical ideas in a more holistic and multidimensional manner. Furthermore, the interactive and collaborative nature of the history-infused lessons encouraged students to approach learning logarithms with a sense of curiosity and adventure, rather than fear and reluctance.

Shifts in Students' Perceptions of Logarithms Through History-Infused Logarithm Program:

Through the history-infused logarithm program, students' perceptions of logarithms underwent a noticeable shift. Before the intervention, students primarily viewed logarithms as abstract and disconnected from real-world contexts. They often perceived logarithms as challenging, even forbidding, due to the complex nature of mathematical manipulations involved. The pretest data showed that students had a limited understanding of logarithmic properties and frequently made errors in their application. Common mistakes included misinterpreted language errors and logically invalid inference errors, suggesting that students' conceptual grasp of logarithms was limited. However, post-test data revealed a marked improvement in students' perception of logarithms. Students began to view logarithms as a valuable tool with practical applications, particularly in the context of historical problem-solving. They expressed newfound confidence in their ability to tackle logarithmic calculations and demonstrated a clearer understanding of logarithmic properties and their applications. This shift in perception can be attributed to the rich historical narratives that were integrated into the program, which allowed students to see logarithms as a dynamic and evolving mathematical concept with a rich cultural and historical significance. Students began to appreciate the versatility of logarithms and how they are rooted in the history of human endeavor.

Discussion

The findings of this study echo the conclusions of previous research and contribute to our understanding of the potential benefits of incorporating history into mathematics education. Previous studies have demonstrated that history-infused mathematics lessons can lead to improvements in students' conceptual understanding and engagement (Alibali & Nathan, 2012; Berezovski, 2008; Howell et al., 2017). The findings of the current study extend this research by focusing on students' emotional responses to history-infused lessons and their impact on attitudes towards mathematics. The theoretical implications of this study align with cognitive theories such as Vygotsky's sociocultural theory of learning and Hmelo-Silver's Problem-based Learning (PBL) theory (Kozulin et al., 2003; Hmelo-Silver, 2004). Vygotsky's theory emphasizes the role of social interaction and cultural context in shaping learning, suggesting that the historical narratives embedded in history-infused lessons can provide students with meaningful cultural tools that facilitate learning. Hmelo-Silver's PBL theory focuses on the importance of problem-solving and authentic, real-world tasks in promoting deep understanding. The historical context provided in history-infused lessons can serve as a rich source of problems and tasks that are relevant and engaging for students. From a practical perspective, the findings of this study suggest that incorporating historical contexts into mathematics instruction can have a positive impact on students' attitudes and engagement. By presenting mathematical concepts in a historical context, educators can make abstract concepts more concrete and meaningful for students, leading to increased motivation and interest. Additionally, the emotional engagement fostered by history-infused lessons can help to alleviate students' fear and anxiety about mathematics, creating a more positive and supportive learning environment.

The limitations of this study should be acknowledged. The study was conducted at a single high school, limiting the generalizability of the findings. Additionally, the study focused on a specific topic within mathematics (logarithms), and the findings may not apply to other mathematical concepts. Future research could explore the impact of history-infused lessons on a wider range of mathematical topics and in different educational contexts. Moreover, future studies could investigate the long-term effects of history-infused lessons on students' attitudes and engagement, as well as the role of technology in enhancing the effectiveness of these lessons.

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DEVELOPING DIRECTED-LENGTH DEFINITIONS FOR THE TANGENT AND SECANT FUNCTIONS

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In this study, we examined preservice secondary mathematics teachers' reasoning as they developed directed-length definitions for the secant and tangent functions. We describe their process of developing the new concept definitions and we identify the trigonometric contexts in which their reasoning was embedded. We then describe how those concept images and definitions were used in the preservice teachers' reasoning. The preservice teachers primarily drew from two trigonometric contexts (unit circle and right triangle) as they developed new concept definitions.

Keywords: Cognition, Precalculus, Preservice Teacher Education, Reasoning and Proof

Trigonometry is important for both practical applications and as a basis for higher-level mathematics. The Common Core State Standards for Mathematics and National Governors' Association Center for Best Practices & Council of Chief State School Officers (2010) recommended that high school students learn trigonometric functions. Despite the importance and variety of purposes of trigonometry (Hertel, 2013), there is consensus from the field that the topic is not learned well (e.g., Brown, 2005; Moore, 2014). Thus, teachers need a deep understanding of trigonometric functions.

Trigonometric Stances

Instructors may have a stance toward trigonometry that "establishes boundaries indicating which mathematical ideas" (Hertel, 2013, p. 105) they consider trigonometry. When discussing different trigonometric definitions, Hertel (2013) described three hierarchical stances: (a) right-triangle trigonometry, (b) circle trigonometry, and (c) analytic trigonometry. A circle stance, for example, includes both right-triangle trigonometry and circle trigonometry contexts. In right-triangle contexts, functions are defined as ratios of sides of triangles (e.g., $\sin\theta = \frac{\text{length of side opposite } \theta}{\text{length of hypotenuse}}$), whereas in circle contexts they are defined in relation to the unit circle (e.g., $\tan\theta = \frac{y}{x}$) such that (x, y) is a point on the unit circle. Using a variety of contexts when introducing trigonometric concepts may lead to a more robust understanding.

In addition to using and connecting different trigonometry definitions, researchers have found that students who used quantitative reasoning (Thompson, 1990) and covariational reasoning (Carlson et. al., 2002, p. 354) were able to reason about trigonometric functions more fluently (Moore 2010, 2013, 2014). Given the promise of quantitative and covariational reasoning and the benefits of strengthening connections between different trigonometry definitions and contexts, we expand on Moore's (2014) work of examining the effects of using directed-length trigonometry definitions to improve the teaching and learning of trigonometry.

Toward a New (Old) Context

In the directed-length context, the six trigonometric functions used in modern-day trigonometry (i.e., sine, secant, tangent, cosine, cosecant, cotangent) are all defined as directed lengths of segments along chords, secants, or tangents in a circle (see Figure 1) that depend on the length of the associated arc. We labeled the sine and cosine functions as the chord and cochord, respectively, to highlight the connection to the historical chord and half-chord functions as well as the mistranslation of the word chord into the word sine (Kennedy, 1969). This choice was also made in the classroom lessons for this study. Thus, there are times when the participants refer to the sine or cosine functions as chord or cochord in our results.

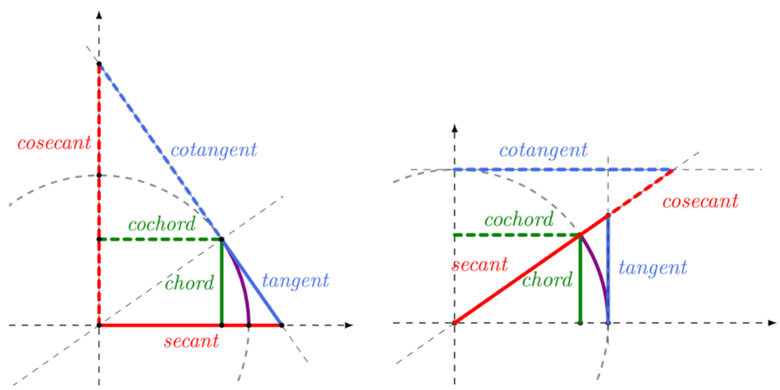


Figure 1: Directed-Length Context of Trigonometry

Hertel and Cullen (2011) found the use of a directed-length approach within a dynamic geometry environment led to a more robust understanding of trigonometric functions. Using a directed-length approach, Cullen and Martin (2018) "focused on the first in a sequence of learning activities in which PSTs reasoned quantitatively about two dynamically changing objects in a circle, and whether those objects could be considered inputs and outputs of a function" (p. 259). In their work, the PSTs engaged in an activity without knowing the task was related to trigonometry. Instead, the PSTs were directed to focus on the concept of function, which they eventually connected to the sine function. Tall and Vinner's (1981) framework of concept image and concept definition was a helpful lens through which to describe how PSTs were making sense of the functions in that study. However, that study's focus was limited to the sine function and the PSTs were not aware that the exploration was related to trigonometry.

In the current study, we tasked PSTs with identifying the needed objects in a dynamic geometric construction (e.g., arcs, lengths of various segments) to define the tangent and secant functions. As the PSTs worked through the activity, they reasoned by determining consistency between their circle stance definitions and a directed-length context. Specifically, we addressed the research questions: Which concept images and concept definitions do PSTs evoke while defining tangent and secant in a novel trigonometric context? How do PSTs reason with their evoked concept images and definitions?

Theoretical Perspective

In this study, PSTs explored a directed-length stance toward trigonometry, specifically seeking to establish consistency between a circle stance and directed-length definitions for tangent and secant functions. Given the focus on the definitions, we relied on concept image and definition (Tall & Vinner, 1981) to guide our analysis. Following Moore (1994) we considered concept image to be "the set of all mental pictures that one associates with the concept, together with all the properties characterizing them" (p. 252). Concept definition "refers to a formal verbal definition that accurately explains the concept in a non-circular way, as might be found in a mathematics textbook" (Moore, 1994, p. 252). In our study, we describe how PSTs leveraged their existing concept images for the tangent and secant functions to support their developing intuitions about a directed-length stance toward trigonometry. Based on Cullen and Martin's (2018) report that PSTs referred to the unit circle when identifying the sine function and Hertel's (2013) findings about the prevalence of right-triangle and circle stances, we anticipated that PSTs' concept images would be primarily anchored in the circle stance. By focusing our analysis on concept images, we hoped to link those concept images to the trigonometry definitions and contexts from which PSTs reasoned during their classroom investigations of the tangent and the secant functions. We identified their concept images as being anchored within two definitions (i.e., right triangle and circle), both from the circle stance provided by Hertel (2013).

Methods

The participants in this study were 23 PSTs, who had completed at least 60 hours toward a degree in mathematics with a focus on secondary education. The setting for the study was a semester-long course—taught by the second author—focused on problem solving connected to the secondary mathematics curriculum and the affordances of technology in those contexts. Full class and small-group discussions were videorecorded. Here, we discuss data from the first and second week of a 6-week instructional sequence on trigonometry. During the sequence, the instructor focused on promoting PSTs' use of quantitative reasoning, encouraging them to focus on objects rather than numeric values.

Prior to this lesson, PSTs determined that a vertical half-chord stemming from a dynamically changing terminus of the arc could be considered an object-centered geometric definition of the sine function. In the lesson examined in this paper, the instructor displayed the same dynamically accumulating red arc (with terminal point C; see Figure 2) in GeoGebra. While gesturing to the dynamically changing arc, the instructor asked the PSTs to identify distinct segments associated with the circle's arc that would represent the tangent function's output and the secant function's output. In this representation, the circle's arc serves as the function's input and the tangent and secant segments, respectively, serve as the function outputs. Although the instructor mentioned this implicit connection between the circle arc and related segments during his lesson, the relationship between two quantities was not repeatedly emphasized. Here, we use the instructor's language that typically refers to the directed-length concept definition of the tangent and secant function as simply the length of the segments along the tangent and secant, respectively.

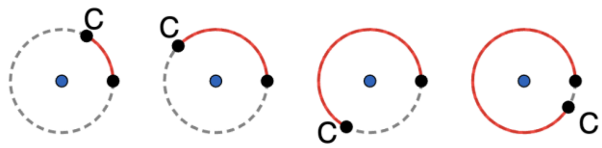


Figure 2: Initial Diagram as Point C Varies

To initiate the exploration into the directed-length approach of trigonometric functions, the instructor asked PSTs to think about the tangent function. To support small-group discussion, the instructor suggested that PSTs consider the following questions.

1. What is “the” definition of the tangent function?
2. When is the tangent equal to zero, undefined, positive, and negative? How do you know?

A similar set of questions helped promote discussion of the secant function after the class negotiated a geometric definition for the tangent function. Our analysis focused on identifying the evoked concept images and definitions during PSTs' reasoning about tangent and secant functions as well as from which context the images and definitions were being drawn. When PSTs referred to sine, cosine, or tangent as ratios of sides of right triangles (e.g., opposite, adjacent, hypotenuse) we classified their concept definitions or images as aligned with a right-triangle context. When PSTs referred to any of the six trigonometric functions defined as ratios of x- and y-coordinates from the unit circle (e.g., $\tan\theta = \frac{y}{x}$), we classified their evoked concept definitions as aligned with a unit-circle context. When PSTs referred to any of the six trigonometric functions defined as ratios involving another function (e.g., $\tan\theta = \frac{\sin\theta}{\cos\theta}$), we examined the way they used these definitions to determine a likely context. When other concept images were evoked, we identified and described likely sources of those concept images. Any disagreements were resolved through team meetings until consensus was reached.

PSTs primarily worked in small groups. The pseudonyms we used for each PST identified their group membership. That is, PST-A1 was a member of Group A. In our analysis we identified the concept definitions and images that PSTs evoked as they worked. We linked those concepts to the trigonometric contexts. Finally, we compared the reasoning used as the PSTs identified definitions for both the tangent and secant functions.

Results

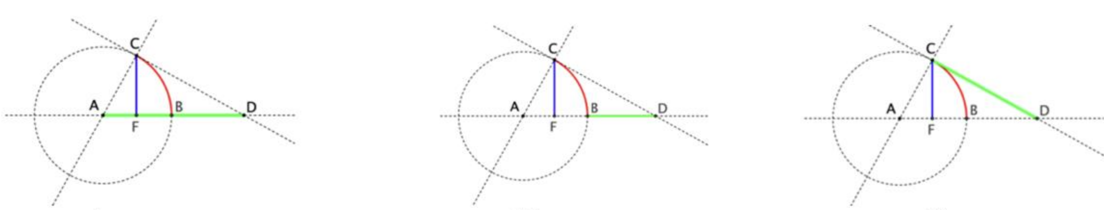
PSTs worked to identify segments that, when paired with the arc, would represent a viable definition for tangent or secant in their emerging understanding of the directed-length context. Their reasoning followed a pattern of exploring in GeoGebra, identifying a segment that might represent one of the trigonometric functions in the directed-length context, then crafting an argument to either justify that their proposed segment was consistent with concept images they held about the tangent or secant, respectively, or discard it because it contradicted their prior knowledge. The definitions PSTs evoked during this process were those native to the right-triangle and circle contexts. However, they also evoked concept images related to benchmark values of each function. Next, we share some of the PSTs’ reasoning in chronological order because the collective argumentation was cumulative. Even so, we observed that when one segment candidate was discarded, the reasoning cycle, starting from identifying a new segment candidate and comparing it to their concept definitions or concept images, was consistent.

Development of a Directed-Length Definition for Tangent

Conversations among members of Group A illustrate the cycle of generating candidates for the tangent function then constructing arguments for or against the candidate segment based on concept definitions and images. An excerpt from a full-class discussion will illustrate their cycle of reasoning. We noted that PSTs drew primarily on the circle stance, encompassing all trigonometry contexts. However, other mathematical concept images were evoked, as well.

PST-A2 proposed that the tangent function ought to be defined as the slope of $\overline{AC}$, rather than by the segment itself. PST-A2 argued "tangent equals sine divided by cosine and so, sine is your *y*-value, cosine is your *x*-value." Here we see the PST reasoning about the tangent moving quickly between and among definitions. When the PST stated "tangent equals sine divided by cosine" we considered $\frac{\sin\theta}{\cos\theta}$ to be evidence that a right-triangle concept definition was evoked with $\sin\theta$ representing the opposite side of the triangle and $\cos\theta$ representing the adjacent side of the triangle. However, it is possible that PST-A2 was simply recalling a memorized definition native to a circle context. Likewise, because the class had previously established that half-chord $\overline{CF}$ (shown in Figure 3) was a representation for $\sin\theta$ in the directed-length context they were exploring, PST-A2 may have been drawing on a component of the newly developing directed-length context as well. When the PST stated "sine is your *y*-value, cosine is your *x*-value," we considered that an indication that he evoked a concept image of slope as "rise over run." Thus, although teasing apart PST-A2's concept images in any given moment could be argued from a variety of perspectives, it does appear that the PST was drawing from several circle-stance ideas and liberally making connections among several mathematical and trigonometric ideas. The PSTs were building their argument by connecting to other concepts images or definitions that they seemed to assume were taken as shared by the group.

The small-group discussion was revisited in a full-class discussion. The instructor asked PSTs to consider when the tangent would be 0. PST-A2 offered "Tangent is zero when sine is zero because tangent is defined as sine over cosine." The instructor recorded the definition at the whiteboard (i.e., $\tan\theta = \frac{\sin\theta}{\cos\theta}$). The instructor then asked the class when the tangent would be undefined. Many PSTs gave a choral response: "when cosine is zero." PST-A3 added "when the denominator is zero something is undefined." This discussion illustrates how PSTs used benchmark values in conjunction with the definition $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to support their arguments about whether certain segment candidates might be viable for the tangent function for a directed-length definition.



A B C

Figure 3: PSTs' Tangent Conjectures

In a subsequent full-class discussion, PSTs from various groups shared the tangent-segment candidates they had considered in their groups. These tangent candidates are shown in green in Figure 3. PST-C2 created line $\overleftrightarrow{CD}$, perpendicular to line $\overleftrightarrow{AC}$ at point C. PST-F1 suggested that segment $\overline{AD}$ (green horizontal segment in Figure 3A), extending from the center of the circle to the point at which it intersected the tangent line $\overleftrightarrow{CD}$, might represent the tangent function. PST-C1 argued that segment $\overline{AD}$ could not represent the tangent function because when the arc has measure zero "the sine is zero, which means the tangent should be zero, but the green segment ($\overline{AD}$) has [nonzero] length at zero." Thus, he used his evoked concept definition, "tangent is sine over cosine," along with a benchmark value for the tangent function to test and reject the first conjecture.

PST-C1 continued his thought process and conjectured that the tangent function could be represented by the green segment $\overline{BD}$ shown in Figure 3B. (We note this segment comprises the exsecant, however, this was not discussed in the class.) This conjecture was abandoned without discussion because PST-F1 conjectured that the green segment $\overline{CD}$ along the tangent line $\overleftrightarrow{CD}$ in Figure 3C represented the tangent function. One small group of PSTs, checking benchmark values, determined that this choice satisfied the requirement that the tangent was undefined at $\frac{\pi}{2}$ because the tangent segment and the horizontal secant are parallel when point C is at the $\frac{\pi}{2}$ position (i.e., the arc was one-quarter of the way around the circle). PST-D1 concurred and added "the length of that green line, as it [point C] gets closer to the point at the top of the circle $\left[\frac{\pi}{2}\right]$, it's going to be increasing to infinity, which is what the tangent line does." In this argument, PST-D1 appeared to be reasoning in a circle context as they spoke about the position of C on the circle in measured terms (i.e., the $\frac{\pi}{2}$ position). We classified this a circle context because they described the position of terminal point C, rather than the measure of the arc, as the independent variable leading to the tangent function output.

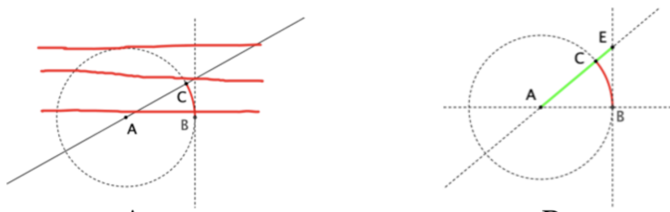
After generating a promising candidate for the tangent function using benchmark values, the PSTs attempted to justify that $\overline{CD}$ constituted the tangent in the directed-length context by showing that it satisfied definitions and concept images from the right-triangle and circle contexts. PST-D2 argued that $\overline{AC}$ and $\overline{CD}$ form a right angle, creating right $\triangle ACD$ with non-right angle $\measuredangle CAD$ (or θ) and reported that $\overline{AC}$ as the radius of the circle could be labeled as one unit in length. Drawing from a right-triangle context, she argued "tangent is opposite over adjacent," the ratio of $\frac{opposite}{adjacent}\left(\frac{CD}{AC}\right)$ must be equal to $\tan\theta$.

In this way, the PSTs' reasoning fluidly incorporated concept images from right-triangle and circle contexts. This pattern repeated in the PSTs subsequent search for an appropriate definition for the secant function.

Development of a Directed-Length Definition for Secant

To help PSTs construct a definition for the secant function, the instructor asked PSTs to consider key features of the secant function, (e.g., when is it zero and when is it undefined?). During small-group discussion, PST evoked two main concept definitions of secant, $\sec\theta =$

$\frac{1}{\cos\theta}$ and $\sec\theta = \frac{hypothenuse}{adjacent}$. These two definitions were central to much of PSTs' reasoning about which objects (segments) might define the secant function in a directed-length context.



A B

Figure 4: PSTs' Secant Conjectures

PST-D2 proposed the first candidate, the interior portions of the parallel red segments in Figure 4A. She argued that the top segment is an undefined secant as it becomes a tangent line to the circle. This candidate was ultimately rejected. However, unlike the first candidates for the tangent function, the red parallel lines were consistent with a concept definition of a geometric object for which it was named, a secant (i.e., a line that cuts a circle at two points). Prompted by the instructor, the class argued that the parallel red segments could not define the secant function because these red segments are bounded by the diameter. The secant function, they argued, grows without bound. Because this candidate was not viable, another PST suggested a different candidate for consideration.

PST-B1 suggested segment $\overline{AE}$ in Figure 4B might define the secant. His argument was based on reasoning from a circle context. PST-B1 argued that when point C moves to point B, segment $\overline{AE}$ will "be one and it will never be lower than one." He explained that as point C reaches $\frac{\pi}{2}$ (circle concept), $\overline{AE}$ will be undefined. PST-B1 compared the length of $\overline{AE}$ to benchmark values of one and undefined to create and support $\overline{AE}$ as a candidate for the secant. With the conjecture surviving the benchmark-checking phase of scrutiny, PSTs worked toward proving that defining $\overline{AE}$ as the secant was consistent with a previously known definition.

PST-B1's argument appears to follow this logic: $\overline{AE}$ could be the secant because $\sec\theta = \frac{hypotenuse}{adjacent} = \frac{\overline{AE}}{\overline{AB}} = \frac{\overline{AE}}{1(radius)} = \overline{AE}$. In Table 2, we parsed PST-B1's logic to identify key elements (in bold) and show their connection to the context from which it was drawn. Even though the reasoning was incorrect, the way he drew on multiple stances was evident.

Table 2: Contexts PST-B1 Used When Reasoning About Secant

Claim	Context	Possible Explanation
"the idea is the **opposite over the hypotenuse**"	Right triangle	This is a description of the sine function based on triangle side lengths.
"that is whatever the **length of the chord (i.e., sine) over one**"	Circle and directed length	Because the PST substituted one for the hypotenuse, the PST seemed to be envisioning a triangle embedded in a unit circle. And "chord" is a directed

		length term for sine.
"So, we want it to be the other way, we want **one over** whatever **the length of our secant** is gonna be."	Right triangle	The PST may have misspoken and meant to say $\frac{\mathbf{1}}{\mathbf{\sin\theta}}$, perhaps because they misremembered the inverse relationship with cosine and secant.
"one right here (referring to $\overline{\boldsymbol{AB}}$); and you have the **hypotenuse** right there (referring to $\overline{\boldsymbol{AE}}$)"	Right triangle (embedded in a circle)	If $\overline{\boldsymbol{AB}}$ has value 1, then the circle is a unit circle. If $\overline{\boldsymbol{AE}}$ is the hypotenuse, then $\triangle\, \boldsymbol{ABE}$ is a right triangle.

Other PSTs contributed to the collective argumentation to complete the argument.

PST-D3: Can't we say its hypotenuse over adjacent? Like 1 over cosine is hypotenuse over adjacent.

PST-B1: That's like kinda like where I am trying to go. We got one (referring to $\overline{AB}$) over the hypotenuse (referring to $\overline{AE}$) and that's why I am thinking that…that's gonna be the length…

PST-A1: So if we are gonna go with PST-B1's idea, then $tan^2\theta + 1^2 = sec^2\theta$.

PST-A1 completed the argument for PST-B1's conjecture (i.e., $\overline{AE}$ is the secant function) by referring to a trigonometric identity, a concept image associated with his circle stance toward trigonometry. Although these identities are not definitions drawn directly from a trigonometry context, they are usually proved using algebraic techniques that draw on equivalent ratio representations of trigonometric functions. From a bird's eye view, one can see that even arguing collectively, the PSTs built the argument while fluctuating between right-triangle and circle concepts to support their reasoning.

Discussion

In this study, we identified PSTs' evoked concept images and definitions as they establish directed-length definitions. PST reasoned from the circle stance (Hertel, 2013), making connections among concepts and definitions from circle, right-triangle, and directed-length contexts as established the new definitions for the tangent and secant functions. Our findings show that having access to the most prevalent trigonometry stance, as well as the sine definition from directed-length context, was sufficient for the PSTs to establish the consistency between the new directed-length definitions and their existing trigonometry stance.

We also described some common ways in which the PSTs used the concept images and definitions. For both the tangent and secant, PSTs used benchmarking with existing concept definitions (i.e., $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\sec\theta = \frac{1}{\cos\theta}$) to generate conjectures about which segments were viable candidates for representing the outputs for the tangent and secant functions. To prove (for tangent) and attempt to prove (for secant) their conjectures, PSTs used concept definitions from right-triangle trigonometry to connect an existing concept definition (e.g., $\tan\theta = \frac{opposite}{adjacent}$) to the directed-length context. For the secant proof, PST-A1 leveraged a Pythagorean identity, part of his concept image from a circle trigonometry context. Although these specific uses may be interesting, we found the flexibility with which the PSTs moved between and among different concept images and definitions coming from right-triangle trigonometry and circle trigonometry contexts to be more important. The flexible use of these concept images supported PSTs' reasoning about the tangent and secant functions embedded in the novel directed-length context. This task gave the PSTs an opportunity to establish or continue to build a connected understanding of trigonometry. These connections may help avoid the isolated trigonometry knowledge about which Brown (2005) has warned.

Finally, the reasoning documented in these activities was somewhat consistent with quantitative reasoning. PSTs reasoned about the different quantities (i.e., arcs and segments), but they also referred to numerical relationships when checking benchmark values. As Ellis (2007) suggested, a focus on quantitative reasoning promoted relating actions. This relating action was demonstrated by the PSTs as they worked to develop new concept definitions for tangent and secant. In this process they engaged in relating a variety of concept images and definitions from their existing trigonometry stance. The PSTs in this study evoked concept definitions from both circle and right-triangle trigonometry contexts to develop their directed-length concept definition of tangent and secant. This opportunity to relate between and among the different images and definitions from different contexts should help them better apply their understanding when they encounter new topics (Tall & Vinner, 1981).

Even though PSTs developed a directed-length definition for the tangent and secant functions, future research should investigate: (1) In what ways can a directed-length stance and accompanying concept definition of trigonometric functions be leveraged to help students reason about additional topics (e.g., graphs of trigonometric functions, trigonometric identities, derivatives of trigonometric functions)? (2) How might secondary students develop a directed-length stance of trigonometric functions as their initial trigonometry stance?

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A PRELIMINARY INVESTIGATION OF CALCULUS STUDENTS' UNDERSTANDING OF GRAPHICAL OPTIMIZATION

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This study reports on students' reasoning about a univariate optimization that involves finding the production level at which the cost per yard is minimized when given the graph of a function that represents the relationship between the cost per yard and the number of yards produced by a factory. Determining the production level at which the cost per yard is minimized was straightforward for all the four students who participated in the study. However, explaining how this production level is related to the first derivative of the given function was problematic for most of the students. Implications for instruction are discussed.

Keywords: Optimization problems, graphical optimization, problem solving, calculus education.

Graphical optimization "…is a simple method for solving optimization problems involving one or two variables" (Bhatti, 2000, p. 47). Bhatti added, "for problems involving only one optimization variable, the minimum (or maximum) can be read simply from a graph of the objective function" (p. 47). Unlike algebraic optimization that uses algebraic methods (that may sometimes be sophisticated e.g., when working with complex objective functions) or numerical optimization (that requires some level of technical skills e.g., proficiency in MATLAB programing), graphical optimization is the simplest method for solving univariate optimization problems (UOPs) as it only requires making sense of graphs of objective functions. A UOP is an optimization problem where the objective function is a real-valued function of a single variable.

UOPs are particularly challenging for first-semester calculus (hereafter, calculus) students (cf. LaRue & Infante, 2015). Furthermore, there is a paucity of research on students' thinking about UOPs (cf. Speer & Kung, 2016). A few studies that have examined students' reasoning about UOPs have found that formulating the objective function is often challenging for many students (cf. Borgen & Manu, 2002; Dominguez, 2010; LaRue & Infante, 2015; Mkhatshwa, 2019). Additionally, all these studies have examined students' ability to solve UOPs using algebraic methods. There is still much to be explored about how students might reason about UOPs when given the objective function in graphical form. The research question investigated in this study is: How do difficulties exhibited by students when engaged in graphical optimization compare with difficulties exhibited by students when engaged in algebraic optimization?

Related Literature

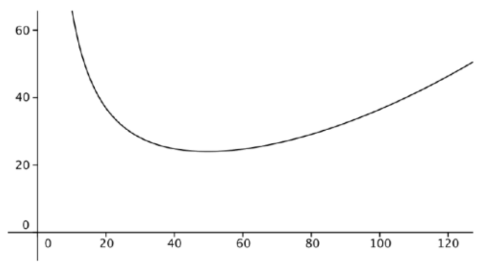
There are three themes that emerge from research that has looked at students' understanding of algebraic optimization when working with UOPs, namely students' difficulties with setting up the objective function, students' difficulties with determining and interpreting critical values and/or extrema, and students' difficulties with justifying/verifying extrema using formal calculus methods (cf. Borgen & Manu, 2002; Dominguez, 201; LaRue & Infante, 2015; Mkhatshwa, 2019; Swanagan, 2012). Findings from a related body of research suggest that difficulties exhibited by students when working with UOPs are directly related to quantitative reasoning

(Thompson, 1993, 1994, 2011). Specifically, these studies have found that students tend to confuse different quantities such as amount quantities [e.g. distance] with rate quantities [e.g. speed] when working with UOPs (cf. Flynn et al., 2018; Mkhatshwa, 2019,2020, 2021; Mkhatshwa & Doerr, 2018; Prince et al., 2012; Rasmussen & Marrongelle, 2006).

Methods of Data Collection and Analysis

Task-based interviews (Goldin, 2000) were conducted with four freshman students (Pseudonyms Adam, Ava, Caleb, and Emily) at a research university in the United States. The interviews lasted for about 14 minutes, on average, and contained two tasks (Task 1 and Task 2). In this paper, we report on how the students reasoned about Task 1:

Task 1 (Cost minimization context): The graph shows $f(x)$, the dollar cost per yard of fabric, given that a certain factory produces x yards of fabric.



a) At what production level is the cost per yard minimized?
b) How can you convince someone that the cost per yard is minimized at the production level you found in part a)? Explain.
c) What can you say about $f'(x)$ at the production level where the cost per yard is minimized? Explain.
d) Why might the cost per yard of fabric increase as the factory produces larger amounts?
e) Why might the cost per yard of fabric increase as the factory produces very small amounts?

The students worked through the task while the interviewer asked clarifying questions about their work. The students were chosen based on their willingness to participate in the study. The students in this study had limited exposure, via course lectures and the course textbook (Haeussler, 2011), to work with UOPs where the objective function is given as a graph such as in Task 1. We further note that the context of cost is commonly used in business calculus courses in the United States. Three students were business majors, and one student was an economics major. Business calculus is a required course for most business or economics majors. Data for the study consists of transcriptions of audio-recordings of the task-based interviews and work written by the four students during each task-based interview session. Data analysis was done in three stages. In the first stage, we used three a priori codes that consisted of the themes on students' difficulties when solving UOPs or difficulties related to quantitative reasoning discussed earlier. In the second stage of the analysis, we used emergent codes that included instances where students reasoned about what they found to be easy or difficult in their attempt to answer the questions posed in the task. In the third stage of the analysis, we looked for patterns in each of the codes identified in the first and second stage of the analysis, respectively.

Results

Determining Critical Values and/or Extrema

Three of the four students correctly determined the critical value of the objective function i.e., the production level at which the cost per yard is minimized. Although not asked, some of these students went on to determine the extreme value of the objective function i.e., the minimum cost per yard. Figure 1 illustrates how one of these students (Emily) responded to prompt (a) of the task whose goal was to assess students' ability to determine the critical value from the graph of the objective function.

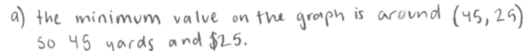


Figure 1. Emily's Solution to Prompt (a)

In her response to prompt (a) of the task, Emily correctly identified both the critical value and extreme value of the objective function. In light of the fact that the objective function is not presented on a grid axis, something that could limit one's accuracy when estimating critical or extreme values of the objective function, critical values ranging from 40 yards to 45 yards and extreme values ranging from $20 to $25 dollars are considered to be correct. Only one student, Ava, incorrectly determined the critical value of the objective function as can be seen in Figure 2.

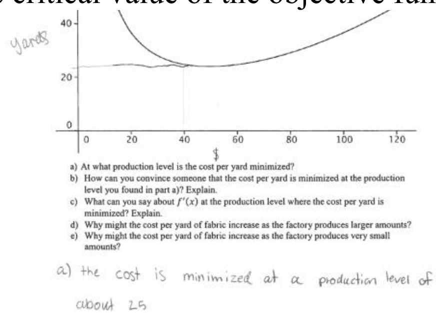


Figure 2. Ava's Solution to Prompt (a)

Specifically, in her attempt to determine the critical value of the objective function, Ava confused two amount quantities namely, the minimum cost per yard and the production level at which the minimum cost per yard is achieved. Another student, Adam, confused a rate quantity with an amount quantity when he confused the derivative of the objective function at the production level at which cost is minimized with the quantity of cost at this level in his response to prompt (c). None of the other students (i.e., Caleb and Emily) confused quantities in their reasoning about Task 1.

Verifying Extrema

All the students provided plausible explanations in response to prompt (b) that asked students to explain how they could convince someone that the cost per yard is minimized at the production level they identified in part (a). The following is a reproduction of one of the students' responses to prompts (b) and (c) on the task. To reiterate, prompt (c) asked students to comment on $f'(x)$ at the production level where the cost per yard is minimized.

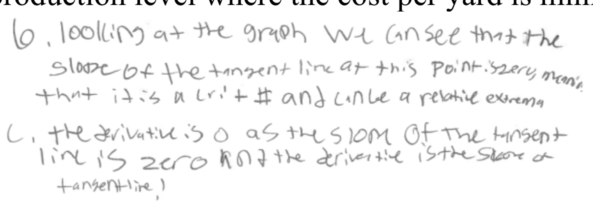


Figure 3. Caleb's Responses to Prompts (b) and (c)

Caleb's claim that "looking at the graph [of the objective function] we can see that the slope of the tangent line at this point [which he identified as $x = 42$] is zero, meaning that it is a critical number and can be a relative extrema" in response to prompt (b) demonstrates an understanding that extrema can be expected to occur where the tangent line to the objective function is zero. This is consistent with his claim that "the derivative [$f'(x)$] is zero…" in response to prompt (c). In response to prompt (c) on the task, the other three students either stated that $f'(x)$ would be positive or that it would be negative, thus exhibiting a poor understanding of concept of the derivative in connection with the critical value of the objective function. Interestingly, only one student (Emily) noted that commenting about $f'(x)$ [i.e., prompt (c) on the task] was particularly challenging when asked about the difficult part in her attempt to respond to the prompts included in the task. It should be noted that Caleb is the only student who used a calculus approach to verify extrema i.e., to explain how he could convince someone that minimum cost per yard occurs at the production level he identified in his response to prompt (a). The rest of the students, and using Ava as an example argued, in response to prompt (b) on the task that "you can look at the minimum point on the graph to find where the cost per yard is minimized."

Understanding the Graph of the Objective Function

When asked about the easiest part when responding to the prompts included in Task 1, all the students made remarks that suggested that making sense of the graph of the objective function was easy for all the students. For example, Adam remarked "…analyzing the graph" while Ava remarked "…finding where the cost is minimized because I looked at the lowest point on the graph." It should be noted that the students' remarks are consistent with their success in identifying critical values and/or extrema and verifying extrema as reported in the preceding subsections.

Explaining why production Cost Might Increase with Small- or Large-scale Production

Only Ava provided a plausible explanation in response to prompts (d) and (e) that asked for possible explanations why the cost per yard of fabric might increase when the firm produces very small or large amounts fabric. This student argued that a very small-scale production could lead to an increase in the cost per yard of fabric "…if there is not enough demand", suggesting that the factory may have to charge more money per yard to make profit. Ava explained that the cost per yard of fabric could still increase even with a large-scale production if the factory "…is not making enough revenue," which is a plausible explanation especially if making or increasing profit is the ultimate goal for the factory. Emily is one student who provided an explanation that we considered to be not plausible. She remarked, "I am not really sure…," in response to prompt (e) on the task that asked about why the cost per yard of fabric could increase when the factory produces large amounts of fabric.

Discussion and Conclusions

A number of studies have reported on students' difficulties with finding critical values or extrema as well as justifying extrema when tasked with solving UOPs using algebraic methods (cf. Borgen & Manu, 2002; Swanagan, 2012). Because of these difficulties, calculus students are often not successful in solving UOPs algebraically. Contrary to these findings, nearly all the students in the present study were successful in finding the critical value or extrema as well as justifying extrema while working with a UOP where the graph of the objective function was provided. To some extent, this may suggest that while students may struggle with solving UOPs algebraically, partly due to lack of facility with some algebraic techniques such as calculating derivatives of complex objective functions, students have better success with solving UOPs graphically not only because they can visualize the objective function, but also because having access to the graph of the objective function supports their quantitative reasoning such as the ease of identifying critical values and extrema. Consistent with findings from previous research on students' thinking about UOPs, two students confused quantities (cf. Mkhatshwa, 2019; 2020). Additionally, three students made remarks that suggested that they had difficulty understanding that the derivative of the objective function ought to be zero at the critical value (i.e., the cost minimizing quantity), something that generally comes easy for students when solving UOPs algebraically. It might be helpful for calculus instructors to expose students to multiple methods of solving UOPs, namely algebraically, graphically, and numerically.

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ANALYSIS OF IMPRECISE LANGUAGE IN UNIVERSITY STUDENTS' TRIGONOMETRY SOLUTIONS

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Vague or imprecise language has been shown to hinder students' mathematical understanding. To investigate how imprecise language contributes to student errors, researchers analyzed the work of college freshmen on a trigonometry problem. Findings showed that students struggled with conceptualizing radians and moving from understanding trigonometric functions in the context of triangles to the context of the unit circle. Imprecise diagrams and formulas evidenced students' lack of full understanding. Researchers recommend the use of precise language that prioritizes coherence, consistency, and conceptual clarity.

Keywords: Mathematical Representations, Precalculus, Undergraduate Education.

Students' understanding of mathematical concepts is hindered by vague or imprecise language. The lack of a shared coherent language in mathematics topics causes confusion and impedes students' ability to build on existing knowledge (Popovic et al., 2023). Intentional, consistent, precise language is essential for students to incorporate complex mathematical ideas into their existing understandings (Karp et al., 2015).

Background and Conceptual Framework

In the realm of mathematics education, researchers have undertaken extensive investigations into various facets of teaching and learning trigonometric concepts. Central to these inquiries is an exploration of the challenges students encounter in understanding trigonometry and the origins of these difficulties. Fundamental to this discourse is the observation that students often fail to recognize the radian as a unit of angle measure, instead favoring a dominant reliance on the degree measure (Koyunkaya, 2016; Moore, 2013). Consequently, students have difficulty in interpreting trigonometric functions' outputs when presented with real numbers as inputs (Akkoc, 2008, Cekmez, 2020). This deficiency is compounded by a lack of foundational knowledge regarding the concept of angles (Koyunkaya, 2016).

Another issue identified in students' understanding lies in their inability to establish connections between the contexts of right triangles and unit circles and their application in defining trigonometric functions (Moore, 2009). This deficit underscores the complex nature of comprehending trigonometric functions, which can be caused by various factors. Weber (2008) explains two primary reasons contributing to students' struggles: the challenge of linking triangles to numerical relationships and grappling with functions devoid of explicit formulas to determine their outputs.

Researchers assert that traditional approaches to teaching trigonometry often fall short in fostering conceptual understanding among students, thereby perpetuating learning difficulties (e.g., Moore, 2009; Orhun, 2010; Weber, 2005). Therefore, researchers have conducted studies aimed at devising alternative instructional approaches, including the design and implementation of novel teaching units or the integration of tools such as Geometer's Sketchpad (Moore, 2009; Weber, 2005).

Departing from previous research, in our study, we investigated the underlying imprecise language contributing to misconceptions and errors. We contend that imprecise mathematical language fosters ambiguity, hindering students' comprehension and concept development (Popovic et al., 2022). Adopting the concepts of concept image and conceptual change, we recognize that students' cognitive structures are shaped by their experiences and exposure to mathematical language. The development of concepts encompasses mental pictures, attributes, properties, and associated processes, as well as the linguistic representation of concepts, and inconsistencies within and between concept images and definitions can lead to cognitive conflicts, impeding the formation of an appropriate conceptual understanding (Jaffar & Dindyal, 2011; Lager, 2006; Siemon et al., 2017; Tall & Vinner, 1981). Thus, we advocate for an approach that emphasizes coherent and shared mathematical language to facilitate conceptual development and mitigate learning difficulties across mathematics education levels. Therefore, in this study, we analyze students' mathematical work and reasoning in their solutions to trigonometry problems and infer what type of underlying imprecise language might have potentially led to such misconceptions and errors.

Data Analysis

The sample consisted of 55 students, who were freshmen enrolled in a Precalculus course at a university in the Midwest portion of the United States. Researchers analyzed student work for one problem on a final exam. The problem, created by one of the researchers, stated:

Given that the $\sin(\theta) = -\frac{5}{6}$, where $\pi < \theta < \frac{3\pi}{2}$, calculate the values of $\sec(\theta)$ and $\cot(\theta)$.

Your work must clearly show how you achieved your answer.

One researcher analyzed student work to determine how the imprecise language might have influenced the appropriateness of the solution approach. Two other researchers then analyzed portions of student work for agreement. Researchers discussed discrepancies in coding until 100% agreement was reached.

Findings

Memorized procedures and mnemonics impeded students in moving beyond their conceptions of trigonometric functions as solely the ratios of the sides of a right triangle. This was seen in various ways in student work. Specifically, in the US, the mnemonic SOHCAHTOA is ingrained in students as a way to remember that, given a right triangle with angle A between 0° and 90°, sin(A) is equal to the ratio of the side opposite angle A divided by the hypotenuse of the triangle (see Figure 1). Teachers accept this written in shorthand as $sin = \frac{opp}{hyp}$. This mnemonic was shown in the work of two students. The use of this mnemonic (i.e., the imprecise language) reflects a common oversimplification in trigonometric instruction, where trigonometric functions are often taught in the context of right triangles only, leading to a limited understanding of their broader applications and concepts.

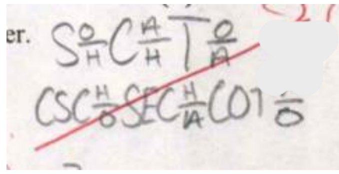

Figure 1: Mnemonic for Trigonometric Ratios

When told $\sin(\theta) = -\frac{5}{6}$, ten students used a right triangle with one side length of 5 and hypotenuse of 6, found the third side length, and used that to determine cos(θ). Students placed a triangle in the plane without relating the hypotenuse to the radius of a circle, even though the angle measurement was provided in radians. Rather than seeing the triangle as having a side length of $\frac{5}{6}$ and hypotenuse of 1, which would have facilitated their understanding of the triangle hypotenuse as the radius of the unit circle, students relied on their understanding of $\sin(\theta)$ as the ratio of the opposite side and the hypotenuse. Given $\sin(\theta) = -\frac{5}{6}$, students automatically drew a triangle with that ratio for one side and the hypotenuse (see Figure 2). This reliance on information from Geometry shows that students were unable to build on their geometric knowledge of triangles in the plane to incorporate trigonometric functions represented in the unit circle. The abrupt transition from using trigonometric functions to describe triangle ratios in Geometry to using the same functions as circular functions in Algebra II caused an inconsistency in the students' concept image (sides of a right triangle) and definitions (unit circle, $\cos(\theta) = x, \sin(\theta) = y$), thus blocking the formation of new understandings about trigonometric functions. This reliance on triangle-based approaches instead of integrating the concept of the unit circle demonstrates a perpetuation of imprecise ratio language used in teaching trigonometry.

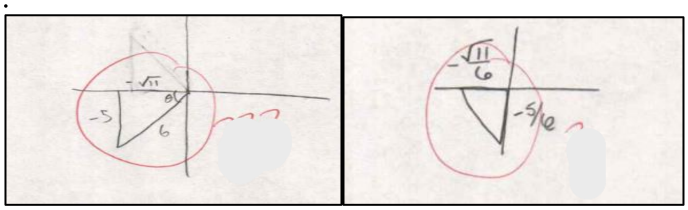

Figure 2: Incorrect Triangle Diagrams

However, the bigger problem here is that there is no triangle with side length of $-\frac{5}{6}$. Rather, an understanding that this is the *x*-coordinate of the point on the unit circle is essential in guiding students to use the equation of the circle. The first triangle shown in Figure 2 also shows evidence that students are taught to see radians as markers on the plane, rather than as angle measures. An acute angle in quadrant three was drawn by 12 students, even though it was given in the problem that $\pi < \theta < \frac{3\pi}{2}$.

Eleven students showed evidence of having memorized formulas for the relationship between trigonometric functions but drew incorrect diagrams. It appears that these students have memorized the formulas but are unable to make a connection to representations of the unit circle (see Figure 2). These students were able to incorporate some definitions into their existing

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

schema ($cos\ (\theta)\ = x,\ sin(\theta) = y$), but an incomplete mental picture of the unit circle impeded their understanding of the hypotenuse as the radius of the unit circle.

Students wrote imprecise equations, such as $sec = \frac{1}{sin}$. This was seen in the work of 11 students (see Figure 3). This is believed to be an acceptable equation in high school, without regard for the imprecision evident in the equation. Acceptance of these imprecise representations encourages the interpretation of trigonometric functions as variables, leading to ambiguity and impeding concept development. This understanding of trigonometric functions as variables leads to difficulties when solving trigonometric equations, such as $\sin(x+\pi)\ = \sin(2x) + \sin(\pi)$, where students divide the whole equation by sin (i.e., cross out sin) to get $x + \pi = 2x +\ \pi$.

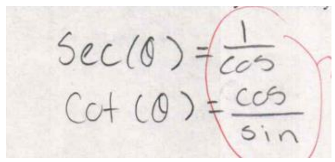

Figure 3: Imprecise Equations

The work and explanations of three students showed that they were logically thinking through their work and understood the connections between the functions. For example, when told $\pi < \theta < \frac{3\pi}{2}$, students explained that the angle would be in Quadrant III, giving a negative value for $sec(\theta)$. However, without a diagram, it is unclear whether these students fully understood the connections to the unit circle. Logically thinking through their work was also evident for four other students, but explanations were not provided to the same extent as for the three students mentioned above. However, these students also drew a diagram to support their thinking, showing evidence of understanding the connection between the unit circle and the trigonometric functions (see Figure 4).

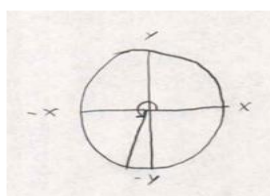

Figure 4: Diagram Supporting Understanding

Conclusions

Our findings underscore the impact of imprecise language on students' ability to navigate the complexities of trigonometry. Examination of student work on the introduction and subsequent study of trigonometric ratios revealed a pervasive reliance on mnemonic shortcuts and memorized formulas, obscuring the conceptual understanding. For example, the use of shortcut formulas such as $sec = \frac{1}{sin}$, without the argument of θ, can lead to insufficient concept development and impede understanding of the concept in a broader or different context. This is evidenced in student representations of a triangle on the plane without relating the hypotenuse to the radius of a circle centered at the origin. The move from the study of triangle side ratios to the study of the unit circle resulted in the use of the same terms in a different context, with new vocabulary. Triangles are replaced with angles that may be equal to or greater than 180°, requiring a restructuring of existing understandings of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$. Merely substituting triangles with circular contexts without fostering a nuanced understanding of radians and circular functions inhibits conceptual growth. Students' struggles to move to envisioning trigonometric functions in the context of the unit circle was also seen in Moore (2009). It is imperative to champion the use of precise language, one that prioritizes coherence, consistency, and conceptual clarity. By cultivating a shared coherent mathematical language grounded in precision and intentionality, educators can empower students to transcend rote memorization and embrace the rich connections in mathematical concepts.

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STATIC AND DYNAMIC: HOW TWO TEXTBOOKS INTRODUCE DERIVATIVE

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Calculus students often struggle to understand the derivative conceptually, even when they can differentiate fluently. Operating from the premise that underdeveloped covariational reasoning skills may be key in understanding the derivative, this study explores how two calculus textbooks vary in promoting derivatives as dynamic and static. The results of analysis suggest that each textbook provides opportunities for both forms of reasoning, but they vary in their emphasis.

Keywords: Calculus, Curriculum, High School Education, Undergraduate Education

Research over the past few decades demonstrates that calculus students have difficulties making sense of the derivative conceptually, even if they can calculate derivatives fluently (e.g., Orton, 1983; Carlson et al., 2002; Thompson & Carlson, 2017; Epstein, 2013). Thompson and Harel (2021) argue that the underdevelopment of covariational reasoning may be the missing link in students' ability to engage deeply with major calculus concepts including the derivative. Covariational reasoning, or reasoning about how quantities vary simultaneously and in relation to one another, is evident in the Mathematical Association of America's (MAA's) recommended conception of the derivative as dynamic. The MAA emphasizes the need to promote an idea of the "derivative as instantaneous rate of change or as a measure of sensitivity of one variable to change in another," rather than the traditional, "very static interpretation" which does not make explicit a covarying relationship between quantities (Bressoud et al., 2015, p. 18).

This study is motivated by the need for students to experience a dynamic conception when learning about the derivative, and the role that textbooks play in the nature of students' calculus learning (e.g., Liakos et al, 2021; Gerami et al., 2023, Porogrelova, 2022). Recent work suggests that textbooks provide different, and often limited, promotion of covariational reasoning related to the derivative (Mkhatshwa, 2022; Chen, 2023), and this has implications for the opportunities students may have to reason about the derivative in a dynamic way. This study adds to this literature by applying Tasova et al.'s (2018) covariation framework to operationalize a distinction between static and dynamic conceptions of the derivative. Ultimately, the purpose of this study is to uncover how static and dynamic conceptions of the derivative are introduced to students by highlighting the ways two calculus textbooks promote reasoning about quantities related to the derivative as static or dynamic. These distinctions have implications for how students continue to engage and apply concepts as they progress through calculus and beyond.

Methods

I compare the introductory material on derivatives from two widely used Calculus textbooks (according to Open Syllabus Project): *Calculus of a Single Variable* (Larson & Edwards, 2018; hereafter *Larson,* for brevity) and *Calculus: Single and Multivariable* (Hughes Hallett et al., 2013; hereafter *Hughes Hallett*). I address the question: <u>How do two calculus textbooks promote reasoning about the derivative as static and dynamic?</u>

I applied Tasova et al.'s (2018) framework (Table 1) to *Larson's* and *Hughes Hallett's* introductory chapters on the derivative. This framework combines Moore and Thompson's (2015) framework for static and emergent shape thinking with Thompson and Carlson's (2017) framework for (co)variational reasoning. The framework is intended to be used "for coding the extent and nature of (co)variation provided in the narrative" of a textbook (i.e., expository sections, figures, and worked examples; p. 1529). I note I have adjusted the wording of the framework to be 'dynamic' instead of 'emergent,' as 'dynamic' characterizes covariation in non-graphical situations, and more clearly aligns with the MAA's conceptions of the derivative.

This framework outlines how textbooks promote the relationships between quantities and the quantities themselves as varying (dynamic) or not varying (static). A textbook narrative element promotes a *static* conception when it represents a quantity or function as a single object that is interpreted through associated facts or properties (i.e., Perceptual Associations) or as an input/output correspondence between its quantities (i.e., Correspondence). It promotes a *dynamic* conception when it emphasizes a quantity as changing (i.e., Variation) or the interprets the relationship between quantities through their coordinated change (i.e., Covariation). For brevity, I share examples of the most relevant codes in the results (for a thorough explanation of each code, see Tasova et al. (2018)).

Table 1: Analytic Framework (adapted from Tasova et al., 2018, p. 1529)

Static	Dynamic	
	Variation	Covariation
• Perceptual Associations		
• Variable as Unknown	• Continuous	• Continuous
• Correspondence	• Gross	• Gross Coordination of Values
	• Discrete	• Coordination of Values

I identified and coded instances in which the textbooks promoted static or dynamic reasoning about quantity/quantities using the elements of Tasova et al.'s (2018) framework (Table 1). I defined multiple units of analysis based on the component of the textbook being reviewed (i.e., expository sections, figures, tables, worked examples). For example, expository texts were coded sentence-by-sentence, but worked examples were coded as a single unit with the most prominent code. Instances of each opportunity were also tagged with the component type through which it was presented to investigate patterns in the ways conceptions were promoted.

Results

Each textbook provides opportunities for both static and dynamic conceptions of the derivative. However, the textbooks differ in their relative emphasis. Table 2 summarizes the frequencies of opportunities identified as static and dynamic across both textbooks. While *Larson* provides more opportunities overall for thinking about quantities related to the derivative, over 75% of these are static. This suggests *Larson* promotes a more static conception of the derivative. *Hughes Hallett* offers slightly more instances of dynamic than static (38 vs 33, respectively), which suggests there is more balance between its promotion of reasoning about the derivative from both a dynamic and static perspective. For brevity, the remaining results explore only the most notable code for each conception of the derivative.

Table 2: Opportunities for static and dynamic conceptions of the derivative

Frequency	Static	Dynamic	Total
Larson	76.4% (n = 84)	23.6% (n = 26)	110
Hughes Hallett	46.5% (n = 33)	53.5% (n = 38)	71

Instances Promoting a Static Conception of the Derivative

I identified the majority of static instances in *Larson* as Correspondence (70.2% of static opportunities). Only 9% of *Hughes Hallett's* static instances were Correspondence. Further, most examples of Correspondence from *Larson* were associated with worked examples. In general, *Larson's* worked example solutions were more perfunctory than *Hughes Hallett's,* and included less explanation and interpretation of the steps and answers.

Most of *Larson's* Correspondence worked examples followed a similar format. Generally, they indicated students should use an established rule to procedurally generate a derivative function, and then use this derivative function as a new rule to address follow up questions. For example, students are asked to "Find the slopes of the tangent lines to the graph of $f(x) = x^2 + 1$ at the points (0,1) and (-1,2)" (*Larson,* p. 102). The given solution applies the limit definition of the derivative, and though it shows intermediate steps for determining the derivative function, there is no attention to what the limit as $\Delta x \rightarrow 0$ means. Once the derivative function is established, it is simply used as an 'input/output' generator to calculate slopes: "So, the slope at *any* point (c, $f(c)$) on the graph of f is $m = 2c$" (*Larson*, p. 102 emphasis in original). This explanation is a clear example of Correspondence, which "simply provide[s] a rule for students to calculate a unique value of a variable or quantity by using any given value of another variable or quantity," (Tasova et al., 2018, p. 1529); in this case, students are given the opportunity to use the rule $m = 2c$ to calculate "*any*" slope at *"any"* point, with no attention to how the values of this slope are related to the original graph's values, to x, or to each other.

Instances Promoting a Dynamic Conception of the Derivative

Though both textbooks promote reasoning about the derivative dynamically, *Hughes Hallett* provided more opportunities to reason at the highest levels of covariation. In *Hughes Hallett,* Coordination of Values and Continuous Coordination were 42.1% and 18.4% of dynamic opportunities, respectively. *Larson* emphasized lower levels of covariation, with Gross Coordination characterizing 53.8% of dynamic opportunities.

A notable difference between *Larson* and *Hughes Hallett* related to promoting a dynamic conception is that *Hughes Hallett* utilized tables of values within its narrative structure and *Larson* did not include any. Further, each table of values present in *Hughes Hallett* coincided with instances of Coordination of Values or Continuous Covariational reasoning opportunities. Though generating a table from a given or derived rule would be considered an example of Correspondence, the use of tables of values in *Hughes Hallett* focuses on interpreting a complete table. This approach promotes reasoning covariationally as Coordination of Values, as it emphasizes the coordination the values of one variable with values of another across discrete pairs (Tasova et al., 2018). With some additional context or explanation, this reasoning can be elevated to Continuous Covariation by highlighting "simultaneous and continuous change" (Tasova et al., 2018, p. 1530). For example, consider the Continuous Covariational worked example from *Hughes Hallett* shown in Figure 3, which asks students to construct a table of values of an estimated rate based on a given table of values.

Example 3 Table 2.7 gives values of $c(t)$, the concentration (μg/cm^3) of a drug in the bloodstream at time t (min). Construct a table of estimated values for $c'(t)$, the rate of change of $c(t)$ with respect to time.

Table 2.7 *Concentration as a function of time*

t (min)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$c(t)$ (μg/cm^3)	0.84	0.89	0.94	0.98	1.00	1.00	0.97	0.90	0.79	0.63	0.41

Figure 3: Worked example prompt coded Continuous Covariation (*Hughes Hallett,* p. 92)

A preconstructed table of values is used in both the problem statement and the solution (not pictured) and instructs students to reason about how both the concentration and the derivative of the concentration change as time passes (i.e., covariationally). The solution explanation draws attention to both the concentration's direction and rate of change through specific values, initially promoting Coordination of Values. However, the solution further states, "we have to assume that the data points are close enough together that the concentration does not change wildly between them" (p. 93). This description uses the small size of the intervals to draw attention to the simultaneous and continuous nature of the coordinated change, thus promoting Continuous Covariational reasoning. Similar language that draws on small or successively smaller intervals is present in many instances of Continuous Covariational reasoning across both textbooks.

Discussion

Addressing the research question, both textbooks promote static and dynamic conceptions throughout their introductory chapters on the derivative. Therefore, they both provide at least some opportunities for students to come to understand the derivative from both static and dynamic perspectives, though their emphasis differs. *Larson* provides more instances of static conceptions and emphasizes a Correspondence approach to understanding and using the derivative to solve problems. *Hughes Hallett* provides more instances of dynamic conceptions, and this is afforded in part by its use of tabular representation interpretation.

It is important to mention that *Larson* is considered a 'traditional' textbook, and *Hughes Hallett* was written during the reform movement of Calculus education (Hughes Hallett, 2006). Though I found differences between the books, these results should not be used to characterize traditional and reform calculus textbooks more broadly. Future research could analyze additional textbooks with Tasova et al.'s (2018) framework to determine whether there are any major differences between traditional and reform textbooks' promotion of derivative conceptions in general.

Given the influence calculus textbooks can have on teachers' planning and practice (Liakos et al., 2022; Gerami et al., 2023), these results suggest the choice of textbook may influence how students ultimately learn to conceptualize the derivative. This has implications on their ability to apply their knowledge of the derivative to other contexts, such as in science and engineering fields in which a dynamic conception is more productive (Bressoud et al., 2015). Teachers should consider supplementing their textbook with materials that emphasize a more dynamic conception of the derivative as well as materials that demonstrate *why* static ideas about the derivative hold true.

Similarly, curriculum developers should consider alternative methods of fostering the dynamic conception of the derivative; the complexity of representing continuous covariation, an inherently dynamic concept, in static mediums like printed textbooks calls for work that makes the dynamic nature of the derivative more explicit and accessible to students. Recent work (e.g., Weinberg & Martin, 2020; Kertil & Dede, 2020) has begun that explore the dynamic capabilities of software, such as Desmos, for investigating and promoting the covariational reasoning of students of different ages. More work is needed to understand how these dynamic activities can be used to supplement the opportunities presented in textbooks to improve students' understanding of the derivative as dynamic.

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MATHEMATICS SELF-EFFICACY IN COLLEGIATE DISCRETE MATHEMATICS

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Mathematics self-efficacy (MSE) is a person's belief in their ability to do mathematics, including beliefs such as "I am bad at math." MSE can impact a students' college major choice, perseverance through struggles, and their success or failure in mathematics, with research generally linking high MSE to beneficial outcomes (Gill, 2019; Hackett, 1985; Multon et al., 1991; Pintrich & de Groot, 1990). MSE exists at both a global and local level (Bandura, 1997). Global MSE includes beliefs about broad topics in mathematics, such as "I am good at Calculus." Local MSE is a narrower belief about specific tasks or single problems, such as "I am bad at proof by induction." Past research on MSE has often not been clear about the level of self-efficacy (global or local) and frequently measured only global MSE (Multon et al., 1991). This is an important limitation because students may have mismatched global and local MSE (e.g., holding high global and low local MSE). In addition, MSE has often been explored in the K – 12 population. These results may not apply to the collegiate population and particularly to discrete mathematics students, a course where researchers argue students' beliefs may change (Sandefur et al., 2022). Discrete mathematics is also often students' first introduction to the "axiomatic formal" mode of mathematics, including proof (Tall, 2008). Given the importance of self-efficacy on students' perseverance and success, more research is needed exploring both global and local MSE of discrete mathematics students.

Research Design and Analysis

This study examined the global and local MSE of collegiate discrete mathematics students, with the goal of identifying and qualitatively describing cases where global and local MSE did not align (e.g., high global but low local MSE) and the causes for that misalignment. Participants were 14 students who participated in a semi-structured interview designed to explore their beliefs. Interviews were transcribed verbatim and coded following Campbell and colleagues (2013). The two authors worked together to ensure reliability, achieving 88% coding reliability.

Findings and Implications

We identified cases of misaligned MSE in discrete mathematics; that is, students with high global MSE sometimes had lower MSE for discrete topics or problems (and vice versa). This suggests potential changes to students' beliefs may have occurred because of discrete mathematics, a result backed up by students themselves who sometimes identified the discrete course as changing either their global or local MSE. Sara for example said, "[Discrete mathematics] definitely affected me… it made me realize … there's gonna be stuff that I'm not always 100% good at going into it." Overall, these results, and other data we will share in the poster, suggest that students in discrete mathematics may have mismatched global and local MSE, and some of this mismatch may stem from the nature of the course itself. This work has implications for our understanding and measurement of students' beliefs as well as for how we, as instructors, can support students through belief changes caused by classes that introduce formal mathematical language and proof. As we envision the future of mathematics education, it is vital that we build on work like this that takes students' perspectives and beliefs into account.

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SIGNIFICACIÓN DEL CONCEPTO RAÍZ REAL DE UNA ECUACIÓN POLINÓMICA MEDIADA POR LA TECNOLOGÍA DIGITAL

MEANING OF THE CONCEPT OF REAL ROOT OF A POLYNOMIAL EQUATION USING DIGITAL TECHNOLOGY

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Presentamos una secuencia didáctica mediada por la tecnología digital para significar y calcular las raíces reales de una función real en un curso de cálculo diferencial para estudiantes de ingeniería. Para introducir las raíces diseñamos un Escenario Didáctico Virtual Interactivo, que simula un problema real, y usamos el sistema tutorial CalcVisual para apoyar el cálculo aproximado de las raíces. Implementamos la secuencia con una población de 45 estudiantes universitarios en México. Los datos se analizaron mediante los modelos emergentes de la Educación Matemática Realista. Mostramos el progreso en la actividad matemática de los estudiantes a través de cada uno de los niveles de actividad de los modelos emergentes quienes mostraron un avance significativo en la comprensión conceptual y cálculo de las raíces reales.

Precálculo, Cálculo, Tecnología, Experimentos de diseño.

Los polinomios son funciones fundamentales en la matemática, en particular en el cálculo, análisis matemático y álgebra lineal. Una de las propiedades más importantes de una función polinómica, son sus raíces reales y complejas, pero determinarlas no es tarea sencilla e incluso en algunos casos se llega a confundir la función polinómica con la ecuación que se deriva de ella (Dede y Soybas, 2011). Significar el concepto de raíz real de una función resulta determinante por tratarse de un concepto fundamental para la matemática e imprescindible para aplicaciones en procesos de optimización, cálculo diferencial e integral, álgebra lineal, cálculo multivariable y método Simplex de programación lineal, por mencionar algunos. La determinación de las raíces simples o múltiples es un problema complejo y vigente que tiene su origen desde los primeros vestigios de la humanidad, y que siempre ha estado asociado a problemas de variación, acumulación y optimización. Hasta nuestros días se mantienen problemas abiertos sobre el cálculo de raíces, sobre todo cuando son múltiples (Cuevas y Madrid, 2013), y cobra relevancia en el problema de cómo introducir desde el plano cognitivo este concepto en la enseñanza a nivel de precálculo y cálculo (Veuillez-Mainard, 2023). Al realizar una búsqueda sistemática de la literatura podemos constatar que existen pocos artículos que reporten las dificultades de enseñanza de raíces reales, minimizando la importancia y dificultad del concepto. Es conveniente recordar que la resolución de ecuaciones polinomiales generó el álgebra (Puig y Rojano, 2004).

Tradicionalmente el cálculo aproximado de raíces reales era un tema importante que tratar en cursos de análisis numérico donde se practicaban diversos métodos como: bisección, regula falsi y Newton-Raphson para aproximarse al valor de una raíz real. Sin embargo, al popularizarse el uso de herramientas digitales con la avalancha de diversos manipuladores simbólicos virtuales, el significado y proceso bajo el cual se desarrolló permanece oculto, dejando la incertidumbre de qué es una raíz real de una función real. Esto debido a que softwares como: Mathematica, Wolfram Alpha, Matlab, GeoGebra y Photomath, resuelven ecuaciones y encuentran sus raíces en cuestión de segundos. Este reto que la tecnología digital ha puesto en la enseñanza y aprendizaje de las matemáticas permanece sin respuesta, y ha creado el paradigma de ¿qué se debe de enseñar? Cuando los estudiantes utilizan cualquier dispositivo o software para calcular el valor de una raíz real ¿sabrán que la mayoría de las veces encuentran un valor aproximado? ¿qué cuando las raíces son múltiples y cercanas pueden confundirse por errores de aproximación? ¿qué significa gráfica y numéricamente una raíz? ¿en qué se puede utilizar el concepto de raíz real, más allá de calcular su valor? Estos significados, se extraviaron al perderse los métodos de aproximación de una raíz y difícilmente se recuperarán algún día. Nos preguntamos ¿cómo recuperar los significados del concepto raíz de una función real mediante actividades mediadas por la tecnología digital? Nuestra propuesta consiste en el desarrollo y creación de actividades didácticas que permitan recuperar los significados de las raíces reales aprovechando los recursos que la tecnología digital ofrece el día de hoy como la capacidad numérica, gráfica y simbólica.

Marco teórico

Cuando un profesor frente a un grupo de estudiantes explica y anota definiciones, fórmulas y ejercicios en el pizarrón, mientras los estudiantes lo observan, escuchan y anotan en sus libretas lo expuesto por él, a esta enseñanza se le denomina enseñanza tradicional, la cual se ha desarrollado durante varios años. Para evitar este tipo de enseñanza y promover una enseñanza participativa con el objetivo de dotar de un significado a los conceptos matemáticos, Cuevas y Pluvinage (2003) proponen una serie de principios – intranet conceptual, partir de un problema en un contexto real, un plan de acción, implementación de operaciones inversas, la articulación de diversos registros de representación, la validación de resultados y la aplicación del concepto en un contexto diferente al enseñado– para la enseñanza de un concepto matemático. Usamos estos principios para el diseño de las actividades.

Los modelos emergentes son una de las tres heurísticas de diseño instruccional de la Educación Matemática Realista (RME por sus siglas en inglés). Esta heurística describe como una serie de modelos puede apoyar el avance matemático de los estudiantes (Gravemeijer, 2020). La heurística de los modelos emergentes destaca la importancia de comenzar con problemas contextuales que ofrezcan oportunidades para desarrollar un razonamiento específico de la situación y con el potencial de crear problemas cuya solución hace necesario el uso de conceptos matemáticos más sofisticados (Gravemeijer y Doorman, 1999). La actividad matemática inicia con el uso o desarrollo de un modelo derivado del contexto y, con el tiempo, este modelo apoya la aparición de formas de conocimiento matemático formal (Doorman et al., 2012). Los estudiantes transitan por distintos niveles de actividad que van desde el uso de estrategias informales hasta el razonamiento matemático formal (Gravemeijer, 1999). Los cuatro niveles propuestos por la RME son:

1. Nivel situacional (actividad en el entorno de la tarea). En este nivel las interpretaciones y

las soluciones dependen de la comprensión de cómo actuar en el entorno (fuera del entorno escolar).

2. Nivel referencial. En este nivel los modelos-de se refieren a la actividad en el entorno descrito en las tareas. En consecuencia, los modelos que surgen se basan en la comprensión de los estudiantes del entorno real y forman parte de las explicaciones en las que los estudiantes describen cómo interpretaron y resolvieron las tareas centradas en los escenarios de partida.
3. Nivel general. Este nivel comienza a surgir cuando los estudiantes empiezan a razonar sobre las relaciones matemáticas implicadas. Por lo tanto, surge cuando el razonamiento de los estudiantes pierde dependencia de las imágenes específicas de la situación. En este sentido, los modelos-para sirven más como medio de razonamiento matemático que como forma de simbolizar la actividad matemática basada en entornos particulares.
4. Nivel formal. En este nivel se trabaja con los procedimientos y notaciones convencionales. Se alcanza cuando los estudiantes ya no necesitan el apoyo de modelos para la actividad matemática.

Usamos estos niveles para mostrar el progreso en el razonamiento de los estudiantes, sobre el concepto de raíz, al trabajar con las actividades propuestas.

Metodología

Este estudio se desarrolló con base en la Investigación Basada en el Diseño (IBD) por lo que esta investigación implica iteraciones de diseño, implementación y análisis mediante las siguientes fases: preparación y diseño, experimentos de enseñanza y análisis retrospectivo (Bakker, 2018).

Fase de preparación y diseño

Se diseñó una secuencia de cinco actividades para introducir de forma gradual el concepto de raíz real (ver figura 1).

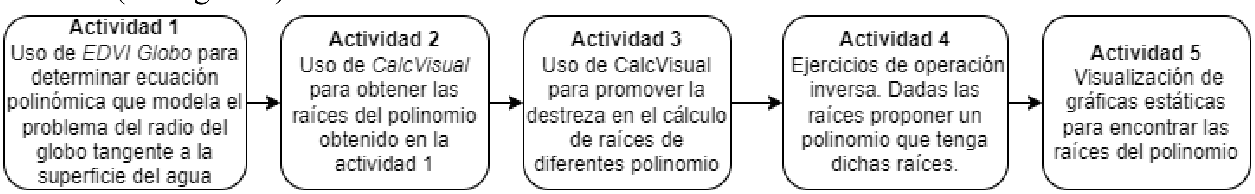
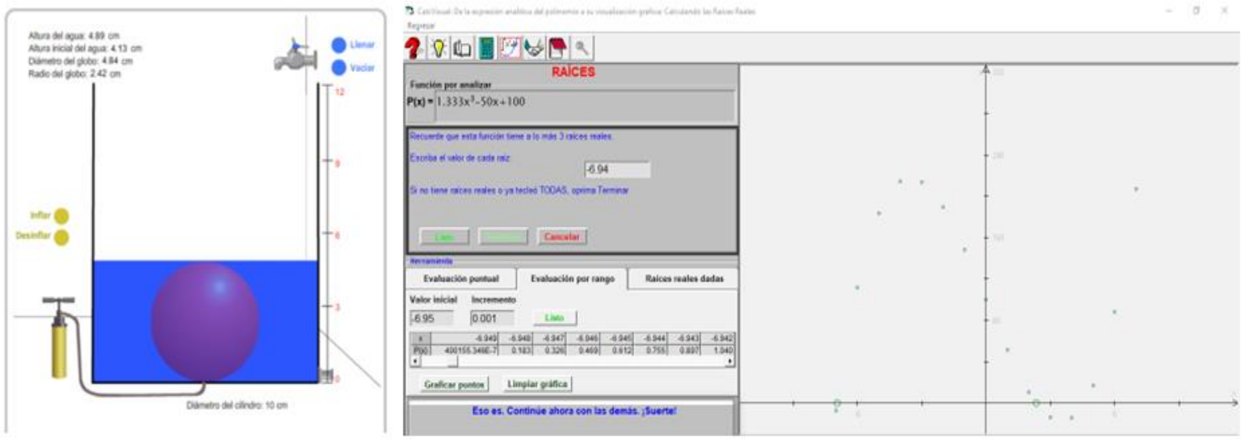


Figura 1: Secuencia didáctica

Como parte de la secuencia, se diseñó un Escenario Didáctico Virtual Interactivo (EDVI), al que denominamos EDVI "Globo" porque simula un recipiente cilíndrico de 10 cm de diámetro con un globo esférico atado al fondo. Este EDVI cuenta con botones para llenar y vaciar de agua el recipiente y botones para inflar y desinflar el globo a partir de los cuales, pueden observar cambios de forma dinámica en parámetros como: la altura inicial del agua con el globo desinflado, la altura del agua al inflar o desinflar el globo y el radio y diámetro del globo (ver figura 2a). En este artículo nos referimos a un EDVI como un manipulativo virtual que permite simular y visualizar diferentes representaciones semióticas de un problema real (Cuevas et al., 2023).



a) b)

Figura 2. a) EDVI Globo; b) Sistema Tutorial Inteligente CalcVisual

Adicionalmente, se utilizó un Sistema Tutorial Inteligente denominado CalcVisual que apoyará a los estudiantes en el cálculo de las raíces (ver figura 2b). El CalcVisual es un software que no calcula las raíces como cualquier manipulador simbólico. Permite introducir el polinomio y mediante herramientas visualizar diferentes representaciones del concepto de raíz. Por ejemplo, su representación tabular y su gráfica sobre un plano cartesiano. Es importante señalar que, CalcVisual no trabaja con funciones racionales y radicales. Asimismo, se diseñaron Hojas de Exploración y Aprendizaje Guiado (HEAG) para cada actividad, las cuales guían al estudiante en la manipulación de las herramientas digitales y en la construcción del concepto matemático.

Fase de experimento de enseñanza

La intervención didáctica se desarrolló en una universidad pública mexicana con 45 estudiantes inscritos en un curso de “Matemáticas aplicadas a la informática”. Las HEAG se enviaron de manera digital a cada estudiante y las actividades se desarrollaron en equipos de 6 integrantes. Después de resolver cada actividad, el profesor seleccionó unas HEAG al azar y realizó una discusión en clase para llegar a las respuestas correctas de forma consensuada. Las actividades fomentan tanto el aprendizaje individual como el colaborativo. Los datos se obtuvieron mediante las respuestas en las HEAG que los estudiantes enviaron al correo electrónico del profesor. Uno de los autores fue el encargado de impartir dicho curso. Los datos se analizaron de manera independiente por los investigadores. Se identificaron estrategias de solución y las respuestas de los equipos se clasificaron en los niveles de actividad (situacional, referencial, general y formal).

Resultados y análisis retrospectivo

En esta sección describimos como progresa el razonamiento de los estudiantes a través de cada uno de los niveles de actividad de los modelos emergentes al trabajar con las actividades propuestas. Debido a la limitación del documento, mostramos las respuestas de dos equipos (T1 y T2) seleccionados al azar.

Actividad Situacional

Como se ha mencionado anteriormente, la actividad situacional implica que los estudiantes trabajen en un entorno real para alcanzar objetivos matemáticos particulares. La actividad en el aula comenzó con la exploración del EDVI "Globo". Clasificamos esta actividad en el *nivel situacional* porque los estudiantes usaron las herramientas disponibles en el escenario para identificar variables, constantes y características cómo: una altura inicial del agua (h_0) con la que el diámetro del globo puede ser igual a la altura del agua (h_a). La tabla 1 muestra las respuestas de los equipos T1 y T2 a las preguntas de exploración.

Tabla 4. Preguntas y respuestas a las actividades de exploración

Pregunta	Respuesta T1	Respuesta T2
¿Qué elementos son variables al inflar el globo?	Radio del globo, volumen del contenido en el recipiente, volumen del globo.	Radio del globo, volumen del contenido en el recipiente, volumen del globo y altura inicial del agua.
¿Qué elementos son constantes al inflar el globo?	Radio del recipiente, volumen inicial de agua y altura inicial del agua.	Radio del recipiente, volumen inicial de agua.
¿Hasta qué valor puede crecer y disminuir el radio del globo (x)?	El radio puede crecer hasta 5cm y disminuir hasta 0cm	El radio puede crecer hasta 5cm y disminuir hasta 0cm.
¿Hasta que altura inicial (h_0) se puede llenar el recipiente de agua?	La altura inicial máxima es de 12 cm	La altura inicial máxima es de 5.47.
Escribe una altura inicial del agua h_0 con la que el diámetro del globo sea igual a la altura del agua h_a.	Si el recipiente se llena hasta una altura de 10cm, el diámetro se expandirá hasta los 10cm.	A una altura de 5.45

En general, los elementos constantes del EDVI son el radio del recipiente, el volumen inicial de agua y la altura inicial del agua. Sin embargo, nótese que el T2 indicó como variable la altura inicial del agua. Inferimos que dieron esta respuesta porque se trata de un parámetro que se puede modificar en el EDVI. Aunque, una vez establecido, al inflar y desinflar el globo este permanece constante. Observe también que, las respuestas a la pregunta 4 son diferentes. Ambas respuestas son correctas, ya que el T1 se enfocó en la h_0 con el globo desinflado, mientras que el T2 primero infló el globo hasta su valor máximo y posteriormente llenó el recipiente con agua. Finalmente, las respuestas a la pregunta 5 nos hacen inferir que los estudiantes confundieron la altura inicial del agua (h_0) con la altura del agua (h_a) aunque se puede observar que sí identificaron valores en los que el diámetro del globo es igual a la altura del agua.

Actividad Referencial

Tras la exploración del EDVI "Globo", la actividad en el aula continuó con el problema de identificar la ecuación polinómica que modela el volumen total del contenido del recipiente, cuando el radio del globo es tangente a la superficie del agua para identificar como solución la raíz de un polinomio. Clasificamos esta actividad como *referencial* porque los estudiantes comenzaron a establecer relaciones matemáticas en el contexto. Esta actividad se dosificó de modo que los estudiantes propusieran ecuaciones para determinar la altura del agua (h_a) en relación con el radio del globo (x), el volumen del globo (V_G), el volumen inicial del agua (V_A) y el volumen total del contenido en el recipiente (V_T). La tabla 2 muestra las ecuaciones propuestas por los equipos T1 y T2.

Tabla 5. Ecuaciones propuestas por los equipos T1 y T2 para modelar el problema del radio del globo tangente a la superficie del agua

T1	T2
$V_G = (\frac{4}{3})\pi x^3$	$V_G = (\frac{4}{3})\pi x^3$
$V_A = \pi(R^2)\, h_0$	$V_A = \pi(R^2)\, h_0$
h_a=$2x$	h_a=$h_0 - x$
$V_T = \pi(R^2)\, h_a$	$V_T = \pi R^2 h_a + V_G = \pi R^2(h_0 - x) + \frac{4}{3}\pi x^3$
$V_T(x) - V_A - V_G = 0$ Donde $V_T(x)$ es el volumen del cilindro con radio x y altura del agua h_a, V_A es el volumen inicial de agua y V_G es el volumen del globo.	$\pi R^2(h_0 - x) + \left(\frac{4}{3}\right)\pi x^3 - V_T = 0$

De las respuestas observamos que ambos equipos identificaron la relación entre el volumen total del contenido en el recipiente (V_T), el volumen sumergido del globo (V_G) y el volumen inicial del agua (V_A). Por ejemplo, los estudiantes del T1 mencionaron que “el volumen total del contenido en el recipiente es igual a la suma del volumen inicial del agua y el volumen sumergido del globo”. Además, señalaron que esta relación se podía expresar mediante la siguiente ecuación “$V_T = V_i + V_g$”. De forma similar, los estudiantes del T2 indicaron que “el volumen total del contenido del recipiente es igual a la suma del volumen inicial del agua con el volumen sumergido del globo”. Sin embargo, ningún equipo llegó a la ecuación polinómica esperada $\frac{4}{3}x^3 - 2R^2x + R^2h_0 = 0$ donde, x es el radio del globo, R es el radio del recipiente y h_0 es la altura inicial del agua. Esta ecuación se desarrolló y explicó en la discusión grupal.

Actividad General

Después de identificar la ecuación polinómica $f(x) = \frac{4}{3}x^3 - 2R^2x + R^2h_0$ que modela el problema del radio del globo tangente a la superficie del agua, se propuso a los estudiantes que usaran CalcVisual para encontrar las raíces del polinomio, con $R = 5$ y $h_0 = 4$, $f(x) = \frac{4}{3}x^3 - 50x + 100$. Clasificamos esta actividad en el *nivel general* porque los estudiantes se enfocaron en representaciones gráficas y tabulares para encontrar las raíces, sin hacer referencia al contexto del globo. Por ejemplo, el T1 respondió “Nuestra función cuenta con 3 raíces, $-6.9493, 2.3430$ y 4.6063. Esto lo conocemos gracias a que al meter la función dentro de nuestro programa graficador, este genera 3 puntos exactos”. Por su parte, el T2 mencionó “Este polinomio tiene 3 raíces, $x1 = -6.94, x2 = 2.34\ y\ x3 = 4.60$. En la gráfica podemos ver tres puntos, lo cual significa que cada uno de ellos es parte de una raíz”. Nótese cómo en ambos casos mencionan la existencia de puntos en la gráfica del polinomio, los cuales asociaron con sus raíces. Al finalizar esta actividad, pedimos a los estudiantes que validaran sus resultados respondiendo la pregunta “¿Qué raíces tiene sentido para el problema del globo?”. El T1 respondió que “las raíces positivas debido a que se no se puede tener un volumen negativo. Los valores de la segunda y tercera raíz el radio del globo es tangente a la superficie del agua". En cambio, el T2 señaló que "todas las raíces son correctas. Sin embargo, para la raíz $x3 = 4.6$ el radio del globo es tangente a la superficie del agua". Las respuestas a esta pregunta se pueden clasificar en el nivel referencial porque los estudiantes interpretaron las raíces encontradas en el problema del globo. El cambio de nivel de actividad va en acuerdo con la afirmación de Rasmussen y Blumenfeld (2007) acerca de que los niveles de actividad no imponen una jerarquía estricta. Sin embargo, es importante aclarar que esta pregunta se hizo con la intención de identificar el significado que los estudiantes estaban dando a las raíces reales en el contexto del EDVI Globo.

Actividad Formal

Después de que los estudiantes realizaron actividades con el uso de CalcVisual para encontrar las raíces de un polinomio, trabajaron con actividades de operación inversa como: "Escribe un polinomio que tenga al menos las siguientes raíces reales: $r1 = 2$ y $r2 = -5$ ¿será ese el único polinomio que tenga al menos esas raíces reales?". Las respuestas de los equipos T1 y T2 se resumen en la tabla 3.

Tabla 6. Respuestas de los equipos T1 y T2 a actividad de operación inversa

Escribe un polinomio que tenga al menos las siguientes raíces reales: $r1 = 2$ y $r2 = -5$	
Equipo T1 Respuesta: x^2+3x−10 1) Se coloca el signo opuesto de las raíces dadas. 2) Multiplicamos los factores que obtuvimos. (x−2)(x+5)= x^2+5x−2x−10 3) Expandimos el producto. $x^2 + 3x - 10$	Equipo T2 Respuesta: (x−2) y (x+5) (x−2)(x+5)= x^2+5x−2x−10 $x^2 + 3x - 10$
¿Será ese el único polinomio que tenga al menos esas raíces reales?	
No es el único polinomio que tiene al menos esas raíces porque podemos multiplicar el polinomio por cualquier otro factor lineal y obtendremos un polinomio que tenga las mismas raíces.	No, también puede ser $f(x)(x-2) = (x^3$ +3x−10)(x−2)= x^3−5x^2−4x+20

Como se puede observar en la tabla 3, los estudiantes escribieron primero los polinomios de forma factorizada y posteriormente realizaron la multiplicación de los factores para proponer un polinomio desarrollado. Clasificamos estas respuestas en el nivel formal porque los estudiantes trabajaron con procedimientos algebraicos desligados del EDVI Globo y el uso de CalcVisual.

La actividad formal también se observó en la actividad 5 que no involucraba el uso de herramientas digitales sino la visualización de gráficas estáticas como la de la figura 3. Esta actividad se diseñó como tarea final para identificar el aprendizaje que los estudiantes adquirieron sobre el concepto de raíz en un contexto distinto al del Globo.

Escala en el eje x es de 1 unidad por marca.

1. ¿Cuántas raíces reales tiene el polinomio de la gráfica?

   a) 0 b)2 c)3 d)No sé

Justifica tu respuesta

____

2. Escribe un valor aproximado para cada raíz:

____

Figura 3. Actividad de visualización de gráficas estáticas para identificar raíces

Los estudiantes iniciaron la actividad 5 en el nivel general porque para responder las preguntas de la figura 3, los estudiantes se enfocaron en identificar los puntos en los que la gráfica corta al eje x como muestran las respuestas de los equipos T1 y T2. Por ejemplo, en T1 mencionaron que "la gráfica está atravesando el cero en el eje de las x tres veces. Por lo tanto, la función de esta gráfica contiene 3 raíces: $-2, 0$ y 3". De forma similar, el T2 contestó "debes contar el número de veces que la gráfica del polinomio cruza el eje x. Esta gráfica lo cruza exactamente tres veces. Entonces, el polinomio tiene al menos tres raíces: $-2, 0$ y $+3$". Posteriormente trabajaron en el nivel formal porque representaron el polinomio mediante una factorización para llegar a su representación desarrollada.

Discusión y Conclusiones

Presentamos una secuencia de actividades que ayuda a los estudiantes a transitar de un razonamiento basado en un problema contextual a uno formal sobre el concepto de raíz. Si bien, Gravemeijer (2020) menciona que los niveles de actividad no necesariamente se observan de manera jerárquica, en nuestra investigación observamos que las actividades guiaron el razonamiento de los estudiantes sobre el concepto de raíz de forma secuencial. Es decir, la actividad 1 se trabajó en el nivel situacional y referencial, las actividades 2 y 3 en el nivel general y las actividades 4 y 5 en el nivel formal. Con lo anterior no queremos decir que los estudiantes no pueden regresar al contexto cuando trabajan en el nivel general y formal. Elegimos el contexto del Globo como un contexto con el que los estudiantes pueden significar el concepto de raíz, por ello, inferimos que se trata de una situación que perdura en la mente del estudiante.

Destacamos que, la actividad 1 con el uso del EDVI Globo fomentó el desarrollo del nivel situacional al identificar variables, parámetros y relaciones funcionales. Posteriormente, fomentó el tránsito al nivel referencial en la actividad de determinar la ecuación polinómica que modelaba el problema del radio del globo tangente a la superficie del agua. Este polinomio se usa en la actividad 2 para que los estudiantes identifiquen sus raíces, mediante el uso de CalcVisual, y les den un significado en el contexto del Globo. La actividad 3 fomentó el desarrollo del nivel general al trabajar con el CalcVisual mediante el tratamiento de funciones polinómicas fuera de cualquier contexto. Las actividades 4 y 5 fomentaron el tránsito al nivel formal al desarrollar un polinomio expresado mediante una factorización y localizar raíces en una gráfica. Los estudiantes ya no usan el EDVI Globo pero pueden usar el CalcVisual para ingresar el polinomio factorizado o desarrollado y visualizar representaciones gráficas, algebraicas y tabulares.

Una de las limitaciones de este documento es que se han presentado los resultados únicamente de dos equipos de estudiantes, aunque el análisis del nivel de actividad se ha realizado con los datos de los 45 participantes. De este análisis observamos que 30 estudiantes alcanzaron el nivel formal al trabajar con las actividades 4 y 5. Aquellos que no alcanzaron el nivel formal se debe a dificultades operativas y el uso incorrecto de procedimientos algebraicos.

Otra limitación del estudio tiene que ver con los problemas asociados a la instalación y uso de las herramientas digitales. A pesar de que se proporcionó a los estudiantes el CalcVisual para que lo instalaran en su computadora, algunos tuvieron problemas para instalarlo y no pudieron realizar las tareas en casa que necesitaban el uso de dicho software. Como solución, acudieron a otra herramienta como Wolfram, GeoGebra, etc para resolver las actividades. Aquí, el problema radica en que al usar herramientas como Wolfram los estudiantes obtienen las raíces sin saber cómo. Prueba de lo anterior es que en la retroalimentación con los estudiantes que trabajaron todas las actividades con el CalcVisual, manifestaron seguridad al resolver el examen del curso. En cambio, aquellos que utilizaron otros softwares, no sabían de dónde provenían los datos. En este sentido, el rol del profesor es importante porque debe intervenir para guiar a los estudiantes en el cumplimiento de los objetivos. En este estudio, durante la discusión en grupo, el profesor mostró a los estudiantes cómo usar el CalcVisual y discutió las desventajas de usar otro software.

Retomando nuestra pregunta de investigación ¿cómo recuperar los significados del concepto raíz de una función real mediante actividades mediadas por la tecnología digital? Sugerimos que el uso indiscriminado de la tecnología digital en la enseñanza de las matemáticas puede contribuir a la perdida de los significados y aplicaciones de los conceptos matemáticos. Por lo que recomendamos que su aplicación requiere de un cuidadoso diseño didáctico previo a su aplicación. Sugerimos iniciar la actividad en el aula con la simulación de un problema en contexto que permita a los estudiantes dar sentido al concepto matemático de interés. La simulación del problema del globo sumergido en un recipiente permitió al estudiante dotar de significado al concepto de raíz que por su propia naturaleza es abstracto. Usar un software no resolutivo como CalcVisual permite manipular diferentes representaciones de forma simultánea para establecer relaciones entre ellas, lo cual lleva a que los estudiantes adquieran significado del concepto de raíz en su representación tabular, gráfica y algebraica.

Destacamos la importancia de incluir actividades sin el uso de herramientas digitales para corroborar el aprendizaje del concepto de raíz. En este caso las herramientas digitales sirven como herramienta de verificación de resultados. Los estudiantes pueden identificar raíces de gráficas estáticas, obtener su polinomio y, posteriormente, ingresarlo en el software para comprobar los valores numéricos de las raíces. La aplicación de las HEAG es fundamental ya que guían la secuencia de actividades, contribuyendo a la comprensión del concepto de raíz de una función real.

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SYMBOLIC VARIATIONS ACROSS MATHEMATICAL SUBAREAS: EXPLORING CHALLENGES IN UNDERGRADUATE STUDENTS' INTERPRETATION OF MATHEMATICAL SYMBOLS

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This research explores how undergraduate students interpret mathematical symbols in new contexts when reading diverse mathematical texts across various subareas. Collaborating with experts in mathematical sciences, we collected proof-texts aligned with their specialized areas. These proof-texts were presented to undergraduate transition-to-proof students who had studied logic for mathematical proof while their experience of proofs in advanced mathematics topics was limited. Task-based interviews were conducted outside their regular classroom. This paper examined student encounters with curly bracket symbols in a graph theory context. Our findings suggest the nuanced relationship students have with symbols in proof- texts. While possessing familiarity with certain symbols, this pre-existing student knowledge could influence their accessibility to symbols introduced in unfamiliar contexts.

Keywords: Reasoning and proof, Mathematical Representations, Undergraduate Education

Introduction

Mathematical symbols serve as a fundamental language for mathematical representations, abstraction, argumentation, and communication (Cobb et al., 2000; Eckman, 2023; Harel & Kaput, 1991; Pape & Tchoshanov, 2001). Conventional symbols particularly play a crucial role in communication among individuals by representing normative meanings of mathematical ideas, formulas, and relationships (Pimm, 1995). Teachers and students can use conventional symbols to engage in a shared discourse in the mathematics classroom (Goos, 2004).

Despite the importance of symbolic representations in mathematics, numerous studies indicate that undergraduate students encounter challenges when confronted with reading mathematical expositions and proofs that include mathematical symbols (Dawkins & Zazkis, 2021; Inglis & Alcock, 2012; Shepherd & van de Sande, 2014; Weber & Mejia-Ramos, 2014). Mathematical texts often employ conventional symbols, especially those presenting theorem statements and their proofs. Moreover, advanced mathematics courses at the undergraduate level introduce new symbols for novel concepts or extend known ones in a different or broader context. Students may find these symbols challenging either because they represent newly introduced concepts or because their meanings are expanded to cover new areas. These challenges, arising from potentially unfamiliar or expanded-meaning symbols, may impact students' comprehension of the theorem statements and their proof-texts. This perspective resonates with the broader concept of 'symbol sense' discussed by Arcavi (1994, 2005), involving an individual's understanding, familiarity, and flexible use of (conventional) mathematical symbols.

In line with this standpoint, we address the following research question: *To what extent do undergraduate transition-to-proof students perceive and respond to mathematical symbols when encountering the familiar symbols in unfamiliar subareas of mathematics while reading proof-texts?* This question reflects earlier concerns about students' potential struggles in interpreting conventional symbols in proof-oriented mathematics courses. By investigating the awareness and responsiveness of undergraduate students to conventional symbols across different mathematical subareas, we aim to provide insights into the challenges students face. This study could also offer valuable implications for instructional practices and curriculum development for transition-to-proof mathematics courses.

Theoretical Framework

Our perspective on students' interpretation of conventional symbols aligns with radical constructivism, positing that symbols gain significance only when individuals attribute meanings shaped by their previous experiences (Glasersfeld, 1995). When facing a familiar symbol in an unfamiliar context, students assimilate, incorporating new information into their existing cognitive structures based on their past experiences. If assimilation proves insufficient, students engage in accommodation, adjusting their cognitive structure to integrate subtle distinctions in the meaning of the familiar symbol in the unfamiliar context. This perspective suggests that students who are not the creators of mathematical symbols may not bring the same meaning to symbols as the creator, especially when those symbols are introduced by authoritative creators, such as mathematicians, their classroom instructors, or textbook authors. In this situation, students may face challenges with what Hiebert (1988) suggested as the procedure of connecting symbols with mathematical objects or operations. Specifically, when students encounter a new conventional symbol for the first time, they may not have a connection with the mathematical objects or operations the symbol represents. Students face the challenge of deciphering the intended meaning behind conventional symbols, often without the opportunity to negotiate their meanings (Eckman & Roh, 2024). Far from indicating deficits, the interplay of assimilation and accommodation in response to these cognitive challenges serve as opportunities for deeper comprehension as students actively construct and expand the meaning of the symbols.

To comprehend students' cognitive processes of interpreting conventional mathematical symbols in proof-texts, we introduce the construct of *Symbol Sensitivity*. Symbol sensitivity involves being aware of and responding to mathematical symbols, requiring a nuanced understanding of the semantic subtleties within mathematical contexts. There are empirical studies illustrating student challenges of symbol sensitivity, where students may not be sensitive to distinguishing various mathematical symbols and, therefore, not perceive the resulting semantic differences the authors of the given mathematical expositions intend to convey through the symbols (Eckman, 2023; Roh & Lee, 2011; Sellers et al., 2017).

In contrast, this paper focuses on another critical aspect of symbol sensitivity that we will call *symbol contextual interpretation* (SCI), which is an individual's ability to perceive and interpret distinct meanings of a symbol in different contexts. In certain instances, the same mathematical symbol is employed to convey different semantic nuances across various sub-areas of mathematics. It becomes crucial for individuals to recognize and interpret these distinct meanings based on the specific context in which the symbol is used. For instance, a student may encounter the equal symbol (=) in a mathematical expression involving two functions, f and g. While the equal symbol itself is not new to the student as they have been using it between two numerical values, its usage in the symbolic expression $f = g$ may introduce a new context. In this situation, students need to be aware that the equal sign in the function context conveys a different meaning than the equality between two numerical values. However, students may not always be sensitive to these variations when encountering a familiar symbol (=) in an unfamiliar mathematical context (functions). In some ways, this parallels McGowen and Tall's (2010) notion of *met-before*. That is, meanings often change in mathematics as new contexts are encountered, and a student's *met-befores* can serve to support or hinder. McGown and Tall (2010) illustrate this with the subtraction symbol (-), which is initially associated with a "take away" meaning; however, that meaning is not conveyed in other contexts, such as when dealing with negative numbers.

This paper centers explicitly on exploring students' symbol contextual interpretation (SCI) across various areas of mathematics. By closely examining students' SCI, we aim to gain valuable insights into student challenges in reading comprehension of mathematical texts involving mathematical symbols.

Methodology

Data Collection

As part of a more extensive project (NSF DUE #2141925) focused on curriculum development for transition-to-proof courses at the undergraduate level, we created twenty-eight (28) proof-texts by collaborating with nine researchers across various mathematical sciences sub-areas. In preparation for implementing these proof-texts in a classroom, we first tested them through task-based clinical interviews (Hunting, 1997) with undergraduate students at two large public universities in the United States during the Spring of 2023. Participants were students chosen from transition-to-proof courses or proof-oriented courses to ensure students' understanding of logic for mathematical proof while maintaining limited exposure to proofs across diverse subareas in mathematics. The students are encountering diverse subareas in mathematics for the first time, with proof-texts authored by experts from these new subareas. This presents a dual challenge, as students not only face unfamiliar subareas but also grapple with challenging and novel proof-texts for the first time. We paired students whenever possible to foster meaningful interaction between students and promote dynamic discourse. Each interview extended over 90 minutes, maintaining independence from the participants' course instructors.

Interview Tasks

In each interview, we provided students with one or two proof-texts, each spanning 2-3 pages, encompassing three main components: background information (e.g., definitions, notations, and examples), the theorem statement to be proven, and a proof of the theorem. The interviews were divided into dedicated sections: background information discussion, theorem statement exploration, proof analysis, and a collective reflection post-reading.

The interviewer initiated each component by inviting students to read independently and collaboratively discuss the proof-text with their peers. Students were encouraged to pose questions and use tablets as scratch paper whenever they wanted. Subsequently, the interviewer posed targeted questions that drew inspiration from the proof comprehension assessment model developed by Mejia-Ramos et al. (2012). These questions encompassed both local and holistic comprehension questions. The former involved inquiries about the meaning of terms and statements, identification of the proof framework, and the explicit explanation of implicit warrants in the proof. The latter focused on summarizing the proof, identifying the modular structure of the proof, transferring general ideas or methods to different contexts, and providing illustrations with examples. The primary goal of the interviews was to investigate ways to support students in making sense of these new and challenging proof-texts.

Data Analysis and Results

Our analysis commenced with a thorough review of video recordings of the interview data. The primary objective of the analysis was to identify instances where students encountered challenges while engaging with reading the proof-texts. Through an exhaustive examination of the entire video dataset, we discerned persistent instances where students observed notational usage within proof-texts, akin to recognizing misuses or typographical errors in the proof-texts.

In this data analysis process, a recurrent theme emerged – several students faced similar challenges with understanding, interpreting, and using symbolic expressions in the given proof-texts. These challenges with symbols introduced in the proof-texts occurred multiple times, as exhibited in one of the universities part of the project (24 students with 14 interviews conducted), especially as students read to understand the background information such as definitions, theorems, and examples preceding a theorem to be proven and its proof. The symbols we focused on were those not unfamiliar to the students, but their appearance in unfamiliar contexts created student challenges.

In this paper, we suggest our construct, *symbol contextual interpretation* (SCI), as a type of symbol sensitivity. We use it to analyze an individual student's perception and responsiveness to distinct meanings of such symbols in varying proof-texts. We further delineated *contextual awareness* and *contextual adaptation* as characteristics of SCI. We refer to contextual awareness as an individual's awareness that a symbol can have multiple meanings in different contexts; and contextual adaptation as an individual's fluency in adapting a relevant meaning of a symbol in varying contexts. These characteristics laid the foundation for establishing three categories of student symbol sensitivity in recognizing and interpreting the same symbol's distinct meanings in different mathematics subareas. Table 1 summarizes the characteristics of each category with the number of instances where students exhibited the SCI category.

Table 1. Three Categories of Symbol Contextual Interpretation (SCI)

SCI	Contextual Awareness	Contextual Adaptation	Description	#(Instances)
SCI.0	X	X	An individual adapts only one meaning for a symbol, regardless of the various contexts in which the symbol is used, without indicating potentially different meanings.	9
SCI.1	O	X	An individual is aware that a symbol can convey different meanings in different	6

			contexts but has not developed the normative meaning in the relevant specific context.	
SCI.2	O	O	An individual is aware a symbol can convey different meanings in different contexts and exhibits fluency in adapting its normative meanings in varying contexts.	9

Results

In the rest of this section, we present an illustrative episode from an interview with Ernie and Sally. These students worked together to comprehend a theorem in graph theory, describing the condition for the degrees of the vertices of a graph that can determine the connectedness of a simple graph. As background information before the theorem statement, the proof-text introduced definitions pertinent to the theorem, such as graphs, vertices, edges, loops, parallel edges, degrees of vertices, etc. The curly brackets, {}, were also presented as symbols for the set of vertices, an edge, and the set of edges of a graph. A diagram of graph was provided as another representation, along with the symbolic expression of an example graph G, its vertex set $V(G) = \{a, b, c, d, e\}$ and edge set $E(G) = \{\{a, b\}, \{a, d\}, \{c, d\}, \{c, d\}, \{b, c\}, \{b\}\}$ (see Figure 1). The diagram illustrated five dots, labeled as $a, b, c, d,$ and e, representing five distinct vertices and 5 segments, representing 5 distinct edges of the example graph. Two of the edges connected the same vertices *c* and d, corresponding to the duplicates of two identical curly bracket symbols, $\{c, d\}$, in the edge set $E(G)$. The example graph G also included a loop, as an edge having one endpoint b, corresponding to the singleton set notation $\{b\}$, and a vertex, e, not connected to any of the other vertices of the graph.

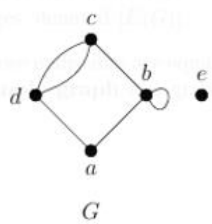
Definition. A **graph** G consists of two sets: a nonempty set $V(G)$ of **vertices** and a set $E(G)$ of **edges**, where each edge is associated with a set consisting of either one or two vertices called its **endpoints**.

Definition. Two vertices u, v are **adjacent** if $\{u, v\}$ is an edge. A vertex u and an edge e are **incident** if u is an endpoint of e. We also say that two edges are **incident** if they share an endpoint (vertex). A vertex is **isolated** if there are no edges incident to it.

Definition. The **order** of a graph is the cardinality of vertices, denoted $|V(G)|$. The **size** of a graph is the cardinality of edges, denoted $|E(G)|$.

Definition. A **loop** is an edge whose endpoints are equal. **Parallel edges** are edges having the same pair of endpoints. A **simple graph** is a graph that does not have any loops or parallel edges.

Example:
Consider the graph G below.



Graph G has vertex set $V(G) = \{a, b, c, d, e\}$ and edge set
$E(G) = \{\{a, b\}, \{a, d\}, \{c, d\}, \{c, d\}, \{b, c\}, \{b\}\}$. The order of G is $|V(G)| = 5$, and the size of G is $|E(F)| = 6$. Graph G has two parallel edges and one loop.

Figure 4 The Excerpt from the Background Information in a Graph Theory Proof-text

The curly brackets, { }, were not new to Ernie and Sally because they had already been acquainted with the symbol when the concept of a set was introduced in transition-to-proof courses that they had taken. However, the proof-text in graph theory introduced the curly brackets in an unfamiliar context to the students, i.e., graph theory. We selected this episode from an earlier moment of the interview to illustrate how Ernie and Sally perceived and responded when encountering the symbol in an unfamiliar context.

Ernie and Sally grappled with the concept of parallel edges represented in the edge set (which uses curly brackets) with repeated pairs. Specifically, unfamiliar with using this symbol to denote "parallel edges" in graph theory, these students found it challenging to interpret instances of the symbol occurring twice in the edge set $E(G)$. Ernie expressed concern about the repetition, while Sally imputed the repetition to two distinct curved segments in the diagram of the graph G, as representing distinct edges, which shared the endpoints c and d. See the transcript below for the students' utterances at the moment.

[1] Ernie: What I don't get, though, is how parallel edges work. If E [$E(G)$] is a set, right, then we can't have duplicate items [$\{c, d\}$] in a [the] set [$E(G)$].

[2] Sally: *(Grabs tablet and begins writing and speaking)* Cause maybe one of them is like pointing from *c* to *d* (*motions writing instrument counterclockwise from the top half of their imaginary circle)* and the other is *d* to *c* (*traces the lower half of the circle in the same counterclockwise direction).*

[3] Ernie: But that's not ordered pairs though (*points to notation of edges on the proof-text)*. I guess it isn't a relation like that, so we don't have a vector, right?

Ernie's SCI regarding Contextual Awareness. Ernie interpreted the letter 'E' in the symbol $E(G)$ for 'the edge set' as the name of a set and extended his understanding of the curly brackets symbol to the definitions and the given example set G (see Figure 1). Ernie was also familiar with conventional rules for using curly brackets in mathematics to denote a set, including the avoidance of repeated elements within the same set or the consideration of repeated elements as representing the same elements (e.g., $\{c, d\} = \{c, d, c\}$). This suggests that Ernie associated the curly brackets with a mathematical meaning, viewing them as a conventional symbol for denoting a set. Despite grasping the mathematical symbol, Ernie encountered difficulties when transferring his principles with the curly brackets symbol to the graph theory context. Specifically, Ernie exhibited a limited awareness regarding specific conceptual nuances within the context. This limited contextual awareness is evident through three distinct instances.

Firstly, from the video recording, we noticed that Ernie directed his attention solely toward the curly brackets symbol in the provided example graph G, while overlooking the accompanying diagram (Figure 1). He did not exhibit any utterances or gestures to establish a representational connection between the two distinct edges in the diagram of the example graph G to the edges in the duplicated symbols $\{c, d\}$ in the edge set $E(G)$. Ernie's exclusive focus on the curly brackets did not position him to leverage the diagram, which may have provided more contextual information about the meaning of the edge set. In this instance, the presence of duplicates of the same symbol in the (edge) set was a barrier to supporting Ernie's comprehension of the concept of edges, rather than aiding his understanding of parallel edges.

Secondly, in the transcript, line 1, Ernie demonstrated a non-conventional principle to the curly brackets when denoting a set. Ernie noticed that in the example graph G, the symbol "$\{c, d\}$" was repeated twice in the symbol for the edge set of G, $E(G)$, and he asserted, "we can't have duplicate items [$\{c, d\}$] in a set." Ernie's utterance indicates that the edge set notation in the proof-text did not adhere to the conventional curly bracket rules for sets in set theory that he was familiar with. He was interpreting the curly brackets in the example not within the graph theory context but rather in the context of the transition-to-proof course where the students at his university initially learned about sets. Ernie is reasonable, bringing in his prior knowledge about avoiding duplicates within set notation. It is unlikely that he had experienced this requirement as a flexible conventional practice aimed at representing unique elements in a set.

Finally, in the transcript, line 3, Ernie responded to Sally's explanation of directional notations involving vertices c and d, by noting that $\{c, d\}$ is not an ordered pair or a vector from point c to point d. This suggests that Ernie expected Sally's directional interpretation to adhere to vector notation conventions rather than the use of curly brackets symbol $\{c, d\}$. Ernie would not allow duplicating an ordered pair, vector symbol, or any symbol within a set notation.

Sally's SCI regarding Contextual Awareness. In contrast to Ernie, Sally exhibited contextual awareness when encountering the duplicates of the same symbol $\{c, d\}$ in the edge set notation. Sally's awareness of the graph theory context was evident in her consideration of both the curly brackets symbol and the diagram depicting the example graph G in Figure 1. By using both representations as resources to understand the parallel edges, she exhibited her nuanced understanding of the symbol in the graph theory context. Her remark in the transcript, line 2, accompanied by hand motions tracing each path in the diagram of the example graph G, illustrated awareness of the context by attending to the curly brackets and bracketed elements in relation to graph theory (and the diagram). Sally recognized that although both edges in the diagram share the same endpoints, they are distinct, the top edge "from c to d," and the other edge "from d to c." That is, they have directionality. Therefore, duplicating the symbol $\{c, d\}$ within the edge set $E(G)$ aligns with Ernie's rule, as each curly brackets symbol represents a distinct edge within the graph G.

Sally's SCI regarding Contextual Adaptation. While Sally demonstrated contextual awareness when encountering duplicates of the curly brackets symbol $\{c, d\}$ in the graph theory context, she exhibited potential interpretative challenges in adapting her interpretation of the symbol to a different graph theory context. Although Sally did not explicitly recognize this potential challenge, evidence of it emerged through her gestures and word choices in this episode. During her examination of the example graph G, Sally's hand motion traced two edges parallel to one another on the diagram for the graph G, attributing a distinct direction to each of them with the same pair of vertices (endpoints). In addition, she correlated these movements with the curly brackets symbols $\{c, d\}$ found in the edge set notation accompanying the diagram of graph G. Thus, Sally interpreted each instance of the symbol $\{c, d\}$ in the edge set symbol as representing a separate edge: one for the top edge and another for the bottom edge in the diagram. Sally's use of the phrase "from [c] to [d] … and from [d] to [c]" indicates that she may conceptualize edges as directed, with each edge having a specific associated direction. Sally appeared to be drawing on the same notions of set as Ernie, but perhaps adding this additional feature made the distinction between the same symbolically represented edge clear. As this is a non-normative distinction, Sally would likely need to continue to expand her contextual meaning if encountering a graph with more than two parallel edges.

A Couple More Examples While a detailed examination was conducted with two students to illustrate contrasting aspects of SCI, Table 2 provides a broader perspective by presenting concise examples across various subareas of mathematics. The table showcases instances of students with different SCI categories, each accompanied by a brief description explaining why their specific case corresponds to the identified SCI. This compilation not only enriches our understanding of SCI but also offers a valuable resource for educators and researchers seeking insights into the diverse manifestations of students' potential challenges with interpreting familiar symbols in unfamiliar mathematical contexts for the first time.

Table 2. More Examples of Students′ SCI

Student	Context	Symbol	Contextual Interpretation	SCI
Patty	Combinatorics	{ }	Perceived the notation within the context of the combinatorics proof-text and described the symbol's meaning using the objects from the combinatorial context, showing adaptivity from her previous transition to proof context to the new combinatorics context.	SCI.2
Nathan	Combinatorics	{ }	Described the use and meaning of the symbol within the contexts of a transition-to-proof course as opposed to the new combinatorics context.	SCI.1

Spencer	Topology	one-to-one	Perceived this term, a symbol, as it was used to describe functions in the context of Topology.	.2	SCI
Cade	Topology	one-to-one	Described this term, a symbol, within the context of a ratio using the symbol colon (:).	.0	SCI
Ronnie	Combinatorics	power set symbol P	Described the symbol script P as a power set as it has been denoted in transition-to-proof contexts.	.0	SCI

Conclusion and Discussion

In the results section, we delve into the challenges experienced by Ernie and Sally as they grappled with a familiar symbol encountered in an unfamiliar context for the first time. Navigating novel situations beyond their prior experiences, the students faced challenges that demanded a nuanced understanding of symbols. We analyzed the students' Symbol Contextual Interpretation (SCI) to understand their sensitivity toward symbols in these contexts. Noteworthy is the collaborative effort exhibited by Ernie and Sally in making sense of these new symbols. This collaborative success suggests the viability of incorporating such challenging proof-texts into a transition-to-proof course. For mathematics education researchers, understanding students' comprehension of notation is crucial for informing the effective implementation of proof-texts in these courses. A key insight from our study emphasizes that introducing students to new symbols extends beyond providing them with texts and definitions; it requires careful consideration of their prior experiences and explicit elucidation of how symbols may take on different meanings. This study highlights the misconception that assumes students in mathematics courses can seamlessly discard prior meanings of symbols, emphasizing the need for a thoughtful approach to incorporating notations when used in new mathematical contexts.

> To emphasize this point further, we reference a quote by Kershner and Wilcox (1950): Whenever nonbasic mathematical words are introduced, they will, of course, be explicitly defined. Whenever technical use is made of these words, the reader must carefully eliminate any preconceptions concerning their meaning and think only of their definitions. This will be difficult, but it is absolutely necessary. Unless all suggestions conveyed by these words from past associations are persistently ignored, a multiplicity of meanings may arise. Our mathematical definitions will be unambiguous and complete (p. 17).

This statement, though seemingly psychologically absurd, reflects expectations placed on students in mathematics courses. It highlights student challenges with isolating definitions from their past associations. Ernie's case exemplifies this challenge as he drew upon his prior understanding of the symbol "{ }" to interpret a new proof-text intending a different meaning. This situation underscores the importance of acknowledging the subtlety and complexity of interpreting symbols across various mathematical subareas in mathematics education literature.

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ARGUMENT-MIRRORING PROOFS: A METHODOLOGICAL APPROACH FOR HELPING STUDENTS RECOGNIZE INCOHERENCE IN THEIR THINKING

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This report discusses argument-mirroring proofs – proof-like texts which utilize students' own thinking to provoke perturbations to their schemes. The aim of the study was to characterize the subject's native understanding of continuity, differentiability, and the derivative, use that subjective understanding to generate argument-mirroring proofs, and examine how her thinking evolved in response to these argument-mirroring proofs. The results suggest that presenting the subject with the conflicts in their thinking directly contributed to changes in their conceptions of relationships among continuity, differentiability, and the derivative. This suggests that argument-mirroring proofs might achieve similarly profound results if applied in other contexts.

Keywords: calculus, research methods, cognition, metacognition

This paper details the way in which I developed and used *argument-mirroring proofs*, texts which formalize the subject's previously exhibited thinking about a topic to formulate and justify a claim. In a case study, argument-mirroring proofs were used on numerous occasions to encourage a student to confront and resolve potential sources of incoherence in their thinking.

Theoretical Perspective

The argument-mirroring proof method is broadly informed by constructivism (Glasersfeld, 1988), and in particular, students' *schemes*. Furthermore, I am interested in *perturbations*, instances where the expected result does not occur, and *accommodations*, adjustment(s) made by the subject to their schemes.

Coherence (Thompson, 2008) is the extent to which s student's schemes are compatible with one another. I specifically investigate the schemes which, in my conjectures, had the potential to give rise to incoherence in the subject's cognition. I used argument-mirroring proofs to evoke discordant schemes and provoke perturbation in order to encourage the participate to engage in accommodation. I henceforth refer to these schemes as *schemes targeted for conflict* because I constructed argument-mirroring proofs and other tasks in effort to cause the participant to recognize the conflict. If the participant indicates that one or more of their schemes has been perturbed, a *perceived conflict* has been triggered.

Research Methodology

Overview

The study employed a specific implementation of a constructivist teaching experiment methodology (Steffe & Thompson, 2000). In this implementation sessions alternated between clinical interviews (Clement, 2000), to glean student thinking about the mathematical ideas, and exploratory teaching interviews (Sellers, 2020), to attempt to perturb the subject using argument-mirroring proofs.

Participant and Interview Tasks

Rachel (pseudonym), the lone participant in the study, was an undergraduate mathematics education student at a large public university in the American southwest. In total, Rachel participated in six interviews, each of which lasted between 60 and 90 minutes.

The purpose of the clinical interviews was to build a baseline for Rachel's understanding of continuity, differentiability, and the derivative. I asked Rachel to consider the function $f: [0,4] \rightarrow R$ defined by $f(x) = \sqrt{k - 1 - x^2}$ given a variety of values of the constant k. For each set of parameters, she was asked to describe where the function was continuous and differentiable. She was also asked to find and interpret the value of the derivative where it was appropriate to do so. Her responses indicated her schemes at the time of the session. From these schemes, I selected those which to target for conflict during the exploratory teaching interviews.

The exploratory teaching interviews consisted of three types of tasks. First, I presented Rachel with the schemes I targeted for conflict to confirm that my model of her understanding was accurate. Next, I asked her to consider a function and discuss it in terms of continuity and differentiability. Finally, I presented her with relevant argument-mirroring proofs and asked her to determine whether the proofs were a) valid and b) representative of her thinking.

Data Analysis

Field notes and transcripts were generated from the recordings of each session. Transcripts from clinical interviews were open coded (Strauss & Corbin, 1998) to capture Rachel's schemes pertaining to the concepts. Axial coding was used to compare the schemes which were identified in open coding. In particular, I identified groups of two or more schemes which I would target for conflict in the subject's thinking if they were presented in an appropriate fashion.

Results

I focus on Rachel's schemes I subsequently targeted for conflict with argument-mirroring proofs. In so doing, I highlight moments when she perceived conflicts, adjusted her schemes, and brought greater coherence to her understanding. This discussion is limited to the first two sessions, since they best illustrate argument-mirroring proofs.

Phase 1: Initial Conceptions of Continuity and Differentiability

The first clinical interview yielded an initial state of Rachel's conceptions of continuity and differentiability as well as a set of ideas to target for conflict in the subsequent session.

Transcript 1

Interviewer: Why is the function continuous on the interval [0, 4]?

Rachel: Because it is defined on the interval.

Rachel (later): A function is continuous if the limit from the left is the same as the limit from the right.

Transcript 2

Rachel: If you had a hole in the function, you can still differentiate there.

Interviewer: If you have a hole, is the function still continuous and differentiable?

Rachel: Yes.

In the first transcript, Rachel gives two different descriptions of continuity of a function, one requiring the function to be defined and another only requiring its one-sided limits to agree. Her description of a hole in the second transcript suggests that a function may be continuous at points where it is not defined. From this session, I identified three of Rachel's schemes, shown below, which could potentially result in a perceived conflict.

a) If f is defined at a, f is continuous at a.
b) If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x)$, then f is continuous at a.
c) Although f is not defined where $x = a$, if $\lim_{x \to a} f(x)$ exists, f is continuous at a.

Items (a) and (b) are indicated by Rachel's responses in transcripts 1 and 2, respectively. Her assertion in transcript 2 that a hole may not preclude differentiability suggests item (c). Since (a) and (b) offer non-equivalent definitions of continuity, I conjectured that targeting these schemes simultaneously would result in a perceived conflict.

Phase 2: Provoking Conflicts among Initial Conceptions of Continuity and Differentiability

When I presented Rachel with items (a)-(c), she agreed that they were representative of the ways she understood continuity and differentiability but indicated no conflict between them. Next, I presented her with the following graph of a function (Figure 1) and asked her to determine where it was continuous, in hopes of triggering a conflict between items (a), (b), and (c) from phase 1.

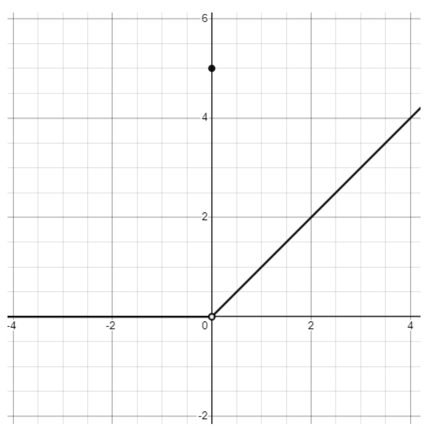

Figure 1: Graph from Task in Phase 2

Claim: f is continuous at $x = 0$.
Proof:
- $\lim_{x \to 0^+} f(x) = 0$
- $\lim_{x \to 0^-} f(x) = 0$
- Since the one-sided limits agree, $\lim_{x \to 0} f(x) = 0$.
- Therefore, f is continuous at $x = 0$.

Claim: f is continuous at $x = 0$.
Proof:
- From the graph, we see that $f(0) = 5$.
- Since f is defined at $x = 0$, f is continuous at $x = 0$.

Figure 2: Argument-Mirroring Proofs from Phase 2

The excerpt below illustrates her perception of the conflict when evaluating the left argument-mirroring proof from Figure 2.

> *Interviewer:* Is the claim [that f is continuous at $x = 0$] correct?
> *Rachel*: I would be inconsistent to deny this.
> *Interviewer:* Is it a problem that the proof doesn't talk about [the point] (0, 5)?
> *Rachel:* It seems like it might be pertinent since the claim centers around $x = 0$…But I can't decide why I think that matters.

Considering the argument-mirroring proof was when Rachel first perceived a conflict. Her certainty that f was continuous at $x = 0$ began to waver during the above excerpt. By the end of the session, Rachel stated that a function is continuous at a point if the value of the function at the point is equal to the limit of the function at the point. As such, I represent the transformation in her schemes below (Figure 3). She accommodated her schemes by striking scheme (c) from her conception and combining the hypotheses from the schemes (a) and (b) to yield a new, unified scheme. After perceiving and resolving the conflict, her conception of continuity became more coherent because it was less susceptible to conflicts.

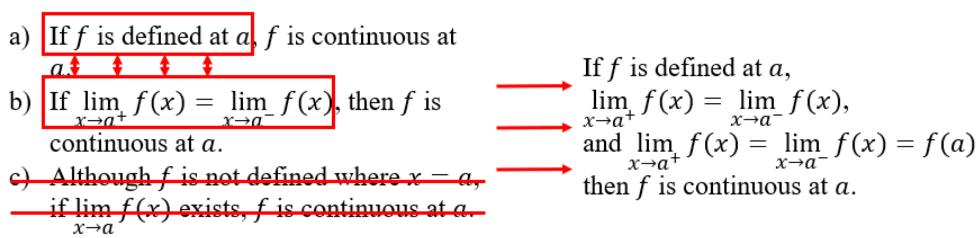

Figure 3: Argument-Mirroring Proofs from Phase 2

Discussion and Conclusion

Rachel's responses to the tasks indicated that her understanding regarding continuity was deeply incoherent. The clinical interviews provided baselines of her understanding. Recall that she affirmed the schemes I presented to her as an accurate initial portrayal of her thinking. The review of the identified schemes, directed tasks, and argument-mirroring proofs in these sessions were designed to trigger perceived conflicts. The argument-mirroring proofs were effective in making the conflicts evident to Rachel. In every occurrence, when Rachel perceived a conflict between her schemes, she immediately worked to resolve it. In the language of Glasersfeld (1988), she responded to the perturbation of a scheme (its inconsistency with some other scheme) by accommodating it (adjusting one or more schemes). Worth noting is that Rachel was never explicitly instructed to resolve any perceived conflicts. When she reconciled her schemes for continuity, her thinking became more coherent (Thompson, 2008). Considering the argument-mirroring proofs was instrumental in illuminating the conflicts in Rachel's thinking. The method of argument-mirroring proofs warrants further investigation with additional participants in different mathematical contexts.

Methodologically speaking, the effectiveness of argument-mirroring proofs as the cornerstone of a teaching experiment (Steffe & Thompson, 200) to trigger Rachel's perception of a conflict was profound. While teaching experiments are common, my protracted execution allows for the argument-mirroring proofs and other activities to be carefully tailored to the participant. This method represents a novel contribution to the field. As such, whether this method yields similar effectiveness with different participants in future studies warrants further investigation.

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EXPLORING THE USE OF HISTORICAL CONTEXTS IN TEACHING LOGARITHMS

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This study investigated the effectiveness of integrating history-infused lessons on students' understanding and attitudes towards logarithms. The theoretical framework drew from sociocultural perspectives and embodied cognition, emphasizing the social and emotional dimensions of learning. Design-based research principles guided the iterative development of history-infused logarithm lessons. Data was collected through pre- and post-assessment tests, interviews, and classroom observations. The findings indicated a significant improvement in students' post-test scores, suggesting a reduction in their fear of logarithms. Additionally, interviews revealed a positive shift in students' perceptions of logarithms, from abstract and intimidating to practical and relatable.

Keywords: History-infused mathematics, logarithms, secondary mathematics education, student attitudes

The research literature emphasizes the importance of logarithms in both advanced mathematics and real-world contexts, including sound measurements (decibels), earthquake magnitudes (Richter scale), star brightness, and chemical properties (pH balance). However, many students struggle to grasp the conceptual underpinnings of logarithms and often resort to rote memorization of rules, as noted by various authors (Berezovski, T., 2008; Kuper & Carlson, 2020; Weber, 2016). The challenges faced by students include interpreting logarithms as the "inverse of exponents" and developing a coherent understanding of logarithmic notation, logarithm properties, and the application of logarithmic functions (Kuper & Carlson, 2020; Berezovski, T., 2008; Chua & Wood, 2005; Gol Tabaghi, 2007; Strom, 2006).

To address these issues, researchers and educators have suggested a variety of strategies. These include using concrete materials (Thompson, 1994), implementing authentic assessments such as project-based learning and computational thinking (Shin et al., 2021), engaging students with game-based learning (Barab et al., 2010), problem-based learning (Hmelo-Silver, 2004), and effective teaching methods (Larmer, 2018). The use of gestures alongside diagrams (Walsh & Hord, 2019), gestures combined with manipulatives (Beilstein, 2019), and incorporating the history of mathematics (Liu, 2003; Poh & Dindyal, 2016; Sampaio & Batista, 2018) have also been recommended. This study aims to bridge the gap in research regarding the teaching of logarithms by utilizing the history of mathematics in combination with gestures. The research questions guiding this study are as follows: (1) How do history-infused logarithm lessons aid in reducing students' fear of logarithms? (2) How do students' perceptions of logarithms change over the duration of the history-infused logarithm program?

Theoretical Framework

This study was grounded in sociocultural perspectives for student learning, a history-infused program, and embodied cognition for evaluating understanding. It focused on social constructivism, which emphasized the role of social interaction in cognitive development and suggested that learning occurred best in a social context. The study also incorporated the dynamic nature of assessment, including formative, diagnostic, prognostic, or summative assessment. Vygotsky's sociocultural perspective underscored the extensive impact of social learning, suggesting that learners did not engage with new knowledge in isolation. Key instructional aspects in Vygotsky's perspective were mediation, scaffolding, and creating a zone of proximal development (ZPD). Building on this, the instructional technique in the history-infused math lessons involved exploratory activities and mediational strategies, including scaffolding based on prior knowledge. This study also emphasized embodied cognition, focusing on how humans used their bodies to express thought processes, like gestures. Students' multimodal approaches, such as gestures, were coded and compiled based on McNeill's typological categories (1992), allowing for a holistic approach to teaching and learning logarithms

Methods

Participants, Settings, and Programs

The study took a holistic approach, incorporating both quantitative and qualitative data collection methods to assess the effectiveness of integrating historical insights into logarithm teaching. This approach was underpinned by design-based research (DBR) principles, enabling iterative refinement of teaching strategies based on observed student interactions and outcomes. The study took place at a private high school in Western New York, United States, with a student-teacher ratio of 12:1. The focus was on 14 students (10 girls and four boys) in Grades 11 and 12, all of whom had prior exposure to logarithms in their mathematics courses.

The curriculum, based on the Precalculus with Limits: A Graphing Approach by Ron Larson (High School Edition, 6th Edition), was adapted to incorporate historical insights into the discovery of logarithms. The goal of the first phase was to develop an initial design of the program, which consisted of three history-infused math lessons. These lessons were developed by integrating the history of logarithms, allowing students to explore historical perspectives on the discovery of logarithms by John Napier, and other mathematicians (e.g., Pythagoras), tailoring students' participation in peer collaboration through the lens of history of mathematics, and improving learner engagement in the instructional process in the form of mini projects on history-infused mathematics. Students were asked to use log tables and watch a video of the process. Students also explored the history of mathematics related to logarithms and did a presentation. In the second phase, the program underwent iterative design to test and refine it. This second iteration, to be conducted in the spring of 2023, involved designing three history-infused modules, each consisting of three lessons with scenario-based problems (e.g., Scenario-based Log) and historical approaches (e.g., Using Log Tables). The historical accounts of the discovery of logarithms will be introduced from existing sources, highlighting how logarithm computations were performed before calculators.

Data Collection and Analysis

Data collection involved pre- and post-assessment tests, interviews, observations, and analysis of classroom artifacts. Pretests assessed students' baseline knowledge, while post-tests measured their understanding after the history-infused lessons. Interviews and observations offered qualitative insights into student engagement, attitudes, and understanding. The study was conducted over two weeks, covering initial assessment, implementation of the history-infused program, and subsequent assessment and interviews. Data analysis was conducted using a mixed-method approach, combining quantitative and qualitative methods. For quantitative analysis, descriptive statistics were computed to compare pretest and posttest scores, and the Wilcoxon Signed-Rank Test was used for inferential analysis to assess the effectiveness of the history-infused lessons. Error analysis of participants' written responses was also conducted. For qualitative analysis, interviews were transcribed using the ELAN annotation tool, and thematic analysis was conducted. Open-coding was done using ATLAS.ti software to categorize data from surveys and interviews. Additionally, gestural analysis was conducted, categorizing gestures based on McNeill's framework, and disagreements between coders were resolved through consensus. The general inductive approach was employed to analyze qualitative data, systematically organizing and summarizing textual data.

Summary of Findings

Analysis of pre- and post-test showed that students' average scores in the posttest (M = 76.2, SD = 17.4) were significantly higher than their average scores in the pretest (M = 50.2, SD = 21.1).

How History-Infused Logarithm Lessons Alleviate Students' Fear of Logarithms:

The analysis of pre- and post-test scores shows that the history-infused logarithm program led to a statistically significant improvement in students' understanding of logarithms. This finding is particularly noteworthy considering the pre-existing fear and apprehension that many students typically harbor towards this complex mathematical concept. Interviews with students provided deeper insights into the impact of history-infused lessons on students' emotional engagement and attitudes towards logarithms. A majority of students expressed that the historical context provided in the lessons made logarithms seem more accessible, relatable, and less intimidating. Many students indicated that understanding the origins and evolution of logarithms gave them a sense of connection to the subject, and a better appreciation for its practical significance. Student L7 articulated this sentiment, saying, "It helped me understand it better because I can be more appreciative of the mathematicians back in the day and it gets me more interested in math, so I will be motivated to learn more about the concepts knowing the philosophers that contributed to it." Incorporating historical narratives and activities into the logarithm curriculum served as a cognitive scaffold for students, allowing them to contextualize complex mathematical concepts within a narrative framework. Students appreciated the opportunity to engage with mathematical ideas in a more holistic and multidimensional manner. Furthermore, the interactive and collaborative nature of the history-infused lessons encouraged students to approach learning logarithms with a sense of curiosity and adventure, rather than fear and reluctance.

Shifts in Students' Perceptions of Logarithms Through History-Infused Logarithm Program:

Through the history-infused logarithm program, students' perceptions of logarithms underwent a noticeable shift. Before the intervention, students primarily viewed logarithms as abstract and disconnected from real-world contexts. They often perceived logarithms as challenging, even forbidding, due to the complex nature of mathematical manipulations involved. The pretest data showed that students had a limited understanding of logarithmic properties and frequently made errors in their application. Common mistakes included misinterpreted language errors and logically invalid inference errors, suggesting that students' conceptual grasp of logarithms was limited. However, post-test data revealed a marked improvement in students' perception of logarithms. Students began to view logarithms as a valuable tool with practical applications, particularly in the context of historical problem-solving. They expressed newfound confidence in their ability to tackle logarithmic calculations and demonstrated a clearer understanding of logarithmic properties and their applications. This shift in perception can be attributed to the rich historical narratives that were integrated into the program, which allowed students to see logarithms as a dynamic and evolving mathematical concept with a rich cultural and historical significance. Students began to appreciate the versatility of logarithms and how they are rooted in the history of human endeavor.

Discussion

The findings of this study echo the conclusions of previous research and contribute to our understanding of the potential benefits of incorporating history into mathematics education. Previous studies have demonstrated that history-infused mathematics lessons can lead to improvements in students' conceptual understanding and engagement (Alibali & Nathan, 2012; Berezovski, 2008; Howell et al., 2017). The findings of the current study extend this research by focusing on students' emotional responses to history-infused lessons and their impact on attitudes towards mathematics. The theoretical implications of this study align with cognitive theories such as Vygotsky's sociocultural theory of learning and Hmelo-Silver's Problem-based Learning (PBL) theory (Kozulin et al., 2003; Hmelo-Silver, 2004). Vygotsky's theory emphasizes the role of social interaction and cultural context in shaping learning, suggesting that the historical narratives embedded in history-infused lessons can provide students with meaningful cultural tools that facilitate learning. Hmelo-Silver's PBL theory focuses on the importance of problem-solving and authentic, real-world tasks in promoting deep understanding. The historical context provided in history-infused lessons can serve as a rich source of problems and tasks that are relevant and engaging for students. From a practical perspective, the findings of this study suggest that incorporating historical contexts into mathematics instruction can have a positive impact on students' attitudes and engagement. By presenting mathematical concepts in a historical context, educators can make abstract concepts more concrete and meaningful for students, leading to increased motivation and interest. Additionally, the emotional engagement fostered by history-infused lessons can help to alleviate students' fear and anxiety about mathematics, creating a more positive and supportive learning environment.

The limitations of this study should be acknowledged. The study was conducted at a single high school, limiting the generalizability of the findings. Additionally, the study focused on a specific topic within mathematics (logarithms), and the findings may not apply to other mathematical concepts. Future research could explore the impact of history-infused lessons on a wider range of mathematical topics and in different educational contexts. Moreover, future studies could investigate the long-term effects of history-infused lessons on students' attitudes and engagement, as well as the role of technology in enhancing the effectiveness of these lessons.

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CONJECTURNING THE DIVERGENCE TEST FOR SERIES: "MAYBE THEY ONLY WIN IF THEY CONVERGE TO ZERO"

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Series are a key part of the calculus curriculum and warrant more research on how students can be supported in using their informal intuitions to conjecture about series convergence. We conducted a teaching experiment with a pair of post-Calculus I students, during which they reinvented claims related to the divergence test using a context problem that we call the Partial Sum Sequence Game. We investigated how the students' collective argumentation for these claims emerged and evolved throughout the sessions.

Keywords: Undergraduate Education, Calculus, Reasoning and Proof

Series convergence is a key part of the calculus curriculum and has applications in computer science, physics, and other disciplines (Earls, 2022). Martínez-Planell et al. (2012) suggest that it is constructive to view series as sequences of partial sums, but this view is challenging for students, and they find it difficult to recognize the connection between sequences and series. This disconnect might explain how students can use informal intuitions to make sense of sequence convergence (Oehrtman, 2009) but might have "mechanical views" of when series converge (Kung & Speer, 2013, p. 428). Rather than viewing series as sequences of partial sums, students sometimes view series as a list of values (Martínez-Planell et al., 2012, Przenioslo, 2006) or as a running total that sums various numbers of summands without coordinating the sum with index values (Eckman & Roh, 2022). These interpretations can support the idea that summands tending towards zero implies the running total will eventually stabilize (Eckman & Roh, 2022).

Textbooks and instruction often introduce series in an algorithmic, decontextualized, and formal way (González-Martín, 2010). We join other scholars who advocate for more innovative approaches (González-Martín, 2010; Morrel, 1992). Scholars have started to take some important first steps in this direction including investigating how students can construct a formal $\varepsilon - N$ definition of series convergence (Martin et al., 2011), create algebraic representations for arbitrary partial sums and infinite series (Eckman & Roh, 2024), and conjecture the comparison test for convergence (Davis & Vroom, 2024). We add to this work by investigating: *How did two undergraduate students reinvent claims related to the divergence test?* Specifically, we investigated how two undergraduates' collective argumentation for two claims emerged and evolved throughout a teaching experiment. These two claims are equivalent to the divergence test and the converse of the divergence test is false.

Theoretical Support

Realistic Mathematics Education's guided reinvention heuristic (Freudenthal, 2005; Gravemeijer, 1999; Gravemeijer & Doorman, 1999) framed the instruction during the experiment (similar to Lockwood & Purdy, 2019). This instruction aimed to support "a process by which students formalize their informal understandings and intuitions" (Gravemeijer et al., 2000, p. 237). A key component to guided reinvention is a context problem in which the problem situation is experientially real to students and evokes students' informal understandings and intuitions about the concepts they reinvent (Gravemeijer & Doorman, 1999).

To investigate how the students' informal intuitions related to the divergence test developed, we studied how their collective argumentation for their related conjectures emerged and evolved throughout the sessions. By collective argumentation, we mean instances when students and/or the teacher-researcher made mathematical claims with evidence that supported them (Conner et al., 2014). Like other scholars investigating collective argumentation (Alzaga Elizondo, 2022; Andrews-Larson et al., 2019; Conner et al., 2014), we adapted Toulmin's (2003) model of argumentation. Toulmin's model dissects arguments with (at least) a combination of *claims* (conjectures being justified), *data* (evidence that supports the claims), and *warrants* (connections between the data and claims). Our adaptation of Toulmin's model included attention to student and teacher-researcher contributions and the possibility of more inter-connected argument arrangements such as sub-arguments. Students' collective argumentation includes but is not limited to valid mathematical proofs (Krummheuer, 1995), and we do not view other forms of argumentation as subordinate to valid proofs.

Methods

Our data comes from a teaching experiment (Steffe & Thompson, 2000) with two first-year undergraduate students Lara and Stella (pseudonyms), a teacher-researcher (the first author), and an observer. We recruited the students from their Calculus I course where Lara earned a 2.5 (on a 4.0 scale) scale and Stella earned a 4.0. Lara was majoring in biological chemistry and Stella double-majoring in psychology and neurosciences. The experiment was eleven 1.5-hour sessions, with the last four focusing on exploring series. We audio and video recorded the sessions, including a synced recording of the students' collaborative digital work and gestures.

The students' exploration of series began with a context problem (Gravemeijer & Doorman, 1999) after students had classified various sequences based on their properties like convergent and increasing (Vroom et al., 2024). The context problem was posed as a game featured in Figure 1. Playing the game involves considering sequences of partial sums (what Lara and Stella called "total distance sequences"). A player wins the game if and only if the sequence of partial sums converges. For instance, $\{n^2\}$ loses the game because it generates a divergent sequence of partial sums. The game paired with the instruction supported a "running total" meaning like Eckman and Roh's (2022) participants with one key difference being that Lara and Stella coordinated the running total with the index (days). The students played the game with several sequences of their choice. As they did so, they voiced predictions about sequences and their properties winning or losing the game. Afterward, the students were tasked to write a "cheat sheet" for the game in which they gave future game players advice about sequences that won or lost the game, as well as wrote some insightful warnings.

Game:
We are conducting an experiment in which we move an object a certain number of feet north per day. A sequence $\{x_n\}$ will tell us how many feet x_n to move the object north on any given day n. To win the game, accurately predict the object's location if the experiment continues indefinitely!

Playing the game with $\{n^2\}$:

n	1	2	3	4	5	…
n^2	1	4	9	16	25	…
Total distance from starting point	1	5	14	30	55	…

$\{n^2\}$ <u>loses</u> the game since we cannot accurately predict the object's location

Figure 1. Partial Sum Sequence Game

Some of the statements on the cheat sheet resembled the divergence test in the game context ("if the original [sequence] converges to any number except 0, the total distance [sequence] will diverge") and the falsehood of its converse ("not all sequences that converge to 0 will win"). We investigated how the students' collective argumentation for these claims emerged and evolved throughout the sessions. We began the retrospective data analysis by re-watching video and re-reading the corresponding transcripts, identifying key episodes that featured the students discussing ideas seemingly relevant to their statements on their "cheat sheet." With these key episodes, we then applied Toulmin's model of argumentation, first identifying claims (C), then data (D) and warrants (W), as well as any other contextual comments that seemed relevant and not captured by the codes. As we did so, we compared new C-D-W to previous ones, noting any connections. Like Conner (2008), we noted who contributed (students, the teacher-researcher, or a combination), and if the warrant was our interpretation of what was implicit in the data. We created Figure 2 through this process, where the symbols ⬚, ▭, ☁, and ➡, respectively denote that student(s) primarily contributed, both student(s) and teacher-researcher contributed, implicit warrant, and a revised claim.

Results

We next present part of the students' conjecturing activity for claims related to the divergence test, focusing on a few key episodes throughout several teaching sessions. Prior to what we will share, Lara and Stella played the game with $\left\{\frac{1}{2^n}\right\}$, which they argued won because "the values [we] are adding on, keep getting smaller, and our total is approaching one." They also claimed that sequences with the "same kind of shape" such as $\left\{\frac{1}{n}\right\}$ and $\left\{\frac{1}{n^2}\right\}$ would also win. Figure 2 summarizes the students' collective argumentation that we will share next.

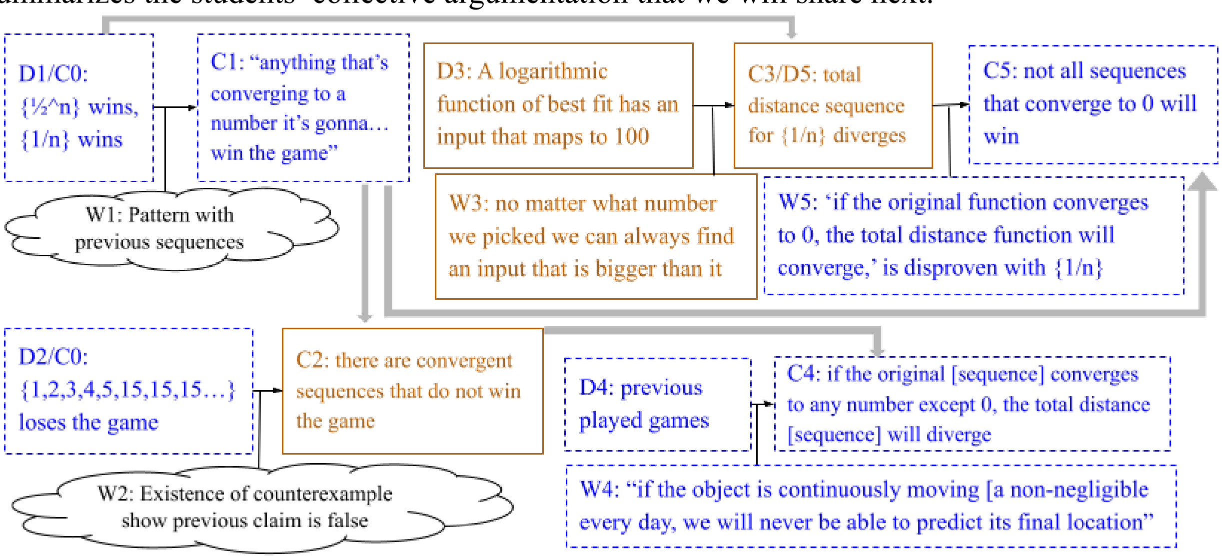

Figure 2. Summary of students' collective argumentation.

After the teacher-researcher asked if they had predictions about types of sequences that would win the game, Lara responded, "anything that's converging to a number it's gonna…win the game" (C1) using the previous claims about sequences that won the game as their evidence

(D1) since this pattern held for them in these cases (W1). The teacher-researcher then asked if they had a converging sequence that would lose the game. Stella pointed to the sequence $\{1,2,3,4,5,15,15,15,\ldots\}$ recalling that they previously "didn't think this one would win" (D2) and the teacher-researcher added, "so we can say that's not always the case" (C2).

Later the students used Desmos to view the first thousand terms of $\left\{\frac{1}{n}\right\}$'s total distance sequence. They started questioning their previous claim that the sequence would win the game (C0). After Stella expressed desire to understand the total distance sequence "pattern," the teacher-researcher supported the students to find a logarithmic function that fit the first thousand terms. The students observed the logarithmic graph, noticing "it's increasing by less" and questioned whether the logarithmic function converged. The teacher-researcher then asked if they would be convinced that it diverged if the function surpassed a certain number. In response, the students set the equation equal to 100 and solved, finding a solution (D3). Lara then responded, "it diverges" (C3). The teacher-researcher questioned, "Do you think no matter what number, y-value, we picked we could always find an x that [maps to something] bigger than that y-value?" The students both responded "yes" (W3) referencing the generality of their algebraic work, and Lara added, "I wish we couldn't… it would make more sense if $\left\{\frac{1}{n}\right\}$ matched $\left\{\frac{1}{n^2}\right\}$."

The students later returned to writing their cheat sheet, discussing a previous claim they revised (C1). Then, they considered sequences to see if a winning sequence converged to a non-zero value. After their unsuccessful search (D4), Stella proposed, "maybe they only win if they converge to 0." She later explained, "if the object is continuously moving every day, we will never be able to predict its final location." After the teacher-researcher highlighted that $\left\{\frac{1}{n^2}\right\}$ required them to move the object every day, Stella clarified that it needed to move "a non-negligible amount" (W4). This resulted in the students writing: "if the original [sequence] converges to any number except 0, the total distance [sequence] will diverge" (C4).

The students continued to discuss how $\left\{\frac{1}{n}\right\}$ converging to 0 meant that "we are moving the object a negligible amount" each day; however, the total distance sequence diverged (D5). Stella explained this was, "not a problem with what we have written [C4]… though it makes it less helpful and more confusing" and elaborated: "if we had written the statement as 'if the original function converges to 0, the total distance function will converge,' it would be disproven by the $\left\{\frac{1}{n}\right\}$ example. [C4] says nothing about functions that do converge to 0, which means $\left\{\frac{1}{n}\right\}$ is not applicable here" (W5). The teacher-researcher suggested that they use this information to write a warning, which the students wrote as: "not all sequences that converge to 0 will win" (C5).

Discussion

Lara and Stella's reinvention of claims related to the divergence test was rooted in their experience with the Partial Sum Sequence Game. Their experiences playing the game with various sequences supported them to revise C1 ("anything that's converging to a number it's gonna…win the game") to C4 (equivalent to the divergent test) and C5 (equivalent to claiming the converse of the divergence test is false). The game with $\{1,2,3,4,5,15,15,15,15\ldots\}$ was important for the students to recognize that convergent sequences could lose (C2). The game with $\left\{\frac{1}{n^2}\right\}$ and $\left\{\frac{1}{n}\right\}$ was also important as they provided evidence that converging to 0 was necessary (C4) but insufficient (C5) for winning the game. We note that although the students claimed that $\left\{\frac{1}{n}\right\}$ lost the game (C3), they were still grappling with why beyond relying on a graph (in a similar way that Eckman and Roh's (2022) participants believed that decreasing summands converge). In our future work, we hope to further support students in this way.

As we continue to analyze our data, we aim to understand a fuller story of how all the students' claims on their cheat sheet emerged and evolved together. For instance, the data we presented here also gives some insight into how the students reinvented claims related to *p*-series convergence/divergence as well as the comparison test (Davis & Vroom, 2024). We plan to expand our analysis to explore further how the students' collective argumentation for their claims related to series tests for convergence emerged and evolved throughout the sessions.

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EXPLORING GRAPHING MEANINGS USING EYE-TRACKING

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Keywords: Mathematical Representations; Research Methods; Technology; Cognition

Individuals' quantitative and covariational reasoning form a critical foundation for their construction of STEM concepts and their ability to make critical, data-informed decisions (Karagöz Akar et al., 2022; Yoon et al., 2021). Graphs form a linchpin representation for quantitative and covariational reasoning (Moore et al., 2022). Moore and Thompson (Moore, 2021; Moore & Thompson, 2015) introduced *static* and *emergent* (*graphical*) *shape thinking* to distinguish between students' ways of reasoning for graphs. They described emergent shape thinking to involve understanding a graph as both the process by which it is made (coordinating quantities' covariation) and the product that is made (a trace of that covariation). A student who reasons about a graph emergently can imagine the reconstruction of a graph as a trace in progress, where the trace records the values of the two covarying quantities at different moments. Static shape thinking involves conceiving a graph as an object in and of itself, imagining the graph to be a piece of wire with particular perceptual characteristics (Moore & Thompson, 2015). Static shape thinking involves indexical associations between particular shapes of graphs and learned facts, and thus can imply properties about relationships that those graphs represent. Those relationship properties are not organic to the graph's emergence (Moore, 2021).

Eye-tracking technology is a tool whose use has grown in the past decade, and it has recently shown promise as a tool to gain insights into the phenomenon of the teaching and learning of mathematics (e.g., Brunner et al., 2024; Seidel et al., 2021; Haataja et al., 2021; Roy et al., 2017). Providing inspiration for the presently proposed approach, both Thomaneck et al. (2022) and Waters (2019) used eye-tracking to investigate participants' covariational reasoning in the context of graphing, with Waters and colleagues drawing on the constructs of static and emergent shape thinking as well. Extending this work, we pair eye-tracking technology with the generalized models of static and emergent shape thinking to address the following research questions: *(a) In what ways are eye movement patterns related to students' graphing meanings? (b) In what ways can the use of eye-tracking technology complement current methodologies (e.g., teaching experiments) for exploring and supporting students' graphing meanings?* We are currently designing and conducting interviews to compare eye movement patterns between instances when participants are reasoning statically versus emergently. If the eye movement patterns associated with particular ways of reasoning are understood to some confidence, then eye-tracking data could be used as evidence for (or as a contraindication of) hypothesized meanings. We also envision eye-tracking technologies as contributing to innovative interventions during a teaching experiment. For example, a researcher might show participants videos from their own teaching sessions. Rather than asking them to solely recall their previous thinking as in stimulated recall interviews, researchers could prompt them to discuss how they might have been thinking during the task and how that relates to any observations they make regarding their attentional focus. We envision such an intervention could prompt rounds of focused reflection, which is critical to mathematical development (Ellis et al., 2024). In our poster, we focus on our methodological design and our preliminary findings, and provide examples of eye movement patterns consistent with both static and emergent shape thinking.

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EMOTIONS IN SOCIAL JUSTICE MATHEMATICS: COLLEGE PRECALCULUS STUDENTS' EXPERIENCE

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This study investigates the interrelationship between emotion and learning in a college precalculus lesson that examined lead poisoning in participants' local community. Through a thematic analysis of students' responses to lead poisoning lessons, the study reveals a range of emotional responses, including empathy, concern, and heightened awareness of social issues. The findings underscore the critical role of emotions in deepening student engagement and commitment to social change, emphasizing the importance of incorporating social justice themes into the precalculus curriculum to foster a more engaged and empathetic student body.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Precalculus, Social Justice, Undergraduate Education

The call to change mathematics instruction in high school and college classrooms has been ongoing (see Bartell, 2013; Gutierrez, 2002; Gutstein, 2003, 2006; Guzmán & Craig, 2019) with arguments from mathematics education researchers for a need to change mathematics instruction from conventional procedural practice to focus more on understanding the world and social issues (Gonzalez, 2009; Wright, 2016). Recent research in the field suggests that students are increasingly engaged in mathematics lessons contextualizing issues of injustices in society (Gutstein & Peterson, 2013; Voss & Rickards, 2016). We imagine the future of mathematics education to lean toward contextualizing students' local environments to make mathematics more relatable to students. In addition to the theoretical arguments and empirical claims in favor of teaching mathematics by contextualizing social injustices, students experience various emotions when they learn about societal inequities (Kokka 2019; 2020). The empirical evidence from Kokka's research necessitates the inclusion of emotions in students' mathematical learning. Our study aims to explore the variable of emotions in students' mathematical learning. Our study is set to explore and inform the mathematics education field about emotions students experience in situ of social justice mathematics lessons.

Mathematics education research has for a long time been focused on reasoning (Roth & Walshaw, 2019), knowledge, and other cognitive factors, with little attention to emotions (Schukajlow et al., 2017), the exception being mathematics anxiety (Zan et al., 2006). A few decades ago, however, there was a shift in focus toward incorporating "affect" into this research. Specific definitions and perceptions of affect vary. McLeod (1992) viewed affect as being connected to cognition and performance. Lewis (2013) perceived affect as a student's attitude toward or enjoyment of mathematics. These slightly earlier perceptions of affect regard students' relationships to mathematics, with less emphasis on other factors that may influence students' emotions in the mathematics classroom. In recent years, mathematics education researchers have also more directly incorporated emotion into their study of affect. For example, students bring emotions with them into the classroom (Kokka, 2019; Valoyes-Chavez & Darragh, 2022). Students experience emotions in a mathematics classroom when learning about injustices (Kokka, 2019, 2022; Valoyes-Chavez & Darragh, 2022). Therefore, it is important to consider students' emotions as a factor that affects their learning in mathematics classrooms. Our study is guided by the research question: *How did learning about a local social justice issue affect students' expression of emotions in a college precalculus classroom?*

Theoretical Framework

We leveraged a combination of two theoretical frameworks for this study: affective pedagogical goals (APG) for social justice mathematics (SJM) by Kokka (2022) and historically responsive literacy (HRL) by Muhammad (2020; 2023). The APG aims to support instructors' preparation of SJM lessons by anticipating students' emotions related to injustices. The idea is to provide a space for students to express their emotions about mathematics and the oppressive systems they interact with; this, in turn, may help students process their emotions to understand and use mathematics to address inequities., For this study, we focused on the affective pedagogical goal (Kokka, 2022) of supporting students' expression of emotions related to the local social injustice issue and ongoing efforts to address the issue. Kokka named this goal "identifying and processing emotions to take action." We consider this the first step in helping students process their emotions to act in SJM lessons.

Muhammad (2020; 2023) defined the historically responsive literacy (HRL) framework as a literacy model with five learning pursuits: identity, skill, intellect, criticality, and joy. Muhammad (2020) defined identity as learning about yourself, and the people around you. Skill refers to the concepts and procedures as outlined in school standards. Intellect is the ability to apply skills to understand social interactions. Criticality is the ability to understand injustices in society and use knowledge to challenge the status quo. Finally, Muhammad (2023) defines joy as experiencing happiness, related to celebration, wellness, and justice. We broadened Muhammad's (2023) framing of joy to include other emotions as students engage in a SJM lesson. For this study, we conceptualize meaningful mathematics learning based on the integration of the five learning pursuits of Muhammad's HRL framework, and Kokka's APG.

Methods

Research Context

The participants were forty-three first-year precalculus students enrolled at a predominantly white university in the Northeastern United States. Students completed a SJM lesson designed from an HRL and SJM framework, that mathematized the lead poisoning issue in Metroville, the city where participants attended university. It was created on Desmos, an online tool for creating and teaching lessons. After the lesson, instructors downloaded and saved the anonymized student responses in a shared OneDrive folder accessible only to the research team.

The SJM lesson had three components: pre-lesson survey, Desmos lesson, and post-lesson reflections and surveys. This paper focuses on analyses of students' responses to the Desmos lesson. The Desmos lesson introduced the students to the lead poisoning issue through a video, showed how lead decay can be modeled using exponential functions, asked students to solve exponential decay problems, and informed students about the ongoing efforts to address lead poisoning in Metroville. We asked students to express and explain their emotions at two instances in the lesson: first after learning about the lead poisoning issue, and second after learning about the ongoing efforts. Students were asked to “Name up to three emotions, if any, you experienced after watching the video and writing about lead poisoning in your hometown. Explain what made you feel these emotions.” Our analyses focused on students’ responses to the two questions that asked students to name emotional responses related to injustices and action.

Thematic Analysis

We employed Braun and Clarke’s (2006) six-phase thematic analysis approach to analyze students’ responses to “Explain what made you feel these emotions” questions from the Desmos lesson. We reviewed students’ responses and the research question, wrote notes/memos about initial impressions, and shared them during weekly research meetings. In the second phase, we coded the data and reconciled coding weekly, using a data-driven approach. The third phase involved developing themes by identifying patterns and grouping similar codes. We created a coding frame document to ensure intercoder reliability, including themes, code definitions, and example data segments. In the fourth phase, we checked theme coherence across the dataset. The fifth phase involved aligning the coded responses of the students with themes to address the research question. In the final phase, we selected compelling data excerpts to support our results and developed solid arguments based on these examples.

Results

Our analyses informed us that students expressed emotions because of lack of awareness about lead poisoning, empathy for children/tenants, knowledge about lead poisoning, and views on existing efforts to address lead poisoning. There were other themes in addition to the four mentioned, but in this paper, we focus on these four themes (see Table 1). Students expressed emotions because of a lack of awareness or experience with lead poisoning, and empathy for families living in lead-poisoned houses. Students also expressed emotions based on their knowledge of the lead poisoning issue, willingness to play a role in helping affected families, and their views on the current efforts to address lead poisoning.

Table 1: Theme Findings and Examples

Theme Name and Definition	Associated Emotions	Examples
Theme 1: Lack of awareness about lead poisoning Students express a lack of awareness about the lead poisoning issue by self or people in general.	Shock, Concern, Sad, Annoyance, Curiosity, Disbelief, No emotion, Disturbed	• *"It is frustrating that people don't put in a effort to help others and fix this issue."* • *"never thought about lead poisoning being a problem"*
Theme 2: Empathy for children/tenants Students express empathy for children, tenants, or families who have lead poisoning or have to live in homes with lead poisoning issues.	Sad, Compassions, Pity, Guilty, Concerned, Anger, Bad, Sympathy	• *"I feel concerned for the welfare of the children still subjected to live in those areas."* • *"It is sad that young children are exposed to major health risks due to their living conditions"*
Theme 3: Knowledge about lead poisoning and/or determination to help Students share their knowledge about lead poisoning and a desire to help solve the cause of lead poisoning issues.	Guilty, Motivated, Shocked, Upset, Disturbed, Sadness	• *"I would say that I feel disturbed about the severity of issue and the potential consequences of lead poisoning"* • *"I feel the emotion of sadness desire to help these people."*
Theme 4: Views on existing solutions efforts to mitigate lead poisoning Students express their views on the current efforts/solutions to solve the problem of lead poisoning.	Optimistic, Happy, Promising, Hopeful, Confidence, Enlightened, Knowledgeable, Grateful, Relieved, Excited, Motivated	• *"I'm happy to see that the problem of lead poisoning is being addressed in the community"* • *"It feels good to know that issue is being addressed and that there is things that are beginning to get done in order to combat the problem."*

Theme 1: Lack of Awareness about Lead Poisoning

The first theme captures students' emotions about a lack of awareness regarding lead poisoning. One student expressed their shock at learning about lead poisoning by stating, "I was unaware that lead poisoning was still a thing and lead poisoning itself is a very dangerous thing to still have around. I can't believe that lead poisoning is still a thing, especially in this day in age." We interpreted this student's response as evidence of shock and lack of awareness about lead poisoning. Some students expressed surprise at how easily one can get lead poisoning.

Theme 2: Empathy for Children/Tenants

The second theme explores the emotions that arise from empathy for the people dealing with lead poisoning. After learning about the lead poisoning issue and how it affects people who live in homes with lead, students expressed empathy for the tenants, families, and children who had to choose between having a roof over their heads or being safe from lead poisoning. Learning that people living so close to them are dealing with these issues brought on a wide range of emotions in the students, such as sadness, compassion, pity, guilt, concern, anger, and sympathy. Notably, students were concerned that children were subject to lead poisoning in their home.

Theme 3: Knowledge about Lead Poisoning and/or Determination to Help

The third theme explores students' knowledge about lead poisoning and their determination to help. This theme includes student responses where they share their knowledge about lead poisoning and a desire to help solve the cause of lead poisoning issues. Furthermore, this theme also includes student responses expressing awareness about lead poisoning health risks and the population that is more vulnerable to being impacted. The theme also includes answers expressing students' willingness to help solve or raise awareness about this issue. As students learned about lead poisoning in Metroville, they demonstrated an understanding of the issue and a willingness to help the cause.

Theme 4: Views on Existing Solutions to Mitigate Lead Poisoning

The theme centers on students' positive emotions when learning about efforts to combat lead poisoning. Characterized by optimism and hope, students feel happy and relieved to know that governments and organizations are actively addressing the issue through specific initiatives, fostering confidence in future solutions. It reassures them that change is possible and that there are viable solutions to the challenges they are studying. Students know that their learning has an impact on the real world, which fosters a sense of self-empowerment, encourages them to take a forward-looking perspective, and allows them to look to the future where their newfound mathematical skills and social awareness contribute to social progress and justice.

Discussion

In summary, this study emphasizes the critical role of emotion in social justice mathematics. By addressing social justice issues, mathematics educators could deepen students' understanding of mathematical concepts and develop empathy, critical thinking, and social responsibility. We argue that a careful study of students' emotional responses to SJM lessons is needed to understand and support their learning. Further research could explore how instructors can attend to students' emotions to support their learning. Making mathematics relatable to students by contextualizing their social environments is the future of mathematics education.

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UNDERGRADUATE STUDENTS' PARTICIPATION IN CALCULUS I COURSEWORK AND PEER-LED, COMPLEMENTARY INSTRUCTION WORKSHOP: CASE OF NEIL

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In postsecondary education, Calculus has been historically recognized as a "gateway course" for students to pursuit STEM fields. Responding to this issue, researchers at Montclair State University designed a model of complementary instruction to engage Calculus I students in collaborative problem solving on groupworthy tasks. This multiple-case study seeks to address the question, "How do undergraduate students experience their calculus learning in the parallel spaces of coursework and inquiry-oriented complementary instruction?" The findings of Neil's case study are presented here and include characterizations of the different forms of agentive participation afforded to students in the two spaces, as well as their complementary nature relative to learning calculus with understanding. Implications for dismantling the persistent barriers imposed by calculus on access to postsecondary STEM fields are also discussed.

Keywords: Calculus, Undergraduate Education, Problem Solving, Calculus Thinking

Calculus has the track record of serving as a "gateway course" that contributes to postsecondary students abandoning their pursuit of a STEM career (Hagman et al., 2017). The calculus reform effort in the 1990s emphasized to include fewer topics and incorporate an active learning and teaching approach aiming to transform calculus education to be "lean and lively" (Johnson et al., 2014). Twenty years later, the President's Council of Advisors on Science and Technology (2012) made a similar recommendation in order to provide students the time they need to develop robust understandings of mathematical concepts in order to succeed. Despite the continuing reform effort, the gate-keeping function of Calculus has hardly changed.

Drawing on the Mathematical Association of America's seven recommendations from the Insights and Recommendations (Bressoud et al., 2015), researchers at Montclair State University designed an inquiry-based *complementary* workshop, called Inquiry-Based Instructional Support (IBIS), facilitated by a peer leader (Roth et al., 2001) to run parallel to students' in class learning. During IBIS, students work collaboratively in small groups on groupworthy tasks (Buell et al., 2016) that are non-routine problems to promote conceptual understanding of calculus concepts.

The literature on peer-led cooperative learning models in postsecondary education confirms their effectiveness on students' academic achievement across different undergraduate mathematics courses (Altomare & Moreno-Gongora, 2018; Trenshaw et al., 2019). However, as the literature mainly focuses on evaluating the effectiveness using quantitative methods, there is a lack of insight into why, how, and what about peer-led cooperative learning models that contributes to these successful outcomes. Hence, this study seeks to address the question, *How do undergraduate students experience their calculus learning in the parallel spaces of coursework and inquiry-oriented complementary instruction?*

Perspectives and Methods

This exploratory (Yin, 2003) multiple-case study (Merriam, 1998) is grounded in a situated perspective (Lave & Wenger, 1991) leveraging the "learning as participation" aspect and utilized the concept of figured world (Holland et al., 1998) to examine the change in students' agentive participation and their identity formation (Vågan, 2011). To answer the research question, all of the observation video recordings were transcribed and analyzed using the grounded theory analytical approach (Corbin & Strauss, 2014). To depict a summary overview of each case study participant's enacted agentive participation in class and IBIS, a word cloud with agentive participation codes as clusters was created for each instructional space.

The participants of this study consist of two cohorts of Calculus I undergraduate students whose IBIS attendance is a part of their course requirement. Each cohort has four participants from the same class and attended the same IBIS sessions. Video recordings and field notes were taken for all 24 classes, six workshops, and three focus group interviews (Creswell, 2012).

Findings

The table in Figure 1 shows the various forms of participation enacted in class and IBIS by both cohorts' participants. These participation actions were organized into *high*, *moderate*, and *nominal* interactivity categories to describe students' participatory interactions with others, material resources, or tasks. Next, Neil's case (pseudonym) will illustrate how the participation codes and interactivity categories are used to characterize his participation in both spaces.

		Cohorts A & B		
Categories of Interactivity	Level 1 Codes	Class A	Class B	Cohorts A & B IBIS Workshop
High	Sharing	(Voluntary *[Answer] [Idea] [Resources] [Work]*) (Upon request *[Answer] [Idea] [Resources] [Work]*) (Solicit *[Answer] [Resources] [Work]*)		(Voluntary *[Answer^{A}] [Idea^{A}] [Resources^{A}] [Work^{A}]*) (Upon request *[Answer^{A}] [Idea^{A}] [Resources^{A}] [Work^{A}]*) (Solicit *[Answer^{A}] [Idea] [Work^{A}]*) (Offer *[Work] [Idea]*)
	Inquiring	(Conceptual) (Procedure)	(Procedure)	(Conceptual^{A}) (Other mathematical) (Procedure^{A})
	Scaffolding			(Scaffolding)
	Explaining	(Concept *[Representation]*) (Mistake *[Peer's]*) (Procedure) (Struggle) (Task) (Technicality)	(Mistake *[Instructor's]*) (Procedure) (Reasoning)	(Concept *[Definition] [Representation^{A}]*) (Mistake *[Facilitator's] [Peer's^{A}] [Self]*) (Notation) (Procedure^{AB}) (Provide Example) (Reasoning^{B} *[Realistic]*) (Struggle^{A}) (Task^{A}) (Technicality^{A})
Moderate	Independent work	(Student initiated *[Task] [Review] [Homework]*) (Instructor initiated)	(Student initiated *[Task] [Review] [Homework]*) (Instructor initiated)	(Student initiated *[Task^{AB}]*) (Facilitator initiated)
	Seeking	(Confirmation) (Help) (Resources) (Time) (Clarification *[About something] [For someone]*)	(Confirmation) (Help) (Clarification *[About something]*)	(Confirmation^{AB}) (Help^{AB}) (Resources^{A}) (Time^{A}) (Clarification *[About something^{AB}] [For someone^{A}]*)
	Responding	(Agree/Disagree) (Answer) (Confirm) (Respond to help request) (Private) (Uncertain) (Unfamiliar)	(Agree/Disagree) (Answer) (Confirm) (Private) (Unfamiliar)	(Agree/Disagree^{AB}) (Answer^{AB}) (Confirm^{AB}) (Respond to help request^{A}) (Uncertain^{A}) (Unfamiliar^{AB})
	Check-in	(Peer) (Self)	(Self)	(Peer^{AB}) (Self^{AB})
	Check (and revise)	(Compare) (Other's) (Self)	(Self)	(Compare^{A}) (Other's^{A}) (Self^{AB})
	Accessing resources	(Lesson) (Notes) (Online resources)	(Homework) (Notes) (Online resources) (Textbook)	(Homework^{B}) (Notes^{AB}) (Online resources^{AB})
Nominal	Agency request unfulfilled	(Public) (Private)	(Public)	(Private^{A})
	Refraining	(Refraining)	(Refraining)	
	(Re)launches task	(Read aloud)	(Read aloud)	(Read aloud^{AB}) (Recite info) (Invitation to work on problem)
	Emoting	(Affirmation) (Confusion) (Frustration) (Success)	(Affirmation) (Confusion) (Frustration) (Success)	(Affirmation^{AB}) (Confusion^{AB}) (Frustration^{AB}) (Relief) (Success^{AB})
	Note-taking	(Note-taking)	(Note-taking)	(Note-taking^{AB})
	General coursework	(Give) (Seeking)	(Seeking)	(Give^{AB}) (Seeking^{AB})
	Non-participation	(Non-participation)	(Non-participation)	(Non-participation^{AB})

(Lvl 2 code *[Lvl 3 code] [Lvl 3 code]*)
(Lvl 2 code *[Lvl 3 code^{ClassA}] [Lvl 3 code$^{Class B}$]*) = Occurred in both spaces

Figure 1: A table of participation actions in class and IBIS for Cohorts A and B.

Neil’s Participation Profile

Neil was a private student both in class and IBIS. He spent most of his time in class *taking notes* and, in both spaces, *working independently* on the task at hand. Regarding the opportunities that the instructor provided for students to participate, Neil refrained from participating 323 times across 23 in-person class observations, for an average of about 14 times per class observation. The class and IBIS word clouds, in Figures 2A and 2B, provide a summary overview of Neil’s participation in both spaces. A comparison of his class and IBIS word clouds shows that his independent participation characteristics tended to be magnified in class.

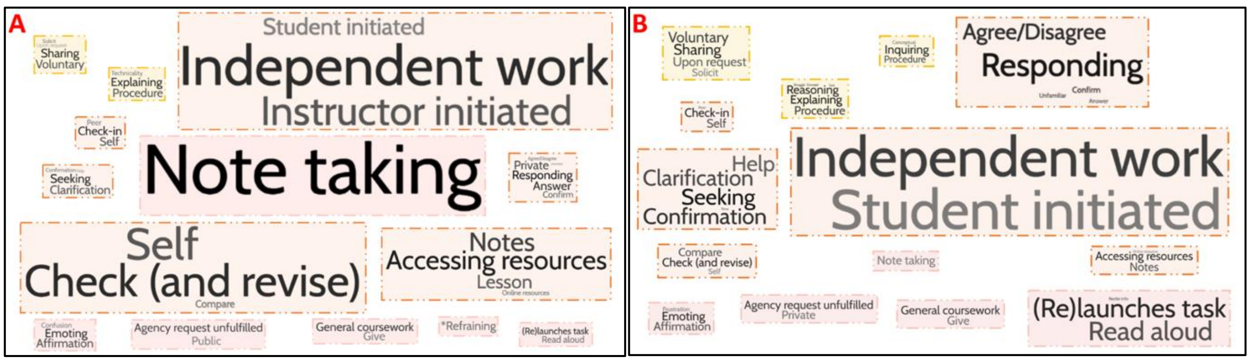

Figure 2: Neil’s class (A) and IBIS (B) participation word clouds

Early in the course, Neil’s participation consisted almost exclusively of *taking notes* and *working independently* on problems posed by the instructor, and then waiting for the instructor or another student to provide a solution. Every so often, as Neil worked on problems independently in both spaces, he would reference a variety of resources, such as his notes and online resources.

The size of the *independent work* cluster in Neil’s IBIS word cloud suggests that though he also tended to work independently in IBIS, groupwork in IBIS offered opportunities and space for him to be a more active and interactive participant. The biggest clusters in his class word cloud are participation actions with moderate interactions with tasks and material resources (e.g., *note taking*, *independent work*, *accessing resources*, and *checking and revising*). In contrast, some of the biggest clusters in his IBIS word cloud are participation actions that have moderate interactions with his peers (e.g., *responding* and *seeking clarification*, *confirmation*, and *help)*. This suggests that more of his interactions in IBIS were with peers than with tasks and material resources. Moreover, as his IBIS word cloud also reveals, Neil was more likely to *respond* to his peers than to initiate interactions with them. He was also more likely to *seek confirmation*, *clarification*, and *help* from his peers than to enact the explainer role. The following excerpt illustrates some of these forms of participation from Neil during IBIS. In this excerpt, his group was discussing a composition function/chain rule problem, given the rates of change of profit per book sale, $p'(s)$, and book sales per month, $s'(t)$.

Table 1: An excerpt of a Chain rule discussion in the third IBIS session.

1	Amelia	Uhm. Well, for [#3] c, I just wrote using the chain rule. I kind of wrote the first part. And then for [#3] b, I just plugged in 4 to s. Like, to $s(t)$ I used 4. And then the answer I plugged it into $p(s)$. And then I just explained that in [#3] c. <...>
2	Neil	I thought you have to put one into the other.
3	Rachel	I did.
4	Amelia	Yea, that's kind of what I did.
5	Rachel	I thought you just have to plug it into the equation it goes with. Like for [#3] a, you just plug it into the $s(t)$. No? For $p(s)$…-cause like it gives you an example.
6	Neil	Yea, you put this portion here into s. <Points to his work on paper>
7	Rachel	So, for [#3] a, you got 16?
8	Neil	Huh? Yea.
9	Rachel	And then for [#3] b, you got 160, too?
10	Neil	Uh huh.
11	Rachel	And then what about [#3] c? What did you do?
12	Neil	You plug this portion into s and that becomes p and then you take the derivative and then you take the derivative becomes $p'(t)$.
13	Rachel	Oh! So, you do 3… <Continues to work on problem independently>
14	Neil	Yea, and then there is…we're done.
Note: <actions>; (unclear utterance); -interruption/cut off-; and [words inserted for clarity]		

In this excerpt, Neil responds to Amelia's (pseudonym) invitation by *sharing* his *ideas* about what to do for this problem (lines 2 and 6). Upon Rachel's (pseudonym) further request for him to share his work with her (line 11), Neil *explains* the *procedures* he took to determine $p'(t)$ (line 12). Even though Neil spent a lot of time in IBIS working independently, in contrast to his class participation, he was also a more active and interactive learner in that space by *sharing* with, *explaining* to, and *seeking* from his peers. As the semester progressed, there was some evolution in how Neil *shared*, *explained*, and what he *sought* from his peers in IBIS.

Overall, even though *note taking* continued to be the dominant form of Neil's participation throughout the semester, as the semester progressed, his participation in both spaces expanded from the predominantly nominal interactions of *note taking* and *working independently* to include both moderate and high interactions (e.g., *seeking*, *explaining,* and *sharing)*. The next excerpt illustrates his participatory expansion trajectory in class from mid-semester. In this excerpt, the class was working on finding the derivative of $f(x) = \sqrt[3]{2x^3 + \sin^2(5x)}$ posed by the instructor. Neil overheard Amelia expressing her confusion and took the initiative to check on her.

Table 2: An excerpt of Neil checking in on Amelia in class.

1	Amelia	I am so confused.
2	Neil	<Overhearing Amelia> Well, what are you confused about?
3	Instructor	<In the background of Neil and Amelia's interaction> Alright. Ah. So, let's quickly differentiate the whole equation.
4	Amelia	Where [does] cosine come from at the end?
5	Neil	<Looks back at his work> When I think about it- When I think about it in 2 pieces, [in terms of] 2 pieces. <Shows Amelia his work>
6	Amelia	Wait. Hold on. Let me finish [writing]. <Looks at Neil's work>
7	Neil	Well, she's talking about quotient (inaudible) first thing (inaudible) the number 2-
8	Amelia	Like, that I got.
9	Neil	Yeah, number 2, then (inaudible) it still has to be opened up (inaudible) so it will be (inaudible) plus (inaudible) sine. The opposite of sine is cosine, so cosine of 5 and then you just grab the last piece, (which is) 5.
10	Amelia	I kind of get it.

In this excerpt, Neil *seeks clarification* on what confused Amelia (line 2). Even when the instructor calls for the class's attention to go over the problem (line 3), Neil and Amelia continue to carry on with their conversation. After Amelia clarifies her confusion (lines 4 and 8), he offers his *explanation* to help her resolve it (lines 5, 6, and 9). This excerpt is one of the examples that illustrates the evolution in the interactivity of Neil's participation. As the semester progressed, Neil also enacted new kinds of *responding*, *sharing*, and *seeking* actions in both spaces.

Discussion and Conclusion

To summarize, this study found a range of agentive participation actions that were further categorized into *high*, *moderate*, and *nominal* interactivity categories based on the quality of their interactions with others, tasks, or material resources. These findings can inform and guide the design and implementation of parallel spaces of coursework and complementary instruction, particularly when the realities of coursework alone impose constraints that do not allow for adequate opportunities for high and moderately interactive participation. Specifically, these findings would be of value to postsecondary calculus educators and program directors who are committed to offering students the kinds of participatory experiences that are productive for their learning of calculus. That way, they can be more mindful in planning, structuring, and designing their calculus programs so as to dismantle the persistent barriers imposed by calculus on access to postsecondary STEM fields.

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Chapter 10:
Pre-Service Teacher Education

EXAMINING AN INSTRUCTOR'S ENACTMENT OF CURRICULUM IN A MATHEMATICS CONTENT COURSE FOR PRESERVICE TEACHERS

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This pilot case study examined how one teacher educator (TE) enacted written lesson plans from a shared knowledge base to support preservice teachers' (PSTs) mathematical content knowledge (MKT) in a mathematics content course. Using classroom observations, reflective interviews and the TE's reflective journal, we explored the modifications made by the TE during the classroom instruction of a unit on percentages. We report that these modifications did not alter the content of the course and additionally created opportunities for PSTs to learn more than the subject matter knowledge (SMK). These reported modifications were prompted by responding to students' responses during classroom instruction and the influences of former instructors' modifications.

Keywords: Instructional Activities and Practices, Mathematical Knowledge for Teaching, Preservice Teacher Education

Much attention has been given to developing the mathematical content knowledge necessary for teaching among preservice teachers (PSTs). The importance of mathematical knowledge for teaching (MKT) is evident in studies examining its relationship with student outcomes. For instance, teachers' MKT was significantly related to student achievement gains even in elementary grades (Hill et al., 2005), and teachers' MKT was significantly related to students' participation in quality mathematics discourse (Wilhelm et al., 2017).

Given its importance, MKT is a primary concern for teacher preparation programs. As a way to improve the quality of teacher preparation, a shared knowledge base for teacher educators has been proposed to document and collect knowledge about effective teaching in university teacher preparation courses (Hiebert, 2013; Hiebert & Morris, 2009; Morris & Hiebert, 2009). In one case, a program using such a knowledge base reported that graduates from multiple cohorts scored higher several years after graduation on measures of MKT on mathematics topics that were covered in the program compared to topics that were not covered (Morris & Hiebert, 2017; Suppa et al., 2018). Although not causal, the results of these studies suggest the possibility of a positive and sustaining impact on teachers' knowledge of mathematics content.

Adding to this work, we report on preliminary findings from a pilot case study in which we consider the ways a TE enacted written lesson plans from a shared knowledge base in the context of a mathematics content course for K-8 PSTs. We wanted to understand the role of the teacher educator (TE) in creating opportunities for PSTs to learn MKT as the TE implemented these lessons. We were curious about how a TE might balance competing interests to follow the lesson plans and to modify the lesson in the moment to be responsive to student thinking. In particular, we examined what modifications emerged in the TE's implementation of the lessons to consider what changes were made, what opportunities to learn MKT were provided by those changes, and why those changes occurred. The research questions that framed this study were: (1) in what ways does a TE modify written lesson plans when enacted to provide PSTs with opportunities to learn MKT ? and (2) what prompted the TE to make these changes?

Mathematical Knowledge for Teaching

Ball and colleagues (2008) specify the kinds of mathematical knowledge needed for teaching. In their framework, they name domains of mathematical knowledge for teaching (MKT) as subject matter knowledge (SMK) and pedagogical content knowledge (PCK). SMK includes common content knowledge which is the mathematical knowledge not unique to teaching and used in various contexts, horizon content knowledge which refers to understanding connections among mathematical topics, and specialized content knowledge which encompasses mathematical knowledge specific only to teaching. In defining these distinct ways that mathematical knowledge for teaching operates, Ball et al. (2008) propose that these domains can inform teacher education and help course designers refine curricula for mathematics content courses in teacher preparation programs.

Methods

In this pilot case study, we examined a TE's enacted changes to written lesson plans to consider how and why these changes occurred and what opportunities the changes provided for undergraduate PSTs to learn MKT. In this research report, we present preliminary results from this case study based on data collected from one unit of teaching during which the second author was observing the first author's teaching.

Context

The semester-long course on rational numbers is the second in a series of three required mathematics content courses for K-8 PSTs. The lesson plans for this course have been developed over several years within a shared knowledge base. They include information about the goals and rationale for each lesson and insight into PST thinking, such as common questions and misconceptions or expected responses. The first author, a doctoral student and new to teacher education, was the instructor of one course section in the Fall 2023 semester. All course instructors met once a week to debrief each week's lessons, get feedback on teaching, and plan for the next week's lessons by discussing the lesson plans.

Data Collection

We used instructional materials, observations of classroom instruction, and reflections from the TE during a unit on the meaning of percentages to explore our research questions. This unit contained three lessons taught over three 80-minute class sessions mid-semester. For each lesson, we reviewed the shared lesson plans, a student course packet containing practice problems, and a slide deck prepared by the TE. The TE kept a reflective journal about the in-the-moment changes she made during instruction and her rationales for those changes. Additionally, we met after each class for a 10-20 minute audio-recorded debrief interview in which the first author reflected on the lesson. The data was prepared for analysis by transcribing all audio recordings and de-identifying all data of student names. Finally, all lesson data was organized by lesson activity such that comparative analyses could be made across the written and enacted lessons.

Data Analysis

The data analysis took place in three phases. In the first phase, modifications of the written lesson plans were identified within the observation data from the enacted lessons. These were coded for each domain of subject matter knowledge within the framework for mathematical knowledge for teaching (MKT) (Ball et al., 2008). Then, these modifications were coded again to identify the source of change (i.e., how it happened) as initiated by the TE or by students and coded to identify the content of the change (i.e., what was changed) using open coding (Saldaña, 2021). In the third phase, we reviewed the recordings from the lesson observations and created event maps (Green and Bridges, 2018). We defined an event as the tasks and subsequent discussions happening in the classroom when a slide was displayed to the students and made notes about the event. These event maps and the other data were coded using the constant comparative method (Charmaz, 2014) to find common themes of how modifications emerged. Then, the data was re-coded according to the themes established and we coded to find what prompted these modifications.

Results

From our preliminary analysis, we found that the TEs enacted modifications of the written lesson plans, providing opportunities for PSTs to develop mathematical knowledge for teaching (MKT) that responded to their engagement in the class by modifying the setup or sequencing of activities. These modifications also created circumstances for new opportunities to learn additional types of knowledge for teaching. The TE was prompted by student questions and informal exchanges of knowledge with previous course instructors to modify the written lesson plans to suit her and her students' needs.

Opportunities to Learn within Enacted Modifications to Lessons

Modifications to the lessons occurred mostly within the setup of the tasks or the sequencing of the tasks but did not change the content of the tasks. In addition, they created opportunities to engage with mathematical content in deeper ways, promoting opportunities to develop the kinds of rich mathematical knowledge specific to teaching.

The following example illustrates a typical way the TE made modifications while teaching. In the second lesson plan, the instructor was directed to give groups of PSTs time to work on tasks and encourage them to represent their solutions in a variety of ways. In the enacted lesson, the TE prompted PSTs to work on each problem one at a time, chunking the setup of the task to support students in unpacking one problem at a time so they could more deeply consider the efficiency of various strategies. These modifications create opportunities for PSTs to engage in more practice problems with a potentially deeper mathematical understanding of the tasks.

Opportunities to Learn More than SMK. Another significant finding was that the TE's modifications to the lessons created opportunities for additional kinds of learning. In the written lessons, opportunities to learn were almost exclusively directed toward mathematical content knowledge, SMK. Within the enacted lesson, modifications created opportunities for PSTs to also learn norms for mathematical activity and practice thinking like an elementary teacher or student.

One way that this occurred was in a reframing of the homework review, which began each lesson. The TE selected a PST in each class to act as a teacher discussing her solution and asked the class to ask questions as if they were fourth graders. In this routine, the class engaged with their peer's ideas to think about ways to improve their representation of mathematical ideas prompted by the TE asking, "This is about ways to improve this solution, what would make it easier for a 4th grader to understand, what would make it more clear?" This modification to the written lesson creates an opportunity for PSTs to learn about connections between mathematical content and teaching and learning.

What Prompted Changes to Lesson Plans

Based on our data analyses, we found that changes occurring in the enacted lessons were prompted by students' unanticipated questions and responses and by communicating with former course instructors to resolve instructional challenges. As a result, instructional modifications included providing additional verbal and written resources, asking probing questions to the students to guide them to a correct answer, skipping planned tasks, and shuffling the sequence of tasks. We highlight two instances to illustrate these findings below.

Responding to students in the moment. The lesson plans included examples of students' responses and possible instructor's responses. However, on many occasions, students responded in ways that were not predicted in the lesson plans. In one such response, the students asked questions about setting expectations for writing answers in the exams multiple times. In one instance, when Reese [pseudonym] asked about the correct way to write the answer for an upcoming exam. Instead of providing one correct way of answering the problem, the TE explained four different options for answering the question. This moment was captured in the field notes of a lesson:

> *The TE said, "I was going to talk about this a little later but since you want me to talk about it now, let's talk about it". She explained the 4 ways of writing number sentences and then told the students that she expected the students to use one of these ways to write in the exam.*

Often, these instances created opportunities for the TE to work towards establishing norms with the students for mathematical practices. As in the example above, Reese's question prompted the TE to communicate expectations for demonstrating learning in the class.

Influences from modifications made by former instructors. The TE made an effort to reach out to the former course instructors, including doctoral students and department faculty. In these conversations, the TE shared her experiences instructing the course and discussed challenges that she was facing, seeking to understand if they faced similar challenges and how they navigated those challenges. As a result, she learned from former instructors' multiple approaches to addressing the challenges and implemented those approaches in her instruction. In one of the debriefing meetings, the TE gave reference to such modifications.

> *"Yeah, I was talking to Carmen [psuedonym] about this- that the students want me to tell them the norms for drawing and writing. Carmen told me that she had success when she asked the students to consider themselves to be fourth graders; this way they set up their own norms."*

The TE implemented this strategy of asking students to imagine that they were fourth graders or asking them to imagine that they were teaching fourth graders when answering her questions.

Discussion

We view this work as an extension of others who report on the implementation of a shared knowledge base for teacher educators (Hiebert, 2013; Hiebert & Morris, 2009; Morris & Hiebert, 2009) by considering how a TE implements shared lesson plans to create opportunities for PSTs to learn MKT. Understanding the use and modification of these materials highlights the role of the TE in interpreting and enacting the curriculum. In this case, we saw that the TE modified the lessons to respond to the needs of PSTs, which created opportunities to learn more than SMK. These changes are potential sources for the TE to contribute to the shared knowledge base as a newcomer to teacher education.

In the next phase of this case study, we intend to examine opportunities for learning that attend to additional types of knowledge, skills and dispositions that may emerge from the in-the-moment changes that TE makes during the enactment of lessons. Also, in addition to the units on percentages, we will collect data from the units that have been informally reported to be particularly difficult by the students during the instruction. Such inquiries will help researchers to understand how TEs interpret and make use of the shared knowledge base.

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HIGH SCHOOL FIELD EXPERIENCE PERSONNEL'S PERCEPTIONS ABOUT MATHEMATICS IDENTITY

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Field experience personnel, such as cooperating teachers (CTs) and university supervisors (USPs), play an important role in supporting mathematics pre-service teachers (PSTs) to learn about equitable teaching practices. We employed case study methodology to explore the perceptions of CTs and USPs about mathematics identity. A group of CTs and USPs participated in professional development during the Fall 2023 semester to learn about ways to develop students' mathematics identity. In this brief research report we share these CTs' and USPs' ideas about their own mathematics identity, their students' mathematics identity, and how these ideas influence their teaching practice. Our findings have implications for redesigning field experience in teacher education programs.

Keywords: Pre-service Teacher Education, Professional Development.

Objectives of the study

Field experience is an important component of teacher education (Butler & Cuenca, 2012). The various personnel involved in the field experience component of teacher education influence what student teachers learn when placed in actual K-12 classrooms. According to Anderson (2007), cooperating teachers (CTs) have a significant impact on pre-service teachers (PSTs). Moreover, Rozelle and Wilson (2012) reveal that PSTs' teaching practices and beliefs are strongly affected by cooperating teachers. University supervisors (USPs) also have a major impact on PSTs' thinking and their practice as they bridge theoretical learning with practical field experience (Cuenca et al., 2011). Thus, it is important to learn about the perceptions, practices, and beliefs of these field experience personnel to support PSTs' learning (Borko & Mayfield, 1995). In particular, when it comes to teaching mathematics PSTs about equitable teaching practices, it is important to develop a cohesive system of supports where PSTs get the same message from their courses and their field experience. Hence, collaborative work between university and school personnel is needed to prepare PSTs to teach diverse populations (Lee et al., 2010; Maher et al., 2022). Given the important role that field experience personnel play in supporting mathematics PSTs to incorporate these practices we wanted to learn about their own ideas about mathematics identity. In particular, our research question was: *What are field experience cooperating teachers' and university supervisors' perceptions about mathematics identity?*

Theoretical framework

Mathematics identity refers to "the dispositions and deeply held beliefs that students develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in powerful ways across the contexts of their lives." (Aguirre et al., 2013, p. 14). This identity can influence the beliefs of a student as a, "competent performer who is able to do mathematics or as the kind of person who is unable to do mathematics." (Aguirre et al., 2013, p.

14). Students' mathematical identities are deeply connected to their other identities making it important to welcome all of their selves into the classroom (Ruef, 2020). The five practices of equity-based teaching can guide the development of classroom environments that allow students to feel welcomed and help build their mathematical identities (Aguirre et al., 2013). These practices are: Going deep with mathematics; Leveraging multiple mathematical competencies; Affirming mathematics learners' identities; Challenging spaces of marginality; and Drawing on multiple resources of knowledge. Various researchers have cited the importance of mathematics PSTs' learning about equitable teaching practices (Gutiérrez, 2002; Mintos et al., 2019). If we are to teach PSTs about these practices it is important to help them experience these practices in action during their field experience (Moldavan & Gonzalez, 2023; Sandoval et al., 2020).

Methods

We used case study methodology to highlight the ways in which CTs and USPs conceptualize mathematics identity (Yin, 2009). Our unit of analysis in this exploratory case study is a member of our field experience team: CT or USP. We aim to share their own perceptions about mathematics identity in order to expand existing understanding about how field experience personnel may influence PSTs' teaching practice.

Setting

The study took place at a university in the Mid-Atlantic region of the United States. The university is situated in a rural community. As part of an extended professional development (PD) program, the first author invited high school mathematics field experience personnel to meet virtually. The group met four times during the Fall 2023 semester and engaged in discussions about equitable teaching practices in mathematics classrooms, reading vignettes of classroom scenarios, sharing classroom activities, and providing feedback on each other's teaching practices.

Participants

The participants were three cooperating teachers (high school mathematics) hosting PSTs from the university's teacher education program, and one university supervisor teaching the field experience course for future high school mathematics teachers. The university supervisor was a former high school mathematics teacher. All participants had more than 10 years of teaching experience. Karla was teaching Geometry, Computer mathematics, and AP Calculus; Ranita was teaching Algebra I & II; and Jake was teaching Algebra 1, Precalculus, and Data Science at the time of the study. Jake had also taught Geometry and Algebra II in past years. Lisa – the university supervisor – had taught Geometry, Computer math, and Algebra I when she was a high school teacher.

Data

All participants were interviewed at the beginning and end of the Fall 2023 semester. Semi-structured interviews were conducted, transcribed, and analyzed. In addition, the participants were asked to select at least one equity-based teaching practice to develop students' mathematics identity. They were also asked to share actionable steps that can be taken in their classrooms to support equitable mathematics teaching and develop their students' mathematics identity.

Analysis

Data were analyzed using open coding (Strauss & Corbin, 1998) to look for emergent themes about participants' perceptions of mathematics identity.

Results

In this section we provide accounts of each participant's perceptions about mathematics identity. Data analyses revealed that teachers' mathematics identity may influence their teaching practice and their teaching goals for their students. Teachers' mathematics identity may also influence their opportunities for learning when engaging in PD.

Karla

Karla is a cooperating teacher hosting our PSTs in her classrooms. Thinking about her own mathematics identity, Karla shared that she always identified as being good at mathematics. She liked seeing mathematics as a language and loved the precision mathematics provided to communicate ideas and concepts. Mathematics always made sense to her and was "straightforward to understand." In addition, she enjoyed teaching mathematics to peers. She described her students' mathematics identity as procedure followers who want to memorize algorithms and follow them rather than think through mathematics problems. Karla's goal is to provide students with mathematics problems that foster their problem-solving mindset as she wants them to become "problem solvers and critical thinkers." She mentioned that "giving them (students) a lot of examples of problems that do not have a straightforward algorithm" is needed to help students develop their mathematics identity. She selected going deep with mathematics as her preferred equity-based teaching practice. We see an alignment between Karla's own mathematics identity, her teaching practice and how she resolves challenges in her practice to help students become successful at doing mathematics.

Jake

Jake, a cooperating teacher, always liked to solve real word problems using mathematics and found numbers comforting. When talking about his own mathematics identity he said, "Well, my mathematics identity goes back to when I was in elementary school, I think about third grade when I was required to learn how to add, subtract, multiply and divide fractions. And so I found success in that, I found comfort in that!" In addition to finding numbers comforting, Jake also saw himself as a problem solver. He liked to play problem solving games and explained, "One of my more recent musings about mathematics is that all mathematicians are Gamesman. And I love games. And I love creating parameters and saying, Now, what can I do with that?" His goal is to help students change the narrative about mathematics, and to help them be successful. Jake did not select an equity-based mathematics practice but based on his conversations he seems to be leaning towards affirming mathematics learners identities and challenging spaces of marginality. We see an alignment between Jake's own mathematics identity, and his teaching practice. Jake wishes for his students to see the world through a lens of mathematics and wants them to have discussions about real world problems.

Lisa

Lisa is a university supervisor in the teacher education program. Up until last year she was a high school mathematics teacher and recently took on the role of a university supervisor. She described herself as a problem solver because she likes to think mathematically when encountering challenges. Lisa said that mathematics is "part of who I am, my identity." She always enjoyed teaching mathematics even when in school and liked to help her friends learn math. In contrast she feels that her students (high school) hate math, they don't have any interest in doing mathematics and don't understand why they have to do it. Her teaching focuses on motivating students and helping them reach aha moments. Her goals to develop students' mathematics identity include having students feel good about themselves when doing math. She selected, affirming mathematics learners' identities as her favored equity-based teaching practice. In her current role as a university supervisor, Lisa encourages her PSTs to work with students who struggle with mathematics. She uses her own experience to guide her PSTs in supporting their students.

Ranita

Ranita a cooperating teacher, recalled that she found mathematics to be challenging and did not feel like she was good at math. She shared, "I do math every day but I still really feel like deep down, I was never good at math." Her challenges learning mathematics earlier on in life allow her to empathize with her students when they say they don't like math. She shared that her students lack prerequisite mathematics knowledge and her goal is to develop students' mathematics identity by helping students feel confident and to help them experience success. Based on her own experiences learning mathematics, Ranita believes that helping students feel confident in mathematics is crucial. This motivates her to focus on supporting students to experience success in mathematics exams. She gives them chances to redo their assignments and provides scaffolded practice exercises because she believes that passing mathematics exams will allow students to experience success and develop a positive mathematics identity. Ranita's teaching practice is based on empathy for her students. She believes in building relationships with her students and learning about them. She selected leveraging multiple mathematical competencies as her preferred equity-based teaching practice.

Discussion and conclusions

There were similarities and differences between the four cases. In terms of similarities, all participants except for Karla shared that their students had a strong dislike for mathematics, that it was important to develop relationships with students and to help them experience success. For all the cases, the participants' interest, perception, and background influenced what they learned from the PD meetings. All the participants were able to learn from each other about strategies, tasks, and norms that worked for their students. They were all similar in their goal to help their students be successful, but their teaching practice, understanding of student success, and perception of their role as a teacher differed from each other.

Our findings have implications for PSTs' field experience. CTs and USPs guide PSTs as they try to connect their coursework-related learning to the teaching practice experienced in actual field experience classrooms. PSTs may learn new pedagogical ideas in their methods courses but it is important that these ideas are reinforced and modeled during their field experience (Matsko et al., 2020). For instance, if we want our PSTs to develop equitable teaching practices that support the development of students' mathematics identity, we must learn about pedagogical beliefs held by CTs and USPs. We noticed that all participants were deeply interested in supporting their students' learning; however, they had different ideas about what it meant to develop their students' mathematics identity. Some believed that experiencing success on state tests would do the trick, while others believed that being able to solve real life problems using mathematics might help their students develop a positive mathematics identity. Work is needed to support field experience personnel in their development of equity-based practices. In addition, continued collaboration is needed between MTEs, CTs, USPs, as well as PSTs.

It is through a cohesive network of support that our PSTs can be successful in developing equitable teaching practices. Field experience personnel can influence PSTs' sense of preparedness as teachers (Ambrosetti & Dekkers, 2010; Hamman et al., 2006). MTEs, CTs and USPs must collaborate, to develop tools and procedures in order to become effective mentors for PSTs. This collaboration can help align PSTs coursework and field related experiences. In particular, such an alignment is needed to support PSTs' development of equitable teaching practices that can support the development of students' mathematics identities.

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IN-N-OUT OF PRODUCTIVE STRUGGLE

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Keywords: Instructional Activities and Practices, Preservice Teacher Education, Teacher Educators, Teacher Beliefs

Productive struggle arises when students persevere in problem-solving, applying their mathematical knowledge in new ways (Hiebert & Grouws, 2007). While teacher educators recognize the importance of productive struggle for mathematics teaching and learning (NCTM, 2014), pre-service teachers (PSTs) may initially have negative perceptions of productive struggle (Warshauer et al., 2021) or try to alleviate struggle for their students (Anhalt et al., 2006; de Araujo et al., 2021; Nicol & Crespo, 2006), rather than seek to create opportunities for productive struggle. PSTs' desire to prevent students from struggling may be rooted in their own negative experiences as mathematics learners (Gellert, 2000).

In this ongoing study, I investigate how direct instruction of the concept of productive struggle, the promotion of productive struggle through rigorous mathematical tasks, and reflections focused on productive struggle, impacts elementary PSTs' understanding of productive struggle and their perceptions of themselves as doers and teachers of mathematics.

Participants were 66 elementary education undergraduate PSTs enrolled in a 14-week course on Algebraic Thinking at a Hispanic-serving four-year liberal arts college in the Southeastern United States in Spring 2022 (21 PSTs) and Spring 2023 (45 PSTs). PSTs regularly engaged in high cognitive demand math tasks. I documented students' problem-solving strategies and behaviors associated with struggle (SanGiovanni et al., 2020; Warshauer, 2015). After completing each task, PSTs wrote reflections on their experiences of productive struggle, using sentence stems (SanGiovanni et al., 2020, p. 151).

This poster reports on PSTs' experiences with a specific math task, the In-N-Out Burger task (Kaplinsky, 2013). End of course reflections asked PSTs to identify: which of the tasks was their favorite, and why; and which task did they find most challenging, and why. 26 PSTs (39%) said the In-N-Out Burger task was their favorite, while 17 PSTs (26%) said it was the most challenging. Compared to the other tasks, this task stands out because of the high percentage of PSTs who both favored the task and found it challenging. For the other tasks, there was an overwhelming majority for either favorite or challenging.

Preliminary findings suggest that some PSTs valued the freedom of the open-ended nature of the In-N-Out Burger task, while others were frustrated by the ambiguity. PSTs enjoyed the In-N-Out Burger task because it was an interesting, real world context; was enjoyable to solve; was challenging; had multiple solution paths; and fostered collaboration and discussion. PSTs who found the task most challenging expressed frustration, confusion, and difficulty getting started. One concern for classroom practice is that PSTs who are uncomfortable with ambiguity may avoid providing similar experiences for their future students. Additional analysis will explore how PSTs' end of course reflections connect to their initial reflection on the task. I will also present the solution paths students pursued, and types of struggle documented in my field notes, seeking connections between observed behaviors and PSTs' perceptions of the task, and their self-expressed experiences with productive struggle.

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BEGINNING TEACHER'S TRAJECTORY OF IDENTITY FORMATION IN THE FIGURED WORLDS OF REFORM AND TRADITIONAL INSTRUCTION

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Transitioning from a teacher education program to autonomous teaching is a complex process, fraught with challenges. This transition involves developing identities and teaching practices that allow novice teachers to reconcile the reformed teaching world of their teacher preparation program with the more traditional world of school teaching. In this paper, we follow the identity formation of one beginning teacher, Olive, by examining her narratives about her pedagogical actions as she transitions from being a pre-service teacher (PST) to being an intern (INT) to becoming a new teacher (NT). As PST, Olive's narratives about her current and desired actions aligned with reform actions; as INT, a gap opened between her current traditional actions and desired reform actions; and as NT, the gap narrowed as she modified her desired narratives to more traditional ones. We discuss our findings and their scientific significance.

Keywords: Novice Teachers, Teacher Learning, Identity, Reform Teaching

Introduction

University teacher preparation programs aim to prepare future mathematics teachers to enact ambitious pedagogical practices that align with the NCTM's vision of effective teaching and the Common Core Standards for Mathematical Practices (AMTE, 2017). As beginning teachers transition from university to school teaching, they need to reconcile between the world of ambitious (hereafter, *reform*) teaching of their teacher preparation program and the world of school teaching characterized mostly by traditional teaching practices (Jacobs, et al., 2006). The reconciliation between the two worlds can be accomplished by integrating reform practices, to varied extents, into the more traditional school world (Thompson et al., 2013). Indeed, some beginning teachers tend to lean toward traditional practices (Gainsburg, 2012), while others hold on to their reform teaching practices (Conner & Marchant, 2022; Smagorinsky et al., 2004).

The extent to which beginning teachers adopt and integrate reform practices has been linked to their emerging teacher identities and more specifically, to "the kind of teachers selves they have developed and seek to create"(Horn et al., 2008, p. 63). We can learn about identity by considering teachers' narratives about their general current pedagogical actions and their desired ones (Heyd-Metzuyanim, 2019). Moreover, it is important to examine the beginning teachers' identity formation over time and across settings. However, such longitudinal studies are rare, and not enough is known about how the processes of the reconciliation of the two worlds unfold.

In this paper, we analyze the case of one beginning teacher Olive (pseudonym), whom we followed for four years: as a pre-service secondary teacher (PST), as an intern (INT), and as a novice teacher (NT). Olive was chosen because she represents a case of a highly successful beginning teacher, in terms of her mathematical knowledge, pedagogical creativity, and productive dispositions, aligned with reform teaching (as evidenced by her undergraduate coursework). This meant that, relative to other PSTs, Olive had a good starting point in terms of integrating reform practices into the school world. We explore Olive's trajectory of teacher identity formation, building on the theoretical notions of "figured worlds" and pedagogical narratives, which we describe below.

Theoretical Framework

Drawing on Holland et al. (1998) and other scholars (Horn et al., 2008; Ma & Singer-Gabella, 2011), we view the traditional mathematics instruction and the reform teaching, not only as two pedagogical approaches but as two "figured worlds". A figured world is a "socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain *acts*, and particular *outcomes* are valued over others" (Holland et al., 1998, p. 52). To conceptualize the reform and traditional figured worlds, and specifically, their valued actions and outcomes, we drew on the National Council of Teachers of Mathematics' *Principles to Actions* (2014), a central document in the discourse on reform mathematics instruction. Table 1 shows our resulted conceptualization of the two worlds.

Table 1: Conceptualization of the Reform and Traditional Figured Worlds

	Figured World of Reform Teaching	Figured World of Traditional Teaching
Valued pedagogical actions	<u>Examples:</u> providing students with opportunities to explore and problem-solve; supporting students without eliminating their challenge; encouraging them to use reasoning and proving when justifying mathematical claims.	<u>Examples:</u> posing tasks which students are expected to solve using a specific memorized procedure; guiding students step by step through problem-solving; encouraging students to give short answers and respond to teacher only.
Valued pedagogical outcomes	<u>Examples:</u> collaborative explorations, open and reasoned discussions, productive struggle, and student authority.	<u>Examples:</u> memorization, correctness of answers, procedural knowledge, and teacher authority.

To be able to investigate processes of identity formation during the reconciliation of the reform and traditional figured worlds, we examined Olive's narratives about her pedagogical actions. Hence, our claims concern teacher's *narratives* about pedagogical actions, rather than the actual classroom practice We conceptualized four types of narratives a teacher produces about their actions, as shown in Table 2.

Table 2: Conceptualization of Narratives about a Person's Actions

	Narratives about *Specific* actions	Narratives about *General* actions
Desired actions	<u>Example:</u> In this specific lesson, I wish I had encouraged more peer discussions.	<u>Example:</u> I wish to have more peer discussions in my class.
Current actions	<u>Example:</u> In this specific lesson, I encouraged students to talk to their peers.	<u>Example</u>: I always encourage peer discussions in my class.

By distinguishing between narratives about *Specific* actions (first column) versus ones about *General* actions (second column), we drew on a distinction made by Heyd-Metzuyanim and Sfard (2012) between a person's communication about their *specific* performance versus their routine or *general* performance. We see these two types of communication as essential to exploring the formation of *identity*, defined as "narratives about individuals that are reifying, endorsable and significant" (Sfard & Prusak, 2005, p. 16). By distinguishing between narratives about *Desired* actions (first row) versus ones about *Current* actions (second row), we drew on Sfard and Prusak's (2005) distinctions between various types of identity narratives (i.e., actual and designated). Building on their conceptualizations, we distinguish between narratives teachers author about their *current* pedagogical actions, and those authored about their desired pedagogical actions. This focus on personal narratives allows us to adopt the teacher's perspective on her pedagogical practice while we tell the story of her identity formation. Based on our four-fold conceptualization, we ask the following research question: *What was the trajectory of Olive's (specific and general) narratives about her current and desired pedagogical actions in relation to their alignment with traditional and reform pedagogical actions?*

Methods

The data on Olive's trajectory as a beginning teacher came from multiple sources, collated over four years. The PST-period data comprised lesson plans, video recordings and written reflections for the four lessons Olive taught to small groups of high-school students, as part of the capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2023). The internship data were collected for two lessons Olive taught in her cooperating teacher's (CT's) classroom. Data sources included video recordings of the observed lessons and debriefing interviews of each lesson. As an NT, Olive was three times observed and interviewed after each lesson; and once, in lieu of an observation, we conducted an extended interview in which Olive shared an activity she enacted in her class and a sample of student work. Supplementary data included Olive's contributions to bi-monthly meetings of the professional learning community (PLC) of all new teachers participating in this study. The video recordings of all interviews and PLC meetings were transcribed for analysis.

To analyze the data, we first identified in the transcripts instances of Olive's narratives about her pedagogical actions. Each narrative was coded as either *specific* to that lesson or describing Olive's teaching in *general*. Further, the narratives were coded as either describing her *current* teaching or what she considered her *desired* way of teaching. This created the four categories conceptualized in Table 2: *current specific, current general, desired specific* and *desired general*. Next, the narratives were coded as aligned with either *reform* or *traditional* teaching practices. The coding scheme was based on content analysis of NCTM's (2014) *Principles to Action*, in which we generated a list of teaching valued actions consistent with reform or with traditional pedagogy (as exemplified in Table 1). Finally, we created a profile of Olive's narratives about her pedagogical actions *in each lesson*. Figure 1 shows one such profile extracted from the debrief interview of Olive's first lesson as NT. The actions aligned with reform practices are shaded in blue and the traditional ones are shaded in yellow.

	Specific	General
Desired	Ensure students leave with deep math knowledge Give students more time to work independently	No narratives about desired general actions
Current	Encourage students to take responsibility for their learning Ask students to provide justifications for their claims Provide activity in which students choose between strategies Let students do the mathematical talking	Encourage students to think independently Provide students opportunities to author their own tasks Teach students in a lecture style Provide activities for students to practice procedures

Figure 1: Profile of Olive's Narratives about her Actions in her First Lesson as NT

Examining these lesson profiles, we tracked changes in Olive's narratives about her pedagogical actions across time points and settings: university (PST), internship (INT), and autonomous teaching (NT). We describe them below.

Findings

PST: Current and Desired Actions Align with Reform Teaching

As a PST, all of Olive's narratives about her current pedagogical actions were specific to the four lessons she taught during the capstone course, and there were no narratives about her current general actions (as she was not yet teaching her own classroom). She did, however, author narratives about her general desired pedagogical actions. All her narratives were coherent with each other and aligned with the reform teaching practices, specifically those related to reasoning-and-proving (Stylianides, 2008). This alignment was evident in all the reflections Olive wrote after her four lessons. For example, in her second lesson, Olive incorporated reasoning-and-proving actions with the mathematical topic of congruent triangles and special segments in a triangle, while introducing students to conditional statements. In her reflection, she wrote:

> Together [with the students] we defined a conditional statement and discussed how they occur in a variety of settings. I also asked for students to provide their own examples of both if/then conditional statements and non-if/then conditional statements, identify hypothesis "P" and conclusions "Q" and determine their truth value, this includes some proofs and some counterexamples.

Olive's narratives about her pedagogical actions in this lesson (current-specific) included reform reasoning-and-proving actions such as providing students with opportunities to author their "own examples"; validate mathematical claims ("determine their truth value"); construct proofs and use counterexamples to refute arguments.

Similarly, Olive's narratives about her general desired pedagogical actions were aligned with reform teaching practices. For example, in her forth reflection, she wrote:

> Indirect reasoning and proving was so much fun to integrate into a lesson, I had never realized before that putting indirect reasoning into any math concept could be relatively easy for a teacher and *absolutely* accessible for students [emphasis in the original]. [...] I should certainly be able to find ways to incorporate it into different types of lessons.

Here, Olive shared her excitement ("so much fun") about the relative ease of integrating indirect reasoning into this specific lesson as well as future lessons ("I should certainly be able to find ways to incorporate it"). She talked about the joy of engaging students with indirect reasoning "in any math concept" and "into different types of lessons." Thus, her desired actions in this specific lesson, and general desired ones (in future lessons) were coherent and aligned with reform actions. A similar pattern was observed in all of Olive's four reflections as a PST.

INT- First Debrief: Traditional Current Actions vs. Reform Desired Actions

As an INT, Olive taught in the classroom of Julia (pseudonym), her cooperating teacher (CT). In her debrief interview, following her first lesson, there were still no narratives about her current general way of teaching, only ones specific to the lesson. These narratives, in contrast to her narratives as a PST, were completely aligned with traditional teaching practices, prevalent in her CT's classroom. This was apparent in Olive's debrief interview when she described her goals for the lesson as follows:

> I would feel accomplished if they [the students] understood how to combine like terms on two sides of the equation, moving chunks with the variable […] I knew that having the variable on both sides would freak them out because they'd only done it with a variable on one side. So that was probably the biggest hurdle to get over today, combining the variable terms.

Olive's narratives in this excerpt (current-specific) were aligned with traditional practices such as making sure that students know how to carry out mathematical procedures ("the biggest hurdle to get over today, combining the variable terms"); and ensuring that students are not too frustrated or confused ("I knew that having the variable on both sides would freak them out"). However, as the debrief progressed, Olive authored narratives about wishing she had taken pedagogical actions that allow students to be more explorative in their learning, saying:

> I'd love to have sometimes an equation up and they'll [the students] suggest something [like] "add 24 to both sides," and it won't make sense, but I wanna just go along with what they're saying […]. [As if saying] "let's just play with the rules for the day" […] and show them why it doesn't make sense. […] They are stuck to doing only moving these [terms], but if you can conceptually understand that we have a scale, and as long as you're doing both things to both sides [of the equation], the answer is going to be the same at the end of the day.

In this quote, Olive contrasted her traditional pedagogical actions in this lesson, with desired ("I'd love to …") actions aligned with reform pedagogical practices of encouraging students to explore mathematical rules and use their own methods for solving problems ("let's just play their rules"), and of proving them with opportunities to establish a strong conceptual foundation ("if you can conceptually understand…"). Thus, as Olive transitioned from PST to INT, a gap opened between her narratives about her current traditional actions and desired reform ones.

INT- Second Debrief: Reform/Traditional Current Actions vs. Reform Desired Actions

In the debrief following Olive's second observed lesson, the gap between her narratives about current and desired pedagogical actions narrowed as Olive's narratives about her current actions became more aligned with reform practices, as the next excerpt shows:

> I had decided to do that little, what I called an exploration in the beginning [of the lesson]. […] I kind of pushed to do that just because in the original lesson plan that Julia had written years prior, the idea of flipping the inequality side is not really explored at all […] I wanted them to see it for themselves and to understand, using numbers, why that was the case.

Olive described in this quote how she took the liberty to modify her CT's lesson plan by introducing in it a short "exploration" activity that would help students make sense of the rule for changing the sign of inequality when multiplied or divided by a negative number. This activity aligned with reform practices such as providing students with opportunities to explore rules and to make sense of and justify mathematical claims ("I wanted them to see it for themselves and to understand, using numbers, why that was the case"). Although Olive's narratives about her current pedagogical actions in this excerpt aligned with reform practices, in the rest of the lesson, she followed her CT's lesson plan that was rooted in traditional practices. Olive shared that she would have preferred to continue the exploration activity for the entire lesson, saying: "I wish I could have given a whole 40 minutes to that, you know, but I'm so glad that it got worked in at all". She added: "I would've liked to take some of the conjectures that they said that I didn't agree with and show them why I didn't agree with them." Thus, Olive's narratives about her current specific actions were aligned with both reform and traditional teaching practices, while her desired ones remained purely aligned with reform practices.

NT: Reform/Traditional Current and Desired Actions

As NT Olive was faced with the responsibility of day-to-day teaching, but also was free to experiment with reform teaching on her own. The first lesson we observed was a group activity where students explored parabolas as projectiles in the Angry Birds ® game. In the debrief interview Olive shared her excitement about this activity by saying:

> I'm really enjoying doing this project because I have been doing a lot of boring, I feel like, lecture-style things. And so, this is like an opportunity for me to stop talking, which is really wonderful. […] It is incredible how they [students] can focus, […] and kind of crank stuff out pretty quickly, and without a ton of wrestling from me, which is really nice.

Here Olive described her current general pedagogical actions ("I have been doing") as traditional ("boring lecture-style") contrasting them with reform-aligned current actions specific to this lesson, including giving students a rich mathematical task and supporting them in taking responsibility for their learning ("without a ton of wrestling from me"). However, when the interviewer inquired about incorporating more conceptually rich prompts, Olive replied:

> I guess my hesitation […] I'm playing devil's advocate in the situation, if I went and tried to go off on a conceptual tangent with each kid, I think that they would tune out immediately […] with so many kids […] to try to circulate that room and have a deep conceptual conversation with each of those 28, I don't even think I'd have time in the block to do that.

In this quote, Olive questioned whether it is realistic to pursue "deep conceptual conversations" "with so many kids." While not rejecting the idea, she *implicitly* positioned this kind of reform action as generally non-desirable. Thus, in her first debrief as NT, Olive's narratives about her *desired* actions began to move away from being purely aligned with reform practices.

In the following interviews, Olive's narratives about her current pedagogical actions presented a mixture of reform and traditional actions. For example, in her second NT interview, Olive described a lesson on operations with radicals in the following way:

> I liked that some of it [the lesson] felt exploratory, but then some of it definitely felt like a very telling way of teaching, like just kind of giving them the information as opposed to letting them figure it out".

Here, Olive communicated narratives about her current specific actions as being both reform-oriented ("I liked that some of it felt exploratory") and traditional ("telling way of teaching", "giving them the information").

As time went by, Olive's narratives about her current pedagogical actions, both specific to a certain lesson and general, continued to present a mixture of reform and traditional actions. Olive explained: "I try to start each unit with like a nice little discovery activity of some kind." She talked excitedly about one activity where students discovered the value of Pi, saying: "I didn't want to give too much away. I was trying to teeter a line of 'figure it out yourself." She shared her joy when students "have the best conversations, and they argue, and they don't want my input." These narratives were all aligned with reform practices. However, Olive admitted that these types of reform activities are infrequent among more traditional ones. She said: "I feel like as the unit goes on, it becomes less exploration and more like, here's the content, get it in your head." Thus, Olive's narratives about her current general pedagogical actions aligned also with traditional practice.

As captured already in her first debrief as NT, Olive's narratives about *desired* pedagogical actions changed as well toward more traditional practices. Notably, there was still some gap between current and desired narratives. On the one hand, Olive still strived to enact reform practices, saying "In an ideal world, I would've absolutely loved to do that activity [Pi-exploration] with every section." But on the other hand, reform teaching seemed to be more of a hypothetical ideal for Olive. The next quote illustrates this duality:

> I definitely don't wanna be a person that's lecturing every day, but I also don't necessarily have some super fun exploration planned every day either. There's gotta be like a healthy balance of those two things.

Olive did not want to revert to traditional teaching practices; however, her questioning of the feasibility of enacting reform actions day in and day out became more explicit and upfront. In one of the later interviews, Olive reflected on her trajectory from the university to classroom teaching:

> [In] undergrad and even as an intern [I] made some lesson plans that were just absolutely ridiculous in terms of what I expected the students to understand at a rate or pace that was absolutely unrealistic for the children […] I've definitely become more, I like to consider it, realistic […] sometimes I feel bad about it and sometimes I feel guilty about it. […] I try to maintain high expectations, but I think I, I'm a little bit more realistic.

In this excerpt, Olive framed some of her ambitious, reform-aligned lessons she developed as PST and INT as "ridiculous" and "unrealistic." She positioned her current mixture of reform and traditional actions as "realistic," although admitting she felt "sometimes bad" and "guilty" about this change. Thus, Olive in her second year as a new teacher identified herself as a realistic teacher who strives for a "healthy balance" between the traditional school world and the teacher-education world of reform pedagogy.

Summary and Discussion

Figure 2 summarizes the trajectory of Olive's narratives about her specific/general, current and desired pedagogical actions as aligned with traditional or reform pedagogical practices.

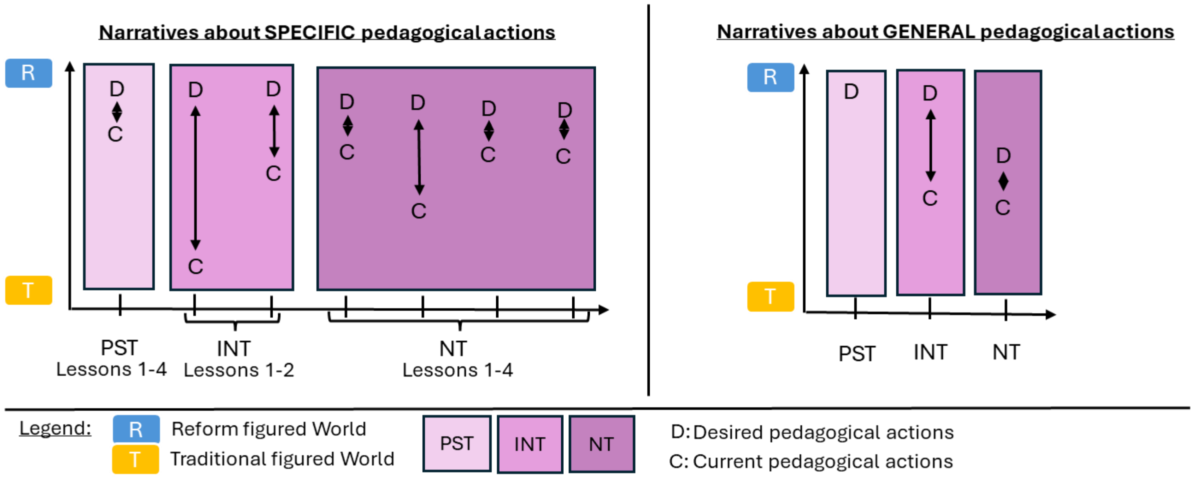

Figure 2: Trajectory of Olive's Narratives about her Pedagogical Actions

We present *specific* and *general* narratives separately, side by side. The vertical axes represent a non-quantified continuum between reform and traditional actions. The position on the continuum is not absolute, but represents general discursive tendencies observed in the data (cf., Truxaw and DeFranco' (2008) Sequence Maps). The narratives about *general* pedagogical actions (on the right) are represented by a single instance per period (PST, INT, NT) based on aggregated data. Also, narratives about *specific* pedagogical actions for the PST stage are collapsed across four lessons. However, each observed lesson as INT or NT is represented separately, to provide greater detail about Olive's narratives during this critical period of her teaching career. The icons "C" and "D" stand for current and desired actions; close placement represents coherence of narratives, while distanced placement represents a gap between current and desired narratives.

As a PST, Olive's narratives about *desired* actions, both specific and general, were aligned with reform practices. The same was true for narratives about her *current* actions in the four specific lessons she taught as PST. This reinforces our choice of Olive as a case of a beginning mathematics teacher, who was both well-prepared and eager to enact reform teaching practices.

Two critical processes followed Olive's promising starting point as a PST. The first one occurred during Olive's internship, when her narratives showed a gap between desired reform actions and current traditional ones (both specific, as seen in lesson 1 as INT, and general). This gap is consistent with previous studies suggesting that novice teachers tend to adopt the traditional teaching practices of their mentors (Bieda et al., 2014; Gainsburg, 2012). In her second lesson as well as the following lessons during her NT period, Olive tried to close the gap by aligning her current traditional pedagogical actions with her desired reform ones (captured in the Figure by the rise of "C" toward "D"). This finding aligns with Horn and colleagues (2008) who foregrounded the link between interns' reform desired images of good teaching and the modification of their traditional teacher identities.

The second process occurred during Olive's NT period. We did not identify a particular turning point in Olive's narratives, but rather a gradual process characterized by two trends. The first one is the fluctuation of narratives about *current* specific practices somewhere in the middle along traditional-reform continuum (captured by changes in the location of "C"). This trend points to Olive's multiple attempts to integrate reform practices in specific lessons revealing the dynamic changes in her narratives about specific actions. These changes are important as they underly the formation of Olive's more stabilized narratives about current general actions aligning with *both* traditional and reform actions. The second trend in Olive's narratives is the gradual shift in the *desired* actions toward more traditional practices (captured by the lowered position of "D" both in relation to specific and general actions). This shift points to Olive's attempts to close the gap between narratives about her current and desired actions by "lowering the bar" and modifying the desired actions to be more "realistic," traditional ones. This finding suggests that even an enthusiastic beginning teacher, like Olive, who is committed to reform practices, may need support in keeping the view of reform practices as desired (Ingersoll & Strong, 2011).

We believe that Olive's longitudinal case of a promising new teacher entering the world of traditional schooling, contributes to a better understanding of the underlying processes of identity formation during the reconciliation of the two worlds. As we continue to investigate trajectories of Olive and other beginning teachers in the larger study, we intend to include further lenses to gain a better understanding of identity formation as interwoven with social and cultural contexts.

Acknowledgments

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EARLY CAREER MATHEMATICS TEACHERS' REFLECTIONS ON THEIR PREPARATION FOR TEACHING

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As mathematics teacher educators, we strive to motivate and inspire our students to engage in *ambitious teaching* as described in Principles to Action (NCTM, 2014) and the Taking Action (NCTM, 2027) series published by the National Council of Teachers of Mathematics. We also address and stress the need for strong "foundations of pedagogical knowledge, effective and equitable mathematics teaching practices, and positive and productive dispositions toward teaching mathematics to support students' sense-making understanding, and reasoning." (AMTE, 2017, C 2, P 3)

Ambitious teaching contrasts sharply with the instruction models that most students have found in their mathematics classrooms. As middle and high school learners, they encountered, for the most part, a routine that included homework review, teacher lecture and demonstration followed by individual practice. This routine was further articulated in the *I do, we do, you do,* model of instruction. This disconnect continues in their college courses. (Nguyen & Munter, 2023)

It then becomes our challenge as teacher educators to overcome that model and help our students see themselves as facilitators of learning rather than engaging solely in the practice of direct instruction that they have been exposed to as students throughout much of their learning of mathematics. In their methods courses and in their practicums, our goal then becomes one to ensure that our pre-service teachers have the opportunity to fully engage with the Effective Teaching Practices in all ways. This has become our most important goal, and our students demonstrate their progress in comprehensive unit plan design and implementation of these practices in field experiences and student teaching assignments during their final year of their college program.

What then, do they take with them into their own classrooms from their methods courses and internships? What do they remember, and what do they put into practice in their own teaching? What support and mentoring have they experienced in their first years of teaching? What do they need and how can we help? What can we learn about re-designing our programs when we answer these questions?

This poster session will answer some of those questions and will include the design and results of an interview study with 15 early career middle and high school teachers. Each of these teacher-participants had completed a few months to 5 years of teaching experience at the time of their interview. All were former students in a methods class with the authors and were supervised by them the following semester in their student teaching. The poster will also highlight important aspects of the participants undergraduate preparation for teaching.

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MATH AUTOBIOGRAPHIES: THE POWER OF STORYTELLING IN MATHEMATICS EDUCATION

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This study explores preservice teachers (PSTs) autobiographical reflections on their experiences with math. Findings reveal distinct trajectories, including positive journeys characterized by enthusiasm and satisfaction, discouraged and disheartened paths marked by negative emotions, empowering shifts triggered by positive encounters, and disruptive shifts leading to negative perceptions due to challenging experiences. We discuss the factors that contribute to these trajectories including relationship with teachers and families and PSTs' perceptions of their own aptitude. The narratives point to a need for supporting positive early experiences for children and tailoring instruction for PSTs to attend to their personal math experiences.

Keywords: Preservice Teacher Education, Affect, Emotion, Beliefs, and Attitudes, Elementary School Education.

Early experiences with math play a crucial role in laying the groundwork for teachers' perceptions and connections with the subject. Autobiographies serve as a valuable tool for gauging someone's inclinations toward subjects and tracking changes in their attitudes over time (Ellsworth & Buss, 2000). Autobiographical storytelling, the practice of recounting personal experiences, holds a significant place in social sciences research (Miller, 2000). Studies exploring storytelling in math education have demonstrated that narratives offer insights into individuals' attitudes, beliefs, and identities regarding math that traditional survey instruments fail to capture (Ellsworth & Buss, 2000).

Perspective(s)

Autobiographies serve as a reflective tool, offering teachers a means to understand themselves both personally and professionally (Connelly & Clandinin, 1999). Crafting autobiographies helps students develop a conscious awareness of the connections between their present beliefs, identities, and emotions and their past experiences with math (Drake, 2006; Hauk, 2005; Machalow et al., 2022). Autobiographical narratives can also unveil teachers' orientations toward math (Machalow et al., 2022), as well as their preference for specific teaching methodologies (Ellsworth & Buss, 2000; Hauk, 2005). In this study, an autobiography denotes an individual's written account of their experiences learning math. We asked preservice teachers (PSTs) enrolled in an elementary math methods course to recall events that had positive, negative, or neutral effects on their interest in, attitudes toward, and emotional responses about math, along with their overall sense of competence. Our analysis focused on examining changes in their experiences throughout their multi-year engagement in math. We focused on answering the following research question, *how do PSTs' past experiences influence their interest, attitudes, emotional responses, and sense of competency related to mathematics?*

Methods

Fifteen PSTs enrolled in an elementary math methods course participated in this study. The course's main objective is to equip future educators with the necessary skills to effectively teach in student-centered ways. The activities were specifically designed to encourage PSTs to actively reflect on their own mathematical thinking, facilitating the development of instructional approaches and strategies with students. PSTs were tasked with writing autobiographical accounts of their experiences with math. Prompts were provided to guide the PSTs to explore their perceptions (e.g., How do you feel about math? Why do you think you feel this way? What factors contributed to this feeling? Describe how that person or event influenced your feelings); however, they were free to construct the narratives that best reflected their experiences.

Data Sources and Analysis

PSTs' autobiographies served as the data source. Employing an inductive analytical approach, we engaged in multiple readings of the narratives to identify the core ideas PSTs conveyed about their experiences. Then we looked across the narratives for common themes (trajectories, as explained in the next section) within their stories (Polkinghorne, 1995). We categorized the experiences PSTs described (label of each column) and the nature of the experience (label of the rows; Figure 1).

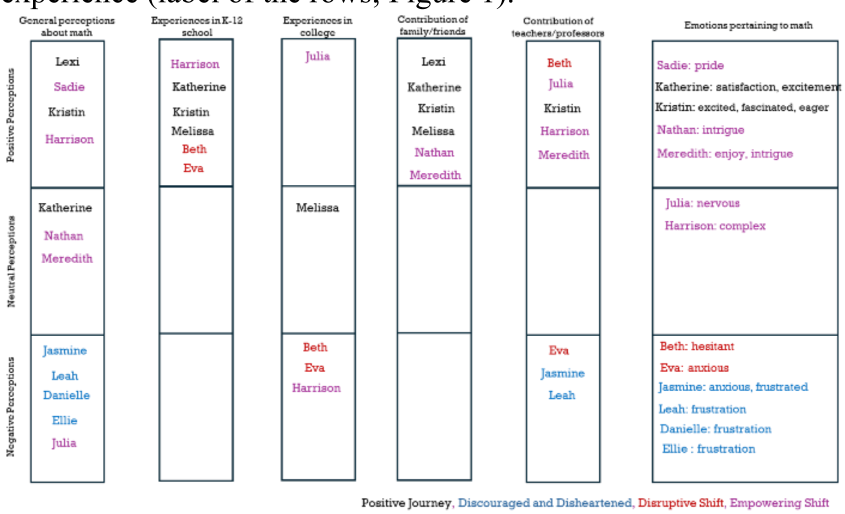

Figure 1: Categorizing PSTs' experiences across distinct areas and associated emotions

Then, we examined each PST's case individually to explore their trajectory more comprehensively. During this phase, our aim was to gain deeper insights into the main influences on the teachers' mathematical journeys over time. To achieve this, we devised a mapping framework that outlined the phases of each teacher's mathematical life and identified perceived influences. This mapping allowed us to track the transitions within mathematical narratives, including positive, neutral, and negative influences/experiences that prompted shifts in their ideas about math. Four distinct trajectories emerged from this analysis (Table 1).

Findings

Table 1 presents the categorization of the PSTs' trajectories accompanied by a description. Next, we describe the distinctive characteristics of each trajectory type.

Positive Journey

Four PSTs – Kristin, Lexie, Melissa, and Katherine - discussed their positive math journeys. Their positive perceptions of their mathematical abilities were based on consistently experiencing success and having empowering experiences in math. Lexie, for instance, described how she effortlessly grasped math concepts and excelled without much difficulty. She attributed this to her innate understanding and confidence in her abilities, stating, "Since I can remember, I've always felt naturally adept at math". Others recounted having exceptionally positive experiences with math during their K-12 schooling. Kristin, for example, expressed her fondness for math, labeling it as her "favorite subject" throughout middle and high school due to her consistent success and appreciation for its logical problem-solving nature. Additionally, they acknowledged the presence of consistent support from parents or teachers. Katherine credited her father for nurturing her curiosity and skills in math, while Kristin highlighted the influence of her 9th-grade algebra teacher, who encouraged diverse problem-solving methods.

Table 1: Types of Trajectories

Trajectory	Description
Positive Journey	Had no adversities in their mathematical journey.
Discouraged and Disheartened	Had predominantly negative early math experiences, interpreting them as indicative of innate mathematical incompetencies.
Empowering Shift	Had predominantly negative early math experiences; then, experienced a critical positive event engendering a positive attitude, belief in mathematical competence, and enjoyment in mathematical endeavors.
Disruptive Shift	Had predominantly positive early math experiences; then, experienced a critical negative event engendering a negative attitude, lack of belief in mathematical ability, and negative emotions about math.

Discouraged and Disheartened

Some PSTs including Jasmine, Leah, Danielle, and Ellie, perceived failure and/or consistently felt discouraged from math experiences, resulting in negative perceptions of math. For instance, despite Ellie achieving high grades in K-12 math, she felt inadequate relative to peers who excelled in mental math which she struggled with. Additionally, this group referenced negative experiences with their math teachers. Leah attributed part of her struggle to male teachers explaining the material in an unhelpful manner, shaping her negative perception of math. A particularly hurtful experience occurred when her high school teacher questioned her work ethic, which made her feel misunderstood and incapable. Consequently, her frustration for math was largely shaped by the teacher's attitude towards her struggles rather than the difficulty of the content itself. Similarly, Jasmine described a decline in her confidence during middle school, feeling overwhelmed as her peers seemingly effortlessly solved algebra equations while receiving little support from teachers. She felt frustrated and abandoned by teachers – "teachers certainly knew I was struggling, but they didn't do anything to help". They all expressed negative emotions, such as anxiety or frustration with no interest in doing any math.

Empowering Shift

Some PSTs - Julia, Sadie, Harrison, Meredith, and Nathan – described negative or neutral experiences until a critical event. A positive experience engendered optimism toward math. Julia, for instance, recounted struggling with math for many years resulting in feelings of shame and anxiety. However, encouragement and support from her middle school math teacher, Mr. S, helped develop confidence in her math abilities by appreciating her unique problem-solving approaches. Nathan described his experiences with math as neutral. A high school friend played a significant role in altering his perception, initially making math intimidating but also broadening his understanding of its potential, ultimately leading to a more positive outlook. These transformative moments instilled a belief in their potential for success and enjoyment in math. This group described a range of emotions related to math including Julia expressing nervousness, Sadie feeling pride, and Nathan and Meredith expressing feelings of intrigue.

Disruptive Shift

Two PSTs, Beth and Eva, described positive perceptions of math, followed by negative experiences that changed their perspective. Before taking pre-calculus in college, Beth considered herself skilled in math, however, struggles in the pre-calculus class altered her perspective, leaving her frustrated and repelled by math, contrasting her previous sense of contentment and competence in math classes. Similarly, Eva recalls being adept at math and enjoying it because it made sense to her. However, her negative feelings toward math arose during college because she felt less capable when it came to abstract concepts. This change in attitude coincided with an experience with a particular math professor who taught at a rapid pace without ensuring understanding before progressing. This encounter induced feelings of anxiety and nervousness towards math. At the time of data collection, both Beth and Eva shared predominantly negative feelings towards math – experiencing nervousness and anxiety.

Discussion

Across the narratives, PSTs described grappling with both challenges and successes in math. We noted that the main influential factors including supportive teachers, familial encouragement, and individual aptitude could have a positive or negative effect depending on the nature of the experience and the PST's interpretation of it. The influence of teachers across K-12 schooling emerges as a significant factor shaping participants' attitudes towards math (Kaur Bharaj et al., 2023), with positive experiences often leading to increased engagement and confidence, while negative experiences served to exacerbate feelings of self-doubt and disinterest. Similarly, as the subject became more complex, depending on the interpretation, PSTs expressed confidence and enjoyment if they made sense of the ideas, while others had feelings of frustration and anxiety as they encountered abstract concepts that they felt inadequate to understand. These feelings lingered and directly influenced PSTs' ideas about math as they were entering the math methods course. The data highlights the complex interplay among PSTs' interpretations of teacher-student math-related interactions, the nature of math and their math-related emotions and attitudes. It underscores the importance of fostering a supportive and inclusive learning environment that empowers all students to engage with and appreciate the beauty of math.

Significance of Study

The findings are significant in two distinct ways. First, they shed light on the experiences underlying attitudes and emotions PSTs have toward math. This deeper understanding of teachers' personal journeys with math underscores the importance of consistent positive learning experiences throughout formal schooling. Second, exploring PSTs' perceptions of math derived from their own math experiences informs our understanding of how these experiences shape how they enter math and math education spaces. With this understanding, teacher educators can better tailor PSTs' experiences in ways that engender a disposition toward math and teaching math that includes student-centered teaching and positive attitudes towards the subject. Ultimately, by acknowledging and valuing teachers' personal experiences with math, education stakeholders can work toward creating effective math instruction that meets the diverse needs of all learners.

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EXAMINING THE IMPACT OF INVOLVING UNDERGRADUATE PRE-SERVICE TEACHERS IN RESEARCH

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Keywords: Pre-Service Teacher Education

The primary role of a teacher education program is to provide experience for pre-service teachers (PSTs) that will support their professional practice in creating supportive and effective learning environments. Achieving this goal comes with challenges including, allocating sufficient time and resources to support PST's pedagogical and content knowledge (Ball, 1990) and addressing pre-service teachers' prior experiences as a student, a phenomenon aptly and often described as an apprenticeship of observation (Lortie, 1975). Mathematics methods classes have developed approaches to address these concerns using approaches such as lesson study (e.g. Leavy& Hourigan, 2016) and mediated field experience (e.g. Horn & Campbell, 2015). Strategies such as these intentionally engage students in the application of theoretical ideas and provide opportunities for PSTs to reflect on themselves as practitioners.

There is a need to be cautious in assuming that field experiences alone will support PSTs in fully integrating theory into practice. "Field experiences may expose student teachers to a limited repertoire of strategies and to a narrow and unrepresentative sample of students. Preservice teachers may easily come to believe that only the strategies they observe are appropriate, regardless of the students they may eventually teach" (Santagata, Zannoni, & Stigler, 2007, p.4). One approach that may aid PSTs in not adopting a myopic view of instructional practice is to involve PSTs in research. This approach has been utilized in teacher preparation programs for over 30 years in Finland. From their perspective "the aim is not to produce researchers, but rather to provide students with skills and knowledge to complete their own studies, observe their pupils, and analyze their thanking (Toom et. al., 2010, p. 333). To be most impactful the research conducted should be purposeful and meaningful to the PSTs (Zeivots, Buchanan, & Pressick-Kilborn, 2023). Providing opportunities for PSTs to engage in research may be one effective way to address concerns related to teacher knowledge of both content and pedagogy and may also help address pre-established beliefs of teaching that are rooted in past experiences.

The study presented here investigates the ways in which conducting research may be utilized as a means impacting teacher knowledge and beliefs. This study examined two groups (n_1=8; n_2=8) of PSTs in their senior year conducting research on the practicing teachers' perceptions on the role of homework. Homework was chosen as a topic as it was an area of interest and concern to this group of PSTs as they transition to in-service teachers. The PSTs in this study identified a research question, reviewed relevant literature, designed a study, gathered data, analyzed data, and constructed a poster presentation of their results. After completion of their study, PSTs were asked to reflect upon the way in which conducting the research study impacted their own views of homework and on their own views of themselves as teachers. Results from the two PSTs' studies as well as the study of the PSTs will be presented. Notable results from the PSTs' studies suggest a pattern where teachers surveyed believed "homework to be ineffective and did not contribute to student success." Results from study of PSTs indicated conducting research on homework reinforced their previously held beliefs. Results related to impacting PSTs' own self-image as teachers were less clear.

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ON NOTICING STUDENT MEASUREMENT THINKING: THE LEARNING JOURNEY OF ALINA

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Teacher noticing of students' mathematical thinking plays a pivotal role in reform-oriented instruction. Hence, it is crucial to support preservice teachers, who do not generally possess well-developed noticing skills, in learning to notice student thinking. However, this line of research is scarce in the domain of measurement, an important topic in mathematics curricula. In this paper, I report the learning path to notice student thinking about length measurement of a preservice teacher when given various learning opportunities to develop her noticing. Findings suggest that with the support of a framework on student measurement thinking, the preservice teacher's attention and interpretation become broader, deeper, and better aligned with research-based knowledge. However, her learning path does not consistently follow an upward trend.

Keywords: Preservice Teacher Education, Teacher Noticing, Measurement.

Teacher noticing of students' mathematical thinking is at the heart of high-quality mathematics instruction, which promotes adaptive and responsive teaching (Jacobs & Spangler, 2017). Previous research (e.g., Jacobs et al., 2010) has indicated an expert-novice difference in teacher noticing, with preservice teachers typically lacking well-developed noticing skills due to limited teaching experience (Sherin & van Es, 2005; Star & Strickland, 2008). Across the literature, there have been many efforts to support preservice teachers in learning to notice. However, this line of research is limited in measurement (Ergene & Bostan 2022), an important topic in mathematics curricula, especially at the elementary school level. To address this gap, I have conducted a research project to explore how preservice teachers learn to notice student measurement thinking over time when given learning opportunities specifically designed to support their noticing. This paper presents a finding from the project, focusing on the learning path to notice student thinking about length measurement of a preservice teacher, Alina.

Theoretical Framework

Across the literature, there have been various conceptualizations of teacher noticing. This paper adopts Jacobs et al.'s (2010) concept of professional noticing of student thinking, defined as a set of three interconnected skills: *attending to students' strategies details, interpreting students' understanding,* and *deciding how to respond on the basis of students' understanding*. For the scope of this paper, I focus on the first two skills, attending and interpreting, as they are the foundation for deciding skill. Attending refers to the extent to which teachers identify mathematically significant details in students' solutions. Interpreting refers to the extent to which teachers use the specific details in students' strategies to reason about their understanding in a way that is consistent with research on students' mathematical development.

In the definitions of these two skills, the phrases, "mathematically significant details in students' solutions" and "research on students' mathematical development", appear. What do they mean in the context of measurement? To answer this question, I synthesize relevant literature and present a framework on student measurement thinking called Big Ideas of Measurement (BIM)[3]. The framework comprises 16 big ideas that researchers identified as foundational for students to develop a deep and robust understanding of measurement across different attributes. Therefore, I believe that these big ideas are "mathematically significant details" that teachers should attend to and interpret so that they can better support their students' understanding of measurement. This framework was presented in detail in Bui (2023). In this paper, I focus on six big ideas that show up most often when students measure an attribute with non-standard units and measurement tools such as rulers (see Table 1).

Table 1: Some important big ideas of measurement

Big ideas	Description
Identifying the attribute	Knowing/understanding what is being measured.
Identical units	Same-sized units should be used to measure an attribute of an object.
Exhaustive measure	All of the object has been measured without gaps between units or overlapping units
Unit iteration	This includes making copies of units and arranging them, accumulating those units to obtain a measure, and eventually being able to reuse/copy a single unit
Partitioning unit	Units can be partitioned into fractional amounts smaller than one unit
Zero-point	Each instrument has conventional zero-point(s), but any point can serve as an unconventional zero-point on the instrument. To use an instrument with understanding and flexibility, students should understand what is counted and what the numbers on the instrument represent.

Methods

The study context is set within a content course for preservice teachers with a focus on measurement and geometry at a large public research university in Southern United States. The case study participant, Alina, is a Hispanic preservice teacher, majoring in Pre-K–6 bilingual education. Data about Alina's noticing was collected over 17 weeks of Fall 2023 when she had various learning opportunities to develop her noticing (see Table 2). To measure Alina's noticing, she was asked to engage with various artifacts of practice and share her noticing in written form. These artifacts include short videos or work samples of elementary students measuring the length of different objects with non-standard units and with a ruler.

Table 2: Data related to Alina's noticing of student thinking about length measurement

Time	Relevant learning opportunities	Data
Week 1	N/A	Pre-noticing assessment
Week 2	Class 03: BIM framework given and discussed	Exit ticket after class

[3] I borrowed the phrase *Big Ideas of Measurement* from Empson et al. (2006) who used it in course materials and interviews.

	Class 04: Watch 4 videos and discuss student thinking	Handout “Analyzing student thinking about length”
Week 3	Class 05: Watch 3 videos and discuss student thinking	Homework 1
Week 4	Class 08: Discuss preservice teachers’ responses to HW1: What counts as good explanation?	N/A
Week 5	N/A	Homework 2
Week 6	N/A	Exam 1
Week 7	Preservice teachers paired up and conducted the measurement interview with a fourth-grader	N/A
Week 16	N/A	Final exam
Week 17	N/A	Post-noticing assessment

Jacobs et al. (2010)’s scoring scheme and the BIM framework in Table 1 was used to analyze Alina’s attention and interpretation of the big ideas of measurement. I first considered if Alina paid attention to a big idea of measurement or not, and if yes, what the level of detail was. The level of detail can receive a score of 0, 1, or 2 where 0 means attending to very little or almost no relevant details, 1 means attending to some relevant details but may miss some important details, and 2 means attending to almost all important and relevant details to the big idea. Next, I considered if Alina interpreted a big idea or not, and if yes, what the level of evidence and level of alignment to research-based knowledge on big ideas of measurement were. Similarly, the level of evidence and level of alignment can receive a score of 0, 1, or 2 in the same manner as to level of details. For example, if Identical unit receives the scores 1 for *Interpret or not*, 2 for *Level of evidence*, and *1 for Level of alignment*, it indicates that Alina interpreted students’ understanding of measurement (e.g., understand or not), she gave almost all important and relevant evidence for her claim; however, her interpretation of students’ understanding of that big idea was not completely align to research-based knowledge. In addition, I noted if Alina explicitly used the technical terms (for example, Identical unit) in her noticing or not. For example, if she wrote “The student understands the big idea of identical unit”, a score of 1 will be assigned; if she wrote “The student knew that we have to use same-sized units when measuring”, a score of 0 will be assigned. Finally, I calculated the sums for each of the six columns in Table 3 (Attend or not, Level of detail, Interpret or not, Level of evidence, Level of alignment, and Use of technical terms). These sums will be referred to as scores for Breadth of attention, Depth of attention, Breadth of interpretation, Depth of interpretation, Alignment of interpretation, and Use of technical terms from BIM.

Table 3: Coding Alina’s attention and interpretation of big ideas of measurement

Big ideas	**Attend or not**	**Level of details**	**Interpret or not**	**Level of evidence**	**Level of alignment**	**Use of technical terms**
…	0=No, 1=Yes	0, 1, 2	0=No, 1=Yes	0, 1, 2	0, 1, 2	0=No, 1=Yes
…						
Sum						

Findings

Table 4 summarizes the scores for Alina's attention, interpretation, and use of technical terms from the BIM framework over time. Overall, with access to the BIM framework and more learning opportunities to unpack the meaning of the big ideas in the framework, Alina's noticing of students' understanding of the big ideas of measurement became broader, deeper, and better aligned with research-based knowledge on students' measurement thinking. She also used more technical terms from the BIM framework. However, the development of her noticing did not always follow an upward trend. For example, in homework 1 (week 3), Alina's noticing significantly improved compared to her performance on the class's handout and exit ticket in week 2. Two weeks later, when Alina worked on homework 2, the depth and alignment of her interpretation slightly decreased before continuing to develop in week 6.

Table 4: Alina's attention and interpretation of big ideas of measurement over time

Time	Data	Attention of BIM		Interpretation of BIM			Use of technical terms
		Breadth	*Depth*	*Breadth*	*Depth*	*Alignment*	
Week 1	Pre-noticing[4]	2.5	2	1	1	1.5	0
Week 2	Exit ticket	3	5	3	4	5	1
	Handout[5]	4.25	7.25	3.75	5.75	6	4
Week 3	Homework 1	6	9	6	9	10	6
Week 5	Homework 2	6	8	6	5	6	6
Week 6	Exam 1	7	12	6	8	8	6
Week 16	Final exam	6	9	6	9	9	6
Week 17	Post-noticing[2]	5.5	6	4.5	5	5	5

Let us look at two excerpts from Alina's analyses of student measurement thinking in the pre- and post-noticing assessments to see how her noticing changed over time.

> Helena was able to take two different objects or mediums and compare them in order to find Speedy's length… Helena has a general understanding of how to compare two different units. (Alina, pre-noticing assessment)
>
> Reid was able to identify the attribute... His question of "Can I line them up like this" helped me see that he attends to orientation and knows what he is measuring...He was able to produce a measurement with identical units for the most part, up until the end. Based on the video, I was able to see that his reasoning behind this was the fact that Reid wanted Speedy to fit exactly into all of the paperclips lined up… (Alina, Post-noticing assessment)

At the beginning of the semester, Alina's noticing was very general: her description of Helena's strategy was vague, and her interpretation did not point out any specific ideas of measurement that Helena understood. In contrast, in the post-noticing assessment, Alina provided many important details about Reid's strategy through verbal quotes and subtle actions,

[4] The average across 2 types of artifacts: video and work samples

[5] Average across 4 videos

and she interpreted these details to make sense of his understanding of the big ideas of measurement (e.g., identifying the attribute and identical units).

Closing Thoughts

This report shows that access to the BIM framework and opportunities to unpack the meaning of the big ideas from the framework supported Alina in noticing student measurement thinking more broadly and deeply. This finding suggests that BIM could be a potential tool to assist preservice teachers' professional noticing. Future research may explore how preservice teachers with diverse backgrounds learn to notice students' measurement thinking over time and what challenges they face during that journey. Findings from this line of research will provide mathematics educators with more insights into how to design courses in teacher education programs to support preservice teachers' professional noticing of student measurement thinking.

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UTILIZING SCRIPTING TO EXAMINE DIFFERENT TYPES OF MATHEMATICS QUESTIONS OF PRE-SERVICE TEACHERS

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Questioning is one core practice mathematics teacher educators (MTEs) create instructional activities to develop. An instructional activity creates visibility around questioning is that of scripting dialog that takes place within a mathematics classroom. Through a qualitative research study, the researcher examined PSTs' mathematics questioning types through two iterations of script writing from one problem situation. Findings illustrated that PSTs commonly used two questioning types: gathering information and probing student thinking style questions. Implications for MTE's instructional activities are summarized and discussed.

Keywords: Preservice Teacher Education, Instructional Activities & Practices, Classroom Discourse

Introduction

Questioning has long been considered a core practice in mathematics education literature (e.g. Vacc, 1993). Preservice elementary teacher (PSTs) approaches to questioning during interviews and discussions have included categorization schemes for types of questions posed (Parks, 2010). Such studies identify how PSTs question in various contexts, yet how PSTs develop questioning through methods instruction is still an open question. One way mathematics teacher educators (MTEs) support questioning development is through approximations of practice (Ball & Forzani, 2009, Grossman & MacDonald 2008; Grossman et al., 2009; Reid, 2011) such as scripting. Herbst (2018) describes scripting as the creation of a dialogue as if for a play. Scripting is a form of rehearsal that requires PST to consider and compose dialogue between themselves and their students. The variety of questions that PSTs elicit during a mathematics lesson is low (Weiland et al., 2014). Engaging PSTs in script writing allows for an opportunity to imagine dialogue between a teacher and students in such a way that MTEs can provide guidance around different questioning types.

Different types of questions such as implicit and explicit questions (Parks, 2010) can be challenging to create while engaging with learners and even in planning lessons. Moreover, when PSTs are developing in their questioning techniques, often times they miss opportunities to foster thinking of their students and their questions are more leading in nature (Weiland et al., 2014). PSTs, with little teaching experience, draw from their interactions as students with their teachers as they plan mathematics lessons. In this paper, I characterize PSTs' questions in a scripting context that provided PSTs with opportunities to draw from their lived interactions while also developing their understandings around pedagogical practices during their methods coursework (Liston, Whitcomb, & Borko, 2006). While there is literature around analyzing questioning of PSTs during interactions in lessons that that they have scripted (Zazkis, Liljedahl, & Sinclair, 2009;) and studies of those who seek to utilize scripting to analyze dialogic teacher moves centered on mathematical tasks (Crespo, Oslund, & Parks, 2011; Campbell & Baldinger, 2022), there are limited findings on utilizing scripting by MTEs to analyze and develop questioning practices of PSTs. To investigate questioning types utilized by PSTs, collaborative scripts of a mathematic lesson were written followed by individual script revisions. I use the following research question to guide my inquiry: *How do PSTs' plan questioning in a script of a mathematics lesson?*

Literature Review

Campbell and Baldinger (2022) suggest to prepare PSTs for teaching, teacher preparation programs must create opportunities for PSTs to participate in activities that mimic strategies in-service teachers utilize upon entering the field. One such way to aid in the development of PSTs' variation of questioning, is to utilize scripting as an instructional activity. In this paper, I first summarize literature on PSTs questioning drawing a distinction between questioning with children and theoretical questioning produced in instructional activities such as lesson planning. I then build upon questioning by utilizing the context of scripting as an approximation of practice MTEs use to create opportunities for PSTs to develop their questioning practice.

Questioning

Lim et al. (2018) described the ability to facilitate classroom discussion and discourse as the "teacher-students' dialogic interactions where students talk about mathematics and develop new understanding of concepts, through teachers' effective use of questioning" (p. 293). Without teacher questions, students struggle to identify the content or practice focus of a learning opportunity. For example, Parks (2020) examined the affects teachers' pedagogical moves had on students access to the mathematics being taught noticing how the teachers' questioned the students correlated to whether the students were probed for correctness or conceptual understanding. Effective questioning supports different phases of a discussion including initiating, orchestrating, and closing (Shaunessy et al., 2019).

Moyer and Milewicz (2002) described questioning approaches PSTs used in interactions with mathematics learners: *checklisting*, *instructing rather than assessing*, and *probing/follow-up questions* (p. 300 – 301). Checklisting occurs when an individual asks planned questions in order, regardless of the response a student provides. Instructing rather than assessing questions, are sequences of questions the teacher uses to lead the students' thinking. This category also includes teacher approaches that abandon questioning in favor of telling students how to approach a situation. Probing/follow-up questions are questions that ask the student to expand upon their ideas. These approaches to questions in the moment of interacting with learners suggest that PSTs need opportunities to build their questioning practice prior to or alongside their interactions with students. In this paper, I focus on questions that PSTs use during their preparation for interaction with students. This work illustrates the importance of MTE awareness of how PSTs typically pose questions and, in turn, create instructional activities to allow PSTs to practice planning and delivering a larger variety of question types.

In contrast to Moyer and Milewicz (2002), Purdum et al. (2015) analyzed PSTs' planned questions in their lesson plans prior to interactions with students. Without interaction with learners, Purdum focused on sorting questions into categories. PSTs planned both closed and open questions. Closed questions are those such that when posed to multiple students, they would all arrive at relatively the same set of responses. An example of this would be if a teacher asked a student to explain what another student had done when presented with a worked out solution to a problem. In contrast, open questions allowed for multiple solutions such as if a teacher asked students to show other ways of modeling their mathematical reasoning for the same problem. Findings aligned with those of Moyer and Milewicz (2002) in that PSTs were limited in the number and type of questions planned as most of the PSTs planned questions that were closed style questions.

Scripting

MTEs use scripting for a variety of purposes. Spangler and Hallman-Thrasher (2014) used scripting to examine PSTs ability to lead mathematical discussion. Zazkis (2017) examined scripts for creativity in mathematics teaching. Campbell and Baldinger (2021) further identified their use of scripting in addition to other instructional activities to prepare PTs for error handling. Such instructional sequences were identified as ways for MTEs to gain insight into PSTs knowledge of students, mathematics, and pedagogy. Lee and Lim (2021) use of scripts to examine PSTs approaches for moving lessons from launch, to exploration, to conclusion. Findings showed PSTs' struggles conceptualizing launches and conclusions of lessons were represented in the scripts Spangler and Hallman-Thrasher (2014) created in an activity sequence using scripts to prepare PSTs for teaching a mathematics lesson to students. PSTs who engaged in thoughtful consideration of the lesson progression, saw sustained improvement in the discussions they led with students (Spangler & Hallman-Thrasher, 2014). Crespo (2018) utilized scripting so PSTs could self-assess hypothesized teacher moves and student responses for revision.

PSTs need opportunities to plan questions if they are to develop utilizing a variety of questions throughout their lessons. Scripting is one opportunity MTEs can use to scaffold PSTs' question variety. MTEs use scripting to support PSTs' learning to plan and facilitate mathematics lessons. In this paper, scripting is used as a context for describing changes in PSTs' questioning during a mathematics lesson. Based on the existing literature focused on PSTs questioning and scripting as a context for development of PSTs practices, scripting might be one way to gain further evidence of PSTs development of questioning practice prior to interactions with learners.

Methods

Participants in the study were enrolled at a small Midwestern liberal arts institution, enrolled in mathematics methods during the spring of their junior year. The context for questioning is situated with the utilization of the five question types within the course text *Taking Action: Implementing Effective Mathematics Teaching Practices K – grade 5* (Huinker & Bill, 2017). During one instructional activity, the participants were divided into four groups of three or four, given a 3rd grade *Illustrative Mathematics* (Illustrative Mathematics, 2016) task, and asked to write a script of a mathematics lesson at the beginning of the semester. The MTE then provided readings and instructional activities to help the PSTs gain understanding of teaching practices as described in the course text. PSTs then independently revised the collaborative script of one of the four groups and provided a rationale for their changes. The collaborative script used for revision and each of the 16 revisions were analyzed and compared as it related to question types posed.

Questions in each script were coded using categories from the course reading: (1) gathering information, (2) probing thinking, (3) making the mathematics visible, (4) encourage reflection and justification, and (5) engage with the reasoning of others (Huinker & Bill, 2017). Examples of these types of questions can be found within the findings and discussion section. Given that only one collaborative group was chosen to be revised, in the analysis, only the revised scripts of those PSTs were highlighted in the findings (scripts 5, 9, and 10).

Findings and Discussion

PSTs' primarily used two question types: (1) gathering information and (2) probing thinking. The third most used question type was (3) engaging in reasoning with others. Two of the 16 PSTs revised the script using all five question types. One PST did not provide any additional dialogue, however, reasoned "the teacher is asking students about their thought process and why they chose the equation they did" or "after students work individually, they will partner up and compare the processes and their answer to a and b". Another PST added the question "Why would that be important?" after each student response. Lastly, when the teachers posed questions to their students, all student responses provided in the scripts were correct.

In examination of the group who provided the initial script for revision, only one of the three PSTs added significant number of additional questions. Script 9 had commentary for all five questioning types and posed a variety of questions. While script 5 targeted four out of five of the questioning types, the PST no dialogue between the student and teacher or student and student. Script 10 also had lacked dialogue of any sort in the revision in that she posed several teacher questions, but no student responses. Both scripts 5 and 10 were similar to their peers in that there was not much diversity in types of questions and the dialogue lacked substance.

Findings illustrate significant differences in questioning activity across PSTs with five students demonstrating four questioning types. PST focused on gathering information such as "What is important from this problem?" (script 16) from students. The second most common question type was probing student thinking which could be seen in script 13 when the teacher asks, "Is there only one way to set up this problem?" In addition, nine of the 16 PSTs included questions engaging in the reasoning of others and how it might support the discussion. One way this was shown was in script three when the teacher asks "Did anyone approach this problem differently and if so how did you approach it?" Only two of the 16 PSTs used a variety of questioning as aligned with findings of Moyer and Milewicz (2002). In script 4, the PST utilized all five question types and posed 11 questions throughout her revised script. In script 9, all five question types were utilized and 17 questions were posed. Both script 4 and script 9 had gathering information as the most used question type which shows similarity to all revised scripts. To build PSTs' questioning MTEs can provide opportunities to pose questions in various circumstances. These opportunities can be designed as approximations of practice (Grossman et al., 2009) in the form of scripting with revisions can support PST questioning development. Future studies could examine the impact of such activities on scripts and the variety of questions posed.

Teacher need to be able to facilitate these mathematical discussions through questioning (Parks, 2010). Findings from this pilot study suggest that PSTs use one or two question types most often. Due to the limited question types, MTEs should consider modeling alternative types of questions so PSTs can increase their own repertoire of questions. The PSTs were idealistic regarding how students respond to teacher questions (i.e. – student responses were always correct). One way to encourage additional questions is to include student errors in the scenario provided to the PSTs as Campbell and Baldinger (2022) did.

Future studies can be more deliberate in choosing to complete this scripting task after having the PSTs engage in discussion around posing purposeful questions (mathematical teaching practice 5) to address the question: *Can scripting tasks help MTEs identify where PSTs need more exposure to different types of questions?* In conclusion, writing a script for a fictional lesson and posing questions is one thing, but examining the questions posed during interactions with learners as Zazkis and colleagues (2009) showed is quite different. Further research can examine questions planned in scripts included in lesson plans and questions posed during the implementation of the lesson. Sequencing activities (Ghousseini & Herbst, 2016) has the potential to impact PSTs questioning.

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WHO'S YOUR FAVORITE MATH TEACHER AND WHY?: INSIGHTS INTO WHO PRESERVICE TEACHERS ASPIRE TO BECOME

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In many ways, preservice teachers embody the future of mathematics education. Guided by a conceptual framework of identity in figured worlds, the purpose of this study was to understand preservice teachers' designated mathematics teacher identities by prompting reflections about their favorite mathematics teachers. Data were collected from fifty-three preservice elementary and early childhood teachers who responded to an open-ended prompt about their favorite mathematics teacher. Thematic analysis of these responses resulted in three themes that preservice teachers described about their favorite mathematics teachers: their character, their pedagogy, and their classroom environments. Understanding how preservice teachers envision their futures may help mathematics teacher educators to support preservice teachers in realizing these aspirations or to reshape these aspirations to reflect best teaching practices.

Keywords: Preservice Teacher Education; Teacher Beliefs; Affect, Emotion, Beliefs, and Attitudes

Undergraduate preservice teachers enroll in teacher preparation programs with previously formulated conceptions of what it means to teach well (Lortie, 1975). These preconceived ideas influence preservice teachers' developing teacher identities as they "create future images of their teacher self and draw on remembered strategies they plan to replicate or avoid" (Miller & Shifflet, 2016, p. 27). These future images may be considered designated identities (Sfard & Prusak, 2005), and in the context of mathematics education *designated mathematics teacher identities*. In this study, I define designated mathematics teacher identity as the ways of being that one envisions of a future role as a mathematics educator (Graven & Lerman, 2020). In not so many words, this is a preservice teacher's answer to the question, who will I be when I'm a teacher? Attending to preservice teachers' designated mathematics teacher identities is especially important for mathematics teacher educators, as these identities may support or hinder preservice teachers' learning during their teacher-preparation programs (Caviness & Masingila, 2023). Furthermore, such designated mathematics teacher identities may become actualized when preservice teachers do transition into the field, especially with appropriate supports (Andersson, 2011; Jong, 2016).

In an overview of mathematics teacher identity research, Hannula et al. (2016) asserted that overall findings from the literature base "suggest that teachers' personal histories, such as those of being a learner, undoubtedly shape and become a part of their teacher identities" (p. 17). Therefore, the purpose of this study was to better understand the personal histories of preservice teachers that influence their designated mathematics teacher identities. Specifically, I sought to understand preservice elementary and early childhood teachers' personal histories related to their favorite previous mathematics teacher. I did so by examining the qualities of those mathematics teachers that were described as reasons why they were considered favorites. With this purpose in mind, the research question that guided this study was: how do preservice elementary and early childhood teachers describe their favorite mathematics teachers?

Conceptual Framework

I draw on Holland et al.'s (1998) theorizing of identity in figured worlds to understand preservice teachers' developing identities. Holland et al. (1998) described an individual's identity as constantly forming and reforming throughout their lifetime, shaped by their actions, personal histories, and social influences of the worlds they navigate. Drawing on Vygotsky, Bakhtin, and Bourdieu's work, Holland et al. (1998) outlined three contexts wherein identity is practiced or "authored" in figured words (p. 271). The first context is the figured world itself, which they defined as "a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others" (Holland et al., 1998, p. 52). Figured worlds are the spaces wherein identities are formed and reformed. In this study, I conceptualized preservice teachers navigating the figured worlds of their previous schooling at the PK-12 level, and the teacher preparation program based at their university.

The second context that Holland et al. (1998) described is one's positionality, with an emphasis on social positions such as gender, race, and cultural backgrounds. Holland et al. made explicit that individuals always exist in multiple figured worlds simultaneously based on their multiple positionalities. Lastly, the third context Holland et al. described was the "space of authoring" (p. 272), which emphasizes that individuals always exert agency when authoring identity. Individuals are shaped by the worlds they inhabit, yet they simultaneously shape their surroundings and do so continuously.

Methods

Fifty-three undergraduate preservice elementary and early childhood teachers at a northeastern university in the United States participated in this study. Participants were recruited from two different contexts:

- A mathematics content course in the fall 2022 semester consisting primarily of first-year undergraduates (n=25)
- A mathematics methods course in the fall 2022 semester (n=11) and spring 2023 semester (n=17) consisting of second-year and third-year undergraduates

Participants were asked to respond to a questionnaire that was designed to better understand their mathematics identities. While all participants voluntarily agreed to have their responses be collected and analyzed as data in a research study, the administration of the questionnaire differed based on the contexts above. In the mathematics content course, my relationship with participants was as a researcher and completing the questionnaire was not a course assignment. In the mathematics methods courses, my relationship with participants was as the course instructor and completing the questionnaire was a course assignment. The questionnaire included several prompts, however, the dataset for this study consists only of responses to the open-ended prompt: *Think about your favorite mathematics teacher (this includes elementary). What about them makes you say they are your favorite mathematics teacher?*

All 53 responses were analyzed in this study following Braun and Clarke's (2021) approach to reflexive thematic analysis. This approach includes six phases beginning with immersion in the data, followed by coding, generating themes, reviewing themes, refining themes, and culminating in writing about the themes developed. Braun and Clarke (2021) described their process as one of getting lost in the data and going back and forth between phases, ultimately, to identify patterns of meaning across the dataset. I report on these themes in the findings section below.

Findings

In my analysis of the dataset, I generated three themes for how participants in this study described their favorite mathematics teachers. These themes are: my favorite math teacher's character, my favorite math teacher's pedagogy, and my favorite math teacher's classroom environment. Each of these themes are elaborated in greater detail below with examples that I selected to best represent the dataset. In addition, each data excerpt presented below represents a different participant, with the goal of presenting a comprehensive look across the dataset.

Character

I found that preservice teachers commonly identified relational characteristics when describing their favorite math teachers. The theme "my favorite math teacher's character" encompasses the personality traits of favorite math teachers and the personable actions these teachers made toward their students. As exemplified in Table 1, the character traits of favorite mathematics teachers included being understanding, supportive, and patient.

Table 1: My Favorite Math Teacher's Character

	Data Excerpts
Understanding	"She was extremely compassionate and understanding…"
Supportive	"She was always available for extra help after and before school and well as [*sic*] during class hours."
	"She also acted as a person I knew I could count on for math and not math related issues."
Patient	"He… was super patient, and he was very personable."

Pedagogy

Many preservice teachers cited the teaching practices of their favorite mathematics teachers as reasons for them being their favorite math teacher. See Table 2 for examples that include differentiating by process to meet the needs of all students (Tomlinson, 2001), and explaining concepts well.

Table 2: My Favorite Math Teacher's Pedagogy

	Data Excerpts
Differentiates by Process	"Some of us could not understand certain methods, so he would teach the one he meant to teach and the ones we would understand. Comparing and contrasting different ways of solving math problems, Mr. [Teacher] was never the type of teacher to get upset with students for doing it their own way."
	"Despite having roughly 100 students, she provided us the opportunity to divulge new material in whichever learning style we preferred."
Explains Concepts Well	"He did a great job of explaining and showing material…"
	"They were really good at explaining things we were confused about."

Classroom Environment

In the prior two themes, participants' responses focused directly on the mathematics teacher. Here the thematic focus zooms out slightly as participants described the classroom environments that their favorite mathematics teachers created. This included how these environments made preservice teachers feel. As displayed in Table 3, the classroom environments of favorite mathematics teachers were safe, fun, and for some, these environments fostered a sense of belonging to mathematics.

Table 3: My Favorite Math Teacher's Classroom Environment

	Data Excerpts
Safe	"I say that he was my favorite math teacher because he didn't make anyone feel dumb for not knowing the answers."
Fun	"They made the class fun and engaging and tried to make us laugh whenever."
Fostered a Sense of Belonging to Mathematics	"I actually felt pretty confident in that class and like I understood the content."
	"My sixth grade teacher was the first person who got me to like math."

Discussion

In this study I sought to better understand the designated mathematics teacher identities of preservice elementary and early childhood teachers by analyzing how they described their favorite mathematics teachers. I based this analysis on the premise that preservice teachers carry their prior experiences as learners of mathematics into their developing roles as teachers of mathematics (Caviness & Masingila, 2023; Lortie, 1975; Ma & Singer-Gabella, 2011; Miller &

Shifflet, 2016). I found that preservice teachers especially valued their favorite math teachers' character, pedagogy, and the classroom environments they created.

In many ways, preservice teachers *are* the future of education. This work takes up the call to envision the future of mathematics education in uncertain times by seeking to understand what preservice teachers envision about their future roles as educators. I have shown one way that researchers might learn about preservice teachers' designated mathematics teacher identities by prompting reflection specifically about their favorite mathematics teachers. Mathematics teacher educators may similarly learn about their students in this way, and it is possible that such learning could be used to foster preservice teachers' developing mathematics teacher identities.

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ENGAGING PROSPECTIVE TEACHERS' IMAGINATION IN LEARNING INQUIRY-BASED MATHEMATICS PEDAGOGY

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Inquiry-based teaching represents a contemporary perspective of mathematics education that is important for prospective teachers (PTs) to learn about to meaningfully engage their future students in learning mathematics with understanding. This paper reports on a study of the use of narrative and imagination in PTs' learning of inquiry-based mathematics pedagogy. Participants were prospective secondary mathematics education majors who were exposed to theory about inquiry-based teaching but had limited experience with it as learners of mathematics. Data consisted of their narratives of imagined inquiry-based lessons for mathematics concepts of their choice. Findings indicated that the narratives have the potential of helping them to imagine and understand key aspects of an inquiry-based lesson that they could apply to their future practice.

Keywords: Narrative; imagination; secondary preservice teachers; inquiry-based teaching

Inquiry-based teaching of mathematics continues to be a challenge for teachers to implement particularly if their experience with learning mathematics did not align with this approach to teaching and learning (Artigue & Blomhoj, 2013; Maaß & Doorman, 2013). Prospective mathematics teachers (PTs) also continue to enter teacher education programs with limitations in their mathematical content and pedagogical knowledge needed to support it (Author, 2023). Thus, ongoing research is important for us to understand meaningful ways to support their learning to improve mathematics education. This paper reports on one aspect of a study to explore the use of narrative and imagination in PTs' learning and use of inquiry-based mathematics pedagogy. The focus is on the question: What understanding of inquiry-based teaching secondary PTs are able to develop through creating a narrative of an imagined mathematics lesson that unfolds in real time?

Theoretical Perspectives and Related Literature

A narrative/story (Egan, 1986; Polkinghorne, 1988) is a way of representing experience for oneself or for others. It involves people, settings, and events that take place in a given time frame. It is established in the broad field of education as an important tool in supporting meaningful teaching and learning. It helps people to remember things by making knowledge more engaging, helps us think and do things more effectively, and enlarges our powers to think and understand (Egan, 2008). It aids in the process of meaning making (Clark & Rossiter, 2008) and teaching mathematics (Zazkis & Liljedahl, 2009). Some ways in which narrative has been used in mathematics teacher education include researching mathematics teachers' trajectories as they enter the profession (Frost, 2010), how focusing on healthier narratives can help teachers work toward liberatory futures (Gutiérrez et al., 2023), teachers' reflection on their practice or learning (Author, 2008), PTs' construction of educational meaning through narratives (Dolk & Hertog, 2008), and use of "scripts" in mathematics teacher education (Zazkis & Herbst, 2018). In general, narrative has been used in teacher education mainly as a research method to obtain and analyze data with a focus on researcher constructed and elicited stories. In this study, narrative is used as a cognitive tool to support imagination in PTs' learning.

Regarding imagination, Egan (2008) explained:

> When we speak of imagination, we are referring to the ability to think about what might be possible. It is the "reaching out" feature of students' minds that picks up new ideas, tries them out, weighs their qualities and possibilities, and finds a place for them amidst the things they have already learned. (p. 5)

Vygotsky (2004) also considered imagination to be a process directly connected with meaning making. He explained that everything that relates to interpretation and construction of something new requires the indispensable participation of imagination. Everything we create, "is the product of human imagination and of creation based on this imagination" (Vygotsky, 2004, p. 7). These perspectives of imagination suggest that engaging PTs to use their imagination to learn ideas about inquiry-based teaching that are new to, or different for, them could help them to create meaning of it for themselves in a way that they could live it in their teaching. Narratives serve as tools for engaging imagination (Egan, 1992). In this study, PTs created narratives based on their imagination of an inquiry-based lesson. The assumption is that if they could imagine a complete inquiry-based lesson, they are more likely to be able to adopt it in their practice.

While there are multiple perspectives of inquiry-based learning (Artigue & Blomhøj 2013), there are commonalities, particularly regarding placing emphasis on learners with their autonomy and understanding as the central focus (Jaworski, 2015). Inquiry-based teaching, then, refers to teaching approaches that support students' individual and collaborative engagement in inquiry-based tasks to "foster students' construction of their knowledge through inquiry, exploring, and finding their own path to solution" (Maaß & Artigue, 2013, p. 782). The teacher's role includes:

> orienting students towards questions and problems …; making constructive use of students' prior knowledge; supporting and guiding when necessary their autonomous work; managing small group and whole class discussions; encouraging the discussion of alternative viewpoints; and helping students to make connections between their ideas and relate these to important mathematical … concepts and methods. (Maaß & Artigue, 2013, p.782)

Features of this perspective of the teachers' role formed a basis for analyzing the PTs' narrative regarding their understanding of inquiry-based teaching based on their imagined lesson.

Research Methods

This qualitative study explored the understanding of inquiry-based teaching secondary PTs were able to develop through creating a narrative of an imagined mathematics lesson that unfolds in real time. Participants were 16 secondary mathematics education majors in a mathematics education course in the third semester of a 4-semester, 2-year Bachelor of Education program. They had either a 3-year or 4-year undergraduate degree in mathematics. Their experience in learning school mathematics was not inquiry based. In the course, they were exposed to theories of inquiry-based teaching/leaning, inquiry-based tasks, and questioning/discourse. They explored examples of inquiry-based tasks but not examples of complete inquiry lessons.

The narrative task was intended for the PTs to draw on the theory and experience in the course to imagine their teaching of an inquiry-based lesson. They worked in groups of four to create the narrative to allow them to pool their imagination of how they would live the lesson. They were required to imagine themselves engaged in inquiry-based teaching of a mathematics concept of their choice from the G7-9 provincial curriculum. The guideline included that the lesson should be from a lived perspective and the narrative should include a preamble of the learning objectives for the lesson, class size and classroom organization. The lesson must include a task and intervention in at least three groups (e.g., one stuck; one with no issues; one thinks no issues but has issues). The narrative must have a temporal flow as lived and include dialogues.

Data sources were the four narratives. Analysis included categorizing and coding the narratives to identify the PTs' understanding of inquiry-based teaching. Each narrative was categorized based on the structure of the lesson including the mathematic concept, learning objectives, main task, lesson introduction, intervention in student groups, whole-class discussion, and conclusion. Each category was then coded to form themes based on aspects related to inquiry-based teaching including learner-centeredness (based on type of teacher's questions/ prompts/guidance and students' questions and responses), the inquiry nature of the main task, and framing of the learning objectives. Themes for each category were compared across the four narratives. The designs of the narratives were also analyzed and compared for the temporal flow of the events in the lesson, the characters, use of dialogues and commentaries, and focus on the central themes. Mainly findings regarding key components of the narratives are presented here.

Findings

All four groups of PTs (PT-GA, PT-GB, PT-GC, and PT-GD) were able to imagine key aspects of inquiry-based teaching. The following are summaries of key components of their narratives. Their class sizes ranged from 24 to 32 with students in groups of mainly four.

PT-GA: *Concept* - Grade 7 divisibility. *Learning objectives* – develop conceptual understanding of divisibility, represent divisibility using multiple approaches, understand how to test divisibility of numbers 2 to10. *Main task* – explore the concept using a pile of cubes. *Lesson introduction* – students asked to think about and share whatever they knew about "divisibility". *Intervention in student groups* – three groups considered: could not start, had an incorrect model, had a correct model. The teacher used open questioning and prompts, e.g., "Explain what you mean?" "How are you connecting them?" "Why did you decide to use a rectangle to represent the number six?" "Think about if there are other useful alternatives." *Whole-class discussion* - groups presented and justified their processes with teacher questions, e.g., "How can you be sure that this "splitting things up" model works correctly?" "What does this model indicate about the concept of divisibility?" *Conclusion* – students asked to write a sentence to describe divisibility.

PT-GB: *Concept* - Grade 8 perfect square/square root. *Learning objectives* - develop conceptual understanding of a perfect square/square root, illustrate/classify perfect square/square roots through multiple representations. *Main task* - explore example versus nonexamples with multiple representations. *Lesson introduction* – individually think about then share "What does the word 'perfect' mean to you?" "Do you think the real-life application of 'perfect' applies to mathematical applications?" *Intervention in student groups* – four groups considered: stuck and asked for help, believed they understood but had misconceptions, did well, could not begin. The teacher used open questioning and prompts, e.g., "What are you noticing?" "Try to focus on the other given numbers." "Do you think it's important what shape they make?" "Test your conjectures and ideas by adding new examples and non-examples." *Whole-class discussion* -

groups presented and justified their processes with teacher's questions, e.g., "How did you resolve this conflict?" "Why do you think the shape of the examples matters?" "What does the shape tell us here?" "Did your group come up with any other insights …?" *Conclusion* – students defined a perfect square and square root and worked on worksheets to complete missing items in two tables of multiple representations of perfect squares and square roots respectively.

PT-GC: *Concept* – grade 8 rates. *Learning objectives - i*dentify and describe rates, convert rates to unit rates, express rates using words and symbols, understand rates in everyday life. *Main task* - explore rate by racing on two racetracks set up in the school yard. *Lesson introduction* – students given one minute to figure out who in their group got to school the fastest then share and respond to "how did you get to school today?" "How long did it take you and how do you know that you were the fastest?" *Intervention in students' groups* – Three groups were considered: thought they were doing okay but were not, stuck and needed help, doing okay. The teacher used questions/prompts that were at times closed or directed to the answer or involved telling, e.g., "think about what the number 0.14 means. 0.14 What? Or 0.125 what?" "That means that speed isn't just about how much time you take but it needs something else. Any ideas what that is?" "Start by figuring out how many meters each person ran." "Another way of saying that is 8.3 meters per second. It's a rate … a unit rate." *Whole-class discussion* - no sharing or discussion. *Conclusion* – teacher summarized what they should know about the concept.

PT-GD: *Concept* – grade 9 multiplying exponents. *Learning objectives* - demonstrate understanding of multiplying powers with integral bases and whole number exponents. *Main task* – explore the concept by sorting 32 cards (3 blank, 8 different representations). *Lesson introduction* – teacher asked what an exponent is and what they think happens with something like $1000000^{1000000}$ x $1000000^{1000000}$ then used $5^2 * 5^3$ to take them through a process with questions leading to predicting the rule to be validated through the main task. *Intervention in students' groups* – three groups considered: confused and asked for help, believed they could do the task but needed help, understood the task well. The teacher used open questions and prompts, e.g., "Why did you put those (2 cards) together?" "Maybe think about writing out and expanding each of the exponents." "Try to see if you can notice what they have in common." "Do you notice any patterns"? "Try to see if you can figure out a way to check whether your answers make sense." "What can you do to justify matching 5^2 x 5^1 to 5^2 and 3^2 x 3^3 to 3^6? Try it out." *Whole-class discussion* - two groups shared but no questioning by teacher. *Conclusion* – summarized what they should know about the concept and they answered $1000000^{1000000}$ x $1000000^{1000000}$.

Conclusions

Each group of PTs was able to create an appropriate narrative of an imagined lesson that unfolded in real time. They were able to imagine a lesson with key features of inquiry-based teaching. They had introductions to motivate students with connection to prior knowledge/ experience. PT-GA and PT-GB were able to maintain a learner-centered focus throughout and PT-GC was mostly student centered with limitations during whole-class sharing. These groups had objectives focused on students' understanding and tasks that involved inquiry through noticing patterns and relationships that seemed to help their learner-centered group interventions and whole-class discussion. They used open questioning and prompts appropriately to expose and engage students' thinking. PT-GA and PT-GB also asked questions to stimulate discussion and thinking. PT-GD used an applied task that was more challenging for students to see the concept for themselves and so had to provide more direct guidance at times to get them there during group interventions, which resulted in no whole-class sharing/discussion since the groups had similar process. All groups were able to imagine appropriate student thinking and responses to their questions or prompts and consider alternative processes to explore the concept.

Since the PTs' narratives are approximations of inquiry-based lessons as they unfold in real time, there are obviously gaps regarding details of the lesson (e.g., whole class discussion), but the focus here is on key features of the lesson that they could build on. The narrative imagination offered opportunity for them to make sense of inquiry-based teaching and to begin to develop an image of it in a way that they could apply and build on in their teaching. Future work on the project will include PTs sharing their narratives to learn from each other and observation of sample of the PTs in their practicum and as beginning teachers to gain further insights of the potential of the approach and how to modify it to be more effective as a tool for PTs' learning.

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EXPLORATORY CASE STUDY: USING OBSERVATIONAL DATA TO STUDY PRESERVICE TEACHER MATH ANXIETY

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Research on preservice elementary teachers' experience of math anxiety has historically utilized narrative, interview, and survey data. Our searches revealed no studies that used observational data, which has the potential to reveal insights that are less reliant on self-perception. Thus, this case study aimed to understand how a highly math-anxious preservice elementary teacher made sense of their math anxiety and how it may have changed in the context of a mathematics content course, and to explore whether or not it was possible to collect observational data that aligned with these understandings and perceptions. Findings reveal that math anxiety was meaningfully observed for this preservice teacher and that these observations provided valuable insight into how her math anxiety changed over the course of the semester.

Keywords: math anxiety, preservice teacher education, interpretation account.

Mitigating the detrimental effects of preservice elementary teachers' (PSETs) math anxiety (MA) has been a persistent concern in teacher preparation programs for decades. Defined as "a feeling of tension, worry, and/or fear in situations involving math-related activities," (Bjälkebring, 2019, p. 1), MA has a well-established link to lower achievement in mathematics (Barroso et al., 2021; Ma, 1999), and is particularly prevalent among PSETs (Hembree, 1990). For these future educators, the negative impacts of their MA may extend to their students. PSETs reporting high levels of MA tend to hold lower expectations for their students (Mizala et al., 2015) and have lower self-efficacy for teaching mathematics (Gresham, 2008). As noted long ago by Dutton (1951), it is vital for math teacher educators to find ways of "breaking this 'vicious cycle'" (p. 89) by supporting PSETs in learning to manage their negative emotions towards mathematics before they become practicing teachers.

To best support PSETs in this way, mathematics teacher educators should understand how PSETs experience MA and how this experience might change through experiences in teacher preparation courses. Researchers have employed interview, narrative, and survey data to both characterize PSETs' experiences with MA and to document shifts in these experiences (e.g., Finlayson, 2014; Hollingsworth & Knight-McKenna, 2018; Olson & Stoehr, 2019). While these data sources are indispensable in studying MA, we were unable to identify studies of PSET MA that further triangulated these data sources with observational data. The purposes of this study are therefore to: (1) understand how a highly math-anxious PSET perceives and makes sense of their MA and how it may change in the context of a mathematics content course, and (2) explore whether it is possible to collect observational data that aligns with these perceptions and understandings.

Review of Literature

Researchers have used survey, narrative, and interview data to characterize and better understand the individualized nature of PSETs' MA. For example, although several studies have found that being in evaluative settings is commonly cited by PSETs as eliciting MA (e.g., Finlayson, 2014; Harper & Daane, 1998; Olson & Stoehr, 2019; Wilson, 2015), Olson and Stoehr's (2019) study suggested that the way PSETs interpret these evaluative experiences may vary greatly. In written narratives, two of the three highly math-anxious PSETs in this study reported experiencing MA in evaluative settings because they had interpreted poor performance on past exams as evidence of their inability to learn mathematics. The third participant, however, experienced MA in relation to being evaluated because of the uncertainty she felt while waiting for the teacher's feedback. Finlayson's (2014) work also illustrated the variability of PSETs' MA experiences. Finlayson analyzed responses to open-ended survey questions from 70 pre-service teachers (many of whom were PSETs) to identify common causes, symptoms, and coping strategies that study participants associated with their MA. Several common themes emerged in each category, but no theme applied to more than half of the participants, and Finlayson's analysis revealed that study participants gave different explanations for similar responses. For example, 23 participants identified a lack of self-confidence as a cause of their MA. Among these, some attributed this lack of self-confidence to a history of inadequate support from teachers, while others described it as a family trait, citing parents who had also struggled with mathematics. These studies suggest that, although many PSETs may report similar MA experiences, the ways they make sense of these experiences can be highly individualized. Because PSETs experience MA differently, it is likely that they also respond to MA-eliciting situations differently, positioning observational data as an underutilized research tool.

Analyzing survey, narrative, and interview data has also allowed mathematics education researchers to develop strategies for reducing PSETs' MA. Some studies suggest that certain kinds of learning activities implemented in teacher preparation courses can positively impact PSETs' reported levels of MA. For example, researchers have reported successfully lowering PSETs' MA by emphasizing the use of manipulatives (Barrett, 2013), implementing inquiry-based learning (Van der Sandt & O'Brien, 2017), and by including a field work course component (Hollingsworth & Knight-McKenna, 2018). Although all of these studies concluded that PSETs' MA was reduced as a result of engaging in these learning activities, none included observational data of how individual PSETs had participated in these activities. Hollingsworth and Knight-McKenna noted this as a limitation of their study, stating that "Much of our analysis relied on self-report. Students may have provided responses they felt the instructor wanted to hear" (p. 324), and then suggested that future research should include observational data to guard against this possibility. However, in our literature search, we were unable to identify any studies of PSETs' MA that utilized observational data.

Analyzing narrative, interview, and survey data has provided researchers with important insights into how PSETs make sense of their MA experiences and helped them to identify effective MA interventions that can be used in teacher preparation courses. Observational data has the potential to serve as a unique research lens, adding depth to current understandings of PSETs' MA experiences. Further, observational data may also uncover aspects of PSETs' experiences that are unattainable when using self-reported data sources alone.

Theoretical Perspective

For this study, we rely on Ramirez et al.'s (2018) *interpretation account* of MA, which argues that "students' development of MA is largely determined by how they interpret … previous math experiences and outcomes (rather than the outcomes themselves)” (p. 151). For example, consider how receiving a failing grade on an exam provokes MA in some students but not in others. One student may interpret the failing grade as evidence of an innate inability to perform mathematically and may become anxious in future mathematical settings. Another student may instead interpret the failing grade as the result of the teacher’s harsh grading policies and become angry instead of anxious. As a theoretical perspective, interpretation account centers student experiences of MA, the ways in which they make sense of these experiences, and how these experiences influence the way they participate in mathematical settings.

Interpretation account suggests that different situations will trigger MA in some PSETs but not in others (e.g., answering questions in front of the class or being stuck on a problem). This means that individual PSETs will have a unique set of *MA triggers*. These triggers, along with the individualized ways in which PSETs respond to encountering these triggers (e.g., disengaging from a task or making jokes) comprise what we refer to as their *MA baseline*. Establishing PSETs’ MA baselines is informed, in part, by their use of *MA aligned language*. Although the language that PSETs use to describe experiences of MA is not uniform, researchers have identified some language that math-anxious individuals commonly use. Stoehr and Olson (2021) compiled a list of MA-aligned language that PSETs used when speaking of mathematical experiences, such as “embarrassed,” “math is not my thing,” and “dreaded math class.” Other researchers have also identified common responses to experiences of MA (e.g, Dowker, 2019), such as rushing to complete a task at the expense of accuracy, avoiding mathematical situations, and disengaging from a mathematical task. Thus, for this study we will adopt interpretation account in tandem with research on MA-aligned language and responses to answer our research questions: How does a highly math-anxious PSET perceive and make sense of their MA and how it may change in the context of their experiences in a mathematics content course? Are these perceived understandings of MA and changes (if any) observable in the PSET’s mathematical interactions?

Methodology

This study followed a single-case study design (Yin, 2009). Because our goal was to explore an unstudied phenomenon (i.e., if MA can be observed and if these observations further triangulate a PSET’s perceived MA), a single revelatory case (Yin, 2009) was most appropriate for our goal. The case was a single PSET, Rose (a pseudonym), and was bounded by time and context; data was collected in a single semester and within a mathematics content course.

Context

This study took place within the context of a terminal mathematics content course for PSETs at a large public university in the northeast United States in Fall 2023. The course content focused heavily on operations with fractions, which has been identified in the literature as a difficult concept for PSETs (e.g., Rosli et al., 2020), particularly those that are math-anxious (e.g., Rayner et al., 2009). The course was student centered and relied heavily on in-class group work and discussion. During whole-class discussions, PSETs compared the different solution strategies that groups had used. The course also used Revision-Reflection Grading (a non-traditional grading system adapted from *Ungrading* by Blum, 2020).

Case Selection

Since our goal was to gain insight into a PSET's MA baseline, their perception of any changes, and how these perceptions may align with observational data, open communication and willingness to discuss MA and related experiences was essential. Thus, of the many highly math-anxious PSETs (as identified by written narratives and validated MA scale), we selected Rose. Rose's candid communication and outgoing personality afforded us a window to explore how her perceived MA aligned with classroom observations of her engaging with mathematics and/or with others while doing mathematics.

Data Collection

The brief version of the Mathematics Anxiety Rating Scale (MARS-B; Suinn & Winston, 2003) was used to select the case as stated above and was also used to determine the baseline MA score for Rose and her group members (for comparison). Rose was identified as highly math anxious (109 out of 150; highly math anxious in this study was defined as having a MARS-B score greater than 90, cutoffs adapted from Sanders et al., 2019), and her group members reported moderate to high levels of MA (48, 85, 85, and 101). Rose and her group members also took the MARS-B at the end of the semester to track any changes in MA level. In addition to the validated scale, we sought to describe their perceived MA. Thus, we asked Rose and her group members to write a short narrative to describe their relationship with mathematics at the start of the semester to serve as a baseline for perceived MA. Similar to the scale, we also requested an end of the semester narrative to track any changes in perceived MA. The prompt read: "Tell the story of your mathematical journey. Describe your relationship with mathematics."

Rose took part in one semi-structured interview. The original goal of this interview was to explore Rose's perceived changes in MA. We thought that the validated scale and narrative would provide sufficient information to describe Rose's baseline MA. However, given the exploratory nature of this case study, her narrative only revealed insight into her triggers but not her typical responses to them. Thus, we altered the design of the semi-structured interview into two parts: the baseline interview and the post interview. In the baseline interview, we sought to corroborate the MA triggers from Rose's narrative (e.g., "Is there a particular kind of classroom situation that you associate with feeling anxious about math?") and identify typical responses to those triggers (e.g., "How has feeling anxious about math impacted your experiences in math classes?"). In the post interview, we sought to elicit Rose's perceived changes in MA across the course of the semester and how she made sense of such changes.

The final data source consisted of two classroom observations (one after the baseline MARS-B and narrative and one mid semester). The goal of the observations was to determine if MA-aligned responses to Rose's baseline MA triggers could be observed. Rose's group was audio-recorded during these observations and field notes were taken with respect to Rose's identified baseline MA triggers.

Analysis

Analysis occurred in two phases: (1) analysis of the baseline data and (2) analysis of the post narrative and interview data. In phase one, the baseline narrative and interview were coded using a priori theoretical coding based on the MA-aligned language from the literature (e.g., Stoehr & Olson, 2021). We then open coded across those codes to extract any potential MA triggers and responses to these triggers. The first observation was used to see if we could observe any of the identified baseline MA triggers and responses during class. We coded the transcript for the presence of any of Rose's identified triggers, and then open coded Rose's responses to those triggers to identify any common themes in her responses to each trigger.

In phase two of analysis, we sought to examine any perceived changes in MA. First, we open coded the post narrative to identify any changes in Rose's responses to her baseline MA triggers. The response themes were open coded for alignment (or not) to her baseline responses. To analyze the post interview, we applied interpretation account (Ramirez et al., 2018) to describe how Rose made sense of any changes in her experience of MA. We used the second observation to corroborate any changes identified in Rose's post narrative and interview regarding her MA triggers and responses. In other words, we aimed to determine if Rose's MA triggers elicited a different response later in the semester compared to earlier in the semester. All code discrepancies were discussed until consensus was reached.

Findings

Rose's MA transformation in the context of a mathematics content course provided an opportunity for us to understand how she made sense of her MA and how it changed, and to see whether or not these changes were observable in her engagement in mathematical tasks and classroom interactions. Rose's MARS-B pre and post-semester scores (109 and 39, respectively) indicated that she began the semester as the most math-anxious member of her group (her group mates scored 48, 85, 85, and 101), and had experienced a drastic decrease in MA by the end of the semester.

In this section, we will discuss how we were able to use Rose's baseline narrative and interview to characterize her baseline MA triggers and responses and will present data from our first classroom observation that was consistent with this characterization. We will then describe how Rose made sense of the reduction in MA that she experienced over the semester, and how these changes manifested in observable differences in the way that she engaged in classroom activity in our second observation. Through these findings we will demonstrate that, at least in the case of Rose, MA was meaningfully observed and that these observations provided valuable insight into how her MA changed over the course of the semester.

Rose's Baseline Math Anxiety Triggers and Responses

When Rose spoke or wrote using MA-aligned language, it often centered on the MA trigger of *being confused by mathematical content.* For example, she wrote in her baseline narrative that she "was always incapable of understanding certain math concepts, which created a lot of anxiety around math class." In her interview, Rose clarified why this was a MA trigger for her, indicating that she interpreted this confusion as a threat to her self-concept as a good student:

> I just never could get it. I would go to extra help. I would stay after. I would study. I would do all the right things… I tried very, very hard. And I just, my grade didn't reflect that… And that was just something that I hated. And I resented math because of that.

Again in her interview, Rose provided insight into how she responded to this MA trigger as she described her experience of confusion with high school precalculus content: "It wasn't going to make sense to me. Like, I didn't want to bother trying because it just, I couldn't wrap my head around it. Like, I felt like it just was a waste of time." Rose's assertion that she "didn't want to bother trying" suggested that one of her baseline responses to feeling confused by mathematical content was disengagement.

Although being confused by mathematical content appeared to be Rose's most prominent MA trigger, her use of MA-aligned language also indicated that *classroom interactions with teachers* were also MA triggers for her. The interactions she described were uniformly negative, and were recounted with an emphasis on the strong emotions that these experiences elicited. For example, Rose wrote in her baseline narrative about a particular occasion when a teacher "called me out in front of the class and not in a positive way." Rose elaborated on this experience in her interview, saying that the teacher "was like, 'I just don't understand why you don't understand it.' Like it, and it was *embarrassing* for me, like, *embarrassing*. … all those kids heard" (italics added to show the emphasis Rose used when speaking). Rose explained that she had interpreted this interaction as an indictment on her ability to understand mathematics: "If she is confused why I don't understand it, like, clearly I should be, I *should* be understanding this." She then described her response to this MA trigger, again in terms of disengagement: "The rest of the year, I just was like … why am I even going to bother trying to understand this?"

During our first classroom observation, we saw evidence of Rose's experience of MA that was consistent with her baseline narrative and interview, both at moments when she expressed confusion and when there was a possibility of interacting with the professor. At one point, when her group had reached an impasse with a mathematical task on fraction addition, Rose expressed her intent to disengage by saying, "I could figure this out. But it's a lot of energy for me that I don't have right now. I'm gonna save my energy." Later, while working on a problem about installing a playground on a fraction of a park's area, Rose used a cooking metaphor her group developed for mathematical thinking to announce her intent to quit working with the group, saying "You cook over there. I'm lost. The kitchen's burning." Then, Rose rubbed her face and, even though the group had not yet completed their problem set, said, "We're done. I've been done." Although she almost always re-engaged and continued to work with her group, Rose's pattern of responding to confusion with disengagement was a consistent and clearly observable phenomenon.

The possibility of interacting with the professor was also an observable MA trigger for Rose. However, rather than responding with disengagement as she had indicated in her interview, we observed Rose responding to this trigger by expressing a desire to keep her work private. For example, at one point, the professor asked the entire class, "One-fourth is how many twelfths?" Rose whispered to a groupmate, "One-fourth is three, three twelfths. I don't want to answer because then she's gonna ask me another question." Towards the end of class, the professor stopped by Rose's group. Although the group was having a productive discussion, Rose responded to the professor's arrival by saying, "you came at the worst time." This desire to hide her thinking from the professor was an MA response that Rose had not identified in either her baseline narrative or interview, but was prominent in the observational data.

Changes in Rose's Math Anxiety

Rose's post narrative and interview were both well aligned with the remarkable drop she reported on her post-semester MARS-B (39 compared to 109 at the start of the semester). In her post narrative, she wrote, "To put simply, my relationship with math has completely changed… [for the] first time (I believe ever) I was made to believe that I am capable of understanding math." In both her post narrative and interview, she specifically addressed her baseline MA triggers. Although she did not directly address changes in the way she responded to these triggers, we were able to observe Rose responding to these triggers in markedly different ways in the second in-class observation.

In her post interview, Rose no longer used MA-aligned language when talking about mathematical content or about being confused by mathematical content. She said: “I was always the person who needed help…And now I'm helping people learn how to solve, like people are asking me, which is something that's never happened before.” Rose also explained that she had begun to interpret experiences of confusion differently:

> People who are good at math were always like, yeah, it's fun. It's like a puzzle. Like, I'm like, what puzzle? Like whatever. And now I kind of understand…because I'm given the opportunity to look at it a different way and look at it in a way that makes sense to me. And I can solve the puzzle all on my own, and I don't need help. And sometimes you need help. And that's okay, like, everyone needs help with math every once in a while.

In contrast to her earlier declaration that mathematics “wasn’t going to make sense,” this statement implied that Rose now viewed being confused as a normal part of the mathematical process, rather than a source of anxiety. Rose credited this transformation to the professor’s practice of highlighting multiple solution strategies, saying, “I always can find a [solution strategy] that's on the board that I know, I understand the way I know how to do it.” This new awareness also seemed to prompt Rose to reinterpret some of her past mathematical experiences:

> Now I know, like, learning from this class, there's a million ways you can explain things like, … me not understanding how to do something in freshman year algebra was not because I was stupid. It's not because I couldn't understand it. It's because I didn't understand it in the way that it was explained to me.

In our second classroom observation, we saw a sharp contrast in the way Rose responded to being confused during group work compared to our first observation. At one point, a groupmate, Ellen (pseudonym), was watching Rose attempt to use fraction strips to evaluate $\frac{3}{10} \div \frac{4}{5}$~~31045~~. Ellen said, “I am genuinely so lost.” Rose replied, “I am too. I’m just gonna model it and see what I come up with here. And kind of just hope for the best here.” After working on the problem for several minutes, Ellen expressed doubt that Rose’s strategy would solve the problem, saying, “I feel like four-fifths isn’t going to go into three.” Rose replied, “Well, here, we’re gonna figure it out. Ready? Let’s figure it out.” This observation was evidence that being confused was no longer the MA trigger for Rose that it had been, as she was now responding to confusion with persistence and encouragement rather than disengagement.

In her post narrative and interview, Rose indicated that her professor had played an important role in her MA transformation. She wrote in her post narrative that the professor had been “a positive light for me this semester.” In her post interview, Rose indicated that the professor had created a classroom environment where she felt valued:

> [The professor] understands, like, a lot of us struggle with math… She makes everybody in that class feel like they’re seen, and feel like they're understood… even if you have times where you don't understand it… [if] you're doing your, she does revisions… So as long as you're putting the effort in, you're getting the grade back, which is what I've always wanted in math…I actually feel like I'm getting the results in this class.

Although this implied that perhaps interacting with the professor may no longer have been a MA trigger for Rose, she did not write or talk about any changes in the way she responded to the possibility of interacting with the professor in class. However, our second in-class observation provided evidence that Rose had indeed experienced a shift with respect to this MA trigger. During a whole-class discussion of the problem $\frac{1}{3} \div \frac{1}{9}$ ~~1319~~, the professor asked students to share their answers and their reasoning. Rose volunteered to explain to the class how she had used fraction strips and said, “So, I took one third … and then I saw how many, or, how many one-ninths fit into the one third, and it’s three, so the answer would be 1/3?” Rose ended her statement in a questioning tone, implying that she was unsure of whether her answer was correct. Her answer wasn’t correct, but that is immaterial in this interaction; her willingness to share her thinking represented an enormous shift in participation for Rose. In contrast to our first observation, when Rose had been reluctant to share an answer that she knew was correct, she now volunteered to share her thinking even when she was not completely sure of her answer.

Discussion and Conclusion

In this exploratory study, we sought to understand (1) how one highly math-anxious PSET made sense of changes in her MA throughout a mathematics content course, and (2) whether these changes were observable in her in-class participation. With respect to our first research question, we found that Rose reported a substantial reduction in her level of MA, a change that was reflected in all of her self-reported data. With respect to experiences of confusion, Rose’s language shifted from reflecting anxiety, resentment, and futility to an assertion that “everyone needs help with math every once in a while.” When speaking of interactions with teachers/professors, Rose described feeling “understood” and “seen” instead of feeling embarrassed. Rose’s new awareness of the existence of multiple solution strategies was pivotal in her MA transformation. Not only did she view this as a source of her newfound mathematical confidence, it provided her a lens through which she could reappraise her past mathematical experiences. She now interpreted her struggles in high school mathematics as a deficit in “the way it was explained,” rather than a deficit in her cognitive abilities.

With respect to our second research question, we found that we were able to observe Rose responding to her MA triggers in ways that both corroborated her self-reported data and provided additional insight into her changing experience of MA. This responded to Hollingsworth & Knight-McKenna’s (2018) call to reduce reliance solely on PSETs’ self-reported MA data. For example, we were able to verify Rose’s self-reported description of disengaging in response to moments of confusion when we observed her saying, “I’m gonna save my energy” and “you cook over there, I’m lost.” In our second observation, we found evidence of Rose’s new interpretation of being confused as a natural part of “solv[ing] the puzzle” when she told Ellen that, even though she was confused, “I’m just gonna model it and see what I come up with here.”

Our observational data also revealed aspects of Rose’s experience of MA that were not present in her MARS-B responses, narratives, or interviews. Although Rose’s interactions with her past teachers and the professor were prominent topics in these data sources, her observed responses to this trigger were not represented in her narratives or interviews. In the first observation, when there was a possibility of interacting with the professor, we saw a desire to keep her answer private when she said, “I don't want to answer because then she's gonna ask me another question,” and “you came at the worst time.” Attempting to keep one’s work private is a

MA response identified by Sanders et al. (2019) and could be viewed as evidence that Rose interpreted such interactions as threatening. During the second observation, when we observed Rose volunteer to share an answer that she was unsure of, it was clear that this was no longer the case. Interestingly, Rose never mentioned how she might respond to interacting with the teacher in her post narrative or interview, which may suggest that she was unaware of how her response to this trigger had changed. If this is the case, it means that, in the absence of observational data, this aspect of Rose's experience of MA was inaccessible, both to herself and to us as researchers. This implies that observational data can not only triangulate more traditional MA data sources, but can also provide unique and otherwise unattainable insights in studies of PSETs' MA.

Our exploratory case study was inevitably limited by the case size. While we gathered robust evidence of Rose's experience of MA that was both corroborated and extended by our observational data, more research is needed to determine if similar insights could be achieved with more reticent PSETs. Also, Rose mentioned that the professor's grading system (Revision-Reflection Grading), unlike traditional grading, enabled her effort to be reflected in her grade. Future research could investigate whether alternative grading systems can reduce PSETs' MA.

The use of survey, narrative, and interview data is well-established as a means to better understand PSETs' experiences with MA (e.g., Barrett, 2013; Finlayson, 2014). We found that there was a symbiotic relationship between these data sources and the observational data we collected. Survey, narrative, and interview data enabled us to establish the baseline necessary to collect meaningful observations of Rose's responses to MA triggers; in turn, the observational data enriched our understanding of Rose's MA transformation. Most notably, the observational data provided insights that were unattainable with survey, narrative, and interview data alone.

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EXPLORING PRESERVICE TEACHERS' FOCI AND STRATEGIES FOR ACKNOWLEDGING COMPETENCE IN A SCAFFOLDED COURSE ASSIGNMENT

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Acknowledging competence through the intentional identification and affirmation of students' mathematical strengths and contributions can foster positive math identities and advance equitable math instruction. Using an interpretive qualitative approach, we examine 68 elementary preservice teachers' (PSTs') attempts to acknowledge competence for four focal students in a scaffolded assignment in math methods courses. To characterize PSTs' efforts, we generated descriptive initial codes, and then refined and grouped codes into categories. Our findings can inform teacher educators' pedagogical approaches to cultivating PSTs' proficiency in equity-oriented teaching strategies like acknowledging competence.

Keywords: Preservice Teacher Education, Equity, Inclusion, and Diversity, Instructional Activities and Practices, Elementary School Education

There is a growing body of work focused on recognizing students' mathematical strengths and positioning students as competent to pursue equity in mathematics education (e.g., Iacono, 2018; Jilk, 2016; Johnson et al., 2022; Kalinec-Craig et al., 2021; Skinner et al., 2019). Within this body of work, scholars have identified equity-oriented teaching practices like *assigning* or *acknowledging competence* (Imm, 2022) in which teachers deliberately name students' mathematical strengths and contributions. Such practices aim to support students' development of positive mathematics identities, broaden their participation in math discourse, and deepen their mathematics learning (Aguirre et al., 2013; Featherstone et al., 2011). These practices also reflect visions of equitable math instruction emphasized by professional organizations (Association of Mathematics Teacher Educators, 2017; Huinker, 2020; Huinker & Bill, 2017; National Council of Teachers of Mathematics, 2014). To make tangible progress towards these visions of equitable math instruction, we must learn more about how preservice teachers (PSTs) take on and attempt equity-oriented practices. In this paper, we explore elementary PSTs' early efforts with the practice of acknowledging competence in a structured course assignment.

Research Focus

Building on previous work of our own (DeFino, 2022) and of others in the field (e.g., Jilk, 2016; Kalinec-Craig et al., 2021), we adapted a mathematics methods course assignment originally developed at the University of Michigan to scaffold PSTs' naming of students' mathematical strengths and contributions. For this assignment, PSTs watch a video clip of a math discussion, and then respond to a series of questions and prompts about four specific students' contributions. We were motivated by a shared concern that PSTs often interpret and respond to students' mathematical thinking as either right or wrong, dismissing answers and strategies that are not "correct," and reinforcing narrow and exclusionary notions of mathematical ability (Louie, 2017). Thus, through the assignment, we purposefully push PSTs to identify students' mathematical strengths and contributions beyond the right vs. wrong binary.

In this study, we examine PSTs' responses to this assignment, exploring what sorts of mathematical strengths and contributions PSTs' focus on and what statements they use to acknowledge students' competence. We investigate the following research questions:

1. What mathematical strengths or contributions do PSTs highlight in their draft statements to acknowledge each student's competence?
2. What strategies do PSTs use to acknowledge competence? (i.e., How do PSTs structure their acknowledging competence statements?)

Our aim is to better understand tendencies in how PSTs attempt to acknowledge competence for specific students and scenarios. This study will inform our own instructional designs and offer implications for other mathematics teacher educators working to develop PSTs' skills with equity-oriented instructional practices like acknowledging competence.

Theoretical Framing

This research focuses on the practice of acknowledging competence. Drawing on Cohen and Lotan's (1995) conceptualization of assigning competence, we define acknowledging competence as a deliberate act of naming and validating students' mathematical strengths and contributions. Following Imm (2022), we use the language of "acknowledging" rather than "assigning" to convey the stance that students already demonstrate mathematical competence in a multitude of ways, and it is teachers' responsibility to look for and recognize that competence.

Acknowledging competence involves conscious attention to how students are positioned through classroom interactions and the use of teacher moves and statements to intervene on patterns of exclusion (Chval et al., 2021; Davies & Harré, 1990; Featherstone et al., 2011; Louie, 2017). A key purpose is to position students as capable contributors to the learning process, supporting and encouraging students' participation (Johnson, 2017). For instance, a teacher might verbally emphasize a student's question in a class discussion, highlighting how that question is bringing the class's attention to an important mathematical idea (Johnson et al., 2022). Another key purpose of acknowledging competence is to broaden what students recognize as "smart" or important in mathematics (Featherstone et al., 2011; Jilk, 2016; Kalinec-Craig et al., 2021). This requires teachers to take a more inclusive and holistic approach to assessing and responding to students' mathematical contributions, rather than reinforcing the conventional emphasis on correctness or speed (Horn, 2007; Louie, 2017; Skinner et al., 2019).

Salient in recent research is the connection between acknowledging student competence and advancing equity in mathematics classrooms (e.g., Boaler & Staples, 2008; Johnson et al., 2022). Specifically, Hand (2012) argues that "it is only when teachers become disposed to *attend* differently to classroom mathematical activity that the field of mathematics education will provide a more even playing field for nondominant learners" (p. 235, emphasis in original). In other words, teachers must learn to actively *look for* and attend to students' competence to create more equitable and inclusive mathematics classrooms. To support PSTs in developing this disposition and skill, we pursued an instructional approach that allows PSTs to approximate (Grossman et al., 2009) the practice of acknowledging competence while initially stripping away some of the complexities of classroom teaching (e.g., drafting statements to acknowledge students' competence without the time pressure of responding in the moment). We envision that

PSTs will gradually incorporate acknowledging competence into their math teaching with additional practice opportunities, such as leading math discussions in field experiences.

Methods

This study utilizes an interpretive qualitative approach (Hesse-Biber & Leavy, 2011) to examine PSTs' draft statements to acknowledge specific students' competence in a structured course assignment.

Data Collection

Context. Data was collected from undergraduate math methods courses at two regional public universities in different parts of the United States, one in the upper Midwest and the other in the Southeast. Both institutions are predominately white, enroll many students from rural areas, and have elementary education classes largely composed of women. The first author teaches a math methods course designed to prepare PSTs to teach kindergarten through ninth grade. The second author teaches a math methods course designed to prepare PSTs to teach kindergarten through fifth grade. All three authors collaborated to design learning experiences focused on acknowledging competence. While course instruction and details varied by institution, both courses emphasized the importance of math identity for equity (Aguirre et al., 2013), worked on asset-based interpretations of student thinking, then introduced acknowledging competence as a strategy to affirm student's math identities and position students as capable.

Participants. The 68 participants in this study are PSTs who were enrolled in elementary math methods courses in a recent semester and consented to have their work analyzed for research. The majority, 55 participants, come from three course sections at the first author's university and the remaining nine PSTs are from one section at the second author's university.

Data sources. The PSTs' written responses on a focal course assignment were collected. This assignment is built around a video clip titled "Mamadou-Half-Rectangle" (Mathematics Teaching and Learning to Teach, University of Michigan, 2010). In this video, a class of rising fifth graders discuss what fraction of a larger rectangle a shaded area represents. The assignment identifies four students in the video (one of whom is Mamadou) and poses questions that guide PSTs to consider each student's verbal contributions from an asset-based perspective. Our analysis centers on PSTs' responses to the following prompt: *If you were the teacher, what could you say to acknowledge this student's mathematical competence?*

Data Analysis

A thematic analysis approach (Braun & Clarke, 2006) was used to identify categories and themes within the data. The PSTs' drafted acknowledging competence statements were coded along two dimensions: the focus [*what*] of the statement and the approach [*how*]. All three authors conducted initial coding (Saldaña, 2016) of ten participants' responses, then compiled and sorted codes into categories to generate a codebook. These codes were applied to a second set of ten responses to refine code definitions and categories. Each PST response could receive multiple codes. Using the revised codebook, the researchers are individually coding sets of ten responses and then meeting to discuss and come to a consensus. This process is still underway. Once complete, descriptive statistics will be tabulated, and overarching themes will be identified.

Preliminary Findings

Our collaborative coding process has resulted in a codebook that begins to answer our research questions, characterizing *what* PSTs focused on in their draft acknowledging competence statements and *how* PSTs went about attempting to acknowledge each student's competence. We have found that PSTs focused on a range of student strengths and contributions, some of which were more distinctly mathematical than others. Coding categories describing PSTs' foci consisted of the following: (a) a specific mathematical idea (e.g., recognizing the whole in the problem), (b) a specific process (e.g., explaining another student's thinking), (c) language (e.g., using the term "equal parts"), (d) a community contribution (e.g., "helping us understand"), (e) compliance with classroom norms or behaviors (e.g., listening, sharing), (f) correctness, (g) affect (e.g., eagerness to explain), or (h) being smart (e.g., "You're smart at math."). Some PST responses did not meet our definition of acknowledging competence. In those cases, we coded the focus of PST responses as "not applicable" because they included questions or *solely* consisted of general praise (e.g., "Great job"). Notably, some of PSTs' foci seem more likely to support broadened notions of mathematical competence than others. For instance, statements highlighting contributions students made to the community could help to portray mathematics as a collective undertaking rather than as an individual effort to arrive at correct answers (Featherstone et al., 2011). In contrast, statements highlighting more compliance-oriented behaviors like listening or paying attention seem likely to reinforce conventional notions of "doing school" (Goldin, 2010).

In characterizing *how* PSTs attempted to acknowledge each student's competence, we first categorized PST responses as being statements or questions. We then further categorized statements according to the "strategy" being used to acknowledge competence: thanking the student, specific praise, describing what the student did to the class, making a teacher-centric statement (e.g., "I like how you…"), framing the student's contribution as helpful to the community, affirming correctness (either explicitly or implicitly), or directly stating that the student is smart. Statements that did not meet our definition for acknowledging competence were categorized as orchestrating moves (e.g., "Let's pause to hear more about Mamadou's thinking") or hypothetical statements (i.e., describing the type of thing the PST *would* say without providing specifics). We did not consider questions as instances of acknowledging competence but still tracked the types of questions PSTs listed as eliciting moves (e.g., "Could you say more about what you're seeing as the whole?") or orienting moves (e.g., "Who can repeat Mamadou's thinking?"). Our thinking was that, though not acknowledging competence on their own, such questions could be used in ways that elevate and value student contributions. Additionally, we noted instances in which PSTs explicitly used student names in their responses with the rationale that accurately pronouncing and using students' names is one way to convey that students are seen and valued (Kohli & Solórzano, 2012). These coding categories and subcategories illustrate the variety of strategies that PSTs used in their efforts to acknowledge students' competence.

Interpretation and Implications

In our interpretation of these findings, we do not view any given focus or strategy as inherently good or bad; we maintain that much depends on the details of how acknowledging competence statements are delivered. For example, we recognize instances where explicitly affirming a student's correctness seems productive for highlighting student contributions that might otherwise be overlooked. At the same time, there are other instances where explicitly affirming a student's correctness could reinforce narrow and exclusionary notions of mathematical competence (e.g., emphasizing one "correct" way to solve). We do not think there is just one right or best way to acknowledge competence. Instead, we see these descriptive codes as offering a picture of what PSTs tend to do in their early efforts at acknowledging competence, which can inform our efforts to build on PSTs' productive inclinations and redirect PSTs away from counterproductive foci and strategies. Additionally, our codes offer a conceptual lens for other mathematics teacher educators to analyze and interpret their own PSTs' attempts to acknowledge competence. We envision a future of mathematics education in which learning to actively see and name students' strengths in ways that disrupt and challenge exclusionary notions of mathematical ability is an essential component of teacher preparation — a future in which acknowledging competence is as salient a practice as selecting cognitively demanding tasks or using and connecting mathematical representations. This study offers some preliminary conceptual tools for mathematics teacher educators to engage in serious and ongoing work on acknowledging competence.

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AN EXPLORATION OF THE RELATIONSHIP BETWEEN INSTRUCTIONAL TIME AND PERSEVERANCE GROWTH FOR ELEMENTARY PRE-SERVICE TEACHERS

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This in-progress study investigated how time elementary preservice teachers (PSTs) spent studying certain mathematics topics during a content course was related to growth in their perseverance in problem-solving. Using a quasi-experimental design, PSTs from two classes taught by the same instructor engaged in 12 problem-solving sessions each to measure their willingness to initiate and sustain, and re-initiate and re-sustain upon impasse, productive struggle during engagement. There were two conditions: over one semester, the treatment class studied 5 mathematical topics and the control class studied 10 mathematical topics. Preliminary results suggest that PSTs in the treatment class show greater perseverance growth over time compared to PSTs in the control class. This suggests that PSTs' perseverance development may be supported by spending more time studying fewer topics during mathematics content courses.

Keywords: Preservice Teacher Education, Problem Solving, Elementary School Education

Rationale, Background, and Theoretical Perspectives

Mathematics education researchers continue to look for best practices by which to structure elementary teacher education programs, yet consensus continues to elude the field (Garner et al., 2023; Masingila & Olanoff, 2022). Professional organizations like the Association of Mathematics Teacher Educators have published standards for preparing teachers of mathematics (AMTE, 2017), which include coursework recommendations and emphasis on developing future teachers' knowledge of mathematics concepts and dispositional practices. Teachers' mathematical knowledge for teaching directly impacts the quality of instruction their students experience (Hill et al., 2008), thus careful coordination over what elementary pre-service teachers (PSTs) have opportunities to learn in their preparatory mathematics coursework is of critical importance to their development as effective teachers. Yet still, without consensus, PSTs across North America have experienced great variance in the mathematics content and practices they study and for how long they study it (e.g., Hrusa et al., 2020; Malzahn, 2020; NCEE, 2016), which jeopardizes the effectiveness of teacher preparation. In fact, a PME-NA working group, *Mathematics Curriculum Recommendations for Elementary Teacher Preparation*, is devoted to such issues. Aligned to the theme of PME-NA 46, the future of elementary mathematics teacher preparation depends on new research efforts amidst such uncertainty across PST education.

At many institutions, PSTs engage in survey courses aiming to cover the complete spectrum of elementary mathematics content topics (An et al., 2021). These courses are often developed from a knowledge-oriented theoretical perspective, with an emphasis on teaching all the mathematical topics PSTs need to know to teach effectively (Li & Howe, 2021). At other institutions, certain elementary mathematics content topics are purposely omitted to focus more time on high-leverage topics and practices, such as number concepts and problem-solving (Chapin et al., 2021). These courses are often developed from a thinking-oriented theoretical perspective, with an emphasis PSTs learning to reason about, explain, and make sense of the mathematics they will teach (Li & Howe, 2021). Although such variance in coursework still exists, recent research has shown the long-term benefits of spending more instructional time on a smaller set of mathematical topics during teacher preparation (Beckmann & Izsak, 2021; Corven et al., 2022; Gibbons et al., 2018). For instance, Corven and colleagues (2022) examined the relationship between the number of instructional minutes dedicated to mathematics topics in teacher preparation and the specialized content knowledge (SCK, Ball et al., 2008) demonstrated by program graduates. Analyses showed that over 400 minutes of high-quality instruction on one mathematics topic were needed to develop the SCK to teach it well years later. This finding suggests that survey courses may not be effective in fostering lasting knowledge and application.

Although such research is promising, more research is necessary to investigate the affordances and constraints related to the instructional time spent (or not spent) on specific mathematics topics included in content courses for PSTs. Existing research has shown how PSTs develop and retain mathematical understandings by devoting more time studying less topics, but we know very little about other outcome measures of PSTs, including outcomes related to the development of mathematical practices like perseverance in problem-solving. Perseverance, or initiating and sustaining productive struggle in the face of obstacles (DiNapoli, 2023), promotes making sense of mathematics (Middleton et al., 2015; Warshauer, 2014). Students make meaning through productive struggle, or as they grapple with mathematical ideas that are within reach, but not yet well formed; it is imperative for teachers to create learning environments for their students that promote such productive struggles (Hiebert & Grouws, 2007). Recent studies have shown that perseverance in problem-solving can be malleable in students and nurtured, depending on the learning environment, to grow and improve over time (DiNapoli & Miller, 2022; Paurowksi et al., 2024). Learning environments that provided consistent opportunities to productively struggle with mathematics content, and thus, opportunities for students to deliberately practice their perseverance have shown promise. In PST education, PSTs must have consistent opportunities to persevere and develop a disposition and willingness to engage in productive struggle during their mathematics content coursework to be able to empathize with and support their future students to productively struggle to learn mathematics (AMTE, 2017). As such, the research question that guides this in-progress study is: *What is the relationship between mathematics instructional time and perseverance growth for elementary PSTs in a content course?*

Context and Methodology

This in-progress, mixed-methods study used a quasi-experimental design (Patten, 2016) to help describe a relationship between instructional time and perseverance growth for elementary PSTs in a content course. I collected and analyzed data from two distinct groups of participants at a public university in the northeast United States: one Fall 2023 section (complete) and one Spring 2024 section (in-progress) of a terminal Mathematics Content for Elementary Teachers II course. There were two class conditions: a treatment condition (the Fall 2023 section) and a control condition (the Spring 2024 section). There were 30 PSTs in each section, resulting in 60 total participants. I was the sole instructor for each class.

Treatment and Control Conditions

In the treatment condition, PSTs engaged with 5 mathematics topics during one semester, averaging about 400 minutes of classroom time devoted to each topic. The treatment topics were

Conceptions of Fractions, Addition of Fractions, Subtraction of Fractions, Multiplication of Fractions, and Division of Fractions. In the control condition, PSTs engaged with 10 mathematics topics during one semester, averaging about 200 minutes of classroom time devoted to each topic. The control topics were the 5 treatment topics, plus Percentages, Ratios and Proportions, Polygons, Angles, and Area. In both conditions, I taught each lesson following the same style lesson plans which emphasized conceptual learning opportunities.

Data Collection

Each PST in each condition engaged in 12 virtual problem-solving sessions, approximately one per week. In these non-graded sessions, I presented PSTs with a challenging task as part of their individual homework related to that week's lesson (e.g., *If 100 stars represent* $\frac{6\frac{2}{3}}{5}$*, how many stars represent 1 whole?).* PSTs video-recorded themselves thinking aloud as they worked toward a solution. The tasks were designed to evoke productive struggle, and if they did not, PSTs were given a different task that did. If PSTs ever reached a perceived impasse (i.e., they were substantially stuck (VanLehn et al., 2003)), they were instructed to say so. PSTs could stop working on a task at any time. Also, each PST engaged with written stimulated recall prompts (Ericsson & Simon, 1993) about specific moments during each of their problem-solving sessions. Responses to these prompts helped reveal and explain any in-the-moment cognitive and emotional activity that PSTs experienced while working on a task, especially around moments of perceived impasse. In some cases, I followed up with participants even further to gain clarifications about specific moments during problem-solving sessions.

Data Analysis

To analyze each PST's problem-solving session, I used the Three-Phase Perseverance Framework (3PP) (see Figure 1) (see DiNapoli & Miller, 2022). The 3PP has been used in several empirical studies (e.g., DiNapoli, 2019; DiNapoli et al., 2021; DiNapoli & Miller, 2022; DiNapoli & Morales, Jr., 2021) to measure perseverance in problem-solving, or the extent to which students initiated and sustained, and re-initiated and re-sustained upon impasse, productive struggle on a challenging task. The 3PP operationalizes perseverance by considering the ways in which a student makes an initial attempt at solving a problem for which they do not already know a solution pathway, and makes an additional attempt at solving the problem if their initial attempt was unsuccessful and led to a perceived impasse. Theoretically, a student could engage in multiple additional attempts as they encounter multiple impasses. In my analysis, I only considered students' experiences around one perceived impasse during the problem-solving.

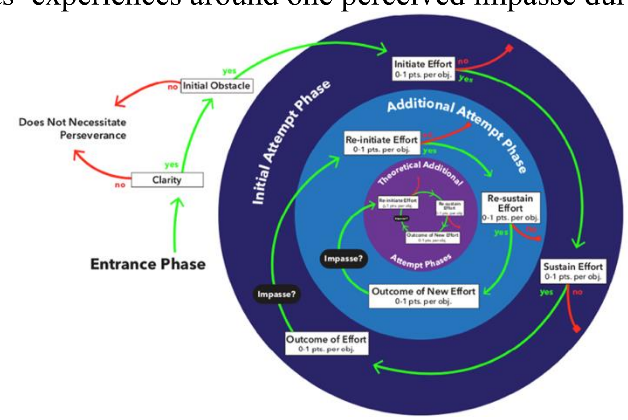

Figure 1: The Three-Phase Perseverance Framework

I used the 3PP to capture if and how a PST initiated (0-1 point) and sustained (0-1 point) efforts toward a solution before an impasse, and if and how those efforts were mathematically productive (0-1 point). After an impasse, I also captured if and how a PST re-initiated (0-1 point) and re-sustained (0-1 point) their efforts toward a solution, and if and how those new efforts were mathematically productive (0-1 point). I relied on PSTs' think-alouds, their written work, and their stimulated recall responses to make scoring decisions. Thus, PSTs could earn 0-6 3PP points per problem-solving session, with 0 indicating no evidence of perseverance and 6 indicating ample evidence of perseverance (i.e., a PST could demonstrate ample perseverance while working with a task through building incremental understanding via effort, yet not completely solve the task). When considering PSTs' improvement, a gain of just one 3PP point is substantial since it could represent perseverance growth in several ways, such as: the difference between not engaging at all vs. initiating some effort (0 points vs. 1 point); initiating some effort but then giving up vs. sustaining that effort (1 point vs. 2 points); sustaining an effort but not making mathematical progress vs. actually making mathematical progress based on that sustained effort (2 points vs. 3 points); engaging in a successful first attempt but giving up upon impasse vs. re-initiating a second attempt after impasse (3 points vs. 4 points); and so on.

Each PST engaged in 12 problem-solving sessions, so each PST earned 12 3PP scores. To help control for pre-existing differences amongst PSTs, my analysis focused on each PST's personal perseverance growth. I used linear regression to model each PST's perseverance growth across their 12 problem-solving sessions. I also used linear models to represent PSTs' perseverance growth per class condition, that is, for PSTs in the treatment and control conditions. My findings focus most on the slopes of these linear models, which represent the average increase of PSTs' 3PP scores per problem-solving session, per class condition. Qualitative analyses are still ongoing.

Preliminary Results and Discussion

PSTs in the treatment condition increased their perseverance by an average of 0.329 3PP points per problem-solving session. This suggests that it took, on average, approximately 3.0 problem-solving sessions for PSTs in the treatment condition to improve their perseverance in problem-solving by one 3PP point. In contrast, PSTs in the control condition increased their perseverance by an average of 0.163 3PP points per problem-solving session. At this rate, this suggests that it will take, on average, approximately 6.1 problem-solving sessions for PSTs in the control condition to improve their perseverance in problem-solving by one 3PP point. Thus, PSTs who spent more instructional time on less topics improved their perseverance growth by more than double the rate of their PST peers who spent less instructional time on more topics.

At this time, it is inappropriate to make any formal claims about the relationship between instructional time and PSTs' perseverance growth since qualitative analyses are still ongoing. These analyses will help reveal some reasons behind why perseverance growth was so different between class conditions. However, these preliminary results might suggest that more time spent on less mathematics topics during PST content coursework can influence the development of PSTs' perseverance in problem-solving, compared to spending less time on more topics. These preliminary findings about developing such mathematical practices in PSTs, alongside existent research about PSTs' development and retention of mathematical understandings under similar conditions (Beckmann & Izsak, 2021; Corven et al., 2022; Gibbons et al., 2018), would strengthen the practical argument for devoting more time on less topics in content courses, and present more comprehensive evidence against the efficacy of survey courses. Furthermore, these findings would emphasize the need for thinking-oriented theoretical perspectives (Li & Howe, 2021) in elementary mathematics teacher preparation. Addressing the theme of PME-NA 46, this could help the field of mathematics education work toward a more certain future of elementary mathematics teacher preparation.

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COUNTING ON JUSTICE: ADDING SOCIAL IMPACT TO ELEMENTARY PRE-SERVICE TEACHER MATHEMATICS EDUCATION

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This brief research report advocates integrating social justice into mathematics education for elementary pre-service teachers (EPSTs). It highlights the importance of this integration in addressing mathematics anxiety and improving attitudes toward mathematics. The report also discusses how negative attitudes and mathematics anxiety can hinder PSTs' ability to teach mathematics effectively, stressing the necessity to incorporate mathematics applications into education method courses. Furthermore, it underscores the relevance of integrating social justice topics into the pre-service teacher mathematics curricula to promote critical thinking and humanization, thereby aiding in addressing educational disparities and promoting confidence in teaching numerical skills. This approach can potentially transform PSTs' attitudes toward mathematics, alleviate mathematics anxiety, and enhance their teaching practices in elementary classrooms.

Keywords: Preservice Teacher Education; Social Justice; Instructional Activities and Practices; Instructional Vision

Negative attitudes toward mathematics and mathematics anxiety can significantly impact mathematics learning by elementary pre-service teachers (EPSTs) (Gonzalez-DeHass et al., 2017). Mathematics anxiety, an adverse physiological reaction to working with mathematics (Luttenberger et al., 2018), is particularly prevalent among EPSTs, who also have the lowest mathematics proficiencies (Novak & Tassell, 2017). However, mathematics methods courses that teach PSTs about mathematics applications have been shown to help reduce mathematics anxiety (Tooke & Lindstrom, 1998). These courses can also integrate other subjects, such as science, social studies, and reading, into mathematics methods courses to positively influence PSTs' attitudes toward mathematics (Bursal & Paznokas, 2006). By challenging PSTs to engage in tasks involving cooperation, problem-solving, personal mastery of mathematical skills, and communication of mathematical facts and relevance (Gonzalez-DeHass et al., 2017), these courses can significantly improve PSTs' teaching abilities and confidence in teaching numerical skills.

Integrating social justice topics into all curricula is relevant since social justice teacher education aims to prepare PSTs to "recognize, name, and combat inequity in schools and society" (Spalding, 2013, p. 284). Social justice education involves critical thinking, humanizing space (Freire, 2003), and creating a space where potentially controversial issues can be discussed and analyzed (Kumashiro, 2000). Mathematics can and should be the tool used to analyze social injustices in a complex world (Greenstein & Russo, 2019). As Gutstein (2006) states:

> [T]he idea of liberation from oppression as the fundamental purpose of teaching mathematics […] Teaching mathematics for social justice flows from the broader notion of liberatory education and has two sets of pedagogical goals; one focused on social justice and the other on mathematics. (pp. 22-23)

While mathematics is thought of as either divorced from the real world (Gutstein & Peterson, 2006), masculine (Rubel, 2016), a disingenuous cognitive domain (Rands, 2013), isolated (Hernandez-Martinez & Vos, 2017) or as a set of rote memorizations (Peterson, 2006), EPSTs should see mathematics as a method to increase student interest in social justice advocacy (Stocker, 2005). In doing so, there could be an increase in PST engagement in mathematical content and methods, thus aiding their understanding of mathematics and possibly lowering their mathematics anxiety.

Incorporating social justice topics into mathematics methods courses for EPSTs not only addresses educational disparities (Fasheh, 1982) but can also aid in reducing mathematics anxiety in EPSTs due to its integrative nature (Bursal & Paznokas, 2006). These courses can offer a more relatable and inclusive approach by emphasizing real-world applications and diverse perspectives on mathematical concepts (Yeh & Otis, 2019), helping future educators build confidence and effectively support students with varied backgrounds (Clark & Brown, 2016). Our research question was: How could learning about social justice mathematics transform PSTs' attitudes and knowledge in teaching mathematics?

Theoretical Framework

The theoretical framework for this study centers around teaching social justice mathematics (Bartell, 2013; Gutstein, 2006) to pre-service teachers in an elementary mathematics methods course. As research has shown (Bursal & Paznokas, 2006; Gonzalez-DeHass et al., 2017; Novak & Tassell, 2017; Tooke & Lindstrom, 1998), EPSTs experience high levels of mathematics anxiety and low enthusiasm for the subject. This, in turn, impacts their students' mathematics learning (Stoehr et al., 2017). Social justice mathematics education allows students of all ages to engage in social justice advocacy while using patterns and relationships taught in their courses instead of a more traditional manner, which mostly focuses on "memorization, regurgitation, standardized testing and reams upon reams of mathematics problems whose content is immaterial" (Stocker, 2005, p. 48). Social justice in mathematics could be particularly relevant in the context of EPSTs who may have their own experiences of mathematics anxiety and may be more likely to pass these anxieties on to their students if they do not see the relevance of mathematics in their lives.

This theoretical framework might also draw on research on reducing mathematics anxiety in learning by promoting a supportive and encouraging educational environment, incorporating real-world examples and applications of mathematics concepts, and addressing the root causes of mathematics anxiety through self-reflection and mindfulness techniques. We also draw on the humanizing framework to "move beyond individual components of mathematical belief (e.g. self-efficacy, confidence, mindset, views of what it means to do mathematics, and others) and challenges us to consider each EPST as a collection of all of these complex views and beliefs" (Skultety et al., 2023, p. 2). Overall, social justice education in mathematics, in the context of this paper, focuses on promoting social justice and equity in mathematics methods courses for EPSTs to equip them with a method that can confront their mathematics anxiety and ultimately improve their attitudes, knowledge, and effective mathematics teaching practices.

Nixing Mathematical Nerves

Mathematics appears to be a subject that elicits either a strong affinity or a strong aversion among individuals (Çetinkaya et al., 2018). This often stems from mathematics being a classroom subject heavily focused on achievement assessment, and the fear of failure often traps individuals in a negative attitude toward mathematics (Çetinkaya et al., 2018). Alkan, Coşguner and Fidan's (2019) study examines how negative attitudes and perceptions of mathematics can impact individuals who go on to teach mathematics. The study indicates that there has been research on how PSTs feel about mathematics and its influence on how PSTs approach teaching the subject. Stoehr, Carter and Sugimoto (2017) also found that PSTs with mathematics anxiety need a mathematics course that connects mathematical content knowledge with teaching knowledge. Therefore, a mathematics methods course which focuses on relieving PSTs' mathematics anxiety should focus on raising confidence in their mathematics content knowledge by focusing on the application (Tooke & Lindstrom, 1998) of mathematical concepts; we contend that social justice mathematics is a way to do so.

Many mathematics educators face the infamous question: "Why do I have to learn this?" (Hernandez-Martinez & Vos, 2017). When mathematics teachers encounter this problem and do not know the reason for teaching mathematical concepts, they may unintentionally perpetuate a negative perception of mathematics. As a result, PSTs should know how to integrate mathematics into various other disciplines, such as social studies or history (Peterson, 2006), emphasizing its relevance to our social world and social justice issues. Changing the views on mathematics for PSTs could lower their mathematics anxiety and directly influence how they teach mathematics to other generations of students. Additionally, this could demonstrate to PSTs and their future students how practical mathematics is in everyday life, making it more approachable, relevant, and engaging (Hernandez-Martinez & Vos, 2017).

Integrating social justice into mathematics methods courses should be more commonly practiced in teacher education programs. Despite common misconceptions, mathematics and social justice education can exist in a harmonized pedagogy (Bond & Chernoff, 2015). If educators of EPSTs deliver this integration of social justice into mathematics to these PSTs, it could impact future generations of teachers and students alike. A shift in the attitude towards mathematics courses for EPSTs has the potential to bring about substantial improvements in the delivery of mathematics education within our system. Teachers and students could realize that mathematics can be used to examine social issues more profoundly and further contextualize their mathematics lessons. This can potentially lead to the alleviation of anxiety experienced by PSTs within mathematics and the likelihood of increased confidence in the teaching of the subject.

Calculating a Confidence Boost

Integrating social justice into mathematics, as proposed by Gutstein (2006), has the potential to alleviate mathematics anxiety and enhance confidence among PSTs, fostering meaningful connections through mathematics. When EPSTs learn to incorporate current social injustices into their mathematics curricula, they may become better equipped to create a more relevant and engaging educational experience for themselves and their students. Establishing connections in mathematics can strengthen the mathematical identity of PSTs and promote a sense of belonging among their students within the mathematical community (Skultety et al., 2023). This approach may facilitate a deeper understanding of mathematical concepts while addressing societal issues pertinent to the students' realities.

The contemporary world presents numerous opportunities for instructing PSTs on analyzing and comprehending social justice issues through mathematics, thereby equipping them to better facilitate their students' mathematical education. Despite the pervasive presence of numerical data in daily life, its relevance may not always be immediately evident (Gutstein, 2006). By teaching PSTs to apply critical thinking skills in mathematics to navigate our society, they may gain greater confidence in addressing complex societal issues and making more informed decisions (Yeh & Otis, 2019). This can be exemplified using news articles on current social issues, such as the increase in the number of gender-based hate crimes in the United States of America (Pollard, 2024), violence rates against Indigenous Peoples in Canada (Lee, 2023), the cost of childbirth in the United States (Rivelli, 2024), environmental racism (DeLaire, 2023), and more. Employing real-world events to establish meaningful connections to mathematics highlights how PSTs can integrate cross-curricular subjects to create impactful experiences for future students.

Equipping EPSTs with the knowledge to integrate relevant global events into their mathematics classes could enhance PSTs' engagement in the subject. By providing a medium for delivering mathematics instruction with greater impact, PSTs may feel more prepared to teach future mathematics lessons, thereby increasing student involvement. This approach ultimately empowers EPSTs, instilling confidence and competence as they enter classrooms where they will be responsible for mathematics education. Consequently, PSTs may be more inclined to embrace the challenges of teaching mathematics in ever-evolving and uncertain global contexts and moments.

Considerations for Social Justice Integration in Mathematics Education for PSTs

Amalgamating social justice into mathematics education for PSTs holds significant potential for reducing mathematics anxiety and improving attitudes toward the subject. Research indicates that PSTs often struggle with mathematics anxiety and negative attitudes toward mathematics, which can impair their teaching efficacy. By incorporating social justice topics into mathematics methods courses, PSTs can appreciate the relevance of mathematics in real-world contexts and engage in critical thinking about social injustices. This approach may address educational disparities and enhance PSTs' understanding of mathematical concepts and their societal significance.

PSTs need to recognize that changes in mathematics teaching pedagogy do not occur instantaneously. For instance, PSTs can begin by integrating activities such as analyzing the number of books authored by People of Colour versus those by White authors. Depending on the students' grade levels, PSTs could analyze the data they discover using various mathematical methods. Integrating social justice into mathematics education could be an important pedagogical strategy, empowering PSTs to become more confident and effective mathematics educators. By equipping PSTs with the tools to analyze social issues through mathematics, they can create more meaningful and engaging experiences for themselves and their future students. Overall, integrating social justice into mathematics education can transform attitudes toward mathematics, reduce mathematics anxiety, and enhance the teaching practices of PSTs in elementary classrooms.

Potential Pushbacks and Limitations

Acknowledging the potential pushbacks and limitations to integrating social justice education into EPST mathematics methods courses is essential. While this incorporation can offer significant benefits, some educators and stakeholders may resist due to differing views on the role of social issues in mathematics instruction and varying values regarding specific social justice topics addressed in the course. Additionally, the differing levels of preparedness and comfort among PSTs in handling sensitive topics can present challenges. Furthermore, limited resources and support from the administration and other educators may hinder the effectiveness and sustainability of this curriculum in PST education programs.

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PRESERVICE TEACHERS' REPLICATIONS OF A COMMON FRACTION ERROR

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Keywords: Teacher Knowledge; Number Concepts and Operations; Rational Numbers.

There are numerous assessments of preservice teachers (PSTs) content knowledge of fractions (Erdem, 2016; Huang et al., 2009; Izsàk et al., 2019). Yet, most do not adequately distinguish between common content knowledge and the specialized content knowledge (SCK) needed to interpret children's work with fractions (Copur-Gencturk et al., 2019). One way to examine SCK for fractions is to study teachers' engagement with common fraction errors. These common errors are well-documented, with students are numerous such errors documented over the past 100 years (Behr et al., 1984; Brueckner, 1928; Schumacker & Malone, 2017), but there are few examinations of how teachers' interpretations of them. We selected four well-documented common errors in fraction arithmetic and assigned each randomly to 95 PSTs enrolled in their first mathematics content for a teaching course. This paper reports on a preliminary analysis of one of these common errors completed by 24 PSTs: using proportional addition instead of multiplication to convert fractions for addition/subtraction. When children demonstrate this error, they add the same number to the numerator and denominator (proportional addition) in an attempt to reach a common denominator between fractions (see center example in Figure 1). PSTs were provided two illustrative examples of the common error and then completed a 10-item with the option to use a calculator.

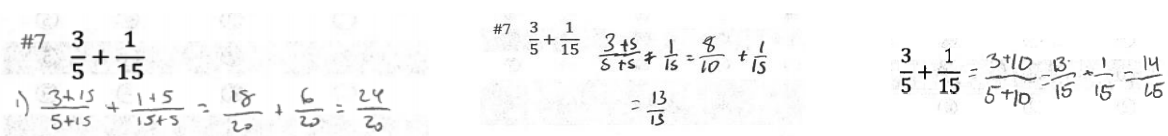

Figure 1: Some PSTs' Efforts in Replicating the Common Error

Independently, the authors coded for the use of addition or multiplication and whether it was implemented proportionally (K=.50) before reconciling codes (see Table 1). PSTs replicated proportional addition about half the time (50.43%), while the 'correct' approach with proportional multiplication was used at 9.57%. PSTs used both addition and multiplication in other ways that deviated from the common error. These attempted replications suggest that PSTs may have wrestled with how a child demonstrating the common error uses addition and does so proportionally (for numerator & denominator). Although PSTs successfully replicated the error half the time, they tended to be quite inconsistent in doing so. Relatively few PSTs actually replicated the common error correctly across all 10 tasks. These PSTs were observed to be able to solve fraction addition/subtraction tasks without difficulty. Thus, results suggest SCK is not a given even if PSTs have the requisite content knowledge. SCK tasks such as this may provide a useful means for assessing and facilitating PSTs' professional knowledge for teaching fractions.

Table 1: Descriptive Statistics for PSTs' Attempts to Replicate the Common Error

Unproportional		Proportional			Mixed
Add for CD	Mult for CD	Add for non-CD	Add for CD	Mult for CD	
10.00%	2.61%	9.13%	50.43%	9.57%	3.91%

Note: CD stands for Common Denominator. Addition & Multiplication are also abbreviated.

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“MICHELLE COMES OUT OF LEFT FIELD”: CLINICAL SIMULATIONS AND PROSPECTIVE TEACHERS’ KNOWLEDGE OF STUDENT THINKING

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To support prospective teachers (PTs) in developing knowledge of varied ways that students generalize from sequences of figures, we designed three theory-driven sets of protocols and training materials for orchestrating cycles of live, clinical simulations of student thinking about linear and quadratic generalizing tasks. We designed the cycles according to principles derived from Variation Theory (Marton, 2014): create opportunities for PTs to experience contrasting approaches within each cycle, and opportunities to construct general understandings of each approach by repeated experience across cycles. We present several case studies documenting how PTs’ knowledge, as represented by the approaches they anticipated prior to each set of simulations, evolved across simulation cycles in way that were consistent with the predictions of Variation Theory.

Keywords: Mathematical Knowledge for Teaching; Pre-Service Teacher Education; Early Algebra, Algebraic Thinking, and Function

The Problem: Preservice Teachers’ Knowledge Base for Enacting Core Practices

The National Council of Teachers of Mathematics (2014) recommends eight teaching practices as core to effective mathematics teaching. Among those are the practices of implementing tasks that promote reasoning and problem solving, posing purposeful questions, supporting productive struggle, and eliciting and using evidence of student thinking. According to the Association of Mathematics Teacher Educators (2017), the capacity to enact those practices is an essential component of being a well-prepared novice teacher.

However, the capacity to enact such practices requires Mathematical Knowledge for Teaching (Hill et al., 2004), including knowledge about how students might approach a particular type of mathematical task and errors or misconceptions that might crop up as students engage in problem solving. According to the tenets of Situated Cognition (Brown et al., 1989), the knowledge that a teacher invokes when supporting students’ productive struggle during problem solving is the knowledge that the teacher has developed from prior experience in similar kinds of situations. The problem is that traditional models of teacher education relegate the experiential component of teacher learning to field placements. Grossman et al. (2009) suggested that separating clinical experience from coursework is problematic and called for teacher educators to make concerted efforts to create better links between the experiences of prospective teachers in field placement and the knowledge gained in their education courses. However, field placements are unreliable as sites in which prospective teachers can implement task that promote reasoning or problem solving – there are multiple types of influences that act to discourage teachers from creating space for reasoning or problem solving in their classrooms (Serrano Corkin et al., 2019). Therefore, in addition to creating stronger connections between field placements and experiential learning opportunities, there is a need for teacher educators to supplement field placements by orchestrating experiences for PTs through which they have opportunities to develop the kinds of knowledge needed to enact the core practices identified by NCTM and others.

We have developed an intervention, theoretically grounded by Situated Cognition (Brown et al., 1989) and Variation Theory (Marton, 2014), for engaging PTs in clinical simulations that we designed to support the development of their Knowledge of Content and Students (KCS) (Ball et al., 2008) for anticipating multiple ways that students might engage in figural generalizing tasks. In this report, we extend from our previous brief report of a single case study (Graysay & Bermudez, 2023) to multiple case studies examining the following research question:

What do PTs learn, by participating in theoretically grounded clinical simulations of student-teacher conversations about student work, about students' approaches to generalizing from figural data?

Theoretical Framing and Simulation Design

Our designs are grounded in a theory of preservice teacher learning that intersects Situated Cognition with Variation Theory. From the perspective of Situated Cognition, to develop knowledge of content and students that will be used in teaching PTs need experiences of engaging in interactions with students in situations similar to those of the interactive work of teaching. From that guiding principle we modeled our intervention after the clinical simulations of Dotger and colleagues (e.g., Dotger, 2010) in which participants converse in a live, interactive setting with an actor trained to present specific statements or actions so that the conversation simulates a particular problem of practice. Because our goal is to support the development of KCS for supporting student reasoning, we chose to simulate students' approaches to generalizing from figural data for two reasons. First, generalizing is an essential mathematical process (Mason, 1996). Second, within the body of research on generalizing, Rivera and Rossi Becker (2008) analyzed students' approaches to generalizing from figural data and described three qualitatively distinct approaches. *Numerical* approaches are those in which the learner, after quantifying an aspect of each figure, proceeds to work with the numerical values without revisiting, drawing, or envisioning additional figures, and without attending to the structure of the figures. In *constructive figural* approaches the learner attends to the structure of each figure as constituted by disjoint components. In *deconstructive figural* approaches the learner attends to the structure of each figure as composed of intersecting components or as the result of the removal of elements from an imagined figure. The latter approach poses a particular challenge for teachers: El Mouhayar and Jurdak (2013) found that preservice and inservice middle grades teachers have difficulty explaining the reasoning behind deconstructive figural approaches.

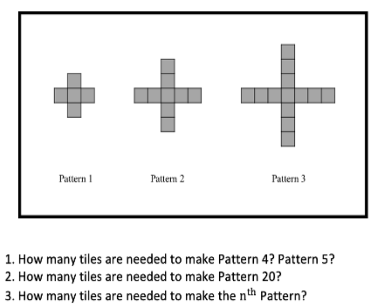
According to Variation Theory (Marton, 2014) learning requires an experience of contrast, across examples, of the defining features of an intended object of learning while aspects of the examples that are unrelated to the defining features should be held constant. Once the learner has experienced contrast, they should then experience variation in non-defining aspects of examples with the defining features of the intended object held constant, to support generalizing across examples toward a context-independent concept. Based on Variation Theory we designed protocols to train actors with statements and gestures to simulate the three contrasting student approaches documented by Rivera and Rossi Becker (see Table 1) in three cycles, each focused on a distinct task (see Figure 1), leading to nine protocols in total (see Figure 2).

Table 1: Dimension of Approach

Numerical Approach (Nell)	Constructive figural (Suzy)	Deconstructive figural (Michelle)
General Approach: Makes a table of values and looks for patterns in numerical values *Example:* In the Tiling Task (see Figure 1), Nell counts the number of tiles in each Pattern, records the values in a table, notes a pattern of increase in numerals in the table, and extends the table iteratively to find totals for Pattern 4, Pattern 5, and Pattern 20.	*General Approach* Perceives patterns as constructed from disjoint components *Example:* In the Tables and Chairs Task (see Figure 1), Suzy perceives each Pattern as two rows of chairs across the top and bottom of the set of Tables, with one chair at each end of the row of Tables.	*General Approach*: Deconstructs patterns into intersecting components Example: In the Theater Seats Task (see Figure 1), Michelle perceives the figure as a large rectangular array of seats with two triangular arrays removed from each side.

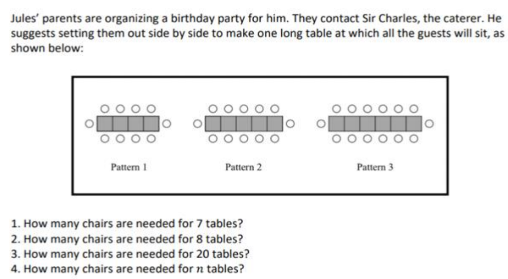
Figure 1. Figural Generalizing Tasks

Tiling Task



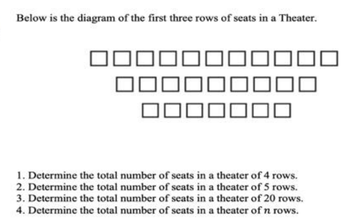
Adapted from Rivera (2007)

Tables and Chairs Task



Adapted from DeMonty et al. (2018)

Theater Seats Task



Adapted from Alajmi (2015)

Table 2. Intersections of Approach and Task.

	Approaches		
	Numerical	***Constructive Figural***	***Deconstructive Figural***
Tiling Cycle	Nell, Tiling	Suzy, Tiling	Michelle, Tiling
Chairs Cycle	Nell, Tables and Chairs	Suzy, Tables and Chairs	Michelle, Tables and Chairs
Seats Cycle	Nell, Theater Seats	Suzy, Theater Seats	Michelle, Theater Seats

Data Collection and Analysis

We recruited seven prospective secondary mathematics teachers (PTs) for this research and assigned pseudonyms. Blake, Denise, and Taylor were each in the seventh semester of their undergraduate teacher education program. Ezra, Kristy, Max, and Sam were graduate students in a parallel teacher education program. All seven were enrolled in a joint math methods course that met once weekly over fourteen weeks. We implemented our simulations within three cycles of activity. The first cycle (the Tiling cycle) began with a *case analysis* phase in which participants analyzed a narrative case of a teacher working with students on a figural generalizing task similar to -- but distinct from -- the Tiling task. In the *anticipating* phase PTs solved the Tiling task and anticipated how students might solve the task. In the *simulation* phase PTs each engaged in three interactive clinical simulations. In each simulation a single PT met one-on-one with an actor who was each trained to simulate exactly one of the three designated approaches to solving the Tiling task. The cycle ended with a *reflection* phase in which PTs reflected on the simulation experience, focusing on contrasting the simulated student approaches. We then iterated those phases for the Chairs cycle and the Seats cycle.

We hypothesized, per Variation Theory, that PTs would notice differences in student approaches within each set of simulations and begin to develop more general, abstract understandings of each approach across sets of simulations. To test our hypothesis we reviewed transcripts of PTs' statements during the anticipating and reflecting phases of our cycles. We posited that PTs' knowledge of the range of student approaches to generalizing from figural data is evidenced by how each PT approaches generalizing and by the approaches that they anticipate others might use. Therefore, we classified each PT's approach to solving each task and the approaches that PTs anticipated they would encounter in the simulation phase of each cycle, using the categories from Rivera and Rossi Becker (2008). We noted when and how PTs referenced Nell, Suzy, or Michelle when describing an anticipated approach. We compared across cycles to examine whether and how their anticipations of student approaches and their associations of those approaches to specific simulated students evolved across their experiences.

Findings

Graysay and Bermudez (2023) reported tentative findings suggesting that Kristy's understanding of students' approaches evolved from anticipating constructive figural and numerical approaches; to anticipating that the Tables and Chairs simulations would involve three approaches (numerical, constructive figural, and a third unspecified approach); to explicitly anticipating each approach for the Theater Seats simulations based on the names of the simulated students. Though Kristy recognized that Michelle's approach would be different from the other two, Kristy explicitly expressed difficulty anticipating what Michelle's approach would be. In this report we present additional cases to suggest that other PTs in the cohort demonstrated a similar trajectory, including difficulties anticipating Michelle's approach.

The Case of Max

In the anticipating phase of the Tiling Cycle, Max worked with Ezra to share their individual solutions to the Tiling task. Ezra described a numerical approach, and Max described a constructive figural approach:

> Ezra: For pattern four, I did five plus four times four minus one, which got me 17. Pattern five, five plus four times five minus 1. Got 21.
>
> Max: That's an interesting way of looking at it, because I just was looking at like, "Okay, so there's four branches, and then the one in the middle."

In the anticipating phase of the Tables and Chairs cycle, Max asked whether the characters would shift approaches or take similar approaches in the second set of simulations compared to the first set. We take this as evidence that Max had experienced the intended contrast across the three simulations in the Tiling cycle:

Max: And we should expect them to be like similar like, like, we should expect Suzy to work out the problem in a similar way . . . in, like, a Suzy way? . . . She's not going to all of a sudden do it the Nell way, right?
DG: She's not going to do it in a way that anybody other than Suzy would. The Suzy character has a way of doing things. . . . There is a consistent through line.

Max and Kristy anticipated two ways of solving the Tables and Chairs task. Max and Kristy described a constructive figural approach (adding front, back, and then adding sides), though Max had used a more numerical method:

Kristy: What are all the ways this task can be solved? I would say I would say call our method the -- Call this method, I would call it the --
Max: Maybe adding front back and then adding sides.
Kristy: Yeah. Adding front, back and sides. Only looking at the numbers. Looking at the ends as three and the middle as groups of two. And how did you think of it, like?
Max: I looked at— I just was looking at the number of tables.
Kristy: You were just looking at the number.
Max: I just counted first. I guess that's a way -- counting.
Kristy: Yeah. Honestly. Which of these methods students would use? I could see them using any of these methods. Honestly, I think none of them are that weird. I do think that counting and the front back ends are going to be the most common.

In anticipating the Theater Seats simulations, Max identified Michelle's approach from the preceding cycles as the most challenging. Max had listened in on a conversation between Ezra and Sam about which character they found most difficult that began with a comment from Ezra:

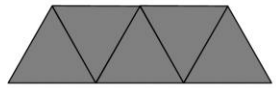
Ezra: I'm going to be honest, every single time, like, I'll think about what I can expect. And . . . it usually is Michelle, where I have the most difficulties. And that's the one where I always have to slow down and like try doing a problem with them at the same time.

This initiated a discussion between Ezra and Sam about which character was most difficult (see the subsequent case of Sam). The first author invited Max, who was listening in, to comment:

DG: [Max], I know that you were listening, and it looked like you were agreeing with part of [the conversation].
Max: I, well, I was agreeing mostly [that] Michelle, Michelle usually comes out of left field for me.

Though Max did not give evidence of anticipating Michelle's way of approaching the Theater Seats task for the third set of simulations, we identified evidence that Max and Sam were able to clearly describe a deconstructive figural approach that they assigned to Michelle in response to a final, post-simulation activity. We provided PTs with the Polygon Problem (see Figure 2) and asked them to respond in pairs or triads, in writing, to the question, "How would each simulated student approach this task?".

Figure 2. The Polygon Problem (Seago et al., 2004)



In response, Sam and Max wrote:

> Nell – make a table comparing number of triangles to perimeter. Notice the difference is always 2. Perimeter = number of triangles + 2
> Suzy – Notice that adding one triangle adds two sides. Perimeter = n + 2
> Michelle – use that each triangle has three sides and subtract inner sides. Perimeter = 3n – 2(n-1).

We note that Max, in collaboration with peers, gradually developed understandings of how each of Nell, Suzy, and Michelle would approach figural generalizing tasks. After experiencing the three sets of simulations, Max and Sam anticipated and described numerical, constructive figural, and deconstructive figural approaches similar to those that they had experienced in their simulations with each character.

The Case of Sam

Sam worked with Taylor and Blake to compare approaches to the Tiling Task. The three PTs agreed on a constructive figural approach:

> Taylor: So I went by the, you know, there's going to be one in the middle each time. And then for pattern one, there was one on each of four sides. In Pattern Two, there are two on each of four sides . . . one plus four [times] *n* being the generalization.
> Blake: Me too, but I did it, I guess a little -- no, I guess I did it in the same way, nevermind.
> Sam: I did the exact same way, 1+4n.

Sam worked with Ezra to anticipate student approaches to the Tables and Chairs task. The pair of PTs gave evidence that they anticipated numerical and constructive figural approaches:

> Sam: Alright what's another way? Can we think of another way.? Let's think of students like just counting, so I bet your way of just focusing on chairs is going to be something that they do.
> Ezra: [Yes] . . . I can't figure it out! I thought I had it . . . and then you're right, it's asking for the tables and each pattern itself is on a table. It's four tables straight to four tables and that increases by one.
> Ezra: I wonder if there's a way we can do it where there's gonna be subtraction. Cause last time they threw me. . . . and there's gonna be three of them. So that means three different ways.
> Sam: Yeah, I don't know what else you're gonna get other than . . . two n plus two.
> Ezra: 2n + 2. And then I doubt they're gonna recognize the n plus three, where n is the number of tables, is two n plus three.
> Sam: Yeah, I feel like I'm just coming up with more complicated ways rather than more logical ways.

Sam and Ezra did not appear to associate each approach any specific simulated student. Though they understood that three different approaches would be simulated in the second round of simulations, they did not anticipate a deconstructive figural approach.

Anticipating the Theater Seats simulations, Sam indicated that they found the constructive figural approach the more difficult to follow. In response to Ezra's statement that Michelle is the most difficult, Sam responded with a different perspective:

Sam: Suzy's always the hardest.
Ezra: Really? You think Suzy is the hardest?
Sam: Because she does everything in her head.
Ezra: True.
Sam: I literally tell Suzy like, write that down. Write that down. . . Sometimes it's more like me figuring out what Suzy is doing, because I'm trying to understand and Suzy's like, "well, this" and I'm like, "okay, like, I guess?"

As we noted in the case of Max, Sam and Max collaborated to anticipate how each simulated student would respond to the Polygon Task. As a team, Sam and Max gave evidence that they were able to describe how each character would respond to that task with a numerical approach (Nell), constructive figural approach (Suzy), or deconstructive figural approach (Michelle).

The Case of Ezra

We have shared data related to Ezra's anticipations of student thinking in the preceding cases. We note that in anticipating the Tiling Task, Ezra and Max expected two different constructive figural approaches. In anticipating approaches to the Tables and Chairs task, Ezra anticipated that the simulations would present three distinct approaches, but did not express what each of those approaches would be. Finally, in anticipating the Theater Seats simulations, Ezra noted that Michelle's approach was the most difficult for them to understand.

The Case of Taylor

Taylor and Blake collaborated in the anticipating phase of the Tiling cycle. Taylor initially perceived the Tiling task constructively as one middle tile in each pattern, with a set of four tiles appended to each end of the "sides" of the pattern. Each subsequent pattern would have another set of four tiles appended to each end:

Taylor: There's going to be one in the middle each time. And then for pattern one, there was one on each of four sides. In Pattern Two, there are two on each of four sides . . . . One plus four [times] *n* being the generalization.

After Blake responded with their approach to the task, Taylor acknowledged the existence of multiple approaches:

Blake: I guess I thought about it a little bit differently. Or probably the same, as I was like, "Yeah, of course, one in the center, and then you add a tile to each side of the squares." . . . .
Taylor: There are multiple ways of looking at it. . . . When you're looking at it individually, you're like, "This is the only one." But . . . I'm sure there's a different one that we're all missing, or something.

Moments later, Taylor anticipated the possibility of a numerical approach:

Taylor: I could see more like the plus four each time thing, like the numerical one.

After the Tiling simulations, Taylor, Blake, and Denise worked together in the anticipating phase of the Tables and Chairs cycle. Taylor approached the Tables and Chairs task using a constructive figural approach, describing the pattern as “six plus two three [times for Pattern 2], four times [for Pattern 3]”. However, Taylor generalized using the pattern number. They then said, “I could also say it's the number of tables minus two that stick out front here.” Taylor then showed evidence that they attributed tabular approaches to Suzy’s or Nell’s characters:

> Taylor: I could also generalize this by saying it's the number of tables minus the two on the ends, times two. And then I get the [unclear]. I also said, you know, you're predicting [Suzy] or Nell will come at it with the table again. Try to just go with it.

We cannot state with confidence whether Taylor’s mention of Suzy, then Nell represents a self-correction regarding the character name or an uncertainty about which character might simulate a numerical approach. However, Taylor provided evidence that they discerned a difference in Michelle’s approach -- specifically that Michelle’s approach was hard to anticipate:

> Taylor: With errors or misconceptions, we have, like, [using] pattern number instead of table number. Is there another misconception? Because [Michelle] came at us with a whole new way of looking at the problem.

At this stage, Taylor approached both tasks using a constructive figural approach. Taylor attributed the use of tabular representation of mathematical insights to Nell or Suzy, without any further distinctions about how each simulated students might approach the task.

The Case of Blake

We have shared some of Blake’s contributions in each cycle in Taylor’s case. We note that in anticipating the Tiling Task, Blake anticipates two different approaches: numerical and constructive figural approaches. Blake also asserted that most students will approach the task numerically when she said: “I feel like most of them actually count though. Once we have that, because that's what I have most experienced seeing students do.” In anticipating approaches to the Tables and Chairs task, it was not clear whether Blake’s approach aligned with a constructive figural approach, though they claimed to have used a similar approach to Taylor’s. Blake also anticipated that Michelle would approach the task in a ”visual” way. Finally, in anticipating the Theater Seats simulations, Blake approached the task with a constructive figural approach and did not verbalize any clear distinctions between all three approaches.

Interpretation and Implications

Variation Theory predicts that the experience of contrast across our simulated approaches will support learners in the first stage of recognizing critical differences among approaches, and that experiencing consistency in approaches across multiple simulations will support PTs in developing general, context-independent understandings of multiple potential student approaches to figural generalizing. Our tentative findings in the case of Kristy (Graysay & Bermudez, 2023) were consistent with that prediction. The additional cases presented in this report provide empirical support for hypothesizing that each case may represent a generalized learning progression related to the experience of variation within and across each set of simulations. However, PTs found a deconstructive figural approach difficult to anticipate until after completing the third set of three simulations. Those difficulties corroborate El Mouhayar and Jurdak's (2013) findings and suggest that PTs need additional experiences to fully conceptualize the deconstructive figural approach.

We propose two directions for future research. First, there is a need for further research to explore conditions under which our findings are generalizable to other PTs, other institutions, and other mathematical topics. Second, the goal of this design is to impact teachers' practices by supporting their knowledge of different student approaches to generalizing. Our data and analysis do not offer insight into whether or how PTs' experience impacted their ways of engaging with authentic students. There is a need for further research to examine whether and how experiencing systematic variation across simulations impacts the ways that PTs interact with authentic students during authentic generalizing.

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RESPONDING TO MATHEMATICAL EMPATHY

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In this ongoing study, we continue to look for opportunities to document mathematical empathy in the discourse of Preservice Elementary Mathematics Teachers. Using a framework for mathematical empathy, we asked: How is empathetic comprehension visible in discourse about mathematical definitions? and How do PSETs respond to public displays of mathematical empathy? Preliminary results indicate that empathetic practice is robust, but difficult to inspire in others. We position mathematical empathy at the intersection of teacher beliefs and mathematical knowledge for teaching with implications for teacher preparation programs.

Keywords: Preservice Teacher Education, Affect, Emotion, Beliefs, and Attitudes

Background

Although definitions serve as foundational elements in mathematics, there exists a scarcity of agreed-upon practices regarding their construction (Torkildsen et al., 2023). Understanding that multiple definitions may exist for the same mathematical concept poses challenges for preservice teachers (Linchevski et al., 1992). When we view of mathematics as a humanistic discipline where mathematics is socially constructed and personal values influence our evaluation of results, it is important for instruction to be participatory. Definitions, in this light, transcend mere mathematical tools; they become teaching instruments facilitating the conveyance of perceived meaning to others. Recognizing meaning as a negotiated construct grants authority to the knower, even as external sources are critically evaluated (Langer-Osuna, 2017).

Teachers acknowledge that the selection of definitions in mathematics classrooms hinges upon pedagogical context (Winicki-Landman & Leikin, 2000). These authors also posit that factors such as curricular approaches, classroom demographics, or the pursuit of clarity and elegance may influence this choice. In order to cultivate a classroom environment conducive to making informed contextual decisions, teachers must be aware of and able to comprehend the perspectives and mathematical thinking of their students; teachers need to see the work of teaching as an empathetic practice. This perspective serves as the focal point of our discussion.

Mathematical Empathy

In a recent study (Cox et al., 2021), we employed a definition of mathematical empathy as "the ability to comprehend another person's ideas and the true meaning or purpose behind them, seeking to utilize the other person’s frame of reference” (Araki, 2015, p. 122). Upon reflection, we question the use of “true meaning” in this definition. We would like to make the learner’s identity and intention more explicit in the interaction, retaining their expertise as primary. Rather than centering the expertise of the listener or the transference of expertise or knowledge, we focus instead on the willingness of the listener to be transparent and cede mathematical expertise to the learner. This is akin to how showing empathy does not grant us the right to claim the emotional experiences and perspectives of those we seek to understand. Therefore, we now define mathematical empathy as *the willingness to cede mathematical expertise to others and see them as someone from whom to learn*.

Teachers engaging in empathetic practice require both an awareness and acceptance that multiple mathematical truths coexist, alongside possessing the pedagogical content knowledge necessary to comprehend these truths. This paper aims to provide a succinct overview of a study exploring the capacity for mathematical empathy among preservice elementary mathematics teachers (PSETs). Through initial analysis, we identified two key empathetic practices: *empathetic awareness* and *empathetic comprehension* (Cox et al., 2021, 2024). *Empathetic awareness* indicates that the speaker believes that there is multiplicity in mathematical perspectives. Awareness can emerge as a belief that others see things differently than we do, or that students will have different mathematical backgrounds or experiences that are worthy of attention. *Empathetic comprehension* indicates the speaker can comprehend from someone else's mathematical perspective.

While instances of empathetic awareness were abundant in the data, observations of empathetic comprehension were comparatively rare. This prompted us to consider whether an expanded dataset could afford more opportunities to investigate this practice. We also pondered how participants might respond to their peers' public demonstrations of empathy. This led us to the following research questions: (1) How is empathetic comprehension visible in discourse about mathematical definitions? and (2) How do PSETs respond to public displays of mathematical empathy?

Methodology

To answer these questions, we expanded our initial data set. We included initial reflections where PSETs were asked to reflect on an instructional sequence and their responses to their classmates' writing.

Instructional Sequence: Defining in a Collaborative Space

Seventy-one participants were recruited from two sections of a geometry course for PSETs (grades PK-3). PSETs explored dynamic quadrilaterals constructed with interactive geometry software (IGS) in a face-to-face environment. The PSETs actively participated in tasks aimed at measuring, describing, and comparing quadrilaterals to support the task of formulating definitions. Initially, the PSETs collaborated in small groups during the class session, then reconvened as a whole class to collectively create definitions for quadrilaterals, kites, parallelograms, rectangles, rhombuses, squares, and trapezoids.

We wanted to capture the firsthand experiences of PSETs creating geometric definitions, while also exploring how this experience might shape their perceptions of the definition's role in a primary education classroom. In order to help PSETs frame their comments as both learners of mathematics and future teachers of mathematics, we first asked them to individually read Keiser's (2000) *The Role of Definition*. The article was chosen as a catalyst for reflection on this experience because it suggested that early presentation of formal definitions can curtail thinking in middle grades classrooms and argued for student-generated, fluid definitions based on concept imagery (Tall & Vinner, 1981) relevant to classroom learning.

Data Collection

Following the in-class IGS activity, the defining discussion and the assigned reading, we asked our PSETs to respond to the following prompt in an online discussion board: "After reading the article, *The Role of Definition*, what new thoughts do you have about the conversations we had in class about defining quadrilaterals? How about using definitions with children?" PSETs understood that both their peers and instructor would review their reflections, with the instructor actively engaged in the discussion. The assignment carried a grade based solely on completion. PSETs were unable to view their classmates' reflections until after submitting their own. Upon completing their reflections, PSETs gained access to classmates' submissions and were encouraged to engage in an online discussion regarding the shared reflections. The data comes from the initial posted reflections (n=71) and responses (n=145).

Framework: Practicing Mathematical Empathy

The initial posted reflections (n=71) were analyzed using grounded theory (Vollstedt & Rezat, 2019) to build and then apply a framework by which to give nuance to what we learned about PSETs' beliefs about the purpose, nature, and origin of mathematical definitions. As we read, we began identifying instances where PSETs expressed something similar to empathy in a mathematical context and coded these practices empathetic awareness and empathetic comprehension (Cox et al., 2021, 2024).

Results

There were twenty-four initial reflections that were coded for evidence of either empathetic awareness (EA) (n=22) or empathetic comprehension (EC) (n=3). One of these initial reflections was coded for both EA and EC, so the total number of initial reflections showing empathy was 24. We gathered all of the responses (n=25) that these initial reflections garnered from classmates and applied the framework (see Table 1).

Table 1: Analysis of Responses to Initial Reflections coded for Empathetic Reflection

Responding to expressions of:	Total Responses (n=25)	Responses showing EA (freq.)	Responses showing EC (freq.)
Empathetic Comprehension	2	0 (0%)	0 (0%)
Empathetic Awareness	23	8 (34.8%)	0 (0%)

Of the 25 responses, few showed evidence of empathetic practice and these were exclusively empathetic awareness. There was no evidence of empathetic comprehension in the responses. Further, initial reflections that showed empathetic comprehension only inspired two responses, and neither showed evidence of empathetic practice at all.

Eight responses, or roughly one-third, showed evidence of empathetic awareness. This is approximately the same frequency of empathy as in the set of initial reflections (24/71 = 31.0%). Consider the following exchange sparked by Lily's public display of empathetic awareness. Note that Natashia showed an empathetic stance in her initial reflection, but this was new for Beckett.

Lily: I enjoyed doing this in class because it gave a broader perspective of how this could be set up in a classroom setting. It is also important to hear what student's peer's ideas are about a definition to think about them in a different way and gain a broader understanding.

Beckett: I agree with what you are saying, especially the last sentence. I concur that it is important understand your peers perspective

Natashia: I agree with that as well. Allowing children to come up with their own definition makes way for their peers to learn how others think as well. For example, at the beginning of the semester when we had to draw the [quick draws] from memory, I found it helpful when our peers discussed how they memorized the [quick draws] versus how I may have. I really enjoyed that activity.

Natashia's response also referenced a Quick Draw activity from the first days of class. The learning objective from this assignment was to "see mathematics through someone else's' eyes." That this activity is referenced in a response where Natashia is honoring her peers' perspectives is an important piece of context.

Inspired by Beckett, we went back to compare the empathetic stances taken by responders on their initial reflection. We found that authors who showed empathy in their responses were three times more likely than not to have also shown empathy on their initial reflection, and those who did not show empathy in their responses were three times more likely not to have shown it in their initial reflection (see Table 2).

Table 2: Empathy across Initial Reflection and Responses

	Initial Reflection	
	Empathetic	Non-Empathetic
Empathetic Response	6	2
Non-Empathetic Response	4	13

Implications and Conclusion

In this follow-up study, we asked how is empathetic comprehension visible in discourse about mathematical definitions. While analysis is ongoing, we have not found any additional instances of empathetic comprehension.

We also asked how PSETs respond to public displays of mathematical empathy. Summarized in Table 2, we are able to say that PSETs do not necessarily adopt an empathetic stance in response to a public display of empathy. Overall, 32% of all responses to empathetic displays showed evidence of empathy themselves. This mirrors the frequency of empathy in the initial reflections. Even though empathy does not seem to inspire more empathy, we do believe that an empathetic stance is robust; PSETs who demonstrated an empathetic stance in their initial reflection maintained that stance when responding to the thoughts of others.

The findings of this study, preliminary though they may be, are important. We are interested in identifying pivotal experiences for PSETs that position them as authors and mathematical authorities and believe that there are many ways to go about that work. As we reflect (as instructors) on the episodes from which our data came, we posit that there is a hurdle to overcome before teachers are able to adopt empathetic stances and thus, teaching practices. A teacher (or PSET) who believes in the sanctity and external completeness of school mathematics, ("someone already knows this, I'm just learning it") may struggle to cede mathematical expertise to children, even as they cheer achievements and mark progress toward learning outcomes.

This study indicates that we can overcome that hurdle with careful attention to creating mathematical opportunities for teachers to claim personal expertise. We support opportunities for PSETs to encounter or experience episodes of teaching that position teachers as learning alongside students. This might include open-ended problem solving with multi-age groups. It might also include exposure to situations where children know more than the adults in the room. The research on math circles might be leveraged to further imagine how we come to learn how to learn from others. We see great potential in using these types of experiences within teacher preparation programs to document and support emergent empathetic practices.

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TYPES OF CONNECTIONS BETWEEN ELEMENTARY MATHEMATICS AND LITERACY TEACHING AND LEARNING: A CONCEPTUAL FRAMEWORK

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Keywords: teacher knowledge; elementary school education; preservice teacher education

In elementary teacher education, there has been increasing interest in making connections across disciplines (e.g., Muhammad et al., 2021; Prough et al, 2022); however, there does not seem to be clarity on what we mean by connections. This poster will share a conceptual framework of types of connections.

The Case Study: Developing the Framework

This framework was developed as part of a case study of the connections between mathematics and literacy by elementary teacher candidates (TCs) who were taking concurrent mathematics methods and literacy methods courses as a cohort. Of the 18 TCs in the cohort, 13 participated in the case study, which included observations and video recordings of class meetings, collection of coursework, and one-hour focus group interviews with 6 of the participating TCs.

The framework was developed by working iteratively with the data from the case study and the scholarly literature. After initial open coding of the data and the literature, I used the most significant or most frequent codes to construct larger categories that synthesized the themes in the data and literature, using the constant comparison method (Charmaz 2014).

The Framework: Types of Connections

I identified three main types of connections made by the TCs in the case study and by scholars in the literature: (a) curriculum integration, (b) language as a basis for learning mathematics, and (c) similarities in teaching and learning literacy and mathematics.

Curriculum integration includes any learning models or lesson structures that draw on the unique ways of knowing from more than one discipline. Interdisciplinary learning, thematic teaching, and project-based learning are examples of this type of connection (e.g., Parker et al, 2012; Zhou & Kim, 2010).

Language as a basis for learning mathematics includes engaging with mathematical texts (e.g., Beaudine, 2022), using mathematical language to communicate ideas (e.g., Armstrong et al., 2018), using writing to make sense of mathematical ideas (e.g., Caputo, 2015), and mathematizing read-alouds of children's books (e.g., Hintz & Smith, 2022).

Similarities in teaching and learning can include learning goals (e.g., constructing arguments with evidence; Cheuk 2012), thinking skills (e.g., monitoring for sense; Halladay & Neumann, 2012), or instructional practices used in both disciplines (e.g., rough-draft thinking; Jansen, 2020).

Understanding the types of connections that can be made between literacy and mathematics can help elementary teachers, and teacher-educators with elementary backgrounds, to leverage the unique strength of multidisciplinary knowledge to teach mathematics in engaging and conceptually-oriented ways.

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VALIDATING DECOMPOSITION OF A TEACHING PRACTICE FOR FORMATIVE ASSESSMENT FEEDBACK

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SimulaTE is studying teaching simulations as formative assessments of pre-service teachers' (PST) practice of eliciting and interpreting students' mathematical thinking. Preparation and protocols that promote reliability and validity of the simulations as formative assessments will enhance their effectiveness and generalizability. Teacher educators who use the simulations document each PST's performance to generate feedback for the PST in nine categories, arising from a decomposition of the teaching practice into specific component skills or actions. A series of coordinated validation studies include research to determine if the nine categories are distinguishable through the use of the simulation assessments, and can benefit from attention beyond other experiences PSTs have in their teacher preparation programs.

Keywords: Assessment, Mathematical Knowledge for Teaching, Preservice Teacher Education, Teacher Educators

Framing and Purpose of the Study

Ideally, teacher preparation develops candidate's skills and abilities for ambitious instruction that promotes student learning and counters inequities in outcomes. We ground our work in the understanding that frequent opportunities to engage in core practices of teaching, with formative feedback, can develop the knowledge, skills, and dispositions necessary for nurturing young learners of mathematics. Formative assessment provides pre-service teachers (PSTs) with feedback to improve their practice (Grossman, 2010), which is considered crucial for teacher preparation (Darling-Hammond et al., 2005; AMTE, 2017). It requires teacher educators to see teaching practices in action, yet traditional field settings afford neither frequent accessibility nor opportunities for deliberate work on specified facets of teaching. Simulations of mathematics teaching practices are an approximation that can provide early, frequent, and substantive formative assessment opportunities while engaging PSTs in particular facets of teaching.

PSTs begin preparation with views on teaching that need to be surfaced and, in some cases, challenged (Boerst et al., 2020; Shaughnessy & Boerst, 2018; Shaughnessy et al., 2020). Work initiated at the University of Michigan has produced multiple simulations to engage and refine PSTs' practice of eliciting and interpreting students' mathematical thinking. By revealing PSTs' knowledge, skills, and dispositions and providing immediate feedback, the simulations are designed to facilitate growth (Shute, 2008; Hattie & Timperley, 2007). This study's dual purposes are to investigate the decomposition of the teaching practice into measurable components for providing feedback, and to consider whether these skills or actions can benefit from concerted attention beyond other experiences typical to teacher preparation programs.

This study is one of a series of studies generating evidence regarding validity arguments (AERA et al., 2014; Kane, 2001; 2013) for using the simulations as formative assessments. It focuses on two sources of validity evidence (AERA et al.): internal structure (specifically test component interrelationships) and relations to other variables. Initial evidence for two specific claims of the validity argument are addressed here: (1) the nine component skills/actions of the teaching practice can be measured distinctly through simulation performances so that feedback can be specifically targeted, and (2) PSTs' other experiences in teacher preparation do not fully develop the component skills/actions of the teaching practice.

Teaching Simulations as Formative Assessments

Using the teaching simulations as formative assessments involves three interacting roles:

- The PST prepares for, engages in, and debriefs what they learn via the teaching practice of eliciting and interpreting student thinking with a Simulated Student.
- The Simulated Student is an adult prepared to follow a provided profile and to respond in specific ways to anticipated questions and prompts. (Student role)
- The Teacher Educator (TE) documents the PST's performance and provides formative feedback based on the performance. (Proctor role)

Figure 1 illustrates the full formative assessment process. The underlined components in the figure indicate the parts of the process investigated in this part of the validity studies.

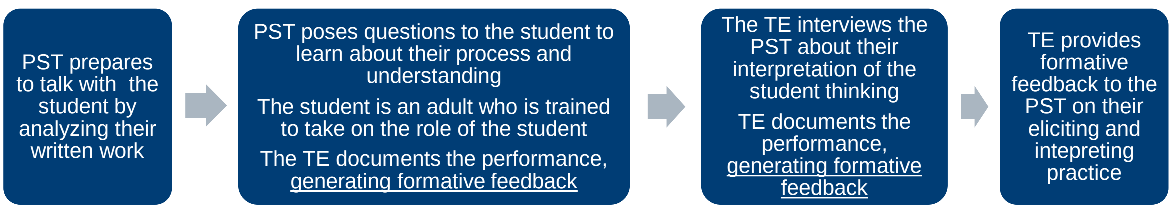
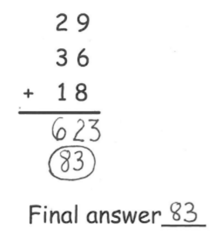


Figure 1: Structure of Teaching Simulations as Formative Assessments

The tasks in the simulations represent core content of elementary mathematics. The student work and specifications of the student role are evidence-based recreations of student thinking about that content (Shaughnessy et al., 2012). Figure 2 illustrates key elements of an assessment.

Mathematics topic: Multi-digit addition



- **The student's process:** The student is using the column addition method for solving multi-digit addition problems, the student is working from left to right.
- **The student's understanding of the ideas involved in the problem/process:** The student has conceptual understanding of the procedure including why combining is necessary (and when and how to combine).
- **Other information about the student's thinking, language, and orientation in this scenario:** The student talks about digits in columns in terms of the place value of the column. The student uses the term "combining" to refer to trading/carrying/regrouping.

Sample PST prompts	Sample Responses
What did you do first?"	"I added the tens: 2 + 3 + 1 and I got 6."
"How did you get from 623 to 83?"	"I had to combine the 6 and the 2."
"Why did you need to combine those numbers?"	"Because they're both tens."

Figure 2: Excerpts from a Sample Teaching Simulation Protocol

The content of the student work in the assessments was purposefully selected to cover mathematics concepts that PSTs are expected to have a strong understanding of and to provide insight into their capabilities. The simulation protocols were designed to reflect non-traditional approaches to solving mathematical problems or student thinking that results in an "incorrect" answer. The four simulation assessments used in this study included:

- Column Addition (CA): As shown in Figure 2
- Common Denominator: Comparing fractions, with an error in creating an equivalent fraction to compare using common denominators
- Common Numerator Correct: Comparing fractions, creating an equivalent fraction to compare using common numerators
- Expand and Trade: Multi-digit subtraction by writing quantities in expanded form and making trades among values before subtracting by place value, with an error in recording the value of a traded quantity

In teaching, "teachers pose questions or tasks that provoke or allow students to share their thinking about specific academic content in order to evaluate student understanding, guide instructional decisions, and surface ideas that will benefit other students" (TeachingWorks, 2024). The work of eliciting student thinking is conceived as: (a) formulating and posing questions to elicit and probe student thinking; (b) listening to and interpreting how students respond; (c) developing additional questions or tasks to pose; and (d) making sense of what students know and can do. Interpreting students' thinking is integral to eliciting, but is a distinct, overarching practice relying on broader information. It is conceived as: (a) sampling from evidence of student thinking and (b) using insight to articulate inferences grounded in the evidence. These practices take place within and across lessons, and in longer cycles of teaching that depends on learning about students to drive instruction (TeachingWorks, 2024).

Drawing on these conceptualizations, the simulation assessment situation and its documentation are based on a the following decomposition of the teaching practice into nine component skills or actions.

- Eliciting Process (EP): Uses questions or prompts to the student regarding their process for solving the task
- Interpreting Process (IP): Describes the student's process for solving the task
- Probing Understanding (PU): Uses questions or prompts to the student regarding their understanding of the mathematics of their process
- PST-Generated Interpretations of Student Understanding (PGSU): Spontaneous description of the student's understanding of the mathematics of their process
- Prompted Core Interpretations of Student Understanding (PCSU): Prompted description of core elements of the student's understanding of the mathematics of their process
- Attending to Student Thinking (ST): Asking questions about the written work and attending to what the student says in response to questions
- Applying Mathematics Knowledge for Teaching (MKT): Generating a task that can be used to confirm PST's understanding of the student's process
- Using Mathematics Knowledge and Skill (MKS): Applying the student's process to a new example, Generalizing about the mathematics/reasoning of the student's process
- Respecting the Student and Their Thinking (RS): Interacting with the student, and describing their work in ways that respect them as learners/knowers/doers of mathematics

To ensure assessment evidence about these specific components arises, the simulated student will disclose aspects of their process and understanding only when the PST deliberately prompts for it. Similarly, the PST is asked during the debriefing interview to recount very specifically what they learned about the student's process and understanding, supporting their claims with evidence they gathered. To further assess their application of mathematical knowledge for teaching and use of mathematics knowledge and skill regarding the targeted content, the PST is also asked to generate a problem to confirm what they learned about the student's process, and to explain the mathematics ideas undergirding the student's process and understanding. An online tool with protocols specific to each assessment (about 75 items) supports the teacher educator in documenting this fine-grained information. This documentation generates a level of performance (1-4) and formative feedback for each of the nine components. The TE can then use the performance levels and feedback to guide a discussion with the PST about areas of strength and potential improvement. The nine components are:

Study Methods and Participants

Data Collection

Data to address the two claims of the validation argument were collected between April 2023 and March 2024. Assessments were administered to 200 PSTs at 14 higher education institutions. Demographic data on participants are shown in Table 1.

Table 1: Characteristics of the Participants

Characteristic (N)	Percent of Respondents
Educational Attainment (199)	
Undergraduate student	96
Undergraduate degree in education or STEM discipline	4
Student Teaching or Internship (199)	
Had not yet begun student teaching or internship	80
Was currently doing student teaching or internship	20
Sex (199)	
Female	93
Male	5
Non-binary/non-conforming	1
Prefer not to answer	1
Hispanic or Latino (199)	
Yes	13
No	87
Race† (199)	
White	87
Asian or Asian American	4
Black or African American	3
American Indian, Native American, or Alaskan Native	1
Native Hawaiian or Pacific Islander	1
Prefer to self-describe	1
Prefer not to answer	1
Age (184)	
Traditional undergraduate-aged student (born 1998 to 2005)	96
Non-traditional undergraduate-aged student (born 1980 to 1997)	4

† Respondents were allowed to select more than one option; therefore, percent of respondents may add to more than 100.

Seven researchers, including authors 1, 2, and 4, prepared to administer the four assessments by learning the student and proctor roles and documenting performances in sample videos. The research team established reliability in both administration and documentation (Boerst et al., 2023; Heck et al., 2023). Researchers were assigned in multiple pairings to conduct site visits for data collection. A pair administered two assessments to each PST, alternating to distribute who served in the student and proctor roles. Each PST completed two of the four assessments, purposefully assigned to ensure equal distribution of assessments. Column Addition was administered to 90 PSTs, Common Denominator to 106, and Common Numerator Correct and Expand and Trade to 102 each.

Researchers' documentation generated a level of performance (1-4) for each component, along with potential feedback for discussion. Descriptive results for performance level scores on the nine component skills/actions are presented in Table 2. For these studies, feedback was not shared or discussed with participants to ensure it did not influence their performance on the second assessment they completed.

Table 2: Component Performance Level Scores

Skill/Action	Min	Max	Mean	SD
EP	1	4	2.95	1.03
IP	1	4	3.12	0.82
PU	1	4	1.85	0.99
PGSU	1	3	1.88	0.79
PCSU	1	4	2.58	1.12
ST	1	4	3.01	0.22
MKT	1	4	3.49	0.89
MKS	1	4	2.89	1.19
RS	1	4	3.79	0.55

Data Analysis, Results, and Findings

The first claim of the validity argument: the nine component skills/actions of the teaching practice can be measured distinctly, was examined in this study using the levels of performance that the documentation tool generates. A lack of correlation among the nine scores would offer evidence supporting this claim. Table 3 summarizes, for the 36 possible combinations of components, the correlations that were statistically significant. All were positive.

Table 3: Significant Correlations Between Components by Simulation Assessment

Comp.	EP	IP	PU	PGSU	PCSU	ST	MKT	MKS
IP	CA CD							
PU								
PGSU	CD CN	CD CN	CA CN ET					
PCSU	CN	CD CN	CA CN ET	CA CD CN ET				
ST	CA CN							
MKT		CD CN		CA	CA			
MKS	CA CN	CA CD CN ET	CD	CD CN	CA CD CN		CA CD	
RS			CD					

No significant correlations were found for 17 combinations of components on any of the assessments, and 5 other combinations produced a significant correlation on only one assessment.

Significant correlations (ranging from 0.20 to 0.55) were found for at least one combination involving each component. However, the most common significant correlations involved one of four components: IP (5 combinations of components, 12 instances across assessments), PGSU (6 combinations, 14 instances), PCSU (6 combinations, 14 instances), or MKS (6 combinations, 14 instances). In fact, only two other combinations—EP with ST in two instances and PU with RS in one instance—produced significant correlations.

Overall, these results provide mixed evidence regarding the claim that the nine components can be measured distinctly via the assessments. The extent to which these components are related has implications for targeting feedback to inform improvement on each component. Four components appear to be related to multiple others, suggesting that providing feedback on these components may be especially important for developing capabilities with the overall practice. Moreover, feedback on these four components might become especially useful by discussing their relevance to other components or the overall practice.

The second claim of the validity argument: PSTs' other experiences in teacher preparation do not fully develop the component skills/actions, was examined by predicting performance levels for each component using information PSTs reported about their progress in their programs. Specifically, the analysis considered their concurrent enrollment (N=96) or completion (N=98) of a mathematics for teaching (MfT) course, as well as their completion of other mathematics courses that are foundational (e.g., college algebra; N=89) or advanced (e.g., calculus; N=108). It also examined their concurrent or completed engagement in a student teaching placement (N=40).

Table 4 summarizes the results of a set of HLM analyses (scores nested within PSTs) predicting the performance level score for each component using data on PSTs' experiences in their preparation programs. Since PSTs were assigned to different pairs of assessments, dummy codes were also included to control for which assessment produced each performance level score. A lack of predictive association between PSTs' experiences and the performance levels on the simulation assessments offers initial supporting evidence for this claim.

Table 4: Positive and Negative Effects of PST Experiences on Components

Experiences	EP	IP	PU	PGSU	PCSU	ST	MKT	MKS	RS
Foundational Math								Neg.	Neg.
Advanced Math									
No MfT			Neg.		Neg.				
Enrolled in MfT				Neg.			Neg.		
Completed MfT									
Student Teaching							Neg.		

For three of the nine component skills/actions, PSTs' experiences in teacher preparation programs did not predict performance level scores. Variations in performance level scores for five of the other components were each predicted by only one type of experience. The remaining component was predicted by two experiences. PSTs who were neither concurrently enrolled in nor had completed a Mathematics for Teaching course with PU (respectively, -0.86 points, p=.008; -1.04 points, p=.002) and PCSU (respectively, -1.27 points, p<.001; -1.39 points, p<.001) suggests that participation in such courses contributes to development of these components of the practice the simulation assessments address. However, very few students who participated in this study (N=5) fell into this category. It is likely that such courses are an early requirement in most elementary education programs, so when participants were considered eligible for this study, they were already enrolled in an MfT course. Two other components showed predicted differences in performance levels between students concurrently enrolled and those who had completed MfT courses: PGSU (-0.18 points, p=0.048) and MKT (-0.24 points, p=0.013). To the extent that this smaller distinction represents differences in progress through teacher preparation programs, the lack of prediction of performance level scores on most components between these two conditions lends support to the validity claim. Further research involving PSTs who have not yet enrolled in MfT classes would be worthwhile.

PSTs' completion of a Foundational mathematics course predicted lower performance level scores on MKS (-0.45 points, p=.010) and RS (-0.13, p=0.036). Rather than calling the validity claim into question, this negative association may suggest that students whose mathematics coursework in college includes foundational content are likely to need more help in using mathematics knowledge and respecting student thinking in their teaching practice. Further research to pinpoint why some PSTs complete these courses and whether it signals something about their general mathematics knowledge would be informative.

PSTs' concurrent engagement in student teaching predicted a lower performance score on MKT (-0.27 points, p=.025). Again, this negative association does not challenge the validity claim. Rather, it might suggest a need to further support PSTs in making use of mathematics knowledge for teaching when they are student teaching. Additional longitudinal research would be informative to understand if entry into student teaching somehow affects PSTs' ability or propensity to apply MKT in the practice of eliciting and interpreting student thinking.

On the whole, these results provide initial evidence that experiences in teacher preparation are not likely to fully develop PSTs' abilities in the teaching practice of eliciting and interpreting student thinking. By extension, the simulation experience and associated feedback on the component skills/actions appears to offer a unique opportunity to support PSTs in more fully developing their capabilities with this practice.

Conclusions and Next Steps

The mathematics preparation of elementary teachers should develop their capabilities to enact teaching practices that support young learners' growth in mathematical knowledge, fluency, and disposition. Coursework and field placements that traditionally make up the bulk of PSTs' experiences in teacher preparation provide opportunities for PSTs to develop foundational knowledge of mathematics and pedagogy and to learn about and engage in these practices to an extent. They do not offer early, frequent, and structured experiences for PSTs to apply what they are learning in low-risk, high-feedback settings to support improvement in their capabilities. Simulations designed for engagement in teaching practices not only offer early, frequent, and structured experiences, but provide a measure of authenticity of PSTs' performance of the practices and opportunities for teacher educators to give immediate feedback to inform learning and improvement (Boerst et al., 2020; Darling-Hammond et al., 2005; Grossman, 2010).

Teaching simulations are resource intensive to develop and time intensive to use for formative assessment in teacher education. Strong validity must undergird their use to justify these investments. Kane's (2001; 2013) recommendations for developing and testing a validation argument require that the specific claims underlying the processes for administering assessments, generating results, and using results be stated and studied. Validation, in this view, is an ongoing process of amassing evidence to support or refute and, if necessary, refine these claims.

Prior work has demonstrated that preparation and support provided in the assessment materials result in consistent enactment of the simulations (Boerst et al., 2023) and reliable documentation of performances (Heck et al., 2023). In this study, the generated results were examined to test two additional validity claims, that (1) the component skills/actions of the teaching practice can be measured distinctly, and (2) PSTs' other experiences in teacher preparation do not fully develop these skills/actions. Analyses of data collected from PSTs in multiple teacher education programs on four different simulation assessments provided evidence supporting both claims, along with some discrepant evidence to be further studied.

Next steps in this validation work include further study of the first claim through an exploratory factor analysis of the items used to generate the component performance level scores. These data will also support analyses to test two additional claims addressing the response process as a source of validity evidence (AERA et al., 2014), within the full validation argument. First, the distribution of the four assessments across PSTs in various programs and their planned administration by multiple researchers serving in the student and proctor role will support a variance components analysis to examine the claim that the performance levels scores are mainly due to variations in the performance and not due to the effects of the specific assessment or the individuals playing the student and proctor roles. Second, data from this process were gathered from back-to-back performances on simulation assessments without sharing or discussing the generated feedback in between. Other data gathered within the larger project offer cases of the same pairs of assessments being administered with sharing and discussion of the generated feedback. These two situations will be contrasted to study the claim that engaging with the generated feedback promotes learning and improvement in performance.

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HOW DO INSTRUCTORS DESCRIBE STUDENTS' MATHEMATICAL WORK AND OPPORTUNITY TO LEARN IN GEOMETRY COURSES FOR TEACHERS?

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We report partial analysis of a survey of instructors of undergraduate geometry courses for teachers, attending to how they described the nature of the mathematical work they engage students in and the opportunities to learn that students had. Analysis of latent construct correlations showed that engagement of students in inquiry into geometry was significantly associated with opportunity to learn about mathematical definitions and conjecturing and engagement of students in the study of geometry was significantly associated with opportunity to learn about axioms and about history of geometry. Latent variable means comparisons showed group differences in claimed opportunity to learn between instructors whose highest degree was in mathematics and those whose highest degree was in mathematics education.

Keywords: geometry, teacher knowledge, undergraduate instruction, inquiry, opportunity to learn, survey

Objectives

We report on an analysis of survey responses from instructors of geometry courses for teachers (GeT) focusing on the curricular choices of instructors. Herbst et al. (2024a) reported on two distinct sets of characteristics of the mathematical work undergraduate geometry students may be engaged in: inquiring into geometry and studying geometry. Here, we investigated whether the type of mathematical work promoted could predict the topics stressed in different classes by looking into correlations between the former and the latter sets of variables and whether the field of highest degree attained by instructors could predict those curricular choices.

Literature Review

The mathematical preparation of teachers is an important component of secondary mathematics teacher preparation. This is so not only because teachers need to know the subject matter they will teach but also because the work of teaching, particularly when teaching for understanding, includes organizing the mathematical environments in which their students will learn and making sense of how students demonstrate their understanding (Manouchehri, 1998). Among the mathematical knowledge teachers need is the capacity to organize and manage mathematical work (Kuzniak & Nechache, 2021).

What knowledge to aim for and what mathematical work to engage prospective teachers in mathematics courses for teachers are important decisions instructors need to make. Though classically secondary mathematics teachers took courses equivalent to the mathematics major, the value of this choice has been questioned (e.g., Proulx & Bednarz, 2008). For a while, mathematics education researchers, mathematicians, and mathematics teacher educators have taken an interest in improving the mathematical preparation of teachers (Bass, 1997; Martin et al., 2020; Wasserman et al., 2023). This interest has been demonstrated in several ways. The promotion of inquiry-based learning in undergraduate mathematics courses has included mathematics courses for teachers demonstrating some learning of mathematical knowledge for teaching (see Laursen et al., 2016; Yoshinobu & Jones, 2013). Also, design research approaches to the teaching of mathematics courses for teachers have endeavored to connect the content of advanced mathematics courses with occasions of use in school mathematics teaching (Buchbinder & McCrone, 2023; Wasserman, et al, 2022). And communities of mathematics teacher educators and mathematicians have worked together to develop shared ownership of the problem of mathematical preparation of teachers as well as create curricular resources (e.g., CBMS, 2001, 2012; Martin et al., 2020; Senk et al., 2004; Usiskin et al., 2003).

Of particular interest to our study is the work of GeT: A Pencil, a community of instructors of geometry courses for secondary teachers (including mathematicians and mathematics educators) who have been working together since 2018 to improve those courses (see getapencil.org; An et al., 2023, 2024). An outcome of the work this group has been a consensual set of 10 essential student learning objectives (SLOs) that are meant to be a common core that diverse curriculum materials and pedagogical strategies could aim to align with. These 10 student learning outcomes are presented in Figure 1.

SLO	Description	SLO	Description
1	Derive and explain geometric arguments and proofs.	2	Evaluate geometric arguments and approaches to solving problems.
3	Understand the ideas underlying current secondary geometry content standards.	4	Understand the relationships between axioms, theorems, and different geometric models in which they hold.
5	Understand the role of definitions in mathematical discourse.	6	Effectively use technologies to explore geometry and geometric relationships.
7	Demonstrate knowledge of Euclidean geometry, including its history.	8	Be able to carry out and justify basic Euclidean constructions.
9	Compare Euclidean geometry to other geometries such as hyperbolic or spherical.	10	Use transformations to explore definitions and theorems about congruence, similarity, and symmetry.

Figure 1. Student Learning Objectives (SLOs)

The pursuit of all of those kinds of improvements can be facilitated by the existence of background information that describes the specific courses which are to be improved. Descriptive studies, such as TEDS-M, have contributed information about the qualities of teacher preparation in mathematics (e.g., Tatto & Senk, 2011) based on surveys and knowledge assessments, aiming to characterize how different nations prepare mathematics teachers. Tatto and Bankov (2018) provided an account of the opportunity to learn mathematics for secondary teachers in the United States based on an analysis of syllabi, noting that a large majority of prospective secondary teacher education programs provided opportunities to learn Euclidean or axiomatic geometry. However, information about what those geometry courses include both topically and in terms of mathematical work was beyond the scope of that study.

As regards geometry courses for secondary mathematics teachers, only two surveys have been conducted in the past. Wong (1970) surveyed leaders of mathematics departments and teacher education programs asking for their level of satisfaction with the geometry preparation of teachers. Grover and Connor (2000) reported on a survey of 108 instructors and were able to describe broad strokes of curriculum choices (e.g., that more than half of the courses emphasized Euclidean geometry from a synthetic perspective and that more than half included lectures with some discussion, but all group work was done outside of class). Herbst et al. (in press) interview study of 32 instructors showed they recognize a tension between sourcing the GeT curriculum from synthetic geometry and from geometry knowledge needed for teaching. It seemed important to develop a new instructor survey not only to update the description from Grover and Connor (2000) after broader emphases on mathematical knowledge for teaching and inquiry-based learning, but also to make more fine-grained claims. Though the survey targets questions about a range of issues on instruction and curriculum, the present report is focused on describing the geometry topics and geometric work students have opportunities to learn and do (for reports on other aspects of the survey see Herbst et al., 2024a, 2024b).

Theoretical Framework

We frame this inquiry using Cohen et al.'s (2003) instructional triangle which considers instruction as a transaction of content among instructor and students. We elaborate the content vertex of the triangle (see Figure 2, lower right) by noting that content is manifest in instruction in two different ways. Content is, on the one hand, a set of instructional goals or knowledge items which are at stake; and content is, on the other hand, the mathematical work that students are asked to do, in the form of problems and other tasks. In particular, different types of mathematical work with the content may be present for the same content. Geometry courses include many theorems about geometric concepts and students may all be expected to know the definitions and be able to prove the theorems. Yet the manner in which they get to attain such learning (the work in which they engage) may vary: In some classes they might participate in constructing the definitions or be given a chance to conjecture the theorems, while in other classrooms the definitions and the statements of such theorems may be given to them. That difference in the kind of mathematical work is an important one to track in geometry. Brousseau's (1997) notion of didactical contract can help distinguish between those classrooms. In particular, the survey as a whole pursues characterizing a contract that we name *geometric inquiry* (inquiry, hereafter) and one that we name the *study of geometry* (study, hereafter).

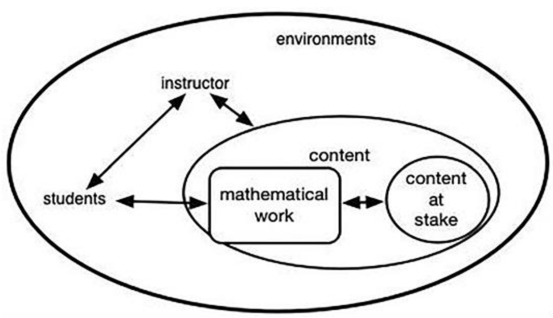

Figure 2. The instructional triangle adapted to include two manifestations of content

Our survey improved upon Grover and Connor's (2000) by taking into consideration the findings from Shultz (2022) about inquiry-oriented instruction (IOI). Shultz (2022) found that instructors' IOI practices were not homogeneous: Whereas some instructors who affirmed using inquiry involved students in conjecturing and defining, others who also affirmed teaching through inquiry interpreted it mostly in terms of student-teacher interaction (e.g., discussions, group work). Our survey built on Shultz's and also included questions that helped gauge the incidence of non-inquiry practices in teacher-student interaction and in the nature of the mathematical work. Our items were designed to answer separately questions about the incidence of different hypothesized factors associated with inquiry and study. Specifically, our survey included items that indicated constructs associated with inquiry and study regarding how instructors interact with students and how students interact with each other, and constructs associated with inquiry and with study concerning the nature of the mathematical work students do, including whether they participate in defining new concepts or rather receive the definitions from instructors. Herbst et al. (2024b) examined the distinctions among study and inquiry contracts in regard to the instructor-student interactions depicted on the left arrow in Figure 2. In this paper, we focus on the horizontal arrow shown in Figure 2 (how students interacted with content, which we call students' mathematical work) and some aspects denoted by the right arrow in Figure 2 (specifically the content that instructors recognized to be at stake).

Methods

To investigate the relationship between the mathematical work students were engaged in (i.e., study or inquiry) and the geometric ideas instructors recognized students had the opportunity to learn about, a survey was designed and distributed among instructors of geometry courses for secondary teachers in the United States. We targeted mathematics departments in all US universities where an undergraduate geometry course is regularly taught and required for students seeking certification to teach secondary mathematics. The survey was sent to all mathematics departments whose website included mention of such a geometry course (n=670). Emails were sent to department heads asking them to forward a link to the survey to the instructor who had taught the course last. We recognize that surveys provide only an approximation of teaching practice (Kennedy, 1999), and that more robust conclusions often need richer data collection. At the same time, a survey affords to see general trends in practice at low cost.

Figure 3 provides a list of the items used to describe students' mathematical work in relation to the constructs study and inquiry. We also used the SLOs (Figure 1) to operationalize what instructors might recognize among the opportunities to learn provided to their students in their courses, creating items that indicated each of the SLOs (see Appendix for some SLO-related items). Items shown in Figures 3 and items associated with the SLOs (see Appendix) were included in a larger survey administered through Qualtrics, which also asked questions about instructor demographics, prior preparation, and experience. The analysis focuses on a portion of the survey about mathematical work assigned to students and students' opportunity to learn the

SLOs, particularly looking at associations between the kind of students' mathematical work and the mathematical content recognized by the instructor to be at stake.

List of items that indicate Study (6-point Likert, from strongly disagree to strongly agree)	
821104	For the theorems whose proofs they had to learn, the proof was fully provided to them.
821105	The corollaries (i.e., consequences) of theorems students were supposed to use were explicitly stated for them.
821106	The constructions students were expected to learn were presented step by step to them.
List of items that indicate Inquiry (same scale as above)	
821204	Students were assigned to write (or improve) definitions.
821205	Students were asked to critique definitions given by either you [the instructor] or the textbook.
821219	Students were asked to critique construction procedures.

Figure 3. Items that indicate study and inquiry

Sample

About a third of the targeted departments had instructors return surveys. Our effective sample consisted of 140 GeT instructors who completed all survey items, including the GeT Instructor survey and a background questionnaire. Our sample participants confirmed they had taught a geometry course required for secondary mathematics teachers in the previous ten years. Approximately 69% had their highest degree in mathematics, while 28% had their highest degree in mathematics education (in both cases, highest degree is a Ph.D. or a Masters); also 35% had prior teaching experience in high school geometry. A sizeable 83% of participants held either tenured or tenure-track faculty positions, while 15% occupied non-tenure roles including lecturers and graduate students.

Results

The consistency of the inquiry and study scales for forms of mathematical work was reported elsewhere (Herbst et al., 2024a). To estimate a measurement model for opportunity to learn, we performed confirmatory factor analysis (CFA) on the items associated with the 10 hypothesized SLO constructs (see Figure 1). Given the limitations of that measurement model, we conducted exploratory factor analysis (EFA) on those same items to construct a new model. We felt the need to go beyond confirming our hypothesized model because some constructs had only two items; inter-item correlations within items indicating some constructs, namely SLO 1 and SLO 3, were too low suggesting poor internal consistency (Furr, 2017), and the item loadings under some of the constructs were low or cross-loaded to other constructs (Worthington et al., 2006).

We initially conducted an Exploratory Factor Analysis (EFA) with a smaller sample (n=118) and subsequently validated our model with additional samples (n=140). Following the identification and removal of items with low loadings, cross-loadings, or loading onto a two-item or one-item construct, we used eigenvalues and Kaiser's criterion to analyze and determine the number of constructs. This process resulted in a new model comprising five constructs with improved inter-item correlations.

Table 1: Correlation between constructs (in each model) and the mathematical work students were engaged in (student and inquiry)

	Study	Inquiry		Study	Inquiry
SLO 1	0.045	0.081*	Axiom	0.263*	0.085
SLO 2	0.039	0.165	Definition	-0.120	0.765***
SLO 3	-0.023	0.083	DGS	-0.252	0.412*
SLO 4	0.266*	0.094	History	0.644**	0.151
SLO 5	-0.105	0.906**	Conjecturing	-0.026	0.284*
SLO 6	-0.245	0.450*	**p<0.05, **p<0.01, ***p<0.001*		
SLO 7	0.372**	0.124			
SLO 8	-0.196	0.241*			
SLO 9	0.270	0.155			
SLO 10	0.019	0.432*			

$^*p<0.05$, $^{**}p<0.01$, $^{***}p<0.001$

In comparison to the fit indices of the hypothesized model (CFI = 0.842, TLI = 0.814, RMSEA = 0.079, SRMR = 0.096), the new model demonstrated significant improvement (CFI = 0.947, TLI = 0.935, RMSEA = 0.062, SRMR = 0.072) (Hu & Bentler, 1999). Examination of individual items under constructs in the new model compared to those in the hypothesized model revealed that SLO 4 (referred to as Axiom in the new model) and SLO 5 (referred to as Definition in the new model) remained unchanged. SLO 6, SLO 7, and SLO 10 differed by only one item from the constructs DGS, History, and Conjecturing, respectively, in the new model. This similarity between the models supports the confirmation that some constructs in the hypothesized model were robust, while others were not.

Given the robustness of these constructs concerning student opportunity to learn, we focused on exploring the relationships between these constructs (in each model) and the mathematical work students were engaged in (i.e., study or inquiry). An item covariance between an item in the Inquiry construct and an item in the SLO 5 construct (or Definition construct in the new model) was added to the model, as suggested by the highest modification index. Adjusting these parameters not only improved the overall model fit but also brought the correlations to standardized estimates (see Table 1). We found a significant association between engaging students in geometric inquiry and giving students opportunity to learn about geometric transformations, digital geometry environments, and the role of definitions in mathematics (SLOs 5, 6, and 10). We also found a significant association between engaging students in the study of geometry and giving students opportunity to learn about Euclid's Elements, and the role of axiomatic systems (SLOs 4, and 7). Conversely, the distinction between study and inquiry did not significantly impact the extent to which instructors claimed their students learned about proof, evaluating arguments, the content of high school geometry, non-Euclidean geometry, or constructions.

Table 2: Latent Variable Mean difference between demographic groups (ME or M)

Received Highest Degree in Field of Mathematics Education (ME) (N=39) or Mathematics (M) (N=96)			
Constructs in hypothesized model	LVM in ME after setting M to 0	Constructs in new model	LVM in ME after setting M to 0
SLO_1	-0.166*		
SLO_2	-0.406*	Argument	-0.284*
SLO_3	0.064		
SLO_4	-0.372*	Axiom	-0.368*
SLO_5	0.336	Definition	0.330
SLO_6	0.418	DGS	0.405
SLO_7	-0.188	History	-0.229
SLO_8	0.116		
SLO_9	-0.535*		
SLO_10	-0.075	Conjecturing	-0.023

* p-value < 0.05

We also conducted a comparative analysis of latent variable means among instructors holding the highest degree in either mathematics (M) or mathematics education (ME) to explore whether their preparation could serve as a predictor for the likelihood of offering opportunities for students to learn content associated with various SLOs. To assess the size of between-group differences per construct, we set the latent variable means in the group with the highest degree in mathematics to zero and estimated the means in the group with the highest degree in mathematics education. Across both models, it became apparent that instructors with highest degrees in mathematics were more likely to engage students in learning axioms (SLO 4), and in learning geometric arguments, such as understanding proofs (SLO 1) and evaluating arguments (SLO 2). Notably, the SLO construct related to non-Euclidean geometry (SLO 9)—whose items indicated the construct named History in the second model—appeared to be associated with instructors holding the highest degree in mathematics.

Conclusion

The results shared provide a glimpse of how instructor claims about the opportunity to learn geometry they provide to their students relates to the kind of mathematical work they organize for them. In turn these results help see a baseline of implementation of the SLOs. Some differences in this implementation are associated with the field in which instructors were prepared. We notice that extending the consensus over the 10 SLOs may require more conversations across the differences among instructors, one of which seems to be their academic preparation. We also notice the need to better measure opportunity to learn; notably, engagement of students in proof (SLO 1), which Ion et al. (2023) showed to be something most instructors agree should be an objective in geometry courses for teachers could not be measured robustly.

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Appendix: Sample items indicating opportunity to learn

SLO	Item	Item statement: Students had the opportunity to …
1	4101	… learn to write geometric arguments (e.g., proofs)
2	4105	… check whether proofs were valid
3	4118	…analyze properties of different two-dimensional geometric shapes
4	4106	…work with different axiomatic systems
5	4132	…write definitions
6	4108	…use dynamic geometry software to explore figures.
7	4110	…learn about Euclid’s Elements
8	4114	… perform basic Euclidean straightedge and compass constructions
9	4117	… learn differences between Euclidean geometry and other geometries
10	4126	… apply transformations to analyze mathematical situations

AN ANALYSIS OF ELEMENTARY PRESERVICE TEACHERS' KNOWLEDGE TO SOLVE AREA AND VOLUME MEASUREMENT TASKS

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This study explored elementary pre-service teachers' (PSTs) content knowledge in area and volume measurements. Written pre-assessments and follow-up interviews were conducted with 26 PSTs to explore how PSTs approached and tackled area and volume tasks using a variety of strategies, including conceptual and procedural strategies. Recommendations for supporting elementary mathematics teacher education classes design are discussed.

Keywords: pre-service teachers, area, volume, problem-solving strategy

Introduction

The National Council of Teachers of Mathematics' (NCTM) "Principles to Action" (2014) outlines highly effective teaching practices. Two recommended teaching practices are implementing tasks that promote reasoning and problem-solving and building procedural fluency from conceptual understanding. Among the many topics in mathematics, area and volume provide foundational knowledge and applications that extend to understanding concepts such as multiplication, fractions, as well as advanced topics like calculus and the sciences (Vasilyeva et al., 2013). To enhance students' performance in higher-level mathematics, teachers need a robust content knowledge that enables them to plan and implement effective teaching practices. Previous studies have highlighted curricular limitations in widely-used U.S. textbooks concerning measurement (Smith et al., 2016) and teachers' limited content knowledge in area and volume measurements (Baturo & Nason, 1996; Gutiérrez & Jaime, 1999; Murphy, 2012). These results make it challenging for educators to plan area and volume measurements lessons that promote reasoning and problem solving and developing procedural fluency from conceptual understanding.

Literature Review

How Students Learn Area Measurement and Common Challenges

Several studies have delved into students' understanding and their pathways to grasping area measurement conceptually (Barrett et al., 2017; Sarama & Clements, 2009). Previous research has indicated that there are foundational ideas in area measurement including covering a region without gaps or overlaps with equal-sized units, equally partitioning a region, counting unit measures, iterating combined units, comprehending row and column structures, linking the number of squares to length and width to make sense of the formula, and understanding that the subdivided whole is equal to the sum of its parts (Barrett et al., 2017; Sarama & Clements, 2009). These foundational concepts align well with an area learning trajectory, and students who grasp area measurement conceptually can apply these ideas effectively (Barrett et al., 2017). Additionally, understanding that a subdivided whole is equal to the sum of its parts and that manipulation and transformations of subdivided pieces can conserve areas is crucial (Kospentaris, Spyrou, & Lappas, 2011; Lehmann, 2022).

How Students Learn Volume Measurement and Common Challenges

Students employ several strategies to solve volume measurement tasks, including visualizing a set of unit cubes as (a) rectangular arrays organized into layers, (b) space-filling structures without organizing them into layers (e.g., column iteration), (c) entities in terms of their faces, and (d) employing the volume formula (Vasilyeva et al., 2013). The development of knowledge in volume measurement aligns with area measurement; being able to fill three-dimensional space, comprehend partial layer structures, and iteratively construct partial structures can lead to deriving the volume formula as similar process in two – dimensional space can lead to verification of the area formula (Van Dine et al., 2017). These conceptual ideas are essential components of students' volume learning trajectory, as students who grasp volume measurement conceptually can effectively apply these conceptual ideas (Van Dine et al., 2017; Vasilyeva et al., 2013).

Teachers' Knowledge in Area and Volume Measurements

Despite the importance of teachers' content knowledge, several studies underscore the challenges that pre-service teachers (PSTs) face in developing the knowledge necessary to plan lessons that promote effective teaching practices in area and volume measurements. PSTs lesson plans often have a procedural focus, and they frequently depend on prototypical images of specific shapes to make sense of formulas (Gutiérrez & Jaime, 1999; Hong & Runnalls, 2020; Murphy, 2012). Additionally, PSTs often equate the ability to use formulas correctly in response to area and volume tasks with understanding area and volume measurements (Hong & Runnalls, 2022; Runnalls & Hong, 2019). These studies indicate that PSTs often encounter challenges similar to those faced by elementary students (Kospentaris et al., 2011; Lehmann, 2023). The purpose of this study is to explore the current problem-solving strategies employed by elementary PSTs regarding area and volume measurement content knowledge. Two research questions guide the study:

(1) What multiple conceptual/procedural strategies do elementary PSTs employ to solve area and volume tasks?
(2) How do elementary PSTs struggle when confronted with problem-solving processes while attempting to employ conceptual and procedural strategies?

Methodology

Setting, data collection and analysis

This study took place in an elementary teacher education program at a large Midwestern public university in the United States. The Geometry and Measurement course, taught by the first author, is a 16-week mandatory course for upper-class students in the program. The research design consisted of two phases: 1) a pre-assessment of content knowledge, and 2) semi-structured interviews. The first phase involved 26 PSTs (25 females and one male), and the second phase included eight of them (seven females and one male). All participants volunteered for this study. The pre-assessment consisted of a total of 18 items, which were completed by the 26 PSTs. These 18 items were adapted from the Trends in Mathematics and Science Study (TIMSS) studies due to the poor performance of U.S. students on these selected items. It was essential for PSTs to have strong content knowledge related to the fundamental conceptual ideas addressed by these items. From the initial set of 18 items, we pinpointed three items—4, 9, and 11 (as shown in Figure 1)—on which PSTs displayed a wide range of problem-solving strategies.

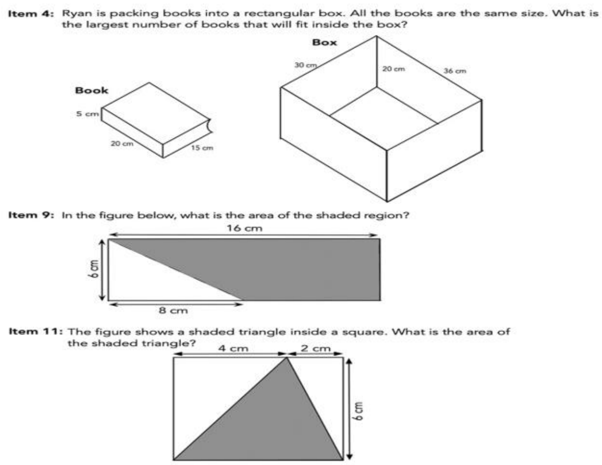


Figure 1: Three items with the greatest response variability (adopted from Foy, Arora, & Stanco, 2013)

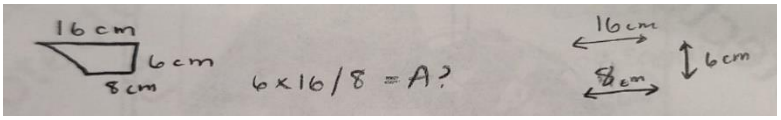
Following the pre-assessment, a semi-structured interview lasting approximately one hour was conducted with each of the eight participants who consented to participate in the second phase. Compared to the pre-assessment, the interview questions were significantly more open-ended as interviewees were encouraged to think about additional possible problem-solving strategies for each item. After all interviews were completed, the PSTs' responses to the three items were analyzed to distinguish among multiple strategies as per previous studies on how students learn area and volume measurements (Table 1) (Barrett et al., 2017; Van Dine et al., 2017; Vasilyeva et al., 2013).

Table 1: Conceptual and procedural responses for area and volume tasks

Problem-solving strategies	
Conceptual	Procedural/non - conceptual
Covering/tiling/filling two and three – dimensions with same sized units (squares or cubes) and linking it to the formula (Item 4) Understanding that the subdivided whole is equal to the sum of its parts (area conservation) (Items 9 and 11). Using array and layer structures (Item 4). Using partial structure of array (squares) or layer (cubes) to iterate or skip counting (Item 4). Recognizing that a triangle covers half the space compared to a square (Item 11). Being able to identify the base and the height of non – prototypical shapes (Items 9 and 11).	Using the formula without conceptual strategies (all three items). Looking for the lengths of the sides of shapes without considering conceptual strategies. Using formulas incorrectly.

We already know a triangle's area is 1/2 times base times height. Because the base of the shaded triangle is equal to the square's side length, which is 6 cm, and the height is also the 6 cm, the area of the shaded triangle is half of a square without doing any calculations. (**Conceptual strategy**)

(**Procedural strategy**)



Book volume is 6×15×20=1800 cubic cm, and box volume is 30×36×20 = 21600 cubic cm. Then 21600/1800 = 12 books. I noticed the height of the book is the same as the height of the box, and the width of the book is exactly half of the width of the box 36/6 = 6, then 2 rows of 6 books is 12 books. (**Conceptual and Procedural strategy**)

Figure 2: Coding Examples

Figure 2 shows examples of coding. The first response pertaining to Item 11 was coded as conceptual. This response shows that this PST was able to indicate why the area formula for rectangle is multiplied by a half and how much space a triangle covers compared to a square. The second written response, from Item 9, was coded as procedural, without considering that the subdivided whole is equal to the sum of its parts, this PST attempted to identify the relevant lengths of sides but struggled to apply the correct formula or use the numbers appropriately. Lastly, the third interview response was about Item 4 and it shows that this PST was able to use the formula to find the volume and at the same time, the idea of filling the box with the book. This can be both procedural and conceptual at the same time.

The two authors independently coded pre-assessment responses and interview responses to ensure the reliability of coding in this study. After the completion of the coding process, consensus between the authors was reached by discussing the responses on which they had had initial disagreements. The final result was 100% agreement for each response.

Findings

Pre-assessment responses to area and volume tasks

Figure 3 illustrates the distribution of PSTs' pre-assessment responses on three items categorized by the type of strategy and correctness. The data indicates over 60 % of PSTs struggle with solving geometry problems using either procedural or conceptual strategies. While it's not surprising to see that more PSTs employed procedural strategies, it is interesting to note that a significant proportion of them were unable to execute procedural strategies correctly (50% on item 4; 43% on item 9; 53% on item 11; and 48% across all three items). When examining conceptual and procedural strategies separately, a similar situation occurred with PSTs who incorrectly used procedural strategies nearly three times as high than those who incorrectly used conceptual strategies (48% vs. 17%). On the contrary, the PSTs were more likely to provide correct answers using conceptual strategies (22% vs. 13%) in overall correctness.

Item 4

	conceptual	procedural
correct	9%	16%
incorrect	25%	**50%**

Item 9

	conceptual	procedural
correct	33%	5%
incorrect	19%	**43%**

Item 11

	conceptual	procedural
correct	19%	19%
incorrect	9%	**53%**

Summary across three items

	conceptual	procedural
correct	22%	13%
incorrect	17%	**48%**

Figure 3: Pre-assessment responses on three items by type of strategy and correctness

For Item 4, we observed that four PSTs employed approaches involving filling, stacking, or using rows with one PST correctly applied these methods, while four PSTs utilized a formula, with two of them applying it correctly. Here are some of the written responses that we were able to find.

> $15 \times 2 = 30\ cm.$ *6 goes into 20 three times. Two rows of books stacked three high = six books.*
> $b \times h\ \ 15 \times 6 = 90\ books.\ 30 \times 20 = 600.\ 600 \div 90 = 6.66$

In the first response above, it's evident that this PSTs attempted to apply conceptual ideas related to filling or stacking three-dimensional space with a reference unit, but struggled to place the books to fill the space without gaps – fundamental conceptual idea of finding a volume. The second response indicates that this PST failed to recognize that the task pertained to the volume concept and instead seemed to apply an area concept. For Item 9 we observed that nine PSTs divided the shape into a triangle and a rectangle (using the idea the subdivided whole is equal to the sum of its parts), with six of them correctly applying this approach. Eight PSTs subtracted the area of the non-shaded triangle from that of the whole rectangle to find the area of the shaded region, and four of them solved it correctly. Three PSTs employed incorrect formulas, such as the Pythagorean Theorem or volume formula, and six made computational errors like $6 \times 16 = 72$ or $6 \times 8 = 24$. Four PSTs identified some relevant lengths in attempts to use the formula but were not successful. Figure 4 displays these responses.

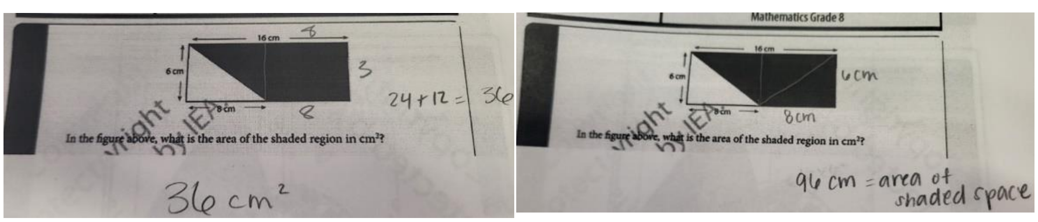


Figure 4: Examples of PSTs' written responses

These responses in Figure 4, including one in Figure 2, demonstrate that these PSTs recognized the need for lengths of sides but were unable to use them correctly. The first response shows that this PST did divide the shape into a square and a right triangle (using conceptual strategy) but was not able to identify correct length. The second response shows that the PST was correctly identify lengths but did not subtract the area of the unshaded part or simply used length × width without considering that the shape is not a rectangle (using procedure without

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considering conceptual strategy of dividing the shape or subtracting the unshaded triangle). The second response highlights that without a conceptual understanding that the subdivided whole is equal to the sum of its parts, PSTs may find it challenging to identify correct procedure to fluently use it, even when they correctly identify the lengths. For Item 11, two PSTs divided 36 by 2, indicating an understanding that a triangle covers half of a rectangle (or square). Other strategies included dividing the square into two right triangles to obtain 18 or simply using the base (6 cm) and height (6 cm) of the shaded triangle to calculate 18 cm² (done by two PSTs – again using the subdivided whole is equal to the sum of its parts). Six PSTs correctly applied the method of subtracting the area of the two corner triangles (6 cm² and 12 cm²) from the area of the square (36 cm²). Two PSTs used the Pythagorean Theorem to find the lengths of the hypotenuse to calculate the area of triangles. Similar to Items 4 and 9, several PSTs provided answers like 24, 36, or 98 without showing their work. The following responses in Figure 5 suggest that some PSTs were unsure about the amount of space covered by an inscribed triangle, uncertain to know why $\frac{1}{2}$ is multiplied for the area of triangle (incorrect use of conceptual strategy of covering) or which formula to apply, unsure about the difference between area of square and triangle (simply using procedure without considering conceptual strategy). Like Item 9, these PSTs attempted to identify lengths but struggled to choose the correct formula.

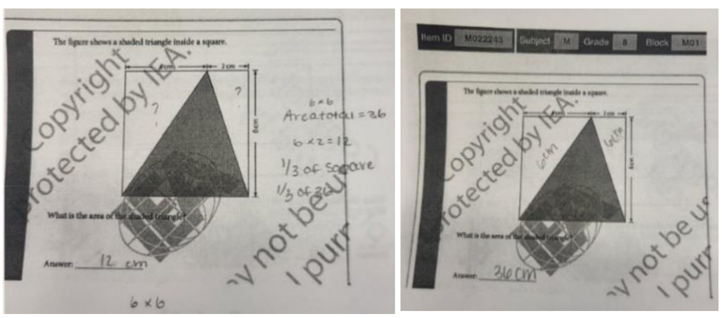

Figure 5: Examples of written response

Finally, another interesting finding is that all PSTs who were unable to provide correct responses to Items 9 (7 PSTs in total) and Items 11 (13 PSTs in total) also couldn't provide correct responses to Item 4. This result may indicate a strong relationship between understanding area and understanding volume, as described in area and volume learning trajectories (Van Dine et al., 2017).

Interview responses to area and volume tasks

With our results from written responses, we were interested in interviewing them to explore their thinking further. The findings revealed their initial struggle and when they were probed further by the interviewer, some of them were able to provide either conceptual or procedural strategies correctly. Consider the following descriptions of their responses on Items 4 and 9:

> *I found the volume of the box first, then the volume of the book, then divided the volume of the box by the book to see how many books can be contained by the box. What I only remember is that I needed to multiply these three*

numbers to obtain the volume, but I found out why it works actually. Maybe because the length and width can be used to determine the perimeter of the outside of the shape measured, and the height indicates the depth of it. So if you multiply all those together, the full shape can be obtained. (Item 4)

I calculated the area of right-angled triangle as 8 times 6, which equals 48 but I don't know what should be done next. (Item 9)

The first response shows that PSTs were able to articulate the procedure (finding volumes of the box and the book and length × width × height) but was not able to clearly state why (mentioning perimeter). Among the 8 PSTs interviewed, four of them eventually used the formula to correctly answer Item 4. The second PST, on Item 9, struggled to explain why and how the formula worked, the formula she mentioned was not correct. For all three items, we were also able to see that more PSTs were eventually able to suggest alternate methods as they were probed by the interviewer. The responses below were not used by the PSTs in the pre-assessment, but they were able to suggest them as the PSTs were asked to think about additional strategies during the interview.

If you stack the books like you would on a bookshelf, it will six across the top because 6 times 6 equal 36. And then 15 is half of 30. It will be two rows of 6 times 2 is 12 (Item 4).

Dividing the shaded region into three smaller congruent triangles, then finding that the area of each small triangle was 24 cm^2 and multiplying by 3 (Item 9).

The area of the shaded region is equal to 75% of the area of the large rectangle (Item 9).

PSTs described additional strategies they used in the interview, such as considering how to fill the box with same-sized unit books for Item 4, counting three unit triangles in the trapezium shape, and the area of the shaded region is 75% of the large rectangle (Item 9), and identifying the base and height of the triangle while considering conceptual strategies for Item 11. Notably, when asked whether the area of the shaded triangle in Item 11 was always equal to half the area of the square without performing calculations, half of the PSTs realized that the triangular area would indeed be equal to half the square's area when they share the same base and equal height. Ultimately, two PSTs on Item 4, five on Item 9, and six on Item 11 proposed correct solutions through multiple conceptual or procedural strategies. In addition to outlining alternative strategies, PSTs were questioned about the rationale behind fundamental formulas, such as volume (volume = length × width × height) and area (area = length × width). Initially, they found it challenging to explain why and how these volume and area formulas worked. For instance, two responses regarding volume for Item 4 included: "*If you multiply the three numbers, you can count the spaces*" and "*Length and width can help to get the perimeter of outside of the shape, and height represents the depth*." Related to area in Items 9 and 11, one PST remarked, "*Area is two-dimensional, so you need to multiply two numbers.*" These responses indicated a lack of clear conceptual understanding. Only a few PSTs provided partial, reasonable explanations when prompted by the interviewer, such as *"Area represents the number of squares, and volume* *represents the number of cubes*". Similarly, when asked by the interviewer which geometry measurement topic was related to each item, fewer than half of them were able to make the correct alignment.

When we asked PSTs to provide definitions of area or volume during the interviews, the majority responded with uncertainty, saying something like, "*I'm not sure what an area/volume formula is, or how the formula can be described*." Others offered definitions such as, "*Area is how many rows of length*." These ambiguous definitions likely contributed to their uncertainty about identifying relevant lengths, and which formulas to use for each task. In summary, Table 2 illustrates the types of incorrect strategies in PSTs' content knowledge related to area and volume, as revealed through pre-assessment and interview findings.

Table 2: PST's incorrect strategies in area and volume measurement

	Pre-assessment	Interview
Incorrect Conceptual	Filling (or stacking) space but with gaps. Not being able to identify area or volume concept. Using a formula that did not identify base and the height	Struggling to understand area or volume conceptually. Struggling to explain why and how a formula worked. Unable to provide appropriate definitions for area and volume measurements
Incorrect Procedural	Using a wrong formula Identifying lengths but not being able to use the appropriate formulas. Making calculation mistakes when using the correct formula Not being able to identify relevant lengths.	Confusing formulas for different shapes

Summary and Discussion

As prior studies have indicated, our PSTs initially preferred procedural-based problem solving strategies (Chamberlin & Candelaria, 2018). PSTs provided unclear definitions of area and volume, unable to identify relevant lengths, and were uncertain about the correct formulas to be used for each task. One implication of this finding is that more PSTs were able to find the correct way to use conceptual or procedural strategies when they were probed, which indicates the importance of providing purposeful questions and scaffolding when implementing lessons, as suggested by NCTM and other study (NCTM, 2014; Wickstrom, 2022). Another interesting finding was that when conceptual strategies were used, it was more likely for PSTs to answer area and volume items correctly. Some of them could not articulate appropriate formulas and even when they were able to articulate them, they were uncertain to use them. Identifying relevant lengths and stating correct formulas for each task is challenging for some PSTs and being able to use them correctly is another challenge that PSTs had. In previous studies, PSTs often equated using formulas for area and volume to understanding these measurements (Hong & Runnalls, 2022). Our results showed that it is not synonymous to think that using formulas can be interpreted as understanding because using formulas is challenging without conceptual understanding. This finding supports finding from previous study that formula might be used without knowing why and for wrong shapes (Vasilyeva et al., 2013; Zacharos, 2006). It demonstrated an interesting relationship between conceptual strategies and procedural strategies; if a PST was unable to use the conceptual strategy correctly, it was more unlikely for them to use procedural strategies correctly. Furthermore, when PSTs were not able to provide correct responses to area tasks (Items 9 and 11), they were not able to provide correct responses to volume task either (Item 4). The conceptual link between area and volume can help them understand that volume can be viewed as an extension of area from 2-dimensional objects to 3-dimensional objects as learning trajectories for length, area and volume are closely related (Van Dine et al., 2017). As shown by the PSTs' responses in both pre-assessment and interview, inadequate understanding of area measurement will very likely lead to challenges in learning volume measurement as described in learning trajectories (Van Dine et al., 2017).

It's not surprising to discover that elementary PSTs often lack content knowledge in area and volume measurement, and their preference for procedural strategies in solving such tasks is expected. However, these findings highlight important implications for mathematics teacher education programs. Considering the well-known limitations in widely used curriculum materials (Smith et al., 2013; Smith et al., 2016), these results become even more concerning. PSTs' limited content knowledge suggests that they face similar learning challenges as their future students (Lehmann, 2023), which can lead to mathematics lessons that may focus mainly on procedural tasks. Even though teachers might prepare procedurally focused tasks, it is also possible that they might not be able to use the procedures correctly by simply memorizing them. This underscores the importance of mathematics teacher education programs enhancing PSTs' conceptual understanding alongside procedural fluency by providing precise definitions of area and volume, and establishing the connections between these definitions and the formulas to explain why each formula works. Also, it will be beneficial to develop lessons to show how area and volume measurements are conceptually related in order for students to be able to link two important measurement concepts.

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EXAMINING ELEMENTARY PRESERVICE TEACHERS' RESPONSES TO AREA AND VOLUME MEASUREMENT TASKS

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Keywords: pre-service teachers, area, volume, problem-solving strategy

Introduction

Teachers need various types of knowledge to plan and implement effective teaching practices that promote reasoning and problem solving. Different types of knowledge can pertain to content or pedagogy. In this poster, one area that we are particularly interested in is area and volume measurements. Area and volume are two important topics that have wide range of applications in mathematics and provide foundational knowledge for multiplication, fractions, and the advanced topics of calculus and the sciences (Vasilyeva et al., 2013). Despite the importance of area and volume measurements, US students have not performed well in area and volume measurements (Lehrer, 2003; Mullis et al., 2012; Muillis et al., 2016). Here are research questions that guided us.

(1) What is the discrepancy in problem-solving strategies of PSTs' responses to area and volume measurement tasks?
(2) How does conceptual understanding are utilized to support PST's problem-solving strategies?

We used items from Trends in Mathematics and Science Study (TIMSS) studies to examine PSTs' content knowledge. Our results show that it is challenging for PSTs to use procedures if they are not able to use conceptual strategies. This finding shows the importance of building procedural fluency from conceptual understanding. It is unlikely that PSTs can simply memorize and use procedures correctly because being able to correctly identify relevant lengths and formulas is challenging for PSTs. They need to first identify that area and volume measurements are relevant concepts for the given tasks and also know foundational ideas of covering and space filling to interpret and solve area and volume tasks correctly. Also, solving volume tasks correctly may enable them to solve area tasks correctly as well. This may indicate that foundational ideas in area and volume measurements are conceptually related as demonstrated in learning trajectories (Barrett et al., 2017; Van Dine et al., 2017) and importance of including tasks that show how area and volume measurements are conceptually related to each other. Challenges that we found in this study are very similar to challenges demonstrated by elementary students (Lehmann, 2023), which is concerning.

With our findings, we can recommend that in teacher education program, PSTs need to have opportunities to be exposed to area and volume tasks that promote reasoning and problem solving. They need to be exposed to how foundational ideas in area and volume are related to each formula to build procedural fluency from conceptual understanding.

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PREPARING FUTURE TEACHERS: USING MENTOR LESSON PLANS TO SUPPORT TEACHING ELEMENTARY MATHEMATICS FOR SOCIAL JUSTICE

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Keywords: Elementary Mathematics, Social Justice, Pre-Service Teachers, Lesson Plans

Objectives and Perspective

In supporting elementary pre-service teachers (PSTs) to enact pedagogical practices that promote teaching elementary mathematics for social justice, mentor lesson plans can be a useful resource to enhance their learning and teaching competence. Bartell (2013) asserted that through teaching mathematics for social justice, students use mathematics to study the world, increase their knowledge, and learn about issues of social injustices toward developing ideas of equitable practices for creating changes. In most teacher education method courses, PSTs learn how to teach the required mathematics methods and strategies before engaging in student teaching experience. While some PSTs may further be introduced to social justice pedagogies (Freire, 2000; Ladson-Billings, 1995a, 1995b, 2014, 2021; Cochran-Smith, 2004; Paris & Alim, 2014; Paris, 2021) that enforce practices to support, enhance, and sustain the learning of all students inside the classroom. However, to effectively apply their learning, PSTs must explore instructional resources and engage in classroom discourse of ways to enact social justice pedagogies when planning and teaching elementary mathematics. It is for this reason, I suggest the use of mentor lesson plans as an instructional tool during mathematics methods courses to support PSTs learning. This practice will enable PSTs to see how they can plan and write mathematics lesson plans that promote teaching elementary mathematics for social justice.

I position this idea of supporting PSTs to teach elementary mathematics for social justice by using three frameworks (a) Freire's (2000) Pedagogy of the Oppressed with emphasis on the banking and problem-posing education models; (b) Culturally Relevant Pedagogy (Ladson-Billings, 1995a, 1995b, 2014, 2021) with emphasis on the three tenets - academic success, cultural competence, and critical consciousness; and (c) Cochran-Smith (2004) six principles to support PSTs in teaching for social justice.

Methods of Inquiry and Summary

To provide PSTs with mentor lesson plans that promote teaching elementary mathematics for social justice, I make use of the textbook created by Bartell et al. (2022) that provides different elementary mathematics lesson plans that incorporate specific social justice standards, mathematical focus areas, and mathematical concepts. By using these mentor lesson plans, PSTs are provided with a guide that connects their learning of social justice and more specifically, teaching mathematics for social justice. Through this approach, PSTs can begin to develop an understanding of how social justice pedagogies can be integrated into mathematics learning for students in elementary classrooms. The overall goal is to provide a scaffold for PSTs to engage with mentor lesson plans that incorporate the social justice standards in preparation for student teaching experiences.

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ARE WE REALLY TEACHING INTEGRATED STEM? EXPLORING UNCERTAINTY IN STEM TEACHER PREPARATION

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National policies and curriculum documents call for STEM integration into K-12 and teacher education. Certainly, studies have shown that engaging in STEM activities for PSTs is important; unfortunately, their implementation may not be uniform or effective. Analyzing the integration approaches of teacher educators and their method courses may better support PSTs' development of STEM integration and use of STEM-integrated activities. Results indicated that participating educators have various approaches to STEM integration within their methods courses. Yet, commonalities of pedagogical techniques were consistent across course syllabi and participants' interviews.

Keywords: Integrated STEM, Preservice Teacher Education

National policies and curriculum documents call for STEM integration into K-12 classrooms. A product of this integration is that students will better understand connections across STEM disciplines that will prepare them for specialized, high-demand skills (National Research Council (NRC), 2012; National Council of Teachers of Mathematics [NCTM], 2014). Furthermore, the Association of Mathematics Teacher Educators (AMTE) has suggested that effective teacher preparation programs "provide opportunities for candidates to make mathematical connections between various approaches to solving problems and opportunities for candidates to make connections between mathematics and other disciplines" (2017, p. 3). Similarly, the National Science Teaching Association (NSTA, 2020) has noted that STEM education provides opportunities for students to engage with "content in authentic and relevant ways." Therefore, preservice teachers (PSTs) must experience STEM integration and develop pedagogical knowledge, even if this occurs within disciplinary-specific courses.

Experiences with STEM integration must come sooner rather than later within teacher education. As an example, in Bartels et al.'s 2019 study, PSTs who had not participated in integrated STEM lessons could only define STEM disciplines and lacked an understanding of authentic STEM integration. After STEM-integrated lessons were modeled, 13 of these PSTs integrated three or more STEM subjects into their lesson plans and, overall, showed an enhanced understanding of STEM integration. Similarly, Bozkurt and Özyurt (2019) reported on 44 secondary, mixed-discipline PSTs who participated in a 12-week STEM program. Before this program, their PSTs understood STEM disciplines but were unfamiliar with STEM integration. At the end of the 12-week program, all 44 PSTs collaboratively developed and implemented integrated STEM lessons. Nonetheless, the participating PSTs reported challenges with finding materials and implementing STEM activities. However, Bozkurt and Özyurt reported that students were interested in the activities, so they anticipated using integrated activities in their teaching. Overall, these results are important as they show that prior to experiencing STEM integration, PSTs may not understand how to integrate STEM disciplines in their classrooms.

Yet, when PSTs engage in STEM activities and design and implement integrated STEM lessons, they show an enhanced understanding of integrated STEM.

As shown above, designing integrated STEM activities for PSTs is important; unfortunately, due to the lack of integrated STEM curricula, the implementation of such activities may be uncertain as they may not be uniform or effective. While many STEM educators agree that STEM disciplines are connected by ideas and skills and that STEM integration should focus on real-world contexts, the ways that disciplines are integrated into STEM education vary (Moore et al., 2020). Additionally, when STEM activities are multidisciplinary, engineering and mathematics may be given less attention. Furthermore, within the activities, if students are not asked to explicitly describe content concepts, such as specific mathematical ideas, they may not recognize the mathematical concepts that were employed. Thus, English (2016) suggested that teachers need support when they implement integrated STEM activities, especially with respect to highlighting the connection between engineering and mathematics. In addition to the literature showing variation in the content of STEM activities, there is also variation in the pedagogies and implementation strategies that PSTs experience. To understand more about the various approaches to STEM integration, finding commonalities is likely to support the development of authentic STEM integration in teaching methods courses. Thus, understanding how methods courses intend to prepare future teachers to implement integrated STEM activities is important for teacher educators. For this study, we explored: How do STEM teacher educators intend to integrate STEM within their teaching methods courses?

Methods

Twelve STEM teaching methods course syllabi and transcripts from six follow-up interviews of those who submitted syllabi were analyzed using qualitative methods. All but one of the syllabi were designed for secondary PSTs, the remaining for elementary PSTs. The data were analyzed using constant comparative methods and thematic coding in which the researchers conducted a line-by-line analysis of the syllabi and interviews (Strauss, 1987). A hybrid inductive and deductive coding method was used as most codes emerged from the data (Hatch 2002). To serve as a foundation, the definitions of levels of STEM integration (Figure 1) were used to code explicit mentions of any of the STEM disciplines and their respective levels of integration.

Type of Integration	Features
1. **Disciplinary**	Concepts and skills are learned separately in each discipline.
2. **Multidisciplinary**	Concepts and skills are learned separately in each discipline but within a common theme.
3. **Interdisciplinary**	Closely linked concepts and skills are learned from two or more disciplines with the aim of deepening knowledge and skills.
4. **Transdisciplinary**	Knowledge and skills learned from two or more disciplines are applied to real-world problems and projects, thus helping to shape the learning experience.

Figure 1: Increasing Levels of Integration (English, 2016, p. 2)

Through an iterative process of individual coding and team meetings to gain consensus on codes and develop a coding dictionary, the codes were refined. The following actions were taken to ensure trustworthiness: all coding was completed using Dedoose (2021), qualitative analysis software; meetings were held to discuss coded excerpts and resolve disparities; all syllabi were coded by at least two researchers; and at times, syllabi and interviews were coded simultaneously by all researchers. Through this process, themes emerged that described levels of STEM integration and emphasized pedagogical techniques across the various courses.

Results

We present our results related to the levels of STEM integration as defined by English (2016) and pedagogical techniques. While there were examples of higher levels of integration such as transdisciplinary and interdisciplinary, the levels of STEM integration varied (Table 1). Most syllabi focused on multidisciplinary or disciplinary levels of integration within the course descriptions, standards, weekly calendars, and assignments. The few instances of transdisciplinary integration included students solving real-world problems through tasks or projects. PSTs would learn about how to teach "solving human and environmental problems through mathematics" (Yu-ri's syllabus). There were many references to a single discipline or two or more disciplines. This was evident through the discipline-specific reading assignments for mathematics or science PSTs. Julie's syllabi course descriptions serve as an example where mathematics and science are explicitly mentioned in which students build capacity to teach the societal uses of mathematics and science as well as "the use of technology across mathematics and science content" In these syllabi, PSTs would make connections across mathematics and science but may not have opportunities to use them in an interdisciplinary or transdisciplinary nature. These examples represent instances that were coded throughout the syllabi.

In addition to levels of integration, the syllabi included several references to student-centered pedagogy. For the purpose of this report, we have included student-centered strategies with at least 50 occurrences in the coding such as equity, discussion or discourse, designing, and opportunities for reflection (see Table 1). As one of our highest occurrences, the equitable practices consisted of a variety of features such as cultural relevance and responsiveness, differentiation of instruction, building a positive classroom environment, and supporting students with disabilities. For instance, 10 of 12 syllabi mentioned that culturally responsive teaching approaches would be studied or discussed. Nine syllabi also included that differentiated instruction would be part of the course. Additionally, we noted that PSTs would learn a variety of teaching methods such as how to design and implement hands-on activities, facilitate discourse and discussion, lead inquiry activities, and appeal to students' interests. For example, eight of the syllabi noted that PSTs would learn methods like how to teach in ways that "all students investigate, collaborate, communicate and defend their explanations" (Julie's syllabi) as well as how to select resources for supporting inquiry and problem solving. Thus, throughout the syllabi, the instructors intended to teach in ways that support STEM Education methods that are grounded in constructivism (NSTA Board of Directors, 2020).

Table 1: Overall Themes in STEM Teaching Methods Syllabi

Theme	Frequency	Theme	Frequency
Level of Integration		***Student-Centered Pedagogy***	
Transdisciplinary	20	Equity	143
Interdisciplinary	23	Discussion/Discourse	57
Multidisciplinary	58	Designing	136
Disciplinary	89	Opportunities for Reflection	69

To further understand the syllabi results, follow-up interviews were conducted with participants who submitted syllabi. The interviews provided further insight into the STEM methods instructors' intentions for STEM integration. Some of the instructors noted that while they wanted to include more of the higher levels of integration, their students were going to be teaching siloed STEM content courses (e.g., algebra, biology, chemistry, physics). Thus, while they did their best to incorporate integrated STEM activities, they also had the tension of preparing their students to teach within their content areas. As evident in Julie's interview: "Sometimes I have to be like, 'Okay, here's the integrated [task]. Now, how would you silo it?' Because I have to be real about what their teaching contexts are." The interviews also revealed that, given the current curricular demands for K-12 teachers (e.g., standardized assessments), they found it best to focus on STEM pedagogies such as how to implement hands-on activities and provide opportunities for authentic problem solving. For example, Lana said, "I emphasize high leverage teaching practices or best practice strategies, but I think any of those right? I mean, like, student-centered. Lots of student voice in the classroom…working with different people every single day, getting kids up and moving. So, you know whether it be in math [or] science, whether doing labs or experiments, but getting kids up, so movement, collaboration." Here we can see how Lana considers best teaching practices focused on discussion and hands-on activities in her STEM teaching methods class while also balancing the need to prepare her students to teach siloed STEM courses.

Discussion

The findings learned from this analysis contribute to a better understanding of approaches to STEM integration and will inform the design of future STEM teaching methods courses. We can see from our research that while efforts are being made to incorporate integrated STEM into teaching methods courses, there are outside factors that influence the levels of integration. While the levels of integration may need some improvement, our findings also revealed that student-centered pedagogies were a main feature of the syllabi and intentions of the instructors. This is a positive finding, as these pedagogies align with preparing students to apply 21st Century Skills (NSTA Board of Directors, 2020). In particular, PSTs are learning how to support the solving of open-ended and authentic problems that require multiple STEM disciplines, working collaboratively with other disciplinary experts, and maintaining perseverance to work with evolving technologies and solve environmental issues.

Due to the small sample size, our results are not generalizable. This sample size was small in part because, after an extensive search for STEM methods courses within teacher preparation programs and broad calls for participation, teacher educators from only six U.S. programs with STEM teaching methods courses responded. However, these results provide the broader STEM education community with ideas on which to build. We recommend that STEM integration continues to increase throughout all levels of education and within teaching methods courses. To address the uncertainty of how students are experiencing STEM integration within classrooms, we also suggest that further research be done within STEM content courses in K-16 education. As a concluding thought, we suggest that as more STEM teaching methods courses are developed, our findings and recommendations can inform course design and implementation.

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AN EXPLORATORY MIXED METHODS STUDY ABOUT TEACHER CANDIDATES' DESCRIPTIONS OF CHILDREN'S CONFUSION, STRUGGLE, AND MISTAKES

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Keywords: equity, diversity, and inclusion; pre-service teacher education; research methods

This poster focuses on data from a program called SEE Math (Support and Enrichment Experiences in Mathematics; Kalinec-Craig et al., 2021; Kalinec & Rios, 2023), which is a revised version of the TEACH Math (Teachers Empowered to Advance Change in Mathematics; Turner et al., 2012) research group's case study of a child's thinking. Elementary teacher candidates (TCs) in SEE Math select a child in their field placement and engage in a series of problem solving tasks that support children's funds of knowledge, their broader mathematical thinking, and their Torres' Rights of the Learner (e.g., you have the right to be confused; to claim a mistake and revise your thinking; to speak, listen, and be heard; to write, do, and represent what makes sense to you; Torres, 2020; Kalinec-Craig, 2017). As a summative assessment for the methods course, the TCs conduct a 15-minute Mock Parent Teacher Conference (MPTC) to share what they learned about the child's thinking over the course of the program. The purpose of this study was to answer the following research question: How do elementary teacher candidates (TCs) describe children's confusion, (productive) struggle, and mistakes in their MPTC in humanizing ways (Goffney & Gutiérrez, 2018) and/or as rights of the learner?

The authors of this exploratory, embedded, mixed methods study began by collecting the saved transcripts from the MPTCs from 64 TCs across three semesters (spring 2020, fall 2020, and spring 2021). Analysis began by applying a natural language process called "topic modeling" to create a collection of words (named as topics) that described the entire data set; 50 topics (with ten keywords words each) were selected (Rios & Kavuluru, 2018). Only topics involving confusion, productive struggle, and mistakes were included in the final analysis. The second author applied a sentiment analysis procedure to the subset of the topic indices to explore the sentiment language (e.g., positive, neutral, and negative) used by the TCs when describing confusion, productive struggle, and mistakes. Finally, the first author applied a systematic, thematic analysis (Saldaña, 2020) to understand the contextual nuances of the TCs' descriptions in the MPTC.

Findings from the study showed that topics associated with confusion, productive struggle, and mistakes were typically assigned a higher negative sentiment than that of the other topics from the transcripts. Yet the qualitative thematic analysis showed, "the TCs used rehumanizing language that centered on children's confusion, (productive) struggle, and mistakes as a normal, albeit messy and complex, part of the learning process that centers on the child's humanity and rights as a learner." For example, one TC stated in her MPTC that her student, "… let me know when he didn't understand something or when he had questions. And this is totally normal. And those are his rights as a learner to ask questions or be confused." The study concluded that there is more to learn about how TCs describe children's confusion, productive struggle, and mistakes in ways that rehumanize how children learn mathematics. Our poster will focus this underexplored, but important mixed methods approach to understanding large bodies of textual data: a process that combines natural language processing models such as topic modeling and sentiment analysis with qualitative research methods.

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EXPERIENCES IN INFORMAL LEARNING ENVIRONMENTS AND SECONDARY TEACHERS' CREATIVITY IN LESSON DESIGN AND ENACTMENT

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This poster session shares the results of a study that examined eight secondary mathematics teachers' use of contextualized situations in their lesson design based on participation in a STEM discovery learning center. Eight secondary mathematics teachers are participating in a six-year project focused on the development of agency and authentic practice (Priestley, et al., 2015; Frost, 2016). Their experiences co-facilitating activity-oriented lessons in summer STEM camps prior to their degree program and STEM fairs throughout their graduate level licensure program influenced their use of realistic and creative mathematics problem situations in their formal classroom settings both as preservice and inservice teachers.

The foundation for this study was derived from two perspectives in mathematics education: contextualized problem situations and creativity in problem and lesson design (e.g. Sriraman, 2009; Sevinc & Lesh, 2022). Among other components, Sevinc & Lesh (2022) described mathematically rich and contextually realistic problems as having opportunities for sense making and creativity (p. 679). Their study provided evidence that preservice teachers increased in their understanding of how to develop mathematically and contextually rich problems over the course of a semester-long methods course but suggested that additional research is needed.

This study answers the following research questions:

1. How did preservice teachers initially respond to written questions about the feasibility of applying activities from the summer STEM camps to formal classroom instruction?
2. What types of contextually rich and creative problem-solving lessons did they incorporate into their classroom instruction?

The summer camp lessons involved hands-on activities centered around themes such as making stained glass from jolly ranchers as part of an exploration of medieval times. The graduate level licensure program required the preservice mathematics teachers to create their own STEM fair project and also guide students in their internship settings in their development of projects. Data sources included observations, reflective essays, lesson plans and focus group interviews. Initial responses to the STEM summer camp experiences indicated hesitancy to incorporate hands-on activities into formal classroom lessons due to perceptions that there might not be enough time or resources to include during daily instruction. However, views began to shift towards the end of their licensure program and into their first two years of teaching as they recognized the positive impacts of mathematically rich and contextually realistic problems on their own students.

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RELATIONSHIPS FIRST HIGH DOSAGE MATHEMATICS TUTORING: WHAT CAN WE LEARN FROM A LITERATURE SYNTHESIS?

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This paper shares a synthesis of the literature related to the application of a relationships-first approach to high-dosage math tutoring. In the context of our research, high-dosage tutoring is delivered multiple times per week during the school day by paraprofessionals who work with students in historically under-resourced schools. We apply a critical perspective to frame the importance of attending to interpersonal relationships during tutoring. We then explain the core ideas of small group interactions, dialogue, relational interactions, care and belonging and provide a synthesis of these constructs. The literature synthesis presented is intended to be applied to research-based efforts aimed at supporting tutors working to increase their skills for cultivating strong interpersonal relationships and enacting equity oriented pedagogy.

Keywords: Equity, Inclusion, and Diversity, Communication, Classroom Discourse, Affect, Emotion, Beliefs, and Attitudes

The opportunity for students to develop positive personal relationships as part of their content-based interactions during math tutoring is often presented as the keystone of a human tutoring model, providing support that cannot be offloaded onto current or forthcoming technologies, even ones using modern Artificial Intelligence. This is especially true in high dosage tutoring contexts where the same tutor works with a designated group of students throughout an academic year. Research demonstrating that strong teacher-student bonds can enhance student motivation and engagement, and positively impact learning outcomes (Davis, 2003; O'Connor & McCartney, 2007) underscores the value of a relationships-first human tutoring model.

One component of the University of Colorado Boulder (CU) and Saga Education research project is focused on contributing to the professional growth of novice tutors learning to cultivate strong personal relationships with their students. Specifically, the CU and Saga research team is studying discourse-based interactions in the tutoring context and incorporating this knowledge into artificial intelligence (AI) models to generate feedback to help tutors and the coaches who work with them improve their effectiveness. Leaning into the keystone benefit of human tutoring, the primary focus for tutors' professional growth centers on improving tutors' use of discourse and dialogue about mathematical concepts and supporting tutors' ability to cultivate increasingly caring and supportive relationships with their students. One goal of this research is to support tutors to learn how to weave interpersonal relationship building throughout the time they spend working with students on mathematical content for the dual purpose of increasing students' math learning and supporting students' general well-being.

Context and Overview

Historically, tutoring has been primarily available to those who hire private tutors to support enhanced academic success for students whose families can afford to pay for this service (Nelson-Royes, 2015). Providing tutoring to a more economically diverse population of students has been shown to be a valuable strategy for reducing achievement and opportunity inequities and supporting students' emotional well-being (Carlana & La Ferrara, 2021). In recent years tutoring, and particularly high-dosage tutoring provided in school during the school day, has been promoted as a way to support students and help address achievement inequities – this effort has increased in the aftermath of school closures and online learning during the Covid-19 pandemic (Carlana & La Ferrara, 2021). This increased attention and momentum has led to a rapid increase in the availability of tutoring during the school day for students who are less likely to be able to afford the cost associated with hiring a private tutor. As of 2015, 32% of US high schools required academic tutoring for at least some of their students (US Department of Education, 2017). More recently the U.S. Education Secretary, Miguel Cardona, has advocated for all students who are academically behind grade-level to receive high-dosage tutoring (at least 90 minutes per week), and COVID relief funds have provided a source of funding to support tutoring services in schools (Stavely, 2022).

The increase in human tutors working with students during the school day has several unique affordances. First, opportunities for students to work in small groups, for an extended period, with a knowledgeable math adult, tend to be rare in traditional classrooms. In an example of a context in which some students may receive small group instruction and where students have opportunities to engage with highly trained adults providing personalized support - students who receive special education services - the supporting teachers may not have the math knowledge necessary to effectively bring about high levels of learning, and not all students receive this support. Tutors who deliver high dosage tutoring can provide unique benefits by bringing high levels of math content knowledge to a small group tutoring context that is provided to all.

Second, tutors provide socio-cultural and relational support that is not always possible in classrooms. The population of people hired as tutors is more diverse than the teaching workforce (Contreras, 2022) and more demographically aligned with the population of students they are working with. The population of tutors whose professional growth our research supports includes 54% people from nondominant backgrounds, 19% of whom identify as Latine. These demographic characteristics increase the likelihood of students working with tutors with whom they share identity group memberships and/or cultural backgrounds, affinity and experiences. The combined scenario of working for extended periods of time in small groups paired with the potential for common experiences associated with shared identity group memberships and cultural backgrounds creates a powerful and unique opportunity for tutors to support students both academically and personally in ways that complement and augment other supports that exist in traditional school settings.

Theoretical Perspective

Our work is grounded in the perspective that incorporating interpersonal relationship building throughout tutoring sessions in ways that are integrated with content instruction contributes to supporting students' well-being and plays a central role in students' increased understanding and knowledge about math. Foregrounding interpersonal relationships supports students' thriving as both math learners and as developing young adults.

A critical perspective casts a powerful lens to understanding these theoretical commitments and thus better supporting this development. A critical perspective focuses attention on how math instruction that supports the "mathematical identities, excellence and literacies of marginalized students" (Gutiérrez, 2008, p. 357) may differ from instruction that leads to increased test scores and reduced participation gaps. Despite a longstanding call in math education research literature for math teaching to incorporate a critical perspective (e.g., Gutiérrez, 2007; Martin, 2003), math tutoring continues to be almost exclusively oriented toward increasing students' knowledge of math content with insufficient attention paid to how tutors and students relate to each other as more complete human beings. Gutiérrez's (2009) provides an initial framework for understanding equity in math that explains the importance of relationship building in high-dosage human tutoring contexts.

Gutiérrez's (2009) framework consists of two axes: the dominant axis includes the dimensions of access and achievement, and the critical axis includes the dimensions of identity and power. Tutoring models align with the dominant axis via their aim to increase students' content knowledge. By providing students with access to learning opportunities tutoring may increase students' achievement of academic success in math which may be indicated on assessments showing absolute increases in learning outcomes or reduced differences in achievement between identity groups. Tutoring models that only attend to mathematical content, without attending to interpersonal relationships, fail to account for the critical axis, including the impact of students' experiences with identity and power dynamics on their learning.

In delineating what is required in equitable math, Gutiérrez calls attention to the need to engage with students' unique identities and the power dynamics at play in math learning spaces. Engaging in math teaching and learning in ways that reflect distributed power structures can support more student-centered learning, students' agency and students' mathematical identities and sense of belonging. Students have been shown to benefit from opportunities to bring their lived experiences to bear on the task of learning math in ways that align with their personal identity group memberships and individual perspectives and experiences (Esmonde, 2009; Battey et al. 2016; 2018; Ford et al. 2014).

Dialogic learning spaces are an example of attention to both the dominant and critical axes. These spaces are built around rich and generative interactions between instructors and students that enable people to get to know each other's unique ways of thinking and doing math. The goal is supporting students to develop conceptual understandings of math that are rooted in their individual perspectives and lived experiences.

Literature Related to Relationship-First Instruction

A commitment to building strong interpersonal relationships in the context of high-dosage math tutoring has been justified using a broad range of evidence from research literature. Gutiérrez's framework provides a scaffold to situate additional components that are especially relevant to the classroom in general and tutoring specifically. Looking at the dominant axis, the incorporation of tutoring into a student's daily practice increases access and provides more support to increase achievement. Since tutoring inherently takes place in small groups it can bring about specific benefits unique to small group instructional contexts including increased opportunities for dialogue. Examining the critical axis, we draw from literature on relational interactions, care and belonging as relevant components necessary for attending to students' experiences of identity and power dynamics, and we observe that each of these elements is interrelated and critical to student success and wellbeing.

Small groups: creating a context for learning and growth

Tutoring constitutes a unique context in which students are working in small groups, but they have the full-time participation of an adult who holds extensive math content knowledge. Research on small group mathematical activity, and equity and inclusion in small groups working on mathematical tasks, may apply to the tutoring context, recognizing that the tutor's full-time participation differs from the role a teacher plays in a classroom setting. The presence of a tutor could potentially undermine the opportunity that is normally available to students working in small groups to exercise agency over how they engage with mathematical content. However, the small group setting offers opportunities for a skilled tutor to facilitate dialogue, engage in relational interactions, convey care, and cultivate a sense of belonging.

Esmonde (2009) proposed a theoretical framework for understanding Opportunities to Learn during small group learning activities. This framework consists of four points about how learning happens: "(a) through participation, (b) in relation to a social ecology, (c) through processes of identity development, and (d) through communicating about mathematical content" (p. 1011). In Esmonde's conceptualization, participation refers to students' opportunities to "move on a trajectory toward more central and competent participation in classroom practices" (p. 1011). Social ecologies in small groups account for the forces that contribute to the social construction of identity within the norms and dynamics of a single small group, along with external influences on identity development such as intersections of race, class, gender, sexuality, language communities and more. Students' identities can be influenced by their experiences engaging in dialogic learning, by their teacher's enactment of relational interactions and expressions of care, and by the degree to which they feel a sense of belonging in the math learning community. Finally, with respect to communicating about mathematical content Esmonde considers the role of shared meaning making in the processes used by small groups working on mathematical activities. Esmonde's (2009) framework provides a way to account for identity and power - Gutiérrez's critical axis - in math learning, paying particular attention to the experiences of students who are members of nondominant identity groups and considering how students can experience agency in math learning that is relevant within their unique life experiences.

Dialogue: connecting student and tutors together

The term 'dialogue' is often used loosely in reference to discussion or conversation between teacher and students or between multiple students. However, the literature on dialogic teaching and learning has some important characteristics worth attending more closely. Dialogic teaching and learning "characterizes an epistemological relationship," (Freire and Macedo, 2003, p. 191), "a process of learning and knowing" (p. 193) in which all participants have agency in the nature of the learning that occurs and how that learning develops. Billings and Fitzgerald (2002) observed that dialogic discourse can generate a "reciprocal flow of ideas involving actions and reactions of group members [that] may lead to new understandings not held by any group member in advance of the discussion" (p. 909). The learning that results from collective sensemaking through dialogue moves beyond transactional learning that flows primarily from tutor to students. In other words, dialogic discourse results in learning that is multidirectional which reflects participants unique perspectives, experiences and ways of thinking.

Development of a dialogic learning space is initiated by a teacher or tutor who is skilled at supporting student agency and who cultivates norms of participation that enable movement from peripheral to central roles of participation while considering students' identities and experience of shifting power and authority dynamics. Webb et al. (2019) observe that a teacher's role in creating inclusive and dialogic small group learning environments includes helping students know how to engage in active listening; ask and answer questions; brainstorm suggestions, ideas and opinions; explain and evaluate ideas; use persuasive talk; summarize conversations, and much else (p. 177). Xu and Clarke (2019) describe the importance of teachers considering cultural differences related to students' identities and life experiences that may influence students' participation and interactions with persons in positions of authority and may subsequently impact how students engage in a dialogic learning space. Langer-Osuna and Esmonde (2015) describes complexities of the shifting authority relations present in collaborative and dialogic math learning communities. The role of a teacher or tutor in cultivating a dialogic learning experience extends well beyond the creation of group-worthy tasks to include how these tasks are enacted through the intentional use of dialogic moves.

Relational interactions: attending to how tutors communicate

Literature on relational interactions specifically describes how teachers relate to and interact with their students (Battey, 2013; Battey et al. 2016; 2018). Relational interactions factor into the establishment of care, as will be described below, and they shape students' math learning experiences. Teacher content knowledge and implementation of instructional practices have been shown to be impactful for student learning (Battey, 2013; Battey et al. 2016; 2018), but relationship interactions are equally impactful. A teacher may have extensive content knowledge and be skilled at using an extensive repertoire of instructional strategies, but if they do not establish and maintain supportive relational interactions, students may not thrive. Consideration of the nature of relational interactions is especially relevant for students who identify as members of minoritized groups in society and who are more likely to be learning math from educators whose identity group memberships and cultural backgrounds differ from their own. If teachers' approaches to relational interaction are rooted in different cultural experiences or expectations this may result in relationships with these students that feel less personal or less familiar (Battey et al. 2016; 2018; Ford et al. 2014). Battey et al. (2016) specifically highlight how teachers perceive their African American and Latine students as more confrontational and spend more time disciplining their behavior as compared to their white peers. Ford et al. (2014) addresses the potential differences in how authority is established between White and Black teachers, and how this may impact their relationships with their students. While relational interactions are integrated with the concept of care, they can be more precisely descriptive of how a teacher connects with their students.

Battey et al. (2016; 2018) identified five components of relational interactions that help build such "caring relationships," with a particular lens on instructing minoritized students. These include framing math ability, acknowledging student contributions, attending to culture and language, addressing behavior, and setting the emotional tone. These dimensions can be tracked by looking at specific forms of dialogic acts or conversational moves. A teacher who takes a positive approach to enacting each of these components can create a supportive learning environment. However, the converse is true as well; if a teacher takes a negative approach, it can diminish the student's willingness to engage and restrict their opportunities to learn. For example, in a case study looking at an elementary math teacher's class, Battey (2013) observed positive framing of ability when the teacher encouraged students on a specific math problem, reminding them of their accomplishments on similar problems. Conversely, negative framing occurred when the teacher was sarcastic and questioned students' basic math skills. Relational interactions highlight how communication can have an immediate impact on students.

Care: establishing authentic relationships

A fundamental component of relationships is the element of care, or attending to the needs of others (Bartell, 2011; Potvin et al, 2022; Maloney & Matthews, 2020). Within an educational context, Noddings (1984, 1988) has applied care theory to describe the teacher-student dynamic. When a teacher shows genuine care and compassion, this can significantly impact the student's educational experience for the better (Bartell, 2011; Maloney & Matthews, 2020).

However, there are many different critical components needed to successfully develop an authentic caring relationship (Bartell, 2011). Caring for a student means attending to their wellbeing at a personal level, and not just caring for their academic success (Maloney & Matthews, 2020). Noddings (1988) describes the need for 'engrossment', meaning the teacher must understand the students' motivations and feelings and provide positive acknowledgment to form a reciprocal relationship. Similarly, Maloney & Matthews (2020) emphasize the need for empathetic care, which is "teacher's authentic expression of identifying with the challenges of their students and prioritizing students' well-being above their own" (p. 408). Maloney and Matthews (2020) found that when students experienced care as transactional or superficial, they were less invested in the class. However, when students felt empathetic care, they felt more connected to the class, that their input was valued, and that math was relevant to them. Whether a teacher shows genuine care for a student as a person, and not just for their academic performance, has ties to greater investment; students want to do better for teachers who care for them (Bartell, 2011; Maloney & Matthews, 2020).

To establish this level of care and understand a student personally, teachers must understand not only the student's identity and background but also their lived experience in the context at large. This is especially important for BIPOC students and any student in a marginalized or systematically oppressed community. To not acknowledge the challenges that have been built into these students' lives is to ignore a major factor in their educational experience (Maloney & Matthews, 2020). This also means teachers should be aware of their own teacher identity and any potential biases they may hold, especially as they are the ones in the position of power within the teacher-student relationship (Bartell, 2011). It is not enough for a teacher to care, the student must also feel cared for to establish an authentic connection (Bartell, 2011). Care in the classroom clearly has many nuances (Bartell, 2011), but is an essential part of maintaining healthy relationships (Potvin et al., 2022).

Belonging: bringing together all members of tutoring groups

Belonging is a multifaceted social construct that typically relates to the perception of inclusion and support within a community and is seen as a basic human need (Allen et al, 2021;

Gray et al. 2018). The sense of exclusion or isolation has been related to poor quality of life and depression (Allen et al., 2021). Thus, establishing a sense of belonging is important for young people, especially in socially interactive academic settings where they spend a significant portion of their time (Barbieri & Miller-Cotto, 2021). Not only is belonging associated with overall wellbeing, but it has also been found that the sense of belonging in a math setting is associated with better academic performance (Gray et al., 2018; Allen et al., 2021; Penuel et al., 2023; Barbieri & Miller-Cotto, 2021). Barbieri and Miller-Cotto (2021) found an association specifically between a sense of belonging in math and subsequent scores. Penuel et al. (2023) found that a student's sense of belonging predicted their level of contributions. This differed by race with White and Asian students contributing more than their Latine peers.

Similar to the findings on care and relational interactions, a sense of belonging is especially impactful for BIPOC students (Barbieri & Miller-Cotto, 2021; Penuel et al., 2023; Gray et al. 2018). Matthews et al. (2021) posits that there are seven key dimensions - 3 interpersonal and 4 instructional - that constitute Belonging Centered Instruction. Educational institutions have historically been a place of exclusion for students who are members of minoritized identity groups, and these experiences can undermine their opportunities to develop a sense of belonging. However, when care is established and positive relational interactions occur, this can bolster a student's sense of belonging. These concepts are interwoven together; when a student feels that a teacher genuinely cares about their wellbeing (often by using positive relational interactions) then the student may feel more belonging and be motivated to contribute and engage (Barbieri & Miller-Cotto, 2021; Penuel et al., 2023; Gray et al. 2018; Maloney & Matthews, 2020).

Relationship Focused Tutoring: Alignment of Constructs with Critical and Dominant Axes

To help us understand the role that each of the previously described constructs can play as components in a relationships-first high dosage tutoring context, we consider the alignment of key aspects of each construct with the dominant and/or critical axis. Notably, each construct contributes to both axes and helps expand Gutiérrez's framework by suggesting how educators can attend to access, achievement, identity, and power. Table 1 shows this alignment.

Table 1: Alignment of Constructs with Dominant and Critical Axes

	Small Groups	Dialogue	Rel. Interact.	Care	Belonging
Dominant	movement toward central participation	student agency	-framing ability -acknowledge contributions	positioning learning content as good for the self	supports academic success
Critical	attention to student's individual needs and experiences	-shifting authority relations -multi-directional learning	-attending to culture and language -set emotional tone	-attend to student wellbeing and lived experiences -empathetic care -potential biases	supports feelings of membership in learning community

Tutoring by nature involves working in small groups. As Esmonde's (2009) framework explains, small group interactions can support students to move toward more central roles of participation which enhances their access to learning opportunities and potentially academic achievement, the components of the dominant axis. Small group learning contexts also provide opportunities for individualized interactions that attend to students' unique identities and lived experiences, the components of the critical axis.

A dialogic learning space increases students' opportunities to engage with teachers' and tutors' ways of conceptualizing and doing math which has the potential to support access and achievement through students' participation in sense-making about mathematical ideas. The multidirectional nature of the learning generated through dialogic discourse depends on students' opportunities to contribute their own unique conceptualizations and lived experiences in relation to the math they are learning. Learning that results from dialogic interactions is reflective of the ideas, insights and perspectives of all participants and relies on shifting power dynamics and authority structures around who holds and contributes knowledge.

Regarding relational interactions, the ways that educators frame math ability, acknowledge student contributions and address behavior support students' access to learning opportunities, while educators' attention to culture and language and strategies for setting the emotional tone incorporate aspects of the critical axis of equitable math.

Educators' expressions and demonstrations of care attend to the dominant axis' components when they are concerned with how learning mathematical content is good for the students. When educators' enactment of care extends to attending to students' well-being, demonstrating empathy, learning about and being responsive to students' lived experiences, acknowledging and correcting potential biases and engaging in deep and authentic connection the critical axis components of identity and power are being addressed.

Finally, educators' who support the development of students' sense of belonging as it relates to established and accepted practices of doing math are attending to the dominant axis components of access and achievement, while educators' who cultivate learning environments in which students' sense of belonging as unique and valued members of a community of learners are attending to the critical axis components of identity and power.

Conclusion

In the context of high dosage math tutoring in which tutors bring a relationships-first approach to their interpersonal interactions with their students it is helpful to have a clear understanding of what is involved in cultivating strong and productive tutor-student relationships. We propose that a more equitable and inclusive form of math tutoring can be achieved by explicitly considering how tutors engage with students in ways that attend to both the critical (identity and power) and the dominant (access and achievement) axes of equitable math by first building strong positive relationships with students and then leveraging those relationships throughout their interactions about math content. This paper contributes a synthesis of research related to how professionals who work in instructional roles can build relationships that support students' well-being as a mechanism to contribute to increased math learning. This synthesis of the separate but related constructs described in this paper helps to explain how tutors can build positive interpersonal relationships in a high dosage tutoring context, supports the design of professional learning opportunities and contributes to the professional growth of tutors working to improve their skill at supporting and engaging with their students.

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ELEMENTARY PRESERVICE TEACHERS' USE OF VISUAL REPRESENTATIONS FOR FRACTION MULTIPLICATION

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How preservice teachers (PSTs) learn to visually represent fraction multiplication with partitioning is an ongoing area of research. Our study investigated what strategies 15 elementary PSTs used to solve and model two non-unit fraction multiplication tasks before instructional guidance. We found the majority of PSTs used algorithmic approaches to provide solutions and often did not provide an explanation or a visual representation to support their reasoning. Two special cases provided insights into challenges PSTs may face when depicting and partitioning the unit within area models. These findings have implications for purposeful sequencing of fraction tasks and intentional instruction around representing and partitioning units.

Keywords: Mathematical Representations, Preservice Teacher Education, Rational Numbers, Teacher Knowledge

Relevant Literature and Purpose

Area models are a core pictorial representation used throughout the scope and sequence of K-12 mathematics (Lischka & Stephens, 2020). Preservice teachers' (PSTs') ability to use representations fluently will directly impact their future students' conceptual understanding and procedural fluency as representations are the vehicle that students use to understand the abstract concepts central to mathematics (Pape & Tchoshanov, 2001). In the domain of rational numbers, area models have been used to investigate PSTs' conceptions of decimal multiplication (Rathouz, 2011), fraction division (Leitch, 2023), fraction multiplication (Gichobi, 2018), fraction addition (Lee & Lee, 2023), understanding of fraction multiplication through problem posing (Yeo & Lee, 2022), and the connections between fractions, geometry, and measurement (Lee & Lee, 2021). For fraction multiplication using area models, PSTs often struggle with producing visual representations of fraction concepts; connecting representations is crucial for the development of their content knowledge (Thurtell et al., 2019). Some of the challenges with fraction multiplication include: (a) insufficient foundational fraction knowledge; (b) interpreting fraction multiplication; and (c) recognizing the unit they are partitioning for accurate division and subdivision of factors (Son & Lee, 2016). This literature recognizes the importance of investigating the development of PSTs' use of area models for fraction multiplication.

For this report, we investigated the following research question: What strategies do elementary PSTs use when multiplying fractions before instructional guidance? This question contributes to a larger ongoing cross-institutional study focused on examining the relationship between the fluency in which elementary PSTs use area models with whole numbers influence the use of area models with fractions. Specifically, we examined PSTs' fluency with area models in relation to their ability to both solve fraction multiplication tasks and communicate their reasoning.

Theoretical Background

The mathematical experience and content knowledge that PSTs bring to their preparation programs must be strengthened to develop their pedagogical content knowledge. Pedagogical content knowledge should include the effective use of representations and the ability to develop procedural fluency from conceptual understanding (Ball et al., 2008). When adding fractions, PSTs often relied on procedural knowledge to solve the tasks and then represent their answers with an area model (Lee & Lee, 2023). When multiplying fractions, PSTs frequently used more familiar, algorithmic approaches as opposed to those that support the development of conceptual understanding (Gichobi, 2018). Although PSTs may have the procedural knowledge to compute the correct answers for fraction operations, they may lack conceptual understanding to transfer their thinking into different representations. Using only algorithms has been shown to limit PSTs' understanding of how the unit is important in fraction multiplication (Izsak, 2008). To make sense of traditional fraction algorithms, PSTs should engage with using area models to build fluency with fraction operations. To become well-prepared mathematics teachers, PSTs need opportunities to connect procedural fluency with conceptual understanding (NCTM, 2014).

Methods

Participants and Settings

Our participants included 15 undergraduate elementary PSTs (14 females and 1 male) who were enrolled in the second of a two-course sequence consisting of elementary mathematics content taught by one of the authors at a small, Midwestern university in the United States. The first course focused on whole number concepts and operations, and this second course focused on rational number concepts and operations. The concepts covered in these courses included a focus on understanding why familiar mathematical procedures work to support PSTs in developing conceptual understanding and pedagogical content knowledge for teaching.

Data Sources and Analysis

To investigate strategies PSTs used for multiplying two fractions before instructional guidance, they were given a paper-and-pencil pre-test prior to course work regarding rational number operations. The pre-test consisted of two non-unit fraction multiplication tasks: $\frac{2}{3} \times \frac{3}{4}$ (required no additional subdivision) and $\frac{2}{3} \times \frac{4}{5}$ (required additional subdivision), and PSTs were directed to solve them using an area model. Additional directions on the pre-test included: (a) in your area model, only use additional subdivision when it is necessary; (b) do not use algorithms to solve the problem; (c) make sure to draw each step of your solution, provide a brief narrative explanation for each step, and write symbolic notations that will help the reader understand how you solved the problem; (d) clearly mark your final solution.

We used an inductive coding by initially identifying key themes then iteratively generated new codes and themes (Creswell, 2013; Saldaña, 2016). First, we independently explored the data to visually inspect for trends in PSTs' strategies, writing analytic memos on initial thoughts regarding how to categorize PST solutions. We debriefed by discussing responses and refined categories for coding: Provision of narrative explanations, accuracy of response (conceptual, computation, and visual aspects), types of visual representations, and algorithmic approaches. Using these categories, we focused on two cases of visual representation for further examination.

Findings

Our preliminary pre-test findings revealed nuances associated with PSTs' use of the fraction multiplication algorithm and their adoption of area models to represent fraction multiplication. First, we briefly summarize those representations and algorithms. Then, we share two cases that allowed us to examine PSTs' emerging use of visual representations with correct solutions but whose models did not fully depict how they arrived at their solutions.

Summary of PSTs' Fraction Multiplication Strategies

We tasked the 15 PSTs with multiplying two pairs of fractions using area multiplication, accompanied by narrative explanations. Only 40% of the PSTs attempted to use visual representations for both tasks, and approximately 25% offered narrative explanations on both tasks. About a third of PSTs relied on algorithms for both tasks, some correct and some incorrect (e.g. "cross-multiplying"). The majority (around 60%) provided incorrect solutions, attributable to conceptual model errors, procedural strategy errors, or calculation mistakes. For instance, two PSTs used cross-multiplication, and some employed inaccurate common denominator strategies.

Variations in Visual Representations

The distribution of PSTs across different representation types for two distinct fraction multiplication tasks is summarized in Table 1. Only six PSTs attempted visual representations for each task, and one PST attempted a visual representation for only Task 1.

Table 1: Types of Visual Representations Used by the Number of PSTs

	Types of Visual Representations			
	One Rectangle with Length and Width Labeled as Factors	Area Model to Represent Each Factor Separately	Rectangular Area Model with Partitioning and Subpartitioning	Strips / Cuisenaire Rod Model
Task 1	3	2	1	1
Task 2	3	2	1	0

From Table 1, we observe that approximately half of the PSTs who attempted visual representations drew rectangles with the length and width each representing a fraction (the two factors). Two PSTs drew separate models for each of the factors and avoided subpartitioning their models to explain the multiplication process. These approaches were accompanied by algorithmic approaches to generate the product of multiplication. In Figure 1, Danielle (all names are pseudonyms) used a variation of this approach with circular area models representing $\frac{2}{3}$ and $\frac{4}{5}$ alongside a "multiply across" method to produce 8/15 as the answer. Danielle proceeded to draw a rectangular area model to represent the solution incorrectly with only 7 of the 15 squares shaded.

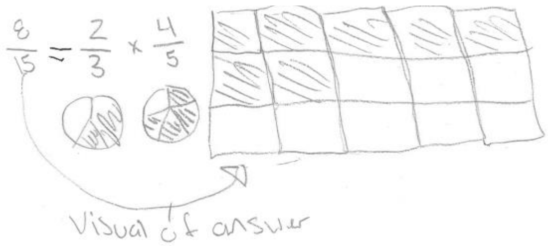

Figure 1: Danielle's Area Model Representation to Show ⅔ × ⅘

Interestingly, one PST drew strips or rods to represent two fractions. For obvious reasons, PSTs who employed two separate models for the two factors appeared to struggle with how to further partition to find the answer. Only one PST used partitioning and subpartitioning in their visual representation when solving the tasks. Amanda seemed to have used two different approaches for Task 1 and 2. Amanda's first model involves partitioning and subpartitioning the unit (Figure 2, left). However, how her model represents a unit (the whole), ⅔, but ¾ is not clear. It is possible Amanda first drew a rectangle and partitioned it into thirds horizontally and shaded two parts to show ⅔, followed by partitioning the whole rectangle again into fourths (one vertical partition and one horizontal partition) and shaded three parts to show ¾. Subsequently, Amanda may have double-shaded overlapping parts and counted these parts (of which some are conceptually unequal but in Amanda's visual representation look about equivalent in area) to generate the solution of 4/8 = ½. With three types of shading and unequal partitioning, it is hard to interpret her process of subpartitioning and finding the answer, even though the solution is correct. Amanda's model of ⅔ × ⅘ clearly shows her process of partitioning and subpartitioning the rectangular area model. It appears that she used the larger rectangle as a unit, initially employing horizontal partitioning to show ⅘ of the unit, followed by vertical partitioning to represent ⅔, thus showing 8/15 of the unit (whole) through double shading. It is interesting to see why Amanda chose to use two potential distinct ways to represent fraction multiplication.

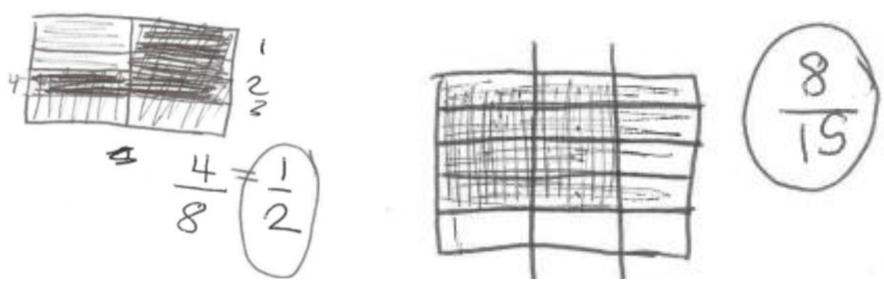

Figure 2: Amanda's Area Model Representations to Show ⅔ × ¾ (left) and ⅔ × ⅘ (right)

These findings highlight the inconsistencies PSTs may have in their current understanding of representing fraction multiplication, s specifically with depicting and partitioning the unit.

Implications

Many of our PSTs did not attempt to visually represent the fraction multiplication tasks and instead utilized algorithmic approaches. Similar to existing literature (Son & Lee, 2016), PSTs erroneously found the common denominator when multiplying fractions. PSTs rarely provided an accompanying narrative to explain their thinking in visual representations. Extending upon prior literature, Amanda's approaches demonstrated the difficulties PSTs face when identifying the unit in need of partitioning. We suggest that subsequent instruction emphasizes the importance of beginning with the unit of reference or involve real-world contextual problems that can assist in identifying the unit of reference. Amanda's inconsistent partitioning could result from the experience of partitioning even-numbered denominators both vertically and horizontally, which is not possible with odd denominators. Math teacher educators should then carefully consider the use of multiplication tasks involving both even and odd denominators with area models. We acknowledge these are preliminary findings drawn from PSTs' pre-tests on fraction multiplication. To further explore PSTs' conceptions of fraction multiplication, we will examine their growth following instruction and how it endured throughout the semester.

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FROM NOTICING TO NURTURING: THE TRANSFORMATION OF PST FEEDBACK THROUGH INTERNATIONAL COLLABORATION

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This qualitative case study explores the impact of Collaborative Online International Learning (COIL) experiences on the ability of pre-service teachers (PSTs) to provide evidence-based noticing and responsive feedback in mathematics education. The research demonstrates that engaging in COIL activities improved PSTs' attentiveness and the quality of their feedback, with an emphasis on conceptual-responses-oriented feedback. Furthermore, the study highlights the influence of group interactions during COIL experiences on these enhancements. The implications of the findings are significant for teacher education programs, emphasizing the importance of offering personalized feedback and cultivating a deeper understanding of mathematical concepts. The study underscores the value of collaborative learning experiences in enhancing teachers' expertise and advocates for further research in this field.

Keywords: teacher education, collaborative online international learning, noticing and feedback

Purpose of the Study

Assessing students' learning progress in a formative manner is essential for educators as it allows them to pinpoint areas that need attention, offer personalized feedback, and prepare for upcoming instructional sessions (Bailey & Drummond, 2006). Analyzing students' incomplete thought processes can provide valuable insights for instructors to tailor pedagogical approaches and improve educational outcomes (Peltier & Peltier, 2020). PSTs who encounter challenges in creating flexible and effective mathematical lessons may lack the necessary preparation to meet the diverse needs of their future students, underscoring the significance of enhancing teacher education programs to better equip PSTs with essential skills and knowledge (Lee & Kim, 2022; Mason, 2002). The study aimed to explore the impact of Collaborative Online International Learning (COIL) experiences on PSTs' capacity to provide evidence-based noticing and responsive feedback in mathematics education. Specifically, it aimed to evaluate the changes in the noticing patterns and feedback approaches of PSTs following their participation in COIL, with a focus on promoting conceptual-responses-oriented feedback. The research questions that guided the study were:

1) How does participation in COIL activities impact PSTs' noticing student thinking and providing feedback?
2) What are the effects of the COIL activity on the alignment between PSTs' noticing levels and the quality of feedback provided?

Theoretical Perspectives

Analyzing students' cognitive processes through observations of their work can provide educators with valuable insights. This approach enables educators to grasp students' understanding of mathematical concepts, facilitate deeper learning, and address any misconceptions through thoughtful feedback (Ball, 1991; McLaren et al., 2012). However, identifying error patterns can pose challenges, and solely evaluating work may not effectively support student comprehension (Peltier & Peltier, 2020). When delivering feedback, teachers should prioritize ensuring student comprehension, promoting alternative strategies, linking current and underlying mathematical concepts, and encouraging critical reflection (Jacob & Empson, 2016). Additionally, it is essential for teachers to highlight student strengths, be responsive by inviting elaboration, introducing diverse perspectives, fostering collaboration, and restating reasoning (Daro et al., 2011). The interdependent relationship between responsiveness, thinking, noticing, and feedback quality underscores the importance of developing noticing skills to enhance interactions and feedback (Jacobs et al., 2011; König et al., 2022).

Social interaction among PSTs plays a crucial role in fostering successful collaborative activities, facilitating learning through discussions, reasoning, reflection, critical thinking, and comprehension (Garrison et al., 2001; Kreijns et al., 2003; Liaw & Huang, 2000; Northrup, 2001). Collaboration positively influences noticing abilities, enabling future teachers to observe, interpret, and appreciate student reasoning from diverse perspectives, thereby enhancing decision-making through an understanding of group dynamics (Abdu & Slakmon, 2023). The use of digital tools has enabled global collaborative learning participation, particularly in virtual exchanges involving intercultural interactions. Engaging in virtual exchanges can stimulate innovation, promote internationalization, and provide networking opportunities (Creelman & Löwe, 2019; Jager et al., 2019). COIL fosters meaningful connections and global collaborative problem-solving for teacher education (Potter & Bragadottir, 2020). In online preparation, composing narratives and receiving feedback assist in identifying learning opportunities for PSTs. Sharing narratives and receiving responses aid in developing discernment and refining practices (Sjöblom et al., 2023). However, instructors involved in COIL should possess knowledge of learning content, pedagogical strategies, and appropriate technology for their COIL experiences. Recognizing and highlighting the differences in learners' cultural backgrounds are essential for creating an inclusive COIL environment (Bae, 2022).

Methods

The research study utilized a qualitative descriptive case study methodology to address the research inquiries and provide a comprehensive account of the case by examining documents (Yin, 2014). The study involved 19 seniors enrolled in an elementary licensure program (the PreK-5 grade range) at a small university in the midwestern region of the United States. The research focused on a COIL activity designed to enhance PSTs' abilities to analyze and comprehend their students' mathematical strategies and errors, and to deliver meaningful feedback and responsive lessons. The PSTs were organized into small groups of 5-6 individuals, comprising PSTs from both a U.S. university and a Spanish university. These groups participated in a collaborative four-phase activity, which included individual work, interactive online collaborations, group report development, and reflection on the COIL experience.

The data collection process aimed to examine the impact of COIL on the PSTs' ability to notice errors/strategies and provide feedback. Prior to the study, the PSTs received instruction on number sense, the four basic operations, and teaching strategies. They were presented with six student solutions involving the four basic operations selected from The Assessment Project (http://map.mathshell.org): Ava (Addition), Jacob (Subtraction), Mason (Division), Aiden (Multiplication), Abigail (Division with numbers including 0), and Mia (Subtraction using expanded form). The initial data collection involved individual written responses from the PSTs during an in-person mathematics methods class. The second set of data included group responses collected electronically after the COIL experience. Throughout the data collection, the PSTs were instructed to: (1) identify error patterns by analyzing the student solutions and (2) provide constructive feedback to enhance student comprehension.

For data analysis, the study employed the general inductive and deductive approach (Creswell, 2013) centered around noticing literature (Jacobs et al., 2011; König et al, 2022). The data analysis process involved reading the data separately by each researcher, discussing the patterns, developing the coding framework, coding the data, and finalizing the coding process. The analysis was guided by a literature review, memos, and ongoing discussions. The analytical framework and coding structure was adapted from Lee et al. (2024) and modified for the research context. The codes include levels of noticing student strategies (Level 0-no attention to student strategies and errors; Level 1-partial or incomplete attention; Level 2-attending to student strategies and errors; and Level 3-fully unpacking student strategies and errors with additional discussion), levels of feedback quality (ranging from Level 0-providing minimal details but planning to correct errors to Level 3-probing guidance and building on student thinking), the mathematical aspect (procedural/ or conceptual), and connectivity.

Results

Unpacking and Supporting Student Thinking

The COIL activity positively impacted PSTs' observational skills and analytical abilities in mathematics education. Before COIL, most PSTs were at Level 1, focusing on basic student strategies. After COIL, there was a notable shift: Level 1 PSTs decreased from 52% to 19%, while Level 2 increased from 44% to 48%, indicating progress towards more detailed analyses. Level 3 PSTs increased from 2% to 31%, showing significant growth in addressing complex student reasoning. Average noticing levels improved post-COIL for Mia, Abigail, Aiden, Mason, Ava, and Jacob, indicating enhanced attention to various mathematical topics especially subtraction and division. The findings suggest that COIL positively influenced PSTs' ability to comprehensively analyze student work, leading to improved support in mathematics education.

The COIL activity positively influenced feedback quality by encouraging more tailored responses based on students' original thoughts. Following COIL, feedback levels increased, signifying a shift towards more meaningful feedback practices that delved deeper into students' reasoning to enhance learning effectiveness. Pre-COIL, feedback predominantly at Level 1 (60%) focused on corrections, with some at Level 2 (34%) offering detailed responses. Post-COIL, Level 1 feedback decreased to 55%, while Level 2 increased to 40%, reflecting enhanced responsiveness. Few PSTs provided Level 3 feedback addressing intricate student reasoning. Notable improvements in feedback levels were observed post-COIL for Mia, Abigail, and Aiden, showcasing heightened quality and specificity. The impact of the COIL activity on feedback quality varied across different math topics and student works, underscoring the significance of thorough noticing for delivering advanced-level feedback.

Before COIL, PSTs heavily focused on procedural aspects in noticing (99.0%) while neglecting conceptual elements. After COIL, there was a shift towards a more balanced approach, with noticing incorporating both procedural and conceptual aspects, increasing to 12.2%. Feedback predominantly emphasized procedural aspects (88.2%) pre-COIL, with minimal attention to conceptual aspects (3%). Post-COIL, feedback on procedural aspects slightly rose to 89.5%, while feedback on conceptual aspects dropped to 0.0%. This shift towards more conceptual understanding, especially in Level 3 responses, underscores the importance of enhancing feedback to address conceptual comprehension in student work. Despite improvements in noticing both procedural and conceptual aspects, a decline in conceptually-focused feedback post-COIL was observed, necessitating further enhancements. Maintaining an alignment between noticing and feedback is vital for effective teaching practices and student learning.

Cross Analysis: Noticing vs. Feedback

Table 1 presents a cross-analysis of PSTs' noticing levels and feedback quality. The comparison revealed that most PSTs were at Level 2 for noticing and Level 1 for feedback. Before COIL, PSTs were primarily at Level 1 for both noticing and feedback. However, after COIL, there was a significant improvement in PSTs' noticing abilities and feedback quality, with an increase in higher levels of noticing and feedback. International collaboration, as exemplified by PST#4 and their group's strategic development of Aiden's thinking, played a crucial role in this enhancement. The shift towards higher levels of attending and quality of instruction post-COIL indicates a positive impact on PSTs' professional development, teaching practices, and student outcomes. The study emphasized moving towards a constructive approach in feedback, focusing on arithmetical errors and alternative correct methods in student work. By fostering Level 3 proficiency in both noticing and feedback, educators can provide a more comprehensive learning experience. The investigation aimed to understand the factors contributing to PSTs' advancements in noticing and feedback, with significant progress observed after COIL.

Table 1. Cross Analysis of Noticing Levels and Feedback Levels Before and After COIL

			Count of Level of Noticing			
			0	1	2	3
Count of Feedback Quality (Level)	0	Before COIL	0	1	0	0
		After COIL	0	0	0	0
	1	Before COIL	0	35	23	0
		After COIL	2	11	29	2
	2	Before COIL	0	12	20	1
		After COIL	0	5	9	17
	3	Before COIL	1	3	1	0
		After COIL	0	0	2	2

Discussion, Implications, and Conclusion

The study found that COIL experiences had a significant impact on PSTs' ability to notice and provide feedback on student work in mathematics education. Before COIL, PSTs mostly attended to student thinking at the procedural level, but after COIL, there was a shift towards conceptual level noticing and higher attention to student strategies and errors. This shift was evident in the feedback provided, with conceptual responses becoming more prominent. COIL discussions facilitated positive changes in PSTs' noticing and feedback, highlighting the value of collaborative learning experiences in developing teachers' expertise.

The study's implications for teacher education programs include the need to incorporate collaborative learning experiences like COIL to enhance PSTs' ability to provide evidence-based noticing and responsive feedback. Teacher educators should emphasize the conceptual aspects of operations and encourage PSTs to move beyond procedural thinking. Providing opportunities for PSTs to engage in collaborative discussions and analyze student work can lead to improvements in their ability to identify error patterns and provide constructive feedback. In conclusion, the study demonstrates the positive impact of COIL experiences on PSTs' ability to notice and provide feedback on student work in mathematics education. Collaborative learning experiences can enhance PSTs' expertise and contribute to the improvement of mathematics education practices. Teacher education programs should continue to leverage collaborative learning experiences to cultivate PSTs' capacity for evidence-based noticing and responsive feedback.

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ELEMENTARY PRESERVICE TEACHER PERCEPTIONS OF USING TECHNOLOGY TO MODEL EQUITY ISSUES

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This research explores the perceptions of 32 elementary preservice teachers (PSTs) in a mathematics methods course when using real-world digital data to design grade 3-5 tasks that elicit mathematical models relating to issues of equity. Specifically, this research explored PSTs choices relating to types of equity issues, types of digital resources, task design to encourage mathematical modeling, and self-reflection on the experience. Results showed PSTs who focused on multicultural topics generally used qualitative digital sources such as images, articles or videos to set a tone or theme for the math tasks rather than encouraging modeling to explore the topic. In contrast, PSTs who explored social justice issues were more likely to support math modeling using quantitative statistical data from online sources. Self-reflections indicated appreciation and awareness of strengths and weaknesses relating to modeling equity issues.

Keywords: Social Justice; Preservice Teacher Education; Modeling; Technology

Elementary mathematics education should reflect the world students live in, to include cultural experiences and issues of social justice (Beard, 2021; Felton-Koesler et al., 2017; Kretz, 2023; Litster et al., 2018; Williams & Roth, 2019; Xenofontos, 2020). Teachers are key to making this happen. However, "while social justice education has been professed, teacher training on social justice education is still not prominent in teacher education programs" (Suriel & Litster, 2022). Thanheiser & Sugimoto (2020) propose that preservice teachers (PSTs) can and should be developing a joint understanding of both mathematical knowledge and social justice issues in the teacher preparation courses to support this effort as PSTs move into their own classrooms. They noted that while most of the PSTs in their courses were able to create meaningful problems that included a social justice focus by the end of their study, only about half provided the source of their data. Without a data source, students may ignore the context or not believe the numbers in the context. Currently, we have a culture of "answer getting" when working with story problems where students are more likely to put on blinders to everything beyond the numbers (Bushart, 2018). An additional problem when working with story problems is the believability of the numbers within the problems themselves. Gary Schulz illustrated this cultural problem in his 1987 Peanuts comic strip when he has Sally note "only in math problems can you buy 60 cantaloupes and no one asks what the hell is wrong with you" (Schulz, 2023). Researchers recommend asking numberless word problems and adding in the numbers later to connect the numbers and the context (e.g., Carle, 2023; Bushart, 2018). Technology can be a great way to add in those numbers and provide a verified data source to extend mathematical understanding to real-world and social justice contexts (Kolb, 2017).

Thus, the purpose of this inquiry was to explore elementary preservice teachers' (PSTs') perceptions of using technology to encourage mathematical modeling to explore various issues of equity.

Theoretical Framework

Mathematical Modeling is the act of using mathematics to answer "big, messy, reality-based questions" (Bliss & Livertini, 2016, p. 7). Modeling real-world tasks can be a powerful tool to help students explore mathematics and issues of equity (Aguirre et al., 2019). This interdisciplinary conjunction of mathematics and other knowledge can help students engage in effective learning opportunities, meet the diverse needs of students, and engage in culture responsiveness (Litster et al., 2023; Williams & Roth, 2019). There are different levels of integration with lower levels focusing on thematic applications to situate the relevance of mathematics within the world and higher levels focusing on authentic applications of mathematics (Litster et al., 2023). Mathematical modeling showcased that authentic application of mathematics to reflect the world students live in.

The number of one-to-one devices in elementary classrooms has been increasing over the past decade, with school closures due to the COVID pandemic exponentially increasing those numbers (Grey & Lewis, 2021). However, an increased use of technology does not always indicate that the technology is being used effectively (Kolb, 2017). Kolb's (2017) Triple E Framework provides one lens for intentional technology use to engage, enhance, and extend student learning. PSTs can use technology to extend student learning for real-world contexts such as multicultural education or social justice issues by accessing real data such as demographics. PSTs can use technology to enhance student learning for these same contexts by using sites that allow students to organize and model real data to explore ideas and contexts.

In their metanalysis of empirical studies utilizing social justice in mathematics education, Xenofontos et al. (2020) found that there is a variety of ways social justice is conceived and what it includes. With so many definitions to choose from, this study chose to utilize Hammond's (2020) Distinctions of Equity framework, which provides a clean and simple framework for PSTs to consider different equity issues and their purposes. In this framework, there are three distinctions of equity: multicultural education focusing on social harmony, social justice focusing on critical consciousness, and culturally responsive education focusing on independent learning. According to this framework, PSTs that wanted to focus on multicultural education could consider tasks that focus on celebrating diversity, creating positive social interactions across differences, or exposing students to other cultures. PSTs that wanted to focus on social justice education could consider tasks that focus on exposing current social or political issues, exploring social or economic inequities, or recognizing historical patterns in society practices.

Methods

This study utilized a three-phase qualitative process (Saldaña, 2013) to analyze perceptions of 32 undergraduate preservice teachers enrolled in a senior mathematics methods teacher preparation course when designing grade 3-5 tasks that elicit mathematical models relating to issues of equity. The specific research questions were:

- What types of equity issues did PSTs choose to explore?
- What types of digital resources did PSTs use to collect or evaluate real world data?
- How did the activity design encourage or discourage mathematical modeling to explore and reason about real-world issues?

- What were preservice teachers' perceptions on their experiences designing and implementing their activity?

Prior to designing their tasks, participants explored examples and non-examples of elementary math modeling (SIAM & COMAP, 2016), purposes of mathematics integration (Litster et al., 2023), and using technology to extend student learning (Kolb, XXXX). Participants also analyzed physical and digital examples of using math to explore multicultural and social justice issues. Finally, participants discussed any societal implications or considerations that should be made when exploring issues of equity in public or private schools. Participants created their task designs within digital slides, using letter dimensions (8.5x 11 in) to allow for easy printing. They were limited to 3 slides for their instructions and worksheets; however, there was no limit to the digital content that could be accessed via links within the instructions or worksheets. The task designs were uploaded to a course discussion board and each participant tested 5 assigned tasks and provided feedback to their peers. Approximately half of the participants chose to implement a variation of their designed tasks with children in local schools in the Southeast region of the United States. Finally, participants reflected on their experience designing, testing, and implementing their tasks.

The researcher collected PSTs' task designs, peer work when testing designs, and reflections. These were qualitatively coded in three phases using structural, process, magnitude, pattern, and thematic coding (Saldana, 2018?). In phase 1, task designs were structurally coded to identify type of equity issues and technology used. In phase 2, peer work was analyzed using process and magnitude coding to identify presence or lack of mathematical models. Task designs were then analyzed using pattern coding to identify features across designs that encouraged or discoursed mathematical modeling. In phase 3, PST self-reflections and peer feedback were analyzed using pattern and thematic coding to identify PSTs' perceptions on strengths and areas of need..

Results

Results relating to RQ1: types of equity issues can be found in Table 1.

Table 1: Types and Categories of Issues Present in PST Task Designs

Multicultural (N= 17, 54 %)	Social Justice (N=11, 34%)	Other (N=4, 12%)
Family Traditions	Income Disparity	Global Warming
Cultural Traditions	Populations Redlining	Weather
Cultural Symbols	Bullying	Air Pollution
Cultural Foods	Women's Suffrage	Animal Shelters
Clothing/Blankets	Black Heroes	

As seen in Table 1, the majority of PSTs (54%) chose to focus on exposing students to multicultural topics such as cultural lifestyles and artistic expressions. About a third of the students focused on present day or historical social justice issues. A small percentage (12%) of students focuses on non-equity issues such as the environment or animals. Results relating to

RQ2: types of technology used can be found in Table 2.

Table 2: Types and Examples of Technology Used in PST Task Designs

Technology Type	Examples
Statistical Data (N= 12, 38%)	Populations, Income, Weather, Temperature, Expenditures, Bullying
Examples & Resources (N=15, 47%)	Images, Blogs, Articles, Videos
Collaboration/ Sharing (N=4, 12%)	Padlet, Google slides/docs, Jamboard
Submit Answers (N=1, 3%)	Google Forms

As seen in Table 2, the majority of PSTs (47%) used technology to access qualitative examples and resources. This is not surprising as these types of technology align to multicultural explorations. The second highest type of technology (38%) was statistical data, which relates well to social justice and environmental issues. Other PSTs used collaborative sites to share simplified data or examples and allow students to set up their mathematical models. One PST used technology to submit answers. Results relating to RQ3:mathematical modeling can be found in Table 3.

Table 3: Math Modeling in Tasks by Equity Types

Modeling in Design	Multicultural (N=17)	Social Justice (N= 11)	Other (N=4)	Total (N=32)
Supported Math Modeling	1 (6%)	9 (82%)	3 (75%)	13 (41%)
Modeling Framework BUT Never Returned to Original Question	1 (6%)	1 (9%)	1 (25%)	3 (9%)
Used Topic as Theme for Math Solutions	15 (86%)	1 (9%)	0 (0%)	16 (50%)

As seen in Table 3, most of the tasks that focused on multicultural issues did not support mathematical modeling (86%). These task designs used the topic as a theme for the math. For example, one PST created story problems with topics from different countries (i.e., pesos in Mexico, sarees in Indiam kimchi in Korea). In another example a PST had students read an article about American Thanksgiving dishes using turkey, potatoes, or pumpkins that were inspired by dishes from around the world. She then asks fractional questions using the article as the theme (e.g., What fraction of the dishes involve turkey?). Most of the tasks that focused on social justice or non-equity issues did support math modeling (82%). For example, one PSTs had students explore population and income statistics for different neighborhoods to explore inequities in redlining. In another example, a PST had students explore the highest priority needs for animal shelters, the cost of those items, and potential ways to fund the purchases (e.g., if everyone donated $1 . .).

Results from the pattern analysis of designs that did support modeling showed that all the designs contained an open-ended original question to explore. For designs that followed with a scaffolded exploration to focus on specific aspects of the data, the last question repeated the original open-ended question. Designs also contained two or more of the following features: social justice or environmental topic, mathematics topic focused on operations or comparisons, used real-time statistical data from various websites or videos, used a collaboration site that required students to show and justify their work using words, numbers, and/or pictures.

Results relating to Q4: PST perception on their experience identified three themes: perceived benefits of this assignment, perceived areas of need when working with equity topics, and perceived areas of need when working with technology. One common benefit PSTs noted from designing and implementing their tasks or testing the tasks for their peers was that they enjoyed being exposed to different cultures, past issues of equity, and current issues of equity they were not familiar with. They enjoyed learning something new in addition to the math. Several PSTs built upon this idea and noted that trying out their peers' tasks helped them revise some misconceptions they previously had about various cultures or events. PSTs noted that they appreciated knowing the information was real, not something the textbook made up. Other PSTs liked the change of pace from "traditional learning." Several of the PSTs who tried out variations of their tasks with K-5 students also noted that their students had these same perceptions of enjoyment from the change of pace, learning new ideas, and using real applications of the math. Many PSTs liked how they were able to use evidence from student work (peers and K-5 students) to see students think critically about different topics and the mathematics associated with those topics. Similarly, PSTs who tried out a task that utilized a collaborative site appreciated how they were able to see the critical thinking and models of that other people created using the same data in different ways.

There were six areas of need that PSTs perceived are essential when designing and implementing tasks relating to multicultural or social justice issues. First, students may need more time to fully explore the topic. Second, students may need scaffolds such as a time limit or worksheet so they don't get too distracted learning about the topic that they never use any math during the lesson. Third, don't assume students understand their own culture. Many PSTs were surprised at issues within their own city, state, or country. Other PSTs were surprised that their peers or students couldn't think of any cultural symbols, traditions, or foods in their family. Fourth, PSTs noted that tasks that were more open engaged students in choices about what they wanted to explore and how to explore it. They found that closed tasks seemed to just focus on the math numbers or shapes. When working with these tasks, they had a hard time identifying what issue of equity the task was trying to explore. Fifth, building on the fourth idea, PSTs recommended that tasks exploring equity issues should use questions and follow-up questions to ensure students are making connections between the math and the equity issue. Many PSTs who did not do this in their initial task reflected that this was something they wish they had added, or in the case of those who taught a variation of their task, was something they changed before trying the task with K-5 students. Sixth, PSTs noted that you should not try to force a math and equity connection. They perceived that not all math works well to explore issues of equity and not all issues of equity are best explored using math.

There were six areas of need that PSTs perceived are essential when working with technology. First, students need clear directions on how to use or navigate the sites. Building on this idea, the second perception is that for complex websites, teachers may need to pull the data from the site and reformat it on a single page for use by younger students, students who struggle with the technology, or students who get easily distracted or overwhelmed. Third, when using collaborative websites, teachers must ensure open access for all participants. Fourth, you may need more that one digital resource to fully explore a topic of issue. Fifth, teachers need to ensure their students understand how to use basic technology features such as copying and pasting, taking a screenshot, using digital drawing tools, font sizes on text boxes, inserting information into a table, creating a graph, resizing shapes, etc. This perception came from both their own experiences trying out tasks as well as seeing what their peers tried or avoided within their own tasks. Building on ensuring digital fluency, the sixth perception was that students may skip an aspect of the task they don't want to do or don't know how to do easily with the technology.

Discussion and Implications

In summary, PSTs who designed tasks to explore multicultural issues of equity were more likely to use qualitative resources such as images and articles to learn more about the culture. They were also more likely to use the topic as a theme without explicitly connecting the cultural aspects to the mathematics, which may explain by most of these task designs did not support mathematical modeling. In contrast, PSTs who designed tasks to explore social justice or environmental issues were more likely to use quantitative statistical data using operations or comparisons to make judgments on inequalities. These task designs were more likely to support mathematical modeling. Regardless of whether PSTs felt their own task design was successful or not, they were able to identify several key ideas relating to benefits of having students create mathematical models to explore multicultural and social justice issues and areas of need when exploring issues of equity or using technology.

PSTs perceptions of benefits and areas of need directly impact teachers who may wish to explore issues of equity in their classrooms or use technology to support mathematical modeling of real data. They may also impact MTEs who want to help PSTs incorporate modeling, technology or equity issues into their math lessons. Teachers may want to start with a multicultural or social justice questions – what they want students to learn about or explore. This can help guide their task design to ensure focus on the topic. They should also examine the topic to determine the inherent mathematics already embedded within their question (comparisons, operations, whole numbers vs decimals, etc.). Doing this can help them determine whether the math can be used to model the issue for an authentic application of the math or whether the topic better situates the relevance of the mathematics within the real world (Litster et al., 2023). For authentic situations, teachers may want to focus on using models and justifications to support their response to an open-ended question. For relevant situations, teachers may want to apply a sequential approach by focusing on one area and then the other, making connections as you go or at the end of the project.

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THE INFLUENCE OF PEERS' THINKING ON PRESERVICE ELEMENTARY TEACHERS' ENGAGEMENT WITH OPEN MATHEMATICS TASKS

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This brief report describes preservice elementary teachers' self-reports of the influence of peers' thinking on their engagement in four mathematics tasks of varying openness. This report draws from a larger study that—in an effort to research certain types of mathematical tasks that are widely used but have not been systematically researched—investigated factors that were influential to preservice teachers' engagement. In the larger context of this study, peers' thinking was an unexpectedly strong and consistent influence on engagement. Preservice teachers' self-reports led to findings that peers' thinking enhanced their cognitive and affective engagement in the tasks. The ways peers' thinking was influential seemed to be related to the relative openness of each task. During discussion of the most open tasks, preservice teachers were more likely to report 1) that peers' thinking inspired them towards further cognitive engagement and 2) stronger affective reactions to peers' thinking.

Keywords: Instructional Activities and Practices; Affect, Emotion, Beliefs, and Attitudes; Preservice Teacher Education

Certain types of mathematics tasks have recently become popular with mathematics teachers and others in the mathematics education field. These educators offer considerable anecdotal evidence of the tasks' positive impact on learners' engagement—and related research suggests the tasks' value—but the tasks and their impacts have not been thoroughly and systematically researched. Our larger study sought to cast light on this Black Hole of research (Matney et al., 2020) by investigating preservice elementary teachers' (PSTs) engagement in "Which One Doesn't Belong?" (WODB, Danielson, 2016), "Notice and Wonder" (N&W, Fetter, 2021; Ray-Riek, 2013) and "How Many?" (HM, Danielson, 2018) tasks. The novelty of these tasks seems to be that they are open to a considerable degree, meaning that learners have a wide range of choices in terms of how they respond to the tasks. Practitioners report that when discussing these tasks with their learners, they notice increased engagement and from a wide range of students (Danielson, 2016; 2018; Illustrative Mathematics, 2021; Newell & Orton, 2019; Ray-Riek; Rumack & Huinker, 2019).

When studying PSTs' engagement in four tasks of varying openness, we asked the questions: 1) "How do elementary PSTs engage with open mathematics tasks?" and 2) "What factors influence PSTs' engagement with open mathematics tasks?" We expected that the PSTs in our study would engage similarly to K-12 students, and that openness would play a significant role in eliciting that engagement. We found that PSTs did engage in a manner similar to that described of K-12 students and the openness of the tasks was a crucial factor in PSTs' engagement. However, being able to hear and interact with peers' thinking as a result of the tasks' openness was also consistently a strong influence. In this brief report, we focus on this unexpectedly strong influence.

Theoretical and Conceptual Framework

In focusing on peers' thinking as one influence amidst a multitude of influences on learners' engagement, we consider the classroom as an activity system (Engeström, 2015) that includes teachers, learners, materials, the physical environment, and their interaction (e.g., Gresalfi et al., 2009). This perspective emphasizes "what someone does in a particular activity is always done in relation to what one has opportunities to do" (Hand & Gresalfi, 2015, p. 191). Peers' ideas affect what one has opportunities to do during mathematical activity, which in turn shapes what one does during the activity.

Understanding the classroom as an activity system is congruent with Middleton et al.'s (2017) conceptualization of mathematical engagement. They considered engagement to be "in-the-moment relationship between someone and her immediate environment, including the tasks, internal states, and others with whom she interacts" (p. 667). The conceptualization includes behavioral, cognitive, and affective dimensions. At PME-NA 45, Middleton (2023) shared that he is increasingly understanding engagement to also consist of a social dimension, as during mathematical activity, "learners play off their peers, reacting to and modifying behavior to support each other and to get value added from the collaboration" (p. 7). The role of peers' thinking connects to a social dimension of engagement.

The larger study investigated PSTs' engagement in mathematical tasks of varying degrees and types of openness. While this brief report focuses on the influence of peers' thinking within that research setting, the ideas shared by peers were shaped by the space created by the openness of the tasks. We regard openness as having three dimensions (see Figure 1). The first two dimensions clarify the 'open-start' dimension of previous frameworks (e.g., Leikin, 2018; Mitchell & Carbone, 2011; Pehkonen, 1997; Silver, 1995) by distinguishing between ways to enter or focus the task (entry points) and the strategy or method to use when engaging with the task (strategies). The third dimension of openness addresses the number of possible endpoints (solutions) of a task. Our framework also incorporates the degrees along which tasks can be open as suggested by Mitchell and Carbone (2011).

Methods

The study from which this data was drawn was conducted across two semesters at a large Mid-Atlantic university in four sections of an elementary mathematics methods course for undergraduate PSTs. All sections were taught by the same instructor. For each task, we identified a section (n=16, 17, 23, 10) whose engagement with the task was generally representative of the other three sections' engagement with the same task. Our overall methodological approach was pragmatic (e.g., Coyle, 2010; Frost & Nolas, 2011; Morgan, 2007) drawing on ethnography's (Macgilchrist & Van Hout, 2011) emphasis on participant voice, case study's (Merriam, 1998) thorough investigation of a clearly defined phenomenon and grounded theory's (Glaser & Strauss, 1967) principle of being data-driven.

Tasks

In collaboration with the course instructor, we chose four tasks of varying openness for PSTs to engage with and discuss. We describe them here in order of relative overall openness. The first task was a word problem (WP) of high cognitive demand (Smith & Stein, 1998) that asked PSTs to find the number of cupcakes and boxes a baker used given parameters for number of boxes, number of cupcakes, and number of cupcakes that could fit in each box. The WP allowed for multiple strategies and had one correct answer. The purpose of including the WP was to provide a comparison to the tasks that had a wide range of possible answers. The second task was a WODB that presented four addition expressions, each of which had multiple reasons it might not belong with the others. The third task was an HM task. PSTs were presented with an image of a large box of chalk and were directed to choose something to count in the image and to choose how to count it. The fourth task was a N&W, in which PSTs were asked to share what they noticed and wondered about three packages of toilet paper that all advertised larger rolls. The relative openness of each task along three dimensions is depicted in Figure 1.

Data Collection and Analysis

I observed each task session in person. During each session, the instructor introduced the task, the PSTs worked independently on the task, and the instructor facilitated a whole-group discussion of the task. Following each session, PSTs completed a semi-structured questionnaire that asked them questions about their engagement with and experience of the tasks. For example, PSTs answered questions such as, “Describe the strategies you used during the task and discussion,” “Describe any emotions you were feeling while work on the task and during the class discussion,” and “Was the way you thought and felt during this task similar of different from when you have engaged with math in the past? Please explain.” Questions varied slightly on between task sessions based on PSTs’ responses to previous questionnaires.

Entry Points		Strategies		Solutions	
	One entry point		One strategy	WP	One solution
WP HM WODB	Multiple but limited entry points	WP	Multiple but limited strategies		Multiple but limited solutions
N&W	Unbounded entry points	HM WODB N&W	Unbounded strategies	WODB HM N&W	Unbounded solutions

Figure 1: Relative Openness of the Four Tasks Along Three Dimensions

This brief report focuses on data collected from PSTs’ questionnaire responses. We inductively coded PSTs’ questionnaire responses for influences on their engagement using a constant comparative approach (Glaser & Strauss, 1967). When an influence appeared in multiple PSTs’ responses for a task, we included it in our findings. For example, if a PST mentioned being excited by the possibility of multiple answers, we coded that mention as evidence for the influence of the task’s openness. If a PST described finding others’ ideas interesting, we coded that description as evidence of the influence of peers’ thinking.

Findings

Peers’ thinking emerged as an influence on PSTs’ engagement across all four tasks. Table 1 shows the percentage of PSTs for each task session whose questionnaires provided evidence of being influenced by their peers’ thinking. This theme was most present in PSTs’ responses to the HM and N&W questionnaires. The theme was also present in more than half of PSTs’ responses to the WODB questionnaire. The theme was present, but less so, in the WP questionnaire responses. Across all four task sessions, peers' thinking was largely a positive influence. The main task session in which peers' thinking had a negative influence was the WODB.

Table 1: Self-Reported Influence of Peers' Thinking Across Tasks

	WP (n=16)	HM (n=17)	WODB (n=23)	N&W (n=10)
Influence Overall	38%	94%	57%	70%
Positive Influence	38%	88%	57%	70%
Negative Influence	0%	6%	17%	0%

The positive influence of peers' thinking generally fell into two categories: appreciation of peers' thinking and peers' thinking motivating further engagement with the task. Table 2 lists the percentage of PSTs who expressed the influence of their peers' thinking in these ways.

Table 2: Self-Reported Positive Influence of Peers' Thinking Across Tasks

	WP (n=16)	HM (n=17)	WODB (n=23)	N&W (n=10)
Appreciation	38%	82%	57%	40%
Further Engagement	6%	18%	39%	40%

When PSTs demonstrated appreciation of peers' thinking, they made statements such as it was "cool to see other ways that people were solving the same problem as me" (Carly, WP) or "I enjoyed seeing how different peers chose different things and ways to count" (Rachel, HM). Evidence of peers' thinking spurring further engagement in the task included responses such as, "[I] was adapting my thinking to others to make sense of things" (Lila, WODB) and "I was constantly looking at the images when people were making notices and wonders to see if I could build off of that in any way" (Andrew, N&W). PSTs were most likely to go beyond appreciation and report peers' thinking as spurring further engagement on the WODB and N&W questionnaires.

An interesting phenomenon that occurred in the responses that expressed appreciation was that one or more PSTs in each of the most open tasks (HM, WODB, and N&W) described their reaction to peers' thinking with unusual intensity. In PSTs' responses to the WP, they described their reactions with words such as "liked," and "interesting." While PSTs also used similar terms in their descriptions of the influence of peers' thinking for the other three task sessions, some PSTs also used words like "amazed," "surprised" and "fascinated." For example, Becca reported feeling "very much in awe" during the WODB task session because "there were so many new ways that other people were coming up with that I… did not see before." During the N&W task session, Liz was "fascinated at how [her] other classmates were really digging deeply into what they noticed and wondered." Lily was "floored" by the creative responses her peers shared during the HM task session.

The main negative influence across any tasks had to do with ways PSTs were influenced by peers' thinking during the WODB task session. This influence involved additional categories that did not occur in PSTs' responses to the other task session questionnaires: being either encouraged or discouraged when comparing their own ideas to peers' thinking. Two PSTs (9% of 23 PSTs) mentioned being encouraged by peers' thinking, particularly in hearing ideas that made them realize their own ideas were valid. Four PSTs (17%) felt discouraged at some point by peers' thinking. Katherine shared that she "did feel a little bit less smart…for focusing on the surface level differences… while my classmates were discussing sums and multiples and patterns." Mae had a similar experience, explaining she "felt like [her] thinking was…not thorough enough" and that she was "trying to keep up… because [she] wasn't seeing these number sentences like a lot of [her] peers did." Notably, all four of these PSTs also described being positively influenced by their peers' thinking at some point in the discussion.

Discussion and Implications

The influence of peers' thinking is an especially striking result as the questionnaire did not directly ask PSTs about peers' thinking. Being asked questions about their engagement with the task and discussion prompted PSTs to bring up their thoughts and feelings about others' ideas. This phenomenon underscores how central hearing peers' thinking was to PSTs' experiences. The openness of the tasks seemed to work in tandem with the influence of peers' thinking, as PSTs were more likely to report the influence of peers' thinking after engaging in discussion of the most open tasks. In the most open tasks, PSTs were also more likely to use more intense affective language to describe peers' thinking and were more likely to report going beyond merely appreciating peers' thinking to being spurred to further engagement.

This brief report pinpoints an aspect of the use of open mathematics tasks that influences engagement. Hearing peers' responses may be a key factor in open mathematics tasks eliciting the kinds of engagement that practitioners purport they elicit. In this case, it is crucial to prioritize discussion as part of the facilitation of these tasks. The significant impact of peers' thinking in this study also supports further exploration of the social dimension of engagement, particularly in terms of PSTs' engagement with one another's ideas. Ultimately, we as mathematics teacher educators want our PSTs to not only experience joy and fascination as a result of others' mathematical engagement, but to feel drawn to be part of that mathematically vital experience themselves.

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FINDING PERSEVERANCE IN EGYPTIAN FRACTIONS

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Keywords: Preservice Teacher Education; Rational Numbers & Proportional Reasoning

Students of all ages, including many adults, have long complained about fractions. The mere structure and definition of fractions have baffled students and caused them to doubt their mathematical abilities for generations, as evidenced by the focus on rational numbers in various guiding documents (e.g., AMTE, 2020; NGA, 2010; NRC & MLSC, 2001) and myriad studies centered on students' ability to reason with rational numbers (e.g., Davis, 2003; Heller et al., 1990; Thompson & Saldanha, 2003). Additionally, perseverance has played a prominent role in mathematics education, so much so that the first Standard for Mathematical Practice, as outlined by the Common Core State Standards (NGA, 2010), encourages students to "make sense of problems and persevere in solving them" (SMP.1). However, perseverance is not a trait that humans are born with; it is a behavior that is developed over time, under the right conditions, and when provided with proper tools (Middleton et al., 2015). Middleton and colleagues outline four core aspects of perseverance: (1) interests and identity; (2) setting goals; (3) utilizing resources; and (4) anticipating consequences.

Early in a math course for elementary education majors taught by the authors, students solve contextualized (e.g., brownies) equal sharing problems involving Egyptian fractions. In class, students work in small groups to find a solution, and report the amount of a share each person will get in the form of an Egyptian fraction. As homework, students are asked to solve two similar problems and find at least two distinct solutions then answer several reflection questions about their experience solving the problems.

There were 91 students enrolled in this course in the Spring 2024 semester, 86 of whom submitted the assigned problems and reflection. After reviewing the submissions and giving students grades and feedback, we compiled the student responses and coded the submissions for common themes based on the four aspects of perseverance outlined by Middleton and colleagues (2015). Though analysis is ongoing, we have found student responses that align with each of the aspects of perseverance. Some responses have discussed how students' identities (Aspect 1) *prevented* them from persevering. Most of these responses center on students not feeling like so-called "math people" and getting stuck after trying one approach. Some students cited a goal (Aspect 2) of getting the job done as their reason for persevering, while others stated that they were motivated by consequences (Aspect 4), like bad grades, if they didn't finish. Finally, several students cited different resources (Aspect 3) they used to finish the task, including referring to previous examples or asking friends for help.

The reflections analyzed in this study have already revealed evidence for each of the four core aspects of perseverance when working on a task involving Egyptian fractions. These findings indicate that problem-solving tasks that challenge students to think outside of their comfort zone, like the Egyptian fraction problems, can be beneficial in many ways. Not only can they help students explore mathematical concepts more deeply, but they can also provide a site for students to reflect on their experiences in ways that point them to productive pathways towards perseverance. Based on the findings in this report, we anticipate future studies will focus on how other activities in this course provide opportunities for students to persevere.

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COMPETING MEANINGS, PERTURBATION, AND ENGENDERING SHIFTS IN (PROSPECTIVE) TEACHER MEANINGS

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Quantitative reasoning's emergence as a foundation for students' mathematical development has generated a need for supporting teachers' capacity to teach for such reasoning. In this paper, we discuss a meanings perspective on working with prospective and practicing teachers in order to support their constructing meanings that foreground quantitative reasoning. Our meanings perspective, referred to as competing meanings, involves a problematization of extant meanings, the construction of alternative meanings, and a critical comparison of each. Here, we present our perspective and informing theories. We also draw on our empirical work to provide tangible and research-based examples of our competing meanings perspective.

Keywords: Cognition, Preservice Teacher Education, Teacher Knowledge, Learning Theory.

Students' quantitative reasoning refers to the ways in which students conceive of and reason with measurable attributes constituting some phenomenon or context (Smith III & Thompson, 2007; Thompson, 2011; Thompson & Carlson, 2017). Addressing number concepts, fractional reasoning, proportional reasoning, algebraic reasoning, rate of change concepts, and function concepts (e.g., Karagöz Akar et al., 2022; Steffe & Olive, 2010; Thompson & Carlson, 2017), researchers have identified quantitative reasoning as a bedrock for students' mathematical development. These same researchers have highlighted that the various factors influencing students' educational experiences do not sufficiently engender or support students' quantitative reasoning. Whether with respect to improved curricular materials, continued knowledge development, or targeted pedagogical practices, a pressing need is better understanding how to prepare teachers in supporting their students' quantitative reasoning.

Over the past decade-plus we have engaged in a research program to understand not only students' quantitative reasoning, but also that of prospective and practicing teachers. Our primary research emphasis has been understanding the relationship between teachers' mathematical meanings and their quantitative reasoning, including how to engender teachers' quantitative reasoning so that it might be leveraged to support shifts in their meanings. We have specifically sought to support shifts reflecting those meanings identified by researchers as critical for K-16 students' mathematics. We report on a perspective for supporting such shifts in this paper.

We term our perspective *competing meanings* due to its simultaneous focus on teachers' extant meanings, the meanings we seek to engender and center when working with teachers, and interactions between those meanings we seek to provoke. In what follows, we first provide background theory that informs our perspective including Piaget's epistemology (Piaget, 2001), Thompson's theory of meaning and quantitative reasoning (Thompson, 2016; Thompson & Carlson, 2017), and Harel's notion of intellectual need (Harel, 2013). Drawing on those informing theories, we outline the perspective of competing meanings as it relates to working with teachers, whether prospective or practicing. We support an operational approach to competing meanings by also providing a tangible example of it, both in the context of task design and research participant themes. We close with potential implications and future work by drawing specific attention to areas of theory left to flesh out or connect with.

Informing Theory and Background

Our perspective is informed by Piaget's genetic epistemology, including von Glasersfeld's (1995) extension of it. We focus here on the constructs of *assimilation*, *perturbation, accommodation*, and *equilibration*, and we point the reader to Dawkins et al. (2024) for an extensive collection of Piaget's theory in mathematics education. Assimilation is the process by which an individual conceives a present experience via their current conceptual structures (von Glasersfeld, 1995). It is a constructive process that shapes an experience so that it affords and is constituted by those structures. In some cases, assimilation to extant conceptual structures results in an unexpected experience, which engenders a state of perturbation (von Glasersfeld, 1995). A perturbation can stem from several causes. For one, an individual might obtain an unexpected result after enacting a conceptual structure, thus yielding a sense of perplexity. As another example, in enacting a conceptual structure, an individual might become aware of some experiential feature that leads to their questioning the efficacy or relevance of that structure.

Having experienced a perturbation, an individual engages in activity to reconcile that cognitive state. One form of reconciling a perturbation involves affective and coping responses, such as anxiety leading to disengagement (Tallman & Uscanga, 2020). Another form of reconciliation is that of accommodation, which can take on several forms. To name a few, the conceptual structure used in assimilation could be modified, an alternative conceptual structure could be enacted, or a novel conceptual structure could be constructed (von Glasersfeld, 1995). Regardless, accommodation is an act of learning via the elimination of a perturbation through a cognitive construction or reorganization. It often entails sustained, and effortful, cognitive engagement. Piaget hence referred to the process of accommodation as one of equilibration that establishes a cognitive state of equilibrium (von Glasersfeld, 1995).

In service of operationalizing the aforementioned Piagetian constructs, Thompson introduced the intertwined theories of quantitative reasoning (Thompson, 2011) and meaning (Thompson, 2016). With respect to the latter, Thompson's (2016) theory of meaning is rooted in Piaget's genetic epistemology and refers to an organization of operations, images, and other meanings. As it relates to the act of teaching, Thompson's theory of meaning is connected to that of Silverman and Thompson (2008), who outlined a developmental process that spans the construction of personalized knowledge to the transformation of that knowledge to incorporate student meanings and pedagogical implications. That is, Silverman and Thompson recognized the importance of teachers' mathematical meanings including teachers' construction of *key developmental understandings*, which are understandings critical to the development of coherent and generative mathematical concepts (Simon, 2006). Before using Thompson's theory of quantitative reasoning to further illustrate this perspective, we note that the perspective emphasizes mathematical knowledge as a dynamic, in-the-moment implicative base of *knowing* for action, as opposed to a static, declarative base of *knowledge* for action (Liang, 2021; Thompson, 2016).

Thompson's (2011, 2012) theory of quantitative reasoning provides one framework for situating Piaget's genetic epistemology, meaning, and key developmental understandings. *Quantitative reasoning* is reasoning that involves conceiving situations in terms of measurable attributes (i.e., quantities) and relationships between those attributes (i.e., quantitative relationships). Quantitative relationships form the basis for the construction and abstraction of mathematical objects (Moore et al., 2022; Smith III & Thompson, 2007). *Covariational reasoning* is a particular form of quantitative reasoning that involves constructing and coordinating quantities that vary in tandem (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). A growing number of researchers have identified important nuances in student and teacher thinking in this area (see Karagöz Akar et al., 2022 for a collection of contributions and researchers). Using the framework by Carlson et al. (2002), one meaning entailing covariational reasoning involves assimilating a situation via *directional* and *amounts of change* relationships. For instance, Ellis et al. (2015) explored students' meanings for exponential relationships in the situation of (magic) plant growth and the quantities height and time. The students' meanings involved their constructing the directional covariation of quantities (e.g., as time increases, height increases), and coordinating additive changes in one quantity with multiplicative changes in the other (e.g., as time increases additively, height increases by increasing amounts while preserving a constant ratio for a constant time period). Here, the operations constituting the meaning for exponential relationships involve conceiving the variation in each quantity, coordinating those two variations to construct and compare changes in each, and considering how the constructed covariational relationship is relevant to different contexts (e.g., a growing plant, a Cartesian graph, or a table).

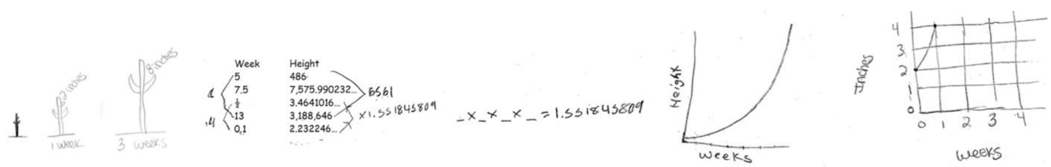

Figure 1: Students' coordinating height and time (Ellis et al., 2015, pp. 143, 147, and 149)

Our work is also informed by Harel's (2013) *intellectual need*. We use intellectual need to clarify the perturbations targeted by our competing meanings perspective. Harel defined intellectual need as "a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, his or her current knowledge" (2013, p. 122). Importantly, Harel's intellectual need refers to a state of perturbation that affords learning, and is thus not merely a state of confusion. A researcher is positioned to claim an individual has experienced an intellectual need when the meanings needed to reconcile an experienced perturbation are within the individual's zone of proximal development, whether that development be in the context of reasoning or domain practices (Harel, 2013; Weinberg et al., 2023). With respect to the work here, intellectual need orients us toward not only seeking to engender perturbations, but also having in mind the ways in which teachers' available reasoning can act as an asset in reconciling that perturbation. Furthermore, intellectual need draws our attention to forms of perturbations that promote their reflective comparison of meanings that caused the perturbations and those that reconciled those perturbations.

In summary, Piaget's and von Glasersfeld's framings of knowing provide us guiding cognitive mechanisms. Thompson's perspective on meaning, with Silverman, Thompson, and Simon's descriptions of how meanings inform teaching, further clarify our attention to the ways an individual's personal meanings may be organized and transformed so that they are generative and flexible during the act of teaching. Theories of quantitative and covariational reasoning provide us concrete constructs by which to specify and differentiate mathematical meanings. Lastly, Harel's notion of intellectual need aids us in clarifying the type of perturbations we seek to engender with teachers. Namely, we focus on perturbations that necessitate the enactment of alternative meanings to reconcile them (i.e., equilibration via accommodation). Furthermore, we focus on the transformative learning experiences that occur when a process of equilibration is accompanied by a subsequent perturbation that motivates reflectively comparing meanings.

Competing Meanings

Our competing meanings perspective identifies one form of learning via particular forms of intellectual need and, hence, perturbation and accommodation. Stated generally, the competing meanings perspective includes an individual experiencing a problematized extant meaning; enacting an alternative meaning; and, through additional processes of perturbation and accommodation, comparing the extant meaning and alternative meaning (Figure 2).

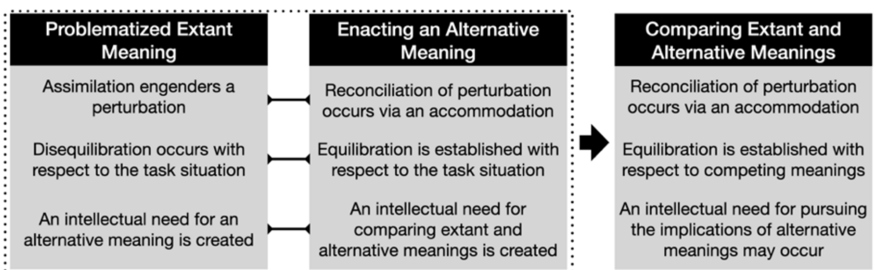

Figure 2: The competing meanings perspective

A problematized extant meaning first occurs via an act of assimilation that engenders a perturbation and an intellectual need for an alternative meaning. Then, via enacting that alternative meaning, the individual reconciles their perturbation with respect to the task situation associated with the initial perturbation. Critical to the competing meanings perspective is that a subsequent state of perturbation then occurs. Whereas the initial intellectual need was respect to the goal-oriented activity of the task, a subsequent round of intellectual need is created at the level of meanings; the individual becomes perplexed as to why their extant meaning results in a perturbation while the alternative meaning does not. The disparate nature of the meanings is thus at the root of the perturbation and associated intellectual need. By disparate, we mean that, *in that moment*, the individual infers that their two held meanings entail important differences and incompatibilities that are not trivial to resolve. This perplexity positions the individual to take each meaning as an object of thought and hold them against each other (i.e., competing meanings) in order to reconcile that perturbation. Yet an additional intellectual need might result from this process, motivating the individual to explore the implications of that reconciliation, particularly if the alternative meaning is novel and viewed as a potentially more productive meaning. We return to this point in the closing of the paper.

To situate and illustrate the perspective, we start with an abstract and abridged example. Consider a hypothetical student or teacher, who we name Blinder. For a particular concept, Blinder has constructed a meaning that we denote by M_a (we remind the reader that a meaning might be composed by a system of meanings), which has served as productive throughout his schooling experience. Entering our class or professional development, we might intend, for a variety of reasons, that Blinder construct an alternative meaning. We denote this alternative meaning by M_b. In working with Blinder, we determine that he holds meaning M_a and that meaning M_a and M_b are disparate; M_a is not a foundational way of thinking for M_b and, in fact, can inhibit the construction of and ability to teach for M_b. This raises the question: how do we engage with Blinder in a way that honors M_a and affords constructing M_b? This is a situation we have been presented with frequently in research, teaching, and professional development settings with students and teachers (e.g., Moore, Stevens, et al., 2019; Tasova, 2021).

Using Linearity to Illustrate the Competing Meanings Perspective

Consider linear relationships as an example topic to contextualize the abstract presentation above. Our work has adopted a quantitative reasoning perspective to center a meaning for linear relationships that involves constructing a constant rate of change. A constant rate of change between two quantities means that as the quantities' magnitudes covary, their amounts of change exist in a proportional relationship. For any arbitrary change x (e.g., Δx), y changes by a scaler factor m of that change (e.g., $m \cdot \Delta x$). If that arbitrary Δx is then scaled by a factor c, the change in y is scaled by the same factor, yielding a corresponding change in y of $c \cdot m \cdot \Delta x$. This is a critical and productive meaning (i.e., M_b), yet our and others' work with teachers and students suggest that this is not always a typical meaning (Byerley & Thompson, 2017; Lobato et al., 2003; Moore, Silverman, et al., 2019; Thompson & Thompson, 1996; Zaslavsky et al., 2002).

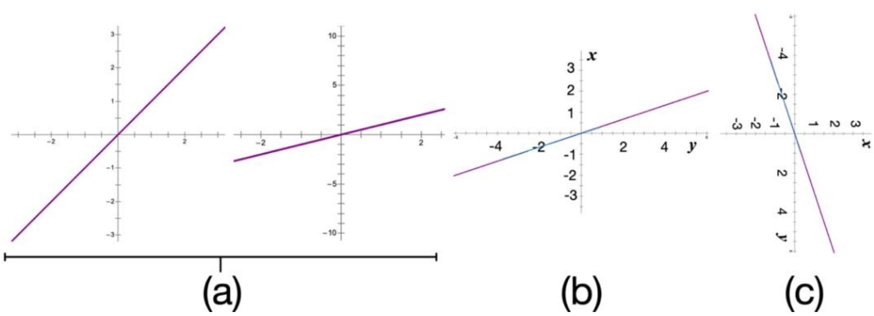

Figure 3: (a) Two graphs of $y = x$ and (b-c) two graphs of $y = 3x$.

A common extant meaning (i.e., M_a) for linear relationships is shaped-based (Ellis & Grinstead, 2008; Moore, 2021; Moore, Stevens, et al., 2019; Nagle & Moore-Russo, 2013; Zaslavsky et al., 2002), which entails reasoning about linear relationships in terms of properties of slope like movement and direction in association with learned formulas (e.g., $(y_2 - y_1)/(x_2 - x_1)$). These associations are forms of declarative knowledge, as opposed to symbolizing abstracted covariational relationships. An example of this is an individual comparing the visual

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

steepness of two lines to conclude the former has a greater rate of change than the latter (Figure 3a). As another example of this meaning, an individual could conceive Figure 3b as having an incorrect rise-and-run and Figure 3c as a negative slope or rate of change because of its downward, left-to-right direction (Moore, Silverman, et al., 2019; Moore, Stevens, et al., 2019).

Returning to the question raised above, in working with individuals holding a shape-based meaning, M_a, as their dominant meaning, we intend to both honor those shape-based meanings while determining how to support their constructing a rate of change meaning, M_b. Our goal is also to support their constructing M_b so that it becomes a meaning they view as important and productive (for them and their students), and more so than that of the shape-based meaning. Before describing an approach that draws on the competing meanings perspective, we recognize one way to support M_b is to use tasks in which M_a is not relevant, but M_b is. Similarly, one might use tasks that target M_b through focused, closed-ended questions and design. In our experience, such tasks are useful to engender M_b and possibly draw connections with M_a. Yet, such tasks can be so contrived as to feel too disjoint from the classroom for teachers. Relatedly, those tasks do not problematize M_a and generate an intellectual need for M_b so that the latter becomes their predominant or habitual meaning. With respect to teachers, for the tasks they envision teaching, M_a remains just as relevant, is more familiar or habitual, and is often more cognitively efficient. Our solution to this issue is to use tasks that not only afford or target M_b, but also draw the meanings M_a and M_b into competition with each other. Doing so requires that the task is designed so that M_a is still relevant to the task. Furthermore, we intend the enactment of M_a to lead to a conclusion that not only invites further thought, but that also stands in opposition to the conclusion derived from enacting M_b. This underscores the competing aspect of competing meanings. The initial perturbation should not leave the teacher viewing M_a as entirely problematic or unrelated, as it is through viewing M_a as still relevant despite some perturbation that the individual is positioned to compare its viability against that of M_b.

We use the graph in Figure 3b to illustrate how we have attempted to target the cognitive process in Figure 2 and draw meanings into competition with each other in the context of linear relationships. When working with teachers, we present this task in two parts. We first provide the graph as illustrated in Figure 3b, but *without* the axes-labels "*x*" and "*y*". We explain that a student provided the graph (without labels) as a solution to graphing "$y = 3x$", and we ask them to consider how the student might have been thinking. After the teacher has exhausted the number of ways they can hypothesize as to how the student might have been thinking (see Moore, Silverman, et al., 2019 for examples), we then provide Figure 3b with "*x*" and "*y*". We explain that the student added the labels to clarify their solution. We ask the teacher to comment on the graph, and we conclude the task asking how they would respond to the student as their teacher. We note that Figure 3c is created by most teachers when making sense of the solution due to their rotating the graph to horizontally orient *x*. If the teacher does not rotate the graph, we rotate the graph and ask them to consider it in that orientation, as well.

The task incorporates the competing meanings perspective by using the following principles: (a) it sensibly affords assimilation to M_a and M_b; (b) in the event that M_a is enacted, it is likely to result in a perturbation, but still be viewed as relevant to the task; (c) in the event that M_a engenders a perturbation, M_b is likely available to the student or within their zone of proximal development; (d) the enactment of M_b can reconcile a perturbation stemming from M_a; (e) the teacher has the opportunity to reflectively compare the affordances and constraints of M_a and M_b;

and, critically, (f) the teacher is likely to perceive the task, M_a, and M_b as relevant to their instruction. Relating these task features to the cognitive account in Figure 2, (a) and (b) occasion problematizing an extant meaning; (a), (c), and (d) relate to accommodation via the enactment of an alternative meaning; and (b), (d), (e), and (f) support reflecting on and comparing extant and alternative meanings. Furthermore, the task embodies the *competing* aspect of competing meanings by using a situation in which M_a and M_b yield sensible, yet different conclusions. Enacted as is, M_a results in classifying the solution and its rotated version (Figure 3c) as inaccurate representations of $y = 3x$ (e.g., the "slope" is wrong in Figure 3b and 3c), while M_b affords accepting both as accurate (e.g., each is the set of points so that y is three times as large as x and for any change in x, y changes by three times that amount). This in-the-moment incompatibility aids comparing the generativity and generalizability of each meaning including weighing which is better viewed as derivative of the other (e.g., slope as an implication of rate of change is more generative and generalizable than rate of change as an implication of slope).

Data Illustrations

Although this is chiefly a theoretical report focused on a particular form of learning and cognitive activity, it represents generalizations from a collection of empirical studies with students and teachers. The studies and their methodologies entailed semi-structured clinical interviews (Ginsburg, 1997) and various forms of teaching experiments (Steffe & Thompson, 2000), and are summarized in Moore et al. (2022) and Moore et al. (2024). Here, we draw from our empirical data with prospective teachers working the aforementioned task.

We use Table 1 to provide emblematic examples of each competing meanings component presented in Figure 2. Due to space constraints, we use quotes and only a brief narrative situating those quotes. We point the reader to our work referenced above for more detailed narratives of the students' actions and meanings. With respect to a problematized extant meaning, the example quote is from a participant, Lizzie, who conceived Figure 3b as having a "positive slope" and Figure 3c as having a "negative slope" due to their direction of rise and run (i.e., M_a). For both graphs, Lizzie checked points to verify the accuracy of the formula $y = 3x$. This, when paired with the slope discrepancy between the given and rotated graph, left her perturbed and calling into question the viability of her thinking on the task ("this is so annoying"). With respect to enacting alternative meanings (i.e., M_b), Tatiana's quote illustrates that by conceiving the graph via quantitative and covariational operations, she determined the graph to be a viable representation of $y = 3x$. This occurred after having not determined what was to her a satisfactory way to produce the unlabeled graph. In attributing a viable way of reasoning to the student solution, it also supported her reflecting on that meaning in terms of its flexibility. This is a key foundation for the reflective comparison of meanings.

The problematization of an extant meaning can occur in a reflexive process with the enactment of alternative meanings. Similarly, the phenomenon of reflectively comparing extant and alternative meanings does not always immediately follow that process. It more often occurs iteratively across a sequence of tasks. With that said, the provided quote is from Ada and it occurred after engaging in several tasks across an instructional sequence. It illustrates that by comparing extant and alternative meanings, she came to view rate of change as a dominant meaning. She conceived slope as a visual property (i.e., M_a) derivative of and thus subordinate to rate of change (i.e., M_b). This enabled her to consider a graph like that in Figure 3c as having a rate of change of 3 and, hence, a positive slope. Furthermore, she could couch her appraisal of the graph in terms of differentiating between the underlying mathematical concept of rate of change and common communicative practices (i.e., conventions). Such appraisals are critical to learning mathematics (Moore, Silverman, et al., 2019). As the participant Thomas described when privileging rate of change, “it’s smart [of a student] to understand that it’s not glued.”

Table 1: Quotes Associated with each Competing Meanings Component

Component	Graph Considered	Quote
Problematized Extant Meanings	Figures 3b-c (with labels)	Lizzie: I’m rising this three…then I’m running negative one…the slope is negative again…this is so annoying.
Enacting Alternative Meanings	Figure 3b (with labels)	Tatiana: Oh…we have a clever kid over here…so it now technically is *y* equals three *x*…not the standard way of doing it…They see the relationship between *x* and *y*.
Compare Extant and Alternative Meanings	Figure 3c (with labels)	Ada: …even though it looks like a negative slope…we call it slope because it’s visual and it’s easy to visualize a negative and positive slope. But that’s only visual on our conventions of how we set it up…slope is rate of change, we can still see that for like equal increases of *x* we have an equal increase of *y* of three. And so for equal positive increase of one we have an equal positive increase of three. And so, it is a positive slope.

Closing

We presented one learning form that identifies how two meanings might be brought into comparison via processes of assimilation, accommodation, and perturbation. We illustrated how such a process involves different forms of intellectual need, including that with respect to solving a task, comparing meanings, and considering the implications of those meanings. We also illustrated the competing meanings perspective through a task and emblematic participant activity. The competing meanings perspective is still in its infancy as a construct. Moving forward, we envision a need for further connecting to other extant constructs and perspectives, the results of which will continue to shape and develop the idea of competing meanings.

We have concentrated much of our research focus on the first two aspects competing meanings and relatively less on the nuanced ways in which teachers compare extant and alternative meanings (cf. Paoletti, 2020). A reflective comparison of meanings is a developmental process that occurs across a sequence of experiences, and it is through such a process that key developmental understandings are constructed and associated pedagogical implications are anticipated (Silverman & Thompson, 2008; Simon, 2006). We envision a fruitful area of inquiry to be more detailed investigations into how the competing meanings perspective might be used to engender such reflective comparisons and, accordingly, the construction of key developmental understandings. Furthermore, we view a need for further relating this process to the development of mathematical knowledge for teaching. Here, we do not make strong claims regarding the development of mathematical knowledge for teaching. There are numerous factors other than a teacher's knowledge that mitigates their instructional practices and the meanings they target in their classroom. But, literature identifies the critical role of meanings and teachers being reflectively aware of them (Liang, 2019, 2023; Tallman & Frank, 2020; Thompson, 2016), and the competing meanings perspective is one potential tool to support the transformation of knowledge to that which informs instructional action.

For the purpose of adhering to the space constraints of the current report, we situated our work in the theories that directly informed its emergence and development. There is significant literature on learning, conceptual change, and perturbation, and thus an additional need is to further situate the notion of competing meanings within that literature. For example, Vinner and Dreyfus (1989) proposed *compartmentalization* as the phenomenon in which a learner has two potentially conflicting meanings. Noah-Sella et al. (2022) have since extended this phenomenon to incorporate Thompson's theory of meaning and explore calculus students' integral meanings. Their perspective foregrounds cases in which a researcher perceives a potential conflict or relationship between meanings, but the participant does not. The competing meanings perspective might contribute a way by which one considers how to support a student or teacher in bringing that conflict to the surface. As another example, researchers have productively pursued characterizing learning using Piaget's forms of reflective abstraction (Ellis et al., 2024; Simon et al., 2010; Tallman & O'Bryan, 2024), including theorizing its role in constructing mathematical knowledge for teaching (Liang, 2021, 2023). We envision that drawing connections with this work will provide insights into how aspects of the competing meanings perspective are related to crucial abstraction processes.

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INVESTIGATING PROSPECTIVE TEACHERS' ASSESSMENT OF STUDENT THINKING AND THEIR MATHEMATICAL KNOWLEDGE FOR TEACHING

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In our mixed-methods study we investigated how prospective elementary school teachers (PSTs) in the early stages of their preparation learn mathematical content and also learn to assess/attend and interpret students' mathematical thinking when analyzing students' written work on a mathematical task in the domain of fractions. We implemented an activity focused on building the PSTs' assessment reasoning. Data from 44 PSTs were used to analyze the PSTs' assessment skills using an adapted framework by Talanquer et al. (2015). Analysis using a linear model was conducted on the PSTs' pre and post Mathematical Knowledge for Teaching using the Learning Mathematics for Teaching instrument and the level of PSTs' assessment skills. Findings provide evidence that there is a significant association between the level of assessment skills and content knowledge at the end of the course than at the beginning of the course.

Keywords: Assessment, Mathematical Knowledge for Teaching, Preservice Teacher Education, Teacher Noticing

Introduction and Background

The practice of formative assessment has been identified as critical for improving teacher effectiveness and student learning outcomes (Black & Wiliam, 2009). However, prospective teachers need support in developing such an important pedagogical skill which involves interpreting students' mathematical thinking (Boerst et al., 2019). This mixed-methods study aims to characterize and differentiate how elementary school preservice teachers (PSTs) interpret students' mathematical thinking when analyzing students' written work on a formative assessment task in the domain of fractions. Additionally, we explore any connections between the intervention focused on building PSTs' assessment skills in the domain of fractions and the development of their mathematics knowledge for teaching (MKT). The mathematics content course which serves as the setting for this study affords opportunities to introduce PSTs to the required mathematics content with an eye towards teaching and interpreting elementary students' mathematical thinking. We were interested to see whether such a content course in general and the activity focused on building PSTs' assessment reasoning in the domain of fraction, in particular, would help the PSTs develop their MKT specifically in the domain of fractions (Ball et al., 2008; Stylianides & Ball, 2008; Thames & Ball, 2010). Research questions that guide our study are: (1) What are the levels of the PSTs' overall assessment skills along the different dimensions within the categories of evaluate/attend and interpretation when analyzing students' mathematical thinking on a written task in the domain of fractions? (2) What connections exist, if any, between PSTs' assessment reasoning skills and their MKT, as measured by the Learning Mathematics for Teaching (LMT, 2004) instrument?

Methodology

This mixed-methods study was conducted in a large Hispanic-Serving public university in the southern United States during the fall 2021 semester. The participants in the study consisted of 44 PSTs enrolled in their first mathematics content course for prospective elementary and middle school teachers. The data collected were (1) PSTs' written assessment of an elementary/middle school student's work on a formative assessment task that focused on identifying fractional values of area shapes. The formative assessment and the sample student work were part of the collection of tasks and student works found in *Balanced Assessment* (Schoenfeld, 1999) and (2) the pre- and post-test scores of the Learning Mathematics for Teaching (LMT) specifically the Content Knowledge (CK) and Knowledge of Content and Students (KCS) in the domain of Number Concepts and Operations (Hill et al., 2008).

To answer the first research question, we analyzed the PSTs' written assessments of student work based on our adapted version of Talanquer's Framework (Talanquer et al., 2015) for evaluating and interpreting student understanding in the assessment of written work. In our framework there are four evaluative dimensions (*evaluation stance, specificity in the evaluation, analysis of coherence, use of evidence*) and four interpretive dimensions (*quality of interpretation, productive thinking, scope of the evaluation, mathematical accuracy*). Talanquer's Framework was originally intended for use within science education, but it was informed by mathematics education, including the Jacobs et al. framework (Jacobs et al., 2010) during its development. The framework provides more layers of differentiation in scoring of 'Attend' and 'Interpret' features of the Jacobs et al. teacher noticing framework thereby allowing to better capture the subtle differences in PSTs' assessment capabilities. The framework also allows for analyzing the interpretations of PSTs' student thinking by levels and with detail that is domain specific. We thus found Talanquer's Framework adaptable to our setting in our mathematics education study. We noticed that PSTs' assessment capabilities vary within each level, so instead of assigning a discrete score by level we decided to assign a score within an assigned interval for each level. We provide one example of an adaptation that we made noted in italics from the Talanquer Framework in Table 1.

Table 1: Interpretive dimensions (*soundness of interpretations characterized along these dimensions*) and levels of sophistication in PSTs' assessment of student understanding.

Interpretive dimensions	Novice [0,1]	Emerging (1,2]	Advanced (2,3]
Quality of the interpretation	Interpretations cannot be supported with the evidence available *or claims are minimally supported by re-stating students' answers without unpacking the meaning in*	Interpretations are built, but the evidence provided is limited, superficial or formulaic.	Reasonable interpretations of student understanding are built given the available evidence.

	students' responses	

Two of the authors first met to establish the modified codes described above and coded six PSTs' written assessment reports together to get a sense of our coding scheme. We then separately coded all of remaining 38 participants' written assessments. After coding, we met to reconcile any differences and came to an agreement. Throughout the coding process we maintained 90% or better reliability. We used a linear regression model to answer the second research question to explore associations between the change in the pre/post LMT scores and the level of assessment skills demonstrated by the PSTs in the domain of fractions.

Findings and Conclusions

Findings of our first research question capture the subtle differences in the PSTs' assessment reasoning skills along four dimensions within the category of evaluation/attend and four dimensions within the category of interpretation in analyzing students' mathematical thinking on a written task in the domain of fractions. Furthermore, we were able to differentiate our PSTs' assessment skills into three levels of sophistication for evaluating and interpreting student thinking along these dimensions. On average PSTs' (N = 44) score within the interval [0, 3] on the construct of overall assessment skills came out to be 1.53, which puts them on average in the middle of the emerging level spectrum. The table below provides a summary of the average scores of the PSTs along different dimensions within the two categories of the overall assessment skills - evaluation/attend and interpretation.

Table 2: PST average scores on Evaluative and Interpretive dimensions

Evaluative/ Attend dimensions	evaluative stance	specificity in evaluation	analysis of coherence	use of evidence	Overall Evaluative/ Attend Skill
Average Score	1.54	1.46	1.51	1.46	1.49
Interpretive dimensions	**quality of interpretation**	**productive thinking**	**scope of evaluation**	**math accuracy**	**Overall Interpretation Skill**
Average Score	1.60	1.54	1.47	1.61	1.56

When differentiating the PSTs' overall assessment skills into three levels of sophistication for evaluating and interpreting student thinking along these dimensions, we found that 34% (15 PSTs) were at the Novice level, 39% (17 PSTs) were at the Emerging level, and 27% (12 PSTs) were at the Advanced level. Furthermore, PSTs' overall interpretation skill average score came out to be slightly more (4.7% more) than their overall evaluative/attend skill average score.

The student work that PSTs were asked to analyze was focused on identifying the fractional parts of a square subdivided into four equal smaller squares and each of the smaller squares divided into fractional pieces. Below we provide sample excerpts of the PSTs' analysis at different levels of sophistication along the 'Quality of the interpretation' dimension using the coding scheme given in Table 1.

Novice Level: "None of the fractions in the square are the actual fraction that it needs to be, but in the mind of the child, I understand how student B got most of the answers." (PST 415)

Emerging Level: "It is clear that they (Student B) had difficulty understanding larger fractions and reasoning for dividing up a whole piece into many smaller and uneven pieces." (PST 428)

Advanced Level: "I noticed one main recurring issue, which was them thinking of it as four individual squares rather than one as a whole. For example, in part one, they incorrectly stated that A and B were each equal to one half. They are correct that A and B are one half of one fourth, however if Student B was looking at the big picture, letters A and B would be equal to one eighth. Those two eighths would be equal to one fourth of the entire square. As you can see, Student B solved the square in four different individual parts. Each fourth of the entire square was equal to one, when in reality the entire square should have been equal to one." (PST 421)

In addressing the second research question, changes in PSTs' MKT were assessed using two versions, Pre and Post, of the instrument created by the Learning Mathematics for Teaching Project (2011). The instrument contains items from two subscales: Content Knowledge (CK) and Knowledge of Content and Students (KCS). We obtained separate estimates of the PSTs' ability scores overall and for each subscale by using an IRT model and item parameters obtained as part of a nationally representative sample of middle school teachers (see Hill 2007). PST's exhibited significant gains overall (0.7, $p < .001$) and in CK (1.2, $p < .001$), but not in KCS (-.1, $p = .499$). To explore the relationship between the level of sophistication of the PST's assessment of student work (L_i) and their MKT (Z_i), we fit the linear model:

$$Z_i^{(Post)} = \beta_0 + \beta_1 L_i + \beta_2 Z_i^{(Pre)} + \beta_3 L_i * Z_i^{(Pre)} + \epsilon_i$$

where $i = 1, \ldots, 44$. The results for the model with CK scale are shown in Table 3 below.

Table 3: Coefficients Linear Model for Post Content Knowledge

Coefficient	Estimate	Std. Error	*p*-value
Intercept (β_0)	-.31	.56	.581
Level of Sophistication (β_1)	.76	.28	.009
Content Knowledge: Pre (β_2)	.43	.30	.160
Interaction (β_3)	.21	.15	.166

Adj-$R^2 = .71$, $F = 35.71$, $df = 3, 40$, $p < .001$

The overall F-test and Adj-R^2 indicate the model is statistically significant and performs well at explaining variation in the post CK subscale score. When controlling for pretest score, the level of sophistication is a significant predictor of post test score. Comparing the estimates and the *p*-values, surprisingly, the level of sophistication of the PSTs' assessment of student work is more strongly associated with the post test score than the pre-test is. That is, the data provides evidence that there is a more significant association between the level of assessment skills and content knowledge at the end of the course than at the beginning of the course. Although we cannot claim a causal relationship between the level of assessment of student work and content knowledge, the results look promising for investigating further this claim. We plan to strengthen the design by including a comparison (control) group in the next phase of the project.

Our findings suggest that analyzing PSTs' assessment capabilities can be enriched by considering different dimensions along the evaluative/attend and interpretive strands of the construct. By capturing the subtle differences in assessment skills teacher educators can better support PSTs' assessment reasoning. The significant association between the level of sophistication of the PSTs' assessment of student work and content knowledge when controlling for incoming knowledge suggests that providing opportunities to learn how to assess students' mathematical work, impacts PSTs' mathematical knowledge positively. This result provides evidence, although taken with caution, that including more opportunities to learn how to assess and analyze student work as part of the teacher education curriculum is an effective instructional practice to improve mathematical learning.

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PROSPECTIVE TEACHERS' USE OF MULTIPLE TEACHER MOVES IN NUMBER TALKS

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Number Talks (NTs) may offer support for prospective teachers (PSTs) to engage in ambitious instruction. There has been a call for more empirical studies on NTs on the part of teachers. There are only a few studies on PSTs' teacher moves during their enacted NTs in field placement classrooms. In this study, we analyzed 22 PSTs' enacted NTs to identify themes associated with teacher moves used by those PSTs in their NT implementations. We found that revoicing and clarification teacher moves were dominantly used throughout 152 strategy segments in 48 NTs enacted by 22 PSTs. We also found how different teacher moves were used in each strategy segment in which individual students shared their strategy one after another and by PSTs, and how additional teacher moves were used with the dominant teacher moves. Discussions and implications are offered about mathematics teacher education and research.

Keywords: Number Talks, Elementary School Education, Instructional Activities and Practices, Preservice Teacher Education

Purpose

In the U.S., there has been an increase in using Number Talks (NTs), brief daily instructional routines (10-15 minutes in length) in which "students mentally solve computation problems and talk about their strategies" (Humphreys & Parker, 2015, p. 5), in mathematics classrooms. Despite such popularity, the instructional routines have not been well-studied empirically. In their extensive literature review, Matney, Lustgarten, and Nicholson (2020) called for empirical studies on the efficacy of NTs on student learning. Some studies have addressed the call by working with (novice) teachers' NTs in terms of, for example, the math-talk learning community (Woods, 2022), relationships with ambitious instruction (Pak, Cavanna, & Jackson, 2023), and formative assessment (Han & Thanheiser, 2021).

In this paper, we investigate teacher moves used by prospective teachers (PSTs) in their NTs in their field placement classrooms. Studies on mathematical discourse (e.g., Arnesen & Rø, 2022; Chapin, O'Connor, & Anderson, 2013; Franke et al., 2015; Herbel-Eisenmann, Steele, & Cirillo, 2013; Kazemi & Hintz, 2014) have shown the positive impacts of teacher moves on students' conceptual understanding. Given that, student learning in NTs also may depend in part on teacher moves, which can offer students opportunities to reason and make sense of strategies. Among the four phases (Introducing, Collecting Answers, Idea Sharing, and Closing) of the NT routines (Pak et al., 2023; Parker & Humphreys, 2018), the Idea Sharing phase, which begins when teachers ask students to share their strategies and ends before teachers conclude the NT, involves multiple teacher moves that foster student reasoning. As such, we focus on the Idea Sharing phase to investigate teacher moves used by PSTs in their enacted NTs. PSTs need to learn how to engage in ambitious mathematics instruction (Kazemi, Franke, & Lampert, 2009; Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010). One way to begin to engage in ambitious mathematics instruction would be learning to effectively use teacher moves in the Idea

Sharing during NTs. As for studies on NTs, little attention has been paid to PSTs' NTs in general and teacher moves used by PSTs in the Idea Sharing phase in particular. In this paper, we would like to contribute to understanding PSTs' use of multiple teacher moves in the Idea Sharing phase. As such, the purpose of this paper is to explore teacher moves PSTs use in the Idea Sharing phase in their enacted NTs. Understanding PSTs' use of multiple teacher moves in the Idea Sharing phase can support mathematics teacher educators with an interest in PSTs' teacher moves in NTs to enhance more effective ways for PSTs to use teacher moves in their NTs.

Conceptual Perspectives

In this paper, we draw on two conceptual perspectives to offer an understanding of how we approach PSTs' use of multiple teacher moves in the context of NTs, especially the Idea Sharing phase. The first perspective is related to the idea that multiple teacher moves work together to facilitate mathematical discourse more productively (e.g., Arnesen & Rø, 2022; Ellis, Özgür, & Reiten, 2019). Ellis et al. (2019) developed the Teacher Moves for Supporting Student Reasoning framework, classifying talk moves into four categories: eliciting, responding, facilitating, and extending. This framework places individual teacher moves on a continuum based on their potential to support student reasoning. Mainly, it underscores the collective interplay of these teacher moves across categories to develop a learning environment that honors student reasoning and sense-making. Drawing on the framework, Arnesen and Rø (2022) found teacher moves with different levels of potential, high and low, to support student reasoning. They offered multiple illustrations of how an experienced teacher used high-potential teacher moves (e.g., requesting justification and indicating relationship) along with low-potential teacher moves (e.g., unraveling student input and acknowledging contribution) to share intellectual authority with students as the teacher facilitated mathematical discussion. Even though we do not fully use the teacher moves identified in their works in this paper, the findings suggest that effective support for student reasoning often involves using multiple teacher moves together. We extend the idea to NTs in this paper because it is also important for teachers, particularly PSTs who need to learn how to use multiple teacher moves together, to support students to make mathematical reasoning more accessible to other students in the context of NTs.

The second perspective pertains to teacher moves used by novice teachers in the Idea Sharing phase during NTs. The Idea Sharing phase is where teacher moves can be used to support students in sharing their reasoning behind their strategies. Two studies (Pak et al., 2023; Murata et al., 2017) entailed findings regarding multiple teacher moves in the context of NTs. In a systematic analysis of 17 videos of NTs enacted by seven beginning elementary school teachers over three years, Pak et al. (2023) identified strategy segments and strategy-plus segments in the Idea Sharing phase. A strategy segment begins when a teacher asks an individual student to share his/her strategy and ends when the teacher moves to another student's strategy. A strategy-plus segment begins when a teacher asks other students to contribute to the initial strategy shared by an individual student and ends when the teacher moves to another student's strategy. To explore what beginning teachers could do to engage multiple students in each other's strategies, Pak et al. (2023) focused solely on strategy-plus segments to find teacher moves used by the beginning teachers. Murata et al. (2017) identified multiple teacher moves used by two beginning elementary school teachers during NTs over five months. They focused on teacher moves used in the whole NTs including the four phases (Introducing, Collecting Answers, Idea Sharing, and

Closing). The teacher moves used in the Idea Sharing phase included feedback evaluation (teacher responding to student talk to offer feedback to the student about the idea), strategy explanation (teachers extending student thinking beyond what the student said), and process questions (questions about strategies and thinking process). Understanding teacher moves by (novice) teachers in NTs, especially in the Idea Sharing phase, is still at the beginning stage. Particularly, teacher moves used by PSTs in the Idea Sharing phase have been under-documented. These two perspectives led us to explore how PSTs used multiple teacher moves together in strategy segments in the Idea Sharing phase.

Methods

Context, Participants, and Data Sources

The research site was in a mathematics methods course for elementary school teachers at a small southwestern university. Data regarding the course submissions was gathered in Spring 2023 as part of a research project that investigated how to support PSTs in learning to implement the eight mathematics teaching practices (National Council of Teachers of Mathematics, 2014).

Twenty-nine PSTs agreed to participate in data collection, including electronic submissions of course assignments, including the Number Talk Project. In their informal weekly reflection, they reported that they had never heard about NTs before and it was the first time for them to learn about NTs. The Number Talk Project consisted of PSTs engaging in a learning cycle to plan, rehearse, and implement their NTs plan. At the end of the learning cycle, PSTs video-recorded themselves enacting their revised NTs plan in their field placement classroom. PSTs also submitted a written reflection paper based on examining their NT video.

In this paper, we only used videos of NTs because the videos allowed us to more clearly see and hear what PSTs did in the Idea Sharing phase than in other written submissions (e.g., NTs plan, reflection/analysis paper). Videos of NTs enacted by 22 out of 29 PSTs were available for analysis for this paper. Some PSTs did more than two NTs, which meant they posed more than two problems. We obtained a total of 48 NTs enacted by 22 PSTs as data sources.

Data Analysis

There were five steps we took to analyze the transcripts of videos of NTs using a thematic analysis (Saldaña, 2015). First, we identified the Idea Sharing phase in terms of two sub-categories: strategy segments and strategy-plus segments. We used the work of Pak and colleagues (2023) to identify the sub-categories. Second, we analyzed 10 NTs enacted by four PSTs to create an initial analytic code of teacher moves in the strategy segments and strategy-plus segments. Since we identified only one teacher move used by one PST in one strategy-plus segment, we decided to focus our analysis on teacher moves in strategy segments because our interest is in PSTs' use of multiple teacher moves. We used both inductive and deductive coding to code the data (Hatch, 2002). We began with teacher moves identified in the prior studies such as Arnesen and Rø (2022), Chapin et al. (2013), and Murata et al. (2017). We particularly drew on Murata et al.'s (2017) teacher moves such as feedback evaluation, strategy explanation, and process questions because they emerged from novice teachers' NTs. We decided to exclude process questions (e.g. "How did you solve to find out what the answer was?" "So now what should we do?) from our potential codes because process questions are used to help students proceed with their explanation rather than helping them reveal their reasoning behind strategies. Third, we applied the initial analytic codes to the rest of the NTs (38 NTs). We went through an iterative process of individual coding and team meetings to reach a consensus on codes and develop codes of teacher moves in strategy segments. As a result of the process, the codes were refined (see Table 1). Fourth, we quantified teacher moves by PSTs, NTs, and strategy segments. Fifth, we compared and contrasted teacher moves within and across teacher moves by PSTs, NTs, and strategy segments to identify salient patterns. We ensured trustworthiness by having weekly meetings to discuss coded transcripts and to resolve disparities and by coding all transcripts by authors.

Table 1. Codes for Teacher Moves Used by PSTs in Strategy Segments

Teacher moves	descriptions	examples
Revoicing what a student said	PSTs repeat what a student said when they shared their strategy.	Student: 3 times 30 equals 90. PST: 3 times 30 equals 90.
Asking students questions for clarification	PSTs ask students to clarify the process of getting the answer.	How did you take away zero? Divided what by 10?
Asking students to justify their reasoning	PSTs ask students to explain the reasoning behind their strategy. This code usually involves a why question.	Why did you divide 70 by 10? Why did you multiply it by 10?
Responding to students' self-correction on mathematical mistakes	PSTs responded to students to support students in realizing their mistakes on their own. PSTs also wait patiently for students.	PST: That is alright. Student: Because I added 3 at the start. You have to take 3 off. PST: Yes, but you added 3 to 67 and made it 70. If you didn't take away 3 from 232, we still have an extra 3.
Mentioning a brief connection between strategies	PSTs briefly mention a connection they see between strategies in the form of questions.	So you took this one and [pointing out another strategy recorded on the board] you kind of did it this way, right?
Recapping a student's strategy for the whole class	PSTs summarize an individual student's strategy for the whole class.	She stacked in her head and she had to borrow. But she knew that borrowing from the zero, there is nothing to borrow from zero.

Findings

In this section, we present four findings to demonstrate how 22 PSTs used multiple teacher moves in the strategy segments. As mentioned earlier, we found only one teacher move used by one PST in one strategy-plus segment, which is not included in the findings. To illustrate the findings, we use frequency tables and an excerpt.

First, we found that seven different teacher moves were observed across 152 strategy segments within 48 NTs enacted by 22 PSTs (see Table 2). The two teacher moves (*revoicing and clarification)* were frequently used by PSTs. We call the two teacher moves as the dominant teacher moves. All 22 PSTs used revoicing 146 times in total throughout 94 strategy segments in 44 NTs. Clarification was used 54 times throughout 43 strategy segments in 27 NTs by 17 PSTs. The other five teacher moves (*asking students to justify their reasoning, responding to students' self-correction on mathematical mistakes, mentioning a brief connection between strategies, recapping a student's strategy for the whole class, and asking students to make connections between mathematical ideas*) were less frequently used by the PSTs. We call the five teacher moves as the additional teacher moves. Asking students to justify their reasoning was used by three PSTs 14 times across seven strategy segments in five NTs. Responding to students' self-correction on mathematical mistakes was used by two PSTs three times in four strategy segments in three NTs. Mentioning a brief connection between strategies was used three times in three strategy segments in three NTs by three PSTs. Recapping a student's strategy for the whole class was used 10 times in 10 strategy segments in nine NTs by eight PSTs. Asking students to make connections between mathematical ideas was used once throughout the entire data.

Table 2: Frequencies of Teacher Moves by PSTs, NTs, and Strategy Segments

Teacher Moves	Instances Total	152 Strategy Segments	48 NTs	22 PSTs
Revoicing what a student said	146	94	44	22
Asking students questions for clarification	54	43	27	17
Asking students to justify their reasoning	14	7	5	3
Responding to students' self-correction on mathematical mistakes	4	4	3	2
Mentioning a brief connection between strategies	3	3	3	3
Recapping a student's strategy for the whole class	10	10	9	8
Asking students to make connections between mathematical ideas	1	1	1	1

Second, we found that the PSTs used at least one teacher move in 114 strategy segments

(75%) and no PSTs used more than three teacher moves in any strategy segments (see Table 3). Among 114 strategy segments, 75 strategy segments (66%) contained one teacher move, 31 strategy segments (27%) contained two teacher moves and eight strategy segments (7%) contained three teacher moves.

Table 3: Number of Strategy Segments by Number of Teacher Moves Used

Number of dominant and additional teacher moves	None	1	2	3	Total
Number of strategy segments	38	75	31	8	152

Third, we found that each PST used different teacher moves throughout NTs. In terms of each PST's use of different teacher moves throughout NTs, three PSTs used only one type of teacher move (revoicing), 10 PSTs used two different teacher moves, five PSTs used three different teacher moves, three PSTs used four different teacher moves, and one PST used five different teacher moves across the NTs (see Table 4). For example, PST2 used both revoicing and clarification moves in the first strategy segment. The same PST did not use any teacher move for the second and fifth strategy segments, used the clarification move for the third and fourth strategy segments, and used the justification move in the sixth strategy segment in her first NT. In her second NT, PST2 did not use any teacher move for the first strategy segment and used the revoicing move in the second strategy segment. So, PST2 used at most two teacher moves in one strategy segment and used three different teacher moves (revoicing, clarification, and justification) throughout the NTs.

Table 4: Number of PSTs Who Used Different Teacher Moves Throughout NTs

Number of dominant and additional teacher moves	1	2	3	4	5
Number of PSTs	3	10	5	3	1

Fourth, we found that a few PSTs were able to use the dominant teacher moves in combination with additional teacher moves in strategy segments. Twelve PSTs only used revoicing and/or clarification without using any additional teacher moves and 10 PSTs used other additional teacher moves at least once along with the two dominant teacher moves across the NTs. Among 114 strategy segments that had at least one teacher move, 106 strategy segments (93%) contained either revoicing or clarification moves and only 8 strategy segments (7%) had teacher moves without revoicing or clarification teacher moves. Among 31 strategy segments PSTs used two teacher moves, 30 of them included either revoicing or clarification. Only one strategy segment had two additional teacher moves (recapping and asking students to make connections) without revoicing and clarification (see Table 5).

Table 5: Number of Strategy Segments With(out) Dominant Teacher Moves

Dominant teacher moves	0 additional teacher move	1 additional teacher move	2 additional teacher moves	Total
Revoice, no Clarify	56	7	1	64
Clarify, no Revoice	12	0	0	12
Revoice and Clarify	23	7	0	30
Neither used	38	7	1	46
Total	129	21	2	152

Table 6 shows PSTs' use of teacher moves when they used three teacher moves in one strategy segment. Among eight strategy segments, seven included both revoicing and clarifying and one PST used the revoicing move but clarifying. PST3 used revoicing, clarification, and brief connection in one strategy segment, and used revoicing, clarification, and recapping in three strategy segments, and used revoicing, clarification, and justification in one strategy segment. PST4 used revoicing, clarification, and justification in one strategy segment and used revoicing, clarification, and recapping in another segment. PST 21 used revoicing, brief connection, and recapping in one strategy segment.

Table 6: PSTs' Use of Three Teacher Moves Across Eight Strategy Segments

PSTs	Number Talks	Strategy segments	Revoicing	Clarifying	Justify	Brief Connection	recapping
PST3	1st NT	2nd	2	1		1	
	2nd NT	2nd	2	1			1
		3rd	2	2			1
	3rd NT	1st	1	3			1
		2nd	4	1	1		
PST4	1st NT	1st	2	1	3		
	2nd NT	3rd	1	2			1
PST21	2nd NT	2nd	2			1	1

We present an excerpt below to illustrate how PSTs used three teacher moves. The excerpt shows how PST3 used three teacher moves such as revoicing, clarification, and recapping in the third strategy segment in her second NT. She posed the second problem (4 x 140), followed by the first problem (8 x 70). In the excerpt, PST3 called on a student to share her strategy.

T: Okay, let's hear from Emma
S: So, what I did is. I figured out that 8 times 70… um... 8 times 70 and 4 times 140 are equal, you put both of those out
T: Okay.
S: If you take away the zero from 70 and 140, then 7 is half of 14, so I do times 2
T: **Times what by 2,**
S: 4 times 2
T: **So you did 4 times 2,**
S: 4 times 2 is 8, and then… (student thinking)
T: Okay, so **how do we get from 140 to 70?**
S: divide it by 2?
T: Okay. And that 70.
S: So it is 8 times 70.
T: Okay. **So Emma doubled our 4 to 8 and halved our 140 to 70 because we already saw that problem, right?** Do we agree that that strategy works?

In this excerpt, PST3 used the revoicing move once ("So you did 4 times 2"). She also used the clarification move twice ("Times what by 2?" "How do we get from 140 to 70?"). The PST used the recapping move once ("So, Emma doubled our 4 to 8 and halved our 140 to 70 because we already saw that problem, right?).

Discussions and Conclusion

Our analyses revealed four findings related to PSTs' use of teacher moves in their enacted NTs, all of which related to teacher moves occurring within the 152 strategy segments in the *Idea Sharing* phase of the NTs routines. Specifically, we observed the PSTs' teacher moves in terms of two dominant teacher moves and five additional teacher moves. We also found that a few PSTs used multiple teacher moves in strategy segments. These findings offer potential insights for the field of mathematics teacher education as we seek to support PSTs to engage in ambitious mathematics instruction. We offer two discussion points in terms of mathematics teacher education and research, respectively.

First, these findings suggest a need for PSTs to learn to use multiple teacher moves in the strategy segments. We agree that novice teachers need to move beyond strings of strategy segments in which students simply share one strategy after another (Pak et al., 2023). As we briefly mentioned in the data analysis section, the data only included one strategy-plus segment in the whole data. The analysis also shows that a few PSTs can only use multiple teacher moves. We conjecture a relationship between strategy segments and strategy-plus segments: Teacher moves in the strategy segments can serve as a springboard to help multiple students engage in each other's mathematical ideas in the strategy-plus segments because multiple students' engagement depends on how much individual students are supported by teachers with opportunities to reveal their reasoning and make sense of strategies. As such, our findings offer implications for mathematics teacher education. Specifically, mathematics teacher educators need to provide opportunities for PSTs to learn to incorporate multiple teacher moves into strategy segments to promote students' reasoning and sense-making.

Building on the first discussion point, second, these findings suggest a yet-uncharted research area regarding the effective use of multiple teacher moves in the strategy segments on the part of PSTs. We highlighted ways 22 PSTs used multiple teacher moves in strategy segments. This paper contributes to our understanding of teacher moves in the strategy segments. Many of the 22 PSTs were able to use teacher moves perceived by researchers as effective for student reasoning (for example, revoicing (Chapin et al., 2013) and justification (Thanheiser et al., 2021)). This paper extends Murata et al. (2017) and Pak et al. (2023) to explore multiple teacher moves in strategy segments. Murata et al. (2017) identified multiple teacher moves in the whole NTs. Pak et al. (2023) did not investigate the nature of strategy segments in terms of teacher moves. Due to a lack of data, however, the paper did not explore the efficacy of using multiple teacher moves in NTs, especially strategy segments, on student learning about mathematical concepts. As Matney et al. (2020) argued, it is essential to understand the impacts of NTs in general and multiple teacher moves in particular on students' reasoning and sense-making. We suggest that further research be on ways to support PSTs to learn to more effectively use multiple teacher moves in the strategy segments in the Idea Sharing phase.

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DEVELOPING ELEMENTARY PRESERVICE TEACHERS' MATHEMATICAL PRACTICE: ATTEND TO PRECISION

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Keywords: Mathematical Processes and Practices, Pre-Service Teacher Education

Purpose of Research and Connection to Literature

This poster proposal represents a research-in-progress examining the Standard for Mathematical Practice 6: Attend to Precision. Mathematics, as a discipline, demands precision as a tool for accuracy and rationalization (Otten et al., 2019). The purpose of this study is to develop elementary preservice teachers' ability to attend to precision through a series of professional learning seminars, written mathematical tasks, and micro teaching demonstrations. A review of literature revealed four key themes demonstrating the benefits to both teachers and students when precision is explicitly emphasized in mathematics instruction. First, attending to precision supports students' learning and understanding (Engledowl et al., 2015; Leatham et al., 2016; Brunn et al., 2015). Second, attending to precision supports the sequential nature of learning mathematics (Kieran, 2007). Third, attending to precision promotes effective communication. Common language demands in mathematics include asking students to analyze, describe, or justify mathematical concepts and ideas (CCSSI, 2010). Lastly, attending to precision supports effective engagement with other Standards for Mathematical Practice (SMPs).

Methods

Participants for the current study were recruited through enrollment in junior and senior block cohorts of an elementary education preparation program. Nine elementary preservice teachers engaged in pre- and post- questionnaires, researcher-led seminars, written discourse tasks, a micro teaching demonstration, and focus group interviews. Data was collected over a span of eight weeks and analyzed qualitatively to answer the research questions: 1) How do ePSTs attend to precision? 2) How do the tasks and experiences within this study influence ePSTs' perspectives and abilities to attend to precision? and 3) How do researcher-led seminars support the development of ePSTs' attend to precision?

Preliminary Findings and Implications

This study aimed to examine how elementary preservice teachers develop the specialized skill, attend to precision, through a series of seminars, engaging tasks, and micro teaching demonstrations. Additionally, this study sought to understand how ePSTs' perspectives toward precision in mathematics changed over the duration of the study. Preliminary findings of the study include the impact of participants' Mathematical Knowledge for Teaching (Ball et al., 2008) on their ability to attend to precision, emphasis on students' prior knowledge in facilitating an attention to precision in teaching, and participants' heightened awareness to the importance of mathematical discourse and use of strategies in their instruction. Additionally, participants were reflective of their teaching demonstrations with respect to knowledge gained through the researcher-led seminars and expressed a need for more exposure and practice with Standards for Mathematical Practices (CCSSI, 2010). Potential implications seek to provide insight for future design of learning seminars within teacher preparation programs and suggestions for integrating Standards for Mathematical Practices (SMPs) into PSTs' coursework and experiences.

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PRESERVICE TEACHERS' CONCEPTUALIZATION AND ENACTMENT OF THE INVESTIGATIVE APPROACH

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Keywords: Preservice Teacher Education; Teacher Educators; Instructional Activities and Practices

Background

Teacher educators have the responsibility of deepening preservice teachers' (PSTs) mathematical content knowledge while also casting a vision of effective mathematics instruction. This research takes place in a middle school math methods course in which the investigative approach is presented as an ideal pedagogical model. The investigative approach "involves purposeful, inquiry-based, and meaningful instruction and, thus, can foster all aspects of mathematical power: a positive disposition, the processes of mathematical inquiry, and understanding" (Baroody & Coslick, 1998, p. 1-27). PSTs were taught about the investigative approach through brief lectures, textbook readings, homework assignments, and modeling. Students were given opportunities to demonstrate their developing conceptualization of the investigative approach by writing an essay, planning and teaching two 15-minute mini-lessons to classmates, and reflecting on their mini-lesson teaching experience (Amobi & Irwin, 2009; Lucero et al., 2023). We wanted to determine which elements of the investigative approach PSTs were able to use with fidelity to inform future instruction and research, so we developed the following research questions to guide the study:

- How do preservice teachers describe the investigative approach?
- What observable practices do preservice teachers use in their mini-lessons?
- How do PSTs' understandings of the investigative approach relate to practices observed?

Methods, Results, and Implications

This poster reports on a case study that examined two PSTs' level of fidelity to the investigative approach through an analysis of their written descriptions of the investigative approach, recorded mini-lessons, and post-teaching reflections. Codes were generated from the data through open and axial coding (Glaser & Strauss, 1967). More specifically, the data were analyzed to highlight what the teachers knew and did not know about the investigative approach, as well as identify observable practices that did and did not match the investigative approach.

The data collected from the PSTs' essays on the investigative approach indicated that they were accurately able to describe its focus, lesson structure, and basic instructional strategies. Their mini-lessons included some observable practices that matched what they had learned and written about the investigative approach such as real-life contexts and effective questioning. The second mini-lesson was somewhat more student-centered than the first for both PSTs. However, the structure of the tasks chosen ultimately limited their opportunity to enact the mini-lessons in a way that we would consider truly investigative. Based on these results, we suggest providing greater support to PSTs in the task selection process. This could be done by providing exemplars and analyzing them in small groups, modeling the task selection and development process for the class, and the instructor or peers offering written or verbal feedback on task selection before lesson implementation.

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A TEACHER RESIDENCY PROGRAM EMPHASIZING STEM EDUCATION IN RURAL COMMUNITIES: WHAT WE LEARNED

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Keywords: Equity, Inclusion, and Diversity; Elementary School Education; Systemic Change; Preservice Teacher Education

Trends in initial teacher preparation have pointed-out that traditional pipelines (i.e., undergraduate programs) found at institutions of higher education in the U.S. are not producing enough teachers to fill the deficit created by teachers leaving the profession (U.S. Department of Education, 2015). Furthermore, the need for teachers in rural communities is especially significant (Villegas et al., 2012) where historically there has been an on-going challenge in the recruitment and retention of teachers in schools located in these areas (Aragon, 2016; Barton, 2012; Monk, 2007). In this poster session, we will report on our findings on a university residency that was designed to address the challenges of recruitment, preparation, and retention of P-12 teachers of mathematics in the rural communities in our state (D'Amico et al., 2022; Roy, et al., 2023).

Carolina Transition to Teaching Residency

The Grow Your Own (GYO) residency was committed to developing and sustaining reciprocal relationships with partner districts in rural communities that met the following criteria: (1) more than 20% of children living in poverty, (2) teacher turnover rate greater than 15%, and (3) located in counties qualifying as Opportunity Zones. The [Blinded] residency program was designed for individuals [henceforth residents] that held an undergraduate degree in a field other than education, and who were interested in transitioning to the teaching profession. During the 18-month program, residents were be provided both professional and financial support as they were immersed in a year-long teacher residency while simultaneously pursuing a Master's Degree in Education. The residency was followed by support for residents in obtaining teacher certification during their first three years teaching in one of our partner districts located in rural communities in our state.

Findings

Carolina Transition to Teaching residency program employed a Grow Your Own (GYO) approach to promote a teacher workforce that is more representative of the rural communities in our state (Villegas, Strom, & Lucas, 2012). At this point, twenty-seven individuals have graduated from the program within the first three cohorts and began teaching in rural, high-needs partner school districts across the state, clustered along a historically significant areas of teacher need in our state. Moreover, across the four cohorts of the program, nearly all residents (i.e., 91.5%) have identified from underrepresented groups in the teaching profession in our state. Lastly, due to a GYO recruitment approach, 60% of all enrolled residents in Cohorts 1-4 previously held an instructional assistant or substitute positions within the partner district in which they were they fulfilled their residency.

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MATHEMATICAL & LINGUISTIC ANALYSIS OF PRE-SERVICE TEACHERS' TWO-STEP PROBLEM POSING MISCONCEPTIONS

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Research has shown that engaging teachers in problem posing is a worthwhile tool for learning how to teach mathematics. There is a rich body of literature focusing on pre-service teachers in one-step problem posing. Despite the difficulty that multi-step word problems present to students and teachers alike, there is less scholarly research investigating misconceptions surrounding two-step word problems. Thus, grounded in prior research on one-step problem posing and linguistic metalanguage, we qualitatively identify pre-service teachers' two-step problem posing misconceptions, and provide implications for practice and future research.

Keywords: Pre-service Teacher Education, Number Concepts and Operations, Elementary School Education, Instructional Activities and Practices

Objectives

In comparison to one-step word problems, two-step word problems provide students with additional challenges (Vershaffel et al., 2000; Van de Wall et al., 2019). First, multi-step word problems require students to identify an unstated *hidden* question and answer it before addressing the *written* question (Huinker, 1992). The *hidden* question provides students with the additional task of deciphering the underlying operations that are needed to solve the unstated question (Verschaffel et al., 2009; Van de Walle et al., 2019). Research has also shown that in addition to solving two questions, two-step problems often include more complex language structures, making them more difficult to understand and solve (Van Dooren et al., 2013; Verschaffel et al., 2009). The complexity of two-step word problems highlights the need for more research on how to support learners in solving these types of word problems.

One approach is to engaging teachers in problem posing themselves, shown to be a worthwhile activity for learning how to teach mathematics (Cai et al., 2020; Cai & Hwang, 2020; Calabrese et al., 2024). Investigating samples of pre-service teachers' (PSTs) problem posing provides insight into their conceptual and procedural understanding (Crespo, 2003). However, teachers can sometimes pose over simplistic problems or, on the other hand, include information that is irrelevant or is mathematically unsolvable (Leung & Silver, 1997; Silver et al., 1996; Stickles, 2011). Our study seeks to identify the linguistic and mathematical features of two-step word problems that present challenges for posing clear, solvable word problems by answering the research question: *In examples of PSTs' posed two-step word problems, what misconceptions were present? What do these misconceptions reveal about the key characteristics of two-step word problems?*

Perspectives

Being able to solve word problems requires a complex interaction between understanding mathematical, linguistic, and contextual features (Daroczy et al, 2020; Lin, 2021). Research has highlighted word problems as the most challenging tasks that students encounter in mathematics (Verschaffel et al., 2020). The language that describes the situations of word problems has been found to pose greater obstacles than the mathematical concepts involved in solving (Kintsch, 1987; Wyndham & Säljö, 1997). Often, clarity in word problems is defined by student understanding and performance (Fuchs et al., 2008); thus, presenting clear word problems enhances students' mathematical thinking and problem-solving skills (Barwell, 2009). Researchers suggest that factors such as comprehension, cultural and linguistic considerations, context, and numerical complexity, can all impact a student's ability to understand and solve these word problems (Daroczy et al., 2015; Daroczy et al., 2020). These studies highlight the complexity of posing a clear word problem, which can be a challenging task for PSTs.

Method

The participant sample for this study were undergraduate PSTs enrolled in a mathematics problem solving course designed for education majors (n=56). The problem solving course is guided by Polya's (1945) steps for problem solving, the Common Core (2010) addition/subtraction and multiplication/division taxonomies, problem posing (Silver, 1994), the written/hidden question (Van de Walle et al., 2019), and linguistic metalanguage (Halliday, 1975). PSTs participated in individual and collaborative one- and two-step problem posing activities to gain a deeper understanding of mathematical concepts. The data source for this study is a structured two-step problem posing task from the final exam. PSTs were given the following prompt for posing: "Write a two-step word problem that utilizes the structures: 1) Take from-Change unknown; 2) Array-Rows unknown." PSTs were also asked to write the hidden question in a separate space from their final word problem, which was also transcribed for analysis.

To analyze the data, the research team inductively identified instances of linguistic and mathematical distractors. We built on our findings from a previous study of the linguistic patterns of one-step additive problems (Author, 2022), to guide us to identify the salient patterns of two-step problems. We drew on Systemic Functional Linguistics (SFL), a social semiotic language theory that has been shown to support students and teachers talk about the functions of language and how it shapes meaning in the subject areas (Fang & Schleppegrell, 2010; Halliday, 1975; Halliday & Matthiessen, 2013). Initially, we coded five sample word problems together and discussed initial patterns and discrepancies to obtain inter-rater agreement. We engaged in constant comparative analysis (Strauss & Corbin, 1997) of PSTs' two-step word problems to code the remaining sample. Once we identified our final coding scheme and no new patterns were identified, we reviewed all of the word problems once again to ensure that the codes were accurately applied.

Findings

PSTs' misconceptions regarding two-step problem posing contained three key characteristics: 1) *requirement* of the hidden question; 2) *alignment* of the written and hidden question; and 3) distinct *linguistic* and *numerical distractors* that could impede requirement and alignment. The *requirement* of the hidden question describes the PSTs' posed problems that require two steps to solve the problem. In other words, one is required to solve for the hidden question in order to solve the second step of the problem. *Alignment* describes the connection between information provided and what the question (hidden and written) is asking for. *Linguistic distractors* describe linguistically unclear characteristics of the posed problems such as unclear referents, inconsistent verbs, atypical phrasing, inference required, and grammatical errors (see Table 1). *Numerical distractors* include characteristics such as an extra numerical referent (i.e., quantity) and missing numerical information (see Table 1). Further, PSTs demonstrated the aforementioned two-step misconceptions across four separate cases. These cases elucidated the moderation of distractors (i.e., linguistic and numerical) on the requirement and alignment of the hidden and written questions when PSTs posed two-step problems that were change (take from)-change unknown and array-number of rows unknown.

Table 1: Linguistic and Numerical Distractors

Linguistic Distractor	**Definition**
Unclear referent	The referent(s) are described inconsistently. Can also be described through multiple synonymous terms.
Inconsistent verbs	Inconsistent use of verbs throughout the word problem (e.g., planted, dead, have).
Inference required	In order to solve the problem, the student must draw implicit conclusions about the actions or conditions of the referent.
Atypical phrasing	Non-standard phrasing (e.g., order of information, novel word choice)
Grammatical errors	Punctuation or spelling errors that may or may not change the meaning of the word problem.
Numerical Distractor	**Definition**
Extra referent	An extra numerical referent is included in the problem that does not change the meaning of the problem.
Not enough information	There is not enough numerical information present to solve the problem.

Case 1 represents problems in which there is not a requirement of the hidden question to solve for the written question, nor alignment between the information presented in the problem and what the written question is asking for. In this case, the misalignment comes from having to solve for Stephanie's organization of ruined stickers; therefore, this question is solvable, but not logical. In addition, the PST did not write their hidden question, which could have contributed to their misaligned problem.

A common misconception for PSTs was posing two separate one-step problems, illustrated by Case 2. In other words, the information gathered was from solving for step one of the problem was not required for solving for step two. On the other hand, there were instances where PSTs posed two-step problems with a requirement of the hidden question, and alignment of the information given and the questions, but numerical distractors that made the problem unsolvable as in Case 3. The PST posed a problem where it was necessary to track Carl's flowers, making the question required and aligned. However, the PST did not provide an important detail: the amount of flowers that Jenny had after giving some to Carl. This leaves the problem unsolvable.

Table 2: Cases of Two-Step Problem Posing Misconceptions

Case	Hidden Question	Written Question	Requirement	Alignment	Solvable	Distractor
1	Did not write one	Stephanie has 42 stickers. While arranging them into equal rows and columns in her sticker book, she dropped some water on them. She now had only 20 stickers that weren't ruined. Saddened, she continued organizing her stickers into 11 equal columns. How many rows of stickers does Stephanie have?	Yes	No	Yes, although unrealistic	Linguistic Distractor-unclear referent
2	How many cookies did I eat?	There were 34 cookies on a baking sheet. I ate some. Now there are only 32 cookies on the baking sheet. If I arranged the cookies equally into 8 columns, how many cookies would be in each row?	No	Yes	Yes, as two separate one-step problems	None
3	How many flowers does Jenny give to Carl?	Jenny has 10 flowers and gives some to Carl. Carl arranges his flowers into 3 columns. How many rows of flowers does he have?	Yes	Yes	No	Numerical distractor-not enough information
4	How many cupcakes did Darcie give to Joseline?	Darcie has 5 cupcakes, she gave some of the cupcakes to Josseline. Now Darcie has 3 cupcakes. Josseline arranged 6 of her cupcakes creating each column from the cupcakes Darcie gave her. How many rows of cupcakes did Josseline arranged them in?	Yes	Yes	Yes	Linguistic distractor-Inference required
5	How many tiles did Jack take?	Jack's mom had 24 tiles she wanted to lay out. Jack wanted to help so he took some tiles leaving his mom with 12 tiles. If Jack laid out his tiles into equal rows with 4 tiles in each row, how many rows of tiles did he lay out?	Yes	Yes	Yes	None

Case 4 represents PSTs' posed two-step problems with the requirement of the hidden question, alignment of the hidden and written questions, but numerical/linguistic distractors. Case 4 posed problems are solvable, however, the linguistic distractors in the problem interfere with the clarity of the problem. In this case, the problem solver must make the inference that Josseline had four cupcakes before Darcie gave her cupcakes to make sense of the array of cupcakes using the cupcakes from Darcie as columns. Last, Case 5 represents two-step problems with the requirement of the hidden question, alignment of the hidden and written questions, and no numerical/linguistic distractors, which represents a clearly posed two-step problems.

Conclusions

Research has shown that engaging teachers in problem posing is a worthwhile tool for learning how to teach mathematics (Cai et al., 2020; Calabrese et al., 2024). Based on our findings, we propose that to pose clearly written two-step problems, the following criteria must be met: requirement of the hidden question, alignment of the information provided to the questions, and absence of numerical and linguistic distractors. Educators can use these findings as guidelines for posing contextually rich word problems that are still accessible from a mathematical and linguistic perspective.

Our study contributes to problem posing literature by providing tools for defining and recognizing PSTs misconceptions with two-step problem posing. Despite the difficulty of multi-step word problems (Verschaffel et al., 2000), there is a dearth of research on two-step problem posing compared to one-step for PSTs. Our analysis focused on two-step problem posing given a structured task; however, evaluating PSTs' two-step problem posing misconceptions given an unstructured or semi-structured task is a future direction of research to further investigate PSTs' understanding of multi-step word problems.

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WHAT HAPPENED IN THE SECOND SIMULATION?: ANALYZING THE FORMATIVE FUNCTION OF FEEDBACK ACROSS TEACHING SIMULATIONS

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We share the teaching simulation as one approach to providing formative feedback in teacher preparation and consider the ways in which teacher candidates (TCs) take up the feedback in subsequent simulations. We hypothesize that TCs' uptake depends on the connections between their own resources, the focus of the feedback provided, and the context of subsequent teaching.

Keywords: Instructional Activities and Practices, Preservice Teacher Education, Teacher Educators, Elementary School Education.

Beginning teachers who are "committed to supporting the mathematical success of each and every student" (Association of Mathematics Teacher Educators [AMTE], 2017) integrate strong content knowledge for teaching, skill with high-leverage teaching practices, adherence to professional ethical obligations, and commitments to equitable teaching and learning, alongside tools for learning and growing as professionals across their teaching careers (Davis & Boerst, 2014) into their teaching. For teacher educators (TEs), this implies designing opportunities for teacher candidates (TCs) to develop integrated knowledge and skills in ways that support TCs to build their capacity for reflection and improvement. We share teaching simulations as one approach to providing such opportunities with immediate feedback and we consider the ways in which TCs take up the feedback to demonstrate growth in their mathematics teaching practice.

Theoretical Framework

At the foundation of our work, we view teaching as involving the interactions between and among teachers, students, and the content situated inside of the school environment (Cohen et al., 2003; Lampert, 2001). We view teaching *mathematics teaching* as involving interactions between and among the TE, the TCs, and mathematics instruction, the content of teacher education (Ball et al., 2009; Ghousseini & Sleep, 2011; Shaughnessy et al., 2022). This implies that learning occurs through supported engagement in teaching, with opportunities to improve.

In this context, formative assessment allows TCs to demonstrate the integration of content knowledge for teaching, high-leverage teaching practices, and commitments to equitable teaching and learning This provides the TE with a snapshot of the TC's current knowledge and skill that can be the basis for providing feedback and additional learning opportunities that can foster subsequent improvement. Wiliam (2010) posited that the utility of formative assessments is measured by the extent to which they allow the demonstration of knowledge and skill paired with the extent to which they support decisions about subsequent work. This is the goal of formative assessment in teacher preparation—to elicit information about TCs' practice and to provide feedback and learning opportunities that impact their future practice.

Our work is situated at the intersection of this interactive view of learning to teach and formative assessment. Simulations are used as contexts to practice and demonstrate capabilities with teaching practice. Simulations are approximations of practice (Grossman et al., 2009) that place authentic demands on a TC while controlling the complexity of the work. Simulations allow the TE to control the mathematical content that the TC encounters, the strategies and thinking represented by students, and the teaching moves necessary to accomplish the goals of the simulation (Shaughnessy & Boerst, 2018a). We use simulations in formative cycles, allowing feedback to TCs to be taken up in subsequent simulations.

These simulations focus on two practices that are essential to the daily work of teaching, eliciting and interpreting student thinking (National Council of Teachers of Mathematics, 2014). Put simply, eliciting student thinking focuses on making students' ideas available through asking questions and posing tasks, and interpreting student thinking entails making sense of the information gathered to drive subsequent instructional decisions (Shaughnessy & Boerst, 2018b).

The TC's engagement in a teaching simulation cycle has four parts. First, the TC examines a piece of student work and plans questions to ask the student to learn about their process and understanding of the mathematics. Second, the TC has five minutes to interact with a simulated student to elicit the student's thinking. The simulated student is a live actor trained to use a set of response guidelines that specify the student's process and understanding of the ideas underlying the process. These guidelines include general guidance and specific responses to anticipated questions (see Shaughnessy & Boerst, 2018a). Third, the TE interviews the TC to learn about their interpretations of the student's thinking and their own understanding of the mathematics. Throughout the simulation and interview, the TE uses an observational tool to capture key aspects of the performance. Fourth, the TE engages the TC in a feedback conversation. Our approach allows us to gather evidence of a TC's knowledge and skills in nine performance areas (Shaughnessy et al., 2025) through an observational tool that is tied to feedback suggestions that the TE can use. We sought to explore whether and how feedback was taken up by TCs in a subsequent simulation involving the same teaching practices but differing mathematics content.

Methods

As part of ongoing work with TEs and TCs at two universities, we are supporting TEs in using the simulations with their TCs and learning about how TCs take up the feedback. We had 37 TCs engage in a pair of back-to-back simulation cycles with their TE (see the student work in Figure 1). TCs were given 5 minutes to consider the feedback and then engaged in the second simulation. After both cycles, we interviewed each TC to learn about their understanding of the feedback that was provided and their perception of and reasoning about the extent to which they did or did not take up the feedback. The simulations, feedback conversations, and interviews were video recorded. We analyzed the feedback conversations to identify the feedback as well as the interviews to identify the TCs' understanding of the feedback and why it was or why it was not taken up. To understand the corresponding uptake, the research team applied an observational tool to each simulation performance (see Shaughnessy & Boerst, 2018a).

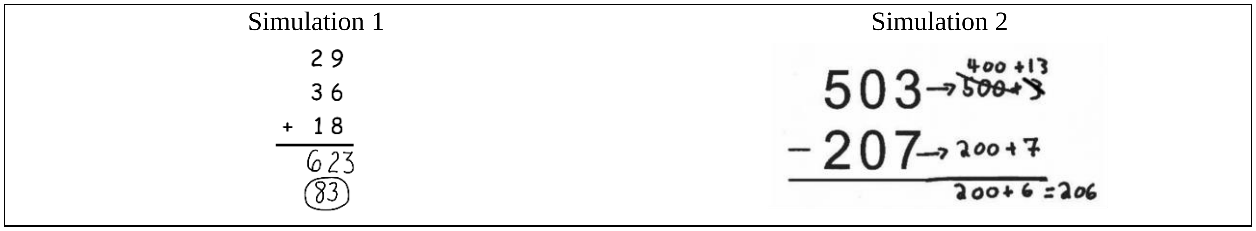

Figure 1: Student Work for Simulation 1 and Simulation 2

How Might a Teacher Candidate Take Up Formative Feedback?

In the first simulation (~3 minutes), a TC, Kendall, began by focusing on where the student had started and why. Kendall then pressed the student on what they had done in the ones place. The student shared how they generated the “23” in 6-2-3. Next, Kendall asked why the student had not carried. The student reiterated that using their approach, they write the answers below the column (and that they do not carry). Kendall then asked the student what would happen if they did carry. Kendall complimented the student on starting with the ones, which was immediately corrected by the student (who had started with the tens). Kendall then (incorrectly) stated that the student needed to start with the ones and asked about the reasonableness of the answer. By the end of the interaction, the student had stated that the answer was reasonable.

As Kendall interacted with the student, her TE used the observational tool to keep track of the sorts of moves she made. The tool revealed that Kendall did not learn about the student’s full process. Further, Kendall did not ask about the student’s understanding of why the 6 and the 2 can be combined, which is crucial given the student’s process. Additionally, the TE noted that Kendall was directing the student to use a different process.

The TE engaged in a 7-minute conversation with Kendall, sharing three main pieces of feedback, which Kendall later described accurately. First, the TE named the importance of learning about the student’s full method and shared that one strategy is rewriting the problem and asking the student to solve it again and to talk aloud as they solve it. Second, the TE suggested that Kendall ask probing questions focused on the student’s understanding. Third, the TE suggested that Kendall stay open to learning about the student’s process rather than imposing her own method to solve the problem or assuming that there is one right approach. Kendall had 5 minutes to consider the feedback and continue to plan for eliciting the second student’s thinking.

For the second simulation (~5 minutes), Kendall began by asking the student to re-solve the problem. She then asked questions about the student’s steps, and after the student talked about the process of the trade, she pressed on where the 10 added to the 3 came from. After getting the student to talk about why they hadn’t crossed out anything in the subtrahend, Kendall returned to probing the student’s understanding of what they were adding to the “3.” The student stated they had taken 100 from the 500 and only added 10 of that 100 to the ones place. The student also expressed that the number (the minuend) is supposed to be “the same” after a trade. Kendall pressed around the reasonableness of the answer and the student stated the answer was not reasonable. Kendall asked about other approaches and the student acknowledged that it was their first time solving a problem “without tens.”

Looking at the uptake of the feedback, first, we see that, as suggested by the TE, Kendall had the student re-solve to support knowing about the entirety of the student’s method. However, Kendall did not have the student talk while reworking the problem. Second, Kendall took up the

TE's suggestion to ask probing questions by pressing on the student's understanding of core ideas. Third, the TE suggested that Kendall stay open to learning about the student's process, which was evident as Kendall did not encourage the student to solve the problem another way.

Contrasting Case: Lack of Feedback Uptake

We next turn to a case in which a TC, Sammie, did not appear to be taking up the feedback provided by her TE focused on probing understanding and posing a follow up problem.

Probing Understanding

In the first simulation, Sammie asked a question about the value of the 6-2-3. Sammie did not ask the student about their understanding of the combining of the 6 (tens) and 2 (tens). Sammie received feedback focused on asking questions to probe the student's understanding. The TE said, "Do you remember how we've been talking about different kinds of questions we ask kids and following up with the whys? That's what I want you to practice doing and continuing to do." In the second simulation, Sammie asked why the student added the minuend and subtrahend together, and the student responded that they "subtracted," but Sammie asked no questions focused on the student's understanding of core ideas. When interviewed by a research team member about how she had taken up the feedback provided by her TE, Sammie said she had heard the feedback and tried to use it but forgot what the student said and filled in her own thinking, noting that she had tried to probe the student's understanding but used the wrong operation. Later, Sammie said, "I think if I had worked with the same style of problem, I think I would've be able to get it right." Thus, Sammie noted that the differences in the approaches used by the two students factored into her ability to probe the student's thinking in the moment.

Posing a Follow Up Problem

As part of the first simulation, Sammie was asked in the follow-up interview to identify a problem that could be posed to the student to confirm their process and understanding. Sammie carefully identified a task and talked through her reasoning. In the feedback, the TE highlighted the care with which Sammie selected the numbers and named that posing another problem can be a useful strategy. In the second simulation, Sammie did not pose an additional problem to the student. Later, she said, "I was like, I don't know what problem I would give this child."

Uptake of Feedback

We note that it is challenging to ask about understanding in a context where there is a wrong answer or an unfamiliar algorithm and that the problem solved was a special case that surfaced a challenge that would not typically arise. Even though the feedback given by the TE could have been applied, Sammie may have needed additional support in learning about the sorts of understanding questions that could be useful in this particular mathematical situation, how to ask those questions in a situation where the answer is incorrect, and how to manage other complexities of interacting with the student in this context. Similarly, Sammie may have needed additional support to think about generating and using a follow up problem in this context.

Discussion

Across the broader data set, TCs varied in their ability or willingness to utilize the feedback provided by their TE. We hypothesize that TCs' uptake depends on the connections between their own resources, the feedback provided, and the context of subsequent teaching. We conceptualize resources as the combination of content knowledge for teaching, skill with high-leverage teaching practices, and orientations to teaching and learning that TCs bring to the work of mathematics teaching. In terms of the feedback, we are examining both the content (e.g., asking questions about student's understanding of core ideas) and the nature of guidance (reminding the TC about something already known, convincing the TC about the importance of something, and/or teaching the TC something new to try or think about). Given that the TC's teaching is influenced by the simulation context in which it occurs, we are considering the demands implied by particular mathematics content (invented/standard process, correct/incorrect answers) and particular characteristics of the student's thinking (degrees of procedural fluency and components of conceptual understanding) designed into the simulation.

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OPPORTUNITIES TO NURTURE EQUITABLE TEACHING THROUGH SIMULATIONS OF ELICITING AND INTERPRETING STUDENT THINKING

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We describe how a teaching simulation cycle focused on eliciting and interpreting student thinking in elementary mathematics can reveal important aspects of a teacher candidate's (TC's) knowledge and skill with eight performance areas relevant to more equitable mathematics teaching. Appraising a TC's knowledge and skill with respect to these performance areas is intended to support formative feedback that is actionable in subsequent teaching.

Keywords: Instructional Activities and Practices; Preservice Teacher Education

Formative assessment of teaching practice allows TCs to demonstrate the integration of content knowledge for teaching, high-leverage teaching practices, adherence to ethical obligations, and commitments to equitable teaching and learning, providing the TE with a snapshot of the TC's current knowledge and skill. We situate our work at the intersection of an interactive view of learning to teach (Ball et al., 2009) and formative assessment, a critical component of teacher preparation (Association of Mathematics Teacher Educators [AMTE], 2017). We use teaching simulations to provide interactive contexts to practice and demonstrate skill with mathematics teaching practice inside of formative assessment cycles that offer opportunities for feedback and improvement. Simulations are approximations of practice (Grossman et al., 2009) that place authentic demands on a TC while purposefully controlling the complexity of the work to allow TCs to encounter appropriate cognitive and practical demands on their budding teaching skills. Teaching simulations can provide early, frequent, and substantive formative assessment opportunities that are embedded in the doing of teaching. Specifically, we focus on the teaching practices of eliciting and interpreting student thinking. These practices are crucial for advancing more equitable mathematics instruction because they facilitate teachers' connection with – and attention to – the children they teach. In this paper, we describe how our teaching simulation cycle provides opportunities for TCs' to demonstrate their knowledge and skills with important facets that comprise the work of eliciting and interpreting student thinking and provide opportunities for TEs to observe such knowledge and skill in action. In turn, the observation provides the basis for formative feedback.

Our Teaching Simulation Cycle

Our simulation situation utilizes a piece of student work (see Figure 1) and a live actor (referred to as the simulated student) trained to use a set of response guidelines that specify the student's process and the student's understanding of the ideas underlying the process. These response guidelines include general guidance and specific responses to anticipated questions (see Shaughnessy & Boerst, 2018). As the TC interacts with the student's work and simulated student, the TE uses an observational tool to capture key aspects of those interactions.

The TC’s engagement in a teaching simulation cycle begins with the TC examining a piece of student work and planning questions to ask the student to learn about their process and understanding of the mathematics. Second, the TC has five minutes to interact with the simulated student to elicit the student’s thinking. Third, the TE interviews the TC to learn about their interpretations of the student’s thinking and their own understanding of the mathematics underlying the student’s process. Fourth, the TE engages the TC in a feedback conversation.

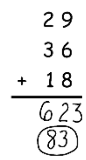

Figure 1: Sample Student Work

For over a decade, we have been designing, using, refining, and studying the use of these teaching simulations with our own preservice teachers. In the context of our current project, we are working with three TEs in different regions of the United States to develop tools and routines that support the using of the teaching simulations to provide formative feedback to TCs on their eliciting of student thinking.

Nurturing More Equitable Mathematics Teaching

Our approach allows us to gather evidence of a TC’s knowledge and skills with respect to eight performance areas that comprise the complex work of eliciting and interpreting student thinking, including why each area matters for nurturing more equitable mathematics instruction (see Shaughnessy et al., 2025 for an overview of how we conceptualize the articulation of performance areas within a more complex teaching practice for the purposes of teacher education). Next, we define each of these performance areas. We include a rationale for focusing on it and describe how the teaching simulation cycle enables us to learn about the knowledge and skills of TCs in that area.

Eliciting the Student’s Process for Solving the Math Problem

Elements of a student’s process may be evident in a student’s written work. However, when teachers look at a student’s written work, they often need to elicit more information about the process because the entire process is not visible in the written work. For example, in the student work shown in Figure 1, it is not clear how the student went from 6-2-3 to 83. Additionally, a teacher might make assumptions that do not correspond with what the student did. For example, TCs may assume that the student added the ones before the tens. This student starts with the tens. By asking about the entirety of the process, TCs can check these assumptions that could otherwise lead to mischaracterization of the student’s thinking and misdirect subsequent teaching. By looking at the interaction between the simulated student and the TC and which steps of the process the TC has the student talk about or otherwise show, we can gather evidence of a TC’s skills in eliciting a student’s process.

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Interpreting the Student’s Process for Solving the Math Problem

Interpreting a student's process is important because sometimes an initial hypothesis and/or in-the-moment interpretation is incorrect. This may be more likely to happen when a student and teacher have different preferences for which process to apply when solving the problem. Inaccurate claims about a student's process and subsequent actions to "correct" or intervene when it is not necessary can be counterproductive to a student's learning. In the follow-up interview, we gather evidence about how the TC interprets the student’s process by having the TC share the student’s process in their own words and apply the process to a new mathematics problem.

Probing the Student’s Understanding

Understandings are crucial to using and recalling a process and a basis for subsequent instructional steps. Teachers need to know a student's reasoning because sensemaking is foundational to doing and learning mathematics. Probing understanding is also a way for teachers to show students that they value the student’s reasoning, not just their process and answer. We can gather evidence of a TC's skills in probing student understanding by looking at the interaction between the simulated student and the TC and the questions the TC asks to uncover the student's understanding of the process they are sharing. We can also gather evidence of the extent to which they probe student understanding when a student’s answer is correct. This is critical because in many classrooms, incorrect answers are interrogated (and assumed to reflect a lack of understanding), and correct answers are affirmed with an assumption of “student understanding.”

Teacher Interpretation Aspects of the Student’s Understanding

Being able to notice and name the understandings that students share and identify corresponding evidence of those understandings is important for leveraging the resources students bring and considering the next instructional steps. In the context of the follow-up interview, we ask the TC to share what was learned about what the student understands about their process and the mathematical ideas underlying that process. By leaving the question open, our simulation allows us to learn what understandings TCs notice and name without support. As a complement to self-identification of student understandings, we also directly ask TCs about pivotal parts of the mathematical processes they encounter in the simulation. Teachers need to know about students’ understanding of these components. These are places where notation or conventions may play an important, but implicit, role in the process. Teachers need to learn about these understandings to leverage students' resources when considering the next instructional steps. When teachers make claims about these understandings, it is crucial that they use evidence to support and check interpretations. This guards against misinterpretations that can arise when the teacher speculates or makes assumptions about what the student understands without information from the student. Misinterpretations of a student's understanding can lead to unnecessary intervention or lack of intervention when one is warranted. In the follow-up interview, we gather information about this by asking the TC to share what they learned about the student’s understanding of pivotal parts of their process and the associated evidence gathered that supports those interpretations.

Attending to the Student’s Thinking

When teachers actively use what a student says and writes as the basis for their next questions, teachers are likely to learn more about the student's process and understanding. It also conveys to students that their ideas are valuable. Helping students to see the value in their own ideas is a core part of building or shifting students' conceptions of what it means to do mathematics. We gather evidence of a TC's skills in this area by examining the interaction with the simulated student and seeing whether there is evidence that the TC is taking up ideas that the student has shared verbally (through revoicing and/or posing questions connected to ideas that have been shared). We can also see whether the TC is attending to written representations of the student's thinking and asking specific questions about what the student has written.

Applying Mathematical Knowledge for Teaching

Teachers use their mathematical knowledge in special ways, such as generating a follow-up problem that supports understanding the student's process. The appropriateness and accuracy of a teacher's communication with the student is important for ensuring the student can access the ideas and preventing misconceptions that language or representations can sometimes create. Through the interaction with the simulated student, we can sometimes gather evidence of a TC's mathematical knowledge for teaching by observing them pose an additional mathematical problem to the student, but this does not always happen. To gather evidence of such knowledge from all TCs, the follow-up interview involves asking the TC to generate a problem that could be posed to the student to confirm the student's process and understanding. We ask the TC why they selected the problem to learn about the features of the problem to which they are attending.

Using Mathematical Knowledge and Skills

Students can share processes for solving problems that are unknown to teachers. Teachers need to be able to use their own mathematical skills to analyze whether a strategy is mathematically sound and how it relates to other mathematical ideas. Determining the extent to which a student's strategy "works" across cases (generalizing) is key for determining the next instructional steps, including helping students see themselves as doers of mathematics. Teachers also need to be able to use mathematical knowledge precisely and with integrity and determine whether answers to problems are mathematically valid. Within the interaction with the simulated student and the interview, we can gather evidence of a TC's use of mathematical language and representations with integrity. We also ask the TC about the correctness of the student's answer and to generalize how the student's process would work across problems.

Conveying Respect for the Student as a Mathematical Thinker and Learner

Equitable mathematics instruction reveals and builds on student sense-making. Critiquing a student's strategy before seeking to understand it or redirecting a student toward a strategy they did not use can communicate to the student that their process is wrong or undesirable and that doing mathematics means deferring to a privileged process, often the algorithms traditionally taught in US schools. Within the interaction with the simulated student and the follow-up interview, TCs can demonstrate skills respecting the student as a mathematical thinker and learner by focusing on the student's process (and understanding of that process) in the interaction and describing and interpreting that process in the follow-up interview.

Discussion

We view the ability to make facets of equitable mathematics instruction concretely visible, doable, and improvable in the work of teaching as a central driver of the design and implementation of the simulation approximation of practice. This paper illustrates how a teaching simulation cycle focused on eliciting student thinking can reveal TCs' knowledge and skill in eight performance domains relevant to eliciting and interpreting student thinking. In our current work, we are using these teaching simulation cycles to provide formative feedback to TCs about their teaching and to understand the ways in which they are able to take up their feedback in a second teaching simulation cycle that immediately follows the first cycle.

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TEACHER CANDIDATES' MATHEMATICS ANXIETY AND EFFICACY BELIEFS

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This study examines mathematics anxiety in preservice teachers, their efficacy beliefs regarding mathematics, and the relationship between these factors. The sample includes 50 preservice teachers at various stages of their undergraduate education. Data were collected using the revised Math Anxiety Rating Scale and Teacher Belief Survey. Descriptive statistics revealed high anxiety related to mathematics assessments and moderate anxiety in learning and performance experiences. Preservice teachers generally held a problem-solving view of mathematics and were neutral about the instrumentalist view.

Keywords: Teacher Beliefs

Mathematics anxiety, defined as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (Richardson & Suinn, 1972, p. 551), is more complex than just test anxiety or general anxiety. Hembree's (1990) meta-analysis of 151 studies found a correlation of .52 between mathematics anxiety and test anxiety but attributed 63% of mathematics anxiety to other sources like mathematics learning and homework. Additionally, Hembree identified mathematics anxiety as a learned condition, more behavioral than cognitive.

Teacher efficacy is perceived as teachers' beliefs in their ability to influence student learning (Guskey & Passaro, 1994, p. 628). Mathematics teacher efficacy includes different belief systems on how students learn. Ernest (1989, as cited in Beswick, 2005) explored three views of mathematics: instrumentalist, platonist, and problem-solving. The instrumentalist view focuses on performance and content, the platonist view emphasizes understanding and constructing knowledge, and the problem-solving view is learner-focused, believing in learning through exploration. Van de Walle et al. (2019) promote a problem-solving approach called "teaching through problem solving" where students learn mathematics through exploring real contexts.

Studies show varying relationships between mathematics anxiety and mathematical beliefs. Beswick (2005) found secondary teachers rarely held an exclusive problem-solving view. Hughes (2016) found elementary teachers leaned towards problem-solving views, with high anxiety corresponding to instrumentalist beliefs and low anxiety to problem-solving beliefs. Haciomeroglu (2013) found that preservice teachers with stronger constructivist mathematical beliefs felt less anxious. However, Uysal and Dede (2016) found no significant correlation between anxiety and beliefs in Turkish preservice teachers. Despite differing findings, many studies indicate a lean towards problem-solving views in mathematics teaching (Hughes et al., 2019; Uysal & Dede, 2016).

Literature reviewed has established that high mathematics anxiety for teachers may have consequences on mathematics learners' anxiety. Literature has also shown the relationship between teachers' anxiety, beliefs, and instructional practices. Investigating preservice teachers' mathematics anxiety and the experiences that are associated with the most anxiety can reveal formative feedback for effective teacher education. Preservice teachers' espoused beliefs about the teaching of mathematics can predict their future instructional practices, giving teacher educators learner-centered tools for teacher education. Furthermore, exploring the possible correlation between mathematics anxiety and espoused beliefs about teaching may show factors that affect beliefs and anxiety, and consequently teaching practices. Thus, the objectives of this paper will answer the following questions:

1. What is the nature and level of mathematics anxiety of preservice teachers?
2. What is the nature and level of preservice teachers' espoused beliefs about the teaching of mathematics?
3. Is there a significant correlation between mathematics anxiety and espoused beliefs about the teaching of mathematics?

Methodology

Preservice teachers were conveniently sampled from one college in the Midwest, consisting of 50 participants at varying stages of their undergraduate education, from first year to senior year, including both elementary and secondary preservice teachers. Participants completed a survey comprised of two valid and reliable instruments: The Mathematics Anxiety Rating Scale-Revised (MARS-R, Plake & Parker, 1982) and the Teacher Belief Survey (TBS, Beswick, 2005). The MARS-R includes 24 items on a 5-point Likert scale from 0 (low anxiety) to 4 (high anxiety), while the TBS uses a 5-point scale where 1 represents strongly disagree and 5 represents strongly agree, with high scores indicating a stronger alignment to the problem-solving view and lower scores aligning with an instrumentalist view on mathematics teaching and learning. Descriptive statistics were explored to summarize efficacy beliefs and anxiety.

Results

Nature and Level of Mathematics Anxiety of Preservice Teachers

The total ratings for MARS-R scores range from 0 to 96, representing no to high mathematics anxiety. In the current study, preservice teachers' MARS-R scores ranged from 13 to 86, with an average score of 50.62, slightly over the moderate stress level. The scores have an interquartile range of 41.5 to 60.25, indicating a symmetric distribution, with the lower 25% of scores between 17 and 41 and the top 25% between 60.25 and 86. Mathematics tests and assessments, particularly final exams and pop quizzes, have higher associated anxiety than other categories, with average ratings of 3.62 and 3.58, respectively (see Table 1). Preparing for a test and waiting for test results have lower anxiety ratings. Mathematics performance experiences, such as reading chemistry formulas and working on abstract problems, also induce moderate anxiety.

The anxiety related to mathematics learning experiences, like having difficult homework due and signing up for a statistics course, is moderate, with an aggregate average of 1.913. Experiences with less mathematics interaction, such as buying a textbook or thinking about mathematics, result in lower anxiety ratings, showing that participants feel less anxiety in such scenarios compared to other categories. Notably, the experience with the least amount of anxiety attributed to it is watching a teacher work on an algebraic equation on the board, with an average rating of 1.48. Overall, mathematics performance does not have as much anxiety attributed to it as the mathematics assessment category.

Table 1
Averages and Standard Deviations of Mathematics Assessment Experiences

Questionnaire Item	Average	Standard Deviation
Thinking about an upcoming mathematics test one day before	3.04	1.009
Taking an examination (quiz) in a mathematics course	3.02	1.116
Being given a "pop" quiz in mathematics class	3.58	0.731
Taking an examination (final) in mathematics class	3.62	0.725
Getting ready to study for a mathematics test	2.4	1.05
Waiting to get a mathematics test returned in which you expected to do well	2.52	1.111

Efficacy Beliefs about Teaching Mathematics

The problem-solving view of the mathematics subset of TBS has a highest possible score of 70, while the instrumentalist subset has a highest possible score of 50. Based on Brechin-Harrison's (2008) method of calculating TBS subset scoring ranges, the scoring ranges for each level of agreement are reported in Table 2. The TBS subscale scores provide insight into preservice teachers' efficacy beliefs. The problem-solving view of mathematics has a mean score of 55.44, indicating that participants generally agree with this view. In contrast, the instrumentalist view has a mean score of 33.56, showing that participants on average neither agree nor disagree with it. Ninety-four percent of respondents agreed that "it is important for children to be given opportunities to reflect on and evaluate their own mathematical understanding." Additionally, 90% agreed or strongly agreed with the statement, "a vital task for the teacher is motivating children to solve their own mathematical problems" (mean=4.3), and 84% agreed that "teachers can create, for all children, a non-threatening environment for learning mathematics." However, only 70% agreed with the effectiveness of having students justify mathematical statements (mean=3.94) or allowing a student to struggle with a problem (mean=3.7). Overall, the problem-solving view has many components with which this sample of preservice teachers agrees.

Table 2
Brechin-Harrison's Scoring Range for TBS Subscales

Sub-scale	Level of Agreement	Scale Range
Problem Solving View of Mathematics	Strongly Agree	70.0 - 63.0
	Agree	**62.99 - 49.0**
	Neither	48.99 - 35.0
	Disagree	34.99 - 21.0
	Strongly Disagree	20.99 - 14.0
Instrumentalist View of Mathematics	Strongly Agree	50.0 - 45.0
	Agree	44.99 - 35.0
	Neither	**34.99 - 25.0**
	Disagree	24.99 - 15.0
	Strongly Disagree	14.99 - 10.0

In contrast, the instrumentalist view of mathematics, which preservice teachers had a neutral view on, had statements with averages ranging from 1.48 to 4.46. The most agreed-upon statement was "it is important for teachers to understand the structured way in which mathematics concepts and skills relate to each other," with 92% of participants agreeing (mean=4.46). Interestingly, while 86% agreed that "it is the teacher's responsibility to provide children with clear and concise solution methods for mathematical problems," only 74% believed in an expository style of teaching mathematics. Furthermore, although participants strongly agreed that there is a set amount of mathematics content to be covered at each grade level (mean=4.18) and that mathematics should be taught in a correct sequence (mean=4.2), there was no strong consensus on whether this sequence should come from a textbook (36.73%, mean=2.959). The responses reflected mixed views about the definition of mathematics, with 55.1% agreeing that mathematics is computation (mean=3.571) and 54% agreeing that mathematics is a beautiful, creative, and useful endeavor (mean=3.48). This overlap indicates that some preservice teachers share both problem-solving and instrumentalist views on mathematics.

Correlation Between Mathematics Anxiety and Efficacy Beliefs

When comparing each participant's efficacy belief statements to their MARS scores, there was no significant correlation between the two variables. The statement with the strongest correlation, -0.298, "Knowing how to solve a mathematics problem is as important as getting the correct solution," is still considered very weak. The correlations of the other efficacy statements to their MARS scores were also very weak, ranging from -0.3 to +0.139. This indicates no significant relationship between the participants' mathematics anxiety scores and their efficacy beliefs. Despite the lack of strong correlations between specific items in the Teacher Belief Survey and the Mathematics Anxiety Rating Scale scores, there is a significant correlation between the TBS subscale scores and the MARS-R scores. The instrumentalist view on mathematics has a correlation of 0.734 with the MARS scores, and the problem-solving view has a correlation of 0.839. This indicates a substantial relationship between mathematics anxiety and holding either an instrumentalist or problem-solving view on mathematics, with higher anxiety corresponding to higher alignment with these views.

Discussion and Conclusion

Mathematics anxiety has been linked to various negative impacts on learners (Kargar et al., 2010; Wang et al., 2015), including the performance of students taught by anxious teachers (Beilock et al., 2010; Schaeffer et al., 2020). In this study, elementary and secondary preservice teachers demonstrated moderate levels of mathematics anxiety, differing from previous studies reporting both high and low anxiety levels (Juanita & Budayasa, 2020; Patkin & Greenstein, 2020). Preservice teachers generally aligned with the problem-solving view in mathematics teaching, which has positive implications for student learning outcomes (Behlol et al., 2018). However, despite overall agreement with problem-solving principles, many preservice teachers hesitated to let students struggle with mathematics problems, a crucial aspect of this teaching approach. Additional training may be beneficial to help preservice teachers embrace productive struggle in the classroom. Interestingly, our study found no clear association between mathematics anxiety and specific teaching views, contradicting prior research (Hughes, 2016; Haciomeroglu, 2013).

Limitations include the small sample size from a single institution, lack of disaggregated data for mathematics teaching endorsement seekers, and failure to distinguish between elementary and secondary education majors. Future research should address these limitations to provide deeper insights into preservice teachers' anxiety and teaching beliefs. Nonetheless, our findings offer valuable insights for modifying preservice teacher instruction to alleviate mathematics anxiety, particularly regarding assessments, and reinforce problem-solving pedagogies.

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EFFECTIVENESS OF AN INNOVATIVE AND COLLABORATIVE PROFESSIONAL DEVELOPMENT EVALUATION DESIGN

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The PrimeD Framework and its Phases

The abbreviation PrimeD stands for Professional Development: Research, Implementation and Evaluation and is a preservice teachers professional development framework as defined in Rakes et al. (2017) and Saderholm et al. (2017). This framework has four phases phase I (Design and Development), phase II (Implementation), phase III (Evaluation) to IV (Research) as illustrated in Figure 1. These phases are dynamic, follow a feedback loop structure (as indicated by arrows in figure 1) and "continuously" build on each other. Together they offer a flexible structure that supports systemic and individual changes and builds leadership and professionalism in preservice teacher candidates.

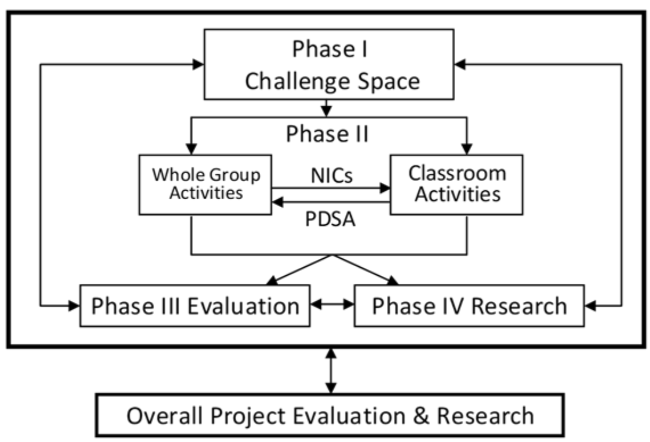

Figure 1. Diagram of PrimeD Phases Interactions with External Evaluation

In Phase I, a common vision is developed, contextual issues are considered, and a plan of action is formed (including goals, outcomes, strategies, and assessments) which drives downstream activity (Bryk, et al., 2015). The Challenge Space from Phase I serves as an overall dynamic monitor to guide the implementation, evaluation, and research phases (indicated by the arrows in figure 1). Phase I is accomplished by preservice teacher ~~candidates~~ along with the project PIs. The next phase is the highly active Phase II where Phase I is implemented in the form of classroom Implementation and Whole Group Engagement (WGE). In other words, NICs and PDSA cycles drive this phase either towards Phase III and IV or back to Phase I. Phase II is accomplished by PIs, preservice teachers, their mentors, program alumni and the Co-PIs of the local and non-local sites.

Figure 2 displays the structure of an observation sequence for the first round of NIC meetings

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

occurring at all four university sites. Each co-PI (solid vertical arrows) observed the NIC meeting at their site. They also observed a NIC meeting at one of the other sites (dashed diagonal arrows). The Lead evaluator observed the NIC meetings at all four sites. This structure ensures three observations for every NIC meeting. The data from each observation form is then entered in a spreadsheet, graphed and analyzed. The results are then discussed at the following evaluation and implementation team meetings. This structure is repeated three times each semester.

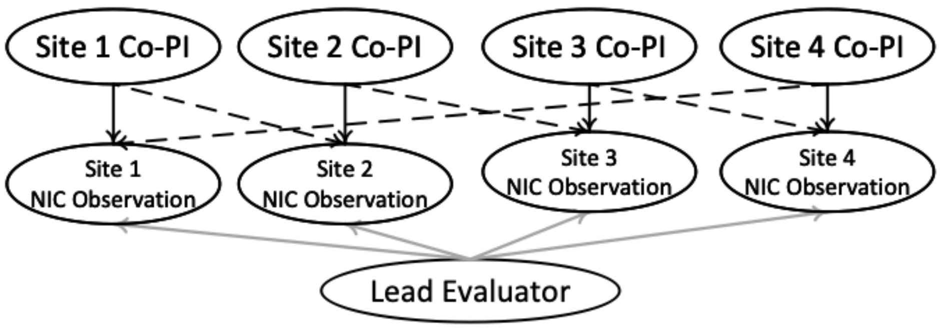

Figure 2: NIC Meeting Observation Structure

Phase III, Evaluation, focuses on the notion that members of a profession participate in establishing what strategies best meet the needs of their student populations. This phase is executed by the preservice teacher and the PIs. Similarly Phase IV emphasizes that it should be common practice that members of a profession participate in generating professional knowledge associated with their practice. The overall project evaluation that differs from the evaluation Phase and is illustrated below the larger rectangle in the Figure 1, should attend to these components as well as other implementation activities. Phases III and IV are interchangeable in their order and can be acknowledged together or individually.

Description of the Evaluation Design

The implementation design includes both evaluation and research components (Figure 1), so the evaluation for this project needed to address that entire structure, that is consist of the project's implementation as well as their evaluation and research efforts within it. Therefore, the research questions are presented in two categories.

Evaluation Questions about the Implementation

- EQI 1: Did the Implementation connect to PrimeD consistently?
- EQI 1a: Was the challenge space vision and design embraced by all participants?
- EQI 1b: Did participants address evaluation and research in their lesson development?
- EQI 2: Did the Implementation team gather sufficient data to answer their research questions?

Research Questions about the Evaluation

- RQE 1: Did the formative evaluation design provide timely and useful data?

- RQE 2: Did the summative evaluation design capture the implementation and impact of the PD?
- RQE 3: To what degree was the evaluation design efficient?

This design builds in multiple collaborative observations which are compared and discussed after the event to calibrate observations and share outcomes as a fundamental part of the process.

Research Questions about the Evaluation Design

- RQ4: Did the evaluation design capture the implementation? Did it provide timely and useful data?
- RQ5: Was the evaluation design efficient?
- RQ6: Was the evaluation design effective; Did it capture the impact of the PD?
-

Results

The project reported improved communication and participation among stakeholders and improved teacher candidate pedagogical content knowledge to engage in research-based teaching strategies for EQI 1. For EQI 2 in the NIC meetings, participants agreed upon innovations to implement in the classroom and evidence to collect regarding the innovation's effectiveness. Participants shared adjustments made to their innovation strategy. Their results directed the conversations and guided further refinements within and across sites. Mentors, supervisors, alumni, and faculty were positioned as a collaborative support team. Finally, the challenge space continued to be discussed at team meetings throughout the project with an explicit focus on converging foci across institutions. Implementation strategies were explored and discussed in detail with a focus on how to maintain coherence across institutions while honoring the necessary contextual differences and aligned with the project vision which addressed EQI 2.

Conclusion

The need for "non-normal" evaluative mechanisms that are adaptive, open to challenges, highly collaborative and participant-centered is urgent, and even more so now, after the COVID-19 period. The PrimeD framework has helped to address some important issues: How collaboration and sense of community benefits all, how to be both structured and flexible in evaluation tasks; How to share formative evaluation results in a timely manner; and how to blend evaluation and implementation in clear and useful ways.

As evaluators, we need to rethink how "professional teachers" are developed. More study is needed to determine what guides change in practice and how that change is key to learning. We need to know how to see the widening gaps that "normal" evaluation in education can create and when "post-normal" evaluation is required (Schwandt (2019). We did learn that PrimeD framework can support "non-normal" evaluation. In today's complex teaching environment, non-normal evaluation will be critical for understanding PD structures and results.

The structure of the PrimeD evaluation team had a number of advantages such as:

1. Open observations (completed observations and recordings were available to all team members as soon as they were completed),
2. A local team member (site PI or Co-PI) was able to share and discuss results within their local team,
3. The non-local team member (the PI or Co-PI at another institution) was able to share new ideas and results with their own team,
4. Local PIs could review the observation results which guided their work and innovations.
5. The PI's (and all team members) could and often did attend regularly scheduled implementation and evaluation meetings in which evaluation results were discussed,
6. Including site Co-PIs as part of the evaluation team reduced the cost of the evaluation and improved the observation reliability,
7. The site PIs or Co-PIs became familiar with the work at all the project sites through the observations and evaluation alignment meetings promoting additional collaboration throughout the project and,
8. Being on the evaluation team provided additional leadership opportunities for site PIs or Co-PIs.

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A CUSTOM MIXED-REALITY SIMULATION DESIGN: MINIMIZING TEACHER BIASES IN MATHEMATICAL DISCUSSION PRACTICES

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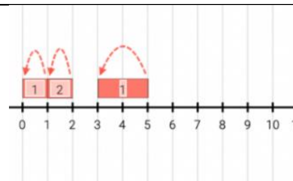
Mixed-Reality Simulations (MRSs), blending real and virtual environments, provide a low-risk environment (Dieker et al., 2014; Piro & O'Callaghan, 2019) to rehearse core teaching practices (Jacobs & Spangler, 2017). Grant and Ferguson (2021) designed a Mursion™ custom simulation to support pre-service teachers' (PTs) mathematics discussion practice. PTs had opportunities to facilitate a discussion of provided student solutions (e.g., drawing, repeated subtraction, creating table) to a mathematics task: "Harry the Dog Problem" (Grant & Ferguson, 2021), where Harry the dog is waiting in a line of 23 dogs to be washed. When a dog is washed, Harry sneaks ahead of two dogs. How many dogs will be washed before Harry?

In this context, PTs encouraged students to share, attend to, repeat, and critique each other's work. However, they less frequently focused on students' mathematical potentials (Kocabas et al., 2024). PTs attend to learner identities (Drake, 2006; Rubel et al., 2022) participation (Dunning, 2022), influencing PTs' discussion moves (Kocabas et al., 2023). We designed MRS to support PTs to attend to the mathematical potential of student solutions by creating PODSim, a new MRS application unlike Mursion™ (Grant & Ferguson, 2021). PODSim features gender ambiguous rainbow avatars and the ability to compare mathematics work (Table 1a & 1b) to encourage PTs to have increased focus on the mathematics. Further, we revised the solutions using virtual number lines to create opportunities for PTs to engage with avatar learners understanding of solutions that enacted the problem using repeated jumps and distances. We hypothesize that PODSim MRS microteaching environment (Grossman et al., 2009; Ledger et al., 2019) will support PTs' foci on mathematics equity and interaction in discussion practices.

Our revised solutions within PODSim foregrounds student work, removes race and ethnicity and positions PTs to directly encounter avatar students' mathematics solutions created using virtual number line models (see Table 1c & 1d). The PTs were expected to use solutions to investigate avatars' mathematical thinking by facilitating discussion that elicits avatar student reasoning, sense making, and ways of knowing to enhance mathematical understanding (Graeber, 1999; Jacobs & Spangler, 2017; Smith & Stein, 2018).

Table 1: Examples of Students' Solutions and the Simulation Interface

a.	b.	c.	d.
			

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SECONDARY PRESERVICE TEACHERS ENGAGING MULTILINGUAL LEARNERS IN MATHEMATICS LEARNING

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As the number of Multilingual Learner (ML) students attending public schools rises, so does the demand for those who can teach them. To address the gaps between theory and practice (Ball, 2000) in professional development (PD) programs, PrimeD (Professional Development, Research, IMplementation, and Evaluation) framework has been proposed which has four phases of Design, Implementation, Evaluation, and Research. The cyclic nature of PrimeD provides a coherent structure to PD activities. The purpose of my study is to examine possible benefits of PrimeD on secondary Mathematics PreService Teachers (MPST) reflecting on their instructional practices that engage ML students.

Introduction

Today, Multilingual Learner (ML) students are the fastest-growing student population group attending public schools nationwide (National Education Association https://www.nea.org/). By 2025, 1 out of four students in the US are estimated to be an ML. With regard to mathematics learning, the achievement gap at different levels between ML and non-ML students has long been recognized. As the number of MLs attending public schools continues to rise so does the demand for those who can teach these students effectively. Based on data from the National Center for Education Statistics (NCES, 2017), the ratio of educators who are dedicated to addressing the needs of this growing population of students is substantially low.

Among the reasons experts mention for the existence of shortage of trained mathematics teachers who are familiar with ML students' needs is lack of robust educator training. Although teacher preparation programs are considered as one of the most important and effective ways to improve the knowledge and skills of STEM teachers and to enhance their performance, research shows that such programs often do not benefit participating teachers in general. To address the gaps between subject matter knowledge and pedagogy while teachers need to be equipped with both to be able to teach their students well (Ball, 2000), a professional development (PD) framework, called PrimeD (Professional Development, Research, IMplementation, and Evaluation), has recently been proposed. PrimeD organizes PD into four phases of Design, Implementation, Evaluation, and Research, which work in a cyclic nature and occur iteratively throughout the program. A key feature of PrimeD that appears to help preservice teacher make stronger connections between field experiences and theories learned in their coursework, as recommended by Gainsburg (2012), is the use of Networked Improvement Communities (NICs) (Bryk, Gomez, & Grunow, 2011; Martin & Gobstein, 2015) to cycle between classroom implementation and whole group engagement in PD sessions.

PrimeD Professional Development Framework

In 2017, Saderholm et al. proposed a Professional Development Framework called PrimeD (**P**rofessional **D**evelopment: **R**esearch, **IM**plementation and **E**valuation) which consists of four phases that are arranged into categories of: Design, Implementation, Evaluation, and Research (Figure 1).

PrimeD first phase (Design and Development) is fundamental in the success of the PD as it focuses on bringing together all stakeholders to agree on a common vision, define the target outcomes, discuss the most critical challenges to reaching those targets, and subsequently develop strategies to overcome them.

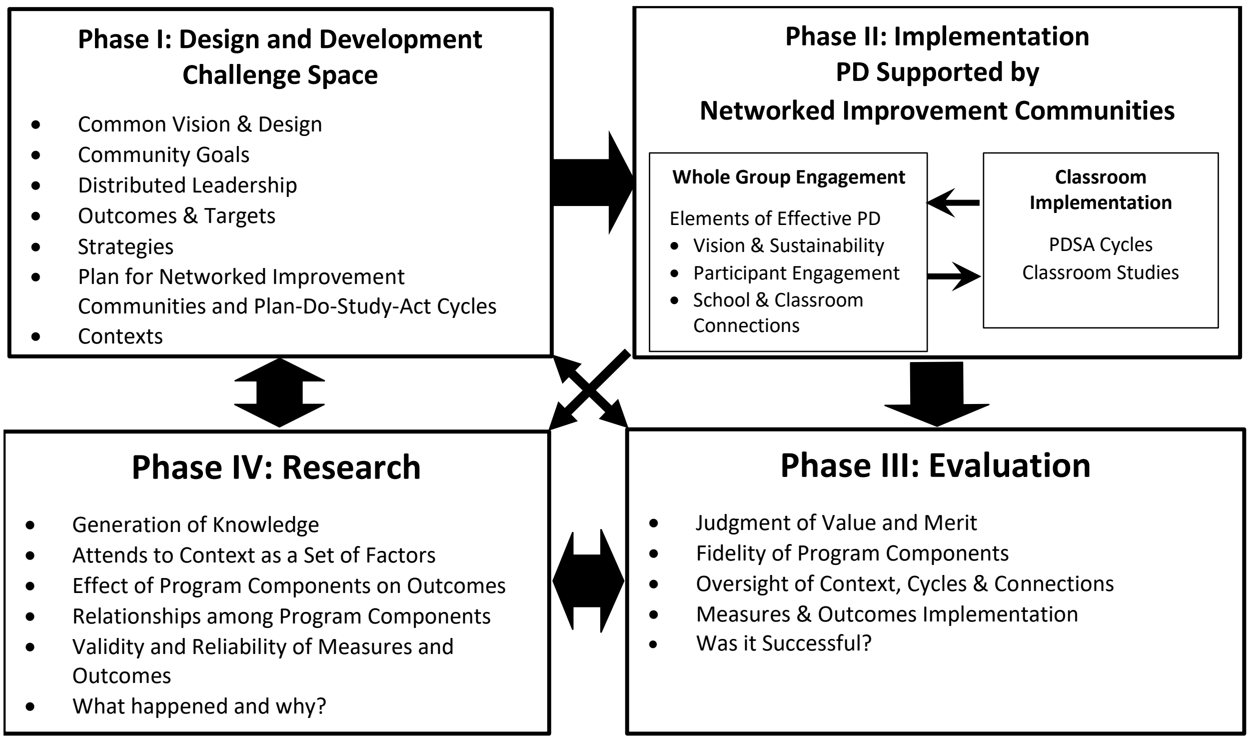

Figure 1: Condensed Model of PrimeD Framework (Saderholm et al., 2017).

The second phase of PrimeD (Implementation), starts with the "whole group engagement" followed by "classroom implementation", both of which are supported through Networked Improvement Communities (NIC). In this phase, the participating teachers will use effective PD elements (planned in the design phase) to come up with strategies that target specific aspects of their teaching through the Plan, Do, Study, Act (PDSA) cycles. The PDSA cycle is an iterative method used for systematic and continued improvement of a process.

The next phase involves evaluation of phases I & II, which should be performed using both formative and summative assessments.

The last phase of PrimeD, called Research, is focused on asking questions that would help better understand "what happened during the PD processes and why", or help "gain insight into effectiveness (or lack of) certain approaches", etc. So, inherently, this phase is closely related to and in constant interaction with other phases.

Methodology

Research Design

I have employed a convergent mixed-methods research design for my study. The design has a single phase of separate quantitative and qualitative data collection and analysis followed by merging and comparing quantitative and qualitative results to see if they confirm or disconfirm the findings (Creswell & Creswell, 2018).

Participants

The participants of my study were chosen by convenience sampling from a mixture of more than eighty-three female, male, and non-binary MPSTs across four institutions (Institutions 1-4) in three Eastern and Southeastern states in the US during the first three years of the program (2020-23). At all four institutions, quantitative data was collected by requiring PrimeD participants to submit two lesson videos (Pre and Post) from the beginning and the end of their student-teaching semester. A teacher evaluation instrument called MCOP^2 (Gleason et al, 2017) was used to code and quantify MPSTs' lesson videos. Excluding the incomplete data, I had access to MCOP^2 scores of sixty-one lesson video pairs. By conducting semi structured interviews with eight MPSTs from Institution 3 and Institution 4, I was able to collect qualitative data.

Analytic Strategy

I used Thematic Analysis (Maguire & Delahunt, 2017) to analyze the interview transcripts and Wilcoxon and t-test to analyze the MCOP^2 scores of MPSTs' recorded Pre and Post videos. The focus of my study is on the nine indicators within MCOP^2 Student Engagement subscale that are presented in Table 1.

Table 1: MCOP^2 Student Engagement Subscale Indicators (Gleason et al., 2017).

Indicator 1	Students engaged in exploration/investigation/problem solving.
Indicator 2	Students used a variety of means (models, drawings, graphs, concrete materials) to represent concepts.
Indicator 3	Students were engaged in mathematical activities.
Indicator 4	Students critically assessed mathematical strategies.
Indicator 5	Students persevered in problem solving.
Indicator 12	There were a high proportion of students talking related to mathematics.
Indicator 13	There was a climate of respect for what others had to say.
Indicator 14	In general, the teacher provided wait-time.
Indicator 15	Students were involved in the communication of their ideas to others (peer-to-peer).

Findings

Table 2 shows that based on the available three-year data from Institution 2, Institution 3, and Institution 4, on average, Institution 3 has the largest percentage of participating MPSTs who improved their observed practices associated with MCOP^2 indicator 1 (investigation and problem solving), indicator 2 (using variety of means to represent concepts), indicator 4 (assessing mathematical concepts critically), indicator 5 (perseverance in problem solving), indicator 12 (mathematical discourse), indicator 13 (creating climate of respect), and indicator 14 (providing wait time). Similarly, on average, Institution 2 has the largest percentage (47%) of participating MPSTs who improved the form of student engagement that is measured by MCOP^2 indicator 3 (mathematical activities in their classrooms). By comparison, Institution 4, on average, has the largest percentage (56%) of participating MPSTs who improved on MCOP^2 indicator 15 (peer-to-peer communication of ideas).

In harmony with the results of Institution3 quantitative data analysis, the emergent themes (Table 3) from interviews conducted with Institution 3 participants focus more on the impacts of offering wait time, mutual respect, and strategies that were implemented in their classroom as PDSA cycles that could increase mathematical discourse. Similarly, the themes emerged from Institution 4 qualitative data are more focused on peer-to-peer communication of ideas which confirm the quantitative results. This answers the first Research Question.

To answer the second Research Question, I only relied on the qualitative data analysis results because the MCOP^2 instrument did not collect any data from teachers' reflections on their practices or with regard to ML students. The results of qualitative data analysis indicate that participating in PrimeD project has made the teachers become more reflective on their practices as they implement interventions, collect data, and study the data within the PDSA cycles. The themes presented in Table 3 that are associated with ML students are results of such reflection.

Table 2: Percentage of MPSTs' who improved scores of each MCOP^2 indicator (highest values shown in bold).

MCOP^2	Institution 2				Institution 3				Institution 4			
	Y1	Y2	Y3	Avg.	Y1	Y2	Y3	Avg.	Y1	Y2	Y3	Avg.
Indicator 1	43	20	40	34	64	38	75	**59**	17	33	33	28
Indicator 2	29	50	40	40	64	50	50	**55**	50	33	17	33
Indicator 3	71	30	40	**47**	64	25	50	46	17	33	17	22
Indicator 4	43	30	40	38	64	50	50	**55**	50	33	17	33
Indicator 5	29	20	20	23	55	63	25	**48**	33	17	67	39
Indicator 12	71	20	20	37	73	25	25	**41**	33	50	33	39
Indicator 13	57	40	40	46	82	50	50	**61**	33	50	67	50
Indicator 14	71	40	20	44	45	75	50	**57**	17	33	17	22
Indicator 15	57	30	40	42	73	38	25	45	67	50	50	**56**

Table 3: Themes emerged from MPSTs' Interview.

Student Engagement Themes	**Participants Frequency**
Higher with activities and real-world problems	3
Having a sense of belonging increases engagement	3
Higher with varying activities	2
Building relationship with students increases engagement	2
Providing wait time increases engagement	2
Impact of PrimeD Themes	
Importance of collaboration	5
Getting new ideas from other participants	4
Introduced different forms of engagement	3
Helped teachers reflect on their practices	2
Receiving valuable feedback from other participants	2
Helped increase engagement	2
Focused on conceptual mathematics learning	2
MPSTs' Reflection on their Instructions Themes	
Building teacher-student relationships help increase math learning	4
Peer support increases participation and math learning	4
MLs engage more in activities within small groups or individually with teacher	4
Less for MLs because verbal communication is a barrier	4
Providing wait time helps MLs participate	3
Looping MLs in conversations helps them participate	2
Building relationships helps MLs' learning	2
Giving a prompt helps MLs get started	1

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Chapter 11:

Professional Development and In-Service Teacher Education

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

RURAL ACTION NETWORK FOR GROWTH AND ENGAGEMENT IN MATHEMATICS: PROJECT OVERVIEW AND FUTURE DIRECTIONS

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Keywords: rural, middle grades, professional learning, teacher leader, university specialist

The Rural Action Network for Growth and Engagement in Grades 6-8 Mathematics (RANGE Math) project is a teacher-researcher alliance for rural middle grades mathematics teachers and teacher-leaders, supported by experienced teacher educators with expertise in job-embedded professional development and school-based research. RANGE studies two models for expanding a successful local professional learning project to a large group of rural teachers in [Blind]. The project will investigate teachers' implementation of Explicit Attention to Concepts (EAC) and Students' Opportunities to Struggle (SOS), two clusters of instructional practices with robust research evidence for increasing students' mathematics achievement (Hiebert & Grouws, 2007).

RANGE tests two models for large-scale professional development with rural teachers of mathematics by attending to the unique funds of knowledge and challenges in rural schools. The project scaffolds teachers' improvement efforts with job-embedded cycles of content-focused professional development and structured action research. The research design includes balanced random assignment of N = 90 rural teacher-participants to either online professional development with university-based specialists (Condition 1) or in-person professional development with a local teacher-leader (Condition 2), accompanied by controls for the quality of professional development, cycles of data-informed instructional improvement, and data collection protocols. Condition 2 includes a teacher-leader program that will build capacity of local teachers to lead instructional improvement, with both conditions benefiting from expert facilitators and a team of researchers offering measurement and analysis infrastructure to assist in data interpretation. A sequential mixed methods research design addresses student and teacher outcomes in the network, allowing for statistical estimates of shifts in teachers' knowledge, beliefs, and instructional practices, as well as growth in students' mathematics achievement and classroom engagement. Importantly, the study will build much needed evidence on the relative benefits of online and in-person models of professional development at scale with rural mathematics teachers.

The purpose of the poster is to share the project design, intentions, and future directions to spur conversation about what it means to provide professional learning opportunities and research said opportunities in rural contexts. This project will improve scientific knowledge of STEM teaching by adding to scholarly understanding of: (a) effective mathematics instruction in the middle grades, (b) the adoption of EAC and SOS in rural settings, and (c) conditions that support rural mathematics teachers' professional development.

Acknowledgments

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VIDEO CLIP SELECTION WITHIN VIDEO CLUBS: RATIONALE AND PURPOSE FOR VIDEO CONTENT

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We explored the videos used to provide professional learning opportunities for coaches taking part in video clubs as part of a three-part professional development project. In year one, professional development facilitators selected the videos, while in the second year of professional development, the coach participants selected videos. We explored the rationale that coaches provided for the selection of their video clips. Findings indicated that coaches selected video clips that included content directly aligned with content-focused coaching, highlighted the practices of coaching, and included content that the selector deemed important for others to notice. We provide implications for those designing video clubs for coaches, as well as teachers, as the findings may be applicable when considering video selection beyond coach video clubs.

Keywords: Professional Development, Teacher Noticing

Video Clubs in Research

Researchers have repeatedly shown that video is a powerful tool for teacher education (Brophy, 2003; Coles, 2019; Christ et al., 2017; Gaudin & Chaliès, 2015; Santagata et al., 2021; Seidel et al., 2013). Of the many uses of video to support teachers, video clubs have been shown to be beneficial in improving teacher learning (van Es & Sherin, 2008; 2010). van Es and Sherin (2008) describe video clubs as the gathering of a group of teachers who "meet to watch and discuss excerpts of videotapes of their instruction" (p. 244). Video clubs have most commonly centered on mathematics or science content (Luna & Sherin, 2017), with the intent to develop teachers' pedagogical development (Kang & van Es, 2019; Luna & Sherin, 2017). Walkoe (2015) notes that the purpose of video clubs is to support teachers to "attend to and reason" about particular content within a video, a purpose mirroring that of professional noticing (e.g., Jacobs et al., 2010; König et al., 2022; van Es & Sherin, 2008)(p. 525). In fact, many video clubs have been intentionally designed to support teacher noticing (e.g., Mitchell & Ariemma-Marin, 2015; van Es & Sherin, 2010; Wallin & Amador, 2019). Researchers have found that what is noticed in the videos in video clubs is consequential for teacher learning opportunities (Borko et al., 2008; Gaudin & Chaliès, 2015; Walkoe, 2015). Given the repeated use of video clubs for teacher learning (Coles, 2019; van Es and Sherin, 2010), and knowing the affordances of the process for learning (Beisiegel et al., 2017; Borko et al., 2008; van Es et al., 2014), we designed and implemented a video club structure for mathematics teacher educators' learning. Situated in a mathematics education context, our video club was designed to help mathematics coaches improve their ability to: (a) facilitate productive planning and debriefing conversations with teachers, (b) notice the impact of their coaching practices on teachers' thinking and instruction, and (c) use evidence from what they notice to make decisions about their coaching practices. The specific focus of the video clubs was to support coaches to notice critical events (see Amador et al., under review) of one-on-one coaching cycles (see Kochmanski & Cobb, 2022).

We designed and implemented the video clubs for a mathematics coaching context because we conjectured that the structure would support coach learning, just as video clubs have supported teacher learning (van Es and Sherin, 2010). However, before studying the outcomes of video club implementation in a coaching context, we considered it important to better understand the design of video clubs, as specific for a coaching context, as a way to delve into the nuances of how the professional learning opportunity was designed. As a result, we worked to answer the following research question: What rationale do coaches provide for how they select clips for video coaching clubs?

Theoretical Framework and Related Literature

We theoretically frame this work with the research literature on noticing. The concept that noticing what is important in educational settings and making decisions based on the interpretation of what is attended to is consequential to learning opportunities (Jacobs et al., 2010; van Es & Sherin, 2008). While the importance of noticing for mathematics educators has gained attention in literature, and researchers have come to recognize the value of noticing (Sherin et al., 2011; van Es & Sherin, 2002, 2021), research on noticing in coaching contexts is limited (Amador et al., 2024). As a result, we frame the video club video selection by considering the opportunities to notice (see Stockero et al., 2017) available in a coaching context.

The coaching approach utilized in our investigation is centered on content-focused coaching (West & Cameron, 2013), differing from instructional coaching or cognitive coaching methods. Content-focused coaching is characterized by its emphasis on disciplinary content during coaching discussions; coaching sessions should delve into the mathematical aspects of the lesson, how these mathematical concepts are incorporated into the task design, and which instructional strategies promote or enhance mathematical comprehension (Callard et al., 2022). The novelty in our professional development initiative was the adaptation of a traditional face-to-face mathematics coaching model into a fully online, video-based coaching model (Amador et al., 2021).

Method

Twenty-three coaches participated in selecting videos as part of video clubs in which they took part during a larger three-part two-year professional development project designed to support coaches of mathematics. The other two professional learning components included an online course and one-on-one content-focused coaching in which a mentor coach (those with more experience who were part of our professional development team) supported the efforts of coach participants (individuals coaching in a school or school district in mathematics).

In the video club structure, four to six coaches met regularly to discuss elements of video of coaching interactions between a classroom mathematics teacher and a coach. These meetings typically lasted from 90-120 minutes and included two focal videos for the coach participants to watch. Prior to the video club meetings, a member of the team selected video clips for participants to watch during the video club. It is the rationale for this selection to which we focus attention. To collect data on the rationale for their selected videos, coaches were asked verbally and in writing to provide an explanation for the video that they selected. Verbal interactions were audio recorded and transcribed verbatim. To analyze data, four members of our research team created a coding scheme that included categories for: (a) content-focused coaching aspects, (b) coach support of the teacher, (c) teacher practices/knowledge, (d) coaching practices, and (e) video coaching club context. Within each of these broad five categories, the codebook had subcategories that further defined various reasons. To code data, all four researchers coded each rationale response with as many codes as were apparent in the data. The team of four then met to reconcile any discrepancies in the coding and arrived at final counts for the rationales coming from each of the 23 coaches. We then calculated frequency counts to arrive at an overall description of the rationale.

Results and Discussion

Analysis of the data reveals three main trends that indicate that participants selected video clips that: (a) included content directly aligned with content-focused coaching, (b) highlighted the practices of coaching, and (c) included content that the coach selecting deemed important for others to notice. The following provides an overview of the findings, and then each trend is described.

Table 1 shows the breakdown of the overall rationale reasons provided. Recall that in some cases, participants provided more than one reason for a particular decision about (König et al., 2022) video choice.

Table 1: Percent in Each Category Provided as Rationale

Content-Focused Coaching Aspects	Coach Supports Teacher's	Teacher Practices/ Knowledge	Coaching Practices	Video Coaching Context	Total
38.61%	5.94%	15.84%	31.68%	7.92%	100%

Content-focused coaching practices were a main reason participants gave for selecting particular video clips to show during the video club. Within this overarching category, participants said they selected their video clips because they: focused on mathematics learning goals, demonstrated the use of instructional practice goals, provided an example of the structure of the lesson design, showed how to talk about anticipating student strategies, examined evidence of student thinking, illustrated the launching of a lesson or task, demonstrated debrief conversation structure, showed the coach doing the mathematics with the teacher, or illuminated collaboration or co-learning. The following is an excerpt of an example of the rationale one coach provided that was coded for aspects of content-focused coaching:

> Yeah, so the first clip is us talking about […] coaching—if you look at the lesson planning document at the bottom there's the debrief coaching cycle. I was just showing her that and letting her know that I was going to use that in the post discussion. That was my own goals. That first section is me setting my goals and making them apparent to her. I don't know, I chose that for that reason. Then the second section, that's kind of long, is her setting some of her goals. I cut out the part for setting content goals, but we focused on her setting the math practice goals. That's why I chose the first one is just highlighted our conversations around goals. That was my number one reason for choosing it.

In this example, the coach explicitly states that the selection was centered on goals, mathematics practice and content goals, which are both aspects associated with content-focused coaching.

Coaching practices were another main theme for why particular clips were selected. Coaching practices that were commonly coded included language related to: the five practices for orchestrating discussion (Stein et al., 2008), mathematics content development, engagement in related professional learning, and lesson content. In the following excerpt, the coach provided rationale related to the five practices, a key component related to coaching:

> As a district we are working on that five practices book […] Every time we do anything with PD, and I work with Arnold, it's always that—what's the goal of the lesson? What's the goal of the teacher? That focus has been there. When I went back and saw the lesson, I'm like, "Woo-hoo, it is in there." I was celebrating that yes, even—because when I did this with her, she's first year teacher too. Being excited that this is being ingrained in her right from the beginning. I guess that's why I picked it.

In this example, the focus is clearly on coaching a teacher to implement the five practices, a theme that was common among participants, with nearly 32% of the reasons given for video selection related to coaching practices.

Finally, a cross cutting theme that was not coded for directly with the codebook but was presented through thematic analysis was that of noticing (i.e., van Es & Sherin, 2008)—coaches selected videos that included content that they thought was important for others to notice. In the verbal description of her rationale for clip selection, Coach Riess shared her rationale behind the clips she chose for her first video coaching club was to include crucial aspects of content-focused coaching. To focus their time during the video club, Reiss narrowed in on learning goals and anticipating student strategies. She stated,

> We felt that those were really important and different to content focused coaching, where you might not see those aspects in other types of coaching, such as cognitive coaching or instructional coaching or student-centered coaching. That was how we kind of chose the topics of our video coaching club. When I went to go look for a clip that showed me having a discussion around learning goals, I first looked for a clip where that was really evident and clear and wasn't chopped up by different conversations that kind of came in and out of that discussion around learning goals. The one I settled on is a clip that, in it, the teacher had provided me with two learning goals for the task, as she had written those learning goals. When I read the learning goals, I felt that one of them really did not match what the task was asking students to do. I knew that the teacher might not have internalized the content as well as I would like. That was one thing I noticed. I also noticed there was other content in the task she had chosen that wasn't encompassed in her learning goals. Those were two things that I thought were interesting as I had looked at the learning goals she provided and the conversation we had.

This example shows that Riess had ideas in mind of what she wanted coach participants to notice in the video club, and she identified a video that contained what she was hoping others would notice during the video club.

These findings illuminate the reasons that specific clips were selected for the video clubs. As researchers have noted, the purpose of video clubs is to support educators to attend to and reason about particular events (Walkoe, 2015). Knowing what events are selected and why provides important information for both researchers and professional development providers, as the videos that are shown are consequential to the possible learning outcomes and can be a powerful tool for educator learning (Brophy, 2003; Coles, 2019; Christ et al., 2017; Gaudin & Chaliès, 2015; Santagata et al., 2021; Seidel et al., 2013). We encourage researchers and professional development providers to be cognizant of video choice during their projects and to maintain awareness of the rationale for video clip selection as the selection evidenced in this research directly related to the coaches' opportunity to notice and learn.

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PEDAGOGICALLY PRODUCTIVE TALK IN PROFESSIONAL DEVELOPMENT: ANATOMY OF A DISCUSSION

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Video-based professional development (PD) supports teacher learning through grounding discussions in representations of practice. We consider PD where teachers, not the facilitator, were responsible for choosing video clips and generating a focal question. Using a framework for pedagogically productive talk, we investigate what contributes to productive video-based discussions. By analyzing two discussions held back-to-back, we unpack how the same group of teachers engaged in more and less pedagogically productive talk. This research has implications for understanding the features that contribute to productive video discussions in PD.

Keywords: Professional Development, Pedagogically Productive Talk, Video-based PD

Video-based professional development (PD) supports mathematics teacher learning through grounding discussions in representations of practice (e.g., Borko et al., 2008; Karsenty & Arcavi, 2017; Santagata, 2009; van Es & Sherin, 2008). In these PD designs, teachers typically view and discuss videos related to a professional learning goal, such as noticing student thinking or facilitating mathematical problem solving. Choosing video clips and focal questions is critical, intentional work for facilitators (Borko et al., 2014; Sherin et al., 2009), as is norm-setting (Borko et al., 2011). Through norm-setting, video selection, and focal question development, the facilitator has great agency to shape teachers' learning opportunities.

However, ceding some of the decision-making from facilitators to teachers may provide teachers more agency in choosing the focus of their learning. Recent research has begun to explore the learning opportunities available when teachers select video clips (Richards et al., 2021). Similarly, we posit that choosing the focal question for discussing a video clip may offer learning opportunities for teachers. Because clip and focal question selection are crucial for supporting teachers' learning in video-based PD, researchers need to investigate the nature of the discussions when they are guided by teacher-selected video and questions.

Pedagogically productive talk in PD consists of "talk that is productive for the development of participants' adaptive expertise and professional judgment" (Lefstein, Vedder-Weiss, et al., 2020, p. 362). This definition is based on Lefstein and colleagues' practical experience and their review of the growing body of literature surrounding teachers' discourse (Lefstein, Louie, et al., 2020). We seek to understand the features of pedagogically productive talk within a video-based PD context where the focal question and video are both chosen by the teachers, rather than by the facilitator. We ask: How do two video-based discussions, where teachers chose both the focal question and the video clip, unfold to produce different kinds of pedagogically productive talk?

Methods

Our research took place in a two-year PD program for early-career secondary mathematics teachers. Designed to promote equitable mathematics instruction (Cohen & Lotan, 2014), the PD included a two-week institute each summer, followed by virtual coaching sessions during the school year. We focused on small group coaching sessions, in which teachers shared videos of their own lessons and discussed emergent issues regarding their practice. For each small-group session, all teachers contributed a video clip, which the other group members viewed and commented on in advance. There were 22 teachers, divided into seven groups, that met three times per year. This resulted in 41 recorded sessions, containing 127 video discussions. Each session had the same structure, led by the same facilitator. Following norm-setting, the groups discussed each teacher's video (20-25 min per video). For each video, first, the video-sharing teacher described the context of the clip and addressed contextual questions (5 min). Second, the facilitator invited each person, beginning with the video-sharer, to name strengths of the video (5-10 min). Third, the facilitator prompted the video-sharer to name a question emerging from the video, which the group then discussed (10-15 min). Following all video discussions, the meeting concluded with the facilitator asking each teacher to name next steps in their practice.

Using a video discussion as the unit of analysis, we wrote memos following the framework for pedagogically productive talk (Lefstein, Vedder-Weiss, et al., 2020). We operationalized the framework using our data to generate indicators, non-examples, and guiding questions. Table 1 briefly presents the six features of the framework and a summary of our guiding questions for analysis. Additionally, we assigned a score for each feature to indicate if it was minimally (0), partially (1), or fully (2) present in each discussion (maximum of 12 for each discussion). Based on these scores, we identified pairs of video discussions that took place in the same session and had different levels of productivity. From five pairs with scores differing by eight or more, we chose one pair for comparison using the analytic memos, video recordings, and transcripts.

Table 1: Features of Pedagogically Productive Talk (Lefstein, Vedder-Weiss, et al., 2020)

Criteria	Summary of guiding questions for analysis
Focused on problems of practice	Is the discussion focused on a clearly articulated problem of practice that is rooted in issues the video sharer has faced in teaching?
Involves pedagogical reasoning	Do speakers offer different ideas, hypotheses, affordances, or constraints related to the problem of practice alongside reasoning?
Anchored in rich representations	To what extent do speakers refer to specific events, words, actions, or moments in the video and use them as evidence during the discussion?
Multivoiced	To what extent are different perspectives offered? How are these attended to, acknowledged, questioned, connected to, and built on?
Includes generative orientations	To what extent do the participants demonstrate orientations toward the problem of practice that are in line with the PD goals and values?
Combines support and critique	Do participants create a collegial and supportive space for sharing? To what extent do they offer critical questions or considerations?

Findings

We present a detailed analysis of two discussions from one coaching session during the first year of the PD. The group consisted of teachers Diego, Rebecca, and Sadie, and facilitator Garima (all names are pseudonyms). The meeting was the group's second of the year. Diego volunteered to discuss his video clip, from a review lesson, first. We found that the discussion of Diego's video scored 2 (present) in all six rubric categories, the highest possible score. The group then discussed Rebecca's video, focused on a novel participation structure Rebecca had tried. This discussion scored 1 (partially present) in the categories of problems of practice and generative orientations, and 0 (minimally present) in all other categories, yielding a total score of 2. This large divergence of scores between two discussions during the same session with the same participants enabled us to interrogate what features of the PD supported productive talk and what features inhibited it. We describe the differences in pedagogically productive talk for each video discussion, focusing on each rubric category individually.

Problems of Practice

The questions Diego posed to the group to frame their conversation were, "At this moment, how could I have gotten more information about what they [the students] do and don't know? What kind of moves could I have done?" These questions focused directly on a problem of practice, how to elicit and understand students' thinking, that is an ongoing concern for teachers across contexts (e.g., Sleep & Boerst, 2012). The questions were explicitly tied to an event in the video and were generalizable and resonated with other teachers. The focus on eliciting and interpreting student thinking was evident throughout the discussion and across contributions from all participants. Rebecca posed the questions, "When you walk around the room and listen to groups, how do you decide when to intervene and what to say? How do I decide when I'm making a useful comment and when I'm interrupting the flow?" While the questions named the problem of practice—when and how should a teacher intervene in groupwork, the question was disconnected from Rebecca's video. This made it difficult for the group to analyze classroom events and justify future courses of action.

Pedagogical Reasoning

During the discussion of Diego's video, teachers engaged in thought experiments (Munson et al., 2021) and reconsidered previous practices. Rebecca, referencing a specific moment from the video, suggested using brainstorming. Sadie extended that thinking, wondering what the goals of that brainstorming might be. Curiosity about classroom events was prevalent, and all participants supported their suggestions with explicit reasoning. In contrast, during the discussion of Rebecca's video, each person gave Rebecca tips and tricks for deciding when to intervene with student groups. These tips and tricks were not linked to specific evidence from the video clip, or necessarily supported with pedagogical reasoning.

Rich Representations

When discussing Diego's video, participants referred to specific moments of the video to gain information and understanding regarding classroom events. They also used the video to create an imagined alternative story, considering how possible actions and interventions could influence outcomes. In contrast, when discussing Rebecca's video, there was only one significant reference to the video, which was offered by the facilitator and not further explored by the group.

Multivoiced

All participants shared ideas during both discussions. However, there were differences in the degree to which participants took up responses and built on ideas. During Diego's video discussion, all participants attended to the ideas shared by others, frequently reflecting on how their suggestions would provide opportunities to deepen connections and student engagement. Conversely, during Rebecca's video discussion, ideas were rarely attended to by any participants. This resulted in a parallel sharing of ideas without dialogism.

Generative Orientations

From the beginning of the discussion of Diego's video, the group showed a generative orientation toward students, problems of practice, and teaching. Diego praised his students' use of classroom resources. Others in the group readily responded to Diego's question, demonstrating a generative orientation toward the problem of practice. At the end of the conversation, Diego indicated that he was planning to try new practices based on suggestions from the group. During the discussion of Rebecca's video, the group demonstrated generative orientations toward students, yet they did not show a generative orientation toward the problem of practice. For example, Rebecca did not identify specific areas for improvement in her activity structure, and the group did not engage deeply with any suggestions for future practice.

Support and Critique

In the discussion of strengths and highlights of Diego's video, the group offered specific examples of the students' mathematical knowledge, their capacity to share their thinking, and the dialogic nature of the discussion. Building from this, Diego positioned himself as open to critique and seeing his practice in new ways. Participants validated his question as shared and thought critically about new ways he could learn about his students' thinking in the moment. In the discussion of Rebecca's video, the strengths identified were largely couched in politeness and were general, including the students' "energy" and Rebecca's risk-taking for trying a new activity structure. Little support was offered to Rebecca in the rest of the discussion, which stayed at a general level about the pedagogical decisions in each participant's own classroom rather than the events and pedagogical decisions in Rebecca's video.

Discussion

Despite similarities in these discussions, including the same participants in the same meeting, the discussions reflected different levels of productive talk. What may have contributed to the differences in how these discussions unfolded? We conjecture that there are three underlying factors distinguishing these discussions. Two factors reaffirm findings in prior literature: video clip selection and focal question selection are pivotal (Borko et al., 2014; Sherin et al., 2009). We found that teachers can select productive video clips for discussion, and with support, they can craft fruitful focal questions. Yet, granting teachers this agency and responsibility comes with certain risks: in response to Rebecca's video clip, participants contributed less pedagogical reasoning and the discussion remained more general. Further, Rebecca's focal question was not directly linked to the video and thus limited engagement with that evidence. This risk does not mean that facilitators must control video clip and focal question selection, only that it is important to note that there is variability in how teachers take up this responsibility.

The third underlying feature that influenced the productivity of these conversations is the degree to which teachers attended to and took up one another's ideas. The presence of *teacher uptake* in Diego's discussion fueled a dialogic discussion in which multivoicedness, critique and support, pedagogical reasoning, and a generative orientation to professional learning and the problem of practice were prominent. Diego repeatedly took up and considered aloud his colleagues' ideas, revising his thinking about the problem of practice he presented. Other participants similarly built on one another's ideas, posed questions, and offered pedagogical reasoning. This orientation, to take seriously other's contributions and be open to revising thinking about one's own practice, was a key lever for productivity in discussing Diego's video and which the discussion of Rebecca's video notably lacked. We argue that the notion of teacher uptake in professional discussions and its connections to other, well-established features of pedagogically productive talk is an area in need of further attention by the field.

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CONSIDERING CONTEXT IN THE AVAILABILITY OF SECONDARY MATHEMATICS TEACHERS' PROFESSIONAL LEARNING OPPORTUNITIES

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Keywords: Teacher Knowledge

Context is a critical consideration in teacher professional learning. When queried about formal professional development (PD), many teachers report dissatisfaction, often citing contextual agnosticism, and, thus, irrelevance (Boston Consulting Group, 2014). In informal learning spaces, teachers are generally freer to exercise choice toward more contextually relevant learning opportunities and resources. Regardless of formal or informal settings, the need for professional learning to be context-specific is not new (Borko et al., 2010; Desimone 2009). In fact, efforts to contextualize formal PD are ongoing (i.e., Fairman et al., 2023), with much work in this area attending to the ways professional learning opportunities can and should embrace specific classroom contexts. Relatedly, this pilot work considers how mathematics teachers' course and geographic context shape their accessed set of professional learning opportunities. We pursue the following questions: *What resources characterize the professional learning opportunities accessed by advanced secondary mathematics teachers? To what extent, if any, do the learning resources vary by mathematics subject and geographic context?*

Data for this study were collected via an online survey distributed through various teacher social media groups and discussion boards, with quotas observed for both the mathematics course taught and school district geographic locale (as defined by the National Center of Education Statistics (NCES) locale codes— *city, suburban, town, and rural*). Survey participants (*N*=14) for this pilot work were U.S. public high school teachers who taught an Advanced Placement® (AP®) mathematics course. The survey instrument had two primary parts, including both qualitative and quantitative questions, asking respondents to first select the learning resources they used for content-based professional learning within their AP® course, and second, to identify where the resources were accessed. Survey responses are currently being analyzed using mixed-methods geospatial analysis (Yoon & Lubienski, 2017). Follow-up interviews with selected survey participants are also in the process of being collected, transcribed, and analyzed for the purpose of further elucidating survey responses.

Early findings reveal that mathematics teachers reported a range of resources (Min = 1, Max = 12) supporting their professional learning, with most resources (53.3%) being digital. Non-digital resources were primarily accessed at home or at school. Locally accessed resources, outside of school and home, included professional learning communities, friends, teacher colleagues, university-affiliated learning opportunities, district-sponsored professional development, trainings facilitated by a local/regional educational cooperative, and national/statewide professional conferences. Despite an equal number of AP® Calculus AB and AP® Statistics respondents, AP® Calculus AB teachers reported more non-digital, locally accessible resources, particularly university-related and local/regional cooperative opportunities. Teachers in rural districts reported little access and the least travel to school-external, in-person learning opportunities. These findings suggest potential inequities in accessed professional learning opportunities across content *and* context. Further inquiry is warranted to understand the mechanisms by which mathematics content and geographic context shape professional learning.

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NAVIGATING EXPERIENTIAL LEARNING: THE ROLE OF TEACHER DISCOURSE MOVES IN AMPLIFYING STUDENT EXPERTISE

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Experiential learning represents a shift in K-12 education that requires teachers to change the ways that they engage students. We created a professional development experience in which teachers learned about the entrepreneurial-based design challenges we developed (Authors, 2019) and practiced implementing teacher check-ins with students participating in our summer camp. In this paper, we conduct a case study to explore how three teachers used teacher discourse moves during their teacher check-ins. We found three types of teacher-student interactions: (a) positioning students as experts, (b) co-designing with students, and (c) pushing students towards an outcome. These findings suggest that teacher professional development for experiential learning should intentionally support teachers in learning how to employ the moves during teacher check-ins in ways that elevate student expertise and advance their thinking.

Keywords: Professional Development, Problem-Based Learning, Classroom Discourse

Experiential learning provides opportunities for K-12 students to work collaboratively and across disciplines to create innovative, actionable, and empathetic solutions (Hashim et al., 2019). Experiential learning represents a transformative approach to education (Slavich & Zimbardo, 2012; Yardley et al., 2012), challenging traditional pedagogical norms by prioritizing hands-on, inquiry-based learning experiences. There are many varieties of experiential learning approaches, ranging from problem-based learning to community-based learning (Haigler & Owens, 2018). Teachers play a pivotal part of experiential learning by adopting diverse roles, from facilitators to co-designers (Grossman et al., 2019; Haigler & Owens, 2018). As teachers navigate experiential learning environments, understanding the nuances of teacher-student interactions becomes imperative for optimizing instructional practices and fostering meaningful learning experiences.

The Design & Pitch (D&P) Challenges in STEM project (Confrey et al., 2019) is an experiential learning curriculum that draws on project-based learning (Krajcik & Blumenfeld, 2006), entrepreneurial-based learning (Lackeus, 2015), and design-based learning (Mehalik et al., 2008) to situate mathematics learning within entrepreneurial pitch competitions. In this paper we report how teacher discourse moves (TDMs; Herbel-Eisenmann et al., 2013) are leveraged when teachers practice facilitating experiential learning during a PD on D&P.

Literature Review

Teacher Facilitation of Student Autonomy

Learning environments that employ these experiential pedagogies are built on a culture of student-centered practices (Haigler & Owens, 2018), where students are provided with autonomy to ideate and create, meaning that each group is often thinking about different topics (Lee & Hannafin, 2016; Wilson et al., 2015). A teacher in this situation needs to check-in with students regularly to assess, support, and facilitate their progress (Grossman et al., 2019; Lee & Hannafin, 2016). This requires teachers to have a deep understanding of the learning goals (Grossman et al., 2019), a willingness to allow students to assume autonomy and authority (Langer-Osuna, 2011), and the flexibility to facilitate student thinking relating to a wide variety of ideas and solutions (Haigler & Owens, 2018; Krajcik & Blumenfeld, 2006).

When managing student-centered classrooms with varying ideas and approaches, Herbel-Eisenmann et al. (2013) found that teachers could effectively facilitate student learning through what they refer to as TDMs. Strategies like these are often used to facilitate whole group discussions (Herbel-Eisenmann et al., 2013; Smith & Stein, 2011), but they can also provide a way for teachers to purposefully engage students in conversations during teacher check-ins in experiential learning environments.

Professional Development

One way that teachers learn to use new curricular resources, especially those based in novel pedagogies, is through professional development (PD; Dingman et al., 2021). For PDs centered on learning about a new curriculum, regardless of format, McDuffie and Mather (2009) suggest first engaging teachers in an experience where they are positioned as the student. Then it is important to shift teachers back to the teacher perspective after thinking as a student, so they can reflect on how their experience informs their approaches to teaching with the resource (Dingman et al., 2021; McDuffie & Mather, 2009). An experiential way to do this is by having teachers approximate the practice of facilitating aspects of the curriculum in conditions that are less complex than a real classroom (Schutz et al., 2018). Approximations of practice allow PD participants to engage with how a novel curriculum might look in a classroom (Schutz et al., 2018), since it may be quite different from their traditional teaching practice.

This study explored how teachers leveraged TDMs while checking in with students during a PD on a novel curricular framework. This research is guided by the following question: *How do teachers approximating the practice of a teacher check-in use TDMs to support, or hinder, amplifying student expertise?*

Methods

The data for this paper was part of a larger study that focused on the design and study of a high school mathematics entrepreneurial curriculum and its associated PD, henceforth referred to as D&P.

Context and Participants

The D&P PD, in which the data was collected, lasted one week, and was held at the same time as a D&P student summer camp. In the first 2.5 days participants acted as students to experience one of the D&P challenges alongside the summer camp students, and then the final 2.5 days the participants acted as teachers during the summer camp. During the D&P PD teachers engaged in teacher check-ins in multiple ways. To experience facilitation moves through the student perspective, teachers experienced multiple teacher check-ins as learners. Teachers also had a chance to debrief their experiences as learners with PD facilitators. Additionally, teachers were given a document that outlined questions to consider asking during teacher check-ins.

Five teachers attended the PD experience. Three were high school teachers, one science and two mathematics, and two were elementary teachers. The two mathematics teachers were new to teaching and the other three teachers were veterans. The two elementary teachers were not able to experience the entire PD therefore their data was excluded from this analysis.

Data Collection and Analysis

All aspects of the summer camp and PD were video recorded. For this paper the videos of interest were the periods in which the teachers were checking-in with student groups. Thus, the video data was reduced to these 20 to 30 minute clips for data analysis. The check-in videos were memoed by the first author, from which a content log with brief summaries of each video was created. Considering each video's memo and the content log, the research team selected one video per teacher participant that was representative of their understanding of teacher check-ins. After reducing the data to three videos, transcripts were created and were coded using a TDMs framework (Herbel-Eisenmann et al., 2013). Additionally, the transcripts were analyzed for the breakdown of teacher talk time versus student talk time (Hennesey et al., 2023).

Results

Preliminary analysis of three teacher check-in videos surfaced three types of teacher-student interaction: (a) positioning students as experts, (b) co-designing with students, and (c) pushing students towards an outcome.

Positioning Students as Experts

When the teacher check-ins began, each teacher spent time orienting themselves to the students' ideas in relation to the D&P challenge. Teachers typically did this through the TDMs of assessing student thinking and revoicing (Herbel-Eisenmann et al, 2013). While all of the teachers had periods of positioning students as experts of their ideas, Teacher B exemplified this interaction type. She employed the TDM of waiting (Herbel-Eisenmann et al, 2013) throughout the interaction, as indicated by the short listening cues she provided to students such as "Yeah," "Okay," and "Nice." The students engaging with Teacher B also had the highest amount of talk time, sharing their ideas and work for almost 60% of the time, in contrast to the 25% of time that students talked during the interaction with Teacher A and almost 50% of the time that they talked with Teacher C.

Teacher C also used the TDMs of inviting participation and orienting to student work (Herbel-Eisenmann et al, 2013) to position students as experts. One instance of this was when she said, "Student S is taking what y'all have mapped out and she is tracing over it in color on the map to show the two routes? What are you doing?," which shows both moves. She first elevated what Student S was currently working on (orienting the other students in the group to that work) and then she invited another student in the group to share what they were working on. These two moves together grounded group interactions in student ideas while facilitating collaboration amongst group members.

Co-Designing with Students

As teachers used the TDM of advancing student thinking (Herbel-Eisenmann et al, 2013) towards the challenge goal, they would sometimes become co-designers with the students, acting as a group member during teacher check-ins. During these co-designing periods, teachers worked to advance student thinking through grounding conversations in student ideas. Teacher A did this when creating a map prototype by directing the students to pull up the technology tool for mapping on one computer and then having everyone (including the teacher) gather around that screen to work together. The co-designing interaction is exemplified in this set of quotes:

> Student: Uh, you could just uh add the whatever it's called, the bottom left corner there, no farther down, yep, like then you can like add them along the route, just like plan it out and then you can hit how long to measure it.
>
> Teacher A: So I guess drop a pin where that thing is. And then you have to find [High School Name]. Or whatever, maybe the movie theater.

Both the student and the teacher are figuring out the technology and which decisions to make together to advance towards the goal of prototyping a mapping app.

Pushing Students Toward an Outcome

The other way the teachers attempted to advance student thinking (Herbel-Eisenmann et al., 2013) was to push students in a specific direction based on their understanding of the student ideas in relation to the challenge goals. This tended to frustrate and constrain students, rather than advance them. For instance, Teacher C was trying to advance the students' thinking to be broader:

> What do you think would happen for your users if instead of giving them a very narrow trip from Dominos to the [location of the camp],...what if you expanded it, say from [close-by town] to the [location of the camp]?

The students became frustrated with her pushes because they felt she was disregarding the work they had already done as well as suggesting that they were not doing the correct task.

Discussion and Conclusion

Within a PD focused on supporting teachers to adopt an experiential learning curriculum, teacher participants had multiple opportunities to engage in teacher check-ins–a critical component of experiential learning (Lee & Hannafin, 2016; Grossman et al., 2019; Wilson et al., 2015). When teacher participants had the opportunity to approximate the practice (Schutz et al., 2018) of teacher check-ins three teacher-student interactions emerged. As they engaged as teachers during the second half of the PD, the teachers naturally employed many TDMs (Herbel-Eisenmann et al., 2013) during teacher check-ins, which led to two beneficial teacher-student interactions (positioning students as experts and co-designing with students) and one concerning teacher-student interaction (pushing students towards an outcome). Our analysis highlights an area of focus for experiential learning PD, the importance of how to facilitate moving towards the goal of the activity while staying grounded in the student ideas. While the literature shows the importance of the teacher engaging with groups during experiential learning to move them towards a learning goal (Grossman et al., 2019), our findings show negative student reactions during these moments. Thus, during these teacher check-ins while teachers are engaging with students around their ideas (a beneficial interaction), teachers must employ TDMs that both support the advancement towards the learning goal while continuing to position students as experts. If teachers push too hard to advance towards the learning goal, as shown above, students will begin to become complacent and lose connection to their idea, thus diminishing their expertise.

Therefore, based on our preliminary analysis, we suggest that experiential learning PD should intentionally support teachers in learning how to leverage TDMs (Herbel-Eisenmann et al., 2011) to engage students as experts while advancing them towards learning goals. This intentionality can be built into debrief sessions that support engaging in the curriculum as a learner (Dingman et al., 2021; McDuffie & Mather, 2009), or side-by-side coaching (Munson, 2018) during teacher check-in approximations of practice.

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GAUGING PROFESSIONAL GROWTH: MOVING BEYOND BINARY ASSESSMENTS BY EXAMINING TEACHERS' PRACTICAL ARGUMENTS IN SIMULATIONS PRACTICE

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Building on the critique of binary frameworks and the need for a nuanced understanding of instructional practices, we offer a methodological approach for gauging teachers' professional growth through a simulation designed to support secondary mathematics teachers in facilitating problem-based lesson discussions. This paper explores data from a pilot online teaching simulation which, by design, avoids feedback reliant on binary labels for instructional practices. The study analyzes responses from ten prospective teachers in a simulation that supplements Milewski & Strickland's (2020) functional framework for responding with subject-generic and subject-specific categories. Our findings demonstrate the viability of this approach in revealing shifts in participants' awareness of the linguistic functions underlying their responding moves.

Keywords: Research Methods, Problem-Based Learning, Professional Development, Classroom Discourse

Mathematics education scholars emphasize the importance of moving beyond narrow perspectives on 'good' teaching to effectively support teachers' professional growth (Chazan & Ball, 1999; Horn et al., 2022). This shift recognizes that framing teaching within binaries such as 'good/bad' flattens the complexities of effective instruction, neglecting the dynamic adaptations required for responsive teaching (Biesta, 2007). Scholars oppose binary frameworks, acknowledging subjectivity and contesting conceptions of teaching framed in such terms (Berliner, 2005; Fenstemacher & Richardson, 2005).

In response, innovative pedagogies guide teachers beyond a singular vision of instructional quality, emphasizing inquiry, practical arguments, and feedback as catalysts for professional growth (Horn et al., 2022; Fenstemacher & Richardson, 1993; Fernandez et al., 2020). However, this raises a methodological question: *How can we assess the potential of professional development programs to support teacher professional growth if those programs do not put forth explicit goals for teachers' instructional practice?* This paper introduces an innovative approach to the practice of responding to students' contributions by developing a measure that does not rely on categorizing specific moves according to binaries.

To illustrate, we delve into data from a pilot online teaching simulation for supporting secondary mathematics teachers in the critical decision-making involved in facilitating problem-based discussions. Participants engage as teacher avatars, making critical decisions (e.g., selecting, sequencing, responding) in simulated discussions. Aligning with the aforementioned calls, we designed the simulation to avoid feedback categorizing participants' moves as more or less 'desirable.' Post-simulation, we grappled with the question of how to assess whether and how teachers were growing in their interactions within the simulation. It felt incongruent to develop a measure relying on a singular perspective of teachers' increased use of 'desirable' practices. How then can we effectively evaluate teachers' professional growth within this innovative and flexible framework of design?

Theoretical Framework

Teachers' discursive moves have often been seen through overly prescriptive lenses. Chazan and Ball (1999) criticized anti-telling rhetoric as too simplistic and agnostic to context, urging the field to move past proscriptions forbidding certain discursive moves. While research has progressed beyond slogans like 'Don't tell' (Ellis et al., 2019; Franke et al., 2009; Michaels & O'Connor, 2015), quick translations from research to practice without context consideration remain common. This is exemplified by reports on the wide-scale dissemination of the Five Talk Moves (Chapin et al., 2009) into diverse professional settings:

> Our focus on the "talk moves" did not always seem to work. While some teachers easily picked them up, other teachers, particularly those less experienced, seemed to find them difficult … using them "robotically." For example, the revoicing move would be used when there was no reason to revoice: a student might state a fact, clearly: "seven is a prime number" and the teacher would query "So you're saying seven is a prime number? Is that what you're saying?" to puzzled looks. Teachers would ask "Who agrees and who disagrees?" in a perfunctory manner, not following up on the reasons. Or "Who can repeat what she said?" when the original utterance was neither complex nor particularly useful for furthering the topic. (p. 117; Michaels & O'Connor, 2019).

These perfunctory uses of talk moves echo the kind of lessons learned by educational researchers in the earlier standards reform movements (e.g., Cohen, 1990). Like their earlier predecessors (i.e., lists of standards), lists of moves, when presented and prescribed as a solution for the problems of practice, tend to overpromise and underdeliver, distorting practice while projecting an unwarranted authority.

The observations outlined by Chapin help to illustrate Berliner's (2004) description of the developmental progression of teachers. Berliner categorizes teachers as novices and advanced beginners, noting that novices often see teaching in terms of absolutes, while advanced beginners begin to recognize and respond to the nuanced contexts of teaching situations. The struggles with using talk moves robotically can be seen as characteristic of novice teachers, who are still developing their ability to adapt their practices to different contexts.

To support teachers to move away from these kinds of robotic performance, it is crucial to support them in gaining better "understanding of and reasoning about practice" which requires both "a language capable of finer distinctions and a stance aimed less at evaluation" (p. 9, Chazan & Ball, 1999). Milewski & Strickland (2020) offer a framework categorizing responding moves into functional categories without evaluating them as more or less "productive." Their framework categories responding moves into two layers of independent functional categories (i.e., Response vs. Rejoinder, Support vs. Confront vs. Invite) which can be composed into the following six functional categories for teachers' responding moves:

- **Supporting-response:** curtail the interaction while supporting the contribution
- **Confronting-response:** curtail the interaction while confronting the contribution
- **Invitational-response:** curtail the interaction while inviting other students to respond to the contribution

- **Supporting-rejoinder:** prolong the interaction while supporting the contribution
- **Confronting-rejoinder:** prolong the interaction while confronting the contribution
- **Invitational-rejoinder:** prolong the interaction while inviting other students to respond to the contribution

More recently, we have been exploring an additional layer—subject-generic and subject-specific moves. By **subject-generic** we refer to responding moves that can be used across a variety of content areas and courses without adaptation (e.g., What do others think); while **subject-specific** refers to responding moves that do make adaptations to various instructional situations (e.g., What do others think about the way he constructed the center of that circle?).

When developing the simulations, we wondered whether gaining a greater understanding for the functional categorization of discursive moves would improve teachers' practical reasoning about the practice of responding. Furthermore, we wondered how such shifts might play a role in shaping teachers' instructional choices. To investigate, we needed to develop measures that moved beyond a categorization of teachers' moves into evaluative binaries and toward measures that attend to the shifts in the kind of practical reasoning teachers' use when providing an account for their decisions. In the background section, we provide more details about the simulation, paying particular focus to the ways we provided opportunities for users to both simulate and receive feedback on the practice of responding to students' mathematical contributions. In the methods section, we describe the measures we developed for assessing the potential of that simulation for supporting teachers' professional growth—that we define as participants gaining a better understanding of the functional role of discursive moves they elect to when responding to student contributions (e.g., when using a rejoinder, they indicate they do so because they want to prolong the discussion of an idea). In short, we describe methods used to pursue the following lines of inquiry: *How does a functional approach to responding, featuring a measure that avoids categorizing instructional moves according to evaluative binary contribute to assessing changes in instructional practices when responding to students' mathematical contributions within the context of an online teaching simulation? Additionally, what insights can be gleaned concerning the alignment of participants' justifications with the underlying linguistic theories embedded in the designed feedback during the responding intervention?*

Background

This paper is part of ongoing efforts to develop and evaluate online simulations supporting secondary mathematics teachers in facilitating problem-based lessons. We created four simulations around the Tangent Segments Problem, which asks students to find a circle tangent to two intersecting lines, culminating in the Tangent Segments Theorem[1]. Each simulation immerses teachers in the complexities of fostering discussions on students' work. Two simulations cover the entire lesson, while two focus on specific phases, offering 'soft' feedback.

The ***Responding to student work*** simulation, a targeted intervention, aimed to impact teachers' reasoning about responding. In this simulation, users encounter:

- 4 storyboarded scenarios with student sharing work publicly (see Figure 1a),

[1] The Tangent Segments Theorem states that two intersecting lines are tangent to a circle if and only if the points of tangency are equidistant from the point of intersection of the lines

- A panel of 18 possible responding moves with differing underlying functions, requiring users to indicate at least one they like or dislike,
    - The underlying functions of the moves are as follows: 9 Responses, 9 Rejoinders; among which 6 Support, 6 Confront, and 6 Invite; and among which 12 are Subject-specific and 6 are Subject-generic.
- 3 prompts for participants to articulate their goals for handling each student contribution, followed by a curated subset of moves that fulfill their goals (see Figure 1b).
- The original panel of choices of 18 responding moves is finally provided without delineation and participants are asked to choose a way to respond, which they can edit.

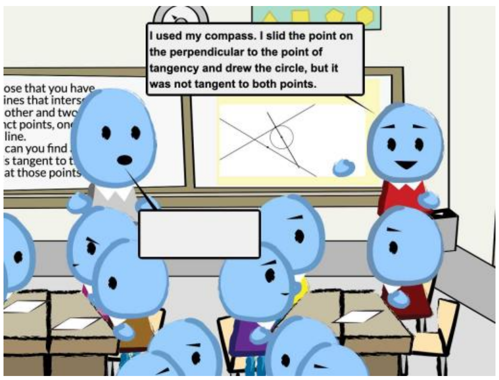

Figure 1a. A scenario embedded in the *Responding to student work* simulation

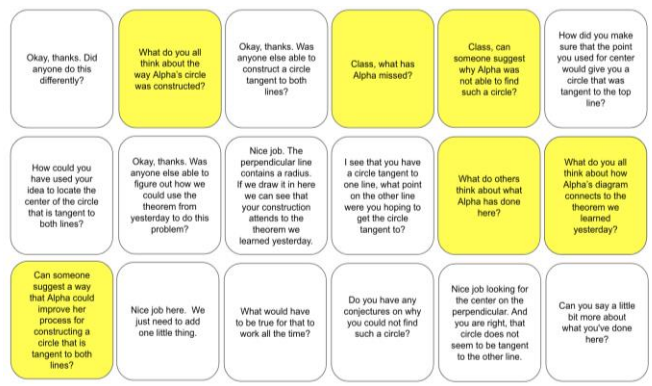

Figure 1b. A panel of 18 possible responding moves, after user expresses a preference for having the class respond, as opposed to responding themselves

Users first select at least one move from the panel that they like or dislike. The second, third, and fourth interactions with the panel provide curated moves based on users' goals (see Figure 1b). This curation is driven by 3 prompts where users indicate (1) whether they prefer to respond themselves or ask the class to respond, (2) whether they prefer to prolong or curtail the discussion, and (3) whether they prefer to support or confront the student's ideas. The fifth interaction asks users to select a single move closest to their preferred response, with the option to revise. These questions along with the curate moves offer participants explicit guidance on linguistic distinctions from the Milewski/Strickland framework but deliberately omitted explicit attention to subject-specific and subject-generic distinctions to investigate the varied impact of feedback on functional distinctions.

The ***Teaching the lesson with student participation*** simulation, serving as a pre-post assessment, presented 12 opportunities for participant responses to students' contributions. Participants shared responses to students' contributions and their reasons for those responses in open-ended boxes. We compared pre-post responses to assess our approach's potential for insights into changes resulting from the ***Responding to student work*** simulation.

Methods and Analysis

The analyzed sample consists of responses collected during the integration of simulations into a Western University methods course in Fall 2023. Our focus is on 10 prospective teachers who completed all simulations (4 males, 6 females; 8 White, 2 BIPOC). These simulations were undertaken individually as homework assignments in the initial half of the methods course, with data collection confined to their simulation performance and subsequent reflective assignments. This study delves into prospective teachers' responses in the pre- and post-assessment, where they engaged in a virtual teaching exercise for a full lesson.

We examined the alignment between the responding moves participants chose and their justifications. We analyzed responses to *"How would you like to respond to the contribution made by…"* (prompt 1) to discern shifts in types of responding moves. For these responses, we used the Milewski/Strickland (2020) framework along with newly added distinctions of subject-generic and subject-specific moves. To further assess the simulation's potential in aiding teachers' understanding of the function of different responding moves, we introduced an additional measure. We sought evidence in participants' responses to follow-up prompts: *"What do you anticipate could happen next?"* (prompt 2) and *"Please explain why you chose to respond to…in the way you did and at this moment (Optional)"* (prompt 3). In our analysis, we looked for participants' recognition of the function of their responding move from prompt 1, ensuring alignment with the functional categorization.

For example, one participant responded to prompt 1 with, *"Why would the compass being smaller make the circle fit better? How would we be sure that it still hit the line and didn't get too small?"* This was coded as a subject-specific confronting-rejoinder move. For prompts 2 and 3, the participant said, "*More discussion about what we need to know about the points. The compass size isn't the problem here it's the location of the two points and that's what I want the conversation to skew towards.*" This was coded as *matching* for subject-specific and linguistic functions because it anticipated a discussion about the location of the points and aligned with the rejoinder and confronting categories by anticipating a prolonged interaction in which the participant desired some problematizing regarding the location of the two points.

Table 1 presents additional coded responses to illustrate our methodological approach. While most responses were coded as matches or non-matches, some instances proved less straightforward. Occasionally, participants did not explicitly address one or more functional categories in their reactions to prompts 2 and 3. In such cases, the data was marked as non-explicit, and this did not count towards the total justifications. We report on shifts in teachers' responding practices and their inclination to provide anticipations and justifications that align with the theories underlying the simulation's design in the findings section.

Table 1: Sample coding for prompts 1, 2, and 3 in the pre- and post-assessments

Prompt 1 Responses	Coding for Prompt 1 Responses	Prompts 2 & 3 Responses	Coding for Prompt 2/3 Responses	Final Code

ask what the class thinks about what was said.	Rejoinder Invite Generic	I would guess because the two circles are both close there would be a majority agreeance [sic] but some might say that it is fine as long as the circle is tangent to the point. I prefer to let the class think about these ideas rather than I giving [sic] them all the answers.	Match rejoinder Match invite Non match generic	Match linguistic Non-Match subject specific
I agree, Iota made some good progress. What do you notice from their work that makes you say that?	Rejoinder Support Generic	I anticipate the student will talk about the arc. I want to hear more and have a contribution from that student deeper than a "good job" while also assessing the student's understanding in the moment.	Match rejoinder Match support Non match generic	Match linguistic Non-Match subject specific
That's great work, Omicron! That angle bisector ensures that our circle's center is equidistant from both sides, so it "fits" inside.	Response Support Subject Specific	I anticipate the class will have more comments maybe on the distance of the points not being equal. I want to ensure Omicron feels validated especially after the comment from their classmate and that the class understands the value of Omicron's contribution.	Non match Response Match support Match SS	Non-Match linguistic Match subject specific
Mu had a similar drawing to Nu, but I like the incorporation of the angle bisector and perpendicular line.	Response Support Subject Specific	I anticipate they will realize they need to edit their picture so the circle is smaller and "fits better."	Match Response Non match support Match SS	Non-Match linguistic

Results

Our methodological approach reveals the potential of simulations to support a multifaceted view of teachers' professional growth. The first six rows of Table 2 demonstrate discernible shifts in participants' instructional moves before and after the intervention. In the pre-assessment, participants employed few rejoinders (29%), which increased to 62% in the post-assessment. Similarly, there was a shift towards supportive moves, with fewer supportive moves used in the pre-assessment (32%) compared to a majority in the post-assessment (67%). Additionally, the use of invitational moves increased from 10% to 16%.

To illustrate these shifts, we share pre-post data from one of the participants, whom we will call Max Zimmerman. The first time through the simulator, Max responded to one of the student contributions with a closed confront, saying,

> *"I would talk about how this is a great start and using perpendicular lines is a good way to think about it. I would then agree with the class and say you cannot just move points so it isn't quite there, but it is getting close."*

In his final time through the simulator, Max responded to one of the student contributions with an open invitation, saying,

> *"I would ask if anyone notices anything about how the circle is made using the points as we have seen examples where the initial point given appears to not work."*

The comparison between Max's pre and post simulation responses to students' contributions illustrates how participants evolved from using more closed and confronting discursive moves to more open and collaborative ones.

Apart from these shifts, we also report there was no substantial change in participants' use of subject-specific moves. Recall, we did not distinguish these moves in terms of more and less productive and even now, we would be hard pressed to make any claims indicating whether these shifts represent improved practice, as our focus is not on changing teachers' behavior, but rather changing teachers' ability to offer more reasonable justifications for their practices.

The analysis of participants' justifications presents a more nuanced picture. While participants did not offer more coherent justifications related to the subject-specific nature of their responding moves, they showed a marked increase in awareness of the linguistic functions of their responding moves. None of the participants' pre-assessment responses aligned with their move in terms of both linguistic functions (rejoinder/response, support/confront/invite), while 71% of post-assessment responses aligned with both functions. Furthermore, 24% of responses had no matches for either function in the pre-assessment, contrasting with 0% of responses in the post-assessment responses had no matches.

To illustrate what this trend looked like in the data, we return to participant Max. Recall, in his first time through the simulator, Max used a closed confront to respond to a student contribution. He justified that decision by stating, *"Encouraging students is always a good idea."* This suggests that Max did not quite understand the ways that his responding move was likely to be interpreted as confronting, rather than supporting, the student's contribution. He offered a second justification by saying, *"Maybe there is a question or two about why using perpendicular lines will help."* Again, at this point, Max seems to not yet understand that because he elected to use an evaluative move, the subsequent interaction about the student's contribution is more likely to be curtailed, rather than prolonged.

In his final time through the simulator, recall Max responded to a student contribution with an open invitation. This time, he offered a more realistic anticipation of that decision stating, *"Hopefully some students begin to notice the pattern that a and b must be the same distance"* and *"Because I think they showed they know what to do and a slight understanding of where these points need to be in order for them to work"*. Unlike the justifications he offered in the first time through the simulation, this time, Max demonstrates an accurate anticipation that his move will lead to students taking the next discursive moves and will do so in a way that promotes a prolonged discussion of the original contribution.

The nuanced shift in participants' justifications underscores the complexity in participants' justifications for their responding moves, highlighting the intricate nature of linguistic functions in the context of the simulation. To be clear, we consider the shift in matched justifications, rather than the shift in types of teachers' responses, to be a more reasonable indicator of professional growth. This approach avoids making value-laden judgments about which responses are good or bad and instead focuses on whether teachers can offer justifications for their decisions that align with the expected outcomes of their chosen actions.

Table 2: Changes in subject-specific moves and justifications across the pre-post assessment (n=sets of responses to prompts 1-3)

		Pre (n=67)	Post (n=61)
Types of Moves	Subject-specific	52 (79%)	49 (79%)
	Response	48 (72%)	22 (36%)
	Rejoinder	19 (29%)	39 (62%)
	Support	21 (31%)	36 (59%)
	Confront	40 (60%)	15 (25%)
	Invite	6 (9%)	10 (16%)
Types of explicit justifications	Matched on Subject-Specific	38 (60%)	34 (53%)
	Unmatched on Subject-Specific	5 (7%)	5 (8%)
	Matched on linguistic functions	0 (0%)	44 (72%)
	Unmatched on linguistic functions	67 (100%)	17 (29%)

Conclusions

The two outcomes concerning participants' ability to align their justifications with the moves they crafted were as expected. We anticipated that given the explicit feedback provided to them on these functions, participants would enhance their proficiency in offering justifications aligning with the linguistic function of their responding moves. Conversely, the absence of shifts in the participants' propensity to provide justifications matching the subject-specific nature of responding moves was predictable since participants did not receive feedback on these aspects.

The results indicating shifts in the function of participants' instructional moves were surprising, as the intervention did not guide preferred responding moves. Nevertheless, these shifts are intriguing, and we contemplate whether similar results would emerge with a larger sample of teachers. The illustration of the evolution in Max's decision making and justifications for his discursive moves demonstrates the potential of the simulator for enhancing teachers' professional growth. Replicating this study on a larger scale with consistent results could strongly support Chazan and Ball's (1999) hypothesis, emphasizing the importance of "a language capable of finer distinctions and a stance aimed less at evaluation" to support teacher inquiry and professional growth. Future research will explore this avenue using the simulation to collect data from a national sample of teachers and employing the outlined methods to investigate whether the simulations help practicing teachers make their responding moves more purposeful.

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READING METHODOLOGICAL CONTOURS IN MATHEMATICS EDUCATION RESEARCH: THE CASE OF NARRATIVE INQUIRY

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In this paper, we begin by mapping major methodological contours in mathematics education research before delving into one methodology: narrative inquiry. We trace the use of narrative inquiry within the disciplinary boundaries of mathematics education noting significantly that it has been underutilized relative to other educational research fields. In order to highlight the potential of the methodology, we share two cases of its use outside disciplinary boundaries. In the first case, 'analysis of narratives' is used to complicate and contextualize mathematics teacher beliefs. In the second case, 'narrative analysis' is used to co-construct voices representing a spectrum of teacher responses to mathematics curriculum reform.

Keywords: Research methods, systemic change, teacher beliefs

Research in mathematics education has a history of maintaining conventional research methodologies and theories; being borne out of the fields of mathematics and psychology (Kilpatrick, 2020) may impact this tendency. When the field has begun to embrace new ways of doing mathematics education research or exploring different ideas as to what might count as mathematics education research (MER) there has been pushback. For example, in a 2010 *Journal for Research in Mathematics Education* (JRME) editorial, Heid asked "Where's the Math (in mathematics education research)?" She contended that the articles published in JRME should have mathematics as an "essential component rather than being a backdrop for another area of inquiry" (p. 103), and her perception was that many submissions had gone astray of this focus. In a later JRME editorial, "A Future Vision of Mathematics Education Research: Blurring the Boundaries of Research and Practice to Address Teachers' Problems," Cai and colleagues (2017) outline a future where MERs and teachers work together to develop, implement, and refine mathematics tasks prior to placing them in a professional knowledge repository: their vision of bridging the theory-to-practice gap endorsed lesson study as a methodology. While this vision indeed bridges theory and practice, it significantly narrows what counts as MER and forecloses many alternative approaches. While we agree with Heid that mathematics should be central, we also ask, "Where is the research (in mathematics education research)?" While mathematics education researchers (MERs) need to center diverse concepts and ideas *within mathematics*, we find it also important for MERs to explore and consider multiple and diverse educational research *methodologies*. Then, complementing Cai's vision for the future of lesson study, we invite mathematics education researchers to consider another methodology to bridge the theory-to-practice gap, another methodology less used than lesson study: narrative inquiry. Our vision for MER is one in which diverse research methodologies, with narrative inquiry (NI) as one example, broaden 'what counts' as MER, what is studied within mathematics, and approaches to research. The following research questions guide our inquiry:

1. How have narrative research methodologies been used in mathematics education research in the last 50 years?

2. In what ways does the narrative inquiry that is being done in MER vary across publication outlets?
3. What unique insights or benefits might narrative inquiry offer to MER?

To answer the first two research questions, we use a combination of (i) reading the *Mathematics Education Atlas* maps (Dubbs, 2021) for the JRME, *Educational Studies in Mathematics (*ESM*)*, and *For the Learning of Mathematics (*FLM*)* and (ii) systematic literature review. The findings of these two analyses together, then, describe the uptake of narrative research methods across time (RQ1) and across publication outlets (RQ2). The details of our analysis and findings are elaborated in the sections that follow. In the discussion, we address RQ3 by elaborating Chapman's (2020) contention that "the field of mathematics education could benefit from more attention to narrative as a way of knowing and narrative analysis" (p. 25). We turn next to an explicit discussion of our theoretical framing and research method before presenting our findings and recommendations.

Theoretical Framing: Discourses and Knowledge Production

Discourses are the taken for-granted narratives which structure our communal understanding of the field. Parks and Schmeichel (2012) argued that the dominant discourses of MER made it difficult to, not only publish research on race and ethnicity in dominant MER journals, but to even think of race and ethnicity together with mathematics education. In a similar way, discourses about MER make it near-impossible to consider MER from a narrative perspective. Thus, as Parks and Schmeichel described the impossibility of considering race and ethnicity in MER, here, we describe the discourses that establish which methods are thought easily and which methods are impossible to be thought. In total, then, these discourses not only have the effect of foreclosing certain research foci (e.g., race and ethnicity) but also certain research methods (e.g., NI).

While our larger project investigates the uptake of the five approaches to qualitative inquiry outlined in Creswell's (2013) *Qualitative Inquiry and Research Design: Choosing among Five Approaches*, in this paper we focus on the specific case of narrative inquiry. As the title suggests, Creswell presented five approaches within qualitative inquiry and, for each, named whose work he relied on to structure each approach: narrative research (c.f., Clandinin & Connolly, 2000; Pinegar & Daynes, 2007; Polkinghorne, 1995), phenomenological research (c.f., Moustakas, 1994; van Manen 1990), grounded theory research (c.f., Corbin and Strauss, 2007; Glaser & Strauss, 1967; Strauss & Corbin, 1990, 1998), ethnographic research (c.f., Van Maanen, 1988; Wocott, 2008), and case study research (c.f., Stake, 2005; Yin, 2009;).

We chose narrative inquiry as our focus because narrative inquiry asks the researcher to come alongside their participants to create more complex and nuanced understandings: this resonates with the future envisioned in 2017 by the JRME editorials. Further, Chapman (2008, 2020) and others (Beattie, 2006; Huber, 2013; Milner, 2007) have established the utility of NI in teacher education both as a research methodology and as a pedagogical tool. As we will show, while narrative inquiry methodology was tentatively being accepted in the 'top-tier' MER journals, it was being utilized more widely outside of MER and had long been an accepted methodology in literacy research. *The Handbook of Reading Research* (Kamil et al., 2001) included an entire chapter dedicated to narrative methods (Alverman, 2001). Subsequently, in 2007, the top journal of qualitative research, *Qualitative Inquiry*, dedicated a special issue (*13*:4) to narrative methods. In addition, *Educational Researcher* composed a special issue (*38*:8) in 2009 and the *Journal of*

Educational Research followed in 2010 (*103*:2). Narrative methods have been clearly established within the field of educational research and its subfields despite MER's relatively slow uptake.

Narrative Inquiry as a Way of Knowing

> Narrative is a scheme by means of which human beings give meaning to their experience of temporality and personal actions... [Narrative] provides a framework for understanding the past events of one's life and for planning future actions. It is the primary scheme by means of which human existence is rendered meaningful. (Polkinghorne, 1988, p. 11).

As indicated by Creswell, Polkinghorne (1988, 1995) is a name often associated with narrative inquiry (NI) as a research method. Within MER writ large and mathematics teacher education in particular, Chapman (2008, 2020)–building on the work of Polkinghorne and Clandinin and Connelly (2000)–is centrally positioned when discussing NI. The purpose of this section is to *briefly* introduce NI as a research method; for a more detailed discussion, we refer the reader to Chapman's (2008) "Narratives in Mathematics Teacher Education," Clandinin and Connelly (2000), and Kim (2016).

For Polkinghorne (1988), narrative is a primary scheme for making meaning of life experiences. Chapman (2008) described five ways that those researchers who accept Polkinghorne's premise have used narratives and NI education research: (i) as a research method, (ii) as a means of collecting data, (iii) as an object of analysis, (iv) as a tool in professional development or teacher education, and (v) as a basis for reflective thinking. Later, Chapman (2020) classified these five ways into the two categories of NI identified by Polkinghorne (1995): narrative analysis (the first way) and analysis of narratives (ways two through five).

Using these two perspectives as a frame, then, our present analysis seeks to understand the ways that NI has been taken up in MER. Those MERs that have taken up NI as a research method will have explicitly named NI as their method and/or cited NI methodologists (e.g., Polkinghorne, Connelly & Clandinin, or Chapman).

Research Method: Mathematics Education Atlas and Systematic Literature Review

Our review of NI literature within MER consisted of two complementary approaches: the Mathematics Education Atlas (henceforth, the Atlas; Dubbs, 2021) and systematic literature review. The Atlas includes a complete mapping of published research and citation relationships in the JRME (1970-2019), ESM (2010-2016), and FLM (2010-2017). We used the database to quickly identify which published articles in these journals cited NI as a method. Since the Atlas' records are limited to these journals and end with research in the late 2010s, we complement the map data with a systematic review of NI published in other research journals (and in ESM since 2017 and in FLM since 2016). Together these approaches comprise a breadth of MER published within disciplinary journals (i.e., JRME, ESM, and FLM). In addition, we highlight two cases from non-mathematics education research journals.

We used the Atlas to identify those articles that cite Clandinin, Connelly, Polkinghorne, and other authors that have developed the theoretical basis for NI as a methodology. In total, our reading of the maps returned one relevant article within the JRME (Wager, 2014), two relevant articles within ESM (Foote & Bartell, 2011; Nardi, 2016), and no relevant articles from FLM. Our definition of *relevance* was 'articles that named and used NI as a method.' For example, other authors (e.g., Remillard & Bryans, 2004) cited Clandinin and Connelly's work on 'teacher as curriculum maker' but did not undertake NI as a research method. We exhaustively discuss the identified articles in the findings section.

Due to the temporal cutoffs of the Atlas maps, we complement our reading of the maps with a systematic literature review (Petticrew & Roberts, 2008). Since our interest is in capturing research that is explicit in naming NI as a method, we searched for articles that cited Clandinin, Connelly, Polkinghorne, and/or Chapman within JRME for articles since 2020, ESM for articles pre-2010 and post-2017, and FLM for articles pre-2010 and post-2016. This systematic search returned no additional articles from JRME, one pre-2010 article in ESM (Lloyd, 2006), and one post-2016 article in FLM (Chapman, 2020). Together with the results from the maps, four of these articles will be discussed in our Findings section related to RQ1. We chose not to separately discuss Chapman's article since it is about NI as a research method instead of an article using NI as a method; instead, we include insights from Chapman through this article to emphasize our affinities with and departures from Chapman's work.

Finally, we chose two articles published outside the disciplinary mathematics education journals that are representative of the two categories identified by Chapman and Polkinghorne: 'analysis of narrative' and 'narrative analysis.' Drake and Sherin's (2006) article in *Curriculum Inquiry* was chosen to represent the first category while Martinie and colleagues (2016) article in *Journal of Educational Research* represents the second. Together, these articles provide the evidence to answer RQ2. We turn now to our detailed discussion of our findings.

Findings

As mentioned above, our use of the Atlas maps and systematic review together returned four articles from the high-quality, disciplinary MER journals included in this review: Foote and Bartell (2011), Lloyd (2006), Nardi (2016), and Wager (2014). This dearth of research evidence, as elaborated in the next subsection, upholds the discursive construction (Parks & Schmeichel, 2012) of NI as an impossible method for undertaking MER. The systematic search of ERIC and Google Scholar, however, shows that NI is indeed a possible method for undertaking MER and we discuss the representative articles next (i.e., Drake & Sherin, 2006; Martinie et al., 2016).

An Impossible Method, Little Uptake within Disciplinary Boundaries

Here, we revisit our research questions and explicitly elaborate our findings in the context of MER and its potential futures. As Parks and Schmeichel (2012) argued, the persistence of particular ideas and relative neglect of others makes it easy to think of particular theories and methods (e.g., lesson study) while others remain difficult to consider (i.e., NI). Indeed, only 15 articles within the three maps–11 within the JRME 2010s map and 4 within ESM–cite primary NI sources. In contrast to both JRME and ESM, in the FLM map, there are no articles that cite Polkinghorne, Clandinin and Connelly, or other authors that have developed NI as a method.

Narrative inquiry as method in JRME. In the JRME maps, there are no articles that cite Polkinghorne while ten articles cite Clandinin and Connelly. Of these ten, nine articles (ranging from 1989 to 2015) cite Clandinin and Connelly's work on 'teacher as curriculum maker' (e.g., 1986, 1992): Averill et al. (2009); Brown et al. (2009); Fraivillig et al. (1999); Gresalfi & Cobb (2011); Huntley (2009); Leinhardt (1989); Oonk & Verloop (2015); Remillard & Bryans (2004); Tarr et al. (2008). The singular article to use NI as a method is Wager's (2014) "Noticing Children's Participation: Insights Into Teacher Positionality Toward Equitable Mathematics Pedagogy"; this article will be discussed in detail shortly.

It is notable, as Creswell explicitly named Clandinin and Connelly as qualitative researchers, that only one article in this group named their study as qualitative research. Tarr (2008) reported their research as a "quasi-experimental design" that "employed both qualitative and quantitative methods" (p. 252). They explained that the qualitative methods were used to 'characterize' curriculum implementation and support the relationships between curriculum, teaching, and learning outcomes ascertained by the hierarchical linear modeling methodology. The first example of a citation that referred to narrative was Oonk and colleagues (2015) in which they explained, "practical knowledge often develops from stories of teaching practice, it is considered to have a narrative character (Clandinin & Connelly, 1996; Lin, 2002)" (p. 561). While we were looking for the influence that Clandinin and Connelly had on the use of NI as a method in MER, we theorize that their work on teachers as active creators of curriculum in these manuscripts helped to shift the positioning of teachers as enactors of fixed curricula to actors who bring stories and experiences that interact and influence curriculum and curricular decision making. This recentering of teachers and their experiences as important to the ways that curriculum is enacted, in our view, laid the groundwork for the entry of NI into the field.

Wager (2014), the only article to explicitly use narrative as a method, looked to the theory of "*identity in practice* (Holland et al, 1998)" (p. 313) in her study of teacher noticing. She describes her use of the theory to "understand how lived experiences shape identity development" (p. 317) including how narratives and discourses position us relative to ourselves and others. Wager acknowledges the importance of teachers' lived experiences and created narratives to "identify a storyline and position the teachers relative to equitable mathematics pedagogy" (p. 319). In her methods section, Wager described her research as an "empirical study" that used "qualitative and, to a lesser extent, quantitative methods to examine narrative data to uncover teachers' actions as evidence of noticing participation" (p. 320). Wager cited Clandinin and Connelly (2000) when describing the "initial narratives" (p. 325) that she wrote to describe the teachers' equity experiences and perspectives. She described these narratives as "inelegant, brief descriptions that were another source of data" (p. 325). Here we see Wager moving from 'narrative analysis' to 'analysis of narratives.' Further based on additional data, Wager refined the narratives to "incorporate the comments made in their reflections" to provide "a snapshot of my interpretation of the teachers' identity… based on the stories they shared (Wortham, 2001)" (p. 325). She then analyzed the narratives to consider the storylines of teacher noticing. While Wager cited Clandinin and Connelly's (2000) seminal *Narrative Inquiry: Experience and Story in Qualitative Research* and used their methods, Wager did not explicitly name her methodology as NI.

Narrative inquiry as method in ESM. In the ESM map, there are no articles that cite Polkinghorne and only four articles that cite Clandinin and Connelly: Darragh (2016); Foote & Bartell (2011); Nardi (2016); and Zazkis & Koichu (2015). Of these, Darragh's review of identity research cited Clandinin and Connelly in reference to the concept of identity and Zazkis and Koichu (2015) undertook a duoethnography that cited Clandinin and Connelly cursorily as an example of the "diverse narrative approaches in qualitative research" (p. 164). The two articles by Foote and Bartell (2011) and Nardi (2016), together with the article by Lloyd (2006), undertook NI as a method and are discussed now.

Lloyd (2006) demonstrated the potential of analysis of narratives as a way "to offer a window into preservice teachers' ideas and sense-making about certain mathematics classroom events"

(p. 81) and to initiate change in practice and thinking. In Lloyd's study, preservice teachers were invited to write fictional stories about mathematics classrooms in a methods course by taking up two opposing perspectives on a response to reform. The purpose was to encourage PSTs to imagine situations in the role of a teacher versus more familiar roles as student or observer. The teachers' stories offered "a wealth of information about their individual identities and classroom experiences" (p. 58) and served as "powerful catalysts for change and development" (p. 59). The researchers theorized that the fictional accounts in particular were catalysts for change because they explicitly asked the teachers to create images of themselves and others outside of their actual experiences. The teachers' fictional accounts were analyzed structurally and thematically. This study highlighted the potential of the use of NI both as a pedagogical tool and a research tool to identify preservice teachers' views and anticipate specific responses to common classroom dilemmas.

In a study exploring the narratives pre- and post-doctoral emergent scholars produced regarding their positionality on equity and diversity, Foote and Bartell (2011), explicitly named "two main forms of gathering stories and then analyzing them within a narrative framework" (p. 49). The first form, in which researchers live alongside participants to gather stories of lived experience, aligns with Polkinghorne's narrative analysis while the second, in which stories are gathered from participants "and the narrative data is analyzed for common themes, metaphors, plotlines, and so on to identify general themes or concepts" (Clandinin, 2007, p. xv, as cited in Foote & Bartell, 2011), aligned with Polkinghorne's analysis of narratives. It is within the latter, the analysis of narratives that Foote and Bartell situated their work. They undertook 26 life-story interviews, "semi-structured interviews designed to elicit accounts of the life stories that the participants felt had influenced their interest both in issues of equity and diversity and in mathematics teaching and learning…[probing] for experiences that may have happened during different periods of the life cycle such as in childhood, young adulthood, and the present time" (p. 51). Through this work, Foote & Bartell found three themes within the narratives–"'othering experiences,' 'bearing witness experiences,' and 'orienting experiences'" (p. 52)–that influenced emergent scholars' choice to engage in equity-focused research. These researchers, however, also identified two tensions that the emergent scholars described in their narratives: "(a) a tension between research seemingly more focused on mathematics or research seemingly more focused on equity, and (b) a tension between research grappling with complex and important theoretical issues and urgent, practice-based considerations" (p. 52). The way these researchers gathered their data narratives and the nature of their findings firmly places the work within the paradigm of 'analysis of narratives.' In this analysis, the researchers provided important findings that "[make] visible how the life experiences of some early career mathematics education scholars impact the stance they bring to their research and, in so doing, reaffirms arguments that socio-cultural perspectives of learning are important for mathematics education research" (p. 65). Furthermore, Foote and Bartell's focus on emergent scholars extended Lloyd's work to show that, in addition to serving as a research tool to identify preservice teachers' views and common classroom dilemmas, NI can be employed to identify emergent scholars' views and common research dilemmas.

Nardi's (2015) work with narrative fits within the broad concept of narrative analysis. Nardi justified the use of re-storying, a process of constructing a "story from the original data (Ollerenshaw & Creswell, 2002, p. 330)" (p. 363) that "assimilates the multiplicity of voices (researchers' and research participants' as well as amongst the participants themselves) without suppressing or eliminating this multiplicity" (p. 364). Nardi's project included interviewing mathematicians and creating narrative accounts that were co-constructed from transcripts of multiple mathematicians and mathematics education researchers to provide insight into university mathematics pedagogy. She argued that the stories she created are a "potent communicative tool which can be deployed by two communities—mathematics and mathematics education research—which often find communication challenging" (p. 364). The stories Nardi wrote provided a common language to discuss teaching and learning between the two communities. She described this method as communicating the *substance* of the research knowledge in *forms* that are amenable to both communities. The use of narrative allowed her to produce a "new form of knowledge about mathematical pedagogy co-constructed by members of two often separated communities …relocated to a novel *third space* which welcomes *non-deficit, non-prescriptive, context-specific, example-centered* and *mathematically focused* discourses" (p. 373-374). Thus, Nardi extended NI beyond *understanding* preservice teachers' (e.g., Lloyd, 2006) and emergent scholars' (e.g., Foote & Bartell, 2011) views and provided a powerful example of the ways that NI can open up other possibilities on *knowing*, on knowledge production (co-constructed multiplicative accounts), and communication across fields/communities (mathematics and MER).

A Possible Method, Differences in NI Uptake outside Disciplinary Boundaries (RQ2)

While NI has had slow uptake within the MER disciplinary journals, there has been MER that employed NI published in educational research journals outside the disciplinary boundaries. In this section, we present two examples of this research to explicate the potential of NI as methodology and to further emphasize the distinction between Polkinghorne's (1995) analysis of narratives and narrative analysis.

Case 1: Analysis of narrative. Drake and Sherin (2006) used NI to explore two teachers' responses to reform-oriented mathematics curriculum. The authors justified their use of narrative inquiry because it "allows for a contextualized and integrated understanding of teachers' beliefs, knowledge, and prior experiences" (p. 157). Drake and Sherin came to believe that understanding the teachers' narratives in relation to their experiences as learners and teachers of mathematics provided clarity to teachers' choices in implementing the curriculum. They stated that "narratives of identity both guide the actions of individuals and frame their interpretations of new information" (p. 158). While Drake and Sherin relied on narratives and specifically math story interviews, they did not employ narrative methodology writ large. Like Sfard and Prusak (2005) before them, they relied on narrative as a way to think about identity. The inclusion of narratives (past and present) in considering teachers' decisions about reform complicates both what teachers say about the reform and *who* a researcher believes the teacher *is*. As Drake and Sharin (2006) explained, narrative inquiry "allows for an understanding of teachers' beliefs not as isolated statements, but as interrelated ideas rooted in teachers' identities-their stories of themselves as learners and teachers" (p. 158). Drake and Sherin (2006) included narrative in the theoretical framing of their article-more than in the methodology. Within the methodology, they named 'mathematics story interviews' drawing on McAdams' (1993) life story interviews. They deemed their method of NI "different from and, in particular, more structured than, many other examples of narrative inquiry (Clandinin & Connelly, 2000)" (p. 162). Drake and Sherin's work with narrative falls into the first of Polkinghorne's (1995) two categories, they analyzed the mathematics life stories and coded to teacher beliefs within the stories. Drake and Sherin justified their use of NI because it allowed them to better understand the teachers' adaptation of curriculum by first considering "their identities as learners and teachers of mathematics, as revealed their mathematics stories" (p. 164). They concluded their manuscript with a recommendation for future research that connects "teachers' narrative mathematics identities using alternative methodologies" that are "more unstructured" (p. 184). Martinie et al. (2016) provides an example of a less structured NI methodology.

Case 2 Narrative analysis. Like Drake and Sherin (2006), Martinie et al. (2016) employed narrative methods to understand mathematics teachers' response to reform. Unlike Drake and Sherin, however, in this case, Martinie and colleagues went beyond 'analysis of narrative' methods and acknowledged narrative *methodology* in general, and 'narrative analysis' in particular, as an "important part of the education research landscape…[that] problematizes the positivist nature of knowledge as the objective and unitary way of knowing" (p. 659). They explained that narrative researchers use stories to "interrogate" and "reshape" (p. 659) dominant views and position their approach as being "alongside." In the decade between Drake and Sherin and Martinie et al., narrative methodologies were more broadly explicated and employed. While *analysis of narratives* was used by Drake and Sherin, Maritinie et al. employed *narrative analysis* to both understand math teachers' experience with common core implementation and to synthesize those experiences using narrative analysis that mingles the voices of the participants and "synthesize[s] the fragmented data and (re)construct[s] them into a coherent story rather than separating them into different categories" (p. 660). The mingling of voices provides confidentiality and "attends to the ethical stance of narrative inquirers while interweaving our ontological, ethical commitments (Clandinin & Murphey, 2009)" (p. 660). In this case, the researchers created four voices that represented the teachers' various responses to adopting the reform (hardcore adopter, anxious adopter, cautious adopter, critical adopter). Rather than providing themes of teachers' responses to mathematics curriculum reform siloed into a monologue, this methodology presented four distinct voices that provided valuable insights into various stakeholders and resisted shutting down oppositional voices as less important forms of knowledge. With these four voices acknowledged, researchers, policymakers, and educational leaders can better address concerns, criticisms, and reservations to enact important change in mathematics education.

Discussion

By mapping the methodological contours of the field of mathematics education across time, we establish what has been done. We then use this sketch to emphasize the ways that this landscape and its boundaries limit what is seen as legitimate within the field, what is seen as proper methodologically and theoretically. To wit, we disrupt the present notion that 'much progress has been made in the adoption of diverse qualitative perspectives in mathematics education research' and imagine a more expansive future by making explicit the call to adopt NI methods. In response to our third research question, What might narrative inquiry offer mathematics education research?, we note the movement in the field of MER due to the positioning of teachers as creators of curriculum when Clandinin and Connelly's work was first taken up, we then point to how analysis of narratives allowed MERs to complicate and contextualize teacher beliefs (Drake & Sherin, 2006) and how narrative analysis created a third space for communication between mathematicians and MERs (Nardi, 2015) and 'voices' that represent a range of responses to mandated curriculum reform so that MERs might better understand how to work with teachers to enact change in mathematics education (Maritinie et al. 2016). In many ways, our findings echo those of Geller, Hernández, and Chapman (2013): "[narrative was] used mainly as a conveyer of teachers' knowledge and experiences and not as a narrative research methodology (which was what was proposed by Clandinin and Connelly, 2000)." We close by reflecting on the overall lack of specificity we found in discussions of methodological work within the MER that we reviewed. We encourage mathematics education researchers to be more methodologically (and theoretically) adventurous and to carefully describe and cite their methodological sources.

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EFFECTS OF STRUGGLE FIRST VERSUS CONCEPTS FIRST INSTRUCTION ON STUDENTS' MATH ACHIEVEMENT

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This study examines the efficacy of two types of instructional strategies - Explicit Attention to Concepts and Student Opportunity to Struggle - for improving middle grades mathematics achievement. We worked with 100 grades 6-8 mathematics teachers, providing professional development on implementing the strategies in their local contexts, then partnering to conduct a large-scale cluster crossover study on two ways of sequencing the strategies, Struggle First vs. Concepts First. Compared to a business-as-usual control group, we found positive effects of both types of instructional sequences on students' mathematics achievement, with greater positive effects for Concepts First than Struggle First. Interestingly, many teachers reported shifts in their beliefs about the strategies. The results highlight the potential for targeted professional development to support teachers' adaptation and use of mathematics instructional practices.

Keywords: Professional Development

Keeping in mind the tensions mathematics teachers face in their instructional choices (Stahnke et al, 2016), we conducted a research study with 6-8 mathematics teachers to study the effects of attempting to adapt new instructional strategies on students' mathematics achievement.

Background

The Explicit Attention to Concepts (EAC) and Student Opportunity to Struggle (SOS) framework focuses on two primary instructional strategies for enhancing mathematics education as articulated by Hiebert & Grouws (2007). Through a synthesis of research indicating positive effects across study design, teaching formats, and contexts, Hiebert and Grouws identified these clusters of instructional practices as supportive of students' conceptual understanding. Stein and colleagues (2017) built upon this work by operationalizing the measurement of the EAC and SOS constructs in teaching practice and examining their relationship to students' skill efficiency and conceptual understanding. Their observational study found that teaching practices with both high EAC and high SOS tended to have the largest estimated effects on students' skill efficiency and conceptual understanding, followed by in order by high EAC-low SOS practices, high SOS-low EAC, and low EAC-low SOS practices. That research, as well as studies that focus on similar constructs (Fennema & Romberg, 1999; Kapur, 2014; Loehr et al., 2014; Schwartz et al., 2011), has demonstrated positive effects of EAC and SOS instruction on students' math achievement. Nonetheless, there are questions about supporting teachers' implementation of EAC and SOS, including how EAC and SOS practices may differentially influence student learning, effective ways to order instructional practices aligned to EAC and SOS (c.f., Schwartz et al., 2011), and how the practices are taken up by teachers.

The 3-year *Researching the Order of Teaching* (ROOT) project, a $3 million research project sponsored by the National Science Foundation (Award No. 1907840) was designed to address some of the open questions about EAC and SOS in the context of middle grades mathematics settings. The ROOT project started in fall of 2019 with the recruitment of 100 Grade 6-8 mathematics teachers from 34 schools within 22 school districts in the western United States. Initially, the professional development activities were designed to engage participating teachers in exploring ways to adapt and implement EAC and SOS as instructional practices in their classrooms (Hughes et al., 2023).

A distinctive feature of the project was our objective to foster idea-sharing and provide support for implementation, without pushing for strict adherence to the researchers' interpretations of the EAC and SOS instructional practices. We view teachers as key stakeholders, experts in their local contexts, and co-producers of knowledge (Kieran et al., 2012). We included activities such as interacting with colleagues around the practice guide as a boundary object (Crawford et al, 2022), planning and implementing a small sequence of lessons, and reflecting on features of the strategies to support teachers' development of personalized routines and practices that fit their preferences, curriculum, and local context. The research questions were:

1. Strategy Efficacy: What are the effects (if any) on student math achievement when teachers are asked to enact EAC and SOS instructional strategies?
2. Sequencing Efficacy: What are the effects (if any) on student math achievement when teachers are asked to enact "Struggle First" or "Concepts First" instructional strategies?
3. Teacher Beliefs: After attempting both Struggle First and Concepts First instructional strategies, which strategies did teachers perceive to be most effective for promoting students' math achievement?

Methods

We recruited 100 grades 6-8 mathematics teachers to participate in the project, of which 99 completed the initial baseline data collection. The teachers all worked in public schools in a single western U.S. state, spread across 34 schools in 22 school districts. Nearly all worked in brick-and-mortar schools, though one teacher worked for a virtual public charter school. Teachers' mathematics instruction often spanned multiple grades (49 taught Grade 6, 44 taught Grade 7, and 44 taught Grade 8) and courses (37 taught one course, 48 taught two courses, and 12 taught three or more courses). The teachers worked in a variety of school settings, both in terms of students' socio-economic status (mean eligibility for federal free or reduced school lunch was 58%, SD = 21%) and locale type (31% rural, 69% suburban or small city). Teachers' demographics indicated substantial variability in mathematics teaching experience (mean = 9.8 years, SD = 7.4, Range = 1 to 32), and they primarily self-identified as female (77%) and white (96%). Educational attainment among the teachers was typically a bachelor's degree (57%), though 40% held a master's degree, and 2% held an Ed.S.

As an optional professional development activity for middle grades teachers, we planned for 20% attrition annually (expected n = 100 teachers in year 1, 80 in year 2, 64 in year 3). A global pandemic began disrupting participating schools during the spring of year 1, causing a shift to primarily remote activities during year 2 and increased attrition as teachers responded to their local crisis contexts. By year 3, the schools had primarily returned to in-person instruction (often with masks) and N = 58 teachers completed the planned research activities. We analyzed teacher-level factors (e.g., grade levels, district types, years of prior experience) to look for indications of differential attrition during the pandemic disruptions, but found no significant evidence of divergence from random attrition.

Research Design

In year 3 we partnered with teachers to implement a cluster crossover research design in which the teachers were divided randomly into two groups – Concepts First and Struggle First during the summer prior to the study. We randomized the assignment of teachers to conditions at the school level to ensure teachers could collaboratively plan with teachers in their building, with students clustered within their teachers. There were 28 teachers in the initial Concepts First group and 30 teachers in the initial Struggle First group. During the fall, the Concepts First group was asked to start each lesson with EAC instructional practices followed by SOS instructional practices, while the Struggle First group was asked to start their lesson with SOS instructional practices followed by EAC instructional practices. In the spring, the groups 'crossed-over', with each group of teachers asked to follow the other groups' sequencing instructions from the fall.

The professional development during years 1 and 2 of the project involved familiarizing teachers with EAC and SOS, as well as supporting their implementation of these practices in their classroom through a summer institute, online modules, and small-scale teaching studies each fall and spring with in-school support from dedicated math instructional specialists. The teaching studies were typically 2-3 weeks in length and focused on whether a particular EAC/SOS strategy was practically useful for supporting student learning. The design and implementation of the instructional unit was supported by one of three full-time instructional specialists working on the project who were directly assigned to the teachers by school. The level of support provided during the crossover study was dependent on teachers' request for support, with an average of $M = 21$ interactions (Range = 11 to 31) with specialists per teacher for an average duration of $M = 31$ hours (Range = 12 to 49).

Data Collection & Analysis

The primary data source was the state mathematics assessment data system, which was developed by a multi-state consortium for use in Grades 3 to 10. Nearly all students completed the exam each spring, which we used as pre and post scores. An interim comprehensive exam was administered during the winter between the "cross-over" of instructional strategies to obtain mid-year scores. We used demographic data for all students in the districts to generate a group of students from participating districts with non-participating teachers using a genetic algorithm to select a balanced sample of a control group with a similar demographic profile along key variables (district, grade level, sex, special-education status, and pre score). The final subsamples included $n_1 = 2202$ students in the Concepts First in fall arm, $n_2 = 1941$ students in the Struggle First in fall arm, and $n_3 = 4246$ students in the control group. The sample was approximately balanced by grade level (Grade 6 = 32%, Grade 7 = 35%, Grade 8 = 33%), with students' primarily reported as White (67%) or Hispanic/Latino (30%). Approximately, 10% of students met Special Education criteria and 11% were identified as English language learners. Demographic distributions were within 1% across the 3 subsamples.

For the two research questions about student achievement, we applied standard procedures for (ordinary least squares) multiple linear regression modeling. The outcome variable was students' gains during periods of instructional intervention, with pre-identified explanatory variables obtained from prior observational studies of student achievement, including students' grade level, study condition, race, and special education status. This included technical verification of model assumptions (e.g., independence, homoscedasticity, non-multicollinearity, omission of outliers, normal residuals) and summary reporting of model characteristics. The primary purpose of the regression models was estimates of effect sizes, relying on the crossover research design for claims of causal effects. For the third question, we visualized teachers' responses to brief survey questions during the final weeks of the crossover study about their pre and post beliefs regarding the relative effectiveness of the Concepts First and Struggle First.

Results

To assess Question 1, we applied multiple linear regression on the outcome variable of students' annual mathematics achievement, with explanatory variables of categorical grade level (6, 7, or 8), binary study condition (control or EAC/SOS), binary race (White or non-White), and binary Special Education eligibility (no or yes). This resulted in a significant model, $F(5, 7379) = 25.68$, $p < .01$, $R^2 = .02$. All of the explanatory variables had statistically significant coefficients, including the intercept = 4.3 ($t = .14$, $p = .03$), grade 7 = 13.6 ($t = 7.04$, $p < .01$), grade 8 = 5.5 ($t = 2.8$, $p = .01$), White = 6.7 ($t = 4.0$, $p < .01$), special education = -17.32 ($t = -6.1$, $p < .01$), and EAC/SOS condition = 7.0, ($t = 4.5$, $p < .01$). The coefficients indicate absolute differences in average annual achievement across the explanatory variables while holding all other variables at baseline levels (listed first above). In particular, holding all other variables at baseline, the model suggests students of teachers participating in the EAC/SOS study had 7.0 points (95% CI = 3.4 to 10.1) greater average mathematics achievement than comparable students of teachers engaged in business-as-usual instruction. For relative comparison, the average estimated effects of grade levels were 13.6 and 5.5 points (for Grades 7 & 8 versus Grade 6, respectively).

To assess Question 2, we applied multiple linear regression on the outcome variable of students' semesterly mathematics achievement, with explanatory variables of categorical grade level (6, 7, or 8), study condition (control, Struggle First, Concepts First), binary race (White or non-White), and binary Special Education eligibility (no or yes). This resulted in a significant model, $F(7, 10389) = 66.25$, $p < .01$, $R^2 = .04$. With the exception of White (estimate = 2.7, $t = 1.9$, $p = .06$), all the explanatory variables in the model had statistically significant coefficients, including the intercept = -15.0 ($t = -7.6$, $p < .01$), grade 7 = 9.9 ($t = 6.0$, $p < .01$), grade 8 = 4.2 ($t = 2.5$, $p = .01$), Spring semester = 27.4 ($t = 20.0$, $p < .01$), special education = -5.8 ($t = -2.4$, $p = .02$), Concepts First = 8.9, ($t = 5.6$, $p < .01$), and Struggle First = 4.5 ($t = 2.7$, $p = .01$). Similar to the other regression analysis, the estimated coefficients indicate absolute differences in average semesterly student achievement across the explanatory variables while holding all other variables at baseline levels. In particular, holding all other variables at baseline, the model suggests students of teachers attempting to enact Concepts First teaching had 8.9 points (95% CI = 5.6 to 12.2) greater average mathematics achievement per semester than comparable students of teachers engaged in business-as-usual instruction. This is approximately twice the estimated effect of Struggle First teaching, which was 4.5 points (95% CI = 1.2 to 7.8) per semester.

For Question 3, during the last few weeks of the crossover study, teachers reported their beliefs about the relative effectiveness of Struggle First and Concepts First. Responses suggested teachers' experiences led many to shift in their beliefs about which way of sequencing EAC/SOS instruction was more effective. For instance, 24 of the teachers reported shifting from believing Concepts First was more effective to believing Struggle First was more effective, compared to only 5 teachers who reported the reverse type of shift from Struggle First to Concepts First.

Conclusions

Overall, the findings support professional development structures that recognize the complex systems within which teachers are teaching, and that providing professional development around well-research strategies and providing autonomy around important aspects of implementation can result in increased student achievement.

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MANAGING PEDAGOGICAL DILEMMAS IN A CONTENT FOCUSED COACHING CONVERSATION

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We explored pedagogical dilemmas in a content focused coaching conversation. We explored how a coach managed the dual obligations of producing viable lesson artifacts while also advancing teacher thinking with respect to content and pedagogical knowledge. We termed these as instrumental and learning orientations, respectively. We describe a representative case selected from a larger sample to characterize: (a) the presence of these orientations, (b) the purposes each orientation served, and (c) how the orientations functioned in tandem. The results show that the coach found strategic moments to advance teacher thinking while fulfilling the instrumental obligation of producing lesson artifacts.

Keywords: In-service teacher education; teacher educators; professional development

Educators face the perpetual tension of conveying conventional disciplinary knowledge to learners while simultaneously eliciting and building from learners' idiosyncratic ways of thinking (Lampert, 1985). While these tensions have typically been explained in terms of teachers and students, there is a parallel to the work of teacher educators whose goal it is to help teachers understand content or pedagogy. We situate an exploration of these tensions in the context of content focused coaching (CFC) (West & Cameron, 2013). An important goal in content focused coaching is to support teachers' development of their content and pedagogical content knowledge (Ball et al., 2008), a goal that may be in tension with more instrumental goals of producing a set of viable lesson artifacts for the teacher to implement in an upcoming lesson. We frame this tension in terms of pedagogical dilemmas, which Windschitl (2002) describes as navigating between "honoring students' attempts to think for themselves while remaining faithful to accepted disciplinary ideas" (p. 133). We translate the notion of pedagogical dilemmas of teachers to those of coaches, a parallelism termed *lifting* (e.g., Prediger et al., 2022) in which experiences from teaching are applied to the facilitation of professional development, including coaching. We posit that the knowledge required to teach is a form of disciplinary knowledge, involving both content and pedagogical knowledge (cf. Ball et al., 2008; Shulman, 1986), and that coaching is intended to develop those forms of knowledge.

Content Focused Coaching

The coaching model employed in our study was CFC (West & Cameron, 2013), which stands in contrast to instructional coaching or cognitive coaching. CFC is distinguished by its focus on disciplinary content in the coaching conversations; coaching sessions focus on the mathematics of the lesson, how that mathematics is addressed in the task design, student thinking that might emerge during the lesson, and what kinds of instructional moves facilitate or advance mathematical understandings (Callard et al., 2022). The innovation in our professional development project was the translation of a face-to-face model of mathematics coaching to a fully online, video-based coaching model (Amador et al., 2021).

Orientations in Coaching Conversations

We characterize pedagogical dilemmas in coaching in terms of two orientations: *learning* and *instrumental*. A learning orientation involves a focus on advancing teacher thinking with respect to mathematical content, student reasoning around that content, and pedagogical strategies that promote student thinking. An instrumental orientation involves a focus on generating a set of tangible artifacts around the lesson, including a well-defined learning goal, a high-quality mathematical task, and a set of anticipated student strategies (West & Staub, 2003). With respect to the presence of these orientations, we posed the following research questions: How were instrumental and learning orientations evident in a planning conversation as part of content-focused coaching cycles, what purposes did each serve in the conversation, and how did they function in tandem?

Methods

We selected a representative coaching conversation from a larger set of 28 conversations, ten of which we analyzed for pedagogical dilemmas in a broader study (Choppin et al., in review). For reasons of space, we restrict analysis in this paper to one planning conversation. The focal coaching conversation involved a teaching team of two teachers, named Larson and Walters, and a coach, named Reiss.

For this paper, we analyzed the planning conversations for the presence of instrumental and learning orientations. An instrumental orientation related to the obligation to produce viable lesson artifacts by the end of the planning conversation, while a learning orientation related to the coach's desire to advance the teacher's content and pedagogical content knowledge. In terms of distinguishing between instrumental and learning orientations in the coaching transcripts, we coded a coach turn instrumentally oriented if: (a) the coach suggested revising the mathematical goal or task without an accompanying explanation, (b) elicited details of a task or student strategy, or (c) described student strategies or mathematical task without explaining connections to the mathematical goals of the lesson. We coded a coach turn as learning oriented if: (a) the coach pressed the teacher to explain the rationale for their choice of goal or task, (b) explained a goal or task in ways that connected to broader mathematical or pedagogical ideas, or (c) explained how a student strategy indicated understanding of a mathematics concept. The key distinction between the two categories was the presence of an explanation or a press for an explanation; explanations entail a connection between two related topics (e.g., between an interpretation and evidence, between a representation and the idea it is intended to convey, between student thinking and the mathematical goal). We interpreted explanations or press for explanations as evidence of the coach's intent to advance the teacher's content or pedagogical content knowledge.

Results

We selected a case to explore the distinctions between the two orientations and how the interweaving of the two characterized the coaching conversation. The teachers (Larson/Walters) sent Reiss a task they had adapted from an online source but communicated to Reiss that they wanted to revise the task. The task required students to generate a graph and function from a table of values, generate a table from a verbal rule, and match multiple representations. Reiss explained that during the planning meeting "we ended up greatly modifying the task … it was very skills-based [and we transformed it to focus on] the relationship between the three different representations." Our analysis revealed three phases in the planning meeting. We characterized the first and last phases as instrumentally oriented and we characterized the middle phase as learning oriented.

Phase One. In the initial phase the teachers provided multiple goals: Larson stated, "They would be able to see the relationships between the coordinates, to be able to write the rules, and be able to articulate how you get the coordinates just based on the rules." The teachers had indicated to Reiss that they had revised the task in advance of the planning meeting to include a series of input / output tables, from which they hoped students would be able to generate rules. Walters explained that the students "have multiple tables with X and Y coordinates and coordinate pairs [to] figure out what the rule might be for those particular tables." Larson added that the rules would be "adding three to X and … adding six to X or something like that." We characterized this phase as instrumentally oriented because the focus was on generating detailed descriptions of the mathematical goal and task. In the second phase, Reiss sought to clarify how students would engage with the task and if the current formulation of the task presented sufficient opportunities for sensemaking, described below.

Phase Two. Reiss initiated this phase with a suggested revision to the task that was more open than the one the teachers had presented, stating "would there be a way to give them … a bunch of tables, and have a bunch of rules and have a bunch of graphs to see if they could match them." Reiss also suggested leaving some blank spaces on the bottom of the table to have students generate their own points. Reiss asked, "Where do you think kids will struggle if you were to give them a set of graphs, a set of tables, and a set of rules? Where do you think they're going to struggle with that?" Larson responded that students might struggle to identify coordinates, to which Reiss responded by explaining that students struggle with identifying coordinates that do not correspond to points visible on the graph. Reiss elaborated on the suggestion to add blank spaces on the table, stating "I think if you left spaces on the table where they had to add in some additional points" so that students would understand that "all those points are still falling on this line." Reiss explained that leaving blanks on the table and matching rules with tables would open up the task and provide more opportunities for students to make connections.

We characterized this phase as learning oriented because Reiss pressed the teachers to explain why they thought that the task would support students to identify coordinates and to make rules from a table. In this phase, Reiss made suggestions about the task and explained how those suggestions would support students to make sense of the mathematical goals of the lesson. The pressing and explaining by Reiss pushed the teachers to provide their justifications for the task and how it addressed the mathematical goal.

Phase Three. The third phase of the conversation focused on finalizing the task features and the summary discussion. Reiss suggested an extension to the matching activity in case some students completed it too quickly. Reiss made this suggestion without explaining how it would support students to make sense of mathematical concepts, so we characterized it as an instrumentally oriented move. Though we characterized this phase as instrumentally oriented, there were moments when Reiss provided additional explanations of how students might struggle with generating points on the line and rules, why her suggestions for revising the task provided opportunities for student sense making, and how to support the students in their efforts. Reiss pushed the teachers to consider how leaving a blank space provided opportunities for sensemaking that a more scaffolded approach would not. We characterized the last part of this phase as instrumental because it primarily focused on logistical details of the summary discussion.

Discussion

Reiss used the first phase (characterized as instrumentally oriented) to clarify the mathematical goal and the task. In the second phase (characterized as learning oriented), Reiss pushed the teachers to consider connections to the big mathematical idea embedded in the task and goals, and to revise the task in ways that supported student sensemaking of the mathematical goals. The third phase (characterized as instrumentally oriented) was focused on finalizing key details of the task and to organize the summary discussion of the lesson.

The analysis shows how the orientations served distinct but complementary purposes. Reiss leveraged an instrumental orientation – making specific suggestions to revise the task – with a learning orientation, explaining why these suggestions would enhance students' sensemaking opportunities. Reiss balanced the obligation of pushing the teachers to expand their understanding of mathematical and pedagogical ideas with the obligation to produce viable lesson artifacts by the end of the conversation. The dynamic between the orientations allowed Reiss to honor the teachers' ideas and their need to produce a viable and well-defined lesson plan with her own desire to advance the teachers' understanding of mathematical content and pedagogical principles.

Regarding the pedagogical challenges coaches face, the findings illustrate how Reiss managed the tension between instrumental goals (e.g., crafting instructional materials for teachers to use in a lesson) and learning goals (e.g., eliciting and advancing teachers' cognitive processes regarding mathematical content, ambitious pedagogy, task design, and student cognition) (Harbour et al., 2021; Russell et al., 2020; West & Staub, 2003). Reiss ensured that Larson/Walters established a clear mathematical goal, a key component to develop teachers' pedagogical content knowledge (Ball et al., 2008). Simultaneously, Reiss leveraged the development of the goal and task to stimulate teacher thinking by prompting them to articulate their thoughts and providing explanations that underscored aspects of the task that afforded connections to significant mathematical concepts. We hypothesize that an excessive focus on instrumental goals might have limited teachers' cognitive expansion, while an excessive emphasis on learning goals could have hindered the creation of high-quality instructional materials for an upcoming lesson.

The three phases showed how the interweaving of the two orientations is an essential feature of coaching conversations. Each orientation was strategically employed in the planning conversation. These phases illustrate Reiss' awareness of the conflicting obligations and opportunities to stimulate teachers' cognitive processes. In summary, Riess coached in a way that navigated pedagogical dilemmas by leveraging the production of viable artifacts as opportunities to stimulate teacher thinking at strategic junctures, providing opportunities for teachers to develop their pedagogical content knowledge.

We argue that the actions of Reiss, to balance pedagogical dilemmas between the instrumental phase and learning phase, all within one coaching planning meeting illuminate coach actions parallel to those of an effective teacher as they would balance teaching phases. The case shows a situation parallel to a teacher's work to hone students' attempts to think for themselves (Windschitl, 2002). At the coach level, this principle was lifted (e.g., Prediger et al., 2022) as Reiss honed the teachers' attempts to think for themselves in the process of planning a mathematics lesson. Building from these findings, the following provides implications for coaching and research on coaching.

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FACILITATING VIDEO COACHING CLUBS

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We explored the facilitation of video coaching clubs to provide professional learning opportunities for coaches taking part in video clubs as part of a three-part professional development project. We lifted a facilitation framework (van Es et al., 2014) from a video-based teaching context to a video-based coaching context to better understand how the facilitators of video coaching clubs drew out contributions from coach participants while simultaneously leveraging their own insights as productive tools to advance the conversation. We further explored how facilitation practices changed over the course of two years. We found that facilitators increased their contributions when the videos came from the participants. The facilitators used the videos to reinforce the principles of content-focused-coaching, to model how to reflect on videos of coaching, and to conjecture about broader issues in coaching.

Keywords: rural, middle grades, professional learning, ambitious teaching, video clubs.

We studied discourse moves of facilitators of video clubs that were designed to support the professional learning of mathematics coaches. Our video coaching clubs consisted of groups of four to five coaches who met regularly to collectively view and analyze videos of coaching practice, similar in structure to video clubs that have been used with teachers (Gaudin & Chalies, 2015; van Es & Sherin, 2008). While there has been considerable literature on the facilitation of video clubs for teachers (cf. van Es et al, 2014; Coles, 2019), there has not been a parallel focus on video clubs for mathematics coaches. Karsenty et al. (2023) define a facilitator as “a professional who manages the PD activities, sets norms for interactions, supports teachers’ exchange of experiences and insights, monitors the discussion, and works with teachers toward the goals set for the PD” (p. 28).

We note the complexities of extracting practices from nested activities to new layers of practices (e.g., nesting and lifting [Prediger et al., 2019]) to analyze the practices of facilitators of teacher educators). We studied the practices of the video club facilitators by adapting the Framework for Facilitation of Video-Based discussions, developed by van Es et al. (2014). This framework includes broad categories such as *orienting group to the video analysis task*, *sustaining an inquiry stance*, *maintaining a focus on the video and the mathematics*, and *supporting group collaboration*, that include specific facilitation moves (e.g., launching, countering, etc.) The literature on video clubs for teachers shows that these clubs provide opportunities for teachers to develop their capacities to attend to how their actions supported student thinking. In turn, we hoped to show how the Video Coaching Clubs–given the name because mathematics coaches are the participants, not teachers–would support coaches to attend to how their actions impacted teacher’s learning. The premise of a mathematics education video club is to create an environment for a group of educators to develop evidence-based reasoning as the basis of teacher growth (van Es & Sherin, 2008); our intent was that video coaching clubs would foster a context for coaches to develop evidence-based reasoning as the basis for coach growth. We answered the following research question: How did the facilitator draw out contributions from the coach participants while simultaneously leveraging their own insights as productive tools to advance the conversation? How did facilitation practices change over the course of four video coaching clubs?

Design Principles for our Video Coaching Clubs

We based our video club design on several principles. First, we wanted our coach participants to draw on videos of coaching to support their evidence-based noticing of relevant coaching incidents; this follows the principle of lifting (Prediger et al., 2019) principles from teacher video clubs to that of coach video clubs.

We also drew from the notion of unpacking content related to teachers' planning practices to serve as a basis for the coaching episodes around which we wanted the coach participants to reflect. In other parts of our project we utilized content-focused-coaching [CFC] (West & Staub, 2013) as the model of coaching we emphasized; CFC is intended to develop teachers' content knowledge and pedagogical content knowledge. Content-focused coaches typically engage teachers in a three-part coaching cycle in which a coach and teacher collaboratively plan, teach, and reflect upon a mathematics lesson (West & Cameron, 2013). The planning phase of the coaching cycle is an opportunity to support teachers to develop new planning practices, while the debriefing phase is an opportunity to help teachers reflect on the ways students engage with mathematics (e.g., Witherspoon et al., 2021). Thus CFC-based planning and debriefing practices were the focus of the coaching episodes coach participants viewed..

Third, we followed the lead of van Es et al. (2014) and Coles (2019) in designing for high quality or productive discussions, which are characterized by four primary purposes for facilitation: orienting the group to the video analysis task, sustaining an inquiry stance, maintaining a focus on the video and the mathematics, and supporting group collaboration (the same key components of the van Es et al. (2014) facilitation model). Our model was intended to manage the tension between providing adequate scaffolding without being too prescriptive (Coles, 2019; Elliot et al., 2009).

Study Context

The Video Coaching Clubs are one of three components of fully online professional learning intervention designed to support mathematics coaches to engage in CFC. Coach participants from rural districts participated in an online course, online video coaching clubs, and one-on-one video-based coaching cycles with a Mentor Coach. Each video club met eight times over two years for approximately two hours each time; in the first year (first four clubs) the facilitator presented a video clip of their own coaching; these clips were chosen as examples, not exemplars of coaching moments, intended to initiate an inquiry into coaching, not an evaluation of the coach or the teacher (Borko et al., 2011). In the second year, each coach participant, rather than the facilitator, presented a video as the basis of group reflection. To facilitate evidence-based reasoning, the coach participants were asked to follow a see-think-wonder sequence for each relevant moment they noticed in the video, meaning they responded to prompts asking: What did you see? What did you think? What did you wonder?. These responses were written independently; these reflections became the basis of the public reflection that followed via dialogue as part of the video coaching club.

Methods

Data Collection

We analyzed transcripts from three groups of coaches who each met eight times, for a total of 24 Video Coaching Clubs. The clubs were all part of the first cohort of our project; consequently, all of the facilitators were new to the roles and to video coaching clubs. The Video Coaching Clubs were conducted via zoom and recorded; these sessions were then professionally transcribed and entered into spreadsheets for analysis.

Data Analysis

We adapted the coding framework from van Es et al. (2014) that was focused on teachers reflecting videos of mathematics lessons. The categories from that framework described the facilitator role of the Video Coaching Clubs in regard to high quality discussions and thus largely aligned with our purposes. However, we made a couple of adaptations to the framework to capture the extent to which the intellectual contributions drew from the facilitator or the participants. We were interested in capturing the ways in which the facilitator drew out contributions from the coach participants while simultaneously leveraging their own insights as productive tools to advance the conversation.

The practices from the Van Es et al. (2014) were: *orienting the group to the video analysis task*, *sustaining an inquiry stance*, *maintaining a focus on the video and the mathematics*, and *supporting group collaboration*. We largely kept the first and fourth roles, but incorporated the second and third into two newly defined practices. Both new categories, *focusing on contributions of participants* and *facilitator interjecting their thinking*, incorporated aspects of an inquiry stance and evidence-based reasoning while allowing us to explore how intellectual authority played out in the clubs. For example, in the category of *focusing on contributions of participants,* the codes *probing participant reasoning*, *paraphrasing*, *lifting up*, and *summarizing and connecting* function to make explicit the reasoning of the participants' reflections on the videos; these promote an inquiry stance. In the *facilitator interjecting their thinking* category, the codes offering an explanation and questioning/wondering focus on the reasoning of the coach, while the code *highlighting / providing evidence* mark moves where the coach focused on the video and the mathematics. See Table 1 for a list of categories, codes, and definitions.

Category	Code	Definition
Orienting to the video club norms and activities	Setting norms / expectations	Setting cultural norms for participating, such as how to formulate disagreements
	Explaining Video Coaching Club activity and directions	Providing details of the activity and how it will be structured
	Contextualizing clip	Provide additional information about the coaching context and mathematics lesson
Focusing on contributions of participants	Prompting participant ideas	Pose general prompts to elicit participant ideas
	Probing participant reasoning	Prompt participants to explain their reasoning and/or elaborate on their ideas
	Paraphrasing	Restate and revoice to ensure common understanding of an idea

	Lifting Up	Identify an important idea that a participant raised in the discussion for further discussion
	Summarizing and Connecting	Make connections between ideas raised in the discussion
Facilitator interjecting their thinking	Offering an explanation	Provide an interpretation of an event, interaction, or mathematical idea, from a stance of inquiry
	Highlighting / providing evidence	Direct attention to noteworthy coaching or teaching moves in the videos
	Questioning/wondering	Coach poses a hypothetical question or wonders about possible alternative actions.
Orchestrating discussion	Distributing participation	Invite participants to share different ideas who have not already participated in a discussion thread. Use in cases where the instructor calls out names to ensure everyone has participated.
	Validating participant ideas	Confirm and support participant contribution

Table 1: Categories, Codes, and Definitions in the Framework

Results

The summaries for each category yielded some consistencies across the facilitators. The facilitators spent roughly one third of their turns orienting the coach participants to the VCCs, with percentages decreasing from the first year to the second for each facilitator. The following quote from Reiss in VCC1 is an orienting move that illustrates how the coaches framed the clubs:

> We've come up with three goals that we're really working towards in these video coaching clubs. The first one is to grow in our ability to make sense of coaching moves and teacher thinking by noticing and naming interesting moments in a planning conversation. Then we also want to work on growing our personal capacity to facilitate content focused coaching planning conversations with teachers. We're going to continue to grow our collaborative community of coaches through rich conversations about authentic coaching moments. (Reiss, VCC1)

Roughly a fifth of the facilitator turns *focused attention on the contributions of the coach participants*; about half of those were *prompting participant ideas* and the other half a combination of the other four codes in that category. Roughly one in six facilitator moves involved *facilitator interjecting their thinking*, though the percentages in this category increased from the first year to the second year, which we explore below. Table 2 displays the overall percentages across both years of the VCCs.

Code Category	Lowrey	Reiss	Whilton
Orienting	32.2	43.5	32.7
Focusing on contributions of participants	20.4	23.0	18.9
Facilitator interjecting their thinking	15.5	13.9	16.1
Orchestrating	20.2	9.7	14.7

Table 2: Results across Categories for Both Years

We noted a number of trends when comparing the first year (the four VCCs where the facilitator presented a video of their own coaching) with the second year (the last four VCCs where the video was from one of the coach participants). These trends offer insights into the nature of VCC facilitation and into the impact of the selection of videos in terms of the conduct of the VCCs. We focus on four codes where we noted differences across the two years of the VCCs, three of which are in the *facilitator interjecting their thinking* category. Three of these codes showed increases across the two years while one did not. We will provide examples of facilitation moves for each of these codes to provide insights into facilitation and why facilitation changed when the source of the videos changed. We first note that the code *setting norms and expectations* decreased for Lowrey and Reiss across the two sets of VCCs from around 7% of facilitator turns to around 1.5%; this can be explained in part because by the second year of VCCs the norms and expectations would already be established. We note this to illustrate that there were some expected changes across the two years, first because the norms of the community had already been established and because the facilitator was no longer presenting their own video. See Table 3 to see the codes and the percentages across the two years.

Code Category	Lowrey VCC1-VCC4	Lowrey VCC5-VCC8	Reiss VCC1-VCC4	Reiss VCC5-VCC8	Whilton VCC1-VCC4	Whilton VCC5-VCC8
Paraphrasing	5.5	1.5	3.3	2.9	4.4	1.8
Offering an explanation	7.6	10.1	3.0	10.3	6.1	11.1
Highlighting / providing evidence	0.5	4.6	0.3	4.9	1.7	3.6
Questioning/wondering	2.2	6.1	0.00	13.2	2.2	9.3

Table 3: Percentage of Facilitator Moves for Selected Codes

The four codes we focus on below showed notable changes across the two years, with *paraphrasing* being the only one of the four that decreased. Here is an example from Lowrey:

> I guess what I'm hearing is that there were—you bumped into some kids that just having that understanding of fractions that we can cut things up and we can still share them and be able to use all of them and share them equally and what that might mean. Then this idea of cutting 'em all into fourths and is that going to give me the same amount as if I had a whole brownie? If I had four of those fourths, would that give me a whole brownie? (Lowey, VCC1)

Here, Lowrey emphasizes the mathematical explanation provided by a participant in a detailed way. This example and others of paraphrasing largely served the purpose of "facilitator modelling the kinds of discourse or social and discussion norms desired in a group" (Coles, 2019, p. 11). Similar to the decline in the norms and expectations code, the incidences of this code likely declined because the norms and expectations were more established in Year 2.

The *highlighting / providing* code increased in part because the facilitator in year two followed the same see-think-wonder sequence to reflect on the video as the other non-presenting participants. For example, Whilton referenced a moment in participant Rice's coaching video:

> One thing, too, from a coaching move that I noticed, I felt like the—I felt like the teacher's explanation in response to the coach's question was … just a broad statement. Then Rice followed up with a very specific, though. "Well, I heard a student say"—she named a very specific moment. (Whilton, VCC7)

This differed a bit from the first year, where the highlighting revealed new insights into a coaching session the facilitator had conducted:

> Just like Stevens said, I didn't even catch the why. The part I caught from the teacher in that same moment though was she said, "I would have asked them this because that would have aimed at our third goal," right? Same moment, but then she named the question and then said, "I would have asked that because that would have gotten us towards the third goal." (Whilton, VCC4)

The two codes that had the highest frequency of these four were *offering an explanation* and *questioning/wondering*. These codes represented the most substantial and detailed insights from the facilitator. As represented in the Reiss quote that expressed the goals of the VCCs, the facilitators' goals included supporting the participants to understand the principles of content focused coaching and to make sense of specific instances of coaching with respect to those principles. Below, we include instances of these codes to show how the coaches accomplished these goals and why they were more prevalent in year two.

The first example of *offering an explanation* is from Reiss connecting her interpretation of the video to the process of supporting teachers to identify a mathematical goal:

> I think what was happening was that the coach didn't want to give away too much about what she—I think she may have been trying really hard not to make the goal for her, so she was trying not to give too specific of examples because she wanted the teacher to self-select her goals. (Reiss, VCC 5)

A second example comes from Whilton, who described how the coach was trying to get the teacher to notice what students were doing:

> [was] the coach picking up maybe on this general nature of, "We talked about, and we did this," and the coach was like, "Wait a second. "We" were doing this stuff. What were the

> kids actually saying?" And pushing for that specificity from what were the kids actually saying versus living in this (Whilton, VCC 6)

These examples illustrate how facilitators used moments in the video to raise essential tensions in content focused coaching; in one case it was about providing opportunities for teachers to contribute to lesson planning and in the second it was about supporting teachers to notice students' mathematical thinking. In both cases, the facilitators' explanations animated the coaches' intentions and actions in ways that foregrounded principles of content focused coaching; they were able to leverage a participant's video to make a point that may not have been as poignant had it been their own video.

The first example of *questioning/wondering* is from Reiss in which she wonders about the outcome of the coaching conversation:

> Then I had a lot of wonderings about it. I wondered what might have happened when the teacher actually taught the lesson, if the students were able to actually make connections between the tiles and, again, it seemed like a rote procedure. Were they able to make a connection between the tile and solving equations or was it really just this procedure of I do this, I do this, I put this tile down. (Reiss, VCC 8)

The second is from Whilton as he wonders about a broader coaching principle;

> My wondering, then, as a coach, is maybe, how do we press—again, back to the "all" conversation. What do we do about the kids who are conceptually challenged versus—what are the coaching moves to not let that just be like, "Oh, they struggle"? It very well could've happened. I'm not saying that Rice didn't do it, but it makes me wonder, how, maybe, do we press in the moment for the "all" piece on that? (Whilton, VCC 7)

These wonderings represent the two most common types of wonderings, one in which the wondering about what happened in the subsequent lesson and the other a wondering about coaching in general. The first kind of wondering creates an anticipatory mind frame to support coaches to envision how their coaching impacts teaching, while the second kind of wondering is connected to general issues encountered in coaching and how to address them.

Discussion

This study explored the design and implementation of video clubs for mathematics coaches who are learning about content-focused coaching. We adapted a framework previously used to study facilitation in video coaching clubs for teachers because our design had considerable overlap with that of the framework's authors. We used that framework to study facilitation moves in 24 VCC sessions across three facilitators in order to better understand how facilitation might differ in video clubs for coaches and to understand how the source of the coaching videos impacts facilitation.

We found that the facilitators utilized some of the norming and orchestration moves documented elsewhere, showing that they were principled in adhering to facilitator roles for video clubs. The most interesting findings related to the differences between year one and year two of the VCCs, when the source of the video changed from episodes of the facilitators' coaching to that of the participants' videos. We found that the norming moves decreased while the moves in which the facilitators injected their insights increased. We saw this particularly with moves associated with explanations and with questioning and wondering. We attribute this to opportunities in which the facilitators used the videos to reinforce the principles of content-focused-coaching, to model how to reflect on videos of coaching, to foster anticipatory thinking, and to conjecture about broader issues in coaching.

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PLANNING TO RESPOND TO STUDENT-WRITTEN WORK IN PROFESSIONSAL LEARNING COMMUNITIES

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This study examines how a Math 1 Professional Learning Community (PLC) collaboratively plans responses to student thinking on formative assessments. The PLC demonstrates awareness of student adaptability, yet there is potential to more fully embrace diverse student perspectives. While the PLC shares insights, there are opportunities to further integrate student work as class-wide learning opportunities, which could enhance the effectiveness of planning responses. Findings underscore the importance of sustained collaborative learning spaces for educators to continuously improve their ability to respond to student work equitably.

Keywords: teacher noticing, formative assessment, Professional Learning Community

When students express their mathematical thinking, they provide valuable insights for teachers to make informed instructional decisions. Making space for this during the process of learning a topic is considered formative assessment (Wiliam & Black, 1996). The use of formative assessments allows educators to monitor student learning and adapt teaching strategies accordingly (Jacobs et al., 2010; Wiliam & Black, 1996). Although there are many proponents of implementing formative assessments in the classroom (Black & Wiliam, 1998), there is limited research on how teachers go about *planning to respond*. When considering *plans to respond*, research is mostly related to individual feedback (Abdulhamid & Venkat, 2018; Jacobs & Ambrose, 2008; Land et al., 2019) and whole group instruction (Abdulhamid & Venkat, 2018; Dunning, 2023; Leshin, 2023; Stockero et al., 2022), with minimal reference to small group interactions. In the articles that reference small groups, no commentary is made to whether these small groups are heterogeneous or homogeneous (Land et al., 2019; Leshin, 2023). Knowing that there are teaching habits of wanting to fix misconceptions rather than building on students' strengths and various strategies (Cohen & Lotan, 2014; Ladson-Billings, 2009), there is a gap in the research on *planning to respond* equitably for heterogeneous groupings. As research supports the benefits of heterogeneous groups for student learning (Boaler et al., 2000), the same is ideal for adult learning; potential benefits of analyzing student work as a PLC include better understanding of student thinking and structured opportunities to ask questions of colleagues related to student thinking and responsive instruction (Jilk, 2016; Little et al., 2003). This study aims to describe the ways that a PLC collaboratively *plans to respond* to student thinking on a formative assessment item.

Background Literature

Formative assessments inform educators of student understanding, allowing for adaptive teaching, as highlighted by the value of students sharing their reasoning and offering teachers a snapshot into their understanding (Baldinger, 2020). Abdulhamin and Venkat (2021) advocate for formative assessments as they establish a two-way feedback loop between students and teachers. According to Bas-Ader and Carlson (2022), educators' respond more adeptly to student ideas when they strive to comprehend students' thinking while also reflecting on their facilitation.

Dunning (2023) found that teachers might adjust their learning goals to align with student strategies. Thus, formative assessments play a key role in cultivating a dynamic learning atmosphere.

PLCs are pivotal in collectively analyzing written formative assessments, offering educators a collaborative platform to merge expertise and perspectives while examining student work (Jacobs et al., 2010). Jilk (2021) says the benefits of PLCs sharing best practices promotes shared learning and continuous improvement, boosting teachers' collective instructional capabilities. Little et al. (2010) shows that collaborative analysis when looking at student work leads to more consistent and holistic evaluation methods, crucial for meeting diverse student needs. Regular PLC meetings allow teachers to coordinate their instruction, address challenges, and create effective responses to student work, fostering equity and inclusivity in education (Little et al., 2010). Such collaborative effort in PLCs ensures that every student's learning journey is understood and supported, enhancing the overall quality of education.

Theoretical Framework

To frame *planning to respond* to students' written work on formative assessment items, literature on teacher noticing of student thinking is referenced. Jacobs et al. (2010) describes teacher noticing of student thinking as consisting of three interrelated phases: attending, interpreting, and *planning to respond*. In the attending phase, teachers document the specific strategies used by students. During the interpreting phase, teachers evaluate and infer the student's level of understanding based on these strategies. Lastly, in the *planning to respond* phase, teachers evaluate the best ways to react, informed by their evaluation of the student's understanding. Leshin (2023) extended this framework to emphasize what each of the components includes when noticing for equity in students' mathematical thinking. Specifically related to *planning to respond* to student thinking on formative assessments, teachers must leverage trends in student understanding to determine multiple *plans to respond* for the entire class and small groups, highlight elements of student work to share, and showcase student work that can elevate an individual's status (Leshin, 2023).

Methods

This research employs a case study approach (Yin, 2014) to investigate a specific PLC's method of analyzing written formative assessment data.

Participants and Context

The case in this study involves a Math 1 PLC, composed of nine high school educators, that regularly analyze student-written work. Among the PLC are two co-teachers and an Instructional Leader (IL). The IL facilitates PLC meetings and coaches teachers.. All members of the PLC are educators of color. In this session, one member was absent, resulting in a gender-balanced composition.

The PLC meets for one hour twice a week. Analysis of student work occurs twice a month. For this session, each PLC member brought one of their class sets of student work to analyze. This PLC's agenda is based on their prior experiences for analyzing formative assessments; no training for this process was provided. This session included 40 minutes to analyze students' work on a formative assessment task from a unit on systems of linear equations and inequalities (see Figure 1). PLC members sat in pairs for this session.

Here is another riddle:

- The sum of two numbers is less than 2.
- If we subtract the second number from the first, the difference is greater than 1.

What are the two numbers?

1. The riddle can be represented by a system of inequalities. Write an inequality for each statement.
2. These graphs represent the inequalities in the system. Which graph represents which inequality?
3. Name a possible solution to the riddle. Explain or show how you know.

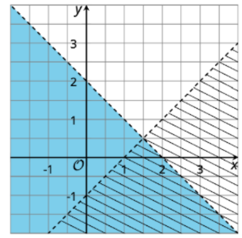

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Figure 1: The Formative Assessment Task

Data Collection and Analysis

Data collection took place during one PLC meeting, involving video and audio recordings of the team collaboration while noticing student thinking of written artifacts, detailed field notes to align conversations with student work being referenced, and pictures of completed student work being referenced.

All talk related to noticing student-written work for student thinking and equity during the PLC meeting was transcribed verbatim. The transcripts were then coded using Leshin's (2023) codebook for noticing student thinking and equity (see Table 1). Other artifacts were used to reference the student work that teachers referred to in conversations with peers. Once the data was coded, the presence or absence of data for each *Planning to Respond* subsection of Leshin's (2023) codebook revealed recurring themes representing common practices that supported or provided opportunities for growth in the PLC while equitably noticing student thinking.

Table 1: Leshin (2023) Sub-codes and Descriptions for Planning to Respond

Planning to respond sub-codes	Description
Patterns shape multiple plans to respond	Teacher uses patterns in understanding to shape plans to respond for the whole class and small groups.
Leverage aspect of work for learning	Teacher identifies aspects of work (including mistakes or partial understandings) to share with the class as a learning opportunity.
Highlight exemplars to elevate status	Teacher identifies aspects of work to share with the class to celebrate or elevate a student.

Results

Based on the framework set by Leshin (2023), one component of *planning to respond* was observed, using patterns to shape multiple *plans to respond*. According to Leshin (2023), effectively planning instructional responses involves recognizing and using patterns in student understanding. Teachers in the PLC discussed multiple patterns across student work, including misreading ordered pairs, misreading scale, and misunderstanding inequality symbols. All trends shared were misconceptions. For misunderstanding inequality symbols, a teacher in the PLC identified a recurring pattern where students replace inequality symbols with equality symbols. As a PLC, this misconception was identified as not being unique to current students but was noted as a common issue observed over several years. The teachers decided on different *plans to respond*, some decided to encourage students to reread the question upon completing their work while other teachers decided to have students box the symbol to bring their attention to the mathematical statement. These strategies were aimed at helping students realize the need to revert the placeholder equality symbol back to the correct inequality symbol.

Two aspects of Leshin's (2023) framework did not occur during this PLC session, leveraging aspects of student work for class learning nor highlighting student work to elevate student status. The PLC did not actively use student mistakes or partial understandings as class-wide learning opportunities. Responses to student work were more individualized, with general soft-skill advice like "read the question out loud" or skill tips such as reminding students to replace equality signs with inequality symbols. Leshin (2023) suggests using aspects of correct student work to elevate student status; however, this practice was not observed in this PLC session. There was no discussion nor selection of admirable student work to be shared with the class, indicating a potential area for growth in acknowledging and showcasing student achievements. On the contrary, students were often referred to in terms of evaluating their abilities, "those tend to be the stronger kids," "these kids can do this," and "you have that data and you can kind of see where kids are stronger and weaker."

As the PLC discussed trends and different possible ways to respond, the discussion became multifaceted, addressing immediate next steps as well as longer-term plans for several days ahead. There was a reality for teachers about being at different places with pacing; teachers ranged from being 0-5 lessons apart from each other in the unit. This created a space for a teacher who had already taught subsequent lessons to share their experiences, including decisions made and where students continue to struggle. For one particular activity implemented, the teacher shared that for "a lot of students, you could tell them from the kids who don't know how to solve with the variables being on both sides, that's where they will struggle, so I need to do a big lesson on that." This exchange provided insights for those who had not yet taught these lessons.

Discussion and Conclusion

The PLC's practices while *planning to respond* show an understanding of the need for student adaptability, yet fall short in fully centering diverse student thinking. While there is collaborative sharing of insights, the lack of integrating student work as class-wide learning opportunities (Leshin, 2023) limits the effectiveness of *planning to respond* in a way that supports a space where students have the right to make mistakes, share ideas, and engage in conversations to deepen their learning (Kalinec-Craig, 2017). Determining how to respond to student work become more natural as educators attempt to understand student thinking more often, suggesting that the craft of *planning to respond* can be learned (Baş-Ader & Carlson, 2022). Since *planning to respond* can be learned, sustained collaborative learning spaces for educators can strengthen a teacher's craft of *planning to respond* (Jacobs, 2010; Jilk, 2016; Little et al., 2003). The findings of this study indicate that PLCs need support with respect to equitably *planning to respond* (Leshin, 2023). Future work is needed to design resources and support learning experiences for PLCs with diverse team members with this focus. It is important to note that a limitation to this study is that only one PLC session was observed; a more in-depth understanding over time is needed to capture the full complexity of current practices. Additionally, since this study only focused on *planning to respond*, future work should aim to understand how PLCs' interpreting practices influence their *plans to respond*.

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ESTABLISHING CONDITIONS FOR ONLINE COACHING: HOW COACHES INTITATE PARTNERSHIPS WITH MATHEMATICS TEACHERS

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Keywords: In-service teacher education; teacher educators; professional development

Coaching literature is replete with guidance on how coaches can lay the foundation for successful in-person coaching partnerships with mathematics teachers (e.g., Knight, 2007; Killion, 2008). With the increased use of virtual coaching (e.g., Carson et al., 2019; Gregory et al., 2017; Matsumura et al., 2019), amplified by the pandemic, it is less certain how coaches create conditions for productive coaching interactions with teachers in online spaces, which arguably are more impersonal as compared to in-person settings. To address this gap, we sought to answer the question: In an initial conversation with mathematics teachers, how do experienced coaches create conditions for future one-on-one coaching interactions?

We adapted conceptualizations from the medical field regarding how doctors seek to build rapport with their patients (e.g., Roter & Larson, 2002) and contend coaches have three central tasks during initial conversations with mathematics teachers: (1) build social and emotional rapport; (2) discuss professional experiences and goals; (3) communicate expectations for future coaching interactions. Our study aimed to understand the ways in which coaches navigated these three tasks given they require a coach to act, respectively, as fellow human, professional colleague, and knowledgeable authority.

We analyzed one-on-one "Getting to Know You" meetings of 18 coach-teacher dyads, conducted via Zoom, that occurred prior to online coaching cycles. All coaches had multiple years of experience coaching online and all teachers taught grades 4 – 10 mathematics. We coded the talk-turns of the coach, parsed at the sentence level, using specific codes nested within the following broad categories: *social rapport building, emotional rapport building, professional experience*, *professional goals*, *conversational goals and logistics*, and *coaching expectations*. Each broad category corresponded with one of the three central tasks previously described.

Preliminary findings revealed that coaches tended to focus on topics related to *professional experience* but used diverse strategies to infuse *social* and *emotional rapport building* into conversations about professional topics. In doing so, coaches managed the obligations to act as both a professional colleague and fellow human by connecting simultaneously at professional and personal levels. Our study contributes to the growing body of research focused on how coaches engage with teachers and productive coaching practices (e.g., Kochmanski & Cobb, 2023; Gibbons & Cobb, 2016; Witherspoon et al., 2021). Specifically, our findings provide new knowledge about how experienced coaches responded to the complexity of supporting mathematics teachers in one-on-one interactions that occurred in online spaces. This knowledge holds implications for future research and points to practical strategies to support coaches as they seek to support the teaching and learning of mathematics.

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EXPLORING TWO TENSIONS IN CONTENT-FOCUSED COACHING PLANNING CONVERSATIONS

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We analyzed the planning conversations of two coach-teacher pairs in an online version of content-focused coaching. We used four coaching discourse moves to explore two central tensions in content-focused coaching: the tension between the intellectual authority of the teacher and the coach and the tension between instrumental and learning orientations. We analyzed the coaching conversations by the number and nature of phases, by how authority was distributed across the coaching conversations, and by how the orientations fluctuated throughout the conversations. The analysis revealed that one coaching conversation was more structured in terms of the progression of content and authority; by contrast, the other conversation was more improvisational and free-flowing. We also noted that other factors, such as the teacher's familiarity with the task and the mathematics content, influenced the conversation.

Keywords: In-service teacher education; teacher educators; professional development.

Coaching is a professional learning activity that holds the potential to improve teaching (e.g., Kraft & Hill, 2020) and student learning (e.g., Campbell & Malkus, 2011). Content-focused coaching, a particular model of coaching, has the primary goal of developing teachers' content knowledge and pedagogical content knowledge (Callard et al., 2022; West & Cameron, 2013). Content-focused coaches typically engage teachers in a three-part coaching cycle in which a coach and teacher collaboratively plan, teach, and reflect upon a mathematics lesson. The planning phase of the coaching cycle has drawn considerable attention from the mathematics education research community. This phase is an opportunity for coaches to cultivate new planning habits for teachers, which are necessary for implementing ambitious instructional practices (e.g., Witherspoon et al., 2021).

The discursive behaviors of a coach shape the learning experiences of a teacher (Costa & Garmston, 2016). Thus, realizing the potential of coaching during planning conversations as a mechanism to improve teaching and learning depends upon the coach's ability to skillfully facilitate the interaction. When facilitating planning conversations, content-focused coaches encounter two central challenges. First, coaches must dynamically manage competing roles of acting as a knowledgeable expert and a collegial partner. Coaches often hold a formal role and title that can elevate their position in relation to the teacher (Mosley Wetzel et al., 2017). Furthermore, content-focused mathematics coaches also tend to possess expertise in both content and pedagogy, as this is a prerequisite for the position (West & Cameron, 2013; Witherspoon et al., 2021). Throughout a conversation, content-focused coaches must carefully balance the distribution of intellectual authority as they operate from a potential position of power (e.g., Ippolito, 2010). As a second tension in facilitating planning discussions, coaches must support teacher learning while simultaneously helping a teacher to generate products needed to teach the lesson. Thus, coaches navigate between two orientations: learning and instrumental. When operating with a learning orientation, content-focused coaches strive for outcomes that can transcend the current interactions, such as helping teachers deepen their knowledge of mathematical content and content-specific pedagogy along with cultivating new planning habits (Stein et al., 2022). When operating with an instrumental orientation, content-focused coaches help teachers develop practical products needed to teach the upcoming lesson, such as a detailed lesson plan or list of anticipated student strategies (West & Cameron, 2013).

Existing studies are beginning to examine the ways coaches manage the tension of acting as an expert and colleague (Gillespie et al., 2024; Witherspoon et al., 2021) and holding learning and instrumental orientations (Choppin et al., 2024). However, there is not adequate research that has explored how coaches manage these two tensions in tandem during planning conversations. Specifically, we explored the following research question: How do mathematics coaches manage the two central tensions of (a) distributing intellectual authority between the coach and teacher and (b) holding both instrumental and learning orientations during planning conversations with teachers?

Theoretical Framework

We identified two central tensions present in content-focused coaching. First, coaches act as both a knowledgeable expert and a collegial partner, which requires coaches to balance intellectual authority through a planning conversation (Gillespie et al., 2024). Intellectual authority refers to whether the coach's or teacher's perspective is privileged at moments in time and relates to how a coach manages the competing roles of acting as expert and colleague. As a second tension, coaches navigate instrumental and learning orientations to support teacher learning while meeting coaching obligations. An instrumental orientation relates to the obligation to produce viable lesson artifacts by the end of the planning conversation, while a learning orientation relates to the coach's desire to advance the teacher's content and pedagogical content knowledge.

To make sense of the ways coaches manage these tensions, we focused on the coaches' discourse moves. We define a discourse move as the way coaches use language within a specific conversational moment to communicate with teachers. In other words, we focused on *how* coaches talked to teachers. Within mathematics education, researchers have analyzed discourse moves to understand how teachers participate in professional learning experiences (e.g., Borko et al., 2008), how facilitators manage discussions in small group settings (e.g., Amador & Carter, 2018; van Es et al., 2014), and how coaches facilitate one-on-one conversations (e.g., Gillespie & Amador, 2024; Witherspoon et al., 2021).

We identified four central coaching discourse moves: *elicit, press, explain, and suggest*. We argue that the coach's use of each move within a conversational moment a) positions either the coach or teacher as the intellectual authority and b) reflects the coach's orientation (learning or instrumental). First, we describe *elicit* and *press* moves, which each position the teacher as the intellectual authority, but reflect different coaching orientations. *Elicit* moves are coaching discourse moves that initiate opportunities for the teacher to share their thinking, making the teacher's thinking about the planned lesson visible to the coach (van Es et al., 2014). *Press* moves also invite the teacher to share their thinking but call for the teacher to elaborate and expand upon previously shared ideas or provide a rationale for why their ideas might be productive (Franke et al., 2009). For example, an *elicit* move could involve the coach asking the teacher to share how they were planning to launch the lesson. A *press* move could involve the coach asking the teacher to explain why they feel their plan to launch the lesson would be effective. With respect to managing the tension between acting as expert and colleague, we consider both *elicit* and *press* moves to position the teacher as the intellectual authority within a conversational moment since both moves invite the teacher to share their thinking or ideas. However, we argue that *elicit* moves reflect an instrumental coaching orientation since teachers are simply asked to report on their existing ideas. In contrast, *press* moves correspond to a coach holding a learning orientation since these moves call on teachers to elaborate on initial thinking, which can support new insights about content or pedagogy (Stein et al., 2022).

Explain and *suggest* are associated with the coach acting as an expert since both moves position the coach as the intellectual authority within a moment in the conversation. In using an *explain* move, a coach shares their interpretation of a mathematical concept or pedagogical principle (van Es et al., 2014). *Suggest* moves involve the coach recommending an action for the teacher to enact (Amador et al., 2024). While explanations and suggestions both position the coach as the intellectual authority, they reflect different coaching orientations. *Suggest* moves relate to an instrumental coaching orientation since the coach is recommending a practical action for the teacher to use when teaching the lesson. *Explain* moves, in contrast, correspond to a learning orientation since the coach is sharing their interpretation of a mathematical concept or teaching principle, which provides opportunities for teachers to deepen their understanding of content and/or pedagogy. Figure 1 visually depicts these four moves in relationship to intellectual authority and orientation.

Coach Orientation	**Instrumental**	Primary Discourse Move: Elicit Description of Move: Coach poses general prompt to elicit teacher ideas. Example: Coach asks teacher how they will launch the lesson.	Primary Discourse Move: Suggest Description of Move: Coach recommends an action for the teacher to enact. Example: Coach tells the teacher how they could launch the lesson.
	Learning	Primary Discourse Move: Press Description of Move: Coach prompts teacher to explain their reasoning and/or elaborate on their ideas. Example: Coach asked teacher why they think their plan for launching the lesson will be effective.	Primary Discourse Move: Explain Description of Move: Coach provides an interpretation of an event, interaction, or mathematical idea. Example: Coach tells the teacher why they think their plan for launching the lesson will be effective.
		Teacher	**Coach**
		Intellectual Authority	

Figure 1: The Four Discourse Moves in Relation to Intellectual Authority and Orientation

Methods

We analyzed two planning conversations involving separate coach-teacher pairs. The conversations took place as part of a larger professional development project that involved three cohorts of teachers; these coaching conversations were part of the third cohort (see Choppin et al., 2021). The two coaches, pseudonyms Reiss and Whilton, were paired, respectively, with middles grades teachers Harris and Jackson. We considered Reiss and Whilton to be mathematics specialists serving as *organization-based mathematics coaches* (e.g., Baker et al., 2022) as they were employed by a professional learning organization external to the participating teachers' schools and districts. Both coaches had coached in at least one of the prior two cohorts and were familiar with the content-focused coaching model employed in the project. All the coaching activities occurred online, with the planning conversations taking place via Zoom.

We recorded the planning conversations and had them professionally transcribed. We then parsed the coaching conversations into stanzas. Each stanza contained at least one coach-teacher interaction as well as text needed to understand the context of that interaction (Saldaña, 2013). Stanzas entailed interactions around a bounded discussion; when the topic shifted, we created a new stanza. Within the stanzas, we coded the coaches' discourse moves at the level of turn using the four codes described in the framework: *elicit*, *explain*, *press*, and *suggest*. Four coders coded each transcript, after which we met to reconcile the codes to ensure consensus.

The transcripts were also coded for the content of each turn, focusing primarily on the following three codes that represented principles of content-focused coaching: mathematical goal; mathematical task; and potential student strategies or misconceptions (e.g., Callard et al., 2022; West & Cameron, 2013). The mathematical goal involved the articulation of the content addressed in the lesson and sometimes the local curriculum standard associated with that content. The mathematical task involved the main problem that would be presented to students during the lesson. Potential student strategies and misconceptions involved likely approaches students would utilize when working on the mathematical task.

We then divided the transcripts into phases consisting of one or more stanzas. Phases were demarcated initially by a change in focus of the discussion (e.g., mathematical goal, mathematical task). Our description of the focus for each phase included the content code and additional details as warranted (e.g., *revising the mathematical task* instead of just *mathematical task*). In some phases, multiple topics were interwoven, so we listed all relevant topics. We then characterized each phase using the categories in the framework. We characterized a phase as *teacher as authority* if most of the coach's discourse moves were either *elicit* or *press* and *coach as authority* if most of the coach's discourse moves were *explain* or *suggest*. Similarly, we characterized a phase as *instrumentally oriented* if most of the coach's discourse moves were *elicit* or *suggest* and as *learning oriented* if most of the coach's discourse moves were *explain* or *press*. There were instances in which the intellectual authority and orientation changed within a stanza, which led us to divide some phases into two. We characterized phases as *mixed* if there were relatively equal amounts of discourse moves associated with each category.

In terms of distinguishing between instrumental and learning orientations in the coaching transcripts, we coded a coach turn instrumentally oriented if: (a) the coach suggested revising the mathematical goal or task without an accompanying explanation, (b) elicited details of a task or student strategy, or (c) described student strategies or mathematical task without explaining connections to the mathematical goals of the lesson. We coded a coach turn as learning oriented if: (a) the coach pressed the teacher to explain the rationale for their choice of goal or task, (b) explained a goal or task in ways that connected to broader mathematical or pedagogical ideas, or (c) explained how a student strategy indicated understanding of a mathematics concept. The key distinction between the two categories was the presence of an explanation or a press for an explanation; explanations entail a connection between two related topics (e.g., between an interpretation and evidence, between a representation and the idea it is intended to convey, between student thinking and the mathematical goal).

Results

The analysis showed how the Reiss-Harris and Whilton-Jackon planning conversations proceeded in quite different ways. The differences were marked by the number and nature of phases, how intellectual authority was distributed across the coaching conversations, and, to a lesser degree, how the orientations fluctuated throughout the conversations. See Figure 2 for a visual summary of the analysis.

The phases in the Reiss-Harris conversation were generally confined to a singular topic (e.g., goal or task) and had a content progression (goal, task, strategies) that was the most common across the larger set of over 30 coaching conversations we analyzed for a larger study. The conversation, however, was unique in the extent to which the coach and teacher discussed the mathematics of the task. The task, informally known as the Locker Problem, can be stated as:

> The first student opens every locker; the second student starts with the second locker and closes every other locker; the third student starts with the third locker and changes the state of that locker and every other third locker; etc. After all the students have gone through, which lockers remain open? (Lappan et al., 2014)

Reiss-Harris	**Whilton-Jackson**

Topic	Authority	Orientation
Math Goal	Teacher	Instrumental
Doing the math	Mix	Learning
Meta discussion around the math	Coach	Learning
Revising math goal	Coach	Mix
Revising the task	Coach	Mix
Finalizing task and student strategies	Coach	Mix

Topic	Authority	Orientation
Mathematical Goal	Teacher	Instrumental
Student strategies	Teacher	Learning
Student strategies	Teacher	Instrumental
Student strategies, advancing questions, task revisions	Coach	Learning
Connections between goal, task, advancing questions, and student strategies	Mixed	Mixed
Student strategies	Teacher	Instrumental
Mathematical Goal	Coach	Instrumental
Lesson features (e.g., group structures, resources)	Mixed	Mixed

Figure 2. Visual of Authority / Orientation Dynamics Across Coaching Conversations

(Note: The length of the boxes roughly corresponds to the relative lengths of the phases.)

Reiss and Harris engaged in extensive discussion about the mathematics in the task (totaling 10 stanzas), in part because the teacher had not solved the problem in advance of the planning meeting and was not familiar with it, as reported by Reiss. During this discussion, Reiss provided numerous explanations about the mathematics and possible student strategies and pressed the teacher to explain her thinking. For example, Reiss stated:

> First kid goes down and opens everything. Then the second kid, I actually kept, okay, he’s going to change and close 2, 4, 6, 8, and 10. Then I went to the third guy. I tried to see in each state what happened. Then I knew after 10 what would still be open after 10, so I just—I broke it down that way to see if that would help. What I found was after 10, locker 1, locker 4, and locker 9 were still open.

Here, Reiss shared her insights into the mathematics content without giving away the pattern, leaving her openings to subsequently press the teacher about her thinking. The phase in which this took place was characterized as mixed in terms of intellectual authority because Reiss provided explanations but also pressed the teacher to explain her thinking. The phase was characterized as learning oriented due to the consistent presence of explanations and press for explanations. Once the conversation turned to the design of the lesson, Reiss made numerous suggestions related to revising the mathematical goal, the design of the task, and how to enact the task. As Reiss made these suggestions, she explained why they would provide or enhance opportunities for student sensemaking. These phases were characterized as positioning the coach as the authority, as Reiss was the primary contributor, and as mixed in orientation because Reiss suggested ways to finalize the details of the mathematical goal (instrumental) and task while simultaneously explaining how these suggestions would support students' mathematical sensemaking (learning).

In contrast to the Reiss-Harris conversation, the phases in the Whilton-Jackson conversation typically involved multiple topics, with an emphasis on making connections between the topics. For example, the fifth phase of the planning conversation consisted of nine stanzas in which Whilton emphasized connections between the mathematical goal, the task, advancing questions, and student strategies. The phase began with Whilton pressing the teacher to explain what kinds of student strategies would indicate their understanding of the mathematical goal. Whilton then explained the connections between the representations emphasized in the mathematical task and the strategies discussed by the teacher. He also explained the connections between different strategies, and suggested to the teacher about how to respond to one of the strategies they discussed:

> In a sense, once they say, 'Wait. I saw one plus three plus one,' if we just capture that for them, like, 'Hey, I'm just going to write down what you said. You said one plus three plus five. Tell me. What did your brain do on the next one?' I feel like we're still trying to unpack the kids' thinking for them as opposed to implanting thinking for them. At least in my interpretation, I feel like we haven't—I can see where it's a real slippery slope, but I think we're still honoring where kids are, their thought process, and just helping bring clarity. What do you think about that? Do you agree with me?

Whilton explained how the question he suggested could simultaneously advance and validate student thinking, connecting to a broader pedagogical principle of building from student thinking. Whilton then returned to explaining the connections between the visual diagram in the task and a potential strategy but then tied that strategy to the mathematical goal:

> I think then you can hear, 'Oh, they're talking about five and then adding three each time,' … I think we can really listen to what kids have to say and help them, but I like how you just added that … that also really helps bring home that other goal you were talking about, too. Right? That seeing this constant rate of change now. Why is it happening, and where is it coming from?

These examples illustrate how Whilton emphasized connections between the task, the strategies that students might produce, the mathematical goals represented in those strategies, and possible questions to pose to students in response to a given strategy. We characterized this as a single phase because there were inter-connections between the stanzas and the topics throughout the phase. In terms of overall characterization of authority, the phases fluctuated, with a trend from teacher to coach; in much of the conversation, Whilton mixed elicitations and presses with explanations and suggestions. The orientations across phases also fluctuated, with some instrumental phases where Whilton pressed for details around the mathematical goal and student strategies and some learning phases where Whilton explained the reasoning behind strategies or instructional moves or pressed the teacher for explanations. The results show how Whilton nimbly moved between topics and between elicits/presses and explanations/suggestions to establish essential understandings and finalize details of the goal, task, and potential strategies.

Discussion and Implications

The framework and visual representations of our findings provide insights into the evolution of and nuanced differences between two planning conversations with respect to the topic of discussion, distribution of intellectual authority, and coaching orientation. In turn, we illustrated the unique ways two coaches managed the two central tensions of distributing intellectual authority, thus serving as both an expert and colleague, and navigating instrumental and learning orientations during planning conversations with mathematics teachers. The analysis showed a purpose in how the Reiss-Harris conversation evolved; first Reiss wanted the teacher to understand the mathematics before she turned the conversation to the more instrumental discussion around the details of how to enact the mathematical task with students. Furthermore, given the teacher's lack of familiarity with the task and the mathematics, Reiss's emphasis on suggestions and explanations (i.e., positioning herself as the authority) was warranted. The trend toward mixed or learning oriented phases also provides insight into her intention to push the teacher's mathematical and pedagogical understanding while producing viable lesson plan. The Whilton-Jackson coaching conversation was more complex because there was no consistent progression across topics, authority, or orientation. This conversation was more free-flowing and improvisational, with Whilton strategically injecting his perspective, thus assuming intellectual authority temporarily to make opportunistic connections between topics.

Our study offers two primary contributions to the collective knowledge of coaching within the field of mathematics. First, prior studies have attended to the ways coaches continually shift the intellectual authority across coaching conversations (e.g., Gillespie et al., 2024; Russell et al., 2020; Witherspoon, 2021). Through our detailed analysis of the Reiss-Harris and Whilton-Jackson planning conversations, our findings suggest that intellectual authority is intertwined with a coach's orientation and topic of conversations, which are related to underlying contextual features of the coaching cycle. Thus, we contend that the ways in which a coach distributes intellectual authority are associated with multiple factors and caution against making inferences about inherent coaching styles or tendencies from the analysis of a single cycle. For example, the discursive actions of Reiss were likely influenced by the teacher's lack of familiarity with the task and underlying mathematics. Other cases, with differing contexts, might look quite different with respect to intellectual distribution, and in fact, we have noted differences in other analyses of Reiss's coaching.

As a second contribution, prior studies have highlighted the need to better understand the interactional patterns of a coach and teacher (e.g., Saclarides, 2022). In response, researchers such as Gillespie et al. (2024) and Baldinger (2014) have developed methodologies and generated visual representations to depict salient features of coach/teacher interactions. Similarly, our framework and methodology weave together three constructs – intellectual authority, coach orientation, and content – to generate visual representations that afford insight into nuanced ways coaches manage two central tensions in planning discussions.

The findings connect to the conference theme - *envisioning the future of mathematics education in uncertain times* – by highlighting the influence of contextual factors on the nature of coaching conversations and the need for both intentionality, as seen most prominently in the Reiss conversation, and flexibility, as seen in the Whilton conversation, to meet the contingent needs of teachers.

Future Research

The analysis illustrates how the framework provides insights into the nature of coaching conversations. We framed the analysis in terms of two central tensions, one involving intellectual authority and the other involving instrumental or learning orientations; furthermore, we connected these tensions to the content emphasized in phases of coaching conversations. The use of the framework leads us to several new questions about the nature of coaching conversations. First, the analysis revealed that the Reiss-Harris conversation was more structured in terms of the progression of content and was influenced by the teacher's lack of familiarity with the task and the mathematics. By contrast, the analysis showed that the Whilton-Jackson conversation was more improvisational and free-flowing. We encourage future research to consider characterizing coaching conversations according to a structure-improvisation continuum and examine what factors influence how coaching conversations play out.

Second, it is plausible that both conversations were productive in terms of supporting teacher development. We encourage future research to use of our framework and methodology to deconstruct the planning conversations into phases, characterized by content, intellectual authority, and coach orientation, as a first step to understanding how a multi-faceted conversation may support teacher development. Our analysis showed how each phase presented different opportunities for teachers' development; future research should examine how the different coaching orientations and intellectual distribution in a phase of a planning conversation results in changes to teachers' knowledge or their subsequent teaching. Finally, in the discussion section, we cautioned against making inferences about a coach's style or tendencies, given the contextual features of a coaching cycle likely influenced the behaviors of the coach. To better understand coaches' tendencies with respect to distributing intellectual authority and holding differing orientations, we recommend future studies use our framework and methodology to examine multiple coaching cycles from a single coach-teacher pair and coaching cycles that feature a single coach working with different teachers.

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EXISTING QUESTIONS AND ANSWERS: A REVIEW OF COACHING LITERATURE WITHIN MATHEMATICS EDUCATION

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The term "coach" carries broad and diverse definitions throughout professional development literature within mathematics education. Such differing definitions consider a coach to be a peer observer, one-on-one teacher support, implementation support, or small group professional development facilitator (Baker et al., 2022; Bengo, 2016; Campbell & Griffin, 2017; Kraft et al., 2018; Kraft & Hill, 2020; Russell et al., 2020). While definitions of coaching vary, the primary purpose of coaches in mathematics education does not. Researchers consistently emphasize the *hope* of coaches supporting teachers' learning to positively impact student learning (e.g. Gibbons & Cobb, 2016) through ongoing, active, and content-focused observation and feedback cycles (Desimone & Pak, 2017; Kraft et al., 2018).

Considerate of these diverse definitions, we have identified a need to synthesize existing coaching literature within mathematics education through a systematic review of related literature in order to organize the contributions based on the specific focus of stated research questions. The intent of this poster will be to share the design of our systematic review and the analysis of the literature to formulate an image of how researchers are further developing understandings around coaching in mathematics education. Our review considers the varying definitions of coaching and diverse research on the impact of coaching in mathematics education.

The following research questions guided our review of the literature: (1) What questions are researchers asking to better understand (define, support, promote) coaching in mathematics education? and (2) What frameworks and methodologies are researchers utilizing to better understand (define, support, promote) coaching in mathematics education?

To answer the first research question, we examined 45 research questions from 21 peer-reviewed journal articles published between the years 2013-2023 that focused on coaching in mathematics education. We organized these research questions in four categories based on focal populations: coaches, teachers, school systems, or students. Preliminary analysis revealed that within these categories research questions related to: (1) coaching behaviors, (2) coach learning, (3) teacher outcomes, (4) system outcomes, or (5) student outcomes. To answer the second research question, we recorded the theoretical framing researchers utilized to position their work along with accompanying methodological approaches to understand trends in how researchers have conducted empirical inquires on coaching within mathematics education. Within our emergent categories, we are finding tendencies in literature favoring certain methodological approaches. Further analysis from this systematic review is underway. Our poster will provide an organized synthesis of recent research questions as well as accompanying methods and theoretical perspectives, thus providing the field with greater clarity about research trends on coaching within mathematics education.

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A COLLEGE ALGEBRA INSTRUCTOR'S TRANSITION FROM PROCEDURAL TO CONCEPTUAL THROUGH THE CODESIGN OF FORMATIVE ASSESSMENTS

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This study explores how an instructor designing assignments in an online homework system for a college algebra course considers the development of procedural fluency in connection with conceptual foundations. Findings reveal shifts in the instructor's perspective, highlighting the importance of connecting procedural fluency to conceptual foundations. The study underscores the potential of co-design activities in reshaping instructors' beliefs and instructional practices, offering insights into enhancing mathematics education in college algebra courses.

Keywords: Instructional Activities and Practices; Undergraduate Education; Technology

The instruction of college algebra courses has long been a concern given the barrier it presents for many students (Tunstall, 2018) – both those for whom it acts as a terminal math course and those who need it as a prerequisite for another course within their major. Mathematics education organizations have long called for undergraduate mathematics instructors to support students' deeper engagement with mathematical ideas and habits of reasoning (e.g., NCTM, 1980; AMATYC, 2006; MAA, 2018). This means attending to students' conceptual understanding, procedural fluency, and how the latter builds on the former. In the Mathematical Association of America (MAA) *Instructional Practices Guide* (2018), the authors noted,

> Conceptual understanding involves knowing what to do and why it works, while procedural fluency involves deciding and knowing how to do it…. When students learn procedures in such a way that they are connected to conceptual foundations, they will have more success in using these procedures, will recall them for a longer period of time, and will be able to use these procedures flexibly and effectively in a problem-solving situation. (p. 42)

In undergraduate mathematics courses, like college algebra, there is widespread emphasis on traditional lecture focusing on procedural knowledge, rather than conceptual understanding (e.g., Duffin et al., 2019; Khasawneh et al., 2023; Veith et al., 2023).

College algebra courses are typically quite large, often having well over 50 students enrolled in a single section. In these contexts, web-based homework systems are often used to support students and instructors by providing immediate feedback to both. Research has shown that these systems can make learning more active and adaptive while also focusing on improving conceptual understanding and problem-solving skills (e.g., Porter et al., 2015; Rochelle et. al, 2016; Twigg, 2009). However, there is little research on how instructors might use these systems to improve their practice – especially as it relates to supporting their students in learning procedures in ways that are connected to conceptual foundations. The purpose of this study is to answer the following research question: How does an instructor who is designing the assignments and supports for an online homework system for a college algebra course consider the development of procedural fluency as connected to conceptual foundations?

Context of the Study

This study is in the context of a larger project in which the instructor is leading the co-design of a set of online homework problems – and their supports – for undergraduate college algebra using the online homework system ASSISTments (www.assistments.org). Online homework systems are commonly used in college mathematics courses (e.g., ALEKS (Hagerty et al., 2005), MyMathLab (Duffin et al., 2023), WeBWork (Roth et al., 2008)), in addition to learning management systems that have mechanisms for creating online assignments (e.g., Canvas, Moodle). These programs are similar in that they make assigning homework problems easy, students can resubmit multiple times until their response is deemed correct, and students are given immediate feedback with respect to the correctness of their final answers. However, in addition to this immediate feedback, students also have the ability to access carefully designed supports. What sets ASSISTments apart are the supports that instructors can add at the problem level (up to 3 different supports), rather than just using publisher-supplied supports available to students as they work on assigned problems. Supports can be in the form of videos, worked solutions, or scaffolding hints. While ASSISTments has been used widely at the middle school level (Feng and Heffernan, 2006; Heffernan et al., 2012; Heffernan and Koedinger, 2012), this is the first use of it in undergraduate college algebra meaning that though the platform exists, the problems and supports for those problems needed to be created.

Methods

This is an intrinsic case study (Yin, 2018) of one college algebra instructor, Michael (the third author of this paper), who is also leading the co-design of the college algebra assignments in ASSISTments at a large southeastern university. This unique context provides an opportunity to learn about how Michael is making sense of "building procedural fluency from conceptual understanding" (MAA, 2018, pg. 42) through co-design (Severance et al., 2016).

The data for this study include a series of interviews as well as analysis of artifacts used in the creation of assignments and supports to use in ASSISTments assignments. Michael took part in 6 semi-structured interviews over the course of a year. The interviews were between 30 and 60 minutes long, took place via Zoom, and were recorded. Artifacts include Google documents and sheets in which Michael kept meeting notes and planned the problems he was going to add to an ASSISTments assignment as well as the associated supports. Given that the rest of the local research team is made up of mathematics education researchers, Michael's practice was to ask for their feedback on all supports as well as problems he had labeled as "conceptual" in nature. As such, discussions about these designs occurred in the shared documents using the "comment" features.

Interview transcripts and design artifacts were coded (by the first two authors) for *attending to conceptual foundations*. All of the quotations assigned this code were then read for emerging themes. Throughout this process, the research team would share the emerging findings with Michael to get his feedback and to make sure we were representing his ideas appropriately. Those emerging themes are what we report on in this study.

Preliminary Findings

In what follows, we share three phases of Michael's ongoing development: his stance that conceptual understanding is not a priority, coming to understand what conceptual understanding of a procedure means, and advocating for making connections between procedures and their conceptual foundations.

Conceptual Understanding is Not a Priority

Early in the project, Michael repeatedly noted that attending to the connections between procedures and their conceptual foundations was not a priority for him. This was evident in the design of both the questions he included in the ASSISTments assignments and their associated supports. For example, approximately 6 months prior to the pilot semester, Michael and his team began identifying problems to include in the first ASSISTments assignments. In one of these early meetings, he shared that he knew he needed to include what he referred to as "conceptual problems", but he was not sure what that looks like. He mentioned many times that he was only including this type of problem because he thought he had to due to the goals of the overarching project. He did not think they were a priority because they are not tested on the course common exam. Michael explained, "I don't think I would highly emphasize any feedback that I give for conceptual questions in the current course format for College Algebra. Conceptual questions tend to not be emphasized on the common final." He also shared that "using space on my tests in order to ask the conceptual questions will take that space away from the procedural questions that are going to be on the final."

Coming to Understand What Conceptual Understanding of a Procedure Means

As Michael started to draft what he referred to as "conceptual problems", there is evidence that he felt he was learning what conceptual understanding of a procedure might mean. When he drafted these items, there was a lot of back and forth with the mathematics educators on the team about what makes a question "conceptual". For example, the first draft of a "conceptual question" for a unit on exponents stated, "When can you subtract two exponents?" After some discussion about the fact that the question is simply asking students to identify when they can use a particular procedure, and was missing the conceptual "why", Michael revised the question to, "Provide an example of when subtracting two exponents is appropriate. Explain your reasoning." This revised question is focused on procedural fluency - recognizing when a procedure is appropriate to use - and by asking students to explain their reasoning it is asking for the underlying concept as well.

Figure 1: Sample Supports for an ASSISTments Item

Support Draft 1	Support Draft 2
Similar to the graph of a parabola, the quadratic formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, can determine the solutions to a quadratic equation. The discriminant, b^2-4*ac*, is part of the quadratic formula. The value of the discriminant indicates how many times the parabola intersects the x-axis and how many real solutions a quadratic equation has.	As a reminder, the discriminant, b^2-4*ac*, is the part of the quadratic formula that is under the square root symbol, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Given a square root can have two (if > 0), one (if = 0), or no real solutions (if < 0), the value of the discriminant determines how many solutions there are for a quadratic equation.

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

If the discriminant is positive, i.e., b^2-4ac > 0, how many solutions does the quadratic equation have?	If the discriminant is positive, i.e., b^2-4ac> 0, how many solutions does the quadratic equation have? Write the answer as a number.

At the same time as he was creating the questions in the assignments, Michael was also creating the supports for those questions. We saw a similar shift in this work. For example, when designing a support for the problem: "Based on the discriminant, state how many real solutions there are to the following quadratic equation: $2x^2 - 6x + 7 = 0$." Michael's first support draft was very procedural, asking students to recall a rule they learned about the discriminant (see the example on the left side of Figure 1). After some back-and-forth discussion with the mathematics educators to identify the foundational concept – that a square root can have one, two, or no real solutions, he revised the support to include the conceptual foundation (Figure 1 right side). Michael later noted that this back and forth was helping him understand what conceptual understanding means in the context of a procedure-heavy course like college algebra.

Advocating Connecting Procedures to their Conceptual Foundations

After a full semester of pilot and design work, there was evidence that Michael not only had a deeper understanding of how to develop procedural fluency from conceptual understanding but also thought it was important. Michael shared that based on his engagement in this design work, he has started to change his practice. He explained, "I started to really think about how I should give an explanation in class…I've gotten better over time, I think my instruction has gotten better." At the same time, there is evidence that Michael has begun to advocate for an emphasis on providing conceptual foundations in the supports for ASSISTments assignments as well. He explained that he reached a point where he thought "those explanations I was giving a class are just so much better than the ones that we were putting into the assignments." This made him realize that,

> If I had to give this to my students, I would want to make sure that they had a support that was the closest thing to what I could provide them if they were actually in person with me…So that's why I started saying that these things need to be better. These things can't just be railroaded through and just kind of shoved down their throats. They actually need to understand what is going on.

Michael is currently redesigning many of the ASSISTments assignments to align with what he now understands about the importance of connecting procedures to their conceptual foundations.

Discussion and Conclusion

The case of Michael provides an interesting example of how engaging in co-design, like designing the ASSISTments assignments and supports, might shift instructors' beliefs about the importance of attending to the conceptual foundations of procedures when developing procedural fluency in a college algebra course. We emphasize the co-design aspect of such work, as it seems as if the interaction with mathematics educators was an important part of Michael's journey. Our results are consistent with research on curricular co-design efforts (e.g., Severence et al., 2016). We recognize that engaging in this kind of design is not something that all instructors get to do, but the findings here suggest it might be helpful to consider ways of possibly engaging them in similar co-design activities. In fact, unlike online homework products that are prepopulated with questions, ASSISTments allows all instructors to add their own questions and supports making such work possible. We are curious about how instructors who engage with the assignments and supports that Michael has created will, or will not, take up his stance on the importance of including the conceptual foundations – consistent with MAA (2018) recommendations – in their own explanations and attention to student responses on conceptual questions within ASSISTments to inform their instruction. This is an area ripe for future research.

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SECONDARY MATHEMATICS TEACHERS' JUSTIFICATIONS FOR SEQUENCING STUDENT WORK

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This study examines secondary mathematics teachers' justifications for sequencing student work (SW) in relation to the different ways they might sequence SW. Existing research recommends selecting and sequencing SW in a way that creates a coherent mathematical storyline (Smith & Stein, 2011), but there is a lack of empirical evidence on how teachers can develop the expertise needed to enact that (Dunning, 2023). We focused on actions teachers might take when considering how to sequence SW in the context of StoryCircles, a lesson-centered professional development program. We found emerging patterns between teachers' heuristics for sorting SW and the obligations they use to justify those heuristics. In particular, the disciplinary obligation was consistently prioritized, and the individual obligation played a crucial role in justifying teachers' attention to making mathematics accessible by scaffolding SW by complexity.

Keywords: Lesson-centered Professional Development, Instructional Activities and Practices, Classroom Discourse, Sequencing

Objective

Some years ago, Stein and colleagues (2008) outlined five fundamental practices for orchestrating productive mathematics discussions: anticipating strategies, monitoring strategies, selecting strategies, sequencing strategies, and connecting strategies within whole-class discussions. According to them, the purposeful selection and sequential presentation of student work increases the likelihood of fostering discussions where mathematical understanding is collaboratively constructed by all individuals in the classroom. Furthermore, teachers' attention on the practices of selecting and sequencing supports their efforts to teach in ways that are responsive to students' mathematical thinking. However, "more research needs to be done to compare the value of different sequencing methods" (Smith & Stein, 2011, p. 11) and little is yet known about how teachers reason about the practices of selecting and sequencing students' work (Dunning, 2023). In this paper, we share about an ongoing data collection and analysis effort in which we seek to gain more understanding by sharing data gathered in the context of a lesson-centered professional development, examining the ways that two different groups of secondary mathematics teachers (one set of algebra teachers, one set of geometry teachers) annotated a set of storyboard frames representing written samples of students' mathematical work. In that context, teachers collectively considered how to leverage and expand upon students' mathematical work in the context of a problem-based lesson. Specifically, we ask:

In what ways do existing heuristics for categorizing teachers' practices of sequencing distinguish the set of interactions about sequencing we observed across two groups of StoryCircles teachers? What justifications for selecting and sequencing student work do teachers *make according to the type of sequencing they want to do?*

Theoretical Framework

While the aforementioned scholars have emphasized the importance of the lesson goal in shaping teachers' decisions regarding selecting and sequencing, empirical research suggests that teachers use a variety of ways to think about selecting and sequencing—some of which prioritize other matters as more pressing than the lesson goal (Ayalon & Rubel, 2022; Dunning, 2023). Researchers have suggested a variety of heuristics for selecting and sequencing students' work, including focusing on misconception (Smith et al., 2008) and building on correct strategies by employing the order of complexity heuristic (Meikle, 2016), but the field has yet to understand the differences in the ways that teachers' reasons about those heuristics. Expanding on this prior work, Ayalon and Rubel (2022) recently reviewed the lesson-centered PD literature and reported four distinct heuristics recommended to teachers for selecting and sequencing: (1) privilege variety of ideas, (2) accessibility and participation considerations require scaffolding by complexity, (3) the importance of attending to errors, and (4) any set of solutions yields particular connections and mathematical ideas. The authors also report that associating with these heuristics, especially the order of complexity, reflects a particular perspective on mathematical ability and teachers were more often observed prioritizing accessibility over the effort to develop a coherent sequence in a mathematical storyline. As such, Ayalon and Rubel (2022) suggests needing to find more strategies that integrate these heuristics:

Privileging a variety of ideas, including diverse strategies, creates the basis for the mathematical conversation that ensues (Dunning, 2023) and helps recognize the important mathematical connections within those ideas (Richards & Robertson, 2016). Such comparison of different strategies has been proven beneficial in enhancing students' procedural flexibility and deepening conceptual understanding (Rittle-Johnson et al., 2009). The variety could be based on the representation type (e.g., graphs, formulas), the underlying problem-solving strategy (e.g., drawing a diagram, looking for patterns), or solutions that produce divergent answers (Ayalon & Rubel, 2022). Teachers also consider strategies in terms of mathematical sophistication, recognizing less sophisticated strategies and increasing progression to the most complex (Ayalon & Rubel, 2022; Stein and Smith, 2011). This *scaffolding by complexity* could be based on "criteria like concrete to abstract, specific to general, or most common to unique" (Ayalon & Rubel, 2022, p. 4). By sharing or *attending to errors*, and then discussing the erroneous solution or contrasting it with another solution could help resolve misunderstandings (Rittle-Johnson et al., 2009; Stein et al., 2008). Meikle (2014) notes that teachers often begin with erroneous solutions. Finally, *yielding particular connections and mathematical ideas* has to do with building a mathematically coherent storyline (Smith & Stein, 2011) that aligns with the lesson goal.

We leverage the ideas from Practical Rationality (Herbst & Chazan, 2011, 2020) to argue that teachers' decisions can be accounted for on more than just individual resources, such as skills, knowledge, and beliefs. In particular, there are professional sources of knowledge that teachers draw crucially on, namely recognition of the (1) norms of instructional situations (Herbst, 2012) and (2) professional obligations of teaching mathematics. In prior work, we have described how teachers' recognition of the norms of instructional situations help account for their decisions related to selecting and sequencing student work (Schwarts et al., 2023), such as letting the lesson goal guide their decisions when there was tension with attending to all students. In this paper, we aim to describe ways that teachers' recognition of the professional obligations of mathematics teaching play a role in informing their decisions about sequencing. While a complete description of practical rationality and all of its components are beyond the scope of this paper, we do take a moment to describe more details related to the professional obligations.

To describe what sources of justification are available for a teacher to justify decisions, we build on the idea that teachers' actions relate to the obligations they need to satisfy as teachers in an instructional system (Herbst & Chazan, 2011). These obligations bind the teacher to the environment of the instructional system which impacts their ties with the students and content in an instructional triangle (Cohen et al., 2003). At the same time, obligations provide sources of justification for a teacher's action, just as much as norms of instructional situations do, sometimes even justifying deviations from the norms as well. Herbst and Chazan (2011) propose four professional obligations which demand teachers' attention, namely the professional obligation to attend to: (1) the discipline that the teacher is meant to represent (*Disciplinary Obligation*), (2) the individual students the teacher is meant to serve (*Individual Obligation*), (3) the socio-cultural world of a given society and its customs and values (*Interpersonal Obligation*), and the (4) institution(s) that create official time, space, and sanction for all those relationships to happen (*Institutional Obligation*).

Methods

Context

The context of this study is Story*Circles*, an innovative professional development model where secondary mathematics teachers collaboratively anticipate a problem-based lesson through iterative phases of scripting, visualizing, and arguing about alternatives (Herbst & Milewski, 2018, 2020). Similar to Lesson Study (Lewis, 2009), Story*Circles* engages groups of practitioners (secondary mathematics teachers in this study) to create a lesson through successive iterations. This activity is animated not only by the processes of scripting, visualizing and arguing, but also the problems of practice that the group collectively identifies and shows a willingness to work on as they work out the details of the lesson (see Figure 1). The group's activities are supported by a facilitator who is not simply another group member who suggests possible ideas for the lesson, but instead takes up the role for directing the discussions toward the instructional goal and orienting practitioners toward one another—encouraging participants to share their professional knowledge through the creation of a common artifact. Distinct from Lesson Study, the group is involved in this process virtually and uses storyboards to visualize their practice and receive feedback on it (Milewski et al., 2018).

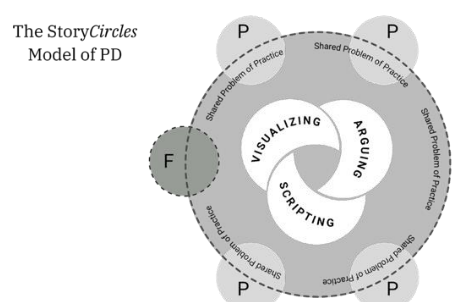
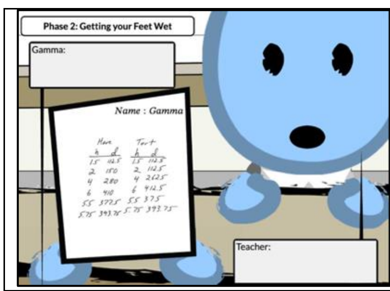


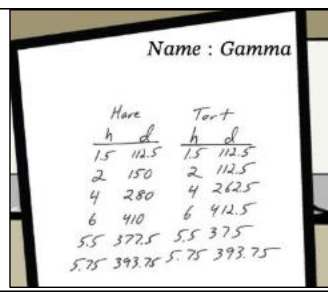
Figure 1: The Story*Circles* Model of PD

The data in this report is drawn from three groups of secondary mathematics teachers (1 Geometry and 1 Algebra) engaged in a six-week long Story*Circles.* All three groups focused on transforming a thinly developed storyboard (which contained only a few key frames) into a fuller representation of the way a lesson could unfold. The geometry group focused on the Tangent Circle Problem and the algebra groups focused on the Walkie-Talkie Difference of Functions Problem. Across these engagements, teachers engaged in asynchronous activities which include the frames for different phases of a lesson: Problem Posed (Phase 1); Getting Your Feet Wet (Phase 2); Whole Class Check-In (Phase 3); Redirecting the Work (Phase 4); Whole Class Discussion (Phase 5); Goal Statement (Phase 6). First, the teachers were expected to review and leave comments on the frames in each phase. Then, they were expected to discuss these frames considering how they could modify and/or use those frames to develop their storyboard.

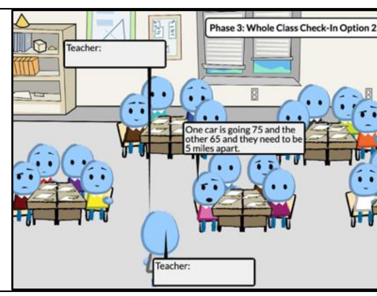
For the purpose of this analysis, we focus on suggestions secondary mathematics teachers from two groups (1 Geometry and 1 Algebra) made about a set of frames representing the second phase: Getting Your Feet Wet. In these frames, student work (SW) examples are shown from the viewpoint of looking over the shoulder of the student. The frames were provided to give resources for the group of teachers to consider for incorporation into their storyboard (see Figure 2a) and discuss during the synchronous meetings. Across the 15 frames, the student work samples varied in terms of their strategies, correctness, normativity to the instructional situation, serviceability for advancing toward the instructional goal, and their responsiveness to the problem statement (Herbst et al., 2023). In their interactions with these frames, secondary mathematics teachers had the opportunity to delve more deeply into given SW examples to notice the features of SW beyond their correctness. The teachers were expected to consider which ideas they would address in the context of a small group setting and which they would feature as part of the coming whole class interactions which refer to the latter phases. For this reason, while the actions are around the decisions of this phase, the justifications may expand on to the other phases if the teachers felt they were relevant to their decisions in Phase 2.



a. A frame from Phase 2: Getting Your Feet Wet



b. A SW from Phase 2: Getting Your Feet Wet



c. A frame of Phase 3: Whole Class Check-in

Figure 2: Examples from different phases of a lesson

Data Sources and Analysis

The data used for this analysis comprise memos and transcripts from synchronous meetings involving two groups of teachers who participated in Story*Circles*. Each group of teachers developed a Geometry or Algebra lesson-centered storyboard over six to seven one-hour synchronous meetings. We identified the specific synchronous meetings from each group where they made the most changes to the storyboard by tracking alterations in their 6 to 7 versions of the storyboard, each created during each of the synchronous meetings.

Table 1: Examples of actions categorized by Ayalon and Rubel's (2022) heuristics

Types of Heuristics	Examples
Privilege variety of ideas	"I like how there's a table, a graph, and an equation."
Accessibility and participation considerations require scaffolding by complexity	"I might start with the work of [student] Epsilon that's at a very basic level. And then maybe go to [students] Delta or Kappa or [Omicron], or Kappa's work to share first, and then, as I progressed through the frames show a more detailed understanding."
Attending to errors	"they [(students whose work was selected)] accounted for the 5 miles, which I don't think any of the other ones had done at that point."

Within the synchronous meeting, our focus was on moments when participants discussed the selection and sequencing of student work. We conducted an analysis of the actions taken by the participants during professional development, employing coding based on Ayalon and Rubel's (2022) framework for guiding teachers in selecting and sequencing students' solutions (see Table 1). Then, we examined the participants' justifications following their actions, drawing on the four professional obligations (Herbst & Chazan, 2011; see Table 2).

Table 2: Examples of justifications based on types of professional obligations

Professional Obligations	Examples
Disciplinary Obligation	"the difference in [Gamma's and Mu's] answers, because they're not wrong. If I did my math right, they're not wrong. They're just representing 2 different things."
Individual Obligation	"the kid that doesn't understand by the time the first thing comes out it's already way over his head. They're checked out, they're done. And so I think it does make a lot of sense to keep it in that order."
Interpersonal Obligation	"because they're both inputting hours. And maybe the kids [can] discuss where that 1.5 differences [are] and what that 1.5 represents."
Institutional Obligation	"I think that we need to be realistic. So how long is, I mean, I don't know of any math teacher that's getting more than an hour of instructional time, do you have? Well, how long is your class?"

Through this analysis, we consider the following questions:

1. How well do Ayalon and Rubel's (2022) heuristics for categorizing teachers' practices of sequencing distinguish the set of interactions about sequencing we observed across two groups of Story*Circles* teachers?

2. What justifications for selecting and sequencing student work do teachers make according to the type of sequencing they want to do?

We found that actions related to selecting and sequencing student work could be classified according to the heuristics described by Ayalon and Rubel (2022). Specifically, we were able to categorize participants' ways of sequencing student work based on the first three categories of action featured in Table 1 as they were more prevalent in our data and easily distinguishable. With more data, we may be able to make use of the fourth heuristic, yielding mathematics connections and ideas.

Results

Across the sets of exchanges about sequencing that we observed, a higher occurrence of the suggestions in those exchanges favored the *variety of ideas* and *attending to errors* heuristics than the *order of complexity* heuristic (see Table 3, number of actions). This could be attributed to our focus on synchronous meetings where most changes were made to the storyboard artifact, and participants began examining individual SW during the "Getting Your Feet Wet" phase. At the same time, focusing on decisions that resulted in the most changes in the artifact lends strength to the actions the participants claimed they wanted to take in sequencing and selecting student work.

Table 3: Relationship between the heuristics argued for and the professional obligations used to support those argument

	Number of Actions	Disciplinary	Individual	Interpersonal	Institutional
Variety of ideas	20	16	1	2	1
Order by complexity	10	5	4	1	0
Attending to error	26	16	8	2	0
Number of justifications:		37	13	5	1

In terms of the justifications made by participants, we found more attention to the disciplinary and individual obligations (see Table 3). Some of the references to the disciplinary obligations were related to the teacher's recognition of the need to sequence work intentionally to build on each other and ensure mathematical correctness (see Table 2, example of disciplinary obligation). Participants also delved into specific mathematical features of student work, such as markings on geometric constructions, providing insights into the students' thinking process and procedure. Individual obligations were also prominent, with a focus on how the individual student who created the work was thinking (see Table 2, example of individual obligation). Another common justification attending to individual obligations involved focusing on students who may be struggling and considering how they might benefit from the student work being potentially shared out. The interpersonal obligation was typically attended to when related to how other students as a whole class might benefit from the student work selected or sequenced and how they might discuss it among themselves (see Table 2, example of interpersonal obligation). Finally, the institutional obligation didn't receive much attention when considering the practices of selecting and sequencing student work. One instance of the institutional obligation was mentioned when participants were attending to time constraints in a typical mathematics class when considering how much variety of student work they would then want (see Table 2, example of institutional obligation).

As we consider the relationship between the types of heuristics participants argued for and the professional obligations used to support those arguments, we generally found that the disciplinary obligation was a consistent priority for participants, regardless of their preferences in selecting and sequencing student work (see Table 3, number of justifications). When using the variety of student work heuristic, participants emphasized conceptual differences among various types. These distinctions were clarified through specific procedures, such as inputting different numbers, and involved prioritizing evidence of students' understanding of the problem over correctness, although the correctness was often considered a distinguishing factor among types of student work. When using the attending to errors heuristic, participants often justified their actions using the disciplinary obligation—providing justifications related to addressing misconceptions and describing specific mathematical features they felt were important for students to learn and for the work to highlight so that there is a progression towards the lesson goal.

In contrast, when using the scaffolding by complexity heuristic, participants often justified their suggestions with the individual obligation—arguing for the importance of understanding how individual students, who completed the work, were thinking and how certain tasks in the earlier sequence would provide opportunities to support struggling students with comprehension. For example, while creating a table might be easier for some, presenting a graph first could offer a visual aid to enhance problem understanding. This reflects Meikle's (2016) general rationale for sequencing solutions, which typically involves the order of complexity heuristic while remaining independent of the mathematical content.

We also acknowledge the important role that the interpersonal obligation played in participants' justifications. References to this obligation involved attending to all students and ensuring that the selected and sequenced student work would contribute to class-wide conversation and all students' learning. This also demonstrates how the participants adhered to the design of the professional development as the latter phases of the storyboard included activities like whole-class discussions, which were observed by the participants in the professional development and connected to initial selection and sequence of student work.

Conclusion

We illustrate a potential relationship between teachers' heuristics for sequencing student work and particular professional obligations. We found that the disciplinary obligation remained a consistent priority for participants, leading to an emphasis on the comparison of strategies, progression of procedures, and learning opportunities by addressing errors. At the same time, the individual obligation was crucial, especially when using the scaffolding by complexity heuristic, as teachers argued for comprehension while maintaining independence from the mathematical content. Purposefully selecting and sequencing strategies and fostering a shared language among teachers to discuss these ideas is an important emphasis in Story*Circles*. This practice of selecting and showcasing a representative solution from each category adds significant value in the eventual whole class discussions (Ayalon & Rubel, 2022). Planning for and leading mathematical discussions is challenging due to its inherent complexity, requiring teachers to notice and interpret students' thinking and make on-the-spot decisions. The attention to heuristics for selecting and sequencing and the extent to which they depend on the professional obligations provides the field with greater opportunities to understand the ways that teachers reason about important instructional practices. Furthermore, providing teachers with opportunities to learn more about why other teachers make the decisions they do has the potential to help support teachers gain greater clarity on their own instructional practices.

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FACILITATING PRODUCTIVE ONE-ON-ONE DEBRIEFING CONVERSATIONS: ANALYZING THE LESSON

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One-on-one coaching often involves a coach and teacher meeting to debrief after a lesson. One function of debriefing conversations is to enable coaches and teachers to look back on the lesson and figure out whether the instructional changes the teacher made were in fact improvements. We examined 25 cases in which a mathematics coach and teacher debriefed after a lesson to understand how coaches can support teachers in accomplishing this function. The findings emphasize the importance of mathematics coaches supporting teachers to explicitly link the instructional changes they made with their implications for students' thinking, learning, or engagement. Doing so can enable teachers to see whether the changes they made were improvements. In further clarifying this process of analyzing a prior lesson, we contribute to research examining effective one-on-one coaching interactions.

Keywords: Professional Development, Teacher Educators, Middle School Education.

Researchers have reached a consensus on the types of instructional practices that can support students in attaining rigorous mathematical learning goals, such as those proposed by the Common Core State Standards in Mathematics (National Governors Association, 2010). Often referred to as *ambitious and equitable instructional practices* (Lampert et al., 2011; Lampert et al., 2010; Author & colleague, 2010), these practices include selecting tasks of high cognitive demand (Stein & Lane, 1996), launching tasks in ways that enable all students to begin working on them productively (Jackson et al., 2013), and pressing students to explain their reasoning and make connections between solution strategies during whole-class discussions (Kazemi & Stipek, 2001; Stein et al., 2008). Ambitious and equitable instructional practices differ significantly from typical instruction in most US mathematics classrooms (Hiebert, 2013), and there is substantial evidence that teachers require sustained support if they are to develop such practices (e.g., Author et al., 2018). Many schools and districts are providing the required support by hiring *mathematics coaches* to work directly with mathematics teachers to aid them in developing ambitious and equitable practices (Mudzimiri et al., 2014; Obara, 2010).

Mathematics coaches often work with mathematics teachers by engaging them in one-on-one coaching cycles. One-on-one coaching cycles consist of three phases: a lesson planning phase, in which a coach and teacher collaboratively prepare for a focal; a classroom instruction phase, in which a coach and teacher implemented the focal lesson; and a debriefing phase, in which the coach and teacher collaboratively analyze the impact of focal lesson (West & Staub, 2003). There is evidence that one-on-one coaching cycles can support individual mathematics teachers' development of ambitious and equitable instructional practices *when coaching cycles are facilitated effectively* (Kraft & Hill, 2020; Russell et al., 2020).

Researchers have made progress in understanding what it looks like for coaches to facilitate one-on-one coaching cycles effectively, especially as it relates to planning conversations (Russell et al., 2020; Witherspoon et al., 2020) and the classroom instruction phase (Saclarides &

Munson, 2021). Yet, comparatively less is known about how mathematics coaches can facilitate debriefing conversations effectively. In particular, little is currently known about how coaches can support teachers to look back on a lesson they taught and determine whether any changes they made in their instruction actually enhanced students' learning opportunities. As we note below, making such a determination is a primary function of debriefing. In this report, we clarify how coaches can support teachers in determining whether a change in instruction enhances students' learning opportunities, and is thus an improvement. We do so by examining 25 coaching cycles in which a mathematics coach and teacher debriefed after the lesson.

Literature Review

In recent years, scholars have begun to investigate how coaches can facilitate one-on-one coaching cycles productively, such that they can support teachers' development of ambitious and equitable instructional practices (e.g., Kochmanski & Cobb, 2023a; Munson & Dyer, 2022; Russell et al., 2020; Witherspoon et al., 2021). To date, researchers have predominantly focused on the co-planning phase (Rusell et al., 2020; Witherspoon et al., 2021) and lesson enactment phase of coaching cycles (Munson & Dyer, 2023; Saclarides & Munson, 2021). Far fewer studies have examined how coaches can support teachers' development of ambitious and equitable instructional practices in the debriefing phase of coaching cycles (Saclarides, 2022). That said, there is evidence that debriefing conversations can function in two ways to support teachers' learning (Gillespie et al., 2023; Saclarides, 2022). First, coaches and teachers can look back on a lesson to figure out whether any instructional changes they made enhanced students' learning opportunities (Hinojosa, 2022; Saclarides, 2022), and thus constituted improvements. We refer to this as the *retrospective function* of debriefing. Second, coaches and teachers can collaboratively identify new instructional changes they might make in future lessons (Kochmanski & Cobb, 2023b; Russell et al., 2017; Saclarides, 2022). We refer to this as the *prospective function* of debriefing.

In our prior work (Kochmanski & Cobb, 2023b), we have clarified how coaches and teachers can collaboratively identify the specific instructional changes they might make in future lessons. We found it is important for coaches to first prepare for a debrief by analyzing the lesson and identifying possible potential instructional changes that the teacher might make. Much as anticipated student solutions can inform class discussions, the instructional changes a coach identifies for a teacher ahead of a debrief can inform how the coach supports the teacher in setting improvement goals when debriefing. Further, we found it is essential for coaches to identify instructional changes that are (a) feasible for the teacher to make with the coach's support and (b) likely to immediately enhance students' learning, if attained (Kochmanski & Cobb, 2023b). We consider changes to be feasible when they build directly from a teacher's current practices. Instructional changes are likely to enhance students' learning when they occur in phases of lessons where students' learning first breaks down. To illustrate this latter criterion, consider a lesson in which almost all students could not start the mathematics task because the launch was confusing. In this case, working to improve how the teacher facilitates whole-class discussions would not immediately enhance students' learning because most students are unlikely to benefit from the whole-class discussion if they have not had opportunities to engage themselves in the mathematics task prior to the discussion. In contrast, it would immediately benefit students to make changes in how the teacher launches the task.

While we have made progress in understanding how coaches can support teachers to accomplish the prospective function of debriefing, there is still much left to learn the retrospective function of debriefing conversations. We are unaware of any studies that have closely investigated how coaches can facilitate debriefing conversations in which they support teachers in determining whether the changes they tried out in the lesson enactment phase actually benefited students' learning and were thus improvements in teaching. Given this, we asked the following research question: *How can mathematics coaches support teachers in determining whether the instructional change(s) they made during the lesson benefitted students' learning, and were thus improvements?*

Methods

Study Context

This study occurred in the context of a multi-year research-practice partnership (RPP) between university researchers and district leaders from a large, urban school district in the south-eastern United States. The district comprises 128 total schools, including 74 elementary schools, 30 middle schools, and 24 high schools. It serves over 85,000 students and is highly diverse, with over 100 languages spoken by students. Over 40% of the district's students are classified as economically disadvantaged and just over 20% of students have limited English proficiency. A primary goal of the RPP was to improve the quality of mathematics instruction in the district by training a cadre of middle and secondary mathematics coaches who could work with teachers in their schools. As part of the RPP, researchers and district leaders conducted a coach professional development (PD) design study that aimed to support mathematics coaches in learning to conduct one-on-one coaching cycles effectively with teachers. In the PD design study, researchers and district leaders collaborated to design and implement a sequence of eight 90-minute PD sessions. Because debriefing is a central part of coaching cycles, a significant portion of the PD design study focused on supporting the coaches to prepare for and then facilitate debrief conversations productively, including how coaches can support teachers to determine whether changes in instruction constitute improvements. Consequently, if the PD was successful, we would be able to collect data that would enable us to analyze the types of debrief conversations in which we were interested.

Participants

Fourteen middle school mathematics coaches and one high school mathematics coach participated in the coach PD. All 15 coaches were school-based and all of them were hired from a district-approved pool of applicants who had demonstrated prior success in ambitious mathematics teaching. We purposefully selected seven focal coaches to be representative of the range of coaching experience in the larger group. We collected data on these seven coaches' enactment of coaching cycles with the same teacher after each of the coach PD sessions. All seven focal coaches had more than five years of experience teaching prior to becoming a coach. Four of the seven coaches were novices in their first year of coaching, two of the seven coaches had extensive experience coaching in the district, and one coach had multiple years of coaching experience but was new to coaching in the district. Six of the coaches worked in middle schools, and one of the coaches worked in a high school.

Data Collection

We documented 25 total coaching cycles conducted by the seven focal coaches. For each of the cycles, we collected three types of data relevant to answering our research question. First, we audio recorded the co-planning conversations between the coach and the teacher. Reviewing the audio recordings of the co-planning conversations enabled us to identify the instructional changes the coaches *intended* to make in the lesson. Second, we observed each lesson and wrote structured lesson observation notes. We also collected students' work from the lesson. These data enabled us to see whether the teacher actually made the intended instructional changes, as well as whether those changes appeared to enhance students' learning opportunities. Finally, we audio recorded the debriefing conversations between the coach and teacher. These audio recordings enabled us to see whether and how the coaches and teachers discussed the instructional changes the teachers made in the lesson.

Data Analysis

We conducted two phases of analysis to answer our research question. In the first phase of analysis, we determined whether the coach supported the teacher in accomplishing the retrospective function in each of the 25 coaching cycles. We considered coaches and teachers to have accomplished the retrospective function when the debriefing conversations satisfied two criteria: (1) the coach and teacher concluded whether or not the teacher made an intended instructional change in the lesson and (2) if the teacher made an instructional change, the coach and teacher accurately determined whether or not the change improved students' learning opportunities, and thus constituted an improvement. In phase two, we answered our research question by comparing cycles in which the coach supported the teacher to accomplish the retrospective function with those in which the coach did not.

Phase 1. We first determined whether each debriefing conversation satisfied the first criterion. To do so, we identified the instructional change(s) the coach and teacher *intended* to make in the lesson by analyzing the co-planning conversations for each cycle. We used inductive coding to record the intended instructional changes in a table. We then determined whether the teacher made the instructional change by analyzing the structured lesson observation notes and students' work. Finally, we identified episodes in the debriefing conversations in which the coach and teacher discussed the instructional change. We compared these episodes with our own assessment of the lesson to see whether the coach and teacher accurately determined whether the teacher made the intended instructional change(s).

We next determined whether the debriefing conversations satisfied the second criterion. This took three steps. First, we identified when in each lesson students' learning first broke down (e.g., launch of the task, small group work time, whole-class discussion), if it did at all by analyzing the structured lesson observation notes and students' work from the lesson. Following a process similar to that described in Kochmanski and Cobb (2023b), this involved asking two analytic questions: "Were all students able to work meaningfully on the tasks?" and "Was the range of student strategies in the lesson rich enough to have a productive discussion?" If we answered no to either of the questions, we considered students' learning to have broken down in the launch of the task or in the small group work time prior to the whole-class discussion because the potential for students' learning in the whole-class discussion wasn't there. If we answered yes to the questions, we considered students' learning to have broken down during the whole-class discussion, if it did at all. For example, in one case, the structured lesson observation notes and students' work indicated that almost all students had blank papers during their small group work time, indicating that students' learning opportunities broke down during the launch of the task.

The first two authors analyzed each lesson separately, and then met to reach consensus on when in the lesson students' learning broke down, if it did at all.

In the second step, we compared the instructional change the teacher made with the results of our analysis from step one to determine whether the instructional change improved students' learning opportunities. We considered an instructional change to have improved students' learning opportunities when the change occurred *in* the phase of the lesson in which students' learning first broke down or in a phase *prior to* the break down in students' learning. For example, if our assessment of a lesson indicated that students' learning broke down in the whole-class discussion, and the coach and teacher made a change in how they organized small group work time, then we considered the instructional change to have improved students' learning opportunities. We also considered an instructional change to have improved students' learning if there was no evidence of a break down in students' learning. In the third step, we identified episodes in the debriefing conversations where the coach and teacher discussed whether the instructional change the teacher made was an improvement. We classified debriefing conversations as satisfying the second criterion when our analysis of the instructional change corresponded with the coach's and teacher's conclusions.

Phase 2. As a reminder, in phase 2, we compared cycles in which the coach supported the teacher to accomplish the retrospective function with those in which the coach did not. We first identified episodes in the debriefing conversations in which the coach and teacher discussed the instructional changes the teacher made in the lesson. We then used open coding to characterize coaches' and teachers' justifications for whether and why the instructional change, if made, was an improvement or not. We open coded these episodes because there is limited research examining why coaches and teachers see instructional changes as improvements, and there was thus no available coding scheme that was adequate for our purposes. When appropriate, we adapted language for our open codes from prior research examining one-on-one coaching conversations between coaches and teachers (e.g., Kochmanski & Cobb, 2023b; Russell et al., 2020; Saclarides, 2022). Table 1 shows the codes we used to characterize the justifications. We then compared the justifications in the cases that accomplished the retrospective function with the justifications in the cases that did not.

Table 1: Justification Codes for Debriefing Conversations (Retrospective)

Coach and Teacher Justification	Definition
Coach and/or teacher attributed changes in students' thinking to changes in instruction	Discussed how an instructional change impacted how students reasoned mathematically in the task or explained and justified their reasoning in whole-class or small-group discussions.
Coach and/or teacher attributed changes in students' engagement to changes in instruction	Discussed how an instructional change the teacher made impacted how students engaged in the task or students' apparent confidence in engaging in the task.

Teacher connected an instructional change to opportunity to learn about students' thinking	Discussed how an instructional change impacted their ability to understand how students were thinking about mathematics in the lesson.
Teacher made the intended instructional change	Based determination solely on whether the teacher actually made the intended change in the lesson.
Coach compared the instructional change with vision of effective, high-quality teaching	Coach tells the teacher that the instructional change is an improvement because it marks a shift toward what the coach sees as effective teaching.

Results

We found that coaches supported their partner teachers in accomplishing the retrospective function in 13 of the 25 coaching cycles. In six of the 12 unproductive cycles, the coach and teacher accurately determined whether the teacher made the intended instructional change. However, their assessment of whether that instructional change was an improvement was at odds with our analysis of the lesson. In one of the unproductive cycles, the coach and teacher did not discuss an instructional change in the debriefing conversation. They therefore could not accomplish the retrospective function. In the remaining five cycles, the coach and teacher did not specify an intended change in the co-planning conversation.

To understand how coaches can support teachers in accomplishing the retrospective function of debriefing, and thus answer our research question, we compared the 13 productive cycles with the six cycles in which the coach and teacher's assessment of the instructional change differed from our own. Table 2 shows the frequency of the types of coach and teacher justifications organized by whether the coach supported the teacher in accomplishing the retrospective function. The types of justifications that coaches and teachers gave in the 13 productive cases differed from the corresponding justifications in the six unproductive cases.

Table 2: Frequency of Justification Codes

Type of Justification	Number of cases	
	Accomplished retrospective function	Did not accomplish retrospective function
Coach and/or teacher connected changes in students' thinking to changes in instruction	10	0
Coach and/or teacher connected changes in students' engagement to changes in instruction	2	0
Teacher connected opportunities to learn about students' thinking to changes in instruction	1	0

Teacher made the intended instructional change	0	4
Coach related the instructional change with vision of effective, high-quality teaching	0	2

In each of the 13 cases with productive debriefing conversations, the coach and/or the teacher connected changes in students' thinking, their engagement in the task(s), or opportunities for the teacher to *learn about* students' thinking to the instructional change the teacher had made. In one coaching cycle, for example, the coach and teacher planned to implement a more open-ended task in the lesson. The teacher implemented the task as intended in the lesson. In the debriefing conversation after the lesson, the coach and teacher identified changes in students' thinking, and they attributed the changes to the teacher's decision to implement the more open-ended task. Specifically, the coach and teacher noticed that the students had to figure out their own strategies for solving the task, rather than using a strategy that was hinted at in the task. Consequently, there was a wide range of solution strategies in the small group discussion time on which the teacher could capitalize in the concluding whole class discussion. The coach prompted the teacher to relate these changes in students' strategies to their decision to implement an open-ended task, asking, "What did you think about [the new task]?" In response, the teacher *attributed the changes she noticed in students' thinking* to her selection of the open-ended task. She noted that students in her class usually "figure out something to do with the numbers," and then come up with an answer without explaining why it might make sense. However, with the new task, the teacher observed that her students "actually had to problem solve" and discuss different solution strategies in their small groups. Then, the groups had to draw on their prior mathematics knowledge to settle on an answer. The teacher and the coach agreed that the new task was an improvement because it resulted in enhanced student learning opportunities.

In contrast, in all six cases with unproductive debriefing conversations, the coach and teacher discussed the change in instruction without considering whether the change made a difference for students' learning and/or engagement. In four of the cases, the coach and/or the teacher concluded that making the intended instructional change was a sufficient reason to consider the change as an improvement. In one case, for example, the coach and teacher worked together to ensure the teacher could "make it through" all parts of her lesson plan. In the co-planning conversation, the coach suggested that she and the teacher make a timeline for the lesson and then use a timer to stay "on track". The teacher used the timer successfully in the lesson. In their subsequent debriefing conversation, the coach and teacher agreed that the lesson was an improvement because the teacher stayed on track. While the coach and teacher perceived this change in instruction to be an improvement, our analysis indicated that this change had limited bearing on students' learning opportunities. Instead, we found that the teacher provided minimal support to students during the launch of the task, and as a result, many students left their papers blank during their work time, indicating they struggled to get started on the task. Completing all phases of the lesson therefore had limited impact on students' learning opportunities because students' learning opportunities first broke down early in the lesson during the launch of the task.

In the other two cases, both coaches used their visions of high-quality teaching to assess the instructional changes their partner teachers had made and then simply told the teachers that the changes were improvements. To illustrate, we focus on a coaching cycle in which the coach and teacher worked to improve students' engagement by making the task more "relatable" to students. To do so, the coach and teacher decided to start the lesson with an introductory task about her recent experience buying strawberries at a grocery store. However, the primary task for the lesson focused on estimating the cost of oranges. In the debriefing conversation following the lesson, the coach and teacher observed that the teacher "kind of lost [students] after a while" because it took a long time to discuss her trip to the store to buy strawberries. Despite this, the coach told the teacher that "making [the launch] personal was really good" She then went on to note that this kind of personal connection is an effective teaching strategy, and thus constituted an improvement. Unfortunately, their decision to include a personal story both increased the time the teacher spent introducing the task and appeared to be confusing for many of the students who then struggled to begin working on the task. Our assessment of whether the instructional change was an improvement therefore differed from that of the coach.

Discussion and Conclusion

The goal of this analysis was to clarify how mathematics coaches can support teachers in determining whether the instructional change(s) they made during the lesson benefitted students' learning and their engagement and were thus improvements. We found that in the cases in which coaches succeeded in supporting teachers in making this kind of determination, they connected changes in students' thinking and their engagement to changes in instruction. In contrast, we found that in the unproductive cases, the coaches and teachers either concluded that making the intended instructional change was a sufficient reason for the change to be an improvement or the coaches used their visions of high-quality teaching to assess whether the changes were improvements for teachers. It therefore appears unproductive for coaches to focus only on the teacher's actions without considering the consequences of those actions for students' thinking and engagement.

The results of our analysis make a significant contribution to research on one-on-one mathematics coaching. Specifically, our findings extend prior research examining one-on-one debriefing conversations in mathematics coaching (Russell et al, 2017; Cross Francis et al., 2021; Saclarides, 2022). Several recent studies of one-on-one coaching have examined the focus and nature of coaches' and teachers' talk during debrief conversations descriptively (Amador et al., 2024; Cross Francis et al., 2021; Gillespie & Amador, 2024; Saclarides, 2022). Our study builds on and extends this primarily descriptive work by clarifying how coaches can accomplish the retrospective purpose of debriefing conversations, thereby advancing our understanding of how coaches can facilitate productive debriefing conversations. Specifically, we highlight the importance of linking students' learning opportunities to instructional changes. While prior research points to the importance of connecting students' learning to instruction during co-planning conversations (Russell et al., 2020; Witherspoon et al., 2020) and when observing a lesson to identify instructional improvement goals for teachers (Kochmanski & Cobb, 2023b), this is the first study to highlight the importance of this connection in debriefing conversations.

Regarding implications for future research, it appears important for coaches to support teachers in connecting students' learning and instruction in all three phases of coaching cycles. Future research might investigate in greater detail how coaches can be supported to see this underlying principle as a central tenet of productive one-on-one coaching cycles. Further, we suspect that coaches who support teachers in connecting students' learning (or lack thereof) to their instruction might also be supporting teachers in coming to view instructional limitations as the primary source of students' learning difficulties (Jackson et al., 2017). This would be a significant development as prior research indicates many mathematics teachers attribute the difficulties of students they perceive are struggling primarily to inherent characteristics of the students (Jackson et al., 2018). Future research might investigate whether this is the case. Finally, while we investigated the retrospective function of debriefing conversations in the context of one-on-one coaching cycles, we anticipate that our findings will prove relevant to other settings in which teachers analyze their instruction with a colleague, such as mentor-mentee relationships in pre-service teacher education and when teachers work with a peer to improve their instruction. However, this conjecture requires further investigation.

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EXAMINING RELATIONSHIPS BETWEEN ELEMENTARY TEACHERS' REFLECTIVE JOURNAL ENTRIES AND MATHEMATICS TEACHING PRACTICES

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Keywords: Instructional Activities and Practices, Professional Development

Professional learning programs are designed to promote teacher learning to foster improved instructional decision-making to achieve better student learning outcomes (Borko et al., 2010; Darling-Hammond et al., 2017). One way to investigate teachers' professional learning experiences is to incorporate reflective journals into professional development. Reflecting in journals affords opportunities for teachers to document their experiences and make sense of their professional learning (Brown et al., 2011; Henderson et al., 2004). In this study, I examined relationships between two elementary teachers' intentions, as documented in reflective journals kept during a professional learning experience, and their mathematics teaching practices one year later. This descriptive case study investigates how elementary teachers made sense of their professional learning in their reflective journals, and then, a year later, which aspects of teaching from reflective journals appeared to translate into their mathematics teaching. The research question that guided this study: *In what ways do elementary teachers' reflective journaling about their professional learning translate into their mathematics teaching practices a year later?*

The main finding is that two elementary teachers' thinking in their reflective journals translated into their mathematics teaching practices, based on both reported and observed data. The aspects of mathematics teaching that teachers intended to improve – identifying the key learning goals of the lesson and creating intentional questions – were key aspects of effective mathematics teaching (NCTM, 2014; Smith & Stein, 2018). Figure 2 summarizes both participants' intentions and their mathematics teaching practices. This study has implications for teacher educators and professional learning designers to consider embedding reflective journaling as part of their professional learning programs. Reflection can be an important tool that supports professional learning and teacher education (Cimer et al., 2013; Schön, 1983).

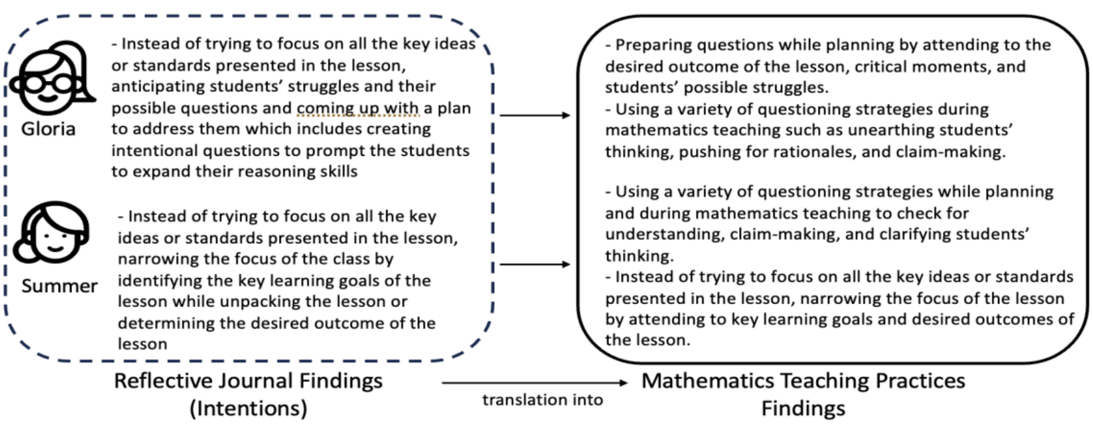

Figure 2: Summary of the Findings

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FRAMING EVAN'S ERROR AS WORTH UNDERSTANDING: HOW TEACHERS NEGOTIATED THEIR NOTICING

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Examining how teachers notice students' errors is one part of understanding how teachers can learn to respond to errors productively. This paper explores how teachers notice student errors. Across one discussion, teachers negotiated their noticing by seeking, offering, and sharpening accounts of video evidence, and by offering, extending, and contesting interpretations of video evidence. Together, these moves constituted their shared framing of the error as worth understanding. This research contributes to our understanding of how teachers notice and can co-construct their noticing in the context of video-based professional development.

Keywords: Professional Development, Teacher Noticing

Mathematical errors — students' contributions that are not mathematically complete, precise, or correct (Baldinger et al., 2021) — are resources that can be leveraged to support students' learning (Borasi, 1987; NCTM, 2000). Teachers can work with students to examine errors to extend and deepen their mathematical thinking (Tulis, 2013). Yet, dominant deficit discourses in mathematics education typically value correctness over meaning, making productively engaging students' errors difficult for teachers (Adiredja & Louie, 2020). Indeed, teachers often ignore or simply correct errors (Santagata, 2005). Even teachers who report believing that errors are resources do not consistently use them as such in the classroom (Alvidrez et al., 2022). How, then, can teachers approach errors in more productive ways?

Every decision and response a teacher makes is, in part, an extension of something they *noticed* (Mason, 2002). Noticing has been conceptualized as the dynamic interplay of teachers' attention, interpretations, and framing (Louie et al., 2021), where framing is understood as the process of activating or developing schemas that both construct and are constructed by what teachers see and their sense of its meaning (Sherin & Russ, 2014). These schemas represent a "framework of frameworks" for a participant in an activity, reflecting a key element of the broader participant group's culture (Goffman, 1974, p. 27). Students running and shouting may be interpreted favorably by a teacher monitoring recess, and less favorably by a teacher monitoring silent reading time. The teacher's understanding of the nature of either activity (her framing) influences her attention to and interpretations of students' running and shouting, and, in turn, her attention to and interpretations of their running and shouting influences her understanding of the nature of either recess or silent reading time.

Teachers can learn to notice students' mathematical thinking (Dindyal et al., 2022), and noticing students' thinking has been linked to improved student learning outcomes (Kersting et al., 2010). Less is known about how teachers notice errors specifically. With this in mind, I unpack a discussion among three teachers about one student's mathematical error, tracing how they negotiated what happened in the video and what it meant to reach a well-supported and shared conclusion, asking: In one productive discussion about one student's mathematical error, how did teachers frame Evan's error as worth understanding?

Methods

Data come from a nine-week, online, professional development course for in-service K-2 mathematics teachers. Nine teachers participated in the course. One goal of the course was to support teachers to notice strengths in students' thinking. On alternating weeks, teachers recorded and uploaded video clips of their students trying new tasks to an online platform, then watched and marked moments of students' thinking in their own and their peers' videos using a video annotation tool (Larison, Richards, and Sherin, 2022). The other weeks, teachers met virtually on Zoom in small groups to discuss their videos. I facilitated the virtual video clubs (Sherin & van Es, 2009), working to maintain our focus on students' thinking in the videos. Here I explore part of one video club between a differentiation support teacher, Bea, and two second-grade teachers, Casey and Glenda (all names are pseudonyms).

A ~four-minute-long video clip from Bea's classroom formed the basis for the teachers' discussion. In the clip, three students shared their thinking about what goes in the blank in the equation 8 + 5 = __ + 7. Evan, a first grader who had historically attained high test scores, originally wrote "4" in the blank. Bea asked, "Why did you put a four there?" He responded, "Because eight plus…" pausing mid-sentence. After ~five seconds, Bea asked, "What are you thinking?" Evan told her, "I got mixed up…I thought the seven was more than eight." He concluded that "six" should go in the blank rather than four.

I examine one ~five-minute discussion segment among the three teachers as a case (Yin, 2009) of a productive discussion about a student's error to unpack teachers' noticing. Productive discussions about students' thinking have been conceptualized as focused on students' thinking, concerned with mathematically substantive ideas, and involving joint sensemaking among teachers (Sherin et al., 2009). I conceptualize a productive discussion about a student's *error* as also reaching a well-supported and shared conclusion about the meaning in the student's error. I do not suggest that teachers reaching a well-supported or shared conclusion is always possible or preferable; rather I use this approach to examine the work teachers did to reach their conclusion.

To begin to unpack teachers' *noticing* (Louie et al., 2021) related to the error in the video, I coded what they *attended* to, or evidence from the video mentioned in conversation, and their *interpretations* of that video evidence. To explore how teachers *negotiated* what happened in the video and what it meant, I used a constant comparison method (Glaser, 1965) to locate and characterize instances in which teachers interacted with video evidence their peers brought forward (a proxy for their *attention*), and engaged their peers' ideas about video evidence (a proxy for their *interpretations*). Finally, I looked across the teachers' discussion of video evidence and its meaning to infer the *framings* that organized and were organized by their attention and interpretations (Sherin & Russ, 2014). In practice, attention, interpretation, and framing are closely related. I separate them here for analytic purposes only.

Findings

Bea, Casey, and Glenda *negotiated* their noticing about the video in six different ways — by seeking, offering, and sharpening evidence, and offering, extending, and contesting interpretations (see Table 1). To understand the relationship between their negotiations, their *framing* of Evan's error, and how together they functioned to support the teachers in reaching shared meaning about the error, consider the following examples within the trajectory of their discussion.

Table 1: How Teachers Negotiated Salient Video Evidence and Its Meaning

	Type of Negotiation	Example
Attending	Seeking Evidence	Casey asked, "I think he says… 'I'm trying to think about how eight plus five, what would make it the same,' *does he say that*?"
	Offering Evidence	Glenda said, "I heard this time, *'would make the same.'*"
	Sharpening Evidence	Responding to Glenda's offering, Casey said, "Yeah he says, *'8 plus 5. I was thinking 8 plus 5 would make the same as 4 plus 7.'*"
Interpreting	Offering	Casey conjectured, "I think he was really thinking…six plus five equals four plus seven..."
	Extending	Adding to Bea's interpretation, Glenda said, "Yeah 'cause then…his pointing of the marker, too…he was saying since eight and five are further apart, seven and six should be closer together in terms of 'the same as.'"
	Contesting	Bea, responding to Casey, asked, "*Or* he got mixed up on which one (seven or eight) was bigger, right?"

In the initial part of their discussion, the teachers established the shared goal of figuring out the meaning of Evan's error by their careful and sustained engagement with it. This goal reflects their framing of Evan's error as worth understanding. They offered and contested a few interpretations about why Evan stated that four should go in the blank. More specifically, the discussion began when I asked, "What did you notice about Evan's thinking?" Casey **offered an interpretation** that the source of Evan's error was related to his sense of the numbers seven and eight, saying, "...he was thinking, 'okay seven is more than eight," so he needed a number more than five." She followed up, **offering another interpretation** that perhaps Even confused the eight with a six, thinking, "…six plus five equals four plus seven." Bea, **contesting Casey's second interpretation and extending** her first, replied, "Or he got mixed up on which one was bigger, right?" After a couple of talk turns between Bea and Glenda, Casey added, "I think he says… 'I'm trying to think about how eight plus five, what would make it the same,' does he say that?" both **offering and seeking evidence** from her colleagues. Bea responded, **sharpening the evidence**, "...He was pointing to the eight plus five, and he would say, 'This would need one smaller number.'" Glenda responded, "I was trying--I still need to listen. I want to think about it more because…I honestly think he has a far more sophisticated understanding of these numbers than I do….I'm like, 'what is he getting that I'm not figuring out?' and I know that it's working in his head," to which Casey and Bea immediately agreed, prompting me to ask if they would like to rewatch the video, to which all three teachers agreed. Together, these moves reflect the teachers' continued framing of Evan's error as worth understanding.

After rewatching the video, Glenda **offered evidence**, saying, "I heard this time, 'would make the same.'" **Casey sharpened the evidence**, adding, "Yeah, he says…eight plus five would make the same as four plus seven." Bea shifted back to the error specifically, **offering a new interpretation**, "...He realized that those were one away from each other and so that's why he had the four at first was because he knew that was one away from five. He just went the other way." Casey and Glenda responded, "Ooh!" in tandem. This series of talk turns signaled an "Aha! moment" for the group. Glenda **extended this interpretation**, offering additional evidence about Evan tapping his marker.

Together, Bea, Casey, and Glenda concluded that Evan, having drawn on his relational reasoning skills, knew that "something one away" from five should go in the blank, and his error was located in his having compensated "the other way" from five when he originally answered six, rather than four. Their conclusion reflects the shared interpretation that Evan had drawn on his sense of the numbers seven and eight, and their relation to five and the blank. The teachers' conclusion can also be understood as the result of their negotiating video evidence and its meaning within their shared framing of Evan's error as worth understanding.

Discussion

This research contributes to our understanding of how teachers notice students' mathematical errors and how teachers co-construct their noticing of classroom videos together in discussion. Here I have begun to explore the social dimension of one teacher group's noticing by tracing how teachers took up, contested (Goodwin, 1994), and otherwise engaged one another's attention, interpretation, and framing of one student's thinking in a classroom video. Note that teachers negotiated video evidence (a proxy for their attention) and its meaning (a proxy for their interpretations), but appeared to frame Evan's error in the video in parallel and seemingly productive ways across the duration of their talk. Future work can explore what happens when teachers bring different frames, perhaps more and less productive, to their noticing together.

Finally, recall that Bea was a differentiation support teacher, not a self-contained classroom teacher. Bea's job was to meet with small groups of (usually three) students, which allowed her to focus more carefully on individual students' thinking, rather than feel rushed to notice a classroom of different students' thinking. She was not bound by a curriculum or accountable to test scores because she was not the students' primary teacher. Evan was selected for differentiated instruction based on his *high* test scores. Indeed, Bea introduced him at the beginning of the course, writing, "Evan consistently scores high…" Framing has been conceived as the coordination of teachers' resources specific to the context, such as their knowledge, beliefs, and perceptions of environmental constraints (Richards et al., 2020). We can reasonably infer that Bea's knowledge (and by extension, Casey and Glenda's knowledge) of Evan's achievement history was salient to framing his error as worth understanding from the outset of the discussion. Teachers also likely perceived *few* environmental constraints for noticing Evan in the context of Bea's classroom, and the video club, where they could slow down to make sense of Evan without the pressure of deciding how to respond.

Let us suppose, then, that the teachers noticed Evan's error particularly carefully because of, in part, his achievement history and their perception of the affordances of the environment of

Bea's classroom and the video club itself. Many questions remain: To what extent did Evan's thinking having been an *error* influence their noticing? How did these teachers engage other students' thinking, with different achievement histories and in other classroom contexts? Did teachers routinely frame Evan's thinking in seemingly productive ways? In our final video club, Bea shared "frustration" with Evan, and "confusion," at his incorrect use of the equal sign, despite having been exploring it for weeks. Future research can continue to explore the complex relationships between teachers' noticing and who and what they notice.

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EXAMINING LEARNING IN A FACILITATOR PROFESSIONAL DEVELOPMENT CENTERED AROUND THE TRU-PG FRAMEWORK

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Keywords: Professional Development, Instructional Leadership

Effective professional development is critical for growth and change within American education (Darling-Hammond et al., 2017). Therefore, those leading the professional development (facilitators) must be prepared to "successfully facilitate newly developed PD models that offer high-quality learning opportunities for teachers" (Borko et al., 2014, p. 14). To do so effectively, facilitators must be adequately prepared and supported (Borko et al., 2014; Koellner et al., 2011; Roth et al., 2017). Facilitators must draw from a rich knowledge base to make decisions and take action in planning and enacting their professional development sessions (Rodgers et al., 2017). Few empirical studies have been explicitly designed to study the facilitation of professional development or facilitator learning. Lesseig et al. (2017) noted that, as a field, we "lack research-based principles to guide the design of leader preparation" (p. 592). Therefore, studying how facilitators learn and enact effective PD is imperative.

This poster aims to describe the emergent findings of a study to address the needs of preparing, supporting, and researching facilitation by investigating facilitator learning in a facilitator professional development (FPD) designed to prepare and support facilitators as they enact a high-quality, effective mathematics PD program. This study focused on five facilitators and one facilitator educator as they participate in an FPD focused on reflective facilitation (adapted from Smith, 2001) and TRU for Professional Growth (TRU-PG) (Schoenfeld, 2015) to support their enactment of the Analyzing Instruction in Mathematics Using the TRU Framework (AIM-TRU) PD model. Each participant had various facilitation experiences and was from the same region in the Northeast United States.

This FPD was centered around Schoenfeld's (2015) Teaching for Robust Understanding for Professional Growth (TRU-PG) framework. This framework depicts five dimensions that characterize powerful learning environments: Professionalism, Room to Grow, Equitable Access, Agency, Ownership, and Identity, and Uses of Assessment. These dimensions have been posited by Schoenfeld as a framework for PD based on abstraction from the Teaching for Robust Understanding (TRU) framework, which has been shown to create powerful mathematics learning environments (Schoenfeld, 2015). Participants meet for seven FPD sessions throughout the school year. Each of the seven sessions focused on one dimension of TRU-PG or reflective facilitation practices. Each of these sessions was video recorded and transcribed. The research questions driving this study are: *How does the facilitator professional development model support facilitator learning?* and *How does the discourse in a facilitator professional development align with the TRU- PG framework?*

Drawing on situated learning perspective and community of practice (CoP) theoretical frameworks (Wenger, 1999), I use frame analysis (Bannister, 2015) to examine participation changes and how the discourse aligns with the TRU PG framework. This poster will report on preliminary findings of facilitator learning in this FPD, including an analysis of the problems of practice faced by participants and how they changed over time. The poster will also report on how the problems of practice align with the TRU-PG framework.

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A SYSTEMATIC LITERATURE REVIEW OF MATHEMATICS TEACHER LEADERSHIP RESEARCH

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Mathematics teacher leadership is promising, yet complex work. This systematic review takes stock of the current research landscape on mathematics teacher leadership, specifically mathematics specialists as teacher leaders (MSTL). This study's findings provide further evidence that mathematics-specific teacher leadership is on the rise. Further, in considering the research design, the trends surrounding the methods, research questions and framing also illuminate future research directions and complexities.

Keywords: Instructional Leadership, Research Methods

Perspective on Teacher Leadership

As early as the 1970s, a teacher-leadership model was established to position teachers as educational change agents (Andrew, 1974). Within this model, the teacher leader position is an "in-school position" that moves beyond supervision responsibilities or those of a master teacher and is a "front line leadership role for improvement of curriculum and instruction" (p. 2). This teacher leader positioning allows for career growth and advancement while allowing teachers to remain in classrooms (Andrew, 1974), a promising aspect given current teacher retention issues (Yow et al., 2021a). Content-specific professional learning has proven to be an essential component of effective professional development (PD; Darling-Hammond et al., 2017; Desimone, 2009). Research indicates teacher leaders also believe that expertise in subject-matter and pedagogical content knowledge is important for them to engage effectively in teacher leadership (e.g., Snell & Swanson, 2000).

We define a mathematics-specific teacher leader as a type of mathematics specialist who has primary responsibilities within a P-12 classroom and provides formal and/or informal support for teaching and learning mathematics to in-service teachers through mentoring and other support structures (Baker et al., 2022). Thus, we refer to this individual as a mathematics specialist as teacher leader (MSTL, p. 31). Mathematics-specific teacher leadership is promising (Yow et al., 2021a), yet complex work (Klein et al., 2018), and research interest in this form of leadership is on the rise (Rigelman & Lewis, 2023).

Purpose

The purpose of this systematic review is to understand the current MSTL research landscape and explore the following research questions: (1) What are the historical publication trends for MSTLs? and (2) What are the various methods, research questions, and framing used in research design for MSTLs?

Methods

We explored the historical publication trends for MSTLs and the various methods, research questions and framings used in research design. Below we outline the systematic procedure we used to investigate the research related to MSTL between 1981-2021 and describe the methodological parameters of our data identification and analysis. We drew on Cooper et al.'s (2019) comprehensive steps to achieve a high-quality literature synthesis (Sandelowski & Barroso, 2007) to inform research and practice.

Data Collection

For the purpose of this analysis, we targeted the articles in which mathematics specialists served as teacher leaders from a larger body of work (Baker et al., 2022), wherein we explored empirical research to examine the positioning of mathematics specialists (see Baker et al., 2022 for a detailed process of inclusion and exclusion criteria). Within that larger analysis on mathematics specialists, 57 articles were identified with the MSTL code. To evaluate study quality, we used the appraisal criteria of Risko et al. (2008). This evaluation tool has seven quality criteria that enables the appraisal of articles from various methodologies. Of the 57 articles, 34 received an overall score of "3", meeting all seven criteria and producing a 60% inclusion rate.

Data Analysis

To answer our research question about historical publication trends for MSTLs, we began by isolating the 34 unique articles that featured a MSTL, and completed counts to determine the frequency of publications between our target dates of 1981–2021. The lower date range was based on the call for mathematics specialists from National Council of Teachers of Mathematics (NCTM) in 1981 (Dossey, 1984), a catalyzing event noted by Fennell (2017), and the upper date range was the last complete year of research available at the time analyses began. We then represented this data in a line graph to help illustrate publication trends over time. Next, we identified the journal title for each article that was published during this time frame and determined the frequency with which each journal published an MSTL article. We then examined the title to determine if the journal was a mathematics-specific education journal or a general education journal. The code "math" was used if only mathematics appeared in the journal title and the code "general" if there was no mention of mathematics in the title, or the journal focused on research across content areas (e.g., literacy, social studies). We then completed frequency counts of the journals that published one of the 34 MSTL articles.

To determine trends in research methods, we coded each article with an overarching description of the methodological approach (mixed methods, qualitative, quantitative) and determined the frequency of the approaches. We first searched the full article to identify how the author(s) described the study. In instances in which the author(s) did not identify a description of the methodology, we made inferences based on the data collection tools identified in the study. For example, if the article mentioned using interviews or observations, we coded the article as "qualitative." If the article mentioned using statistical analysis of scores on an assessment, we coded the article as "quantitative." If an article mentioned both qualitative and quantitative data sources, we coded it as "mixed methods."

To determine trends in research questions, we compiled each article's question(s). While some articles (e.g., Insuander et al., 2019) had multiple research questions, other articles featured a single research question that guided the investigation (e.g., Yow et al., 2021b). Nevertheless, all questions from one article were inserted into a single cell of our matrix. One member of the research team open-coded (Saldaña, 2021) the questions to determine what concepts emerged, and then made a preliminary categorization key. Two research team members then independently categorized the research questions into one or more of the following categories: "teacher leader characteristics", "teacher leader interactions", and "teacher leader professional learning". Only the presence of a category was recorded for each research article. Upon coming together to obtain consensus, a fourth category "systems" emerged. We also further refined "teacher leader interactions" to encompass the interactions a teacher leader has with any K-12 partner and "teacher leader professional learning" to illuminate that the PD was *for* the teacher leaders.

Beginning with a list of the headings and notes about codes for concepts and theories of the studies compiled by four authors, one author created an initial coding framework. A second author then referred to the articles themselves to apply the coding framework as a validation of the conceptual topics and to confirm the interpretation of the headings. As with the research questions, two authors met to come to consensus on the coding. Based on the research question coding and reviewing the literature review sections, a distinction between PD for teachers and PD for teacher leaders was made in the framework.

To determine the type of framing of MSTL research, we identified the articles with a clear heading that outlined a theoretical or conceptual framework. The next step was to consider the introduction and literature review sections to identify a conceptual or theoretical framework. This process yielded additional articles where the framework was identified by the authors embedded within text (often the literature review) and not within an independent section, which meant most of the articles had an identified theoretical or conceptual framework. The remaining articles were deemed to use a practical framework and their literature review constructs were analyzed to determine the practical frameworks for these studies.

Results

Research Question 1: Publication Trends

We examined the 34 unique studies that were coded as MSTL. Overall, there was an increasing trend in publications featuring an MSTL. Although the period we examined spanned over 40 years, the publication range was 2004-2021, with the majority published in the last decade. Between 2012-2021, there were 30 articles published which represent 88% percent of the total MSTL-focused subset, and 24 (71%) of the 34 articles being published during the most recent five years span of 2016-2021.

MSTL articles were published within 23 journals. Of those 23 journals, eight were mathematics-specific education journals and 15 were general education journals. Seven journals published more than one MSTL article. In the sub-set of seven journals, there were 16 MSTL articles of which four were housed in mathematics-specific education journals and three were in general education journals. The mathematics-specific education journals housed 10 of the 16 articles (63% of this subset; 29% of the entire data set).

Research Question 2: Research Design Trends

Methods

The 34 research articles include 24 qualitative studies, one quantitative study, eight mixed methods studies and one article focused on tool development. Common approaches to data collection included surveys (e.g., Yow et al., 2021b); interviews (e.g., Webel et al., 2018); and observational data (e.g., Gerstenschlager & Barlow, 2019). Many studies used case study approaches (e.g., Roberts, 2020) to analyze teacher leaders' development individually (e.g., cross-case analysis) or collectively over time.

Research Questions

Four categories of research questions emerged from the analysis: (a) leader characteristics, (b) leader interactions with others, (c) leader reactions to PD programs, and (d) systems-level analysis. The first three categories characterize aspects of the MSTLs' experience, roles, practices, and leader development. These questions capture their transition from teacher-to-teacher leader, the roles they play in schools, the kinds of knowledge and practices they need, and the influence of PD on their development as leaders.

While each article could include multiple questions with a specific categorization, we only coded for the presence or absence of the category. It was somewhat arbitrary as to the number of research questions as well as whether specific categorizations were present across one or multiple research questions. For instance, Cassata and Allensworth (2021) stated "RQ1. What were the practices teacher leaders used to support instructional change? RQ2. In what ways did school-level factors shape the practice of the teacher leaders?" (p. 2). In this example the first question is about the influence of MSTL on others and the second question is a systems question because it centers on the influence of the school on the MSTL. However, other articles might have multiple, related questions about a MSTL's interactions. For example, Borko et al. (2021) state "(1) How did the Teacher Leadership Preparation sessions evolve over time? (2) What key Problem-Solving Cycle ideas did the Teacher Leaders adopt and how did they adapt them over 3 years of planning and facilitating workshops?" (p. 130). The greatest number of articles are about PD programs for teacher leadership. Fewer studies are about MSTL characteristics or their interactions with others. While a clear pattern of research question categories did not emerge, it is important that the field is studying MSTLs in concert with the influences of either PD for them or their work within a system.

Framing

There was a total of 18 articles (53%) with a clearly identified theoretical framework with two specific theories used in two different articles: (a) professional noticing, and (b) situative learning. Three of the 18 articles did not have an identified section outlining the theoretical framework but instead embedded the framework within the literature review (e.g., Cwikia, 2004; Lu et al., 2020) or the introduction (e.g., Insulander et al., 2019). As we documented the theories used to study mathematics-specific teacher leadership, we also analyzed each article to determine if certain themes were evident across the 18 articles. Within some articles, researchers leveraged more than one theory and sought to answer multiple questions. For this reason, an article may be represented more than once within the identified themes. There were seven articles (21%) with a clearly identified conceptual framework. While one of these articles had a clearly labeled conceptual framework heading, it was also referred to as a theoretical framework elsewhere in the article (Boylan, 2018). All seven articles also used their conceptual frameworks in the analysis of their data. The conceptual frameworks used by studies of teacher leadership within mathematics education included frameworks about conceptualizing and defining informal mentors, communities of practice, induction, images of a teacher, and systems of teacher leadership. Others used formalized conceptual frameworks like Three-Tetrahedron (3-T) Model of Professional Development (Prediger et al., 2019) and The PRIME Leadership Framework (National Council of Supervisors of Mathematics [NCSM], 2008).

After determining whether an article had an author-identified theoretical or conceptual framework, we identified ten articles (29%) that used a practical framework to frame their study within known and successful practices (Eisenhart, 1991). For this analysis, we examined each article's introduction, literature review headings, and background sections to identify the practical frameworks. These were compiled into the following categories: mentors/ mentoring, mathematics coaching/specialists, teacher leadership, and instructional practice. The framing of mathematics coaching/specialists included background information, responsibilities, roles and their development and support. An instructional practice framing included exploring flipped classrooms, synchronous online teaching and blending learning, and data informed practices. Articles with a mentoring framing explored mentoring relationships and influence. A teacher leadership framing included influence, learning environments, data-informed decisions, and program components. One feature that seemed to distinguish the conceptual frameworks from the practical frameworks was that conceptual frameworks used a more general construct of mentoring, while the practical frameworks explored mathematics-specific literature around coaching and specialists, which was highlighted as an individual theme.

While the previous section explored the frameworks more holistically, this section examines the concepts that make up the background literature portions of each study and which are used to situate the studies in the research literature. Four categories of framing concepts emerged from the coding process: (a) practices, (b) positioning, (c) purpose, and (d) PD. Due to the complexity of the research questions surrounding mathematics-specific teacher leadership, an article could be assigned more than one framing category. The first framing category included types of practice and how studies described the type of actions, behaviors, strategies or approaches MSTLs should employ. The second category included two different positionings for leadership. One positioning is "teacher leadership" referring to general teacher leadership framed as: the influence on other teachers, informal or formal leadership roles, and taking up activities beyond classroom teaching. The second positioning was specific framing as content-specific leaders (in this case, mathematics) where the practices, standards, content, and knowledge of the discipline is positioned as critical to the role. The third category was about the purpose of the study itself including the intervention the authors might suggest or the teacher leadership itself. This category included teaching induction, teaching practice, professional knowledge, and instructional change as what might be influenced or changed by the teacher leadership process. The final category included two framings of PD: teacher PD and teacher leadership PD.

The three practices included formal and informal leadership practices and roles. The codes were used to describe the language authors used to situate their research via the literature or other frameworks. So, while Knapp (2017) was an often-cited article about mathematics coaching and the findings include coaching as a term, the literature review is framed around teacher leadership. Other articles framed specific aspects of teacher leadership practice in their findings, thus providing a window into how the research was theorized and supported.

The two categories of content-specific/mathematics leader and teacher leaders displayed how the role might be defined in the school, how the leadership work might be framed, and the complexity of teacher leadership overall. Content-specific teacher leadership (i.e., mathematics-specific) might emphasize mathematics classroom practices, mathematics standards, or how the leadership is situated in the content domain. For instance, Borko et al. (2021) described a model driven by a problem-solving cycle using mathematics tasks. Myers et al. (2020, 2021) and Webel et al. (2018) described elementary mathematics specialists focused on the teaching and learning of mathematics. In contrast, Boylan (2018) described “adaptive leadership” as a teacher leadership framework that could be applicable for multiple kinds of teacher leaders or school leaders. Smith et al. (2017) used ecological systems theory to understand teacher leadership as part of a network of influences and interactions. We have described this as positioning because it is about how the article positions the work and the research that supports it.

The purpose category of codes described the kinds of literature that frame the purpose of the leadership or the study. Teacher leadership was desirable in these studies for some purpose or goal. So, the goal might be to improve teacher induction, spur instructional change, describe teaching practice(s) or professional knowledge without a particular PD or intervention.

Professional knowledge included different types of knowledge for teaching and may also include teacher leadership knowledge. It was difficult to draw a clear distinction in the literature review sections between these, so it remained one category of professional knowledge. In addition, many of the teacher leaders were continuing to teach in classrooms so their knowledge of teaching and knowledge of leadership were likely intertwined. The final category came about after the coding of the research questions where PD for leadership became a clear category. These articles emphasize how they are drawing on findings about teacher PD to inform teacher leadership PD. However, a large number of articles distinguished features of teacher leadership that required different PD (e.g., adult learning, conducting research about teaching practices) than might be expected of a teacher.

We examined how the categories might be related to research question types, specific theoretical frameworks, and other categories. We also looked for patterns in how the categories might connect to one another. While there are consistent categories of the types of concepts and research that articles reference in their studies, there is not a clear relationship about when the categories might be connected to one another. For instance, we expected some connection between teacher induction and mentoring. However, articles might refer to both (n=12), mentoring only (n=5), or induction only (n=2).

It is important that studies are making a distinction between PD for teachers and PD for leadership. Some articles (e.g., Borko et al. 2021; Myers et al., 2021) referred to both. The literature review sections that focus on the design or goals for PD were complex because the design of leadership PD understandably draws upon the design of teacher PD. These sections brought in findings about the duration, structure, and organization of PD that have been long-standing. Despite the complexity, it is an important finding that the specific needs and interests of teacher leadership were recognized as needing different content, such as adult learning theories, adaptive school leadership (Boylan, 2018), and the perspective on content.

Discussion

For the current investigation, we analyzed 34 research articles that centered the work of MSTLs. We remind the reader that we had initially identified 57 research articles but excluded 23 because they did not meet some aspect of the Risko et al. (2008) appraisal criteria. In particular, some articles were eliminated because of inadequate participant description, not linking findings back to research, and/or not linking the findings to the research questions. Some articles had more than one of the Risko et al. (2008) criteria missing from their study.

Our findings revealed that research on and about MSTL is increasing. Nearly 90% of the MSTL research originated in the last decade and 70% of the articles were published during the most recent five years span (2016-2021). Two-thirds of the MSTL articles were published in general education journals. These publication trends highlight a need for the field to consider how to expand research on MSTLs and depict a real tension that scholars face when choosing publication outlets. From our experiences, we have learned that if the mathematics in the manuscript is not prominent enough, scholars tend to shy away from mathematics-specific journals and instead submit to general education journals to face less reviewer push back about "where is the math?" The implication is that mathematics-specific journals are not the predominant outlet for published work about MSTLs. We encourage the research community to consider the ways in which we might elevate work about MSTLs and MS more broadly in mathematics-specific or STEM-specific journals.

Future directions are also evident in the results regarding the research design, the trends surrounding the methods, framing, and research questions. The most frequent research design across the 34 articles was qualitative. Because MSTL is a growing field (Rigelman & Lewis, 2023; Yow et al., 2021a) and there are very few MSTLs working in schools, this makes sense. As a community we are trying to understand the boundaries of these individuals as they navigate their unique P-12 contexts.

To study everyday leadership and support for P-12 school initiatives (Fennell, 2017), it will be essential for future research to continue to address the contextual nature of both P-12 education and teacher leadership. There is then a parallel needed to clearly define the roles, responsibilities, and contextual nuances of MSTL work. Similar to research calls to action around mathematics specialists (Baker et al., 2021), more specificity is required in how we speak about and define MSTLs so that "the field of mathematics education can not only come to a common understanding, but advance both policy and practice" (p. 9). MSTLs can be one part of a coherent system for PD (Cobb et al., 2018). Further research is needed about the interactions between leadership at multiple levels: classroom, grade-level, school, and district.

To design research questions around MSTLs, it is important to recognize the complexities of teacher leadership broadly (Klein et al., 2018) as well as the need for MSTLs to possess specialized knowledge around leadership (e.g., AMTE, 2013) and the teaching, learning, and leading of mathematics (NCTM, 2014). This is amplified by the fact that most of the research questions in the current body of research center on leader interactions with others and leader reactions to PD programs, speaking to the complexity of the work and the need to understand an MSTL's influence and how that might inform MSTL professional learning. It also signals that the mathematics and teacher leadership research communities are wondering about the influence of these positions and the professional learning required. Further research on the MSTL positioning will be essential to understanding how these individuals take on or reject particular roles or responsibilities (Hunt & Handsfield, 2013) and the ways they position themselves or are positioned by others (Davies & Harré, 1990). Not surprisingly, there were fewer research questions around systems-level analyses. System-level work is even more complex, dynamic, and multifaceted as there are additional layers within and beyond an individual school community. It will be essential for future research to consider how MSTLs "develop long-term goals advocating for systematic instructional change and/or short-term goals providing small, measurable successes that serve as milestones" (Hjalmarson & Baker, 2020, p.555).

When analyzing the framing of MSTLs, we noticed a variety of theories and constructs. Although potentially alarming, this is logical as the field of mathematics-specific teacher leadership is emerging (Rigelman & Lewis, 2023; Yow et al., 2021a). The field is also multifaceted as there are many theories in play beyond those associated with teaching and learning: leadership theories, adult learning theories, mathematics-specific theories, and systems-level change theories to name a few. These theories are alluded to within the *Teacher Leader Model Standards* (2011) and the *Mathematics Specialist Standards* (AMTE, 2013) and illuminate the layers of complexities teacher leaders and MSTLs face.

However, while past research has illuminated that teacher leadership is either atheoretical (York-Barr & Duke, 2004) or partially theoretical (Wenner & Campbell, 2017), we see this space of mathematics-specific teacher leadership framing as a potential foundation from which to build. Considering that the field of mathematics-specific teacher leadership is new, the findings from our study illuminate four possible categorizations from which to frame future studies: (a) practices, (b) purpose, (c) positioning, and (d) PD. There is even a similar distribution of construct categorization across the research questions asked. This is a positive sign for the field. If we were focused too heavily on one or two of these categorizations, then we would likely be missing important aspects of mathematics-specific teacher leadership. However, we are not stating that these are the only categorizations, just we are seeking the foundation from which future research can build and new theories can emerge for a fledgling field.

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AN ARGUMENT FOR VALIDATED INSTRUMENTS FOR MATHEMATICS SPECIALISTS RESEACH AND PRACTICE

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Research is needed about mathematics specialist and their work. There are questions of policy and practice that need instrument design and development to advance research about the nuanced work of mathematics specialists. This systematic literature review will identify the gaps and need for the development of rigorous, quantitative measures.

Keyword: Measurement, Mathematics Specialist

With the increased use of mathematics coaches and specialists and an increase in related research (MCSs; Baker et al., 2021; Rigelman & Lewis, 2023), a need for valid measures is at the forefront. The goals of MCSs are to influence teacher practices and beliefs to improve student learning (e.g., AMTE, 2013; Campbell & Malkus, 2011). With the current measures available, however, there is fragmented evidence to make the connections among these goals. Without measures that accurately capture specific information pertaining to MCSs, there is a lack of empirical evidence about their work in schools and influence in the system of mathematics teaching and learning.

Purpose

The purpose of this systematic review is to understand the current instrumentation used in MCS research with the objective of identifying specific needs for validated instruments. This work was guided by the following research question: What are the current tools and instrumentation used for MCS research? For our theoretical framing, we draw on Gutiérrez's (2009) conceptualization of equity in mathematics education. Gutiérrez identifies four equity dimensions: (1) access; (2) achievement; (3) identity; and (4) power for which we will organize the MCS instrumentation.

Design Process

We draw on Cooper et al.'s (2019) comprehensive steps to achieve a high-quality literature synthesis (Sandelowski & Barroso, 2007) to inform both research and practice. In 2020, we analyzed a subset of our data corpus (n=130) that were tagged with a mathematics specialist code. We applied an open coding process to attend to how the mathematics specialist was presented within research. Our primary code identified the role or position of the mathematics specialist (e.g., coach, teacher leader) and our secondary code indicated the context (e.g., school, district). Last, we took this data set and identified the tools and instruments used in order to highlight the need for validated instruments for MSC research.

Implications

The implications for this work include stronger methodological practices for MCS research, in addition to using validated instruments to support MCSs in their practice.

Disclaimer

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COMPLEXITY, TEACHER AGENCY, AND THE FUTURE OF RESEARCH ON MATHEMATICS TEACHER LEARNING

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Extensive research has been conducted on the characteristics of effective professional learning (PL); with many scholars asserting a consensus has been reached. However, empirical studies into the outcomes of PL initiatives do not consistently align with this consensus. In this paper it is suggested that a key element, namely teacher agency, has been overlooked. We advocate for the use of new conceptual frameworks that contextualize teacher learning as occurring within a complex learning system. This allows for the investigation of processes and interactions between parts of the system that support the emergence of learning. Our findings highlight the essential mechanisms that emerge from these systems when mathematics teachers are positioned as active participants in the design and facilitation of their own learning.

Keywords: Professional Development

Perspective of the Study

The characteristics of effective professional learning (PL) have been well documented, with some arguing that a consensus has been reached (e.g., Darling-Hammond et al., 2017; Desimone, 2009). An examination of literature spanning several decades indicates that the essential attributes for successful teacher PL include: a focus on subject matter content and pedagogy, collaboration and interaction with colleagues, engagement in active learning tasks for teachers, coherence with existing curricula and policies, and extended duration of PD programs (Desimone, 2009). What research has found however, is that even when PL programs for teachers are designed using these features they often produce conflicting results (Goldsmith et al., 2014). In fact, more recent critiques make note of the absence of teacher choice within discussion of PL models (see Boylan, 2021; Boylan et al., 2018). Without teacher involvement in designing and directing these learning experiences, PL programs may perpetuate the deficit view of teacher learning in which teachers are seen as passive receivers of knowledge (Davis & Renert, 2014; Bruce et al., 2010). To move away from the deficit view of teacher learning, researchers have begun to consider the role that shared leadership, time and space for PL, and high levels of teacher agency play in effective PL (Day, 2017; Hauge & Wan, 2019).

Building from the assumption that teachers are professionals with the ability to seek out, develop, and lead their own PL experiences, Sachs (2003; 2016) encourages the field to design and study PL experiences that position teachers as professionals. In fact, studies of PL models that sustain teacher learning found that high levels of teacher agency are required (Campbell et al., 2016; Day, 2017; Hauge & Wan, 2019). Therefore, a design and study of PL should go beyond the consideration of widely adopted key features (Biesta et al., 2015; Calvert, 2016) and investigate when teachers have agency, what emerges as explanatory mechanisms that support learning in these communities. By mechanisms we are referring to the processes and interactions between these key features that support learning.

To this end, this report investigates the following research question within a regional university-supported, multi-site PL initiative: When mathematics teachers are positioned as active participants in the design and facilitation of their own learning, what emerges as essential mechanisms of the PL initiatives that support this learning?

Theoretical Framework

As discussed, the consensus on key features of PL have provided the field with standards for how teachers should be supported in their learning. Missing from these features is the role that teacher agency plays in the design and implementation of PL for teachers.

Teacher Agency

Teacher agency is enacted when teachers "exert influence, make choices, and take stances on their work and/or professional identities" (Eteläpelto et al., 2013, as cited in Day, 2017, p. 37). When teachers possess agency, they can exert control and determine the actions they will undertake. Teacher agency, therefore, plays a crucial role in what is learned through engaging in PL of any type. Teachers bring their own experiences and expertise to the practice of teaching, and this is a valuable resource that should be part of any model of teacher PL (Guskey, 1986). Campbell et al. (2016) suggest that teachers are and should be leaders in their own PL and that a priority in designing and supporting teacher learning is the inclusion of teachers' voice in influencing the methods, goals, and content of that learning (Campbell et al., 2016).

Looking Beyond Individual Features of Teacher Professional Learning

We align with other researcher who view education as a complex learning system from which learning emerges (e.g., Cochran-Smith et al., 2014, Jacobsen et al., 2019; Opfer & Pedder, 2011). Research in teacher learning has often focused on specific features of PL rather than considering its complex nature and the ways that learning is socially constructed (Cochrane-Smith et al., 2014). Considering these features separately does not allow for an understanding of how the pieces interact and influence one another (Opfer & Pedder, 2011). Some wonder why some initiatives, designed using these effective strategies, are unsuccessful while others, with none of the characteristics of effective PL, are more successful (Opfer & Pedder, 2011). The issue may lie in the framework used to analyze these studies as current theoretical models of teachers' PL tend towards a linear and reductionist perspective (e.g., Cochran-Smith et al., 2014; Strom & Viesca, 2020).

Complexity Theory

Current research on mathematics teacher PL has begun to consider its complex nature and how the use of complexity science can help the field gain a deeper understanding of ways to support and possibly trigger this learning. (e.g. Cochran-Smith et al., 2014; Strom & Viesca, 2021). Complex systems are open, non-linear, not predictable, and not defined only by the sum of their parts but also by the ways that these parts interact with, and influence, each other and the environment (McMurtry, 2008). Complex systems are defined by key characteristics including self-organization, adaptation, nestedness, and emergence (Cilliers, 1998; Davis & Simmt, 2003). Self-organization suggests that there is no hierarchical authority imposing instructions on the system and is achieved through the non-linear interactions within the system and between the system and its environment (Strom & Viesca, 2021). Adaptation means that a system can change its structure, but involves more than a predictable, linear response to a system's environment (Davis & Simmt, 2003). Complex systems often involve systems nested within self-similar systems, with smaller components of a system resembling the system as a whole (Suurtamm,

2020). Emergence refers to the ongoing evolution of a system that arises through the interaction, self-organization, and adaptation of the nested systems (Davis & Sengupta, 2018).
Emergence also relies on a set of conditions which provide a framework to study what and how systems learn (Cochran-Smith et al., 2014; Davis et al., 2012). These different conditions include specialization and trans level learning. Specialization creates a constant tension between redundancy and diversity. In a professional learning community (PLC), redundancy is represented by the shared practices, assumptions, and expectations of mathematics teaching. Diversity manifests in the different beliefs, attitudes, experiences, and knowledge each teacher brings to teaching mathematics. Trans-level learning involves neighbor interactions in a system with decentralized control (Davis & Sumara, 2006). Neighbor interactions in a complex system involve individuals' "ideas, hunches, queries, and other manners of representation" (Davis & Sumara, 2006, p. 142) bumping up against and colliding with one another. In a PLC, these interactions could include sharing solutions to problems, co-planning lessons, discussing student and student thinking, or engaging in a pedagogical book study. Decentralized control suggests that these interactions occur as individuals and groups find their own reasons to share and listen to each other. Moving forward these characteristics and conditions are referred to as mechanisms that interact to support teacher learning.

Methods

This study was a qualitative multiple-case study (Creswell & Poth, 2018; Yin, 2016) combined with a complexity science framework. This research design was influenced by Anderson et al. (2005) who suggested combining the case study approach with complexity science as a way to gain insight into systems of learning. They argue that analyzing a system by simply breaking it into parts does not provide an adequate description of the mechanisms at play within that system (Anderson et al., 2005). This study moved beyond simply describing key features of PL and towards understanding the mechanisms that support educators, with high levels of teacher agency, as they work in a collaborative system.

Context and Participants

This study was part of a larger study that focused on the Ontario Association for Mathematics Educators Grade 9 Applied Math Inquiry Project (McKie, 2023; Suurtamm et al., 2017), from here referred to as the project. This project involved 10 high school PLCs from across the province. PLCs applied to participate in the project by submitting an application. The application needed to include a list of the PLC members and their roles, and a description of the PLCs self-defined problem of practice (McKie et al., 2017). This self-defined problem of practice highlights the way that teacher agency was purposefully integrated into the project design. The PLCs were supported to meet monthly within their own schools as well as multiple times each year at project wide meetings. The design of the project was influenced by the perspective that, similar to researchers, teachers assume an inquiry stance when examining their practice (Cochran-Smith & Lytle, 2009; Suurtamm & Koch, 2019). This inquiry stance involves the active, iterative process of engagement in, and reflection on, practice which results in the emergence of new ideas and actions (Suurtamm & Koch, 2019). The collaborative nature of the project, the high level of teacher agency, and the use of PLCs was informed by complexity theory (Suurtamm & Koch, 2019). This supported the focus on the role that nested systems, in this case the 10 PLCs in this project, played in influencing the emergence of learning. The condition of diversity was intentionally built into the project through the requirement that PLCs consist of individuals with varying roles such as administration, special education, and mathematics teachers (McKie et al., 2017). The 10 PLCs also represented a diversity of contexts and experiences. Some PLCs were situated in large urban settings and others in more remote settings or serving smaller populations. Some PLCs were established collaborative learning communities and others were newly formed.

Participants in this study were original project participants. Years of experience of these participants ranged from 12-32 years and different roles of the participants included principals, mathematics teachers, mathematics department heads, special education teachers, and research team members.

Data Collection and Analysis

Data for this study was collected in two phases, the first being a survey sent to all individuals who participated in the original project (N≈80). The survey served two purposes, the first to collect information related to the participant's position, role in the PLC formation, how the PLC operated, and the perceived influence the project had on the participant's learning. The second purpose was to identify, and recruit individuals for phase 2 of the study. The survey asked participants if they were interested in being contacted for further participation in this study. Those that indicated 'yes' were contacted to participate in semi-structured interviews during phase 2. These interviews (N=13) provided a deeper examination of the individuals' learning and the shared learning experiences within the nested systems of the PLCs and the project as a whole. Questions in each phase focused on teachers' experiences with the project, the influence the project had on the participant's learning, the collaborative nature of the project, and the sustainability of their learning. A theory-driven codebook (DeCuir-Gundy et al., 2011) was created using the mechanisms of complex learning systems described earlier, self-organization, adaptation, nestedness, diversity, redundancy, neighbor interactions, and decentralized control (Davis & Sumara, 2006). These codes were applied to survey and interview responses, next the results of this analysis are presented.

Results

Analysis of the data from the study revealed that four key mechanisms emerged that supported teacher learning. These include neighbor interactions, decentralized control, self-organization, and nestedness. These four mechanisms are not independent, rather they interact and influence each other.

Neighbor Interactions

Neighbor interactions in the project included time to interact with resources and other individuals and their ideas. At project-wide meetings the PLCs were introduced to different resources that the research team felt supported the identified issues and challenges. One particular resource, *5 Practices for Orchestrating Productive Mathematics Discussions* (Smith & Stein, 2015) was provided to each individual and multiple PLCs chose to incorporate this resource as part of their learning. Neighbor interactions with individuals and their ideas also were identified as an essential mechanism. These individuals included other educators, researchers, and students. Neighbor interactions included PLC members interacting with student thinking, PLCs discussing and negotiating new practices and ideas within their own context, and neighbor interactions across PLCs at the project-wide meetings. Responses to the Phase 1 survey overwhelmingly point to neighbor interactions as influencing teacher learning. 86% of survey respondents self-reported that interacting with resources and with others either from their own

PLC or across the project was either the most influential component on their learning or the highlight of the project for them. For example, one participant shared "the other teachers that we were able to network with and learn alongside with was amazing!! Really challenged our thinking on spiraling and rich tasks" (PLC 1, phase 1 survey) and from another participant, "a highlight of the project was meeting and sharing experiences, perspectives, and resources with colleagues all over the province" (PLC 7, phase 1 survey). Neighbor interactions were not a stand-alone feature, they were supported through the next key feature identified, decentralized control.

Decentralized Control

Although the project schedule of activities and overall goals were determined by project leaders, each PLC was in charge of determining their context specific goals, ways of working, and their focus throughout the project. In this way the project demonstrated decentralized control. At individual PLC meetings each PLC determined their schedule, focus, and activities. At project-wide meetings time for individual PLCs to work together was scheduled as was time for groups to collaborate across PLCs in ways that were meaningful to them. A PLC 1 participant commented that the "the teacher led approach to this was great!" suggesting that providing space for teacher voice and choice was a quality she identified as influential. Decentralized control supported neighbor interactions and created opportunities for individuals who may not have been supported in their own PLCs in ways that were meaningful to them. For example one participant shared how interacting with PLC 7 supported her own learning and that of her PLC:

> We learned about using rich tasks to drive instruction from William [PLC 7] and [the PI] suggested a Jo Boaler book for our group to read… our group continued to learn about Thinking Classrooms and spiraling curriculum mostly being influenced by the group from PLC 7. (PLC 1, Phase 1 survey)

Decentralized control supported neighbor interactions, these two mechanisms interacted to influence and support how the individuals and PLCs self-organized, the third essential mechanism that emerged from analysis of the data.

Self-organization

Self-organization was apparent at multiple levels in the project. This self-organization was not pre-determined at the beginning of the project but rather it was based on needs that arose through the neighbor interactions that were enabled in a network with decentralized control. The research team members from different PLCs were interacting, sharing information and experiences, and adopting different strategies to share with their assigned PLCs. Other ways the project demonstrated self-organization was PLCs seeking out, and making connections with, one another. Different PLCs created shared document folders online for sharing resources and also planned visits to each other's schools to collaborate. This self-organization came from neighbor interactions at the project-wide meetings and the PLCs sharing with one another. One participant explained that as the project progressed more opportunities to connect with other schools were created and the participants self-organized visits with at other school sites:

> Every time [the project] extended I seemed to be connected with new people along the way and that led to all kinds of exchanges with other teachers and other schools. It also led to some dates where we…went to the [other] school and we had [other PLCs in the project] coming to our school as well. (PLC 7, phase 2 interview)

During project-wide meetings different PLCs would seek out connections with people and resources that they felt could support their work and their learning. During the final project-wide meeting at the end of Year 2 different PLCs self-organized into working groups to develop presentations for the project's annual conference as well as for the Grade 9 project conferences that were organized to share the learning from the project with educators across the province. Self-organization, which was supported by neighbor interactions in a system with decentralized control was also made possible by the fourth essential mechanism, nestedness.

Nestedness

The nestedness of the project provided opportunities for individuals and PLCs to connect with others both within their own school and with others beyond their own PLCs. Several individuals commented on the fact that the project allowed them to get to better understand their colleagues professionally. Teachers being nested within PLCs in their own school provided opportunities for neighbor interactions:

> It was an interesting thing because working with [teacher], she and I have been friends for a million years right? And like we fished together, and we hang out together as much as we can right? But I think [the project] made me realize how she thought too, like it not only helped me identify the strengths of myself but of the people that were working with me in the project (PLC 4, phase 2 interview)

Nestedness at the project level introduced new, diverse perspectives and experiences as well. Individuals who were seeking new ideas beyond their own PLC were buoyed by the opportunity to learn with others across the province. One participant shared:

> I got a lot of energy from people from other schools that had the same kind of excitement about things as I did, but within my own board I don't know whether I felt that same level of excitement by others…I really enjoyed being able to get that perspective, like I really enjoyed meeting people from other schools and thought about "wow everybody should, we all should know what's going on out in all of [the province]" so we do have a better perspective [about] where we are at compared to them and where we should be at or what we should be, you know, aiming to be at. (PLC 10, phase 2 interview)

The nestedness of the project enabled the other essential mechanisms that emerged and together these four mechanisms; neighbor interactions, decentralized control, self-organization, and nestedness influenced teacher learning. Next, a discussion is presented on how these essential mechanisms of complex learning systems connect to the research on the key features of effective PL. Through this discussion we argue that the mechanisms of complexity offer insight into how and why these characteristics can support learning and why teacher agency is essential in PL.

Discussion

This study aimed to explore the mechanisms facilitating the emergence of learning in a PL experience where teachers actively participated in designing and directing their own learning. Within this study the emergence of teacher PL was considered using complexity theory. The findings suggest the importance of neighbor interactions, decentralized control, self-organization, and nestedness within PL experiences. These identified mechanisms connect to previous literature on key features of PL and also spotlight the importance of teacher agency to promote

PL. In what follows, we first make connections to previous literature on PL and then highlight mechanisms within PL experiences that promote teacher agency.

Connections to Literature: Neighbor Interactions

Participants in this study overwhelmingly stated that interacting with colleagues in their PLC and with others across the project had the biggest influence on their learning. This is in line with numerous researchers who have reported on the crucial role that teacher collaboration plays in supporting teacher PL (e.g., Borko & Potari, 2019; Schleicher, 2015; Stigler et al., 1999; Vescio et al., 2008). Collaboration has been previously identified as a key feature to PL experiences for teachers.

Neighbor interactions with resources such as rich tasks and the curriculum were also mentioned as key to teacher learning. This aligns with researchers who suggest that a key feature of effective PL should focus on content and be aligned with the curriculum (e.g., Ball et al., 2005; Cohen & Hill, 1998; CPRE, 1998; Darling-Hammond et al., 2017; Loucks-Horsley & Matsumoto; 1999). When teachers engage with content and curricula, specifically mathematical content, they are provided opportunities to develop a deeper understanding of concepts and even to change their own identities as learners of mathematics (Anderson et al., 2018).

Lastly, interacting with student thinking and student work emerged in this study as a widely adopted activity. Several researchers have identified a focus on student thinking as an effective strategy for teacher PL (Darling-Hammond et al., 2017; Roth et al., 2011; Smith & Stein, 2015). Focusing on student thinking can provide teachers opportunities to better understand student ideas in order to better anticipate and plan responses and prompts to students questions or misunderstanding (Roth et al., 2011; Smith & Stein, 2015). This study found that neighbor interactions promoted the emergence of learning within a PL experience as a mechanism that is essential to support the previously identified key features of professional development: collaboration, and a focus on content and pedagogy.

Mechanisms of Professional Learning that Support Teacher Agency

While neighbor interactions connect to previous literature on key features of PL (Darling-Hammond et al., 2017), decentralized control, self-organization, and nestedness highlight the importance of teacher agency, an emerging key feature of PL (Biesta et al., 2015; Calvert, 2016).

Decentralized control. Decentralized control implies a shared leadership where no individual makes decisions and controls the system. Davis and Sumara (2006) go so far as to say that a single leader choosing the direction in a system could eliminate the potential for learning to emerge from a system. Similarly, research suggests that for collaborative PL to be effective a PLC requires a shared leadership approach (Hipp et al., 2008; Leithwood & Riehl, 2005). Shared leadership implies that teachers participate in determining the focus and direction of PL activities (Calvert, 2016). Shared leadership aligns with teacher agency, an important characteristic of effective PL cited in the research literature (e.g., Biesta et al., 2015; Day, 2017; Hauge & Wan, 2019; Sachs, 2016). The mechanism of decentralized control is critical within the PL experience to allow for shared leadership to promote teacher agency.

Self-organization. Self-organization manifested across different levels of the project, including ways the project self-organized in response to PLC's needs and the ways that different individuals adapted their practices in response to neighbor interactions across the nested systems of the project. These examples of self-organization are similar to a claim by Davis and Simmt (2003) that systems in education are adaptive self-organizing systems. Self-organization is an ongoing phenomenon that arises from the interactions of parts of a system with decentralized control and the "process of self-organization changes the relationships among the elements of the system as new relationships emerge" (Cochran-Smith et al., 2014, p. 8). These new relationships were apparent in the ways the different PLCs established connections with others within, and beyond their own PLCs. Teachers choosing the tasks to engage with, the content to focus on, the models of PL to employ, and the learning experiences they chose to share with others in the project all influenced the self-organization of the learning systems involved in the project. All of these actions were determined and negotiated by the participants in the project rather than through top-down instruction. Through self-organization, teachers had the agency to influence their learning opportunities on multiple levels.

Nestedness. The nestedness of the project supported important neighbor interactions between individuals and PLCs. The opportunities for individuals in PLCs that lacked the necessary balance between conditions such as diversity and redundancy to find this balance with others across the nested system supported teacher agency and the emergence of learning. Studies that focused on collaboration and supporting communities of practice (Wenger, 1998) generally focused on what an individual learned through their involvement in a project rather than how the project influenced their learning, and the learning of a larger system (e.g., Zehetmeier, 2015). Through this study nestedness emerged as an essential mechanism of complex learning systems as it was critical to support teacher agency by allowing teachers, and PLCs, to seek out learning opportunities across PLCs in the project.

Conclusion

Supporting and establishing the essential mechanisms of complex learning systems aligns with much of the research on the characteristics of effective PL (e.g., Anderson et al., 2018; Hord, 2008; Liljedahl, 2018, & Loucks-Horsley & Matsumoto, 1999). Yet, establishing the key features of PL is not sufficient to support teachers in their learning (Day, 2017; Hauge & Wan, 2019). Understanding how mechanisms of complex systems interact and align with the features of PL provides insight into how and why these features support teacher learning. This study demonstrates that considering the ways to establish a balance of essential mechanisms necessary to support learning is crucial. Thus, whether designing PL explicitly or considering how to establish environments conducive to teacher learning, the field should consider, and strive to put in place, the mechanisms of complex learning systems. Particularly, those in positions to influence learning systems should be conscious of the ways that decentralized control, self-organization, and nestedness are manifested in their context and strive to support these factors in order to support neighbor interactions.

It is important to note that when a learning system is unable to achieve a balance of these mechanisms, teacher agency becomes critical to make connections across the larger educational system to support a balance. Thus, organizing teacher learning within a nested complex learning system can provide opportunities for teacher agency across levels within the system to support the emergence of learning. We recommended that teachers are provided opportunities to meet and interact with others beyond a teacher's own context. This can provide the necessary perturbations to encourage teachers to continually push their thinking and respond in novel ways to new ideas, leading to the emergence of learning.

Teacher agency has emerged as an important area of focus in research and centering the voice of the teacher was an important component of this study. This study highlights the importance of conducting research that centers teachers' voices because teachers' perspectives, expertise, and experience are valuable assets that enrich the research process and contribute to more meaningful understanding of the complexity of mathematics teachers' collaborative PL. As the world continually changes, for both students and educators, research must move beyond the linear models used to study teacher learning. Teaching and learning mathematics has always been a complex endeavor and in order to better support mathematics teacher learning the field must look beyond individual key features towards essential mechanisms that support learning in complex systems.

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LEVERAGING CONNECTIONS ACROSS CONTENT AREAS TO SUPPORT ELEMENTARY MATHEMATICS CLASSROOM DISCUSSION

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Facilitating classroom discussion is complex work, especially for elementary classroom teachers who must make sense of classroom discussion practices across content areas. We examined two teams of elementary teachers and teacher educators as they engaged in professional learning to understand what connections among classroom discussion practices across content areas emerged from their conversations and how these connections were leveraged to support mathematics classroom discussions. We elaborate on two key ways that cross-content practices were leveraged: student participation and teacher facilitation moves.

Keywords: Teacher Educators; Classroom Discourse; Elementary School Education

Mathematics classroom discussions can support students' deep mathematical learning and provide opportunities to make sense of new content (Kazemi & Stipek, 2001; O'Connor & Snow, 2017). However, literature concerned with supporting mathematics discussion tends to be siloed to practices intended only for mathematics content (e.g., Bishop et al., 2020; Webb et al., 2014). Classroom discussion is a valued dimension of instruction across content areas starting in early elementary grade levels (Fitzgerald & Palincsar, 2019). Elementary classroom teachers are uniquely tasked with facilitating classroom discussion across multiple content areas. As such, elementary teachers are expected to draw on a vast range of practices to balance social and content goals which vary by content area (Rainey & Moje, 2012).

In this study, we look specifically at how teachers consider the similarities of classroom discussion across content areas. We hypothesize that the similarities in elementary discussion practices across content areas can be leveraged by educators to create richer discussion in mathematics lessons. As such, we ask: what and how do teachers and teacher educators identify connections within classroom discussion practices across content areas?

Theoretical Framework

We are interested in examining how educators leverage their experience and expertise to support classroom discussion practices across content areas and make sense of high-quality instructional practices. As a theoretical framework, teachers' collective sense-making frames how educators collaborate to make connections in their discussion facilitation practices in mathematics and ELA. Coburn (2001) argues that teachers' sensemaking is social, as it is "rooted in social interaction and negotiation [and]... is deeply situated in teachers' embedded contexts" (p. 147). Collective sensemaking involves the interactions and negotiations among individuals to establish meaning of particular events or shared experiences (Coburn, 2001; Kelly, 2006). In other words, it is the opportunities teachers have to interact with one another in communities of practice (Wenger, 1998) that allow space for their professional learning.

Elementary teachers are expected to engage in high-quality instruction in multiple content areas, and that instruction is often expected to include discussion. While attention to researching teachers' discussion practices has focused on content-specific analysis (e.g., Bishop et al., 2020), we see opportunities in a collective sensemaking framework to examine how teachers use their different levels of expertise to negotiate meaning of classroom discussion across content areas. Sensemaking can thus occur in the interaction of talk (Bakhtin, 1981) where teachers' shared engagement with ideas prompts new or refined interpretations of ideas.

Methods

We draw on data collected from a larger study of professional learning (PL) that focused on supporting elementary teachers' classroom discussion practices. Our analysis focused on a subset of educators in one district in the northeastern United States, from Parks and South Elementary Schools (pseudonyms). Teachers met in grade level bands for full day job-embedded PL with teacher educators, including instructional coaches and a facilitator from the research team. The PL followed the Learning Labs structure (see Kazemi et al., 2018) and focused on developing classroom discussion practices in mathematics and ELA. Learning Labs consist of a cycle of (1) introducing new ideas related to content and pedagogy, (2) co-planning an instructional routine, (3) enacting the routine in one of the teachers' classrooms, and (4) debriefing the experience to make plans for their future instruction.

Data Collection

We focus specifically on the PL sessions with the third (N=3) and fifth (N=5) grade teachers because they engaged with a wide range of ideas about leveraging the classroom discussion practices in one content area to support another. In the 2022-2023 school year, each grade level team participated in a total of four Learning Labs (two mathematics and two ELA) with a PL facilitator and their instructional coaches. Among these teams, the teachers and teacher educators are primarily white women and come from a wide range of teaching experience. Each PL session was video and audio recorded.

Data Analysis

We used the conversations of teachers as they participated in the series of Learning Labs as our primary data source. Analysis focused on understanding what teachers and teacher educators said about connecting discussions across content areas during the PL sessions. We reviewed the introduction and enactment debrief portions (phase 1 and phase 4) of each Learning Lab (mathematics or ELA, eight Labs total). Using an inductive coding process aligned with a grounded theory approach (Charmaz, 2000), we reviewed these portions of the Learning Labs for moments where teachers reflected on their classroom discussion practices. Ultimately, we documented moments where educators talk about discussions in mathematics compared to other content areas (63 instances total). Then, using a constant comparative method (Charmaz, 2014), we identified larger themes of how teachers leveraged their understanding of classroom discussions across content areas. For this paper, we focus on instances where educators leveraged non-math content thinking to make sense of mathematics discussions.

Results

In our analysis of the third and fifth grade teams' interactions during their respective PL sessions, we identified common ways that the educators leveraged discussion practices across content areas to develop rich mathematics discussions. These four cross-content connection types were (1) similarities in classroom discussion norms, (2) adapting tasks, (3) similarities in student participation, and (4) similarities in teacher facilitation moves (see Table 1). Below we detail two of these types of connections with examples from the teachers and teacher educators.

Table 1: Cross-Content Connections to Support Classroom Discussion

Theme	Description	Example
Classroom Norms	Recognizing ways to develop the norms of discussion for mathematics as a form of developing social skills	Esther: [After our last ELA LL] I've been really trying, across all areas, to get kids to explain other students' thinking. In math, [...] "Why did this person do what they did?," getting kids to repeat what others said… to see if there was active listening. [Grade 3, ELA LL2]
Adapting tasks	Recognizing a need to adapt the goals or structure of a task to be more like tasks from a different content area	Greer: ELA tends to be a little bit more open with our questions when we're asking them during the read aloud. Olivia: How do we then open up like, we have to think of open tasks for kids, right, in math? [Grade 3, MATH LL1]
Student participation	Understanding the ways that students can participate in a discussion that work to create a comfortable environment	Greer: Students are more confident in math than reading. In math, there is a right answer and you can come to it in many ways. Reading is vague and there are multiple answers. It's harder to make a safe space with text because there is more variation and vulnerability. [Grade 3, ELA LL1]
Teacher facilitation moves	Naming facilitation moves that support discussion across content	Elliott: I'll put sunglasses on and they know I'm not there. They can't ask questions, they've got to figure things out on their own. [Grade 5, MATH LL2]

Student Participation

Regardless of content area, the teams developed an understanding of the ways that students can participate in a classroom discussion that work to create a "comfortable" environment. Teachers emphasized the importance of student participation and discussed ways to support students to feel comfortable to share their ideas during discussions, noting differences in student participation across content areas. In one such example, Esther, a third-grade teacher, described her observations about student participation, "It shifts throughout the day if they are comfortable with the content." She explained that in science and social studies students feel more comfortable and excited to respond. The teachers extended this idea by sharing their own observations and collectively came to associate student participation with confidence. Another third-grade teacher, Winter connected this to mathematics and ELA as she described,

> It is important to look at who is participating and who is not participating in all discussions. In math, a student may have an answer but not feel confident sharing it. In reading, students don't know how to say a word, so [they] won't try.

Teachers recognized that student participation was important across content areas but named differences in the ways that students might hold back from participating. They leveraged characteristics of the spaces in which students are comfortable to participate as they talked about how to support student participation in mathematics classroom discussions.

Teacher Facilitation Moves

Teachers also considered the specific pedagogical moves they might make in one content area for discussion that could easily be used in another, not just mathematics teaching. The connections they made focused on patterns of participation that could support classroom discussion across content areas. For example, the PL facilitator in a fifth grade Learning Lab asked what feels similar or different between leading discussions in reading and mathematics. Cait and Janet brought up the desire to move away from the typical discussion structure of initiate, respond, evaluate as a goal in both mathematics and ELA discussions, drawing on their experiences facilitating mathematics classroom discussion. Cait added that she wanted to “shift to other ways to get student responses and student discussion” such as having the teacher stand back to let students collaborate. In this example, Cait and Janet co-developed a shared goal to use facilitation moves that increased student voice in classroom discussion, drawing on the prior success the 5th grade team had with this particular teacher move in an earlier mathematics Learning Lab. We hypothesize that connections among pedagogical moves allowed teachers to extend their experiences facilitating classroom discussion across content areas.

Discussion

In their conversations during PL, educators made connections among discussion practices across content areas to leverage support of mathematics classroom discussion. We primarily focused on how these PL discussions support mathematics and identified ways that the teachers leveraged successful mathematics practices to support other content area discussions.

Each theme emerged from the conversations among teachers and teacher educators through collective sense-making (Coburn, 2001) of classroom discussion practices in elementary classrooms, grounded in their own discussions and collaboration. Collective sensemaking was evident in how educators listened to the reflections of their peers to refine their thinking about discussion practices. This work highlights how teachers draw on their experiences and interactions with one another to make meaning of instructional practices within their own contexts (Kelly, 2006; Wenger, 1998). Although we investigated these ideas with a small sample, this study sets a foundation for future work to examine teachers’ reflections in different settings to support teacher learning in their unique contexts.

Elementary teachers, who teach across content areas, are being asked to incorporate classroom discussion, but often without support for how these practices may be similar. The changing landscape of education and uncertainty in children's mathematical learning indicates an urgent need to support teachers to leverage their expertise in other content areas to support mathematics instruction. Thus, we argue that leveraging teachers' experiences in the areas of classroom discussion norms, adapting tasks, student participation, and teacher facilitation moves provides an opportunity for enriching mathematical discussion practice in the classroom.

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ENHANCING TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING PROBLEM SOLVING SKILLS THROUGH LESSON STUDY

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In mathematics education, significant attention has been given to understanding the knowledge teachers need to effectively teach mathematics. While much research has focused on the knowledge of pre-service and beginning teachers in the context of classroom mathematical problem-solving, there is limited insight into the knowledge in-service secondary teachers use to develop and teach mathematics through problem-solving. This participatory action research via lesson study reports the knowledge nine secondary math teachers in sub-Saharan Africa used to craft problem-solving lessons. The data revealed a collective understanding of problem-solving as involving high level, open middle tasks grounded in students' everyday experiences.

Keywords: Mathematical Knowledge for Teaching, Professional Development, Teacher Knowledge, Problem-Solving.

Mathematics Problem-Solving Knowledge for Teaching

Teachers are essential components of the education system and are pivotal in achieving a nation's educational goals. In mathematics education research, significant attention has been given to understanding the knowledge teachers need to teach mathematics effectively. A key contribution to this discourse is the Mathematical Knowledge for Teaching (MKT) framework (Ball et al., 2008), built on Shulman's (1986, 1987) seminal work advocating for a professionally oriented knowledge for teaching mathematics. Ball et al. conceptualized MKT as "mathematical knowledge needed to carry out the work of teaching mathematics. . . [emphasizing] the tasks involved in teaching and the mathematical demands of these tasks" (p. 395).

Teaching Through Problem Solving

The MKT framework laid the foundation for a new line of research focusing on Mathematics Problem-Solving Knowledge for Teaching (MPSKT) (Chapman, 2015; Foster et al., 2014; Clivaz et al., 2023). The MPSKT framework (see Table 1) underscores the importance of problem-solving skills, as emphasized by the National Council of Teachers of Mathematics (NCTM, 2000).

MPSKT includes understanding the need for authentic, non-routine tasks that challenge students' existing knowledge and skills, encouraging them to develop deeper understanding and proficiency (Bailey, 2022; Lester, 1994; Masingila et al., 2018). Mathematical problem-solving tasks should be meaningful, stimulating, and aligned with students' abilities and interest to promote effective problem-solving. Students should engage in solving tasks for which they do not have an immediate or obvious solution (Schoenfeld, 1985).

However, teaching problem solving poses significant challenges such as finding and selecting appropriate tasks or overcoming the persistence of traditional teaching methods (Masingila et al., 2018). As Lester (1994) notes, problem solving is a complex human endeavor that involves more than recalling facts or following routine or memorized procedures. As

Henningsen and Stein (1997) assert, teaching through problem solving involves selecting and implementing high-level tasks (Stylianides & Stylianides, 2008) while maintaining the level of cognitive demand.

Table 1: Mathematics Problem-Solving Knowledge for Teaching (Clivaz et al., 2023, p. 23)

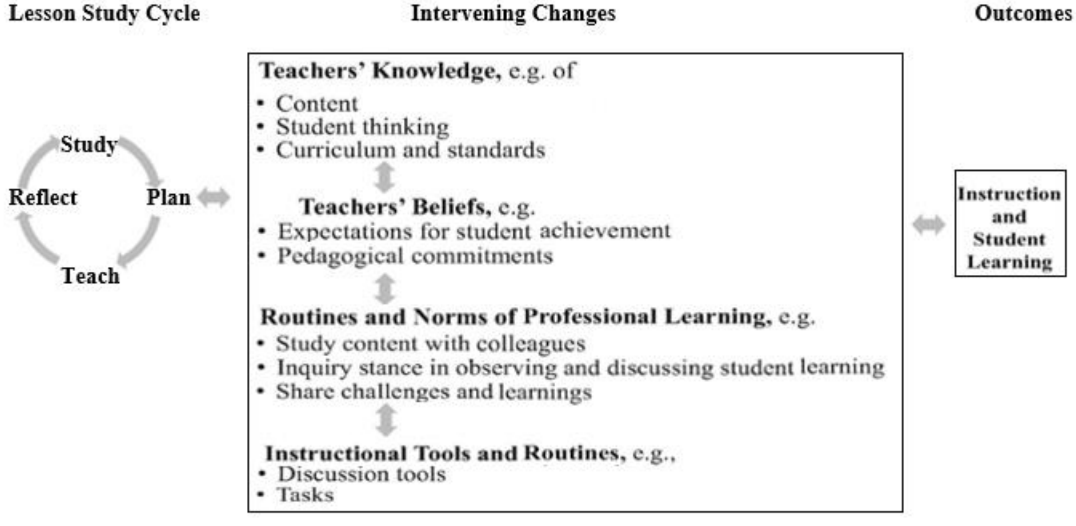
PS content knowledge	Knowledge of mathematical problems	Understanding of the nature of meaningful problems; structure and purpose of different types of problems; impact of problem characteristics on learners
	Knowledge of mathematical problem-solving (PS) *[do/meta]*	Being proficient in PS *[do]*. Understanding of mathematical PS as a way of thinking; PS models and the meaning and use of heuristics; how to interpret students' unusual solutions; implications of students' different approaches *[meta]*.
	Knowledge of problem posing	Understanding of problem posing before, during and after PS.
Pedagogical PS knowledge	Knowledge of students as mathematical problem solvers	Understanding what a student knows, can do and is disposed to do (e.g., students' difficulties with PS; characteristics of good problem solvers; students' PS thinking).
	Knowledge of instructional practices for PS	Understanding how and what it means to help students become better problem solvers (e.g., instructional techniques for heuristics/strategies, metacognition, use of technology and assessment of students' PS progress; when and how to intervene during students' PS).
	Knowledge of affective factors and beliefs *[teacher/student]*	Understanding the nature and impact of productive and unproductive affective factors and beliefs *[of the teachers/of the students]* on learning and teaching PS.

Overview of Study

Through a lesson study-based professional development program, this study examined how nine high school math teachers in sub-Saharan Africa comprehended and planned for teaching through problem-solving. This study addresses the question: How did nine high school math teachers in sub-Saharan Africa collectively comprehend problem-solving tasks and teaching through problem solving? The study contributes to our understanding of knowledge of mathematical problem-solving tasks, fostering a problem-solving-oriented classroom environment, and facilitating professional development on this approach.

The study employs Lewis et al. (2019) model of lesson study impact (see Figure 1), suggesting that "lesson study can influence instruction and student learning through intermediate changes in teachers' knowledge and beliefs, professional norms and routines, and instructional materials." (Lewis et al., 2019, p. 15). This model underscores the potential of lesson study to enhance teaching practices and student outcomes, supported by several other studies (Dotger et al, 2023; Lewis & Perry 2015, 2017; Pernilla & Henrik 2018). George Pólya's heuristic methods provided practical steps for problem-solving (Pólya, 1945), which were further reinforced by the National Council of Teachers of Mathematics (Cai & Lester, 2010; NCTM, 2000) standards emphasizing problem-solving as essential for mathematical understanding.

Figure 1: Theoretical model of lesson study impact (Lewis et al., 2019, p. 15)



Method

The study opted for participatory action research via lesson study to emphasize active participation and collaboration among all stakeholders involved (Lewis & Tshuchida, 1999; Stigler & Hiebert, 1999; Townsend & Taylor, 2022). The study took place within the Kenyan education system, with the goal of investigating the impact of lesson study on teachers' understanding of problem-solving tasks. Nine mathematics teachers from a Kenyan secondary school took part in a series of eight meetings, approximately 80 minutes each. The meetings focused on comprehending problem-solving teaching methods and planning and implementing lessons that followed this approach. Data sources included a pre-survey, recorded meetings, lesson plan, individual reflections from participants and researcher fieldnotes. The first author served as a knowledgeable other, offering support to participants throughout the process.

The first author conducted an initial focus group discussion with participants to identify a focal topic for lesson study. The cohort identified *similarity and enlargement* as a challenging yet essential topic for their curriculum. The cohort discussed what lesson study is and how it works, using resources from the Lesson Study Group at Mills College. The first author introduced participants to Polya's (1945) problem-solving framework: understanding the problem, devising a plan, carrying out the plan, and looking back. During discussion, the cohort considered reasons for teaching through problem solving, the role of teachers in problem-solving classrooms, the nature of problem-solving tasks, and how to orchestrate a problem-solving classroom.

Analysis, Results and Discussion

We used Reflexive Thematic Analysis (RTA), an analysis process that emphasizes active engagement and reflexivity of the researcher during analysis (Braun & Clarke, 2022). The unit of analysis was described as segments of communication reflecting participants' understanding of problem-solving tasks or pedagogy, which included individual sentences, detailed paragraphs, or group conversations. We generated initial codes that we merged into four major themes that describe how the cohort conceptualized problem-solving tasks and teaching through problem solving. We describe each theme and provide illustrative examples of the data that support each theme.

Open middle tasks. Teachers discussed creating tasks with multiple entry points, allowing students to approach problems in various ways and encouraging diverse problem-solving methods. This multidimensional approach ensures students can explore different solutions and deepen their understanding. For example:

> Mr. Mirumbe 24:59: The problems that we wish to have should be multi-dimensional. That a number of students should come with different ways that can be related to or not necessarily related, but they follow a certain rule that at the end of it, we can come up with a conclusion. (Day 3)
>
> Mr. Ambere 12:59: Whatever I highlighted from the lesson is that the students were using the various methods to solve the problem. . . . There are those who after being given the materials, they went directly by measurement and got the various dimensions. But there are those who didn't use the measurement, they just did the calculation to get the length and the width. (Day 6)

Challenge students' thinking. Teachers designed problems to challenge students' thinking and reasoning, avoiding tasks that they perceived as rote application of knowledge. Mr. Okello asked about the complexity of the questions, while Mr. Mirumbe emphasized the importance of problems that require deep thinking and decision-making. For example:

Mr. Okello 36:23: How about. . . if I may ask, the question, should it be complex or just a simple question? (Day 3)

Mirumbe 14:30: We give a problem, something that will require them to think and reason and solve the challenge that will be given to them. Not an exercise that they can copy the knowledge that they had learnt from the previous lesson to solve. Something that is quite challenging that can involve them using good decision. (Day 4)

Real-life relevance. Teachers emphasized the importance of connecting mathematical problems to real-life situations, such as measuring the height of a tree using shadows, as stated by Mr. Kasyoki. Md. Filomena noted that students often perform calculations without understanding the underlying concepts or real-life applications, which hinders the development of true problem-solving skills.

Mr. Kasyoki 30:20: Measuring the height of a tree using shadows. You see we cannot measure it directly. We cannot tell them to go and measure the height of a tall tree or a story building like our new academic block. Instead, they can find the height in terms of linear scale factor. They can measure their heights themselves by comparing [length of] shadows of the tree and a shorter object whose height they can obtain. (Day 4).

Md. Filomena 04:00: Like we said, most of the students can do the calculation of similarity and enlargement but they never think more about what they are doing. It's like they look for the formulas. When you are given this, you do this and leave but now they have never thought about it and the application. ...Are they[students] going to develop the problem-solving skills or it's just a matter of coming up with the formulae and using them in an exam. We talked about the application part of the similarity and also be able to relate in real life situation. (Day 5)

Students' ability. Teachers tailored tasks to match students' abilities, ensuring problems are challenging yet accessible. Mr. Kasyoki challenged his colleagues to consider the students' perspective and avoid overly complex tasks that might confuse them. The goal was to engage students meaningfully without causing frustration.

Mr. Kasyoki 53:15: So I think if we are the ones thinking this much, we have to ask ourselves how much our students will think about it. (Day 3)

Mr. Kasyoki 50:51: You may confuse our students if you give them money in fraction form. . . . how many times have you used currency in terms of fractions? How often? (Day 5)

Discussion

In a region where access to resources and opportunities is limited, a students' ability to think critically, creatively, and analytically can mean the difference between empowerment and disempowerment. Teaching through problem solving supports students to develop problem-solving skills. The MPSKT framework outlines knowledge that teachers need to effectively teach problem-solving skills. This study informs our understanding of how lesson study can support teachers to engage – and potentially develop -- their MPSKT, particularly in the areas of Knowledge of Mathematical Problems and Knowledge of Students as Mathematical Problem Solvers.

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DEVELOPING ADAPTIVE EXPERTISE: EXAMINING OPPORTUNITIES FOR REFLECTIVE PRACTICES DURING A STUDIO DAY CYCLE

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Engaging teachers in reflective practices is recognized as a crucial component of their adaptive expertise development. Drawing on this perspective of adaptive expertise development, we qualitatively examined how the design and structure of a Studio Day professional learning cycle afforded opportunities for reflective practice for secondary in-service mathematics teachers. We found that small group reflections, immediate reflections-on-action, and the use of videos afforded notable instances of reflective practices throughout the Studio Day Cycle that supported teachers' development of adaptive expertise of equity-based, language-responsive teaching. We suggest that Studio Day Cycles are one avenue to better support in-service teachers' development of adaptive expertise of mathematics language routines and multilingual learner core practices.

Keywords: Professional Development, Middle School Education, Equity, Inclusion, and Diversity

Supporting adaptive expertise development is one way the field can better prepare mathematic teachers to face the evolving challenges of teaching and attend to the diverse and emergent needs of students (Anthony et al., 2015). To effectively respond to the increasing linguistic diversity and growing proportion of multilingual learners in K-12 classrooms (Meyer et al., 2020), we contend that an adaptive expertise of equity-based, language-responsive pedagogy positions teachers to support all students in mathematics, especially multilingual learners (Roberts & Olarte, 2023). Despite the growing focus on equity-based pedagogies and curriculum for multilingual learners (e.g., de Araujo & Smith, 2021), the existing scholarship base on adaptive expertise has generally focused on pre-service teachers (e.g., Anthony et al., 2015). We argue that it is equally important to examine and identify best practices to support in-service teachers' development of adaptive expertise of mathematics instruction for multilingual learners. In the present work, we report on how teachers' participation in a professional learning cycle, Studio Days (Von Esch & Kavanagh, 2018), focused on mathematics language routines (Zwiers et al., 2017) and multilingual learner core practices (Roberts & Olarte, 2023) supported opportunities for the development of adaptive expertise of equity-based, language-responsive mathematics teaching.

Engaging teachers in reflective practices (e.g., Cavanagh & Prescott, 2010) – broadly referring to opportunities that teachers can be reflective and metacognitive about their teaching practices – is recognized as an integral dimension of adaptive expertise development (Anthony et al., 2015; Wetzel et al., 2015). Tsui (2009) asserted that "the process of reflection and conscious deliberation in which practical knowledge is theorized and theoretical knowledge is interpreted in practice" (p. 437), is how teachers develop adaptive expertise. Moreover, effective professional learning programs have been characterized as those that allow teachers to explore, inquire, experiment, and reflect (Wise et al., 1999). Although many scholars have reported on professional learning interventions that prompt mathematics teachers to engage in reflective practices and inquiry (e.g., Gningue et al., 2014), there is still limited research examining professional learning efforts that specifically encourage teachers to reflect on mathematics instruction for multilingual learners (de Araujo et al., 2018). The research question that guided this study was: How did opportunities for reflective practices within a Studio Day Cycle support mathematics teachers' development of adaptive expertise of mathematics instruction for multilingual learners?

Conceptual Framework

We draw on reflective practice (i.e., Muir & Beswick, 2007) and adaptive expertise (Yoon et al., 2015) to share how the design and structure of the Studio Day Cycle (Von Esch & Kavanagh, 2018) afforded teachers opportunities to engage in reflection that supported their adaptive expertise development.

Reflective Practice

Existing literature has widely emphasized the importance of teachers being metacognitive about their practice and how looking inward and reflecting on that practice is crucial to their development and change (Cavanagh & Prescott, 2010). Hayden et al. (2013) wrote, "Reflection on critical incidents in teaching and on feedback received can become the catalyst for transformative change in teaching practice" (p. 144), highlighting the salience of teachers both considering important events in their classrooms and receiving support to unpack those events. Encouraging teachers to reflect on their students' use of language in mathematics classrooms and their current language-responsive practices within a professional learning community can be conducive to developing their adaptive expertise. We align our work with Muir and Beswick (2007), who conceptualized reflective practice as "reflection that is deliberate and can be focused on events or incidents, and personal experiences" (p. 77). They offered a three-level model to examine in-service teachers' reflective practices: (1) *technical description,* or teachers recalling general accounts of classroom practices, focusing on technical aspects, and omitting value judgements to the experiences; (2) *deliberate reflection,* or teachers identifying 'critical incidents' and providing rationales for past and future actions; and (3) *critical reflection,* or teachers moving beyond identifying 'critical incidents' to consider others' perspectives and offer alternatives. We consider these forms of reflective practices to uniquely support teachers' development of dimensions of adaptive expertise.

Adaptive Expertise

Adaptive expertise broadly refers to the process of teachers' recognizing and identifying emergent needs, making sense of multiple perspectives, and orchestrating multiple teaching approaches to meet the demands of different situations (Hatano & Inagaki, 1984; Yoon et al., 2015). We utilize Yoon et al.'s (2015) characterization of three dimensions of adaptive expertise to describe teachers' mathematics instruction more explicitly for multilingual learners (See Table 1). Like other scholars who have brought the theories of reflective practice and adaptive expertise together (e.g., Anthony et al., 2015; Tsui, 2009), we consider opportunities of reflective practice as conduits of teachers' adaptive expertise development. With our goal of supporting the development of their adaptive expertise of mathematics instruction for multilingual learners, we aimed to engage in-service teachers in a myriad of reflective practices within a Studio Day Cycle.

Table 1: Adaptive Expertise of Mathematics Instruction for Multilingual Learners

Dimension	Description
Flexibility	Awareness of students and context, particularly multilingual learners. Ability to constantly adapt practice and respond to unexpected issues as related to students' needs, particularly multilingual learners.
Deeper Level of Understanding	Able to assimilate information and to implement or make connections that builds or addresses deeper level of knowledge. Able to bring in variations from outside the present system of activity as related to instruction for multilingual learners. Able to describe the affordances and constraints of mathematics language routines. Considers contexts in which to apply and integrate instructional practices for multilingual learners.
Deliberate Practice	Demonstrates an ability to show motivation, focus, and repeated effort to monitor their practice, and devises and subsequently attempts revamped attempts to improve implementation, as related to multilingual learners. Improves, assesses, and reflects on their own and others' implementation of language-responsive practices. Explicit evidence of reflecting on how to improve as related to mathematics language routines and other language-responsive practices. Describes how they are motivated to continue to develop their practice.

Method

Context

The present study is part of a large, multi-year funded project focused on supporting the development of mathematics teachers' adaptive expertise of mathematics language routines and data science instruction in a school district in the West Coast of the United States. We designed our professional development intervention around Studio Days (Von Esch & Kavanagh, 2018; See Figure 1 below). Adapted from Lesson Study, two teachers develop and study a single lesson (not necessarily the same lesson/content). However, their lessons are focused on the same focal mathematics language routine (MLR) paired with a multilingual learner core practice (Roberts & Olarte, 2023). Other teachers observe a live enactment of the lesson and reflect on the observed lesson. In the 2023-2024 academic year, we planned three Studio Day Cycles with participating teachers. Each Studio Day Cycle began with a Pre-Studio Day where the research team introduced the focal MLR and multilingual learner core practice. We outlined the stages of the routines, highlighted considerations for enacting the routines, and participating teachers experienced the routines as students. About one week after the Pre-Studio Day, two teachers volunteered to enact the focal routine in their classroom during the Studio Day, and the research team and other participating teachers observed these classroom enactments. The teachers then debriefed their enactment following their lesson, briefly sharing their experience, and receiving feedback from the observers. Then, about a week after the Studio Day, we reflected on that experience in a Post-Studio Day, where teachers reflected on video clips of the enactments, copies of student work from the Studio Day, and the MLR and multilingual learner core practice.

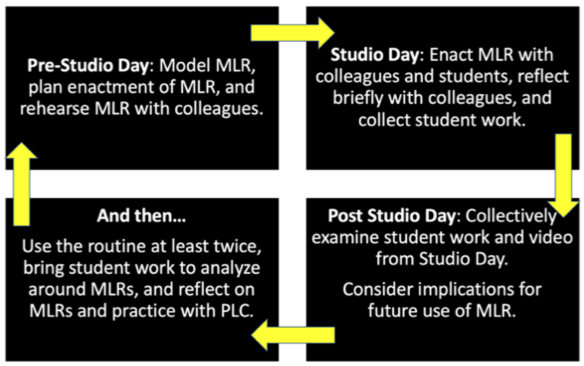

Figure 1: Studio Day Professional Learning Structure

Participants

The research team designed the Studio Day Cycle for the 2023-2024 academic year. Purposeful sampling (Miles et al., 2020) was used to recruit the district's mathematics instructional support specialist, three Math 7 teachers, and two Math 8 teachers from the three district junior high schools (See Table 2). For the present study, we report on the first Studio Day Cycle, where we focused on the mathematics language routine *Collect & Display,* paired with the multilingual learner core practice: *identifying disciplinary language demands and supports.* In Collect & Display, teachers capture students' oral words, ideas, phrases into a stable reference. The intent of the routine is to stabilize students' language in order to use their output as a reference for developing their mathematical language (Zwiers et al., 2017). The multilingual learner core practice of identifying disciplinary language demands and supports, refers to teachers employing or identifying language supports for students. They also adequately scaffold or produce language while attending to aspects of language that may be challenging for students (Aguirre & Bunch, 2012).

Table 2: Participant Profiles

Name (pseudonym)	Grade Level	Race/ Ethnicity (self-described)	Years Teaching Mathematics	Bilingual/ Multilingual (self-described)
Ms. Ruth	7th grade	White	26	–
Ms. Severn	7th grade	Caucasian	4	Semi Fluent in Spanish
Ms. Taylor	7th grade	White	34	–
Ms. Foster	8th grade	White	3	Russian
Ms. Penny	8th grade	White	2	–
Ms. Hope	Instructional Specialist	Caucasian	25	–

Data Collection and Analysis

We video- and audio-recorded each meeting of the Studio Day Cycle, and utilized the videos and transcriptions of each meeting to examine the types of reflective practices that teachers engaged in. First, we created content logs of the videos to identify notable instances of reflective practices. We then coded (Miles et al., 2020) the transcriptions for the type of reflective practice using Muir and Beswick's (2007) three-level model. This allowed us to describe how the participants reflected during the Studio Day Cycle. Next, we drew on Yoon et al.'s (2015) characterization of the three dimensions of adaptive expertise (again, see Table 1) to make sense of how the moments of reflective practice supported teachers' development of adaptive expertise. We met as a research team to discuss themes that we observed in the data and wrote analytic memos (Miles et al., 2020) to better understand how the opportunities of reflective practices within the Studio Day Cycle afforded or constrained teachers' adaptive expertise development.

Findings

To illustrate how the reflective practices within a Studio Day Cycle supported teachers' adaptive expertise development, we describe the structure of each day of the cycle and highlight the notable instances of reflective practices taken up by the participants.

Pre-Studio Day

Our goal for the Pre-Studio Day was to introduce teachers to the focal mathematics language routine Collect & Display, and to the multilingual learner core practice, identifying disciplinary language demands and supports. Although we observed instances of all three types of reflective practice during the Pre-Studio Day, we found that teachers primarily engaged in *technical descriptions*. This was expected given that this first day of the Studio Day Cycle was designed to introduce teachers to the core practice and MLR, as well as to get a sense of how teachers noticed their students' language use in the classroom and to discern what they already did to support students to read, write, and speak about mathematics. For example, Ms. Foster said,

> We've identified disciplinary language demands and supports, like making sure kids truly understand the words that we're saying mathematically. Like, just making sure if I'm saying "solve", what does that mean?...So, just making sure kids truly understand the words that we're saying and using their language to help bridge the gap [motions bringing hands together] between academic [language] and their every day [language].

This *technical description* of her students' language demands and her practice of building on students' language also revealed Ms. Foster's *flexibility,* or awareness of students' needs.

Reflective practices in small groups. One exemplar opportunity for reflective practice during the Pre-Studio Day was placing participants in small groups and having them reflect on the following questions: (1) Why do students use language in mathematics classrooms? (2) How do your students use language and communicate in the mathematics classroom? (3) What tools do your students use to use language and communicate in the mathematics classroom? The questions were purposefully posed to get a general sense of what teachers presently noticed and understood about their students' use of language. They were first given time to think individually, and this afforded opportunities to engage in *technical descriptions.* However, once the teachers were discussing in the small groups, we observed instances where teachers engaged in *deliberate reflection* as they identified critical incidents of students using language in the classroom, and *critical reflection* as they expanded beyond identifying critical incidents and considered the perspectives and experiences of the members of their small group. They used a Jamboard to record their ideas and display them to the whole group, where they shared such ideas as: "They use hands or drawings to clarify because they don't have formal language, so we try to get them to have language to explain" and "language is used to explain their thinking and to clarify their understanding." These ideas illustrated that the teachers were able to work in a group with each other to develop generalizable ideas in their *critical reflections*.

During the small group time, teachers went back and forth sharing their ideas, and it was in these rich discussions that we observed teachers engage in both *deliberate reflection* and *critical reflection.* For example, in one small group we observed teachers collectively reflecting on critical incidents of students' language use, such as students gesturing or asking each other questions in the classroom. Then, Ms. Hope demonstrated evidence of *critical reflection* as she considered the perspectives of the other members of the group, synthesized their group reflections, and articulated that students' language could be broadly categorized as "input and output." In these reflective practices, we again primarily saw evidence of *flexibility,* as evidenced by awareness of students' language in mathematics.

Studio Day

The Studio Day occurred one week after the Pre-Studio Day. On this day, Ms. Ruth and Ms. Taylor enacted the routine Collect & Display in one of their class periods. The other teacher participants took on participant-observer roles during the lessons, walking around, taking observation notes, and interacting with students. The Studio Day began with a pre-brief of Ms. Ruth's lesson, where she provided details of her lesson plan, her classroom, and her expectations of what the research team/other teachers should do during her lesson. After Ms. Ruth enacted the routine, the participants and the research team met to debrief Ms. Ruth's lesson. After this debrief, we held a similar pre-brief for Ms. Taylor's upcoming lesson. The other teachers took on similar participant-observer roles during Ms. Taylor's classroom enactment, and at the end of the day, we debriefed Ms. Taylor's enactment of Collect & Display. The reflective practices of the Studio Day privileged *deliberate reflection* and *critical reflection* because teachers observed actual classroom enactments through which they identified and reflected on critical incidents shortly after each teachers' enactment.

Immediate reflection-on-action. Key reflective features of the Studio Day were the pre-brief sessions that oriented teachers to details of the upcoming lesson, classroom dynamics, and student engagement prior to observing Ms. Ruth and Ms. Taylor enact the routine with their students, and the debriefs immediately after the lesson that engaged teachers in reflection-on-action (Manrique & Abchi, 2015). Reflection-on-action refers to teachers' purposeful reflection after their practice, and in our debriefs, we prompted teachers to reflect on Ms. Ruth's and Ms. Taylor's enactment of Collect & Display. For the enacting teachers, we asked them questions such as: (1) How did you feel about the lesson?; (2) Did you consider multilingual students during your lesson?; and (3) How did you adapt in real time? For the observing teachers, we asked them questions such as: (1) What did we see students doing during Collect & Display?; and (2) How did students engage with disciplinary language demands? The debriefs allowed the teacher who just enacted the routine to reflect on their teaching practices immediately after class, and provided the other participating teachers opportunities to share insights and feedback based on their observation notes. Moreover, we purposefully oriented teachers' reflections using the MLR and paired multilingual learner core practice.

Again, we observed that teachers were able to take up all three reflective practices during the debriefs. For example, Ms. Ruth engaged in *technical descriptions* and *deliberate reflection* within her own enactment of the routine and shared that the students were "very engaged and [for] kids who have difficulty accessing [the problem], it [Collect & Display] gives them opportunities to access, because there's no penalty for getting it wrong." During this moment of reflective practice, we found that Ms. Ruth exhibited the dimensions of adaptive expertise *flexibility* and *deeper level of understanding*, because she demonstrated an awareness of her students, and she articulated an affordance of Collect & Display – mainly that it was a routine that allowed students to access the mathematics content and language. The immediate reflections-on-action in the debriefs also afforded the observing teachers valuable reflective practices that developed their adaptive expertise. For example, Ms. Severn engaged in *technical descriptions* and *deliberate reflections* as she praised Ms. Ruth's ability to connect students' informal language with the formal mathematics language. She explained, "Highlighting the ways that informal and formal language related to each other was, I think, a good way to marry the different levels of language that the kids need." In this reflection, Ms. Severn exhibited *flexibility* because of her awareness of students' language use, and she also exhibited a *deeper level of understanding* of practice of identifying disciplinary language demands and supports. Through her reflection of Ms. Ruth's lesson, Ms. Severn shared those explicit connections between students' informal and formal language supported their language needs in mathematics.

Post-Studio Day

We held our Post-Studio Day one week after the Studio Day and in between this time, teachers were encouraged to continue to use the MLR in their classrooms. Additionally, to prepare for the Post-Studio Day, the research team purposefully selected video clips from Ms. Ruth's and Ms. Taylor's classroom enactments of Collect & Display. We selected clips from two types of videos: a video from an iPad turned towards the front of the classroom (i.e., focused on the teacher), and videos from Ordro headband cameras that students were wearing during class. We selected moments that would allow for broad reflection, as well as those that would allow for purposeful reflection on teachers' language-responsive mathematics instruction. We found that because a goal of the Post-Studio Day was to discuss how teachers can build on the enactments of the Studio Day and how they can implement the routine in their own classrooms, the moments of reflective practices on this last day primarily encompassed *deliberate reflections* and *critical* *reflections* and afforded development of the *deeper level of understanding* and *deliberate practice* dimensions of adaptive expertise. Participants were able to articulate how they might integrate the routine in their classroom and demonstrated motivation and desire to improve future implementations of the MLR.

Reflecting on videos enactments. The notable opportunity for reflective practice of the Post-Studio Day was when teachers were shown the video clips of Ms. Ruth's and Ms. Taylor's enactment of Collect & Display. Overall, the teachers reported that this was an extremely valuable opportunity not only to see themselves teach, but seeing the videos from the student headband cameras provided new insight into how the students worked with their peers, how they spoke about mathematics, and how they made sense of the task. Both Studio Day focal teachers had utilized the Desmos curriculum (Amplify Education, Inc., 2024) to enact Collect & Display, and the teachers demonstrated evidence of *technical descriptions* and *deliberate reflection* as they articulated details of their enactments and provided rationale for features of their Desmos activity. For example, Ms. Ruth explained, "I don't think I've ever gotten as, as rich variety… What's different? But it's really nice that we have a way for kids to share their thinking that's safe because you can anonymize it." In this moment, we also observed Ms. Ruth's development of a *deeper level of understanding* of the routine Collect & Display, because she described an affordance of facilitating the routine specifically through Desmos – integrating her understanding of the goals of the routine with what Desmos affords for the students. Mathematics language routines are flexible and adaptable, and we observed that teachers developed their adaptive expertise because they described the affordances and constraints of using technology to enact the routine instead of traditional paper/written work.

Discussion and Conclusion

We found that the Studio Day Cycle afforded valuable opportunities for reflective practices that supported teachers' development of adaptive expertise of language-responsive, mathematics pedagogy. Over the course of the cycle, we found notable instances of participants engaging in all three types of reflective practices that supported their development of adaptive expertise. For example, we found that participants most often engaged in *technical descriptions,* consistent with how this reflective practice is considered a lower-level reflection (Muir & Beswick, 2007). Additionally, in the moments of *technical descriptions,* we found teachers to most exhibit and develop *flexibility* as they demonstrated increasing awareness of students or adapted their practice in response to students' needs. Importantly, we found that design features of the Studio Day Cycle privileged specific types of reflective practices – such as viewing videos of classroom enactments encouraging *deliberate reflections*, because participants were oriented to specific critical incidents, and small group reflections encouraging *critical reflections* as teachers considered each other's perspectives and ideas. This is consistent with existing literature on the value of supporting teachers to engage in reflection-on-action (Manrique & Abchi, 2015) as well as the use of video in teachers' professional development (i.e., van Es & Sherin, 2010). The present work demonstrated that professional learning interventions can curate *catalysts of reflective practices* that can specifically support teachers' development of adaptive expertise of language-responsive mathematics instruction. With reflective practices a critical component of adaptive expertise development (Anthony et al., 2015), we suggest that Studio Day Cycles are flexible, adaptable models for interventions that can provide in-service mathematics valuable opportunities to develop their adaptive expertise of equity-based, language-responsive pedagogies for multilingual learners.

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IN-SERVICE TEACHERS' PERFORMANCE ON FRACTION OPERATIONS

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This study explored 236 in-service elementary and middle grades mathematics teachers' knowledge on fraction operations across the U.S. The research questions were as follows:

1. To what extent do teachers solve problems that address fraction operations correctly?
2. What relationships exist among teachers' choices of fraction operations?

For each of the four problems, teachers were asked to select one of the following six choices:
i) $\frac{2}{3}$ x $\frac{4}{5}$ ii) $\frac{2}{3} + \frac{4}{5}$ iii) $\frac{4}{5} - \frac{2}{3}$ IV) $\frac{2}{3} \div \frac{4}{5}$ V) $\frac{4}{5} \div \frac{2}{3}$ VI) None of these

Table 1: Fraction Operations Problems

Key Concept	Problem
Fraction Partitive Division	**1)** A 4/5 of a package of butter weighs 1 pound. How much does a 2/3 of a package of butter of that same kind weigh?
Fraction Subtraction	**2)** Emily has a rope that is 4/5 meters long. However, she needs a rope that is exactly 2/3 meters long for a project. What part of her rope should Emily use for her project?
Fraction Multiplication	**3)** If 2/3 of a gallon of yogurt weighs 1 kilogram, then how many gallons are 4/5 of a kilogram of yogurt?
Fraction Measurement Division	**4)** You have 4/5 cups of honey. A batch of brittle calls for 2/3 cups of honey. How many batches of brittle can you make?

Results & Discussion

One main result was that only about a third of the sample of teachers responded to the fraction partitive division, fraction multiplication, and fraction subtraction problems correctly. On the other hand, teachers performed better on the fraction measurement division problem with about two-thirds of the sample's correct answer. Moreover, there was a statistically significant relationship between teachers' responses to each fraction operation and the remaining three operations (e.g., for fraction partitive division and multiplication; $\chi^2(25) = 108.6$, $p = .00$).

Table 2: Summary of the Results

	2/3 x 4/5	2/3 + 4/5	4/5 – 2/3	2/3 ÷ 4/5	4/5 ÷ 2/3	None
Problem 1 [2/3 ÷ 4/5]	66	3	9	75	25	58
Problem 2 [4/5 – 2/3]	38	3	89	28	16	62
Problem 3 [2/3 x 4/5]	91	1	6	48	41	49
Problem 4 [4/5 ÷ 2/3]	17	5	24	14	158	18

As consistent with prior research (e.g., Izsák et al., 2019; Ma, 2010), this study reveals that teachers had limited understanding of fraction operations. Contrary to the most findings in past research (e.g., Timmerman, 2014), teachers in the present study performed much better on the fraction measurement division problem than the fraction partitive division problem.

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[2]CULTIVATING MATHEMATICS TEACHER CURIOSITY: AN EXPLORATORY ANALYSIS

CULTIVANDO CURIOSIDAD DOCENTE: UN ANÁLISIS EXPLORATORIO

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This paper explores the notion of teacher curiosity and connects it to the pedagogical work of K-5 mathematical inquiry. We explore teacher interviews from professional development (PD) designed to foster curiosity among elementary mathematics teachers to consider how teachers reflected on curiosity and the objects of their curiosity. Findings show that within the PD context, teachers: (a) expressed the importance of cultivating curiosity in their students, their colleagues, and themselves, and (b) expressed curiosity about: (1) children's mathematical thinking, (2) their own math instructional practices, and (3) their own affective state. We end with directions for future work.

Este artículo explora la noción de curiosidad docente y la conecta con el trabajo pedagógico de la investigación matemática K-5. Exploramos entrevistas con maestras participando en un desarrollo profesional (PD) diseñadas para fomentar la curiosidad para considerar cómo reflexionaron sobre la curiosidad y los objetos de su curiosidad. Encontramos que dentro del contexto del PD, los maestros: (a) expresaron la importancia de cultivar la curiosidad en sus estudiantes, sus colegas y en ellos mismos, y (b) expresaron curiosidad sobre: (1) el pensamiento matemático de los niños, (2) sus prácticas de instrucción, y (3) su propio estado afectivo. Terminamos con indicaciones para trabajos futuros.

Keywords: Professional development; Affect, emotion, beliefs, and attitudes; Elementary school education

A stance of curiosity is often assumed in teachers who center student thinking and inquiry in the mathematics classroom (e.g., Engle, 2013), though the field knows little about the construct of teacher curiosity or its cultivation. In the field of psychology, research has variously characterized curiosity as (a) a trait, with individuals being either more or less curious by nature, or (b) a state, a transient experience stimulated by the environment over which the individual has little control (Litman, 2005; Schmitt & Lahroodi, 2008). In either case, curiosity is viewed as a condition that fuels inquiry and, ultimately, learning. However, from a sociocultural perspective (Lave & Wenger, 1991), who we are in interaction with one another and our environment shapes and is shaped by the social world, rather than being purely internal or individual. In our work focusing on teacher curiosity, we frame curiosity as neither purely an individual trait or a state,

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but rather as a deliberate stance teachers can learn and then take on in their classrooms. Further, we differentiate teacher curiosity from related practices such as teacher noticing (Van Es, 2011) and inquiry, framing curiosity as a *stance* or *orientation* toward activity, others, and/or self that can make such connected practices more likely to occur. As such, we conceptualize and investigate teacher curiosity as a *learnable professional stance*, one that can be *fostered in communities of practice* by sharing in collective inquiry into and curiosity about the work of teaching and learning.

While the construct of teacher curiosity, in particular, has been underexplored, research into developing teachers' capacity to center, elicit, probe, and design instruction based on student thinking has shown that the most skilled teachers approach student thinking with curiosity (Franke et al., 2001). Indeed, Franke and colleagues (2001) described curiosity about students' mathematical thinking as one of the characteristics of teachers who were able to continue to learn from their daily practice after the end of formal professional development. Building on this work, Ananthanarajan (2020) recently operationalized teacher curiosity about student thinking in the context of mathematics teachers as an instance "where teachers recognize something as unknown, unfamiliar, puzzling, uncertain, or new in the context of teaching and learning, and feel motivated to initiate inquiry into that instance" (p. 31). She described how developing and making space for teachers' curiosity offers them the agency to follow their interests and ask questions that motivate inquiry in their own teaching practice. This operationalization focuses solely on teachers' curiosity about student's mathematical thinking, which we seek to broaden by more openly exploring the possible objects of teachers' curiosity when engaged in mathematics professional development (PD).

This paper explores the notion of teacher curiosity within the context of a PD program designed to cultivate curiosity as a component of inquiry-based mathematics teaching. The program took up the view that curiosity could both be learned by teachers and could in turn support their professional learning. In this context, we draw on interviews with participating teachers to explore the following question: When reflecting on PD and coaching designed to foster elementary mathematics teacher curiosity, to what degree did teachers express curiosity and what were the objects of that curiosity?

Methods

Professional Development Context

The PD explored here took place within a research-practice partnership (RPP) between university-based researchers and a small elementary school district in Northern California that serves a culturally and linguistically marginalized community. The district's mathematics coach, in collaboration with our research team, established a math teacher cohort of five teachers, all of whom volunteered to participate in two years of PD during the 2020-2021 and 2021-2022 school years. In the fall of 2020, the mathematics coach established a second math teacher cohort of four teachers, all of whom volunteered to participate in one year of PD during the 2021-2022 school year. This district PD built on previous work in the RPP (Osuna & Munson, 2023).

The PD, which was co-designed by the mathematics coach and the research team, included two-hour after-school meetings, twice per month, totaling 10 sessions each year. The PD program sought to support teachers in student-centered, responsive teaching in mathematical inquiry, along with one-on-one coaching sessions with the coach. With the onset of the COVID-19 pandemic in March 2020 and an abrupt shift to remote instruction, the need to support teachers and students in this context of incredible uncertainty came to the fore. With these shifts, curiosity became a prominent focus throughout the PD and coaching, as both a lens for considering student mathematical thinking and reflecting inward toward their own experiences and feelings.

PD activities and coaching sessions revolved around cultivating teachers' curiosity about mathematics and about students' mathematical thinking, such as analyzing students' mathematical work. Opportunities to reflect on and discuss their own affective states as it pertained to teaching during the pandemic were also regularly folded in. Table 1 summarizes the areas of focus and offers examples of PD activities within each area.

Data Sources

Several data sources were collected throughout the PD. The present analysis focuses on interviews with teacher participants. 15 semi-structured interviews were conducted on Zoom with six of the nine participating teachers, ranging from one to four interviews per teacher. Complications from the COVID-19 pandemic, including limited teacher bandwidth, impacted the number of interviews the research team was able to collect. Interviews were 30-60 minutes in length. Table 2 shows the timeline and distribution of interviews across teachers. All names are pseudonyms.

Table 1: Overview of PD areas of focus

Area of Focus	Description and Examples of Activities
Cultivating an inward-focused curiosity	Sessions included activities, discussions and reflection questions that prompted teachers to turn their curiosity inwards. The first session began with sharing what brings one joy, followed by an activity where they wrote their mathography. All sessions ended with individual reflection time, where teachers were prompted to articulate their lingering curiosities about themselves and their students, parts of their math instruction they're most proud of, or applying a curiosity lens to their own mathographies.
Cultivating curiosity about children's mathematical thinking and experiences	Sessions included activities and discussions looking at student math work or watching videos of students doing math engaging a curiosity lens. One such discussion began with the prompt of asking teachers to identify what they had found fascinating about a piece of student work. This was followed by sessions analyzing student work with a curiosity lens then reflecting on the experience.
Doing mathematical activities	Most sessions included an inquiry-based, open-ended math activity, such as Notice and Wonder, Which one doesn't belong?, or the Paper Folding task. These activities often culminated with a debrief discussion about how to adapt the activity for their students.

Learning and practicing teacher talk moves to elicit student thinking	Sessions included exploration of teacher talk moves, in particular moves to elicit student thinking. This was followed by watching videos of two teachers eliciting student thinking and discussing what was noticed in the videos, ending with a reflection.
Discussing readings	Discussions were primarily based on the book, *Becoming the math teacher you wish you'd had* by Tracy Zager, which offers ideas and strategies for teaching student-centered mathematics.

Data Analysis

Teacher interviews (n=15) were transcribed for analysis and used to inductively develop a definition of curiosity in the context of our data. We operationalized curiosity as: "Expressing an orientation of openness, inquiry, seeking learning, wonder, perplexity, and marvel, discussing trying to cultivate such an orientation, or naming curiosity as a goal or experience; promoting curiosity among colleagues or students; reflecting on being the object of another person's curiosity; or expressing the desire to seek information." This definition was refined through discussion and then applied to the interview and reflection data. Using our operationalized definition, we identified all instances of curiosity across the interviews.

We then coded across all the instances of curiosity (n = 63) and inductively classified the objects of teachers' curiosity. This approach revealed three distinct objects of teachers' curiosity, as well as expressions of the importance of cultivating curiosity among others. These 4 codes were then applied across the dataset. We then returned to the coded data and explored themes within the two most frequent codes: expressing curiosity about one's instructional practice and expressing curiosity about children's mathematical thinking. Below, we describe the overall findings and zoom into the themes within these two most frequent expressions of curiosity.

Table 2: Timeline and distribution of interviews across teachers

	Math Leads	Interview #1: Teacher Story	Interview #2: End of Year 1	Interview #3: Start of Year 2	Interview #4: End of Year 2
First Cohort	Alejandra	x	x	x	x
	Adeline	x	x		
	Ashley				x
	Roya	x	x	x	x
	Eujin				
Second	Nina	x	x	Teachers in the second cohort	

Cohort	Dolores	x	x	participated in the PD for one year.
	Soyoung			
	Nadia			

Findings

Across 15 interviews with six teachers, we recorded 63 instances of curiosity across all teachers as it related to their participation in the PD. As they reflected upon their learnings from PD and coaching, teachers: (a) expressed the importance of cultivating curiosity in their students, their colleagues, and themselves; and (b) expressed curiosity about various domains, in particular: (1) children's mathematical thinking, (2) their own math instructional practices, and (3) their own affective state.

Teachers expressed the importance of cultivating curiosity in others

Teachers expressed the importance of cultivating curiosity in others, in particular their own students' mathematical curiosity, their colleagues' curiosity about student thinking, and their own curiosity. For one, teachers expressed the importance of cultivating students' own curiosity about mathematics and each other's thinking. For example, Alejandra reflected on the importance of inspiring students to be curious about their own mathematical processes and thinking and being excited about mathematical possibilities:

> And what I have learned from this math cohort is that the most important thing is not that the students get the right answer, but the process behind… getting to the answer, even if their answer is right or wrong, is not as important as understanding their process and making them feel excited about math, helping them get excited and curious about math. (Alejandra)

Another teacher discussed connections between student curiosity and learning, noting how an orientation of curiosity supports sense-making and building mathematical understanding:

> 'Cause that's what their brains are trying to, like, make connections about, you know, that's what they're trying to understand. So like, uh, letting them guide that make sense 'cause then they're- they're already curious about it. They're already trying to understand it. So we just need to help kind of guide them and give them more tools and resources to build that understanding. (Ashley)

Teachers also expressed value in cultivating curiosity among their colleagues and themselves. For example, one teacher noticed the ways in which the PD coach fostered curiosity among teachers and remarked on its importance as "a really good approach" that ought to continue with more teachers in continuing years:

> [Coach], really, she, even when she comes to do like a math talk at a staff meeting. She's really just like, 'Okay, I'm showing you this and, hopefully, you are curious. If you are, you contact me, and I can show you more.' You know, like, she's trying to bring that curiosity out of the teachers, so that more people are interested in this work. So thinking about our site, I think that's a really good approach, maybe next year, just continue this work." (Adeline)

Another teacher remarked on the importance of the emerging curiosity among the fourth grade teaching team, who were not part of the PD, but were showing signs of being "very fascinated" by students, noting that this spread of curiosity was the most noteworthy commentary:

> Fourth grade [teaching team] seems to be very interested in how we're doing math because our kids are going into fourth grade and they're seeing like the results of inquiry, and they're like very fascinated by the strategies that they're seeing. So, I would say the biggest like commentary around it has been between fourth grade third grade and like within third grade. (Roya)

Teachers expressed curiosity about various domains

Teachers expressed curiosity about various domains, in particular: (1) children's mathematical thinking, (2) their own math instructional practices, and (3) their own affective state. While it was the least frequent code, teachers expressed curiosity about their affective state. Teachers at times expressed how curiosity generated feelings of excitement, vulnerability, and even hopefulness. Additionally, while the PD was explicitly structured to cultivate teachers' curiosity about students' mathematical thinking, teachers' curiosity about their own math instructional practices was the most frequent code across the data set. Below we focus on these two types of expressions of curiosity, in particular, illustrating how this lens of cultivating teachers' curiosity about students' mathematical thinking seemed to also promote curiosity about their own practice.

Teachers expressed curiosity about children's mathematical thinking. There were 16 instances of curiosity about students' mathematical thinking across the data set, and every teacher interviewed expressed this form of curiosity. As prioritized in the learning goals of professional development, teachers reflected on the importance and practice of being curious about children's mathematical thinking, shifting away from an evaluative stance. For example, Adeline described working to understand, rather than judge, her students' work:

> The biggest thing is really just be curious about student work and be curious about their thinking before you judge them—[laughs] I used the word judge—before you make assumptions, that's the biggest thing I think I learned from this work is you try to understand them. (Adeline)

Here Adeline reflected on the importance of trying to understand, rather than make assumptions about, her students' thinking. In addition to describing this shift from an evaluative stance to a curious one, teachers discussed what it looked like for them to enact this curiosity lens in their classrooms. Adeline elaborated in a later interview on how she worked to put this curiosity lens into practice when conferring with her students about their thinking:

> [In the past] I was kind of listening for their mistakes and then try to like help them to fix it [laughs]. So it's like kind of a different perspective…like, are you making sense of the problem? Does this student have a plan? And then, does this student know like -- like 'the plan' meaning like what kind of strategy to use--and then, does this student know how to pick the right tool? (Adeline)

In this quote, Adeline contrasted her past tendency of listening for mistakes in order to "fix them" with her curiosity lens of trying to understand students' sense-making. Similarly, Alejandra connected curiosity about what students are saying to staying present in conversations with students in-the-moment, stating:

> It is important that, first we need to have some questions in mind, some basic questions, and then from the students' responses build upon that and then ask them more questions that are just based in the present moment. We need to be 100% present, listening to our students and then being able to question them more, no? So that's how curiosity looks like to me, lots of questioning (Alejandra)

In these quotes, Alejandra and Adeline discussed the ways in which being curious when engaging with students supported them to be responsive to students' mathematical thinking. Although teachers acknowledged the challenges of staying curious about students' thinking in their classrooms, they also marveled at the joy that it brought them, as Ashley reflected:

> I wanted to enjoy math time more and like have more like fun with it and have it be more interesting and like not just like worksheets and be, uh, uh, like really curious and like thoughtful about what students were doing and like really like allow myself to-to be like that curious (Ashley)

Teachers expressed curiosity about their own mathematical instructional practices. There were 32 instances of curiosity about teachers' own mathematical instructional practices across the data set, and every teacher interviewed expressed this form of curiosity three or more times. In particular, teachers expressed curiosity about (a) a different way of teaching, (b) specific pedagogical approaches, and (c) their own professional growth journey.

Across interviews, teachers expressed curiosity about teaching mathematics in a different way than they currently were or had experienced as a student. For example, Ashley noticed the way she was teaching math was not working for her students and wanted to learn about how to shift her math instruction to better support her students. She stated:

> Um, so I think at that point I was looking for that because I could see students struggling with math and I was like, okay, this is not-- Like I know this is not the best way to teach them. So I was looking for other-other ways to teach it. (Ashley)

In addition to this general curiosity about changing math instructional practices, teachers also expressed curiosity about specific pedagogical approaches to teaching mathematics, including the role of students within those practices. In describing a learning experience in coaching, Alejandra talked about how she had specific questions about and then received support in implementing a conferring practice, explaining:

> If I had questions about like," Oh, am I doing the launch? Okay. Like, how am I doing conferring or how should I do?" So she helped me with conferring, with the launch, with the solving math problems and with the debrief. (Alejandra)

Roya also reflected on the particular pedagogical approach of re-engagement lessons and her curiosity and excitement about trying it with her students:

> I'm excited, I think I'm, I want to learn more about how to use student work more effectively in my classroom. Like one thing that I learned through the PDs last year were to try reengagement lessons and that was like, 'Wow I need to get on that level, but like I'm not

> there yet.' So I think learning how to use student work for the entire class to reengage them in like some big ideas. (Roya)

Finally, teachers described across interviews how their curiosity about the processes for their own learning or improving practice drove their learning in professional development and their work in coaching. Adeline gave multiple examples of how her curiosity lead the focus of her learning with her coach, and how powerful those learning and coaching experiences were. She elaborated:

> I could always talk to [coach] this year, just like you know, like, 'I'm really confused about this hundreds chart, because it's kind of, you know, like, there's 15 and then there's 25, kids are, you know like, could be a little confusing.' Like I always have questions about that, and then I just go to [coach], and we talk about it, and we sometimes we, just you know, we search, you know, we do some research together and just like a lot of, it's really helpful. That's one and second is kind of, I think it was helpful like for her to come and model for me like how to do certain routines and observe me. She was observing my, you know, conferring sessions and then we'll like—I'll confer with one student and then we go back to the main room, talk about it, and then go to the next group. We only tried that for two times, but that was very powerful too. We also tried, you know, recording lessons, so like I'll just record my lesson and then we kind of really think about like, what did I say and what did the students say? And then I realized, 'Oh I talk a lot' [laughs]. Just kind of, you know, that are, just really based on like my own practice and reflection was really helpful, I think. (Adeline)

Conclusion

In this analysis of a PD designed to cultivate teacher curiosity, teachers expressed curiosity about their students, their practice, and their own affective state while teaching student-centered mathematical inquiry, as well as expressed value in cultivating curiosity in others. While teachers described the act of being curious as complex and at times challenging, being curious and cultivating curiosity among students was also described in relation to feelings of joy and excitement. Curiosity also seemed to motivate teachers to get to learn more about a student and their mathematical thinking and wonderings, as well as to dig into their own teaching and its evolution toward more student-centered pedagogies. Our findings build on Anantharajan (2020), whose work focused specifically on teacher curiosity of students' mathematical thinking, to show that cultivating such curiosity (a) can occur collectively in a teacher professional learning community and (b) can lead to other forms of curiosity, including curiosity about instructional practice and even one's own affective experience teaching.

All in all, curiosity seems to hold strong potential as a lever toward ambitious, reflective, and even joyful instruction. A key implication is that efforts to stimulate curiosity may be an essential feature of high quality PD not yet referenced in commonly cited frameworks, such as Desimone (2009). New work is needed to explore the pedagogical, humanizing, and even wellbeing possibilities of cultivating curiosity among teachers and, in turn, students. Our exploratory work is limited by the exclusive focus on teacher interviews, rather than curiosity in-situ or other contexts. We see this small study as a jumping off point for further theorizing.

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Factors Contributing to Instructional Shifts at the College Level

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Despite copious evidence of the effectiveness of inquiry-oriented and student-centered instructional practices, many college instructors do not implement these instructional strategies. We report on a three-year project aimed at shifting instruction in College Algebra at one institution. This project established a professional learning community (PLC) of instructors around an incremental instructional improvement framework to guarantee instructor buy-in and increase the practicality of the development materials for use in the classroom. Preliminary results indicate that structural factors such as course coordination, dedicated PLC time, a lesson-study-like framework for improving course curricula and materials, and video clubs contributed to changes in both instructors' thinking and practice of inquiry-oriented teaching.

Keywords: Professional Development, Undergraduate Education, Instructional Activities and Practices

Inquiry-oriented mathematics instruction can lead to increased student learning and conceptual understandings (Freeman et al., 2014; Deslauriers et al., 2011; Kogan & Laursen, 2014). Additionally, there is evidence that students who report experiencing more student-centered techniques in their classes are less likely to switch out of a STEM degree (Ellis et al., 2014), and some studies suggesting that the benefits for underrepresented students in STEM are even higher (Kogan & Laursen, 2014). However, didactic lecture remains the most common form of instruction in STEM courses across the United States (Stains et al., 2018). Reasons for this include instructors' lack of personal experience with student-centered instructional practices (e.g., Andrews et al., 2015), fear of losing control of their classroom (e.g., Hayward et al., 2015), and particular beliefs about teaching and learning (e.g., Aragón et al., 2018). Further, mathematics instructors at the college level may have had few opportunities to participate in focused professional development around teaching. As such, it is challenging for many instructors to make lasting instructional changes that focus teaching on student-centered practices and leverage inquiry-based materials. Although instructional change can be difficult to catalyze, professional development through professional learning communities (PLCs) can be one way to support instructors through this process (Hayward et al., 2015; Lee & Lee, 2018; Tam, 2015). Specifically, PLCs create opportunities for instructors to reflect on and refine their practice and to generate new knowledge (Harris & Jones, 2010). This can result in transformational change when the PLCs possess high levels of professional capital, which refers to "the capacity to transform existing resources and constraints into opportunities through collective action" (Lee & Lee, 2018, p. 466), as well as when they focus on student learning (Bolam et al., 2005).

Description of Professional Learning Community

The goal of the federally-funded project described in this report was to empower College Algebra instructors at a large, Hispanic-serving southwestern university to improve instruction. The Practicality Ethic framework proposed by Doyle and Ponder (1977) helped us (the investigators) to structure the project, as it was originally designed to describe factors teachers consider when deciding whether or not innovative curricula was practical or deemed realistic for implementation in the context of an actual classroom. These factors are: 1) congruence: how compatible the change is with the instructor's classroom, setting, and instructional goals, 2) cost: if the potential benefits (e.g., student outcomes, student attitudes) that outweigh the effort and other costs of implementation, and 3) instrumentality: if the changes consist of clearly articulated procedures for ease of implementation in the instructor's classroom. Informed by the Practicality Ethic, we prioritized instructors' agency in choosing the content and and form of their curriculum (re)development project. In order to encourage instructors to think carefully in advance of their facilitation of lessons about students' opportunities to actively participate in the class, we leveraged the Continuous Improvement (CI) cycle (Berk & Hiebert, 2009), the incremental lesson improvement strategy informed by lesson study.

Continuous Improvement Framework

For each of the five "active" semesters of the project, during the time protected for PLC meetings, instructors chose specific focus lessons from the curriculum and implemented the CI cycle: (1) design a task that targets a particular student misconception or deepens understanding of a particular mathematical idea; (2) develop hypotheses about anticipated student responses; (3) collect data and analyze in the form of student work and classroom recordings; and (4) revise the task for use in subsequent iterations of the course. Our choice to use CI to guide course improvement was to seed gradual transformation made with smaller changes over time for sustained instructional improvement (Hiebert & Stigler, 2004), while also leveraging the knowledge, experience and priorities of instructors to guide these changes. We collected data in the form of instructor interviews, video-recorded class observations, recordings of PLC meetings (including each step of the CI cycle and video club meetings), and participant lesson plans.

Structural Factors Contributing to Instructional Change

We posit that the following structural factors contributed to meaningful instructional change in our context:

1. Course coordination and vertical alignment of curriculum: two years before the start of this project, the department embarked on a concerted effort to coordinate large multi-section courses. Instructors teaching the same course were strongly encouraged to meet regularly, align assessment across sections, and develop a list of student learning outcomes (SLOs) for their course. Course coordinators were similarly encouraged to meet with each other and discuss the progression of SLOs across subsequent courses.
2. Dedicated PLC time: this project provided each College Algebra instructor (all of whom were instructional faculty with 100% teaching loads) with a course release in order to participate in the PLC. Instead of teaching four courses each semester, participating instructors only taught three. Additionally, after the first semester of the project, which was utilized as an establishing and planning semester, all instructors of College Algebra participated in the PLC.

3. Continuous Improvement framework: Using the CI framework allowed us as facilitators to provide a structure for discussions about student thinking and developing active learning materials while still allowing the instructors' own priorities and ideas to guide the instructional improvement process.
4. Video club: embedded into the PLC and as part of the CI cycle, instructors participated in a video club where they observed each other's facilitation of the focus lessons for the semester. This gave instructors the opportunity to open up their practice to each other in a safe and structured way. As advocated by Berk and Hiebert (2009), we encouraged instructors during these video clubs to focus on *instruction* in their observations, not on the personal styles or quirks of the *instructors*.

We report below on preliminary findings that support the importance of each of these factors in catalyzing instructional change.

Preliminary Results

As a result, we have multiple sources of evidence of instructional change on a number of levels. We summarize the results of three preliminary studies below.

Study 1

We investigated the instructors' perceived barriers and drivers for implementing evidence-based instructional practices, drawing on the work of Shadle, et al. (2019). A thematic analysis of the interviews with instructors revealed that most of the barriers to implementing evidence based instructional practices (EBIPs) identified by Shadle et al. (2019) did not resonate with this group of instructors. Specifically, we found that certain departmental policies (e.g., course coordination) mitigated some of the barriers and that the experience of teaching online during the COVID-19 pandemic resulted in the creation of video resources. These resources alleviated the time pressure to "cover" all the required material, making instructors more open to trying out EBIPs. Additionally, we found that the PLC central to this project served as a driver, enabling instructors to implement more EBIPs. For example, the PLC included opportunities for instructors to observe their peers, which provided some accountability and helped instructors to identify and (continue to) implement more innovative strategies in their teaching (Gehrtz et al., 2022).

Study 2.

Next, we looked at instructors' attention during the PLC meetings when each instructor showed video-clips from an observation video of another participant teaching one of the lessons collaboratively developed as part of this project (i.e., video club meetings). Informed by the work of Kelley & Johnson (2022), we used the Instructional Triangle (Cohen & Ball, 2001) as an analytic tool to characterize each instructors' foci during the discussion. Preliminary analysis suggests that each instructor had a component of the instructional triangle that they tended to focus on for the initial observation, but then after participating in multiple video-club meetings and seeing what other instructors focused on in their presentations, the discussions tended to shift to focus more on teacher moves. We also noticed a shift from focusing on explanations of the content at the beginning of the semester to showcasing more clips that highlighted teacher actions to engage and support student learning at the end of the semester (Jones et al., 2023).

Study 3.

Then, we looked at what was happening in each instructor's class by using the Classroom Observation Protocol for Undergraduate STEM (COPUS; Smith et al., 2013) to analyze class video data, documenting what instructors were doing and what students were doing throughout the class period. We then grouped related codes and created radar plots with the percentage of class time spent with: Students Talking, Students Working, Students Receiving, Instructor Guiding, Instructor Presenting, and Instructor Guiding (Smith et al., 2014). This analysis revealed distinct changes within and across semesters with respect to how class time was spent. Specifically, we saw instructors spend less time lecturing while students were listening and taking notes. We also saw more opportunities in class for students to work individually or in groups and for students to talk in class. Additionally, this analysis allowed us to triangulate what was happening in class to what instructors were describing during interviews about their efforts to implement changes to their teaching by incorporating more evidence-based instructional practices (Gehrtz et al., 2024).

Discussion

Put together, these three studies show clear evidence of instructional change, at least during the period of the project. Moreover, interviews with instructors after they had been reassigned to other mathematics courses and were no longer participating in the PLC suggest that they wish to implement some of the lessons learned in the PLC with other courses. However, the dedicated collaboration time that was protected by the project is not available to instructional teams in other courses and instructor buy-in for implementing EBIPs varies across the department, making it difficult for the former College Algebra instructors to overcome the systemic barriers for inquiry-oriented learning that the structure of the PLC and the College Algebra course temporarily eradicated.

However, instructors in the PLC are themselves thinking about the sustainability of the instructional improvements. At the end of the project, the College Algebra curriculum will have been completely revamped based on instructors' interpretation of the evidence-based instructional practices. Additionally, College Algebra course coordinators have started to use the video club recordings of classes in the pre-course orientation meeting for new instructors. The current PLC members are also working on an instructor's guide to the course that can preserve their accumulated knowledge and disseminate it to subsequent course team members. These actions speak to the dedication of these instructors to preserve and sustain the work of the project.

Further Questions

We are currently pursuing a number of other research questions and logistical considerations based on this project:

1. What is the effect (if any) of these instructional shifts on student outcomes?
2. Has participation in this project shifted instructors' beliefs about or use of EBIPs? Is it possible to track these shifts by referencing their interviews, the PLC meeting recordings, or other sources of data that we have collected?
3. How do we preserve the progress made during the project and sustain the work going forward, given the importance of the dedicated PLC time in shifting instructional practices?

Although the evidence of structural factors that influence instructional change may be of use to other investigators and facilitators of professional development for college mathematics faculty, the question of the lasting impact of this project remains open.

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DEMONSTRATING ADAPTIVE EXPERTISE ACROSS A COLLECT AND DISPLAY STUDIO DAY CYCLE

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We examined three elementary teachers' participation in a mathematics language routine (MLR) Collect and Display Studio Day, a modification of Japanese Lesson Study. Our study was guided by a theoretical framework of adaptive expertise. We collected video data from a Studio Day to understand the following: How did elementary teachers demonstrate adaptive expertise of the Collect and Display MLR within their participation during a Studio Day? We analyzed the data using three dimensions of adaptive expertise: flexibility, deeper level of understanding, and deliberative practice. We found that teachers demonstrated dimensions of adaptive expertise during the Studio Day Cycle and we share implications for practice and research.

Keywords: professional development; equity, inclusion, and diversity; elementary school education

Few mathematics teachers have had professional learning experiences that bridge both multilingual learners and mathematics (Ballantyne et al., 2008). This paper focuses on a professional learning experience that use Studio Days (Von Esch & Kavanagh, 2018), a modified version of lesson study, which allowed teachers to focus on language and mathematics simultaneously through the use of mathematics language routines (MLRs; Zwiers et al., 2017). These routines are meant to engage multilingual learners in rich instruction and curriculum; they allow students to access mathematics texts and to communicate mathematical reasoning. MLRs are scaffolded routines intended to lead to students' independent participation in the mathematics classroom through supporting sense-making, optimizing output, cultivating conversation, and maximizing linguistic and cognitive meta-awareness (Zwiers et al., 2017). For example, students might engage in the MLR Collect and Display, the focus of this paper, in which teachers capture students' oral words, ideas, phrases into a stable reference. The purpose of the routine is to stabilize students' language in order to use their output as a reference for developing their mathematical language (Zwiers et al., 2017). While MLRs are becoming more ubiquitous in curricular materials (i.e., Illustrative Mathematics, 2019), the field's understanding of how teachers make sense of these materials and use them with their students, such as within the Studio Day professional learning experience, is still unclear. As teachers make instructional decisions about how to teach MLRs, we should understand how they are making sense of this work. This paper seeks to describe how teachers made sense of how to work with a MLR, including how they made sense of how to work with the content they were teaching their students and how they made sense of how to attend to language they were working with within their classroom to engage in content simultaneously while working on that MLR. Our research question was: How did elementary teachers demonstrate adaptive expertise of the Collect and Display MLR during their participation within a Studio Day?

Adaptive Expertise

Our theoretical framework uses the construct adaptive expertise (Hatano & Inagaki, 1984). We defined adaptive expertise as the ability to implement MLRs, with the flexibility to navigate a localized context without sacrificing ambition or complexity (Hatano & Inagaki, 1984). Hatano and Inagaki found that adaptive experts differed from routine experts in that that were: able to adapt to a desired outcome; able to demonstrate the usefulness of their skill, displaying the context in which the skill was used; and able to find value by the group members. Others have continued to build on this fundamental theory regarding adaptive expertise, in education and beyond. Von Esch and Kavanagh (2018) noted that adaptive expertise is being able to draw on and retrieve relevant existing knowledge. For example, citing Bransford et al. (2007), Von Esch and Kavanagh explained that "adaptive experts organize their knowledge within a conceptual framework of key concepts that guide that their thinking and facilitate use of that knowledge" (p. 241). In this way, adaptive experts are able to flexibly use knowledge that they have developed around a framework they have developed. Additionally, Schwartz noted that people who are adaptive experts are able to rearrange their environments and their thinking, allowing them to acquire and access skills and knowledge. Notably, Schwartz et al. (2005) also explained the importance of being able to move away from efficiency (i.e., learning a single routine, to learn multiple routines and be flexible with them—this takes time and effort).

Teachers with adaptive expertise are able to use their knowledge of their students as they adapt their practices and curriculum (Beltramo, 2017). Teachers who possess adaptive expertise are able to scaffold students' mathematical development through the use of effective instruction and appropriate assessment tools, based on the content they are learning and the context of the students and school environment (Heinze et al., 2009). In this study, we are particularly interested in teachers' development of adaptive expertise as related to MLRs, so as to better attend to multilingual learners. To operationalize our definition of adaptive expertise, we drew from Yoon et al. (2019), who identified three dimensions of adaptive expertise: flexibility, deeper understanding, and deliberate practice. These categories of adaptive expertise will be used for understanding how elementary teachers demonstrated adaptive expertise of the Collect and Display MLR within their participation during a Studio Day.

Method

Our study was situated in a school district in California that included a substantial number of multilingual learners. Teachers from a single elementary school, Alhambra Elementary School (pseudonyms are used for all proper names), participated in a single-year professional learning program organized around multilingual learner mathematics Studio Days (Von Esch & Kavanagh, 2018)—we focus on a single cycle for this paper, organized around the MLR Collect and Display, described above, examining just the Studio Day aspect of this cycle. A third grade teacher and two fifth grade teachers participated in the study, all self-identified as female, and two identified as white and one as Hispanic. This paper is a descriptive case study (Merriam, 1998) of one Studio Day, and we examine the two teachers who conducted Studio Day lessons: two fifth grade teachers: Ms. Moreno and Ms. Butler.

Data Collection and Analysis

The larger project collected multiple sources of data; because of the brevity of this report, we focused on videotaped meetings with teachers, as well as videotaped classroom enactments of the Studio Day. We transcribed these videos using Otter.ai and then cleaned the transcriptions. To analyze our data, we first coded data for instances of MLRs. We then coded these instances of MLRs using a priori codes of Yoon et al.'s (2019) three dimensions of adaptive expertise: flexibility, deeper understanding, and deliberate practice to code our multiple data sources. We then examined each occurrence within the Studio Day, as well as each adaptive expertise dimension to make sense of how elementary teachers demonstrated adaptive expertise of the Collect and Display MLR within their participation during a Studio Day.

Findings

We share key findings of the dimensions of adaptive expertise from the Studio Day. Our Studio Day model is organized around a common MLR. On this particular day, the teachers also planned to teach the same lesson. Following the first lesson, the second teacher, Ms. Butler, changed her lesson based on the observations of Ms. Moreno's lesson. We found that the teachers' lessons tended to include more instances of flexibility and deeper level of understanding. Opportunities to reflect on improving practice were present in the debriefs following the lesson included more instances of deliberate practice, which, we do not describe in this paper.

Lesson 1 – Ms. Moreno

Ms. Moreno's enactment in the classroom of her Collect and Display MLR included evidence of the adaptive expertise dimensions of (a) flexibility because of her awareness to students, particularly multilingual learners, and her attending to students' needs in real-time; as well as (b) deeper level of understanding because of her bringing in variations to the MLR. Ms. Moreno's flexibility occurred as she engaged with the MLR and noted aloud to students what she was doing, such as, "So, I heard a couple of you say that you were multiplying," she was aware of her students through her revoicing of students' ideas (Moschkovich, 1999). With regards to a deeper level of understanding, Ms. Moreno explained to the students regarding the Collect and Display, "Remember, this was just a little refresher." In this way, Ms. Moreno made the MLR a review for students and decided how she would implement the MLR with her students, bringing in a variation for how it would be used in her class. This illustrated a variation from how the MLR was originally shared in the Pre-Studio Day.

Lesson 2 – Ms. Butler

Ms. Butler similarly demonstrated the two dimensions Ms. Moreno exhibited in her lesson: (a) flexibility and (b) deeper level of understanding during her enactment of Collect and Display. Ms. Moreno and Ms. Butler had planned their lessons together, as noted above, and had originally planned to teach the same lesson. Additionally, because Ms. Butler made changes in the moment, her practice demonstrated (c) deliberate practice. with instances of revamping her practice in the moment. Students had difficulties with some of the unit fraction concepts in Ms. Moreno's class, which were meant to be a review for students (the content had been taught earlier in the year). Ms. Butler's changes illustrated her deeper level of understanding of the MLR because was able to assimilate information from an outside source (her observation of Ms. Moreno's class) and make connections that built a deeper level of understanding (change the lesson to better meet the needs of her students). Ms. Butler, instead of teaching the second lesson in the unit, as Ms. Moreno had, taught the first lesson in the unit as a refresher for students.

Ms. Butler's attention to Ms. Moreno's lesson also changed the mathematical terms that Ms. Butler highlighted in her Collect and Display. For example, within her Studio Day lesson, Ms. Butler shared the following with her students:

> I was noticing some different words that you were saying. And, we were talking about the area of these fractions…But, I'm going to write on there the formula for area, because that's really important…We were also talking about shaded. I heard that word a lot.

Ms. Butler made sense of Ms. Moreno's lesson and adapted her Collect and Display and highlighted being able to draw on applicable knowledge during the use of the MLR (Von Esch & Kavanagh, 2018). She also demonstrated flexibility in this instance through valuing students' contributions, noticing what they said and writing those ideas up for other students to see that they were valued.

In another instance, Ms. Butler shared her flexibility with her students, explaining,

> As you guys were working, I was listening to what you were talking about, and I was writing them down. And, so now, we have this ongoing, and this is going to be an ongoing chart of vocabulary that we're going to use as we continue learning about fractions.

She illustrated that she was listening and had an awareness of her students while they were working, which demonstrated her flexibility during the MLR. Her flexibility was illustrated through noting that she was "listing to what you were talking about."

Discussion

We found that elementary teachers who participated in a Collect and Display MLR Studio Day demonstrated two of Yoon et al.'s (2019) dimensions of adaptive expertise—flexibility and deeper level of understanding. The Studio Day provided space for teachers to demonstrate these dimensions of adaptive expertise within the context of their classrooms. For example, Ms. Moreno and Ms. Butler were able to demonstrate flexibility and deeper level of understanding while highlighting particular terms on their Collect and Display MLR. These MLRs provided a context through which teachers could attend to the needs of their multilingual learners in real-time, such as by revoicing their students' ideas and by valuing their students' contributions. Working with other teachers allowed the teachers to consider these future pedagogical practices, highlighting the value of working in the Studio Day space. For instance, Ms. Butler was able to make changes to her practice in real-time (deliberate practice). These examples of adaptive expertise highlight the value of the Studio Day as a supportive environment for teachers to demonstrate adaptive expertise of Collect and Display and to attend to and consider the needs of their multilingual learners.

Implications

This study provides both theoretical and practical implications to our understanding of adaptive expertise, MLRs, and Studio Days. We provided an application Yoon et al. (2019) to MLRs within a Studio Day context, bringing these three constructs together to help add to the limited research on multilingual learners, mathematics education, and professional learning research (de Araujo et al., 2018) while also helping the field understand how teachers make sense of MLRs, through the use of a frame of adaptive expertise. Research on MLRs is still quite scant even if they are becoming more visible in practice, and this research helps to add to the field's understanding of their use. Practically, this research provides information about MLRs in practice in elementary schools. Currently, research (i.e., Zahner, 2021) has focused on the use of this pedagogy at the secondary level. This research provides a practical image of elementary teachers using MLRs, especially as they become ubiquitous in curricular materials. Additionally, the Studio Day Cycle becomes a potential space for developing the adaptive expertise of MLRs, as seen with Collect and Display and other MLRs, with this study providing a practical image for doing so.

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USING GALLERY WALKS FOR COMPARE AND CONNECT: DEVELOPING ADAPTIVE EXPERTISE OF MATHEMATICS LANGUAGE ROUTINES

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We examined teachers' development of adaptive expertise of mathematics language routines (MLRs) as they engaged in Studio Day professional learning focused on the MLR Compare and Connect. We collected video data from pre- and post-Studio Day meetings, as well as debriefs and their lesson enactments. We analyzed the data using three dimensions of adaptive expertise: flexibility, deeper level of understanding, and deliberate practice. We share a case study of a teacher exhibiting dimensions of adaptive expertise during the Studio Day Cycle through the use of a gallery walk. The teacher's enactment of the MLR Compare and Connect provides an image of a teacher's adaptive expertise of this MLR and helps us understand these MLRs and how teachers use and make sense of them in their instruction.

Keywords: Professional development; equity, inclusion, and diversity

This paper focuses on a professional learning experience for secondary mathematics teachers that used Studio Days model of professional learning (Von Esch & Kavanagh, 2018), which is a modified version of Japanese Lesson Study. Our Studio Day experience for teachers was focused on language and mathematics simultaneously through the use of mathematics language routines (MLRs; Zwiers et al., 2017), a unique experience for most mathematics teachers, as few mathematics teachers have had professional learning experiences that bridge both multilingual learners and mathematics teaching (Ballantyne et al., 2008). MLRs are scaffolded routines intended to lead to students' independent participation in the mathematics classroom through supporting sense-making, optimizing output, cultivating conversation, and maximizing linguistic and cognitive meta-awareness (Zwiers et al., 2017). In this Studio Day Cycle, teachers learned about the MLR Compare and Connect, which engages students in comparing and contrasting different mathematical approaches through examining different mathematical representations, approaches, examples, or language (Zwiers et al., 2017). Students are meant to develop meta-cognitive and meta-linguistic through their conversations with peers (Zwiers et al., 2017). Teachers need more than a cursory understanding of these MLRs to help their students to use and make sense of these routines in their classrooms (i.e., more than reading directions of how to use them off a page). Additionally, as a field, we need to understand how teachers use and make sense of these routines. Therefore, we saw Studio Day professional learning experiences as a space for teachers to develop adaptive expertise with MLRs. We define adaptive expertise as the ability to implement MLRs with the flexibility to navigate a localized context without sacrificing ambition or complexity (Hatano & Inagaki, 1984). Our research question was: How did a teacher make sense of the MLR Compare and Connect during a Studio Day Cycle in ways that demonstrated their adaptive expertise?

Adaptive Expertise

Our theoretical framework uses the construct adaptive expertise (Hatano & Inagaki, 1984), as defined above. Adaptive experts differed from routine experts because they were able (a) to adapt to a desired outcome; (b) to demonstrate the usefulness of their skill; and (c) to find value in their work from their group members. Schwartz et al. (2005) explained the importance of being able to move away from efficiency. For example, it takes time and effort to move from learning a single routine to learning multiple routines and being flexible with them. Von Esch and Kavanagh (2018) also developed on Hatano and Inagaki's concept of expertise and noted that adaptive expertise is being able to draw on and retrieve relevant existing knowledge. For example, adaptive experts are able to flexibly use knowledge that they have developed around a framework they have developed.

Teachers who possess adaptive expertise are use their knowledge of their students as they adapt their practices, curriculum, and instruction (Beltramo, 2017), scaffolding students' mathematical development through the use of effective instruction and appropriate assessment tools (Heinze et al., 2009). In this study, we were particularly interested in teachers' development of adaptive expertise as related to MLRs, so as to better attend to multilingual learners. We operationalize our definition of adaptive expertise, drawing from Yoon et al. (2019), who identified three dimensions of adaptive expertise—flexibility: exhibits an awareness of students, particularly multilingual learners and context, as related to MLRs; deeper understanding: brings in variations related to the MLRs and considers affordances and constraints of the MLRs; and deliberate practice: demonstrates motivation, focus, and repeated effort to monitor their practice and devises and subsequently attempts improved implementation. These categories of adaptive expertise will be used for understanding how the secondary teachers demonstrated adaptive expertise of the Compare and Connect MLR within their participation during a Studio Day Cycle.

Method

Our study was situated in a school district on the West Coast that included a substantial number of multilingual learners. This paper focuses on the second of three Studio Day Cycles during the 2023-24 school year.

Context: Studio Days Enactment of Multilingual Learner Principles and MLRs

Each Studio Day Cycle involved three professional development meetings and targeted a single MLR, with this cycle attending to the MLR Compare and Connect, described in the introduction. During the pre-Studio Day, teachers learned about the MLR and prepared to implement a lesson that included this MLR. Teachers then enacted this lesson at their school during the Studio Day, with teachers observing each other's implementation. During the final day of the cycle, the post-Studio Day, teachers examined and analyzed student work and video clips from the implementation, shared challenges and successes, and considered implications for their future practice.

Participants

Four junior high school teachers from the three junior high schools in the district participated in the study. This paper is a descriptive case study (Merriam, 1998) of one of these teachers, Ms. Severn.

Data Collection and Data Analysis

The larger project collected multiple sources of data; because of the brevity of this report, we focused on videotaped pre-, post-, and Studio Day meetings with teachers, as well as videotaped classroom enactments. We transcribed these videos using Otter.ai and then cleaned the transcriptions. We examined the transcripts, and first coded for instances when there were instances of MLRs occurring within the data. Next, we coded these MLR instances for adaptive expertise related to the MLR within each aspect of the Studio Day Cycle transcripts. We used a priori codes of Yoon et al.'s (2019) three dimensions of adaptive expertise: flexibility, deeper understanding, and deliberate practice to code our multiple data sources. We then examined each occurrence within a single Studio Day component (i.e., only pre-Studio Day), as well as examined each adaptive expertise dimension (i.e., only flexibility) to make sense of how participants demonstrated adaptive expertise of the Compare and Connect MLR within their participation during a Studio Day Cycle. We looked for themes within the Studio Day components and the adaptive expertise dimensions and share these themes within our findings.

Findings

We found that one teacher, Ms. Severn, used a gallery walk (e.g., sharing student work) to make sense of and enact the Compare and Connect MLR with their students, thereby making the MLR their own and exhibiting their adaptive expertise of the MLR. We share key dimensions of their adaptive expertise from aspects of the Studio Day Cycle to illustrate the teachers' process for this work.

Pre-Studio Day

Part of the initial work during the pre-Studio Day is to provide teachers an overview of the MLR. During this initial overview, the first author shared that there were multiple ways for teachers to "Compare" work during the Compare and Connect MLR. The author noted, "You could do a gallery walk and just have students put up work." A few minutes later, Ms. Severn asked, "How would a gallery walk work in our classroom with filming students?" Because this is a research study, there are considerations regarding moving non-consented students out of sight during lessons. However, we were very mindful of keeping lessons flowing as normally as possible and let Ms. Severn know this. We highlight that Ms. Severn was already considering how to enact Compare and Connect before she had experienced the routine—simply after a brief overview of the routine. This is the beginnings of Ms. Severn's deeper level of understanding, as she considered contexts in which to apply the MLR within her own classroom. Ms. Severn was using knowledge of her own students to begin to make the lesson her own (Beltramo, 2017).

Studio Day – Ms. Severn's Gallery Walk

Ms. Severn taught the first lesson during the Studio Day. She provided an overview of her lesson during the pre-brief, then taught her lesson, and then had a debrief of her lesson.

Pre-brief of Ms. Severn's enactment. The group began the day with an overview of Ms. Severn's lesson. Ms. Severn explained that the lesson would be focused on a proportional relationship, with each group solving with a specific representation. She explained: "So, I'll assign them to use an equation and a unit rate, a table, or a graph…Then we'll have them look at each other's representations and compare how they solved the problem against each representation." This task considered how Ms. Severn brought the MLR into her own classroom practice, within a specific mathematics task, illustrating a deeper level of understanding. Further, Ms. Severn shared that the MLR was meant to get the students talking to each other about mathematics, because the class was "really hesitant to talk at all." Enacting the MLR as she did employed Ms. Severn's flexibility, or her awareness of her students, particularly her multilingual learners, as she worked to adapt to their needs, particularly getting them to talk about the mathematics in her class.

Ms. Severn's enactment of the gallery walk. Ms. Severn introduced the task and reminded them of the work they had already done related to the content:

> We've talked about tables this unit. We've talked about equations this unit. We've talked about graphs this unit. Each of your table groups is going to get randomly assigned one of those representations and then you're only going to be able to solve the problem with that representation.

She then told the students that "each group needs a poster." Students then knew what was required of them within the task but notifying students of the work they would be doing also illustrated, as noted above, how she had made the MLR her own, highlighting her deeper level of understanding of the MLR. The structure of the MLR as enacted went beyond the first author suggesting, "You could do a gallery walk."

Approximately 22 minutes into the class, Ms. Severn further explained the class' work to complete within the MLR, noting,

> We're doing to do what's called a 'gallery walk'….I'm going to give you this graphic organizer here, and it says 'ratio,' 'table of values,' 'graph,' and 'equation.'…And you are going to have four minutes to walk around the classroom and look at how other people solved this problem and write in the box describing how groups solved the problem.

Ms. Severn, in providing students with a graphic organizer, attended to students' linguistic and mathematical needs during the Compare and Connect MLR, highlighting her flexibility. A graphic organizer, such as this, provided a scaffold for multilingual learners, which allowed access to the content for multilingual learners (Echevarria et al., 2006). Further, Ms. Severn made the gallery walk her own, providing guidelines to students, moving from the broad strokes provided in the pre-Studio Day to fine-tuned details needed for enactment with students, exemplifying a deeper level of understanding. She further provided students with sentence frames and direction for how to engage in their discussions in pairs, for example, "If you're the partner that's sitting closer to the back wall…you are the partner that's going to speak first." These language supports provided ways for multilingual learners to engage in the discussions that she had wanted to support and had noted in the pre-brief, marking, again, her flexibility, particularly related to her multilingual learners.

Discussion

We found that a teacher made sense of the MLR Compare and Connect during a Studio Day Cycle in ways that demonstrated their adaptive expertise through the use of a gallery walk, exhibiting both her deeper level of understanding and her flexibility (Yoon et al., 2019). Ms. Severn was able to use this MLR to engage students in mathematical conversations around proportional reasoning through the gallery walk and more specifically through her graphic organizer—supporting students both mathematically and linguistically (Zwiers et al., 2017). Her flexibility illuminated her attention to multilingual learners. Ms. Severn's enactment of the MLR Compare and Connect provides the field an image of a teacher's adaptive expertise of this MLR. Such an imagine helps us understand these MLRs and how teachers use and make sense of them in their instruction. Future research can then examine how students, particularly multilingual learners, use and make sense of these MLRs in their learning.

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BUILDING A TEACHER COMMUNITY OF PRACTICE THROUGH STUDIO DAYS

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Our work uses Studio Day Cycles (Von Esch & Kavanagh, 2018) focused on the integration and development of mathematics language routines (MLRs; Zweirs et al., 2017). Our conceptual framework draws on two key ideas: communities of practice (Lave & Wenger, 1991) and teacher communities (CoP; Grossman et al., 2001). We discuss each and how they interact with each other. Therefore, our research question was: How, if at all, did a Studio Day Cycle establish a teacher learning community to support teachers to engage in reflection around use of the MLRs?

Method

Our study was situated in a school district on the West Coast that included a substantial number of multilingual learners as part of a larger, multi-year study focused on Studio Day Cycles. We focused on a single cycle that occurred during Fall 2023. We worked with five junior high school teachers and qualitatively analyzed their Google Form reflections using the dimensions of CoP: mutuality, joint enterprise, and shared repertoire.

Findings

Our study found evidence of all three dimensions of CoP. First, Ms. Ruth felt very supported by the "team of teachers" and welcomed their questions, evidencing mutuality. Second, teacher participants considered each other's practice, thought about, and made plans to take what they had learned into their instruction (moving from the mutuality to considering their joint enterprise). Finally, established a shared repertoire around the MLRs by providing examples of what the MLRs looked like in practice and developing joint definitions and understandings of the MLRs that could be used for further reflection.

Discussion and Conclusion

The Studio Day Cycle supported the development of a teacher community of practice by supporting teachers to reflect on their use of the MLRs. As we seek to continue supporting teachers in their work of considering the needs of multilingual learners, it is important to establish spaces and communities where teachers are able to rely on and learn from each other.

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CONNECTING VISIONS OF EQUITABLE MATHEMATICS INSTRUCTION TO PRACTICE: A FIRST-YEAR TEACHERS' COMMUNITY OF PRACTICE

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Keywords: Instructional vision; professional development; equity, inclusion, and diversity

One way to support first-year teachers (FYTs) to navigate tensions between their instructional visions and the realities of school culture while developing as critical educators is through a community of practice (CoP), which serves as the theoretical framework for this study. A thriving CoP can be a space for FYTs to reflect upon and share their experiences as they work to create elementary mathematics classrooms that honor all students' mathematical thinking and strive to center equity and justice (Woods & Rupe, 2024). Through shared curiosity FYTs can think critically about their students as mathematical doers within the daily realities of teaching. Yet, we wondered how do FYTs' visions of equitable and high-quality mathematics instruction serve as filters for the equity-based instruction they enact during their first year of teaching?

Methods

There are three CoPs in this study, with a total of 15 elementary FYTs. Two cohorts of FYTs graduated from the same Pacific Northwest university in 2022 or 2023. One cohort of five FYTs graduated in 2023 from the same university in the Midwest. Each FYT was individually interviewed before the CoP began and at the end of their first year teaching. They described their visions of high-quality and equitable mathematics instruction using the VHQMI and VEMI interview protocols (Munter, 2014; Haines et al., in preparation). Other data sources included transcripts of monthly CoP meetings and individual reflective journals. Thematic analysis was used to identify patterns in the data (Braun & Clarke, 2006). We began analysis by coding interviews and monthly CoP transcripts for semantic themes and then through a recursive process, moved to the latent level of analysis and identified (or examined) the underlying ideas, assumptions, and conceptualizations (Saldaña, 2021).

Summary of Results

Initial findings have shown numerous examples of FYTs grappling with and expanding on ideas from their interviews throughout the school year. For example, Halley shared in August,

> At the beginning of a lesson, I would look for ways the teacher is making the math lesson accessible for all students to start. Are there multiple entry points into the task or different ways for students to access the same idea? Are there places students know they can look for help, like across the room or among peers? I'd also look for a lot of sense making conversations and talk. Including the children's voices a lot, not just the teacher talking…

These ideas of accessibility, emphasizing sense-making, and including students' voices were revisited often by Halley in meetings, journals and other artifacts. This is only one example of how FYTs' instructional visions acted as filters for enacted equity-based instruction. Analysis across the three cohorts is ongoing, yet the implications highlight the tensions FYTs faced (and the supports that they need during their coursework and in the first year of teaching) so that they continue to critically think about their students as mathematical doers within the daily realities of classroom teaching.

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AN INDUCTIVE EXPLORATION OF TEACHER MOTIVATION AND PERCEIVED LEARNING IN THE COACHING PARTNERSHIP

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We partnered with nine elementary teachers who were engaged in coaching cycles with their school-based coach to improve their mathematics instruction. We conducted 16 semi-structured interviews to better understand why they were motivated to engage in coaching, what they reported learning from the experience, and the extent to which there was parity between teachers' motivating reasons to engage in coaching and their perceived learning. Qualitative analyses indicated that teachers frequently reported improved understanding of mathematics content and pedagogy as both the motivation to engage in coaching and their perceived learning. Furthermore, analyses showed five overarching trends when exploring motivation and perceived learning parity. Implications for practice and future research are discussed.

Keywords: Instructional Leadership, Professional Development, Teacher Educators

Since its inception in the US in the 1980s (e.g., Joyce & Showers, 1981), coaching has become a widespread mechanism to support improvements to mathematics teaching. We define a coach as someone who works directly with a teacher, or group of teachers, on instructional improvement issues in service of improving student learning (Baker et al., 2022a). The proliferation of coaching within mathematics education has spurred a body of research focused on understanding how coaches and teachers work together (e.g., Gillespie et al., 2024) and the outcomes of these interactions (e.g., Kraft et al., 2018).

Existing research has tended to center on coaching outcomes such as enhanced student achievement (e.g., Campbell & Malkus, 2011), instead of teachers' learning interests who initiate coaching interactions. As coaching is a responsive professional learning method, it is critical to understand what motivates teachers to engage in coaching and their desired growth. Furthermore, a teacher's practice is a complex interaction of knowledge, skills, identities, and beliefs (Grossman et al., 2009). Coaching holds the potential to develop mathematics teachers' instructional practice in unique and nuanced ways since a coach can engage a teacher in customizable interactions involving collaborative planning, teaching, and reflecting. Thus, coaches can support teachers in ways that extend beyond student achievement measures and deductive evaluation of pre-determined teaching practices which have been the focus of existing coaching studies.

To this point, there has been no systematic analysis exploring (a) teachers' motivation to engage in coaching to improve their mathematics teaching and (b) teacher's perceptions of how they benefited from engaging with a coach. Furthermore, the field knows little about how teachers' motivation to engage in coaching aligns with their perceptions about what they learned. In response, we designed an exploratory study in a context in which teachers had open access to one-on-one coaching cycles with school-based coaches to improve their teaching of mathematics. We pursued answers to the following research questions: (a) What motivates teachers to engage in coaching?, (b) What do teachers perceive they learn from their engagement in coaching?, and (c) To what degree is there parity between teachers' motivation to engage in coaching and what they perceive they learn?

Method

Context and Participants

This study took place in Midtown and Southampton school districts, which had instructional and content-focused coaching programs (respectively). Across these two school districts, we partnered with nine elementary teachers who were currently engaged in one-on-one coaching cycles with their school-based coach. In both contexts, a coaching cycle was a three-part activity in which the teacher and coach collaboratively planned, taught, and reflected upon a mathematics lesson. All teachers identify as White females, and had 1-23 years of teaching experience across grades pre-kindergarten through 5^{th}.

Data Source

This analysis rests upon 16 semi-structured interviews that were conducted one-on-one with each participant before and after their coaching cycle. In the interview before the coaching cycle, we asked participants questions to gauge their motivation to engage in coaching. In the interview after the coaching cycle had ended, we first asked our participants to remind us what motivated them to engage in coaching. Next, participants were asked questions to better understand what they perceived they learned. All interviews were audio recorded and transcribed.

Analytic Technique

Motivation and Perceived Learning

All 16 interview transcripts were read multiple times to support the researchers in developing a holistic understanding of the teacher participants' emic perspectives regarding their motivation to engage in coaching, as well as what they perceived they learned. Next, the broad codes of *motivation* and *perceived learning* were applied to interview excerpts to identify instances in which the participants discussed what motivated them to engage in coaching or what they reported learning. Then, all interview excerpts that had previously been coded with the broad codes of *motivation* or *perceived learning* were read again, and a sub-code was added through an open coding process to capture the specific motivating factor or learning that teachers noted (e.g., differentiation). Then, these sub-codes were inductively clustered into broader categories (e.g., pedagogy) based on their relationships with one another, until all sub-codes could be accounted for. Last, matrices with counts based on the total number of teachers who reported a particular motivation or perceived learning were generated to facilitate pattern detection.

Parity

Parity refers to the extent to which there was congruence between a teacher's motivation to engage in coaching with what that teacher perceived they learned. We began by creating individual teacher profiles using a template. In the template, we recorded the specific motivation and perceived learning sub-codes (e.g., differentiation), as well as the broad category (e.g., pedagogy) under which each sub-code was nested, for each teacher. Next, we coded for parity, or congruence, and assigned one of two codes to each motivating and perceived learning sub-code: (a) paired, and (b) unpaired. Paired was assigned when a motivation sub-code reappeared as a perceived learning sub-code for an individual teacher. Unpaired was assigned when a motivation sub-code did not reappear as a perceived learning sub-code (Unpaired-M), or when a perceived learning sub-code did not appear as a motivation sub-code for an individual teacher (Unpaired-PL). Hence, we assigned the codes paired and unpaired to each motivating and perceived learning sub-code for each teacher and recorded these codes in the templates for each teacher profile. After this round of coding, we looked across all teacher profiles to develop generalizable cases of teacher parity to understand how teachers' motivation to engage in coaching aligned with their perceptions of their learning.

Findings

Teachers' Motivation and Perceived Learning

Teachers described eight different topics when discussing what motivated them to engage in coaching (RQ1), as well as what they perceived they learned (RQ2). In Table 1, we report the number of teachers who mentioned these topics and highlight that the most common topics for both motivation and perceived learning were mathematics content and pedagogy.

Table 1: Overall Teacher Motivation and Perceived Learning

Topic	Definition	Motivation	Perceived Learning
Content	Teachers were motivated to engage in coaching and/or reported learning about math content, which included deepening their own and/or students' understanding of math, enhancing their own and/or students' beliefs and attitudes towards math, changing their own beliefs about students' math capabilities, and/or anticipating students' thinking and/or misconceptions about math.	6	7
Pedagogy	Teachers were motivated to engage in coaching and/or reported learning about pedagogical tools, including: strategies for differentiation, questioning, formative assessment, engagement and incorporating kinesthetic movement; new learning routines; and pacing issues.	5	7
Curriculum	Teachers were motivated to engage in coaching and/or reported learning about curricular issues, including deepening their understanding of the district-provided curriculum, or finding high-quality curricular resources.	3	3
Behavior Management	Teachers were motivated to engage in coaching and/or reported learning about how to improve student behaviors in their classroom.	3	1
Professional Development Opportunity	Teachers were motivated to engage in coaching and/or reported learning about professional development opportunities, in particular coaching, they could pursue to improve themselves as educators.	2	1
Policy	Teachers were motivated to engage in coaching because of policy-related reasons, such as being on an improvement plan and/or wanting to improve their evaluation scores.	2	0
Coaching Knowledge	Teachers were motivated to engage in coaching because they perceived it would enable coaches to deepen their coaching knowledge by interacting with different teachers, thereby becoming more effective coaches.	1	0

Relationship Building	Teachers were motivated to engage in coaching because they wanted to strengthen their relationship with their coach.	1	0

Parity Between Teachers' Motivation and Perceived Learning

For third research question, we explored the extent to which there was parity between teachers' motivation to engage in coaching and what they reported learning. We conceptualized five potential configurations relating the parity of motivation to perceived learning and then located each teacher profile within one of these five configurations (see Figure 2). The most common configuration involved partial parity (n = 5) in which (a) certain aspects of teachers' motivations were also highlighted in their perceived learning and (b) teachers reported perceived learning beyond what initially motivated them to engage in coaching. For example, Barbara described improved questioning in both her motivation for coaching and her perceived learning. Barbara also reported learning about differentiation and anticipating students' mathematical thinking which was not mentioned as a motivation prior to her interactions with her coach.

	Teachers	Definition	Motivation	Perceived Learning
Complete Parity	None	Only paired motivation and perceived learning	X	X
Lack of Parity	Lindsey Michelle	Only unpaired motivation and perceived learning	X	Y
Partial Parity	Mackenzie	Paired motivation and perceived learning AND unpaired motivation	X and Y	Y
Partial Parity	Brianna	Paired motivation and perceived learning AND unpaired learning	X	X and Y
Partial Parity	Barbara Caroline Cathy Cecilia Jennifer	Paired motivation and perceived learning AND unpaired motivation AND unpaired perceived learning	X and Y	X and Z

Figure 2: Parity Trends

Discussion and Implications

One overarching contribution of this study is that it centers teachers' voices in the coaching partnership. Given calls to dignify teachers as professionals by providing them with choice and agency regarding their own professional learning needs (Lieberman & Pointer Mace, 2008), we contend that coaching research must examine teachers' voices, wants, needs, and perspectives. This exploratory, relatively small-scale interview study takes an important first step in that direction as we focus on teachers' motivation to engage in coaching, as well as what teachers report they learn through. Through this work, we hope to pave the way for future coaching research to similarly center teachers' voices in the coaching partnership.

Overall, we found that mathematics content and pedagogy were prevalent motivating factors and perceived learnings for teacher participants. The prevalence of these categories highlight the importance of providing coaches with ongoing professional learning opportunities that will enable them to deepen their content knowledge as well as equip them with rich pedagogical tools to engage diverse student learners. Oftentimes, coaches' professional learning needs are overlooked once they assume their new positions as they are "anointed and/or appointed without the proper background related to their content, pedagogical, and leadership knowledge and skills" (Fennell, 2017, p. 9). Hence, school districts must consider how they are providing job-embedded support for coaches so they have opportunities for professional growth. Findings regarding partial parity between motivation and perceived learning provide new insights about the responsive nature of coaching as a professional learning activity. Mainly, coaches were able help teachers grow towards outcomes named at the outset of coaching interactions while also supporting professional development beyond these predetermined outcomes.

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TEACHER LESSON STRUCTURE AND UPTAKE OF INSTRUCTIONAL NUDGES

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Keywords: Professional Development, Instructional Activities and Practices

Teachers often have discretion in how they format their lessons (e.g., whole-class, groups), even if they are required to use specific textbooks, technology, or common assessments. Additionally, teachers have a choice in using combinations of whole-class discourse, independent, or group work time to engage students in mathematics (Otten et al., 2022). We hypothesize professional developers need to be cognizant of teachers' lesson formats when offering instructional suggestions. For example, if a teacher does not use group work regularly, they may be hesitant to implement interventions designed for group work. These suggestions should be close to teachers' current practices, small in grain size, and be something a teacher could easily take up (Litke, 2020; Star, 2016). We asked, how do teachers' lesson formats impact how they take up instructional nudges provided in a professional development setting?

We observed seven Algebra I teachers' lesson formats as part of a larger study (Candela et al., 2024). We utilized Year 1 observations to identify the percent of time dedicated to whole-class discourse, independent work, and group work. In Year 2 we deployed 16 instructional nudges that are small instructional suggestions, closely aligned to teachers' practices, and have potential to be high uptake (Authors). We aligned the design of the instructional nudges with different lesson formats. For example, One Paper is an instructional nudge that encourages a teacher to use group work and Rate and Review is designed for independent work time where students rate responses to worked problems, much like they would reviewing a product. Teachers selected instructional nudges and we used observations and interviews to capture which one's teachers enacted.

We identified relationships between the time spent in each lesson format and the instructional nudges enacted. One teachers' instruction was typically whole-class discourse, short independent work time, more whole-class discourse, and finally independent work time. This teacher only implemented instructional nudges that fit in during whole-class discourse, and as the teacher spent most of their class time in this format, it follows they would take up nudges that fit in this format. This finding was similar across teachers. We will share visual displays of our results for all teachers and discuss the instructional nudges with the most uptake in relation to format. Implications suggest teachers are more likely to take up practices that are more closely aligned with their current practices and suggest those providing professional development should keep this in mind when planning interventions.

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A CASE OF HOW AN ELEMENTARY MATH TEACHER ATTENDED REFERENT UNIT THROUGH PROFESSIONAL DEVELOPMENT

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Referent units in fractions are often overlooked by elementary and secondary math teachers. Research highlights the challenges teachers face in presenting fractions effectively (Copur-Gencturk & Ölmez, 2022; Izsák et al., 2019; Wang et al., 2023). This study, grounded in Knowledge in Pieces (KiP) (diSessa, 2016, 2018), aims to provide insights into PD initiatives designed to enhance teachers' attending of referent units, crucial for future fraction teaching.

This intrinsic case study (Stake, 1995) focuses on Marcus, a middle school teacher with six years of experience. He participated in a weeklong PD including solving fraction problems and discussing referent units. Analysis centered on Marcus's intuitive (pre-PD interview data) and developed understanding of referent units (post-PD reflection data).

Before the PD, Marcus intuitively partition a garden to rows and did not realize he used a row as his referent unit. Post-PD, he began connecting these intuitive methods with a clearer understanding of the referent unit. For example, in the Mathtopia problem, Marcus initially misinterpreted the referent unit (e.g., "Okra of 3/16"), but adjusted his thinking with facilitator guidance, leading to a better attending of the where the whole appears (e.g., "Okra is 3/64").

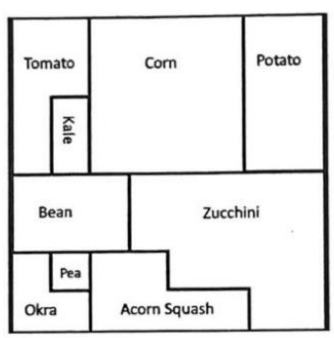

Figure: Mathtopia problem used in PD

Our study aligns with existing research, revealing a notable issue among teachers struggling to keep track of referent units in fraction operations. This highlights the necessity for targeted support to help teachers like Marcus establish stronger connections between partitioning strategies and a consistent awareness of the referent unit.

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EMBEDDING TEACHER LEARNING OPPORTUNITIES IN UNIT ASSESSMENTS

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Keywords: Professional Development, Systemic Change, Teacher Knowledge

One empirically supported way of improving elementary mathematics teaching is engaging teachers in developing and using knowledge of student thinking in their instruction (Carpenter et al., 2000). A significant challenge, however, is embedding these kinds of learning experiences in the day-to-day work of teaching, especially given the many constraints and pressures elementary teachers must navigate. In this project, we sought to understand how unit assessments might function as a site for such learning experiences, and how facilitated grade level team meetings might support teachers in developing and using knowledge about student thinking in their planning and teaching.

The study was guided by a theory that acknowledges both individual and situated aspects of knowledge (Cobb & Bowers, 1999); in particular, we are interested in how knowledge development occurs within a group when the object of learning—knowledge of student thinking—is situated in an artifact of instruction—the unit assessment—that has substantial salience for elementary teachers. The study was conducted in a school district that had adopted commercially published textbook with seven to nine units in each grade level. A team of teachers and leaders worked to adapt the unit assessments so that they would be open to a variety of strategies and created strategy rubrics based on the Cognitively Guided Instruction research project (e.g., Carpenter et al., 2014). Four schools were selected to participate in the study over the course of a school year. The grade level teams for third, fourth, and fifth grade engaged in a series of meetings facilitated by an instructional coach in which they discussed the strategies students used on the assessments when given prior to the start of a unit (pre-test) and at the end of a unit (post-test). Between these two meetings, there was a third meeting in which the group collectively planned a lesson, with the intention to use information about student thinking to guide their planning. Finally, the district also had purchased an online assessment program that ranked students in terms of their grade level (e.g., two grade levels behind). These were used to identify students, and eventually teachers, for interventions. We created charts showing the distribution of student strategies (Kazemi et al., 2016) on the unit tests and juxtaposed these with results from the district assessment program and shared these with teachers between meetings.

We recorded all teacher meetings and analyzed them in terms of "teacher talk paths" (Murata et al., 2012). Specifically, we attended to how participants incorporated the information about student thinking embedded in the assessment materials into their conversations about students and about teaching. We also recorded meetings between the team facilitators.

Early results show that teacher teams spent a substantial portion of their time in initial meetings discussing their students' work on the assessments—determining what counted as evidence of which strategies, etc. Facilitators played a key role in connecting these conversations to information about student thinking more generally (sometimes in response to teachers' statements about student deficiencies) and also prompting teachers to connect information from the assessments to their planning/teaching practices. In addition, the charts with the distribution of student strategies provided a context for discussing differences in the how the unit assessments and the online assessment program supported instructional decisions.

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IMPACT OF TEACHERS' PERCEPTIONS ON INSTRUCTIONAL NUDGE UPTAKE

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Keywords: Professional Development, Research Methods, Instructional Activities and Practices

The idea that professional development should be done with teachers, rather than to them, means professional developers should account for teachers' perspectives as they design professional development (Desimone, 2009; Feiman-Nemser, 2001). Drawing on this, we know mathematics teachers have aspects of their instruction they are and are not pleased with. For example, a teacher may be pleased with how they go over homework, or they may not be pleased with how group work is facilitated in the classroom. It made us wonder, do teachers take up ideas from professional development based on self-identified areas of their teaching strengths, areas for growth, or somewhere in between? This study is set within the context of a professional development focused on teachers' uptake of instructional nudges (de Araujo et al., 2022), where teachers were given choice as to what they engage with. Instructional nudges are small suggestions closely aligned with teachers' practice that we hypothesize will have high uptake (de Araujo et al., 2022). In order to determine which of these nudges the teachers are taking up, we investigated the question of what instructional nudges do mathematics teachers select (or not select) and how do their choices relate to their identified areas of strength and growth?

We surveyed 7 Algebra teachers to identify aspects of their instruction they were pleased with or not, from a list of 15 (e.g., going over homework, facilitating discussion, group work). During the school year, we provided a set of 16 instructional nudges aligned to various aspects of teachers' instruction and provided choice as to which one(s) teachers might engage with. At the end of the school year, we interviewed each teacher about which instructional nudges they enacted in their classroom and the extent to which they loved or hated each instructional nudge. Preliminary findings suggest teachers were more apt to implement instructional nudges which aligned with aspects of their instruction they were pleased with. On the other hand, aspects of teachers' instruction they were not pleased with had more in common with nudges that were only implemented once. During our poster presentation, we will share visual displays of our results and discuss the types of instructional nudges that had the most uptake in relation to aspects of teachers' self-identified areas of strength and areas of growth in their classrooms. Discussion is welcome as we envision a future for mathematics professional learning opportunities for teachers and professional developers.

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Chapter 12:
Statistics, Probability, and Data Science

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EXPLORING THE AUSTRALIAN GENERAL PUBLIC'S NUMERACY CAPABILITIES REGARDING CLIMATE CHANGE

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Numeracy has played an increasingly important role in schooling in recent years, with the recognition that individuals need to be numerate in order to cope with the mathematical demands of everyday life. One societal issue involving a great deal of mathematical information is climate change. We conducted a study to understand the Australian general public's (n = 144) climate change views, practices, and numeracy capabilities. Based on our analysis of questionnaire data, we found that many participants lacked understanding of climate change information presented in stacked bar graphs. Furthermore, several participants seemed unable to focus on the mathematical content of the numeracy questions; instead, they responded in emotional, non-mathematical ways. Consequently, we argue that numeracy education needs to include a focus on the dispositional aspects of numeracy, per the Goos et al. (2014) model.

Keywords: data analysis and statistics; mathematical representations

It has long been argued that one of the purposes of mathematics education should be to prepare individuals to cope with the mathematical demands of everyday life (Paulos, 1988). In the past decade, increasing emphasis has been placed on the role of numeracy, or mathematical literacy (See Forgasz et al., 2017, for a discussion of terminology), in schooling. For instance, in Australia, numeracy is one of seven general capabilities that must be incorporated across the curriculum, in order to "equip young Australians with the knowledge, skills, behaviours and dispositions to live and work successfully" (Australian Curriculum, Assessment and Reporting Authority, 2023, para. 1). One key aspect of being a numerate citizen is the ability to understand data that are presented in popular media and other contexts. These data pertain to a variety of aspects of everyday life, such as health (e.g., COVID-19 data) and politics (e.g., polling data).

Climate change is another topic for which data are presented using a variety of mathematical models and representations. The role of numeracy in the general public's understanding of and responses to climate change issues is complex and often has implications for climate change-related communication, governance, policy, and community education (Kahan et al., 2012). Numerate individuals can typically interpret and transform data into informed decisions and responses, aligning their perceptions with scientific concepts (Nurse & Grant, 2019). This capability, however, notably deviates in the context of climate change-related data, where individuals frequently resort to preferred interpretations that align with their ideological predispositions (Gilden & Peters, 2017; Nurse & Grant, 2019). Therefore, effective climate change education strategies need to focus not only on enhancing numeracy and scientific literacy but also on addressing the motivated reasoning influenced by psychological and ideological factors (van der Linden, 2021).

Recognising the significance of these climate change challenges, Monash University has identified climate change as a major focus area and has provided funding for research projects on this topic, such as our project. Given the limited research concerning the general public's numeracy capabilities in the context of climate change, we focused on the Australian general public's climate change views, practices, and numeracy capabilities. Here, we focus on our participants' numeracy capabilities, sharing findings from our analysis of questionnaire data.

Objectives

As mentioned, the purpose of our study was to understand the Australian general public's climate change views, practices, and numeracy capabilities. We also sought to understand participants' views of climate change and education, as well as whether there were any differences between non-teacher and teacher participants' responses. The research question that will be addressed in this paper is "What numeracy capabilities does the Australian general public demonstrate when interacting with climate change data?"

Theoretical Framework

We view numeracy in accordance with Goos et al.'s (2014) 21st Century Numeracy Model. Namely, context is central to numeracy: That is, numeracy involves mathematics being applied to a context, such as citizenship, work, and personal and social situations. However, numeracy is more than simply the application of mathematics to a real-world context: To be numerate, people need to have mathematical knowledge (e.g., problem-solving, estimation), positive dispositions toward the use of mathematics (e.g., flexibility, confidence), and the capabilities to use tools (e.g., representational tools like graphs, physical tools like rulers). The final aspect underpinning the model is critical orientation, which refers to the ability to consider the veracity of mathematical information, use mathematics to make decisions and support arguments, and question whether answers/outcomes make sense, particularly given the context. We used this model in the design of our numeracy questions, as well as our analysis methods.

Research Design

In the following sections, we describe the data collection instrument and issues encountered during data collection, share researcher positionality statements, provide participant information, and discuss our analysis process.

Data Collection Instrument

Our questionnaire, which contained both closed and open-ended questions, was posted on Qualtrics. There were approximately 20 questions (There were a few questions that were only asked of teacher participants), and the questionnaire took 10 to 15 minutes to complete.

In the first section, Demographic Information, participants were asked to indicate their state/territory, age range (Note: Participants had to be 18 years of age or older), gender, and highest level of education completed (multiple-choice questions). The gender question also had a textbox where participants could enter their gender if the options provided (i.e., woman, man, non-binary, and prefer not to say) were not suitable. In the second section, Climate Change and Education, participants were asked about their views regarding teaching climate change generally, as well as about their own experiences as students. Teacher participants were asked about their experiences teaching about the topic. In the third section, Climate Change Views and Practices, participants were asked more general questions regarding their views (e.g., re:

government policies) and practices (e.g., re: information sources).

The final section, Climate Change and Numeracy, is the focus of this paper. Participants were asked three levelled questions about a stacked bar graph (Figure 1) regarding youths' views of the severity of climate change compared to 12 other global concerns (Kessler, 2019).

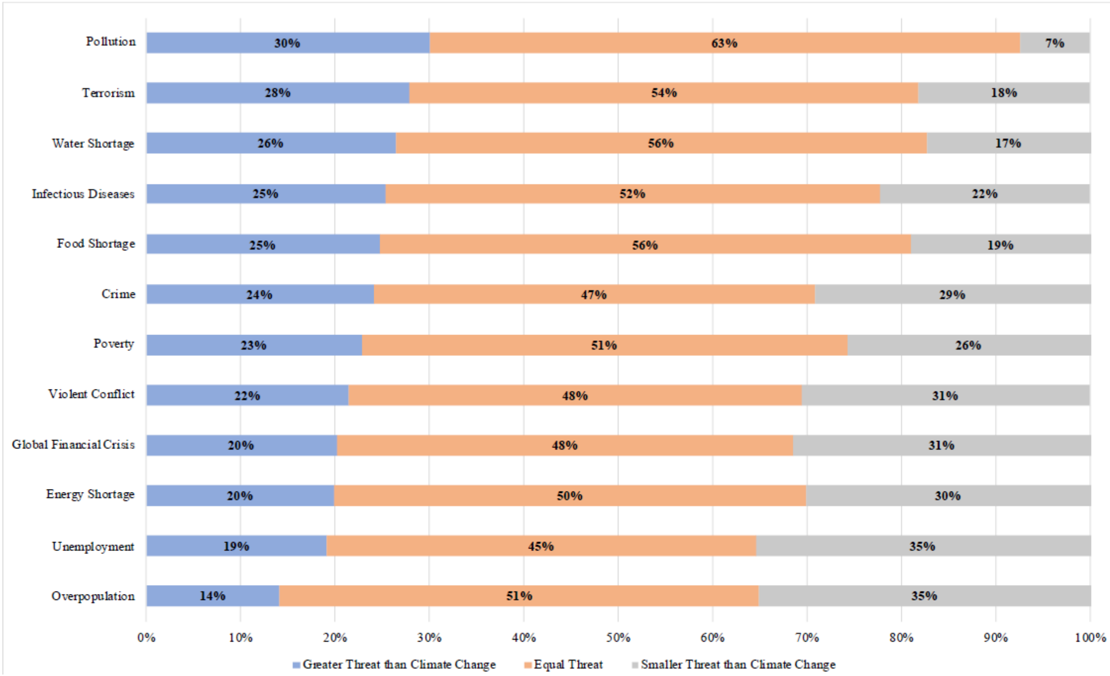

Figure 1: Comparison of Global Threat Severity and Climate Change (Kessler, 2019)

All three questions had multiple-choice response options, and the 'medium' question also had a text box where participants were asked to explain their answer.

Data Collection Issues

We advertised the study on Monash University's Faculty of Education Facebook page. Soon after the advertisement was posted, climate change deniers bombarded the post with negative comments, expletives, and calls for their contacts to complete the survey and skew our dataset. We held an emergency meeting within a day of these issues arising, during which our faculty media contact was notified by Facebook that the advertisement was being taken down due to complaints that it was 'political.'

Researcher Positionality

Our research team comprises academics with expertise in mathematics and numeracy education (Hall), environmental education (Almeida and Arachchige), and STEM education (Kidman). We combined our knowledge and experience to design this multifaceted project. Hall and Almeida led the data collection phase, and thus faced the reactions from the climate change deniers firsthand. We reacted very differently to the trolling, based on our relationships to climate change. Namely, Hall had a generalized reaction of anger and frustration, whereas Almeida felt personally attacked, given her experience working in environmental education.

Participants

In total, 144 participants completed the questionnaire. As hoped by the climate change deniers, the data were indeed very skewed: Most participants were older men (74.3% men; 67.4% over the age of 60). Fewer than 10% of participants were under the age of 50. Despite the skewed age and gender profiles of the participants, the educational qualifications of the participants were broadly nationally representative (Australian Bureau of Statistics, 2023).

Data Analysis

The multiple-choice questions were analyzed via descriptive statistics (counts, percentages). More complex statistical analyses could not be completed due to the skew in the dataset. Responses to the open-ended question were analyzed through via emergent coding (Creswell, 2014). That is, the responses were read several times to get a sense of the dataset. Initial codes were then created and applied. Following this process, codes were combined and refined.

Results

In the following sections, we discuss the findings for the three numeracy questions. Although 144 participants completed the questionnaire, far fewer people responded to the multiple-choice numeracy questions, which may be indicative of people's discomfort with mathematical content, or response fatigue, as the numeracy questions were near the end of the questionnaire.

'Easy' Question (*n* = 78)

The 'easy' question involved participants selecting which of four provided options had the lowest percentage of respondents (from Kessler, 2021) who selected 'equal threat'; hence, only one piece of data from each bar on the graph was required to answer the question. Approximately two thirds of participants (65.4%) responded correctly.

'Medium' Question (*n* = 76)

The 'medium' question involved participants indicating which of the 12 threats was viewed most similarly to climate change (by the participants in Kessler's study). Not only were there more response options than there were for the 'easy' question; the question's wording was less direct. Consequently, a lower proportion of respondents (47.4%) selected the correct response. Participants were provided with a text box in which to explain their multiple-choice selection. Several participants took this opportunity to make non-mathematical comments, such as "STOP SCARING AND DEMORALISING OUR YOUTH WITH THIS RUBBISH!!" (P91). Notably, 14 participants who had not responded to the multiple-choice question still wrote something in the textbox. Of the participants who answered the multiple-choice question correctly, half provided a mathematical explanation in the text box, compared to only 20% of the participants who answered the multiple-choice question incorrectly.

'Hard' Question (*n* = 75)

The 'hard' question was "Which of the following global threats has the **most similar** percentages of respondents who answered **greater threat than climate change** OR **smaller threat than climate change**?" An example of the percentages for these two categories was then provided for Overpopulation. As with the 'easy' question, participants had to select from four options; however, they had to utilize two pieces of information from each bar to respond to the 'hard' question. Participants struggled with this question, with only 42.7% responding correctly.

Discussion and Conclusions

Numeracy is a crucial capability for adults to possess and develop, particularly as society becomes more data-drenched. In our study, we sought to understand adults' capacity to analyze graphs involving climate change data. Despite the questions being at an elementary school level, fewer than half of the participants were able to correctly identify the correct response to the 'medium' and 'hard' questions. Concerningly, many participants seemed unable to focus on the mathematical concepts at hand due to their emotional responses to the topic. At the conference, we plan to share additional findings by participant group.

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STUDENTS' PERSPECTIVES TOWARD DATA SCIENCE

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Data literacy promotes understanding of statistics about society (Engel, 2022) and as educators, we should arm students with data tools to form their opinions about phenomena in the world (Gal, 2002, 2023). An NSF project, Data Science Infused for Undergraduate STEM Education (DIFUSE), seeks to integrate data science into college introductory courses and develop modules that provide students with visualization tools that offload timely computation. These modules allow students to apply mathematics and statistics concepts to gain an understanding of data. A module was adapted for use with high school students. Our study investigates students' interests, experiences, and resources related to data science.

Keywords: Data Analysis and Statistics; High School Education; Instructional Activities and Practices; Affect, Emotion, Beliefs, and Attitudes

The 21st-century workforce is increasingly data-centric, emphasizing the importance of exposing students to data science, ideally starting from K-12 education (Franklin & Bargagliotti, 2020). In an era dominated by data, the modern student needs education to focus on being both a producer and a consumer of statistics (Gould, 2017). Researchers suggest that media reports can motivate students and visualizations coupled with proper scaffolding aid conceptual understanding (Budgett & Rose, 2017). To address this imperative, we adapted a data science module developed in a college environmental studies class for use in high school classrooms. This original module was to cultivate students' interest in data science, by showing practical applications of data science across various fields. We intentionally selected a module that did not require mathematics and statistics content beyond the high school level.

Objectives

We enacted a lesson with a module (DIFUSE Team, 2023) in high school classes to evaluate its use in high school and students' responses. Our research question is how are high school students' interests, experience, and resources related to data science impacted through their work with a data science module? The module uses Louisiana COVID data with a focus on death rates and includes demographic, geographic, and environmental data such as air quality. It provides students with three visualizations (interactive map, correlation matrix, and regression) to aid their understanding. We made modifications to the lesson as appropriate for high school students. Recognizing that many students struggle with mastering key mathematical concepts such as proportional reasoning and geometry, we started the lesson with an activity aimed at clarifying the difference between the number of deaths and the death rate. We used CDC's Data Visualization which allows worldwide analysis of COVID deaths and death rates. We discussed adding a third quantitative variable to a scatterplot by using a bubble chart. We used The Learning Network's [What's Going on in this Graph: Earthquakes] to evaluate whether bubbles should be scaled by area or diameter. Given that Data Scientists vary in technical expertise and most high school students are not proficient in computer programming, we gave them the option to either review and modify the Python code or create a change request form to scale the bubbles correctly. Although lesson modifications were made for high school students, the data and visualization tools used remained the same.

Framework

As we designed the lesson, we used the DIFUSE framework to provide coherence throughout the data investigation (see Figure 1). Like many other data science frameworks, this simply granularizes some of the broader categories of the GAISE II statistical problem-solving process (SPSP) of (1) Formulate Statistical Questions, (2) Consider/Collect Data, (3) Analyze the Data, and (4) Interpret the Results (Bargagliotti et al, 2020).

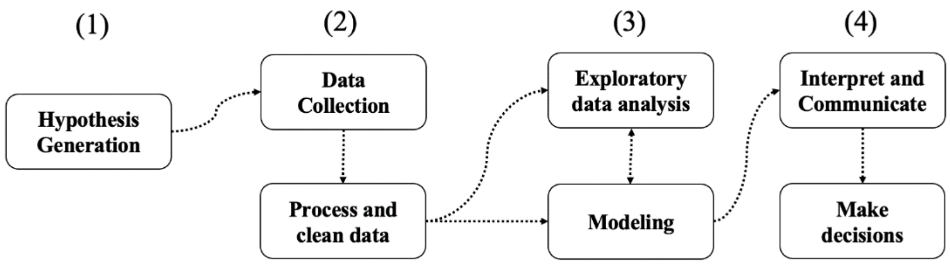

Figure 1: [Report] and [Project] Framework relationship

SPSP was initially developed as part of the original GAISE Pre-K–12 guidelines (Franklin et al., 2007) to infuse statistics in Pre-K–12 education. These efforts to create data-literate high school students were used as a basis for probability and statistics within the Common Core math standards. The SPSP is not necessarily a linear process and curiosity should remain throughout. Analyzing data might prompt new questions that require additional data to be collected or considered. With the increased availability of data, one of the changes to the original GAISE SPSP was the addition of "consider" data, allowing an investigation to start with reviewing data before formulating a statistical investigative question.

This data investigation allowed the students to start with the data and use some visualization tools to analyze data, continually question, and formulate a statistical investigative question. We scaffolded their work to encourage them to think more deeply and communicate the data in a way that answers the statistical investigating question they created. To complete the SPSP and communicate their results, the final product for each group was to complete a poster and submit it to [ASA's high school data visualization contest].

Methods

We utilized a mixed methods approach, blending qualitative and quantitative research techniques. This approach enabled us to thoroughly comprehend the research subject by incorporating diverse data types and viewpoints (Creswell & Creswell, 2017). This study focuses on an AP statistics class of 23 students in the southeast where students used the modified environmental studies module to understand relationships of quantitative variables. They had prior instruction on analyzing one quantitative variable and used CODAP, calculators, spreadsheets, and some applets to visualize data.

To assess students' interests, experiences, and resources and how they were impacted by this lesson, we created an adapted version of the DIFUSE Survey of Attitudes toward Data Science (SADS, Li et al, 2018). This included 14 items, both qualitative and quantitative questions (e.g., "I feel capable when measuring the relationship between two quantitative variables" and "Explain what you would consider when looking at the relationship between two quantitative variables"). Survey data were gathered as pre- and post-module deployment and the initial survey was given prior to any instruction on two quantitative variables, but after the students had worked with one quantitative variable (e.g., center, variation, distribution, z-scores). In the post-surveys, students reflected on their experiences and responded whether they were interested in participating in an interview. We selected students based on their work and classroom discussions within their groups. We interviewed three students using a semi-structured approach, creating recordings and transcripts. Interviews included open-ended questions (e.g. "How was the process of working with the data science module?"). We employed emergent coding due to the broad and exploratory nature of our research question (Blair, 2015).

Results

The students held positive views regarding their attitudes and beliefs about data science both before and after the module. Most consistently valued data science, displaying interest in working with data and recognizing the importance of data science skills for their future. Most students report not getting too frustrated when working with data. An interesting shift from pre- to post-surveys was related to components of the SPSP: (3) Analyze Data and (4) Communicate Results. Although the students' strong desires to communicate with data increased substantially (by roughly 80%), their inclination to analyze the data decreased (see Table 1).

Table 1: Pre- and Post-Survey Results for (3) Analyze Data and (4) Communicate Results

	Pre-Survey	Post-Survey	Percent Increase (Percent Decrease)
Strong desire to communicate with data	31%	56%	81%
Inclination to analyze data (likely)	40%	54%	35%
(very likely)	44%	31%	30%

This was a similar finding to what had been witnessed in DIFUSE modules at the college level. They appreciated having the computation offloaded and using the results of the visualization tools to gain insights and communicate findings.

Our survey also included some open-ended questions and from a qualitative perspective, students were glad to have multivariate datasets and work through the entire statistical problem-solving process. They specifically mentioned that it was "interesting" and "enjoyable" to analyze complex, multivariate data with visualization tools to complete an in-depth investigation. The frustration that students felt arose from not understanding certain aspects, making errors, or experiencing difficulties with tools and resources. There was a shift towards a more positive attitude and increased confidence in working through the statistical investigative process with a large multivariate dataset. From a skills perspective, students noticed an improvement where they no longer conflated analyzing one quantitative variable with analyzing more than one quantitative variable. Additionally, their descriptions of relationships were more refined, for example including strength, direction, and form.

Our interviews confirmed the survey findings and provided some interesting feedback about their interests, resources, and experiences which students were asked to self-rate on a scale of 1-10. Interest increased by 78% on average and was driven by the novelty of the interactive map and correlation matrix, working with multiple variables, and having to communicate results formally. One student stated, "I felt like a real data scientist". Experiences increased on average by 133% and can be attributed to grappling with the new visualizations, seeing actual computer code, and completing an entire investigation (SPSP) using real data. Resources showed the largest increase of 162% on average. This was due to students continually questioning and iterating to include more variables in the explanation of the real-world phenomenon and gaining a better understanding of r-squared and p-values and how they connect to a graph. One student commented on his opportunity to (4) Interpret the Results:

"…this project helped me to better communicate a couple of different statistics like the p-value. I learned how to interpret and communicate the data better".

Based on the findings of the analysis, it is evident that the data science module had a significant positive impact on students' interests, experiences, and resources related to data science. The majority of students valued data science, showing increased interest in working with data and recognizing its importance for their future. Overall, the data supports the effectiveness of the data science module in enhancing students' interests, experiences, and resources related to data science.

Discussion and Conclusions

Overall, high school students were able to work with a modified DIFUSE module which not only deepened their understanding of the content but increased their engagement and made them feel like real data scientists when completing an investigation. The positive shift in students' attitudes and beliefs towards data science highlights the importance of incorporating practical, hands-on experiences in data science education. By engaging students in real-world data analysis projects using the SPSP, educators can not only increase students' interest in the subject but also enhance their skills in data analysis (Hicks & Irizarry, 2018). By exposing students to a variety of data visualizations and analysis techniques, educators can help them develop a more comprehensive understanding of data science concepts and improve their analytical skills

(Bargagliotti, 2020; Keller et al., 2020). This study highlights the importance of incorporating practical, hands-on experiences in data science education to enhance students' interests, experiences, and resources in data analysis. By doing so, educators can better prepare students for careers in data science and related fields. Modules that provide students with data alongside tools that offer visualizations and offload computation might help students. This could aid their conceptual understanding and allow them to navigate throughout the SPSP asking more questions to tell more intricate stories. Future work might consider studying students' conceptual change, implementing this module in other classes, and developing different modules for high school classes based on teacher and student interests.

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DEDUCTION, INDUCTION, AND ABDUCTION: STUDENTS' WAYS OF REASONING ABOUT SAMPLING DISTRIBUTIONS

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Statistical reasoning is inherently different from mathematical reasoning because reasoning about data involves reasoning with uncertainty. In mathematics, deduction and induction are privileged, but abductive reasoning can be powerful when reasoning about data, especially when making inferences from sample data. This study examines three classic forms of inferential reasoning—deduction, induction, and abduction—novice statistics students employed during two task-based clinical interviews. All three forms of reasoning were evident, though not all were productive. Findings suggest that inductive reasoning is productive when estimating sampling variability and abductive reasoning is useful when estimating an unknown population parameter.

Keywords: cognition, data analysis and statistics

Understanding ideas of sampling, sampling distributions, and statistical inference are important for improving students' statistical literacy and productive citizenship. Data are everywhere. Data collected from samples are often reported in the form of polls, medical studies, and advertisement information and an understanding of sampling distributions and statistical inference is important for evaluating data-based claims (Bargagliotti et al., 2020; Ben-Zvi & Garfield, 2004; Garfield et al., 2015; Saldanha & Thompson, 2007). Consistent with the conference theme of "envisioning the future of mathematics education," it is important to acknowledge the growing prevalence of conflicting reports in the media and support students in becoming "critical consumers" of these statistically-based results (GAISE College Report ASA Revision Committee, 2016, p. 8).

The concept of sampling distribution is a foundational concept in statistical inference, which is the main focus of introductory statistics courses, whether at the secondary, undergraduate, or graduate level (Ben-Zvi et al., 2015; Lipson, 2003). Despite the importance of understanding statistical inference and sampling distributions, research suggests that students struggle with these ideas (Saldanha & Thompson, 2002, 2014; Sotos et al., 2007). Although many students can carry out the calculations involved in formal inference procedures, such as confidence intervals and hypothesis tests, they often struggle to understand the underlying process and logic behind statistical inference (Chance et al., 2004). One reason for this difficulty stems from the complex and abstract concept of sampling distribution, which requires students to coordinate multiple ideas such as sample, population, distribution, variability, and repeated sampling (Noll & Shaughnessy, 2012). I argue that another source of difficulty is that the forms of reasoning that are expected and highlighted in statistics differ from those that are standard in mathematics.

Theoretical Framework

Statistical reasoning is inherently different from mathematical reasoning. Mathematical reasoning is often deterministic, emphasizing deductive reasoning and proof. In contrast, statistical reasoning *lacks* definitiveness and includes probability, randomness, and uncertainty (Groth, 2015). Reasoning is typically defined as a process of transforming given information to develop a conclusion or make an inference (e.g., Galotti, 1989; National Council of Teachers of Mathematics, 2009). Deduction, induction, and abduction are three classic forms of inferential reasoning and can be modeled with a triadic structure involving a case, rule, and result that are linked in a particular order (Peirce, 1878; Reid & Knipping, 2010). A *case* is a specific observation that a condition holds. A condition describes an attribute of something, or a relation between things. A *rule* is a general proposition that states that if one condition occurs then another one will also occur. A *result* is a specific observation, similar to a case, but referring to a condition that depends on another one linked to it by a rule. The order in which one links a case, rule, and result determines the kind of inference—deduction, induction, or abduction—needed to make a conclusion. In deductive reasoning, a rule and a case conclude a result. In inductive reasoning, a case and a result (or often, many cases and many results) lead to a rule. In abductive reasoning, a result and a rule infer a case. Consider the different ways case, rule, and result can be linked in Peirce's (1878) original example about a pile of beans found on a table: (1) If I know that all of the beans in the bag on the floor are white (the rule) and that a pile of beans on the table is from the bag on the floor (the case), then I can *deduce* (with certainty) that all of the beans in the pile on the table are white (the result); (2) If I know that a pile of beans on the table is from the bag of beans on the floor (case) and that all of the beans in the pile on the table are white (result), then I can *induce* that (probably) all of the beans in the bag on the floor are white (rule); and (3) If I know that all of the beans in the pile on the table are white (result) and all of the beans in the bag on the floor are white (rule), then I may *abduce* that (possibly) the pile of beans on the table is from the bag of beans on the floor (case).

Although mathematics and statistics can share similar forms of reasoning, the emphases in each field differ. In mathematics there is an emphasis on deduction and proof, arriving at conclusions with certainty, and establishing truth. Although deduction can occur in statistics with the application of general rules to particular sets of data, such as using a rule to determine outliers in a set of data, the emphasis in statistics is not on deduction. Instead, it is on reasoning with uncertainty, a feature of both inductive and abductive reasoning. Thus, it is productive to examine how students reason about foundational statistical ideas, such as sampling distributions, from this lens.

Methods

Participants for this study were eleven undergraduate students who recently completed an introductory statistics course. The data come from two clinical interviews, each 60-75 minutes in length, in which participants worked through a series of statistical tasks related to sampling distributions. I investigated the question: What forms of reasoning do novice statistics students employ when reasoning about sampling distributions?

In the first interview, I gave participants information about students at a large university with a known parameter of interest (20% of the undergraduates are business majors). I asked them to do each of the following: provide an interval estimate for the outcome of any single random sample, predict the sample proportion for many hypothetical random samples of size 100, physically drew many samples from a large box of beads with proportions corresponding to the population, examine the collection of sample outcomes, and compare their predictions to their actual sample outcomes. Participants then used a web-based applet to simulate taking 500 random samples of size 100 and provided an updated interval estimate for the outcome of any one single random sample. In the second interview, participants drew one sample from a second box of beads, simulating all students at a second large university with some unknown parameter of interest (the box was covered so the contents were hidden from the participants). Based on one sample outcome, participants made an initial prediction for the unknown population parameter, tested and refined their prediction using the web-based applet, and ultimately provided a range of plausible values they believed captured the true population proportion.

After checking and timestamping interview transcripts against the video recordings of the interviews, I identified reasoning excerpts—instances in which participants provided justification or explanation for a claim. For each reasoning excerpt, I identified case, rule, and result, then examined how the participant linked them to infer the form of reasoning they employed. At this time, I have analyzed interview data for nine participants. This paper reports preliminary results from my analysis for this subset of participants.

Results

When giving a range of percentages they would expect to get in any one single random sample of 100 drawn from a large university with 20% business majors, participants reasoned deductively *prior* to drawing any physical or simulated samples and inductively *after* drawing many physical and simulated samples. Prior to drawing any samples, six participants gave an interval estimate with a margin of error between 5% and 10% and provided various justifications for their choice of margin of error. Tami explained that "5 is a good number," whereas Eric drew on his previous experience in chemistry: "If you're assigned a study and they have this number as the assumed...like when you cook this compound it should weigh about this…as long as you're within 5% of that, then you're not wildly off." Although their reason for choosing 5% as their margin of error differed, both Tami and Eric applied their rule (random sampling produces outcomes that vary from the population parameter by about 5%) to a case (a random sample of 100 was drawn from a large university with 20% business majors) to *deduce* a result (the sample outcome should be between 15% and 25%). After drawing many physical and simulated samples, all but one participant updated their interval estimate based on similarities they observed in the samples they had drawn. For example, Jess updated her initial range (11% to 29%) to 15% to 25% and referenced both the physical and simulated samples she drew: "I'm going off of the curve here and kind of off of the samples that I did…[there are] a lot of dots gathered around 15 and then around 25 and after that it kind of drops pretty dramatically." Similarly, Lyla justified her range of 10% to 30% based on similarities she observed in the physical and simulated samples she had drawn: "From the samples we've taken so far…we've taken a lot…but I don't think we've gone above 30 and we haven't really gone below 10." Although participants provided different ranges, they used multiple cases (all of these random samples were drawn from a population of undergraduates with 20% business majors) and multiple results (most of these samples produced outcomes between [lower bound] and [upper bound]) to *induce* a rule (I expect that any one single random sample of 100 drawn from this population to produce a sample outcome between [lower bound] and [upper bound]).

In the second interview, participants made a prediction for the unknown population parameter based on their one sample outcome from the second box of beads. When asked how they could test their prediction, six participants reasoned inductively when they proposed drawing more samples from the box of beads to look for a pattern. However, when asked how they could use the web-based applet to test their prediction, all but one participant reasoned abductively to hypothesize and evaluate multiple populations (with different parameters) from which their single sample could have been taken. Initially, Lorraine said, “Take a ton of samples to see what the actual proportion is. [Look for] the center of [the distribution], so the mean or the median, so we could see where the true proportion is.” Tyra also wanted to collect more samples from the box of beads to see “where they group up…where the center is.” Lorraine, Tyra, and others proposed using multiple cases (multiple random samples drawn from the population with an unknown parameter) and multiple results (observing where the sample outcomes are centered) to *induce* a rule (the true population proportion). Later, Lorraine tested 60% by constructing a simulated sampling distribution using the web-based applet, ultimately deciding it was a plausible value for the unknown population parameter: “It looks like [my sample outcome of] 52 would be reasonable [because] there's a pretty significant number of results that are 52. And then past that is where [the distribution] tends to kind of taper out.” Lorraine used a result (one random sample of 100 produced a sample outcome of 52 business majors) and a rule (a population with 60% business majors produces several sample outcomes of 52) to *abduce* a case (the sample I drew could have come from a population with 60% business majors). Lyla reasoned similarly to decide that 30% was *not* a plausible value for the unknown population parameter based on her sample outcome of 46%: “Looking at this [dot plot], 46 doesn't seem very likely to occur…that probability is looking very low. I would expect a number not to exceed 42. Based from this, I would say 0.3 isn't a good prediction.” Both Lorraine and Lyla continued to test multiple values, ultimately abducing a range of plausible values for the unknown parameter. Lyla’s abduction was particularly powerful for her, enabling her to make sense of the process and logic behind constructing an interval estimate for the unknown parameter:

> That was just really cool to go backward. When you take stat, it’s like oh, we have this confidence that this number is going to fall in blah, blah, blah, blah. But working backwards shows the logic behind it even more. I thought it make sense to me before…our rationale for drawing these conclusions. But going backwards, it’s another level of understanding.

Discussion

This study provides insight into novice statistics students’ existing ways of reasoning about sampling distributions. As reported in the results, each form of reasoning was evident in at least one participant’s thinking about sampling distributions. However, not all three are productive ways of reasoning. Inducing an interval estimate for a single random sample was productive because participants were able to estimate how far a sample statistic varies from the population parameter. In addition, participants later used their interval estimate to determine whether particular sample outcomes were surprising. Although productive in such cases, inductive reasoning may not be productive when making inferences from sample data to an unknown population of interest. Lorraine and others proposed drawing multiple samples from the population with the unknown parameter, and observing where the sample outcomes clustered to determine the true population parameter. Reasoning inductively in this way is productive if the population of interest is a box of beads from which samples are easy to collect. However, the box of beads represented students at a large university; it is not feasible to draw multiple random samples of students from a large university. Instead, reasoning abductively is productive in this scenario. Participants who reasoned abductively to hypothesize and evaluate multiple values for the unknown population proportion were successful in making an inference from one sample to the larger population from which it was drawn by estimating the unknown parameter.

Supporting students in reasoning abductively is especially important because this kind of reasoning differs greatly from the forms of reasoning students have experienced in their K-12 years in mathematics classes that privilege deduction. Providing students with opportunities to make and test multiple predictions when drawing conclusions from data is one way to support students in reasoning abductively. Thus, this study has the potential to inform task development, specifically designing tasks that promote abductive reasoning. Supporting students to reason in this way can help them develop strong statistical reasoning skills to critically evaluate evidence and claims based on data, a crucial skill needed for living in a data-driven society.

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REFUTATION TEXT OFFERS MIXED EFFECTS ON NIGERIAN PRE-SERVICE TEACHERS' USE OF THE REPRESENTATIVENESS HEURISTIC

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This study enlisted a refutation text to foster conceptual change among 71 preservice teachers of mathematics education in situations that might elicit the use of the representativeness heuristic. Statistically significant differences were found between respondents' pre- and post-tests, with more than half exhibiting normative reasoning after prior use of the heuristic. Analysis of their written explanations revealed differential patterns in their reasoning across two effect categories. Implications for misconception research and practice are discussed.

Keywords: Conceptual Change, Probability, Refutation Text, Representativeness.

Probabilistic misconceptions have been documented to differ in the extent to which they are amenable to change (Jones et al., 1997), with the representativeness heuristic being a prominent one. As the name suggests, it involves quantifying the occurrence of a random phenomenon "by the degree to which it: (i) is similar in essential characteristics to its parent population; and (ii) reflects the salient features of the process by which it is generated." (Kahneman & Tversky, 1972, p. 430). A variety of studies have been conducted to ascertain the prevalence of the heuristic among different populations, especially preservice teachers (PST). In Wilkins (2007), 11 out of 15 PSTs relied on it. The rate was 17.5% in Kustos and Zelkowski (2013).

Several efforts have been made for the jettisoning of misconceptions for normative ideas. They have mostly included programs involving predictions and data analyses through engaging in experiments and simulations (Fischbein & Gazit, 1984; Jones et al., 1997). Although these interventions have returned modest gains, the fruitfulness of conceptual change (CC) strategies in overcoming mathematical misconceptions (Lem et al., 2017) may offer a commensurate path to forging acceptable conceptions of probability. Consequently, this study adopts one CC approach—refutation text—as a probable mechanism to overcome the misconception involving the use of the representativeness heuristic among PSTs.

Theoretical Framework

Refutation texts: offering a switch from S1 to S2

Misconceptions are known to be resistant to change because they portray beliefs about chance phenomena. And since beliefs are personal, subjective truths, the feeling of agency conditions the holder to stick to them irrespective of their source. With an alternative conception offering valid explanations in certain contexts (Savard, 2014), the automaticity and effortlessness (see properties of S1, Kahneman, 2003) characterizing alternative conceptions purports to be valid across contexts. Consequently, getting the holder toward CC will involve a radical shift from existing to new conceptions, abled due to at least three reasons: dissatisfaction, intelligibility, and plausibility (Posner et al., 1982).

A refutation text (RT) offers an avenue for enabling this switch. By exposing the misconception, a cognitive conflict is created that might cause a person to attend to salient aspects of the problem they would otherwise not have. Based on the strengths of the belief, this encounter with such explicit information prepares the ground for alternative explanations that might help resolve the conflict, setting the stage for an activation of a cognitive ecology that decontextualizes the problem, a feature of S2 thinking (Kahneman, 2003).

Commonly used RTs are characterized by three properties: (1) description of a misconception, (2) refutation of the misconception, and (3) subsequent provision of a normative conception (Chi, 2008). Refuting a misconception comes with reasons why such beliefs are not viable, generalizable, or consistent, to—as Leron and Hazzan argue—bring "S2 to intervene in its role as critic of S1" (2006, p. 109). Consequently, it may be valuable to enlist a RT to answer the following questions: (1) What are the effects of a RT on PSTs' use of the representativeness heuristic? (2) What qualitative affordances exist within and across the pattern of effects?

Methods

The study involved 71 PSTs of mathematics education at a university in south-western Nigeria. Following informed consent, they were issued a pre-test containing a question to elicit their use of the representativeness heuristic which was immediately followed with an issuance of the RT (without being told about a post-test which was administered after 2 hours).

Table 1: Pre-test and Post-test on the Representativeness Heuristic

Pre-test	Post-test
An experiment requires you to flip a penny 100 times and record whether the penny comes up <u>heads</u> or <u>tails</u>. On the first 10 flips the penny comes up <u>heads.</u> After flipping the penny 90 more times, how many <u>heads</u> would you expect to get out of the total 100 flips? Answer………..… (Please explain your answer below).	Suppose you toss a fair coin six times, recording the result of each toss. For instance, if you toss a <u>head</u> and then five <u>tails</u> in a row, you would write H T T T T T. Which is the *least likely* result? (a) H T H T H T (b) H H T H T T (c) H H H T T T (d) T T T H T T (e) All are equally likely. (Please explain your answer below).

Following the plurality of indicators of the representativeness heuristic (see Tversky & Kahneman, 1974), I adopted the hermeneutical position of content analysis (Mayring, 2015) to develop a RT—that largely attended to the misconception of chance indicator of the heuristic typified by less runs and more switches—with a goal of influencing S1 thinkers to interrogate their reasoning to fit into S2. The RT was content-validated by a professor of mathematics education who teaches a graduate-level probability methods' course. The content of the RT is as shown in Table 2.

Table 2: Refutation Text on the Representativeness Heuristic

Refutation Text

In a game of Ludo[3] between Ghufran and Imran, Ghufran threw three double sixes and a 3,4 (i.e., 6-6, 6-6, 6-6, 3-4). When it was Imran's turn, he threw a 4-5. After counting his scores, Ghufran again threw almost the exact same sequence (i.e., 6-6, 6-6, 6-6, 1-5). In protest, Imran said the game was rigged, threatening to opt out of it. While it may be difficult to convince him otherwise, the scenario does not in any way suggest cheating but a valid random process. Specifically, underlying such thinking is a misconception called *representativeness*. Because the throw of the pair of dice represents a random process, individuals often expect the outcomes to appear in a strictly random manner however short or long the sequence. Had Ghufran thrown a sequence like 6-6, 5-4, 3,2, 1-3, 4-4, 1-6, 1-1, 4-2 rather than 6-6, 6-6, 6-6, 3-4 and 6-6, 6-6, 6-6, 1-5 on his first eight throws, Imran might not have alleged rigging. But both sets of outcomes represent valid random processes. As such, they have the same chance of occurring.

To analyze the data, I coded the keys 1—for both pre- (*55*) and posttests (option *e*)—and other options 0, making the data suitable for analysis using the McNemar (Binomial) Test. To answer RQ1, the test was used to determine if an intervention effect existed as measured by changes either in the positive or negative direction. To answer RQ2, two respondents' written explanations were described across positive and negative effects.

Results and Discussion

Quantitative

The RT had a major influence on 48 PSTs (see Table 3). Specifically, 40 respondents switched from the heuristic use to normative conception. In contrast, a fifth of those that were positively influenced were impacted negatively by the intervention. The RT also had no effect on 18 respondents. 5 respondents were consistent in their exhibition of the normative conception at the pre and posttests (reinforcing effect). Collapsing the rows (pretest) and columns (posttest) shows the prevalence of this misconception before (81.7%) and after (36.6%) the intervention. Similarly, 18.3% and 63.4% were without the misconception before and after. Overall, I documented a statistically significant change in PSTs' responses (N=71, p<.001) associated with pre and post-test, suggesting that the RT had an effect in getting PSTs to undergo CC.

Table 3: Prevalence and Changes in the use of the Representativeness Heuristic Before and After Intervention

	All		
	0	1	T
0	18	40	58
1	8	5	13
T	26	45	71

[3] Ludo is a boardgame played by 2 or 4 people. A turn is initiated when an opponent fails to throw a double six on a pair of dice.

Qualitative Differences

In this section I provide excerpts from two respondents who made switches between S1 and S2 thinking as inferred from results of the quantitative analysis.

Table 4: Excerpts Across Positive and Negative Effects

Effects	Excerpts	
	Pre	Post
Positive	Since each flip of a fair coin is independent, the probability of getting heads remains 0.5. Therefore, you would expect to get heads approximately 50 times out of 100 flips, regardless of the outcomes of the first 10 flips.	They are equally likely to have the same result because the probability of getting head or tail in each outcome is equal.
Negative	The chances of getting heads and tails are equal because you cannot determine how many heads will come up after the total experiment but since we have 10 flips already and the chances of the remaining 90 times flipping is equal, you can say 10+½(90) = 55.	The least likely result of the tossing of a fair coin six times is getting three heads and three tails in a row as a possible outcome of an event.

The explanations from the respondent (Opebe) who made a positive switch show an understanding of randomness at the pre and post-tests. Therefore, evidence for the influence of the RT in causing Opebe to engage in the slow, systematic thinking of S2 at the posttest lies in his jettisoning of the need to balance out the outcome, causing him to conceive of all the options as equally likely. In contrast, a scrutiny of Moladun's excerpt presents unclear results even as I classified its effect negative. While she avoided the heuristic at the pretest by applying the base-rate frequency, she initially claimed that the probability was indeterminate. It is possible her answer at the post test was more of an outcome approach (see Konold, 1989) than a desire to map the sample to population. For this reason, it is not clear if the RT really had a negative effect on her because a similar reasoning that masks misconception of chance (see Tversky & Kahneman, 1974) undergirds her explanations both before and after she engaged with the RT.

Conclusions

This study set out to ascertain the effects of a RT on PSTs use of the representativeness heuristic. Based on quantitative analysis, the intervention promoted CC although posing differential effects on the respondents. Qualitative analysis showed that the intervention was mostly effective when the RT matched with the specific representativeness indicator. Owing to the demonstrated potential of the intervention, it may be valuable to enlist CC strategies into the design of instructional programs for fostering normative probabilistic ideas. Therefore, textbook authors may find the structure of RTs useful as a complement, or alternative to the existing writing style that largely involves mere presentation of normative ideas. Overall, this study offers preliminary evidence for an approach other than experiments and simulations in building probabilistic conceptions, one that may especially be worthwhile in countries where students have little access to materials to conduct such experiments.

Limitation and Future Research

Restricting the RT to a single indicator of representativeness (misconception of chance) is a major limitation of the study. For this reason, the RT better attends to the post test than the pretest whose normative reasoning owes to attending to the base-rate frequency. Future research might incorporate more indicators that would lend the RT to a broad array of chance situations.

Acknowledgments

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RAZONAMIENTO INFORMAL DE ESTUDIANTES DE BACHILLERATO SOBRE LA RECTA DE MEJOR AJUSTE

INFORMAL REASONING OF HIGH SCHOOL STUDENTS ON THE BEST-FIT LINE

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En esta investigación se analiza cómo estudiantes de bachillerato avanzan en la conceptualización de la recta de mejor ajuste usando tecnología. Veintidós estudiantes participaron en parejas en un taller con tres actividades centradas en la relación entre dos variables. Las tareas incluyeron construir y analizar diagramas de dispersión, interpolar y ajustar una recta, y manipular una recta en GeoGebra para minimizar la distancia a los puntos. Los resultados se agruparon según similitudes en las respuestas, destacando el desarrollo de una comprensión global de los datos. Se identificaron tres fases clave en la conceptualización de la recta de mejor ajuste: usar el diagrama de dispersión, interpolar una recta en la nube de puntos, y entender los residuos y minimizar la distancia entre la recta y los puntos. Se concluye que el uso de dispositivos tecnológicos permite a los estudiantes formar conceptos para concebir la recta de mejor ajuste como un agregado.

Palabras clave: Informal Education; Data Analysis and Statistics; Technology.

Introducción

La enseñanza de la estadística es crucial en la sociedad actual, donde la abundancia de datos requiere ciudadanos capaces de analizarlos y tomar decisiones informadas. La correlación y regresión son tópicos fundamentales de la estadística, y presentan desafíos educativos debido a la necesidad de articular aspectos matemáticos y estadísticos. Aunque se desarrollan en cursos universitarios, también se abordan en niveles educativos básicos y medios, con enfoques diferentes. En el nivel bachillerato, la correlación y regresión aun requiere enseñarse desde un enfoque informal por la carencia por parte de los estudiantes de las herramientas matemáticas necesarias. Para un enfoque informal los recursos tecnológicos, como CODAP y GeoGebra, ofrecen la oportunidad de apoyar a los estudiantes en dichas circunstancias para que adquieran las ideas centrales del tema.

La idea de "agregado" es muy útil en la educación estadística para interpretar el nivel de comprensión de los estudiantes de conceptos estadísticos, pero también para señalar un objetivo de aprendizaje. Algunas investigaciones revelan que los estudiantes tienden a enfocarse en detalles individuales de los datos en lugar de considerar propiedades del conjunto completo de datos (Bakker y Gravemeijer, 2004; Konold y Higgins, 2002). Esta tendencia se observa en varios conceptos estadísticos, entre ellos, el de la recta de mejor ajuste, donde al trazarla con sus recursos intuitivos los estudiantes suelen centrarse en algunos datos específicos en lugar de buscar combinar todos los datos para que estén representados por la recta resultante. Las actividades propuestas en este estudio utilizan la tecnología para propiciar que los estudiantes formen conceptos que les permitan entender como combinar los datos para dar lugar a la recta de mejor ajuste. El artículo propone actividades con CODAP y GeoGebra para responder la pregunta ¿cómo los estudiantes de bachillerato conceptualizan la recta de mejor ajuste con ayuda de la tecnología?

Antecedentes

Los estudios con estudiantes universitarios revelan interpretaciones erróneas del coeficiente de correlación (Truran, 1995) y una mejor capacidad para estimar la correlación a través de diagramas de dispersión (Sánchez-Cobo et al., 2000). Sorto et al. (2011) identifican tres estrategias intuitivas que utilizan los estudiantes universitarios (tecnología industrial) y futuros profesores para trazar la recta de mejor ajuste: 1) una que divide los puntos de datos por la mitad, 2) una que pasa por puntos medios, 3) una que une el primer y el último punto. Tales estrategias muestran que el criterio de minimizar las distancias de los puntos a la recta para obtener la recta de mejor ajuste no es intuitivo. En niveles de secundaria y bachillerato, Hourigan y Leavy (2021) hallaron que estudiantes de sexto grado a menudo tienen una comprensión local de la asociación estadística y enfrentan dificultades al crear gráficos de covariación, mientras Moritz (2004) y Watson y Moritz (2007) identifican distintos niveles de razonamiento en la interpretación de estos gráficos. En cuanto a la concepción de la correlación, Estepa y Batanero (1996) identifican visiones variadas, incluyendo visiones deterministas, locales y causales. Casey (2014, 2015) y Nagle et al. (2017) observan estrategias similares a las de Sorto et al. (2011) con estudiantes de nivel medio básico. Varios estudios utilizan tecnología para enseñar estos conceptos (Medina et al., 2019; Gil y Gibbs, 2017). Biehler et al. (2018) proponen dimensiones para diseñar entornos de aprendizaje estadístico, enfatizando el uso de datos reales y herramientas tecnológicas. Este estudio retoma la centralidad del papel del software que enfatizan dichas investigaciones para apoyar la comprensión de la recta de mejor ajuste.

Marco Conceptual

El *razonamiento covariacional estadístico* aborda la relación entre dos variables estadísticas e incluye conceptos como datos bivariados, diagramas de dispersión y correlación y la recta de mejor ajuste (Zieffler y Garfield, 2009). Este estudio se centra en la recta de mejor ajuste y otros conceptos relacionados. Los datos bivariados son pares de observaciones de dos variables numéricas diferentes (cada pareja de datos corresponde a medidas tomadas sobre una misma unidad de análisis), mientras que un diagrama de dispersión es la representación de los datos bivariados en un plano cartesiano. La covariación estadística describe cómo varían conjuntamente dos variables, con el diagrama de dispersión proporcionando una visión general. La correlación mide la relación lineal entre dos variables, mientras que la recta de regresión de mínimos cuadrados es un modelo de cómo cambia una variable de respuesta en función de una variable explicativa (Moore, 2000).

Un *enfoque informal del razonamiento covariacional* busca introducir los conceptos asociados a la regresión y correlación a un nivel de elaboración simbólica y matemática accesible para los alumnos. Se propone como una estrategia para introducir los conceptos desde niveles básicos y medios como antecedente para un tratamiento más formal en niveles universitarios. Makar y Rubin (2009) delinearon un marco para la inferencia estadística informal (IEI), que se puede aplicar de manera análoga al razonamiento covariacional informal. Este último implica juicios, razonamientos y procedimientos basados en los datos bivariados a la mano, pero con generalizaciones (ir más allá de los datos) y reconocimiento de la incertidumbre involucrada en tales generalizaciones. La evaluación de actividades se guía por esta caracterización, distinguiendo entre enfoque intuitivo e informal; el primero es espontáneo mientras que el segundo es resultado de un aprendizaje e implica procesamiento de datos para la resolución de problemas (Rossman, 2008).

La *agregación* es el proceso de combinar múltiples observaciones o medidas individuales para formar un objeto estadístico que los asimila y engloba; el objeto así obtenido es un agregado (Stigler, 2016). Aunque implica simplificación y generalización, la agregación revela tendencias y patrones generales en los datos, permitiendo hacer inferencias y afirmaciones sobre variables complejas. Desarrollar una "vista agregada" implica comprender y analizar los datos como un todo colectivo, pero esta perspectiva depende del tipo de agregado utilizado. La construcción de un agregado específico es crucial para obtener una visión global de los datos, ya que diferentes agregados revelan distintas propiedades globales del conjunto de datos.

La revolución *tecnológica* ha transformado la educación estadística, proporcionando recursos para fortalecer el razonamiento de los estudiantes en el análisis de datos (Biehler, et al. 2013). La representación dinámica de datos bivariados en diagramas de dispersión y sus medidas estadísticas facilitan el desarrollo del razonamiento covariacional informal en las aulas. La tecnología permite explorar la relación entre la forma de una nube de puntos y su coeficiente de correlación, además de investigar el efecto de variar un punto en el conjunto de datos. También permite introducir conceptos como residuos y minimización de la distancia entre una nube de puntos y una recta de ajuste de manera comprensible para los estudiantes. Plataformas como CODAP y software como GeoGebra se utilizan para crear dispositivos educativos que promueven la comprensión sin requerir a los estudiantes habilidades matemáticas avanzadas, haciendo el tema accesible y dinámico para el aprendizaje estadístico

Metodología

El presente estudio empleó un método cualitativo y exploratorio que fomenta la resolución colaborativa de problemas con tecnología y guía del profesor. El análisis de relaciones entre variables con datos bivariados es un desafío para los estudiantes al carecer de un procedimiento establecido. Optamos por datos de nutrición, relevantes para la salud y la alimentación, lo que puede aumentar la motivación y la comprensión (Garfield y Ben-Zvi, 2008; Neumann et al., 2013). La resolución colaborativa de problemas en parejas potencia el proceso, permitiendo el intercambio de ideas, negociación y expresión conjunta (Yackel et al., 1991; Webb y Mastergeorge, 2003), reflejando la dinámica del aula (Cobb et al., 2003). La tecnología, como CODAP y GeoGebra, ofrece nuevas oportunidades de aprendizaje, permitiendo manipular datos y visualizar conceptos en tiempo real (Biehler, et al. 2013). Esto facilita la comprensión de la relación entre variables y la construcción de la recta de mejor ajuste. Las actividades están diseñadas para que las parejas de estudiantes dejen evidencia de su razonamiento sobre la correlación y la recta de ajuste, con intervenciones del profesor para explicar el funcionamiento del software y aclarar la intención de la actividad, pero mínimas con respecto a la solución del problema en turno.

El estudio involucró a 22 estudiantes de bachillerato, con edades entre 16 y 18 años, dirigidos por la autora y la profesora titular. Aunque no tenían conocimientos previos sobre correlación y regresión, contaban con experiencia en el plano cartesiano y funciones lineales y cuadráticas, proporcionando una base útil para el enfoque informal del tema. Se diseñaron e implementaron dos actividades.

Los datos se recopilaron de las hojas de trabajo de los estudiantes en parejas, con la autora como administradora de los instrumentos. La primera actividad utilizó datos nutricionales de productos de McDonald's y Burger King, obtenidos de las páginas web oficiales (https://www.mcdonalds.com, https://www.bk.com/menu/search-by-nutrition). La segunda actividad se basó en un escenario del libro de texto *The Practice of Statistics* de Starnes et al. (2010), con un applet digital de GeoGebra diseñado para su aplicación. A continuación, se describe cada actividad y el análisis previo de los elementos que las componen.

La actividad 1 presenta una situación y preguntas diseñadas para explorar el razonamiento de los estudiantes sobre la relación entre variables (consulte la Figura 1). Destaca por su contexto familiar, preguntas abiertas y fomento del uso de tecnología. Los estudiantes enfrentan la tensión entre sus creencias previas y la información del diagrama de dispersión, lo que dificulta observar los datos de forma puramente cuantitativa. Se busca comprender los juicios que los estudiantes establecen sobre la relación entre variables en un contexto específico. La herramienta CODAP facilita la creación y visualización de la nube de datos bivariados.

Actividad 1: *Situación:* Quizás, oyes o creas que las grasas y las calorías son dañinas para tu salud. Es verdad, que algunas personas ingieren más grasas y más calorías de las que necesitan, pero sin ellas no tendrían suficiente energía para realizar sus actividades y crecer. ¿Qué has escuchado sobre las grasas y las calorías que contiene un alimento? Muchos alimentos contienen grasas y calorías, mucha gente cree que la grasa influye notablemente en la cantidad de calorías totales que contiene ¿Tú qué opinas? ¿Cómo podrías saber si la cantidad de grasa influye en la cantidad de calorías? A continuación, los datos de la tabla 1 muestran la cantidad de grasa en gramos y el número de calorías en algunos productos de McDonald's y Burger King.

Tabla 1. Grasa y Calorías de algunos productos de McDonalds y Burger King

	Producto	Grasa(g)	Calorías (Kcal)
McDonald' s	Big Mac	27	540
	Cheeseburger	12	300
	Quarter Pounder with Cheese	26	510
	McDouble	18	390
	Double Quarter Pounder with Cheese	45	780
	Bacon Ranch Grilled Chicken Salad	13	300
Burger King	Hamburger	10	240
	Cheeseburger	16	320
	Bacon y Cheese WHOPPER	51	790
	DOUBLE WHOPPER	58	900
	Double Pretzel Bacon King	60	920
	Chicken Club Salad-Grilled	41	610

a) Observa los datos de la tabla 1 ¿Cuál crees que es la relación entre los gramos de grasa y las calorías? Explica tu respuesta.

Utilizando tecnología digital: Abrir la plataforma digital *CODAP* y crear el diagrama de dispersión de los datos bivariados (grasa, calorías) presentados en la tabla 1. Sigue las indicaciones del profesor.

b) ¿Qué características se podrían tener en cuenta al interpretar un diagrama de dispersión?
c) ¿Qué puedes decir de la relación entre los gramos de grasa y el número de calorías?

Figura 1: Actividad 1

La actividad 2 presenta una situación y preguntas centradas en el razonamiento sobre la recta de mejor ajuste en un contexto de nutrición (consulte la Figura 2). Las primeras preguntas de la actividad 2 buscan que los estudiantes establezcan la idea de informar que una relación se dice que es lineal si los datos en la nube de puntos se conglomeran formando una línea. Se utiliza un applet de GeoGebra para que los estudiantes comprendan las ideas detrás de la regresión lineal, enfocándose en la optimización de distancias.

Actividad 2: *Situación.* Un estudio investigó por qué algunas personas no aumentan de peso incluso cuando comen en exceso. Los investigadores se preguntaron lo siguiente: ¿los cambios en la inquietud y otras actividades no relacionadas con el ejercicio explican el aumento o disminución de peso en las personas que comen en exceso? Deliberadamente sobrealimentaron a 12 adultos jóvenes sanos durante 8 semanas y midieron el aumento de grasa y el cambio en el uso de energía a partir de actividades no relacionadas con el ejercicio (ANE). Midieron el aumento de grasa (en kilogramos) y el cambio en el uso de energía (en calorías) a partir de otra actividad no relacionada con el ejercicio ya sea inquietud, vida diaria y cosas por el estilo (tabla 2).

Tabla 2. Medidas de Cambio en ANE y Aumento de Grasa en los 12 Adultos Jóvenes

Cambio ANE (cal)	-94	-57	-29	135	143	245	355	486	535	571	620	690
Aumento de Grasa (Kg)	4.2	3	3.7	2.7	3.2	2.4	1.3	1.6	2.2	1	2.3	1.1

a) Discute con tu compañero y describe la relación entre el cambio ANE y el aumento de grasa.
b) Realiza el diagrama de dispersión en CODAP y describe el comportamiento de la nube de puntos.

Abre el archivo de GeoGebra (el estudiante visualiza la figura B). Lee cuidadosamente lo siguiente: El archivo muestra el diagrama de dispersión de los datos (puntos azules) de la tabla 2 junto con una línea verde la cual puedes mover con los símbolos en rojo ◆ . El segmento punteado de color rojo muestra la distancia que va desde el punto azul a la recta verde, su valor corresponde a la diferencia entre la ordenada del punto azul (y) y la ordenada del punto que pertenece a la recta verde (ŷ), a este valor lo llamaremos residual, y la "Distancia de los puntos a la recta" es la suma de todos los residuales.

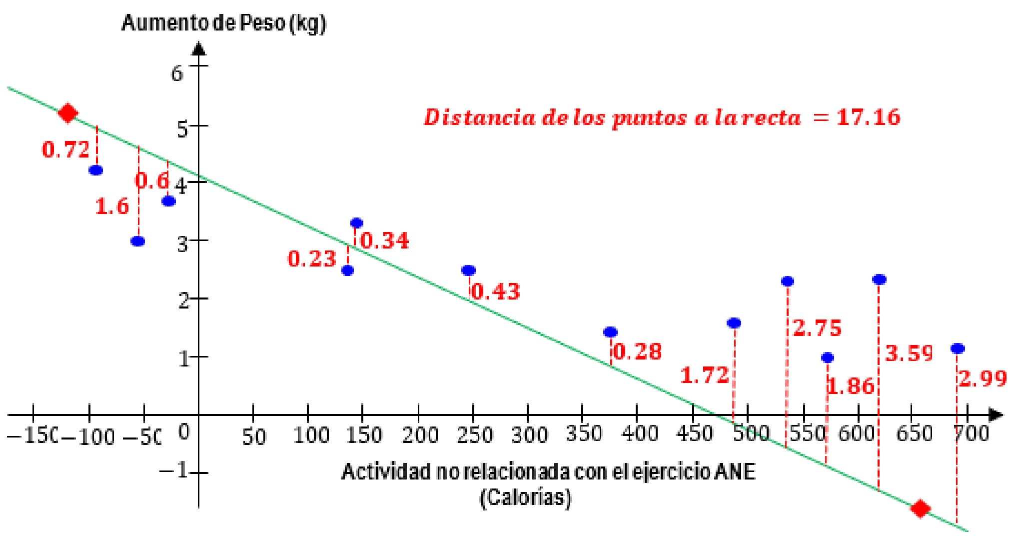


Figura B. Captura de Pantalla de la Vista Gráfica de la Tarea en GeoGebra

Mueve la línea verde, explora lo que ocurre con el valor de cada residual y responde:

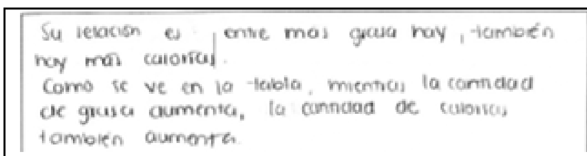
c) ¿Dónde colocarías una recta para que corresponda a la recta que mejor se ajuste a los puntos? Explica el criterio que utilizaste.
d) ¿Qué ocurre con el valor de cada residual si mueves la recta cerca o lejos de la nube de puntos?
e) ¿La manera como ubicaste la recta movible es igual a la recta arrojada por GeoGebra? ¿Sí? ¿No? ¿En qué se diferencian? ¿Cuál crees es el criterio que utiliza GeoGebra para determinar la recta de mejor ajuste?

Figura 2: Actividad 2

Conviene notar que se evita el uso del concepto de distancia cuadrática para no complicar la construcción de la medida de distancia de un conjunto de datos a una recta. El applet de GeoGebra permite visualizar una recta movible y calcular automáticamente la distancia de la nube de puntos a la recta. La función definida muestra el error o residual de un punto como la distancia vertical entre el punto y la recta, se define la distancia de la nube de puntos a la recta como la suma de los residuales. Se compararon las respuestas de parejas de estudiantes para identificar patrones de respuesta, destacando los rasgos comunes en los procedimientos o razonamientos similares. La codificación de estos patrones simplifica la descripción y análisis de las respuestas.

Resultados y Discusión

Este análisis ofrece una visión general de las respuestas grupales y analiza sus procesos de razonamiento, complementándolos con comentarios analíticos para una mejor comprensión. En la actividad 1, los estudiantes abordan la relación entre gramos de grasa y calorías en dos enfoques: *global-cualitativo* y *particular-cuantitativo*. El primero evalúa la tendencia general de los datos y su descripción cualitativa (ver Figura 3-a); el segundo busca expresar los cambios específicos en cada dato a través de secuencias numéricas o funciones, aunque sin éxito (ver Figura 3-b). Ocho parejas ofrecen una valoración cualitativa, mientras que tres intentan un análisis cuantitativo.



"su relación es que entre más grasa hay también más calorías. Como se ve en la tabla mientras la cantidad de grasa aumenta la cantidad de calorías también aumenta"

(a)

"…en 5 productos calculamos la pendiente de las calorías respecto a las grasas y llegamos a la conclusión que por [cada] 1 g de grasa hay 20.7 Kcal.

$$m = \frac{x}{y} = \frac{540}{57} = 20$$

$$m = \frac{y}{x} = \frac{300}{12} = 25"$$

(b)

Figura 3: Ejemplo de respuestas que involucra juzgar la relación entre las dos variables en la actividad 1

Utilizando CODAP, las respuestas se dividen en *casi-linealidad* e *irregularidad* en la distribución de datos (ver Figura 4). El uso del diagrama de dispersión facilita la percepción de una tendencia general y la irregularidad. Aunque los estudiantes prefiguran el agregado, no saben cómo combinar los datos para construirlo. En resumen, el diagrama de dispersión y la tarea de ajustar una recta permiten a los estudiantes pensar en combinaciones de puntos para construir un agregado, aunque requieren un método más preciso.

En la Actividad 2, se repiten preguntas similares a la relación entre las variables como en la Actividad 1. Cinco parejas describen una tendencia global entre cambio ANE (actividades no relacionadas con el ejercicio) y grasa, mientras tres reconocen una dependencia sin establecer una regla definida y tres basan sus respuestas en un modelo contextual sin obtener conclusiones de los datos. En la pregunta (b), ocho parejas identifican una relación inversa entre cambio ANE y grasa, mientras tres se centran en el contexto sin datos concretos.

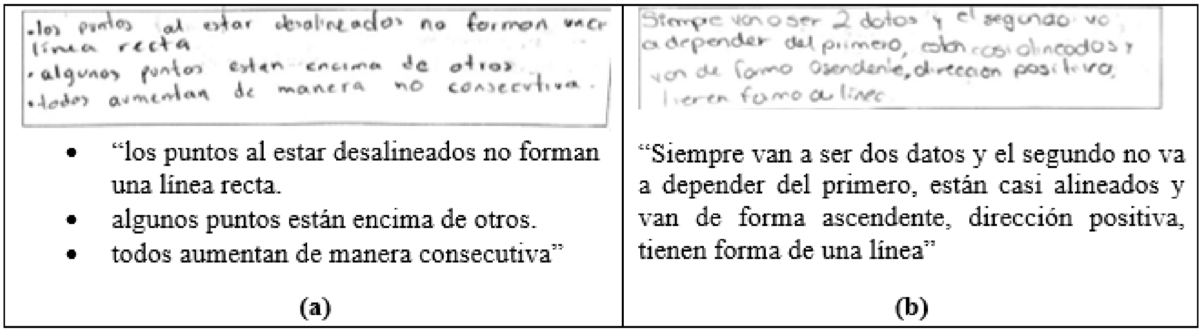


Figura 4: Ejemplo de respuestas al analizar la relación en el diagrama de dispersión en la actividad 1

La diferencia con la actividad 1 es que ahora la tendencia es negativa y la variable cambio ANE es menos familiar para los estudiantes. Probablemente por esto, los resultados no son mejores que en la actividad 1, además de que hubo respuestas basadas en creencias previas sobre el contexto. El objetivo de esta parte es introducir a los estudiantes al problema y proponerles utilizar GeoGebra para centrarse en la idea de distancia de un conjunto de puntos a una recta y minimizarla. Por lo tanto, se presenta la actividad donde los estudiantes utilizan el programa GeoGebra para representar los datos del problema y una recta móvil que permite observar los residuos y su suma. La pregunta (a) es similar a la actividad anterior, pero ahora los estudiantes pueden mover la recta preconstruida en la pantalla y observar las distancias verticales de cada punto a la recta, así como la suma de estos residuos. Las respuestas de los estudiantes se clasifican en tres categorías: "Puntos en la recta", "Cercanía" y "Minimización de residuos". Las respuestas varían desde asociar la recta de mejor ajuste a la minimización de la distancia entre la recta y los puntos hasta simplemente medir la cercanía visualmente sin usar los recursos del software (ver Figura 5-a). Algunos estudiantes basan sus respuestas en la minimización de los residuos, relacionándola con la distancia de los puntos a la recta de ajuste (ver Figura 5-b).

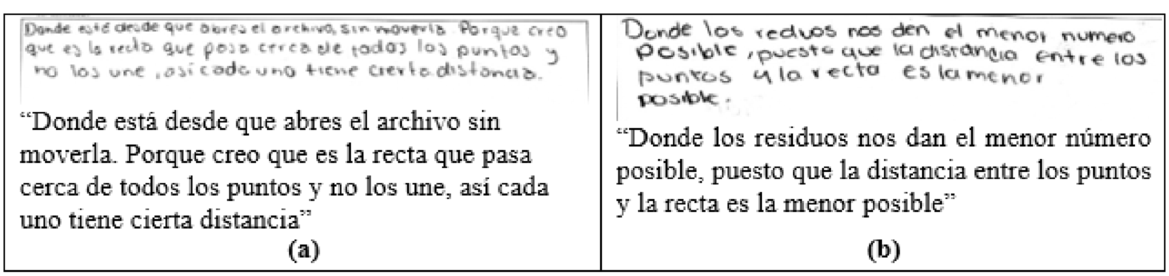


Figura 5: Ejemplos de respuestas del criterio para la recta de mejor ajuste en la actividad 2

Varios estudiantes muestran dificultad para entender la construcción del agregado al buscar minimizar cada residuo de manera particular y asumir que esto resulta en la distancia mínima de los puntos a la recta; esto no funciona, pues cuando un residuo se reduce otro aumenta. Es significativo que no encuentren natural minimizar la distancia (total) y entender que esto produce una distribución óptima de residuos. Ellos pretenden controlar los residuos particulares para minimizar la distancia global, pero no comprenden que este proceso funciona en sentido inverso. No obstante, más tarde, con la intervención oportuna del profesor conseguirán comprenderlo mediante preguntas que los invitan al descubrimiento y reflexión cómo: ¿Qué pasaría si hacemos que un residuo sea cero? ¿Creen que la distancia global sería mínima? Por favor, hagan que un residuo sea cero ¿Qué pasa? Una vez que se dan cuenta que no se logra minimizar ¿Qué pasa si la suma global de los residuos es mínima?

En la pregunta (b), se busca que los estudiantes reflexionen sobre la relación entre la distancia de la recta a los puntos y el valor de cada residuo. Las respuestas se dividen en tres grupos: descriptivo, enfoque en lo particular y la recta centrada. Algunos estudiantes comprenden que al minimizar los residuos se minimiza la distancia de la nube de puntos a la recta, mientras que otros consideran que la posición central de la recta resulta en una menor suma de residuos. En la pregunta final, los estudiantes comparan la recta móvil con la recta de regresión obtenida por GeoGebra, destacando diferencias en la posición relativa de las rectas y en las distancias a los puntos. Se observa que las respuestas demuestran una comprensión de las propiedades generales de las rectas de mejor ajuste y muestran un avance hacia una vista agregada de las mismas. Sin embargo, no se aborda el criterio específico que utiliza GeoGebra para determinar la recta de mejor ajuste, es decir, el método de mínimos cuadrados. La actividad resulta difícil para los estudiantes porque deben coordinar dos variables para obtener la recta que minimiza la distancia. Las operaciones realizadas en GeoGebra corresponden formalmente a la optimización de una función de dos variables, lo cual se logra con precisión mediante técnicas de derivadas parciales. Es importante que el profesor comprenda la relación entre el enfoque informal con GeoGebra y el enfoque formal para abordar las dificultades de los estudiantes.

Conclusiones

La pregunta que impulsó esta investigación fue ¿cómo conceptualizan los estudiantes de bachillerato la recta de mejor ajuste con ayuda de la tecnología? Nuestra respuesta es que el uso de dispositivos tecnológicos permite a los estudiantes formar conceptos para concebir la recta de mejor ajuste como un agregado. El uso del diagrama de dispersión, a diferencia de la simple presentación de datos en una tabla, les permitió percibir tanto la tendencia general de los datos como la irregularidad entre los puntos. La tarea de interpolar una recta enfrenta a los estudiantes a estrategias diferentes para fijar una recta representativa, aunque estas siguen siendo intuitivas y visuales. La introducción de los conceptos de residuo y distancia de una recta a la nube de puntos, mediante el programa GeoGebra, permitió a los estudiantes operacionalizar su idea intuitiva de cercanía de una recta a los puntos y entender cómo se combinan los datos para dar lugar a la recta de mejor ajuste.

En resumen, los estudiantes conceptualizan transitando por experiencias que incluyen nociones de irregularidad, tendencia de la nube de puntos, residuos, distancia de una recta a un conjunto de puntos y minimización de dicha distancia, facilitadas por el uso de la tecnología. Aunque responden a diferentes niveles, la tendencia es captar la recta de mejor ajuste como un agregado. Además, este trabajo propone una trayectoria que ayuda a los estudiantes a comprender los conceptos que les permiten construir la recta de mejor ajuste, especialmente la idea de distancia basada en el concepto de residuo, facilitando así una vista agregada del conjunto de datos en relación con la recta de mejor ajuste. Queda pendiente desarrollar las componentes de variabilidad e inferencia en relación con la recta de mejor ajuste y explorar en futuras investigaciones una actividad similar, pero con distancias cuadráticas en lugar de lineales.

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TEACHERS' KNOWLEDGE OF STUDENTS' REASONING: THE CASE OF SIMULATED SAMPLING DISTRIBUTIONS

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Short-term professional development courses with specific domain learning outcomes are useful to revise concepts that have changed and inform in-service teachers how they can exploit the affordances of technology to enhance their classroom practices (Ling, 2014). We investigate how teachers develop knowledge of students' reasoning about sampling distributions within a 20-hour on-line course. Our research question is: how teachers' understanding of student reasoning through simulated sampling distributions emerges?

Conceptual framework

Informal Approach to Significance Tests for Proportions

We utilize simulated sampling distributions (SSD) to model scenarios assuming the null hypothesis is true, thereby estimating p-values without complex formulas. This approach is feasible for high school students without advanced mathematical tools (Case & Jacobbe, 2018).

Students' Understanding of SSD

High school students' conceptions evolve from interpreting SSD as real samples to viewing SSD as a pattern of variation. This is an abstraction process (Sepúlveda & Sánchez, 2023).

Methodology and Results

The research was conducted with 16 experienced high school teachers during a 20-hour on-line professional development course. Teachers analyzed typical student responses to significance testing problems to interpret their reasoning. Here we compare two activities, both designed to improve teachers' ability to understand student reasoning in the context of SSD.

Problem 1: Teachers analyzed a student response where students misinterpreted SSD as representing real samples and based their conclusions on frequency without considering uncertainty. The analysis revealed that teachers' initial responses were mostly descriptive and evaluative, lacking deeper inference into students' reasoning.

Problem 3: Teachers assessed a response where students correctly used SSD to estimate the p-value but made an incorrect decision about repairing a machine. Here, teachers showed progress by making more detailed inferences about student reasoning, though still struggled to fully explain students' incorrect decisions.

Discussion and Conclusions

The study shows that teachers' ability to interpret students' reasoning improves through targeted professional development. Initially, teachers focused on what was absent in student answers. However, with guided activities, they began to attend more to students' reasoning. This progress is attributed to the design of the course, which emphasized interpreting students' implicit judgments (Brandom, 2000). Two main lessons emerged: a) Interpreting students' reasoning with SSD is challenging for teachers due to a tendency to prioritize their own content knowledge because of their lack of experience with this approach. b) Teachers can improve their ability to interpret student reasoning, but more extensive effort is needed to develop this habit.

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DIRT DON'T HURT: HOW RELEVANT SOIL DATA CAN SUPPORT LEARNING AND MOTIVATION AT A HISPANIC SERVING INSTITUTION

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To create opportunities for meaningful applications of data-science for diverse students, we developed and implemented an online learning module focused on engaging students at a Hispanic Serving Institution (HSI) in an analysis of authentic soil data. Development of the module occurred over three design iterations involving interviews with 10 undergraduate STEM students. We then implemented the finalized module in three undergraduate microbiology classrooms (N=118) using a pretest, posttest, comparison group quasi-experimental study design. Findings revealed that, after adjusting for key variables, the intervention group demonstrated significantly greater microbiology knowledge than the comparison group. Path analyses revealed indirect effects of the intervention through value and interest in STEM.

In today's world, data is ubiquitous and data literacy is essential across industries (Börner et al., 2019). However, traditional data science courses are not meeting students' needs (Baumer, 2015), and racial diversity in STEM is lacking (Cruz et al., 2018; Fry et al., 2021; NSF, 2015). This project aimed to enhance STEM learning among Hispanic students by leveraging their cultural resources (Gonzalez et al., 1995; Wilson-Lopez et al., 2016). We developed an online module on soil microbiology, utilizing data science tools to engage students at a Hispanic Serving Institution (HSI) and assess their impact on STEM learning and motivation.

Theoretical Framework

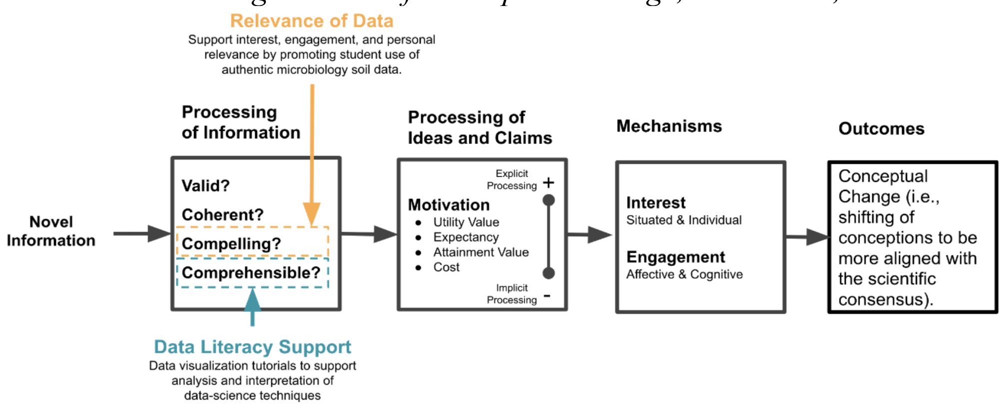
To frame how data visualizations can support science learning for Hispanic students, we integrate theories of Conceptual Change, Data Visualization Literacy, and Expectancy Value. *Conceptual change* theory posits that presenting people with novel information can shift their conceptions about science topics to be more aligned with the scientific consensus (Dole & Sinatra, 1998). For example, the Plausibility Judgments for Conceptual Change model (PJCC; Lombardi et al., 2016) suggests that people process information better if it is *comprehensible*, *coherent* with prior experiences, *compelling*, *relevant*, and stems from *credible* sources. People then judge the plausibility of associated claims and restructure knowledge as a result. Plausibility judgments involve more explicit processing depending on learners' motivation, engagement, and emotion, which then predicts the likelihood that conceptual change will occur.

The *Expectancy Value Theory* (EVT) helps frame motivational factors, and proposes that it is driven by learners' expectancy for success and task value (Eccles et al., 1983; Wigfield et al., 2017). According to EVT, there are four different types of task value: *intrinsic value* is when a learner values a task because they find the activity enjoyable for its own sake, *attainment value* is perceived personal importance of a task as it relates to one's identity, *utility value* refers to perceptions that a task may be useful to a learner to achieve their present or future goals, and *cost* is the extent of time and effort that is perceived to complete a task. Utility value interventions—especially those that are personal and relevant to students—can significantly enhance value, interest, and learning, particularly for underrepresented students (Harackiewicz et al., 2016).

The *Data Visualization Literacy framework* (DVL-FW; Börner et al., 2019) suggests that data visualizations can help ground abstract concepts. Accordingly, a central process required to interpret data from visualizations is *translating* relevant problems of interest into problems of data. As such, our project aimed to leverage students' knowledge to contextualize data.

We synthesized these theories in a process model (see Figure 1) to inform our intervention development and that we also tested using path model analyses. The model predicts that an intervention intended to expose students to compelling and comprehensible microbiology data would trigger motivational processes (i.e., improve utility value, expectancy, attainment value, and reduced cost), which would activate mechanism variables (student interest and engagement), which would predict increased learning outcomes (i.e., microbiology knowledge). Furthermore, we sought to facilitate change by centering learning around compelling topics that students found relevant, and supporting data literacy to improve data comprehensibility.

Figure 1. *Intervention Logic Model of Conceptual Change, Motivation, and Data Literacy*



To test this theoretical model, we designed and tested an interdisciplinary learning experience for undergraduate students at an HSI. We addressed the following research questions:

- **RQ1.** How can a learning intervention be developed to leverage undergraduate students' motivation for the learning of soil microbiology and data literacy skills?
- **RQ2.** To what extent will such an intervention support students' microbiology knowledge, data literacy skills, engagement, interest, and task value in STEM?
- **RQ3.** Will the hypothesized relationships between task value processes and achievement outcomes be mediated by mechanisms of interest and engagement? (See Figure 1)

This project addressed these research questions through two studies. The first was a formative study focused on creating an online module and the second was a comparative study testing the effectiveness of this intervention using quasi-experimental research.

Study 1: A Design-Based Research Study

To answer the first research question, we used a design-based research (DBR) approach to guide the development and revision of an interactive online intervention (Hoadley & Campos, 2022) on soil microbiology. The re-design, implementation, and revisions occurred over several iterations, resulting in an open-source module for undergraduate microbiology students. The module introduces students to the Tiny Earth Initiative (Hurley et al., 2021), which focuses on identifying new antibiotics in soil. It includes information on the antibiotic resistance crisis, microbial ecology, data visualization tools, and interpretation of soil data visualizations.

We conducted 10 recorded cognitive interviews (Desimone & Le Floch, 2004) via Zoom with a convenience sample of undergraduate students at an HSI in Summer and Fall 2023. Students self-identified as Female (70%), Male (20%), Nonbinary (10%), Hispanic (50%), White (40%), Black (10%), Asian (20%), and English Learners (30%). Interviews focused on student feedback to guide revisions of the module and surveys. Zoom recordings were transcribed and open-coded (Corbin & Strauss, 1990) to examine student engagement and learning.

RQ1 Results: A Module for Soil Microbiology Data Exploration. Revisions focused on improving interactivity, engagement, visual appeal, and ease of data and text interpretation, especially for English learners. The finalized intervention and surveys can be accessed using this link: https://www.softchalkcloud.com/lesson/serve/34E2zGcmQxaWZl/html. Ultimately, this version of the module was used in the second phase of the study.

Study 2: A Quasi-Experimental Study Testing Effects of the Design

To answer the second research question, we recruited 118 undergraduate students from an HSI in a southern U.S. state. Students reported their year of study (1% first year, 13% second year, 38% third year, 38% fourth year, 10% other), gender (76% Female, 21% Male, 1.7% Nonbinary, 1.7% prefer not to say), ethnicity (56% Hispanic), race (1% American Indian/Alaska Native, 13% Asian, 6% Black/African-American, 10% Two or more races, 58% White/Caucasian, 11% Other race), and whether they were enrolled in a STEM major (78% STEM major, 15% not STEM, 3% plan to enroll in a STEM major, 4% Other).

The intervention group consisted of 101 students from two undergraduate microbiology courses and the comparison group consisted of 17 students from a separate course. All participants first completed a 12-item pretest questionnaire on microbiology knowledge and an item that captured students' perceptions of the relevance of data science.

After the pretest, learners either completed the module (treatment group) or continued with "business as usual" (comparison group). All participants then completed an identical post-test of microbiology knowledge and data literacy. Participants also completed a microbiology-specific interest scale (adapted from Hulleman et al., 2010), a Data-Science-specific interest scale (adapted from Hulleman et al., 2010), the Cognitive Engagement scale (Greene, 2015), and a Task Value scale (Kossovich et al., 2015). Internal reliability for all scales at pretest and posttest were judged using Cronbach's alpha, and are reported in Table 1.

Table 1. Descriptive Statistics By Condition and Intercorrelations Between Key Variables

	Total								Control			Treatment		
	items	α	n	Mean	SD	Min	Med	Max	n	Mean	SD	n	Mean	SD
Knowledge (pre)	12	.62	117	0.7	0.2	0.2	0.7	1.0	16	0.5	0.2	101	0.7	0.2
Knowledge (post)	12	.72	117	0.7	0.2	0.2	0.8	1.0	16	0.4	0.2	101	0.8	0.2
Data Relevance (pre)	1	NA	116	4.0	1.0	1.0	4.0	5.0	16	3.6	1.0	100	4.1	0.9
Data Relevance (post)	1	NA	116	4.2	1.0	1.0	5.0	5.0	16	3.4	1.2	100	4.4	0.9
Initial Interest	8	.96	117	5.5	0.8	1.8	5.5	6.8	16	5.1	0.8	101	5.5	0.8
Situated Interest	5	.92	117	5.9	1.1	1.0	6.2	7.0	16	5.2	1.3	101	6.0	1.0
Utility Value	4	.91	117	5.8	1.1	1.0	6.0	7.0	16	5.3	1.3	101	5.9	1.1
Data Science Situated Interest	3	.77	117	4.5	1.3	1.0	4.7	7.0	16	4.7	1.1	101	4.5	1.3
Data Science Utility Value	3	.73	117	5.0	1.0	1.0	5.0	7.0	16	4.9	1.1	101	5.0	1.0
Cognitive Engagement	16	.88	117	3.9	0.6	2.4	4.0	5.0	16	3.7	0.7	101	4.0	0.6
Expectancy	3	.95	117	5.9	1.1	1.0	6.0	7.0	16	5.5	1.3	101	5.9	1.1
Attainment Value	3	.95	117	6.0	1.0	1.7	6.0	7.0	16	5.5	1.2	101	6.1	0.9
Cost	4	.86	117	3.4	1.4	1.0	3.5	7.0	16	4.2	1.5	101	3.3	1.3
Affective Engagement	9	.83	117	3.4	0.6	1.8	3.4	4.9	16	3.1	0.5	101	3.4	0.6

	Intercorrelations												
	k.pre	k.post	dr.pre	dr.post	in.int	sit.int	uv	ds.sit	ds.uv	cog	exp	value	cost
Knowledge (pre)													
Knowledge (post)	.62***												
Data Relevance (pre)	.37***	.28**											
Data Relevance (post)	.30**	.44***	.55***										
Initial Interest	.28**	.38***	.31***	.33***									
Situated Interest	.30***	.33***	.39***	.45***	.88***								
Utility Value	.34***	.33***	.36***	.35***	.67***	.71***							
Data Science Situated Interest	.07	.11	.21*	.12	.41***	.44***	.42***						
Data Science Utility Value	.18*	.24**	.26**	.28**	.47***	.44***	.51***	.62***					
Cognitive Engagement	.14	.23*	.30***	.34***	.63***	.61***	.59***	.27**	.29**				
Expectancy	.17	.25**	.28**	.36***	.80***	.80***	.59***	.34***	.40***	.68***			
Attainment Value	.31***	.38***	.37***	.41***	.82***	.83***	.76***	.35***	.49***	.67***	.82***		
Cost	-.29**	-.27**	-.27**	-.17	-.39***	-.49***	-.25**	-.29**	-.15	-.30**	-.48***	-.38***	
Affective Engagement	.06	.12	.26**	.35***	.40***	.48***	.30***	.45***	.36***	.47***	.43***	.36***	-.22*

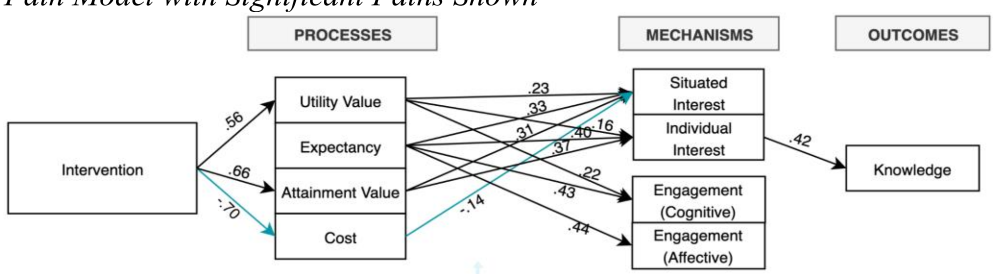
Note: *$p < .05$; **$p < .01$; ***$p < .001$

We assessed if baseline measures differed by condition. Chi-Squared analyses showed gender, ethnicity, race, English speaking status, and STEM status were independent of condition (all $p > .117$). However, year of study ($p < .001$) and pretest knowledge ($p < .001$) were significantly lower in the control group. Thus, we included these as covariates in all analyses. Raw means, standard deviations by condition, and intercorrelations are in Table 1.

RQ2 Results: Module Effects on Learning & Motivation. To assess the module's effects on knowledge, data literacy, engagement, interest, and value in STEM, we used multiple regression with robust standard errors. Predictors included the treatment condition, pre-test scores, and covariates. We predicted the module would improve outcomes due to greater comprehensibility, compellingness, and engaging information (Dole & Sinatra, 1998; Lombardi et al., 2016). Findings showed significant effects on posttest knowledge ($\beta = 1.67$, $p < .001$) and data science relevance before adjusting for pretest knowledge and year ($\beta = .66$, $p = .012$). The intervention promoted situated interest ($\beta = 0.73$, $p = .014$), value of science ($\beta = 0.66$, $p = .029$), and reduced perceptions of cost ($\beta = -0.70$, $p = .013$), but effects on initial interest and utility value were marginal. No significant moderation effects of gender or ethnicity interactions were found. A summary of the standardized regression coefficients, standard errors, and p-values for analyses can be found in the Supplemental Materials, Table S1 and Table S2.

RQ3 Results: Path Analysis. We tested a model predicting process variables (utility value, expectancy, attainment value, cost), followed by mechanism variables (STEM interest), engagement (cognitive, affective), and academic outcomes (microbiology knowledge). Pretest knowledge and year of study were covariates. The model had satisfactory fit (RMSEA=.081, SRMR=.073, CFI=.982, TLI=.943, AIC=2209, Chi-Square=35, df=20; Hu & Bentler, 1999).

Figure 3. *Path Model with Significant Paths Shown*



Note. Only paths that are significant at the .05 level are shown, blue paths are used when coefficients are negative. All variables shown represent values at posttest. All coefficients represent standardized βs.

Figure 3 shows the full path model with all coefficients. The intervention influenced motivation processes, significantly affecting reported utility value, attainment value, and cost. These motivational process variables were associated with mechanism variables: utility value was positively associated with situated interest, individual interest, and cognitive engagement; expectancy positively predicted situated interest, initial interest, cognitive and affective engagement; attainment value positively predicted situated and initial interest; cost negatively predicted situated interest. Of the mechanism variables only individual interest significantly predicted microbiology knowledge.

Significance

We aimed to develop and test an online learning module for microbiology students at an HSI to enhance STEM learning and motivation. Students using the module had greater posttest knowledge, perceived data-science relevance, science interest, situated interest, utility value, decreased perceptions of cost compared with a comparison group. Findings support prior research showing that interventions supporting perceptions of utility can enhance motivation and achievement in STEM. We also tested predictions by Hulleman and Harackiewicz (2021), finding that value perceptions significantly predicted psychological mechanisms, which predicted achievement outcomes. The intervention had marginally significant indirect effects on achievement through attainment value and individual interest, indicating that interest and engagement may be important mechanisms for expectancy and value.

In summary, this study supports a long-term agenda focused on interdisciplinary STEM applications for Hispanic students, contributing to theory and practice by testing conceptual change models, exploring mathematical reasoning in science learning, and producing a shareable intervention for science instructors and the public.

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STATISTICAL LITERACY OF UNDERGRADUATE STUDENTS IN MEDIA CONTEXTS

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Statistical and mathematical products in the news take on many forms, requiring inter-related skills to appropriately comprehend and evaluate (Gal & Geiger, 2022). This study used task-based interviews to investigate how students in an introductory statistics course utilizing a simulation-based curriculum apply statistical concepts when critiquing statistics in the news media. Analysis identified students wanting more information, using intuition over statistical thinking, and challenges in connecting statistical ideas to critical questioning. Findings support the inclusion of media items in discussions and activities in introductory statistics courses.

Keywords: Data Analysis and Statistics, Undergraduate Education

With the pervasiveness of data in everyday life and ongoing uncertainties about the quality of information found in the media, statistics educators are concerned about how to prepare students to engage with data in meaningful ways (Burrill & Pfannkuch, 2023). Simulation-based curricula (SBC) have grown in popularity in introductory statistics courses (Tintle et al., 2015). Example applications in an introductory course include generating confidence intervals from bootstrap distributions and simulating null distributions to calculate p-values for hypothesis tests instead of using theoretical distributions (Lock et al., 2021). Advocates of simulation-based methods assert the methods allow students to see the logic and power of statistical interference (Tintle et al., 2015). Researchers identified consistent improvements in students' statistical literacy when they engage with SBC compared to engaging in more theory-based curricula (e.g. Chance et al, 2022; Maurer & Lock, 2016).

While research calls for increased use of SBC in introductory statistics courses (Tintle et al., 2015), research is less clear about how simulation-based methods help prepare students to engage with statistics in the increasingly complex media environment. This study brings together research on statistical literacy, SBC, and statistics in media to understand how students in introductory statistics courses engage with statistical information in media settings. Specifically, this study addresses the research question: *what statistical topics do students in an introductory statistics course utilizing a simulation-based curriculum apply when critiquing statistics in the news media and at what level of statistical literacy?*

Methods

This study adopted Watson and Callingham's (2003) construct of statistical literacy, grounding statistical literacy in the need for understanding, interpreting, and communicating in real-word contexts such as the news media. Watson and Callingham developed a six-level statistical literacy construct. Each level is characterized by the types of task steps students take while engaging with statistical tasks. Progressing to higher levels requires more detailed use of statistical terminology and topics in connection with deeper engagement with the task's context. Levels 1 and 2 involve mostly non-statistical thinking, instead relying on personal experiences and intuition. Levels 3 and 4 utilize statistical terminology and arguments with increasing amounts of justification and connections to the task's context. Level 5 and 6 incorporate critical questioning and engagement concerning the data and conclusions.

Five undergraduate students (one male, four female), representing diverse academic programs, participated in one, 25 to 45-minute task-based interview. Participants were enrolled in an introductory statistics course utilizing materials which incorporated simulation-based methods (Lock et al., 2021). Each participant completed three tasks; each task had two parts. First, participants received a mock news headline and were prompted to consider how someone might arrive at the stated conclusion and discuss any questions or concerns. Headlines were provided separately because previous research suggested headlines are particularly important in communicating news and can influence the reading of an article (Adams et al., 2017). Following discussion of the headline, participants received a short article containing information about the statistics and evidence behind the claim made in the headline. Participants evaluated the statistics presented and considered questions or concerns they had.

Designed tasks focused on statistical topics prevalent in the news (e.g., sampling and data collection) and topics related to simulation-based methods (e.g., variation, p-values, and hypothesis tests) (Chance et al., 2022). Each task incorporated multiple categories of statistical and mathematical products (StaMPs) in the news identified by Gal and Geiger (2022) to ensure tasks represented a range of statistical demands present in the media. Tasks incorporated five of the nine categories of StaMPs: (a) descriptive quantitative information; (b) models, predications, causality, and risk; (c) data quality and strength of evidence; (d) demographics and comparative thinking; and (e) critical demands.

All interviews were transcribed and coded by the researcher. First-cycle coding involved a combination of descriptive coding to identify statistical topics, In Vivo coding to highlight participant thinking, and magnitude coding to assign a level on Watson and Callingham's (2003) construct of statistical literacy. Second-cycle coding first identified ideas for each participant and then used pattern coding for cross-case analysis to develop major themes.

Findings

Participants (all names pseudonyms) mentioned the statistical topics of sampling, study design, and hypothesis testing at various levels of statistical literacy across the three tasks. Within the topics, additional themes included: wanting more information, relying on non-statistical arguments, and struggling to connect statistical ideas to critical questions. Assigned statistical literacy levels are included in parentheses following any quotes.

Participants frequently noted the lack of data, details, and numbers in the headlines and articles. For example, in the Blue Light task, which described two experiments with conflicting results, students had trouble evaluating the article's claim with limited information. Requests for more details often focused on a lack of specific numbers, such as when Erich stated, "because the only statistic here is 'over 200 participants,' they've come to no conclusion" (1). Participants did not always recognize or acknowledge non-numerical information about study design and data collection as relevant to evaluating the statistical claims. Furthermore, participants rarely indicated the information they wanted. For example, in the Birth Rates task, Violet noted "I think they just have lack of data" (2). When asked about what additional data would be helpful, Violet responded "I think… probably more specificities like… I don't know" (2).

Participants' initial reactions to the headlines often drew on intuition and personal experience. For example, Emma said "I wear blue light glasses when I'm studying" (1) before sharing she thought they helped her. In most instances, these initial statements about the headline aligned with participants' subsequent task responses after reading the article. For the GPA task, several participants were willing to accept the results being reported without engaging in statistical thinking. All participants demonstrated higher levels of statistical literacy when prompted by task questions and/or researcher but connections to initial reactions continued.

Participants relied on their intuition more than statistical reasoning when evaluating the difference in GPAs between the two groups reported in the GPA task. Many quickly wrote off this difference as not significant without statistical justification (3), such as when Alex said, "point two is like one better exam basically" (3). Other participants used the word *significant* more colloquially as opposed to in a statistical way, even after being asked what significance meant to them. No participants questioned how much variation existed in the measures or connected significance to other conversations around hypothesis tests. Since participants did not find the difference to be meaningful to them, they appeared less critical of the results.

While participants rarely used explicit statistical terminology, each participant touched on the topics of causation and confounding variables by offering alternative explanations for the reported relationships. For example, in response to the GPA task's claim that breakfast improves student GPA, Claire pondered "does this have to do with when people actually wake up? Does this have to do with the nutritional aspect of actually eating breakfast… or just the internal factor of having the motivation to do things" (3). Alex (5) and Erich (5) demonstrated higher levels of statistical literacy on this topic by connecting their concerns about causation to the design of the study. Although participants appropriately questioned the causal claim made in the GPA task, participants used similar strategies for the Birth Rates and Blue Light tasks even when no causal claims were made or the study utilized a randomized controlled design.

Without prompting, participants rarely commented on specific methods of statistical analysis such as conducting a hypothesis test. When asked how the conclusions in the headlines could be determined, participants offered comments about data collection such as "take a regular person without blue light glasses, and then take someone that uses blue light glasses and determine the effects" (Violet, 4) but comparison groups were not always discussed when prompted about hypothesis tests. Emma, Erich, and Alex described an appropriate null and alternative hypothesis when prompted, with Erich also connecting their hypothesis tests to the idea of a p-value, describing "when you really look at how big that difference is, it seems like it would really fall well within a standard of error of just an average null hypothesis" (5).

Participants rarely mentioned ideas of simulation. One notable instance occurred with Violet's work on the GPA task. After the researcher asked Violet about assessing the difference in GPAs from a statistical approach, Violet mentioned statistical significance, sketched a bell-curve, and marked an area that could represent a p-value. However, Violet was unable to connect her statistical knowledge to the context, stating, "it's different because it's on a graph and everything though" (3). Alex was the only participant to fully utilize ideas of simulation when during the Blue Light task, she explained a randomization distribution "accounts for all of the statistics that appear by random chance or coincidence if the null hypothesis is true….and if it's outside the certain area, then it would be unlikely that is due to coincidence" (5).

All five participants consistently implemented critical questioning when addressing sampling but not always backed by statistical reasoning. Many remarks addressed sample size, such as Claire noting "so you have 200 participants. That seems like a pretty good number" (3). While most participants stated larger samples were better, not all could articulate why. For example, Erich explained, “it’s just a good rule to go with. When you can, always have more data to help support the evidence” (3). Alex responded at Level 6 regarding sampling, using it during the Blue Light Task as a possible explanation for the different findings, stating “so if [researchers] previously used less participants, then they had less data and the data that they received was probably more coincidental.” Alex did not utilize this reasoning during the birth rates tasks while Emma and Claire both explained how a smaller number of births could explain the difference in birth rates. Other comments on the sample related to how representative or generalizable the sample was, such as Emma stating, “because it’s at one college, it’s kind of dependent on just students at that college” (4). However, none of the participants mentioned the randomization of a sample in relation to representativeness.

Table 1: Range of Statistical Literacy Levels by Task

	Alex	Claire	Emma	Erich	Violet
Birth Rates Task	2-6 (3)	2-6 (4)	3-6 (5)	2-5 (3)	2-4 (2)
Blue Light Task	1-6 (5)	2-4 (4)	2-5 (4)	1-4 (4)	1-4 (2)
GPA Task	1-6 (4)	1-5 (3)	1-5 (3)	1-5 (4)	1-4 (3)

Table 1 identifies the range in the levels of statistical literacy for each participant’s task steps for each task. The number in parentheses following the range represents an overall level for the task determined by considering the levels of all the task steps across statistical topics.

Discussion and Conclusion

The study found students in an introductory statistics course demonstrated varying levels of statistical literacy across multiple statistical topics when engaging in critiquing statistics in the news media. Participants consistently demonstrated higher levels of statistical literacy regarding sampling. However, participants were far less consistent in connecting their understanding of study design, hypothesis tests, and causation to the context for critical questioning. Instead, participants frequently relied on intuitive, non-statistical thinking. Additional statistical reasoning was often made after specific prompting, suggesting participants had understandings of relevant statistical topics that were not initially applied. This finding aligns with previous research identifying that students demonstrate lower statistical literacy on assessment items that heavily incorporate relevant contexts (Phadke et al., 2022).

Notable aspects of the media-based tasks contributed to participants’ display of statistical literacy during the interviews. Participants’ views about a headline frequently carried over into the critiquing of the article, sometimes including participants being less critical of information that aligned with their initial ideas. Adams et al. (2017) similarly noted how headlines can have an ongoing influence on readers. Participants identified the lack of details and specific numbers as a challenge to evaluating the statistical evidence behind the articles’ claims. Dealing with the uncertainty resulting from missing or vague information highlights the critical demands of statistical and mathematical products in the news (Gal & Geiger, 2022).

The results have implications for future research and instruction. Researchers could consider adjustments to media-based tasks for richer discussion of statistical topics. Larger differences between groups may elicit richer discussions since participants will not immediately dismiss results. Task development should consider balancing the amount of numerical information and vagueness so that tasks allow for application of statistical thinking while still representing realistic media contexts. Instructors and course developers of introductory statistics courses should consider incorporating media-based tasks into class activities and assessments. Doing so can highlight the use of statistics in real-world environments and provide opportunities to critically question claims using statistical thinking. Developing statistical literacy in media contexts is crucial to preparing an informed, critical citizenry ready to tackle a data-filled future.

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SUPPORTING TEACHERS’ CRITICAL READING OF A RELEVANT DATA VISUALIZATION

APOYAR LA LECTURA CRÍTICA DE LOS DOCENTES DE VISUALIZACIONES DE DATOS RELEVANTES

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In this paper we present findings from a design research study of an activity designed to engage teachers in critically reading a relevant data visualization. To help us capture the ways that the teachers were reading the data visualization we created a new framework for reading data visualizations from a critical statistical literacy perspective building from prior research. The two-dimensional framework is designed to capture types of reading (i.e. reading the data, reading between the data, reading beyond the data, and reading behind the data) intersected with layers of reading the word and the world with data (i.e. reading the word, reading the world personally and culturally, and reading the world socio-politically). We found participants engaging in every type and layer of reading data visualizations from our framework. However, they most frequently engaged in reading at the sociopolitical layer.

Keywords: Data Analysis and Statistics, Teacher Educators, Design Experiments.

Choice of Problem

The reading of graphs of data has been often associated with statistical literacy (Gal, 2002), which has been included in the school mathematics curriculum of the U.S. over the past several decades (Scheaffer & Jacobbe, 2014). Past scholarship (i.e., Curcio, 1987; Friel et al., 2001; Shaughnessy, 2007) synthesized different ways that people engage in reading graphs and also shaped curriculum and guided scholarship on people’s understanding of reading graphs. However, the ways data is visualized has changed significantly over the past few decades due to technological advances and new forms of media. Wilkerson and Laina (2018) describe data visualizations as those that “use context rich and interactive methods to create narratives and allow users to explore data for themselves” (p. 1). This definition expands beyond traditional graphs to include new forms of data visualization, such as dynamic and interactive spatial displays of data or scrolly-telling, where data visualizations change as a person scrolls down an article on a device. With such advances in how data is visualized and presented to the public, we see a need to revisit old frameworks with new data and lenses to consider the realities of how people encounter data in the world today.

Teachers, who are entrusted to enact the mathematics curriculum that students experience directly (Remillard & Heck, 2014), are at the forefront of curricular shifts such as updating how we teach and learn about data visualizations. As a result, in this study, we focus on a design research study of supporting mathematics teachers’ in critically reading data visualizations. Studying mathematics teachers reading of data visualizations is particularly important not just because they can shift the types of experiences students have with statistics, but because they themselves have often had few if any prior experiences learning statistics themselves (Shaughnessy, 2007). Additionally, past work has found that mathematics teachers are not confident in their ability to teach the statistics concepts required in their state standards (Lovett & Lee, 2017, 2019). The emergence of data science entering the K-12 curriculum in many states (Drozda et al., 2022) has only increased the demand on mathematics teachers to incorporate data visualization into their curriculum.

In this paper, we study the research question: How do mathematics teachers read a relevant data visualization? To help us capture the ways that the teachers were reading the data visualization, we created a new framework for reading data visualizations from a critical statistical literacy perspective (Author, 2017) building from prior research. We discuss implications for the design of data visualization activities for teacher education. This study addresses the theme of the conference, envisioning the future of mathematics education in times of uncertainty, by considering new ways of engaging in the reading of data visualizations, which is increasingly a crucial practice as a member of democratic societies that have become increasingly dominated by data in our current information age.

Theoretical Framework

Our framework is composed of two dimensions. One dimension is types of reading (i.e. reading, reading between, reading beyond, and reading behind), drawing from past scholarship that focused on people's reading of graphs and supporting graph comprehension (Curcio, 1987; Friel et al., 2001; Shaughnessy, 2007). We found this dimension alone was insufficient to capture the different ways people read data visualizations, as past scholarship only took a disciplinary and objective look at graphs. From our critical statistical literacy perspective, what was missing was a critical epistemological perspective, which more recent literature has considered. Drawing from Lee et al.'s (2021) Call for a Humanistic Stance in Data Science Education, where they put forth three layers of such an education including personal, cultural, and sociopolitical layers, we added a second dimension of layers of reading practices where one layer captures the more technical disciplinary practices that we refer to as reading the word drawing from Paulo Freire's (1970) literacy work and then two more layers that center on reading the world with one layer focusing on personal/community practices and the other is focused on sociopolitical factors. We also liken this perspective to the key aspects of culturally relevant pedagogy, where academic excellence maps to reading the word, cultural competence maps to reading the world with personal/cultural practices, and sociopolitical consciousness maps to reading the world sociopolitical practices (Ladson-Billings, 1995).

To unpack the specific practices of our two-dimensional framework we first drew upon past scholarship on reading graphs (Curcio, 1987; Friel et al., 2001; Shaughnessy, 2007). Curcio (1987) created three types of reading graphs for educators to consider as they formulate tasks and questions aimed at improving students' graph comprehension – reading the data, reading between the data, and reading beyond the data. The descriptions of these can be found in Table 1. Shaughnessy (2007) extended these reading levels to incorporate a fourth level – reading behind the data – to highlight the causes for variation in data represented in graphs and to make connections between the context of the data and the graph. Since then, numerous studies have used the reading levels framework to analyze news stories and student work with graphs, statistical tables, and maps (da Silva et al., 2021; Rubel et al., 2016). There is also important work from statistical literacy on reading in statistics such as considering how data is collected to determine what type of claims are warranted, when common forms of bias are present in the methods and other common methodological issue such as sample size (Bailey & McCulloch, 2023; Gal, 2002; Utts, 2003). These practices focus on disciplinary practices which we mapped to the reading the word layer of the framework.

Reading the world facilitates an individual's understanding of themselves, their culture, and their community (personal/cultural layer) as well as analyses of power, oppression, and structural inequities (sociopolitical layer; Freire, 1970; Gutstein, 2006). For example, Rubel et al. (2016) discuss how students often try to locate themselves in the data they are investigating. Rubel et al. (2021) extends their previous work by further considering the practices of narrating, formatting, and framing involved in taking critical reads of data visualizations. These practices describe considering the author's message in a data visualization and how they have highlighted certain aspects of the visualization to convey a message. Bailey and McCulloch (2023) also discuss practices such as acknowledging alternate explanations of the data and recognizing gaps in one's knowledge of the context being explored that is needed to interpret the statistical message. Kahn et al., 2022) also went beyond previous work and additionally included the consideration of feeling and emotions in the reading of data visualizations. It is important to note that though we developed categories in our framework that we differentiate we see them as deeply interrelated where expertise consists of coordinating between different types of reading of data visualizations to read the word and the world.

Table 1: Framework for Critically Reading Data Visualizations

Reading Types	Reading the Word Practices	Reading the World Personal/Community Practices	Reading the World Sociopolitical Practices
Reading the data **Extract information from the data (Friel et al., 2001)**	To recognize the components of graphs, the interrelationships among these components, and the effect of these components on the presentation of information in graphs (Friel et al., 2001) To speak the language of specific graphs when reasoning about information displayed in graphical form (Friel et al., 2001)	Looking for oneself in the data (Rubel et al., 2016)	Look for source of data Look for author/affiliation of visualization Questioning why an author has highlighted particular aspects of a graph or left them absent
Reading between the data **Find relationships in the data (Friel et al., 2001)**	To understand the relationships among a table, a graph, and the data being analyzed (Friel et al., 2001). Finding relationships or trends in the data visualized Recognizing the types of relationships (correlational or causal) can be claimed based on the data collection methods (Utts, 2003)	Making sense of the data visualization in relation to one's own personal experiences	Questioning how and why the author has highlighted particular relationships in the graph (Rubel et al. 2021)

	Identifying the relationships highlighted in the graph (Framing; Rubel et al. 2021)		
Reading beyond the data **Move beyond the data (Friel et al., 2001) to consider interpretations, inferences, and predictions/ extrapolations**	Interpreting information in a graph and answering questions about it (Shaughnessy, 2007) Predicting outcomes based on reasonable claims made from the graph (Shaughnessy, 2007) Making claims/inferences based on patterns and trends in the data to a population beyond what is represented in the data	Making predictions/ claims/ inferences from the data visualizations by drawing upon one's own personal experiences Recognition of one's bias and its impact on interpreting data (Bailey & McCulloch, 2023; Author, 2017) Acknowledging possible Alternate Explanations (Bailey & McCulloch, 2023) Drawing upon personal experiences facing inequities in the interpretation of the data visualization (Bailey & McCulloch, 2023) Connecting to one's feelings/emotions related to the data visualization (Kahn et al. 2022)	Making connections to alternate explanations of others (Bailey & McCulloch, 2023) Recognizing the story, the author is trying to tell with this data (Rubel, 2021) Questioning the author's motives for telling this story (Rubel et al., 2021) Identifying structural inequities at play in the interpretation of the data visualization (Bailey & McCulloch, 2023)
Reading behind the data **Making connections between the context and the data (Shaughnessy, 2007)**	Looking for possible causes of variation (Shaughnessy, 2007), based on the context being measured and the way the data was collected Looking for relationships between variables based on the context Recognizing appropriate graphs for a given data set and its context (Shaughnessy, 2007) Recognizing Appropriate Statistics & Appropriate Representations (Bailey & McCulloch, in 2023)	Using your knowledge of the context of the data to interpret why particular patterns exist in data as well as data generation process Using knowledge of one's community to interpret why particular patterns exist in the data to question aspects of the data generations process Questioning the investigative process undertaken based on personal experiences/identity Recognition of the gaps in one's knowledge of the context needed to interpret the statistical message. (Bailey & McCulloch, 2023)	Questioning sample size and methods (Bailey & McCulloch, 2023) and their impacts on inferences (i.e. practical significance vs. statistical significant; effect vs. no effect) (Utts, 2003) Recognizing when common sources of bias are present in the data collection (Utts, 2003) Recognizing and questioning the source of the data including what is quantified and how it was measured (Rubel et al., 2021; Author, 2017)

Mode of Inquiry

Research Design

This qualitative study is part of a larger design-based research project (Bakker, 2018; Cobb et al., 2003) studying mathematics teacher's development of critical statistical literacies for doing and teaching statistics. The work reported here is our fourth iteration of this framework. The seven participants in our study are high school math teachers recruited from a school district in the southeastern U.S. One of our participants was a district administrator. Six participants identified as a woman, and one identified as a man. Five participants identified as Black or African American, and two identified as white. The teachers' years of experience include: two people with 4-7 years of experience; one person with 8-10 years of experience; and four people with 16+ years of experience.

Design Activity

The activity the participants engaged in was a modified notice and wonder activity on a data visualization that was published by the North Carolina Department of Public Instruction (see Figure 1).

Figure 1: Data visualization published by the North Carolina Department of Public Instruction.

Data Collection and Procedure

Data sources included video recordings of the professional development session, daily written reflections by participants, and ongoing work samples. This comprehensive approach allowed us to capture the nuances and variations in how teachers engage with and interpret data visualizations. The data was collected over a two-week period of the professional development during the summer of 2023. This pilot study focuses on the first 15 minutes of a single introductory activity where the participants are given three questions to consider about a data visualization (see Figure 1): What do you notice? What do you wonder? How does this impact your community? Three members of the research team were present and helped facilitate the professional development and took on a researcher/participant role during the activity.

Analysis

Participant and researcher utterances during the session were coded based on the critical reading data visualizations framework outlined in Table 1.. Three of the authors independently coded the responses, and a comparison was conducted. Differences in coding were discussed until the coders reached 100% agreement. This rigorous coding process enhances the trustworthiness and dependability of our analysis, providing a solid foundation for understanding the diverse ways in which teachers read and interpret data visualizations (Lincoln & Guba, 1985). After coding, we then looked at the frequency of each reading type and layer represented in the data. We identified the patterns in the frequencies for further exploration. We arrived at our findings by looking at themes in the data analysis across reading type, layer, and participant. Once we identified patterns across the themes to develop our findings, described in further detail below.

Findings

We identified two main findings that answer our research question: How do mathematics teachers read a relevant data visualization? In our first finding, we discovered that teachers exhibited engagement across all reading types, with a noticeable emphasis on reading behind the data. Additionally, our analysis revealed that teachers engaged with all three layers of reading, with a predominant focus on the sociopolitical layer. For our second finding, we noticed that different participants engaged in different frequencies.

Table 2 demonstrates the ways teachers engaged with the reading types and layers. Some reading types were taken up less often. For example, reading between the data was underrepresented at the word and personal layers, and no teachers engaged in the reading beyond the data at the word layer.

Table 2: Counts and Precents of Code Occurrences for Each Dimension of the Framework for Reading Data Visualization

	Reading The data	Reading Between the Data	Reading Beyond the Data	Reading Behind the Data	Row Total
Word	9 (16%)	1 (2%)	0 (0%)	3 (5%)	13 (23%)
Personal	1 (2%)	1 (2%)	5 (9%)	5 (9%)	12 (21%)
Sociopolitical	3 (5%)	9 (16%)	10 (18%)	10 (18%)	32 (56%)
Column Total	13 (23%)	11 (19%)	15 (26%)	18 (32%)	57 (100%)

Note: *Percentage values have been rounded to the nearest whole number for clarity and ease of interpretation. The values in parentheses are the counts of each code's occurrence. All percentages are out of the total of 57 occurrences.*

In analyzing the various reading types, regardless of the layer (considering one dimension of our framework), we observed that teachers notably engaged in reading behind the data (18/57; 32%), surpassing the percentages of other reading types such as reading the data (13/57; 23%), reading between the data (11/57; 19%), and reading beyond the data (15/57; 26%). For example, Leona's statement, "you see that important. It's also the vocabulary of median and average because if the community doesn't understand the differences," serves as one instance of reading behind data (from sociopolitical perspective) and that reflects her acknowledgment of the significance of statistical terminology and the importance of understanding small differences in data analysis. By emphasizing the need for clarity and comprehension among community, she is making a connection between the data and its relevance to the community's knowledge and understanding in a way that it is important for everyone in the community to understand the numbers, so they know how that affects them.

In examining the different layers, irrespective of the reading types (considering the alternate dimension of our framework), we found that teachers engaged in reading the data visualization in sociopolitical ways more than any other layer (32/57; 56% overall), compared to the personal layer (12/57; 21%) and the word layer (13/57; 23%). The reading practices in the sociopolitical layer focus on considering the sociocultural context of the data visualization, which the teachers have firsthand experience and background knowledge of. To illustrate this, consider the following statement from Nancy where she questions the relationships the author of the data visualization is highlight, “But glossing over the fact that we have the small print that says average and then, yeah, median and you're wanting me to compare those two.” Nancy goes on to read the data in a sociopolitical way combining her knowledge of the content with her knowledge of her community to point out that this approach is taking advantage of a common misunderstanding of the differences of means and medians and how the shape of a distribution impacts them, “they say the middle income the middle of this and they don't realize that, that middle is usually the median because we know the distribution is not going to be a symmetric or roughly symmetric right.” Teachers also read beyond the data unpacking the story they thought the author of the data visualization was trying to make and questioning the authors motives. For example, Anna stated, “I think this is designed to show that teachers are making more than most North Carolina incomes.” Some participants also began to make connections between the data visualization they were reading and how they could use it in their own teaching. For example, Melody said, "So I would have used this as a perfect example to my students of how we can make statistics say anything we want," where she is connecting issues of the story being presented how she could use this in her teaching.

We also found that the teachers engaged in reading the data visualization in different ways. Three of the teachers did not engage in verbally describing their reading of the data visualization throughout the fifteen-minute activity. Interestingly, these were the teachers with the least prior teaching experience in the group, though all three had at least four years of prior experience so they were by no means novices. Of the five teachers that did verbalize their reading of the data visualization, three out of the five used sociopolitical more than any other reading type. Table 3 provides a breakdown of each participant’s reading type and layer.

Table 3 Instances of teacher engagement with data visualization activity

		Nancy	Anna	Leona	Natalie	Melody
Reading the Word	Reading the data	2	3	2	2	0
	Reading Between the data	1	0	0	0	0
	Reading Beyond the data	0	0	0	0	0
	Reading Behind the Data	1	0	0	1	1
	Reading the data	1	0	0	0	0
	Reading Between the data	1	0	0	0	0

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

Reading the World Personal	Reading Beyond the data	2	0	1	1	1
	Reading Behind the Data	3	0	0	1	1
Reading the World Sociopolitical	Reading the data	1	1	0	1	0
	Reading Between the data	3	4	0	2	0
	Reading Beyond the data	2	3	1	2	2
	Reading Behind the data	0	0	3	2	5

Note. *The columns in this table exclusively represent data for teachers who actively engaged in verbal communication during the activity. There were also three teachers who did not engage in verbal discourse during the specified period.*

Out of all the participants, Nancy used the most variety of reading types throughout the entire activity. Nancy’s engagement is significant because she engaged in almost every type of reading at every layer, which was not typically as evidenced in Table 2. For example, she was the only participant that engaged in reading the data and reading between the data at the personal layer evidenced in her statement:

> That's the question, does the data represent you? So you subscribe to an identity or something of that nature and but when you look for the data on that identity or whatever it is that are you representing it in that, like does it represent you, It's supposed to represent population or something like that.

Nancy’s ability to clearly verbalize her reading of the data visualization was found across the activity and contributed to why so many of her utterances were coded from our framework. This also points to a limitation of our study in that we don’t know how the participants who were not as good at communicating their reading of the graph or chose not to communicate at all were engaged in reading the data visualization. This has implications for our design and pedagogy, which we discuss in the next section.

Conclusions and Implications

In this study we sought to explore how mathematics teachers read a relevant data visualization. Drawing from data collected from a larger design research project, we were able to begin to investigate our question and further refine a framework for reading data visualizations. Our framework allowed us to find interesting patterns in the video data we analyzed from the first 15 minutes of a design activity where teachers were engaged in a modified notice and wonder activity with a data visualization we specifically selected because it’s sociopolitical relevance to the teachers in our study. One finding was that the teachers engaged in reading both the word and the world in many different ways and in particular heavily engaged in sociopolitical readings of the world through the data visualization. This finding has implications for future design in that we hypothesize the reason for the participants’ reading predominantly through the sociopolitical layer was because of the relevance of the data visualization itself. The teachers demonstrated a desire to critically examine data beyond its surface level and show interest in uncovering underlying patterns, causes of variation, identifying biases in data collection methods, and understanding contextual factors.

Another finding from this study was that the teachers engaged in reading the data visualization in different ways. Of particular concern for us was that three participants did not engage in verbalizing their reading of the data visualization. As a contextual note this activity did occur early on in our larger study while we were working on building community amongst the participants and the participants that did not engage in this activity did so in future activities. However, moving forward, we see it as important to consider how to engage more of the participants in the discussion. We also noticed related to this that finding that of the five teachers engaged in the activity – four had 16+ years of experience. The people with the least experience engaged the least. This points to possible power dynamics that might be at play in the discourse, which was beyond the scope of our analysis, but we believe should be considered in future work.

As data visualizations have become increasingly common in the media today, we see an increased need for teachers to engage in such data visualization activities and to translate them into meaningful experiences for their students as well. Furthermore, we believe our framework can be useful in helping research not only design activities in the future but to analyze the ways in which people engage in reading data visualizations.

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THE ROLE OF PRIOR KNOWLEDGE IN EFFECTS OF EMBODIED PEDAGOGIES ON LEARNING

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Embodied learning pedagogy emerges as a promising approach in STEM education, but there remains a gap between experiments in the lab and practices in real classrooms. In particular, when learners come into the class for an embodied learning experience, their understanding of the concepts differs. We posit the Perform First Hypothesis, suggesting that performing physical activities benefits learners with less prior knowledge more than those with more knowledge. Results from a longitudinal classroom experiment that compares the effects of observing versus performing embodied learning activities in students with different levels of prior knowledge supports this hypothesis. We found a significant interaction between the levels of embodiment and learners' prior knowledge. The findings shed light on the design of embodied learning experiences for learners with different levels of prior knowledge in real classrooms.

Keywords: Curriculum Development, Embodied Learning, Statistics and Data Science Education

Objectives of the Study

Embodied learning pedagogies has emerged as a promising approach to fostering transferable knowledge. Yet, questions remain about the practical application of embodied learning in real-world classroom settings. First, even if there is laboratory evidence that embodied pedagogy facilitates better learning of a single construct, what is its applicability and impact in a longitudinal course, encompassing a multitude of concepts? Second, with an increasing number of students traditionally outside the STEM sphere enrolling in STEM-related courses, could embodied learning be an effective strategy to help students with low prior knowledge?

To begin answering these questions, we build up a theoretical framework based on current literature looking for clues about how embodied pedagogy might impact long-term learning of diverse learners. Then we detail the results of an in-class intervention experiment that exposed students to embodied learning activities for nine weeks.

Theoretical Framework

Embodied cognition frameworks assume that concepts are grounded in action and perception (Barsalou, 1999; Borghi & Pecher, 2011; Clark, 2008; Golonka & Wilson, 2012). Physical actions observed and performed during learning and encoding influence our mental representations as well as subsequent retrieval, reasoning, and problem-solving (Barsalou, 2008; Fu & Franz, 2014). Thus, it is crucial for classroom instruction to consider embodied pedagogies.

Although past research has demonstrated that an embodied pedagogy (e.g. gesture) is more effective than a non-embodied one (e.g. no gesture), we are early in our understanding of how and when embodied pedagogies influence learning (Goldin-Meadow & Wagner, 2005; Johnson-Glenberg & Megowan-Romanowicz, 2017; Zhang et al., 2021, 2022). One important element is the treatment of prior knowledge, which is often controlled by random assignment or treated as a covariate in the lab. However, when learners come into a class, their understanding differs. This raises a critical question: How do interventions with different levels of embodiment impact learners with different levels of prior knowledge and experiences in the domain?

To answer this question, we posit the Perform First Hypothesis: performing bodily actions will benefit learners with no or low prior knowledge in the domain more than learners with high prior knowledge. In other words, whereas traditionally in educational interventions, students' prior knowledge of the concepts positively correlates with their performance after the intervention, performing embodied activities might mitigate or even eliminate this correlation.

Our hypothesis is grounded in the perceptual symbols system theory: when people first have a perceptual experience, it activates neurons that are mentally stored as multimodal frames of perceptual symbols (Barsalou, 2008). Later, these frames can be activated and function as simulators that enable people to reactivate the same neural connections and simulate the original experience without bodily actions (Glenberg, 1997; Pezzulo & Calvi, 2011). However, if learners lack the prior experience to develop the simulator, they need to first perform the actions to gain the perceptual experience necessary for the multimodal representations to develop.

Although how prior knowledge and the type of embodied intervention interact over time has not been directly examined for performing versus observing actions, past research has shown that low-prior-knowledge learners benefited more (1) from physical than virtual manipulation (Zacharia et al., 2012); (2) from concrete than abstract gesture (Congdon & Goldin-Meadow, 2021); (3) from object manipulation than gesture (Congdon et al., 2018). In all cases, the more embodied pedagogy seemed to be more contributive for learners with low prior knowledge. However, those are highly controlled lab experiments. Because developing expertise in STEM domains is a prolonged process, a single intervention may not fully elucidate this interaction.

The Current Study

In the current study, we use embodied pedagogies to develop a lab curriculum of a college-level introductory statistics course. Participants were randomly assigned to a partner. Within each dyad, one was randomly assigned to be a performer, who would perform hands-on activities, and another was assigned to be a recorder, who would observe and record their partner. We hypothesized an interaction between the types of embodied intervention (i.e. condition) and participants' prior knowledge such that low-prior-knowledge performers would outperform low-prior-knowledge recorders, but high-prior-knowledge performers would not.

Methods

Participants

Participants were students enrolled at a large public research university, taking an introductory psychological statistics course. The course taught statistics from a textbook employing a modeling approach with R programming (Son & Stigler, 2023), delivered through a technology platform called CourseKata.org. The class was structured as two weekly lectures and a lab session. The experimental interventions took place during the 50-minute lab sessions for nine weeks. Students were allowed to miss one session without losing participation points.

Among the final sample size of 227, 171 students self-identified as female (75%), 49 as male (22%), and 7 as non-binary (3%). The self-reported race and ethnicity were as follows: 100 Asian or Asian American (44%), 8 Black or African American (4%), 47 Hispanic, Latino or Spanish origin (21%), 13 Middle Eastern or North African (6%), 46 White (20%), and 13 mixed/multi races (6%).

Design and Procedure

Each randomly assigned dyad were further randomly assigned to either the perform condition (n = 113) or the observe/record condition (n = 114). The performers were expected to perform embodied instructional actions designed to enhance learning. The observers observed and recorded these actions using their phone or tablet. Their role and partner stayed the same for the entire course. If a student's partner could not make it to a lab, the student would be temporarily paired with another student while maintaining their role. Using a cover story, students were informed that we wanted to use their hand movement video to design a pedagogical agent.

Starting Week 2, each lab starts with a set of "practice questions." Students were given 6-10 minutes to complete the questions through a Qualtrics survey. Half questions functioned as a delayed posttest, to measure what students have learned from the previous week. The other half functioned as a pretest to assess what students might already know about the to be taught concepts. Week 2 only had pretest questions, because Week 1 was a general introduction to the lab without embodied activities. After taking the survey, students engaged in a lesson that incorporated embodied activities consisted of object manipulation, gesture, and drawing. One activity example is cutting out a piece of paper with a dataset and used the pieces to perform "shuffling" and "resampling" (i.e. sampling with replacement). Based on assigned condition, they either performed or observed and recorded their partner performing these activities.

Materials

Each week, students received a paper worksheet and a Jupiter Notebook with coding activities. A description of the lab schedule and activities is available through Open Science Framework: https://osf.io/ntsr2/?view_only=ce650267f407451a9ea26abcb428d8f7.

Measures

Pretest performance / Prior knowledge. Participants' pretest performance was measured using half of the practice questions administered to students at the beginning of each lab. There were four trained coders, and every question was coded by two coders to ensure inter-rater reliability. The rubric was jointly determined with the four coders and the lead researcher. Each question was worth one point, with half points given to correct but incomplete answers or those with minor misunderstandings. All coding was conducted blind to condition.

Delayed post-test performance. Participants' delayed post-test performance was measured using the other half of the practice questions. It was coded the same way as the pretest.

Results

Transparency in Data, Analysis, and Materials

The deidentified data and analytic syntax are available through the Open Science Framework. The study design, hypotheses, and analytic plan were not pre-registered.

Missing Data Handling

When students did not come to a lab, or if their partner was switched during a lab, their data for that week was treated as missing. We imputed the missing data using the Markov Chain Monte Carlo (MCMC) algorithm as implemented in the Blimp application (Enders, 2017).

Three-Level Multilevel Modeling (MLM)

We specified a three-level MLM. Level-1 was the repeated measures (the intraclass correlation coefficient (ICC) for the post-test performance = 68.8%). Level-2 was the students (ICC = 23.8%), who are nested in partners/dyads (i.e. Level-3, ICC = 7%). Below, we show the equation for the overall model (Raudenbush & Bryk, 2002):

$$Posttest_{ijk} = \gamma_{000} + \gamma_{010}Condition_{jk} + \gamma_{020}\,Pretest_{jk}^{b.cgm} + \gamma_{030}Condition_{jk} * \,Pretest_{jk}^{b.cgm} + \gamma_{100}\,Pretest_{ijk}^{w} + \gamma_{110}\,Pretest_{ijk}^{w} * Condition_{jk} + \;\gamma_{200}Time_{ijk} + \gamma_{210}Condition_{jk} * Time_{ijk} + u_{00k} + r_{0jk} + e_{ijk}$$

In above, $Pretest_{jk}^{b.cgm}$ is the variation between each student's average pretest performance, and $Pretest_{ijk}^{w}$ is the variation within each students' pretest performance. Because the variation in pretest has both level-1 and -2 variation, the interaction between pretest performance and the condition was partitioned into the interaction between condition and each student's average pretest performance (i.e. γ_{030}), which we will refer to as the level-2 interaction, and the interaction between condition and the variation within each individual's pretest performance (i.e. γ_{110}), which we will refer to as the cross-level interaction.

We used maximum likelihood estimation and fit the model using Blimp 3. There was no significant level-2 interaction between condition and pretest performance (median = 0.09, 95% credible interval = [-0.15, 0.34], yet there was a significant cross-level interaction between condition and pretest (median = -0.12, 95% credible interval = [-0.22, -0.01]).

We next probed the significant cross-level interaction at each condition. For observers, the correlation between pretest performance and delayed posttest performance was statistically significant (median = 0.11, 95% credible interval = [0.03, 0.18]). However, for performers, the correlation was not (median = -0.01, 95% credible interval = [-0.08, 0.07]).

Discussion

In the current project, we designed and implemented an embodied lab curriculum in a college-level introductory statistics course, testing how different embodied interventions impact learners with different levels of prior knowledge. We put forward the Perform First Hypothesis, which posits that learners who had low prior knowledge need to perform the activities to reap the most benefit out of the hands-on instruction whereas high prior knowledge learners can benefit similarly from observing a hands-on demonstration. The results supported our hypothesis: there was a significant correlation between prior knowledge and posttest performance in the Observe condition, but this correlation was not significant in the Perform condition.

The findings also suggest a new research direction for the field of embodied learning. Beyond investigating merely the efficacy of embodied interventions, the current study informs the field to ask nuanced questions of what and when - what type of embodied pedagogies are the most effective, and are they the most effective?

The contrast between performing and observing hands-on activities is not the only important comparison. We started with this distinction because they are both commonly seen in research and practice. The idea that "performing" is unique and foundational is not new. Piaget argued that the sensorimotor stage of development is foundational to later higher-order thinking with abstract concepts (Piaget, 1983; Piaget & Inhelder, 1969). However, embodiment came likely through a continuum from purely abstract to highly embodied instead of distinct categories. We urge future studies to investigate other forms of embodied pedagogies to fully elucidate the question of "what" and "when."

Lastly, our findings have crucial implications for instructional design. Although teacher demonstrations are easier to implement in class, novices seem benefit more from performing physical activities. After giving students meaningful hands-on experience, teachers can then give demonstration or even more abstract instructions. Future research should continue exploring the nuanced ways in which different embodied pedagogies impact learners at different time points of their knowledge development, catering to the diverse needs of learners in in STEM domains.

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CONNECTING DATA SCIENCE AND ARTS: EXPLORING DATA-ART INTEGRATION IN A DATA-ART INQUIRY PROGRAM

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Keywords: informal education, integrated STEAM

In this era of big data, data science education involves immersing students in the context where data originates, profoundly influencing their interaction with data (Wilkerson & Polman, 2020). As students encounter data not only within the confines of the classroom but also in the real world, this contextual immersion renders data science education inherently interdisciplinary.

One way to explore this interdisciplinarity is to integrate data science education with the arts. Previous research has shown the potential enhancement of students' data science learning through the integration of arts (Bhargava et al., 2016; Matuk et al., 2022). Despite this, the specific intersection of data and arts remains underexplored. Our research seeks to address this gap by investigating how students connect data and arts when creating artistic data visualizations.

In light of the promising intersection of data science education and arts education, we designed a 14-week data-art inquiry program in an afterschool setting. 23 middle and high students, referred to as "data artists" (DAs), participated in this program. The program was structured in three phases: weeks 1 to 3 involved DAs reimagining and reinterpreting existing data visualizations to learn fundamental data science concepts; weeks 4 to 7 saw DAs creating their initial artistic data visualizations using data from local agencies; in the last segment of this program, DAs selected their topic, collected data, and created their own unique data visualizations.

To answer our research question, we interviewed six groups of DAs to investigate how they incorporated data into their art creation after the program ended. Using an inductive coding method (Saldaña, 2009), we analyzed the interview transcripts to gain insights into their methods of integrating data and art. Our preliminary findings revealed three key insights. Firstly, DAs employed color coding to represent their data in their data visualizations, such as the left picture of Figure 1, where a group of DAs investigated people's preferences for school lunch items, calculated preference scores, and used different colors for each item. Secondly, the sizes of artistic elements were used to signify differences in data sets, as seen in the middle picture of Figure 1, where a group of DAs explored the salary disparities between baseball coaches of two local universities and used logos with different sizes to show the difference. Lastly, the format of source data influenced the format of their artistic data visualization, exemplified in the right picture of Figure 1, where a group of DAs transformed qualitative data on text messages during an active shooting into a visualization format that resonated with its source.

Figure 1: School lunch, college coach pay, and gun violence

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Chapter 13:
Student Learning and Related Factors

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CONCEPTUAL ACCOMMODATION AND EXPERIENCING COGNITIVE DISEQUILIBRIUM: MAGIC OR MATHEMATICS?

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This paper explores how two undergraduate students come to the existential moment of expressing mathematics as magic. The "Experience" of a mathematical transitional moment might include feelings; however, our experience is normally considerably richer in content than simple feeling. This study investigates how undergraduate students face disequilibrium when they are involved in problem solving situations that call for exponential modeling. Further, it explores whether there is an alignment between experiencing disequilibrium and a conceptual accommodation to a new contextual situation. The analysis is guided by a concept projection perspective (diSessa & Wagner 2005; Wagner 2010). The results suggest that there is alignment between experiencing a cognitive disequilibrium and the process of conceptual accommodation as a form of learning.

Keywords: Phenomenology, Knowledge Elements, Cognitive State, Concept Projection

Introduction

"What is mathematics?" This is a deceptively simple question that does not have a simple answer. To understand the philosophy of mathematics, we need to dig into the philosophy of the mind and the philosophy of language. The main goal of this research is to explore the epistemological foundations of the understanding of mathematics within a specific mathematical topic, focusing on individuals' internal mathematical and linguistic reasoning processes. In this study, I aim to observe the moments in which individuals make sense of their activity and come to understand mathematical ideas. By understanding these moments, we might comprehend the epistemic forms of knowledge that learners activate and perceive through specific linguistic cues embedded in the context.

The impetus for my dissertation study originated in my experience over six semesters of teaching *Algebra II* at a mid-western university. I observed that students were struggling to model exponentially in half-life type problems. However, I observed that in some problems that required a deeper conceptual understanding of the problem, there appeared to be a greater possibility that students would appropriately model exponentially. For example, students appeared more open to appropriately using exponential modeling in situations that involved interest rates or population growth/decay. These observations made me interested in studying how contextual variation can influence learners' understanding of exponential modeling.

Background

Exponential functions are a difficult, yet essential, mathematical concept that play an important role in the study of advanced mathematics (Ellis et al. 2016; Weber 2002). Weber (2002) focused on students' initial understandings of exponential functions. Weber's main result from the work was that, although all students in their study could compute exponents in simple questions, only few students could reason about the process of exponentiation as a pattern of change. Confrey and Smith (1994, 1995) introduced the idea of "*splitting*" structure as a multiplicative pattern, which opened an avenue for its application as a rate-of-change perspective and "*covariation*". Building upon the notion of "covariational" thinking, Ellis et al. (2016) introduced an Exponential Growth Learning Trajectory (EGLT), which identified three stages of students' conceptual development: pre-functional reasoning, the covariation view, and the correspondence view.

Recognizing covariational thinking, which originated from the idea of *split,* is important for designing tasks to explore students' understanding of exponential functions. However, a major gap in the study of exponential functions concerns how we can develop an understanding of exponential growth specified through contexts that are *only* associated with exponentiation. For example, as it is explained above, Ellis' stages of developing an understanding of exponential growth lacks specificity with respect to the type of function. Covariational thinking is important for the learning of any function, not just exponential functions, and thus the question is: What features of a context can provoke *exponential modeling*?

The results of Weber's (2002), Confery and Smith (1994, 1995), and Ellis et al.'s (2016) work creates some essential questions in understanding exponential functions: Why do students encounter difficulty with reasoning about the process of exponentiation? Can contextual variations steer learners' thinking specifically towards understanding the use of exponential models? How does students' thinking in this domain confront cognitive conflicts (disequilibrium)? How do they move through the disequilibrium and equilibration of a new understanding? And how do learners experience these transitional moments?

In this study, I attempt to investigate the moment at which students perceive *mathematics as magic* as a phenomenological experience. Phenomenology, not as a method but as a theory of studying experience in the stage of consciousness, is the study of phenomena as they appear in our experience of things and thoughts. Phenomenology studies the structure of various types of experiences, including embodied actions and linguistic activity, as different forms of social activities (Smith & Thomasson, 2005). For exploring students' experiences of special transitional moments, we need to explore the appearance of knowledge resources as learners activate them. We need to investigate moments at which learners find mathematical validation for their reasoning, as well as identify and investigate different forms of embodied cognition to explain how students reach mathematics as magic.

In the field of mathematics education, the epistemological descriptions of moments of disequilibrium, resolution, and satisfaction after a conceptual accommodation and new equilibration should be mathematical. Studying how students activate the knowledge resource of exponential modeling helped me to observe some moments in which, I believe, students experienced a certain type of cognitive disequilibrium and conceptual accommodation. Wondering about the following two questions led me to perform this study:

1. How do students react to a possible disequilibrium between prior experience with linear modeling and their new understanding of exponential modeling?
2. How do students undergo a process of conceptual accommodation when they are establishing or modifying a new existing scheme? What is the role of cognitive disequilibrium in the process of conceptual accommodation?

Theoretical Perspective and Conceptual Framing

I sought a modern theory of learning that fits in with my overarching constructivist perspective, in which individuals' internal process of actively constructing knowledge is foregrounded. In addition, I attempted to cover a phenomenological perspective to capture the whole mathematical experience resulting in the experience of cognitive disequilibrium and conceptual accommodation. Knowledge in Pieces (KiP) is a heuristic epistemological framework that is informed by Piagetian constructivism and cognitive modeling (diSessa, 1993; 2018). Moreover, knowledge activation and use in the KiP perspective is *highly contextual*, based on students' perceptions of features of contexts.

In further development of the KiP perspective, diSessa and Wagner (2005) describe the knowledge as a construct of *concept projection.* Wagner (2010) provided an extension of the theory of concept projection, which demonstrated Piagetian processes of *assimilation* and *accommodation* as compositional stages of concept projection. "Concept projections are collections of assimilatory and interpretive knowledge elements associated with some concept, and the construction of new concept projections reflects a process of conceptual accommodation to new contextual situations" (Wagner, 2010, p.451). I used this conceptual frame to investigate how students activate different resources of knowledge, especially when they attempt to reflect on a process of conceptual accommodation. In exploring how students come to model exponentially, it appeared that learners could build new knowledge elements—exponential understanding—which was the result of passing through multiple disequilibrium and accommodating the knowledge elements to particular contexts (Allahyari, 2023).

Specifically, in this study, I analyze different cognitive states that include several transitional moments as well as show how context variation can influence learners' knowledge activation. The unit of analysis in the current study are moments at which learners perceived context as an external cue and experienced different cognitive states, which are the collection of assimilatory and interpretative knowledge elements and a process of conceptual accommodation to new contextual situations. In other words, the core of my conceptual framework is to observe students' mathematical experience through specific types of knowledge activation that might pass through disequilibrium and new concept projections that reflect a process of conceptual accommodation.

Method

This study is part of a larger study in which 14 students participated in at least one round of interviews. They participated in a problem-solving situation with four problems in the first round, which was the main problem-solving session. Data for this research was collected through the first rounds of interviews (45-100 minutes of problem solving). The interviews included screen sharing a specially designed set of slides I created through Desmos Activity Builder (DAB). Participants were undergraduate students who enrolled in an Algebra II at a large public midwestern university. I recorded with audio and video while they worked in DAB or a paper-pencil situation in which they described their thoughts aloud as they worked on contextual problems involving exponential modeling.

I observed and analyzed moment-by-moment thinking and reasoning processes in order to see fine-grained ideas of how students moved through experiencing a certain type of cognitive harmony, from experiencing a possible disequilibrium, to the possible process of conceptual accommodation. There was special attention paid to capturing the moments in which learners' patterns of knowledge activation were changed, the type of knowledge resource, the observation of confronting possible disequilibrium, a moment-by-moment report of facial expressions during the process of concept projection, and the recording of gesture expression.

I asked participants to solve the problems in Table 1 and to explain their reasoning aloud. An analysis of explanatory language used during the interviews enabled me to recognize the pattern of reasoning and how learners experienced assimilatory cognizing moments or conceptual accommodation of different knowledge resources. I recognized a possible alignment between experiencing disequilibrium and new concept projections in context that call for exponential modeling.

Table 1. Exponential Problems That Participants Worked on During the Interviews

Title of Tasks	Description of Contextual Tasks
Baseline problem	What does the function $f(x) = b^x$ mean to you? What do you think of when you see this function?
People Diagnosed with Covid-19 in City A	In City A the number of people who have Covid-19 increases by 11% every 10 days. The initial population of people who are diagnosed with Covid-19 in City A is 1500. Create an equation that models this situation.
People Diagnosed with Covid-19 in City B	In City B the number of people who have Covid-19 decreases by 11% every 10 days. The initial population of people who are diagnosed with Covid-19 in City B is 1500. Create an equation that models this situation.

Analysis

Coming to solve the problem of City A is the heart of this study, where we can see how students face multiple disequilibrium, attempt to find a local resolution through returning to the context and genuinely playing with the linguistic cues of the mathematical context, then accommodating a new understanding of the context. Out of fourteen participants of the larger study, three students stuck with linear modeling in all problems and did not experience any disequilibrium. Three other students already knew the exponential model (without deep reasoning, they modeled the problem as an exponential model, more like memorizing the exponential model for the specific population growth context). Eight students experienced some transitional moments, including a certain type of disequilibrium, when they ended up modeling exponentially in City A and City B problems. I observed that the first reaction to the City A and City B problem was expecting a linear model:

Ken: After 10 days the population is 1500 times 0.11 plus 1500 which is 1665
Interviewer: Great! How about after 20 days?

Ken: It should be the same, but 2 times zero point 11; like this $1500 + 2 \times 0.11 \times 1500$

She continued to do the problem the same after 30 days before making a linear model. Before I felt that she was completely satisfied with the model, I asked her a question: "I just have one question, I can see that in each step you are going back to the original population (1500), but is it like 11 percent of the new population or the original one?" After she received the question, she stopped, looked around, then started to read the problem one more time, crossed out whatever she wrote, and then started from a new page, rewriting the information of the problem from the beginning and whispering important information as she was writing. As she was finding the new pattern for calculating the population growth, she was talking with me (or it was a laud self-talk):

I can see that it should be the 11 percent of the new population…because it is increasing every 10 days, **it** means it should increase on an already increased population, not the original one. For example, the new increase for 20 days should happen on an already increased population.

She eventually found the base of 1.11 by doing some mathematical procedures; eventually, she could generalize to an exponential model of $f(x) = 1500 \times (1.11)^x$. In the City B problem, I did not create a contradictory question to see if she could figure out the exponentiation process more independently. I could see that she was facing multiple disequilibrium when it was a declining situation. I believe she was experiencing two different cognitive state simultaneously. She was recognizing the similarities between the City A and City B problems. She knows that for 20 days she should start from 1335—recursive dependency knowledge elements (Allahyari, 2023); however, a declining situation as multiplicative/exponential process was violating her expectation. It seems that there is expectation of growth for a multiplicative situation for her. From one side she was activating assimilatory and interpretative knowledge elements (recognition of recursive dependency), form the other side she was experiencing a new disequilibrium:

I know the population after 20 days should be 1335- 0.11 times 1335 which is 1188…I'm lost a little bit! I am not sure I am right, are we multiplying? Can we multiply, then it decreases? But the pattern should be multiplying by 0.89; there is no other way! It is 0.89 square times 1500, so it is like the inverse of the City A problem, but a square is increasing or decreasing? I do not know! I think my brain is processing still!

She was simultaneously reacting differently; like experiencing different feeling at the same time: Happy to recognize the similarity that has been explored enough in the previous context, while, not expecting multiplicative situation in a declining process. For a moment she closed her eyes and she said, "I am totally lost!". I could not let her to stay in the very contradictory cognitive states. I realized that she needs a reliance point, a form of confirmation from the similarities she is recognizing.

Interviewer: So how is that multiplying after 20 days?

Ken:	It is another 0.89 times the new population (1335)
Interviewer:	And what is that?
Ken:	It is 1188
Interviewer:	Do you want to check your model for 20 days?
Ken:	You mean $(0.89)^2 \times 1500$?
Interviewer:	That's the model you found, isn't it?
Ken:	Yes! Let's see! It is 1188! Oh Gosh! **It feels like magic!** Learning math is so interesting, why is it like this? For a moment I feel completely lost, but in a moment one thing makes sense then everything makes perfect sense and proves each other. I am serious! This is a major breakthrough. (Covering her eyes with both her hands, then whole her face).

I have observed that sometimes it is hard for students to expect a declining population to be modeled exponentially. Not only should they conquer their linear expectation, but they need to overcome expecting an exponential model that cannot show a declining situation. That is why I call this situation facing multiple disequilibrium. As I observed and watched students' videos over and over, I realized experiencing disequilibrium is actually experiencing a "violation of expectation" that is usually experienced when learners attempt to accommodate a new concept projection. Here, the already settled expectation is a linear expectation, yet it is not satisfied by the linguistic cues of the contexts–City A and City B problems that should be modeled exponentially. I called this violation of expectation, when the context does not fit, experiencing a cognitive disequilibrium.

Tom was a student who showed the *manifestation of experiencing disequilibrium***.** When he was very close to seeing the exponential model in the City A problem, he instead strongly expressed his proportional expectation. In the middle of making sense of the structure, he left all his findings and faced a huge *disequilibrium* with satisfying a systematic and already established proportional scheme. He even saw 1.11 times 1.11 times 1500, but then instead of seeing 1.11 squared times 1500 he stopped and said wait a minute and started to work on his paper:

Tom:	Am I doing so wrong? (very sad and disappointed)
Interviewer:	Why? What did you find?
Tom:	I found 1.321 when I divided 1848 by 1500
Interviewer:	What about the first 10 days
Tom:	That works well. $\frac{1665}{1500}$ is 1.11 which makes a lot of sense.
Interviewer:	What did you expect?
Tom:	I thought $\frac{1848.15}{1500}$ should be 1.22 because it increases 11% each time

I asked him to believe in himself because he was going to be surprised how smart he was. He found the population after 30 days and he multiplied 1.11 three times; then he could see the exponential model. "The *Multiplicative Recursive Dependency* well-imbedded in the context task appeared to help Tom to prefer working with his understanding of context rather than sticking with his proportional expectation" (Allahyari, 2023). He eventually could coordinate his perception of the situation with his strategy for calculating an increasing Covid19 population in each step and to resolve an apparent contradiction in his thinking. He was so happy after seeing the model; it was like a cry of joy, but I went even further! I needed him to find a way to compare the exponential process with his expectation.

Tom:	So! f (x) would be: $f(x) = (1.1)^x \times 1500$ Right? Cool.
Interviewer:	Wonderful! Now, do you want to calculate what you find for 1.11 square?
Tom:	Sure! It is 1.321! What? What is that? ***Is it magic****?*

He saw 1.321 again, but instead of being devastated like the first time, he was joyfully asking if it was magic! This time, 1.321 was affirmation that his understanding of the context fit very well with the model that he created. When he found the exponential model, he stood up, and then sat; he shook his head three times and said: "Oh - that is so cool! I made it - I found it!"; then, seeing 1.321 completed his joy. He was laughing loudly, asking, "Is it magic?" I call this moment an *existential moment* resulting from a mathematical change. He eagerly got validation for what "he found" from the second problem, City B.

For the City B problem, it seems to me that Tom was in an assimilatory situation in which he needed to assimilate the situation to his vision about exponentiation. He did not experience any major disequilibrium; he was productively struggling to find 0.89 as the base, but that struggle was not like an "earthquake" to violate his expectation. He was expecting an exponential situation, but it was challenging to find the base. Eventually, he could find the model in this problem too, but there was not a cry of joy after he modeled exponentially. He was clearly happy, his eyes were shining, but he did not ask about any magic; it was more an assimilation of a projected concept.

Discussion

This study is part of a larger study in which eight undergraduate students came to experience productive disequilibrium, accommodate new understandings of the situation, and generalize problems to make abstract models. Among all eight undergraduate students, Tom and Ken stated a specific form of expression in the final steps of modeling situations: *"Is it magic?"* They both showed specific types of cries of joy and happiness in their facial and body gestures. I believe they were experiencing the moment of mathematical eureka through coming to understand and activate knowledge resources of exponential modeling. Tom participated in one round of interviews, while Ken participated in two rounds of interviews.

Tom was the first student in whom I observed a special *harmony* in his resistance to the population growth problem (Table 1). For the City A problem, Tom experienced a disequilibrium (expecting a linear proportion), resolved the disequilibrium by genuinely working with context and refining the linguistic cues, accommodated a new understanding of the situation, and finally generalized the problem to make the abstract models. Ken was one of six students who went through problem solving with seven problems and two rounds of interviews. There is enough evidence that she experienced cognitive states that resulted in expressing mathematics as magic, including conceptual accommodation of a new concept projection at the end of the City B problem. After facing multiple disequilibrium, she said the same as Tom: "Is it magic or something? It seems I am doing magic, and you just light up to show me how I did it."

When the concept projection and the interpretative knowledge element are more in the construct of new reflection as a process of conceptual accommodation, there is more possibility for a learners' cognitive pathway that include facing cognitive disequilibrium. In other words, there is strong *alignment* between experiencing disequilibrium and a new concept projection. I believe we need to further explore what makes mathematics so interesting that students might joyfully express, "Is it magic?" One thing that this study proves is that passing through a cognitive disequilibrium seems a necessary step before a conceptual accommodation to the new contextual situation and realizing the beauty of mathematics.

All eight students of the larger study who could model exponentially at the end of the City A or City B problems (in the first interview) first expected a linear model, but the linear model did not fit well with the context they were understanding. This was the moment at which they experienced disequilibrium, and it was manifested in Tom's interviews very transparently: He was devastated when he divided 1843 by 1500. He needed to accommodate the new concept projection (exponential model); and a well-embedded Multiplicative Recursive Dependency (Allahyari, 2023) helped him to resolve the disequilibrium locally. Tom played genuinely with the context until he could model in a way that was confirmed with the context (conceptual accommodating to new contextual situations).

Implication

In the mathematics education field, such revelations about the interplay between context and students' experience of context (the whole experience, which involves the knowledge resources participants activate and not just students' feelings) is an important line of work that has practical significance with respect to teaching and learning. Besides exploring a dialectical movement between the epistemological investigation of learners' cognitive development and the ontological exploration of specific features of linguistic contexts, this type of study can underscore the need for more epistemological investigations across disciplinary topics that may lend insight into the organization of learners' perceptions and inferences.

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A NARRATIVE INQUIRY ACROSS RACE/ETHNIC GROUPS: PARENTS' MATHEMATICS LEARNING EXPERIENCES AND KINDERGARTENERS' MATHEMATICS INTEREST

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Learning and teaching mathematics are crucial yet challenging, with racial and ethnic disparities persisting in academic success. White and Asian-American children often outperform minorities. Early math education is vital for national development, but many low-socioeconomic children lack numeracy skills. Framed by sociocultural theory, this study investigates parent-child math experiences across four ethnic groups. Interviews with 20 pairs reveal themes like parental support and communication impact on children's interests. Parental experiences profoundly shape children's math interests, passed down through generations. Diverse parental attitudes prompt efforts nurturing mathematical curiosity. Communication, beyond verbal, bridges linguistic barriers and enriches parent-child bonds. This study adds to literature on parental involvement in math education and emphasizes collaboration among parents, children, and schools. Insights offer a basis for future research, promoting effective communication and equitable math proficiency.

Keywords: Early Math Interest, Mathematics Learning Experiences, Parent Impact on Student Learning, Parent's Mathematics Learning Experience.

The foundation of a child's initial math experience and identity is established within the home (Gozel & Toptas, 2018). According to the report by the National Mathematics Advisory Panel (NMAP, 2008), mathematical abilities in children develops during their early years, even before they enter kindergarten. Notably, when cultural values and parental involvement are acknowledged and integrated into classroom teaching methods, students, particularly those from marginalized cultures, tend to perform better (Hill, 2018). Consequently, parental involvement is a variable, among others that have a positive impact on children's mathematics education (Dinkelmann & Buff, 2016; Kung & Lee, 2016; Temur et al., 2018). However, despite these factors, students in the United States continue to struggle with mathematics learning, as reported by the National Assessment of Educational Progress (NAEP, 2015). The strength of mathematics knowledge during kindergarten and a positive self-perception of mathematics serves as strong predictors of students' career choices in mathematics, science/math teaching, and STEM fields of study (Cribbs, 2012; Morgan-Smith, 2019). Understanding the variables that contribute to positive mathematics learning, particularly at a young age, remains crucial in fostering students' engagement in mathematics and related fields of study (Cribbs, 2012; Hren, 2015; Kiss, 2018). Thus, this study aimed to examine the role of parents' mathematics learning experiences in how they engage their young children in mathematics learning at home.

Theoretical Framework

Drawing from various theoretical perspectives, including Vygotsky's Sociocultural Perspective, Culturally Responsive Theory, and the Concept of Interest and Learning, this study aimed to explore how parental discussions about mathematics shape young children's mathematical interests (Bikner-Ahsbahs & Presmeg, 2015). Research suggests that parental attitudes and past experiences with mathematics significantly influence children's engagement with the subject, with memories and feelings about mathematics learning dictating parental views of themselves as mathematics learners (Antolin Drešar & Lipovec, 2017). Moreover, children's interest in learning mathematics is influenced by the opinions conveyed by their parents, highlighting the importance of parental communication in fostering a positive learning environment (Hand, 2003; Franke et al., 2007). Through a qualitative design, this study sought to delve deeper into what parents are saying about mathematics and how these messages are interpreted by young students, particularly in light of parents' past mathematics learning experiences. Furthermore, this study assumed that young children's interest in mathematics can be influenced by their perception of their parents' attitudes toward the subject.

Grounded in Vygotsky's sociocultural perspective, the Concept of Interest and Learning, and culturally responsive theory, the research aimed to uncover the dynamics of parent-child interactions regarding mathematics (Rosa & Tudge, 2013; Eccles, 2014; Gay, 2018), these theories acceded to the crucial and significant role the home environment plays in a child's early life development and learning (John-Steiner & Mahn, 1996; Rosa & Tudge, 2013; Vygotsky, 1978); supporting the trajectory that human development, growth, and learning are 'formally and informally' impacted by their environment (Artigue et al., 2007, as cited in Lester, 2007, p. 1025). Within these structures, social and cultural practices and identities are formed. Knowledge of mathematics that gradually crystalize in human/cultural practices and interactions as they occur in homes, (Artigue et al., 2007, as cited in Lester, 2007, p. 1025) give credence to how mathematics understanding is shaped during early life and development (Mohr-Schroeder et al., 2017). Mathematics educators have the responsibility therefore, of understanding parents' mathematics learning experience, how this experience is communicated, and its impact on young students' mathematics interest; creating a deeper enabling environment for young students' mathematics learning (Hand, 2003, as cited in Franke et al., 2007).

However, despite the existence of the abundance of empirical studies on parental involvement of different constructs, little to no study exists in the body of literature on the construct of parental mathematics learning experience. The broad spectrum of the pedagogy on parental involvement captures the "home-based and school-based activities through which parents transmit their own skills, knowledge, attitudes, and values to their children. As a result of the complexity and multidimensional nature of the concept of parental involvement, parental experience appears absorbed in this construct. These constructs in research can be fully understood and captured with narrative (in-depth semi-structured or structured) interviews (Antolin Drešar & Lipovec, 2017). By understanding the role of parental communication in shaping children's mathematical interests, the study sought to contribute to the existing literature on mathematics education and inform strategies for promoting positive attitudes towards mathematics among young learners (Krapp, 1999; Renninger & Hidi, 2017).

Methods

Narrative inquiry allows for a nuanced exploration of parent-child interactions within the familial context (Bates, 2004; McMullen & de Abreu, 2011; Mruk, 2006; Mueller, 2019). Adopting a qualitative narrative design approach, drawing from a socio-cultural perspective and employing narrative inquiry methodology, this study sought to examine how individuals construct meaning from events in their lives, through meticulous examination of parents' stories about their experiences with mathematics. The use of episodic interviewing techniques enables participants to recount specific events and experiences, providing detailed insights into the dynamics of parent-child discourse regarding mathematics. Narrative inquiry offers a comprehensive exploration of parent-child interactions within the family microsystem. Additionally, narrative inquiry addresses gaps in prior studies by offering a qualitative exploration of parent-child discourse, which quantitative measures alone cannot capture (Mohr-Schroeder, 2017). Grounded in theoretical frameworks such as Bronfenbrenner's Ecological Systems Theory and Vygotsky's socio-cultural perspective (John-Steiner & Mahn, 1996; Rosa & Tudge, 2013; Vygotsky, 1978), and by analyzing narratives, this research aimed to deepen the understanding of the factors that shape young children's interest in mathematics, thereby contributing to the literature on mathematics education.

Building on a previous pilot study, this research employs both deductive and inductive approaches to understand the relationship between parents' mathematics learning experiences and their children's interest in mathematics (Alli-Balogun, 2022; Bates, 2004). The pilot study identified key themes, providing a framework for the analysis in the present investigation. By addressing limitations identified in the pilot study and deepening our understanding of parent-child dynamics in mathematics education, this research also aimed to contribute to the existing literature and inform strategies for promoting positive attitudes towards mathematics among parents and young learners (Smith & Johnson, 2010).

Participants. Kindergarten experiences can shape future math attitudes, impacting career success in STEM fields. Early math interventions benefit all children, especially those from underrepresented minorities and low socioeconomic backgrounds. To ensure diversity, this study recruited kindergarten children aged 4-6 and their parents. Participants included parents and children from African-American, White, Asian, and Hispanic/Latinx backgrounds. Forty participants, including 20 parents and 20 kindergarten children, participated in the study (see table 1).

Table 1: Gender Distribution by Race/Ethnicity

	Parents		Children	
Race/Ethnicity	Female	Male	Female	Male
Asian/Pacific Islander	2 (5.0%)	3 (7.5%)	3 (7.5%)	2 (5.0%)
Black or African American	4 (10.0%)	1 (2.5%)	4 (10.0%)	1 (2.5%)
Hispanic/Latinx	5 (12.5%)	0 (0%)	1 (2.5%)	4 (10.0%)
White	3 (7.5%)	2 (5.0%)	5 (12.5%)	0 (0%)

Data Collection. This qualitative study employed narrative data collection methods to delve into the complex dynamics of parental experiences and children's perceptions of mathematics. The research methodology centered on semi-structured interviews, strategically designed to elicit rich narratives from participants. To ensure diverse representation, a stratified purposeful sampling approach was adopted, targeting participants from four racial and ethnic backgrounds. Efforts to recruit participants involved both word-of-mouth referrals and targeted outreach in public spaces frequented by families, such as parks and eateries. Interview sessions, conducted either in-person or over the phone, were tailored to accommodate the attention span and comprehension level of both parents and children. Children's interviews were relatively brief, lasting approximately 10-15 minutes, while parents engaged in more extensive discussions lasting around 30 minutes. Throughout the interview process, meticulous attention was paid to maintaining participants' comfort and confidentiality. Audio recordings were used to capture participants' responses verbatim, allowing for thorough analysis. Additionally, follow-up prompts and post-interview communications were employed to clarify any ambiguities and ensure the accurate representation of participants' perspectives. Anonymity was preserved through coded identification numbers assigned to each participant, safeguarding their privacy throughout the research process.

Instruments The interview protocol employed in this study adhered to established qualitative research methodologies, drawing on semi-structured questioning to facilitate in-depth exploration of participants' experiences and perspectives. A primary instrument utilized is the episodic narrative interview technique, comprising six steps outlined by Mueller (2019), which guides researchers in eliciting detailed narratives. This protocol, validated through a process involving parents and kindergartners not included in the study, enabled the collection of nuanced accounts regarding parental experiences with mathematics learning and children's interpretations of these experiences. The interviews are carefully designed to ensure clarity and depth, incorporating prompts and questions adapted from previous research studies and validated instruments such as the Mathematics Interest Inventory (MII) by Stevens and Olivárez (2005). Interviews with parents and children are conducted with sensitivity to their comfort and engagement, with children's interviews deliberately kept shorter to accommodate their attention spans. Thematic analysis, conducted using NVivo coding software, facilitated the identification and exploration of patterns and themes within the collected data, contributing to a comprehensive understanding of the research questions at hand.

Data Analysis The analytical framework of this study was anchored in thematic analytical techniques developed by Braun and Clarke (2006), emphasizing the generation of in-depth data to foster a deeper understanding of early childhood mathematics learning. Through semi-structured interviews and meticulous transcription processes aided by Wreally transcription services and Nvivo coding software, each interview was thoroughly analyzed to identify emerging patterns and themes. The study's objective was to amplify the voices of both young learners (kindergarteners) and their parents, exploring the impact of parents' mathematics learning experiences and discussions on their children's interpretation and development of an interest in mathematics. Thematic analysis, following Braun and Clarke's phases, allowed for the identification and exploration of significant themes within the collected data, offering insights into parental roles, children's interpretations, and the dynamics of math-related conversations within families. By engaging in this comprehensive analysis, the study sought to address overarching research questions regarding parental experiences, intergenerational transmission of attitudes towards math, and its implications for children's attitudes and interests in the subject. During the coding phase, meaningful patterns emerged, such as the importance of math education, the use of manipulatives, and parental involvement in understanding math concepts. Table 2 provides an excerpt of some of these codes generated from the data.

Table 2: Sample Codes

Codes	Quotes
Math was taught as important	"I think my parents teach me that [math] is very important."
Using toys as manipulatives	"Counting blocks, toys sometimes."
Nurturing children's strengths and working on weaknesses	"My parents were very supportive and nurtured us where our strong suits are, [and] recognized where maybe we were weaker."
Parents helped in understanding math	"My dad would help me. He would help me try to figure out the equation."

These codes were clustered based on similarities, leading to the identification of initial themes. An example of an initial theme is shown in Figure 1.

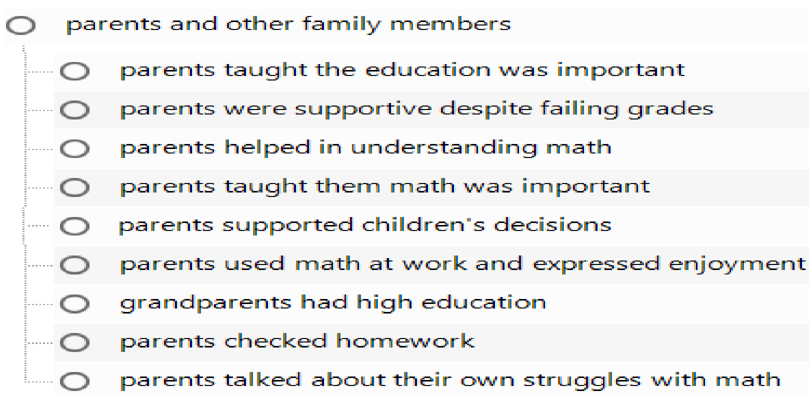

Figure 1: Sample Cluster

Subsequently, the initial themes underwent a higher level of abstraction, forming broader categories of meanings reflective of common experiences among participants. The hierarchical organization of codes facilitated by NVivo software enabled the formation and management of initial themes. Thematic maps were created to visually represent the relationships among identified themes, contributing to a cohesive narrative of participants' experiences. Through this rigorous coding and thematic analysis process, the study aimed to provide clarity and precision in understanding the complexities of parent-child interactions regarding mathematics learning, shedding light on the factors shaping children's attitudes and interests in mathematics.

Summary of Findings

An analysis of the data after repeated reading emerged seven themes. Through semi-structured interviews, participants shared their personal narratives with mathematics learning. Braun and Clarke's (2006) phases of thematic analysis were used on the interview data, from which the themes emerged. The first research question resulted in three themes: (a) Supportive Environment, (b) Own Math Ability, and (c) Perceived Usefulness. The second research question revealed two themes: (a) Indirect Conversations through Parental Motivation and (b) Indirect Conversations through Parental Support. The third research question resulted in two themes: (a)

Parental Influence and (b) School Environment Influence. The themes and their definitions are presented in table 3 below.

Table 3: Themes and Definitions

Themes	Definition
Supportive Environment: Having support contributed to parents' experiences in learning math	Parents who received support from parents, teachers, tutors, other family members and peers while learning math typically enjoyed the learning experience, while parents who did not receive or lacked support typically did not enjoy learning math
Own Math Ability: Math ability contributed to parents' experiences in learning math	Parents who perceived themselves to be good in math generally enjoyed learning math as a student, while parents who perceived themselves to lack analytical skills generally expressed disliking math when they were students
Perceived Usefulness: Perceived usefulness of math contributed to parents' experiences in enjoying math	Parents who perceived math was useful for their future careers and continued to apply math in their current careers generally liked math, while parents who did not find math useful generally disliked math
Parental Motivation: Parents motivate their children to learn math	Regardless of racial and cultural backgrounds, parents typically wanted their children to do well in math and help them become encouraged to learn math
Parental Support: Parents support their children's math learning	Regardless of racial and cultural backgrounds, parents generally provided the same or more support for their children while learning math than they received when they were students
Parental Influence: Children's interest in math is influenced by parents	Children typically expressed feeling good about math when their parents helped them with homework and integrated math activities in their everyday lives, and when they perceived that their parents liked math
School Environment Influence: Children's interest in math is influenced by their experiences at school	Teachers' support for children in understanding math while at school influenced the students' drive to remain engaged in learning math

Discussion and Implications

It is widely understood that decisions, policies, curriculum are designed and developed in response to research findings and the significance of parents' own mathematics learning experience on students' mathematics learning and identity is not an exception which hitherto has little to no evidence in literature. The thematic findings of this study supported the theoretical notions of Vygotsky and Bronfenbrenner on human development, sociocultural perspective and learning (Bronfenbrenner, 1992; Vygotsky, 1978; 1995), and Epstein's theoretical Model of School-Family-Community Partnership (Epstein, 1995). Vygotsky's theory stipulated that, though a child can be "spontaneous", meaning to understand or learn something on their own, there are other learnings a child gets only within an instruction, also known as "nonspontaneous" (Clarà, 2017, p. 55). Vygotsky described this type of child-adult collaboration as 'Intellectual Imitation', in which the adult forms a meaning, new for the child, and the child imitates that meaning (Clarà, 2017, p.55; Vygotsky, 1987, p. 210). Inferring to the type of parental involvement in kindergarteners' mathematics learning as suggested by the thematic findings, this study draws attention to how teachers and the mathematics community can be guided in support of parents in advancing mathematics learning of young scholars.

Further, the themes that emerged revealed that parent's mathematics learning experience influenced their role in their young children's development of mathematics interest. Parents' own positive experiences of learning mathematics in school and their own parents' influence on their mathematics learning impacted their perceptions of the need to develop their children's mathematics skills. Intellectual imitation does not always result in the formation of non-spontaneous meaning (p.55). Subsequently, in this study, when the parent participants acknowledged that their role in encouraging and supporting mathematics learning of their child originated from their own positive mathematics learning experiences, influenced by their own parents, it could be identified as intellectual imitation. Further, it was stipulated by Vygotsky (Clarà, 2017) that a child attains a higher level of development when a series of spontaneous meaning is formed by the child from a non-spontaneous meaning that the child gathered from an intellectual imitation of adult-child instruction. This higher level of development is referred to in Vygotsky's theory as the Zone of Proximal Development (ZPD) (Clarà, 2017) within which lives the possibility of the child attaining a higher level of development from a non-spontaneous encounter.

The parents in this study generally perceived that their children needed to learn mathematics and that mathematics was an important life skill. The themes generated from the data revealed that parents may not necessarily orally communicate their mathematics learning experiences to their kindergarten children, however parents often helped their children become familiar, interested, and comfortable with mathematics.

Limitations and Areas for Future Study

The investigation into parents' discussions about math with their kindergarteners was framed within a socio-cultural lens, drawing on key theoretical perspectives such as Vygotsky's sociocultural theory and Bronfenbrenner's ecological systems theory. These frameworks provided a rich foundation for comprehending the dynamic interactions between parents, children, and their socio-cultural contexts. However, the generalizability of the research findings is inherently constrained by several factors, including geographical location, sample size, and the subjective nature of participants' responses, which may be susceptible to bias. While the study's sampling strategy was purposive and incorporated a stratified random selection method to ensure representation of the socio-economic and ethnic diversity within the target population, its scope is inherently delimited to the specific context of kindergartners and their parents, rendering expansion of the results to broader populations questionable. Future research endeavors of this nature should consider designs integrating parent-child observations to mitigate the impact of subjective bias arising from self-reported accounts.

The implications of this study extend to various stakeholders, including parents, educators, policymakers, and researchers. For parents and educators, fostering engagement in math learning through collaborative efforts and culturally responsive approaches is essential. Parental attitudes, both conscious and subconscious, are undeniably shaped by their historical encounters with learning mathematics, and enhancing parental involvement in their children's mathematical education could be achieved through rudimentary training programs. Additionally, educators should develop and implement culturally responsive mathematics curricula that recognize the impact of socio-economic disparities on students' math interest and integrate diverse cultural perspectives to create an inclusive learning environment. Policymakers should support the establishment of parent engagement programs and allocate resources for comprehensive professional development programs for teachers to address the diverse needs of students within a math classroom. Furthermore, future research directions should focus on promoting mathematical literacy, incorporating diverse math role models, and developing early intervention initiatives to nurture math interest among young children from various socio-economic backgrounds. Through collaborative efforts, stakeholders can create a supportive ecosystem that nurtures mathematical curiosity and enthusiasm, ultimately creating a more inclusive and equitable mathematics education environment.

Conclusion

This narrative inquiry, rooted in Vygotsky's sociocultural perspective, culturally responsive theory, and the nexus of interest and learning, delved into the intricate dynamics shaping kindergarteners' mathematical trajectories. Conducted with 20 parent-child pairs from diverse ethnic backgrounds, the study explored parental discussions about mathematics and their impact on children's perceptions and interests. It sought to bridge racial and ethnic disparities in these conversations, aiming to nurture curiosity and reverse declining minority participation in STEM fields. Key themes emerged, highlighting the profound influence of supportive environments, intergenerational transfer of mathematical skills, and the perceived usefulness of mathematics. These insights emphasized the critical role of parental engagement in fostering robust educational foundations and equal opportunities, illuminating pathways for collaborative synergy between parents, children, and schools to shape effective learning experiences and scholarly achievement. Moreover, the study revealed how parents creatively communicated their mathematical experiences to their children, surpassing verbal exchanges and embedding a legacy of positive mathematical attitudes for future generations.

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FRUSTRATION AND POTENTIAL: EARLY ELEMENTARY SCHOOL STUDENTS' FEELINGS TOWARDS MATH ERRORS

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"Sometimes I feel disappointed in myself, but I know I can always try again and get better." Here, third grader Torrin described contradictory feelings about making mistakes in math; he felt disappointed and saw potential for improvement. His nuanced take reflects a dichotomy not often seen in adolescents' relationships with mathematical mistakes (Bray, 2013; Cohen, 1990; Radatz, 1980; Santagata, 2004). Teachers' perceptions and use of errors in math class often lead middle and upper grade students to internalize errors as reflective of their capabilities (Son & Sinclair, 2010). In turn, mistakes become detrimental to students' mathematics identities (Martin, 2006; Steele, 1997) instead of being opportunities to learn. Research shows that when teachers do leverage mistakes as valuable sources for inquiry (Borasi, 1987; Kramarski & Zoldan, 2008), adolescent students are more willing to engage with difficult problems and explore their own errors publicly (Leatherwood, 2022). However, teachers often shy away from this model and instead avoid error exploration in fear of damaging students' self-esteem or encouraging the reproduction of incorrect computations (Bray, 2013; Ma, 1999; Schleppenbach et al., 2007). Though literature has explored how young children learn mathematics (e.g. Kilpatrick et al., 2001) and their dispositions towards mathematics (e.g. Beyers, 2011), young children's feelings towards errors have not been extensively studied. This led us to ask, "How do early elementary school students feel about making mathematical errors?"

The data for this study come from a dissertation project (Altshuler, 2022). Thirty 1st-3rd grade Chicagoland students participated in semi-structured interviews (Spradley, 1979) over Zoom in November 2020. As identified by their caregivers, 18 of the participants were female, and 12 were male. Five were African American/Black, 5 were Asian, 2 were Hispanic/Latinx, 15 were white, 2 were multiracial, and 1 self-identified as Middle Eastern. Our conversations addressed students' experiences with and feelings towards math; this study involved analysis of interview questions that focused specifically on students' responses to mistakes. Using the identify-and-eliminate (Radatz, 1980) and error-for-inquiry (Borasi, 1987) theoretical frameworks, we engaged in qualitative analysis, which included line-by-line investigation of students' quotes, clustering of students' responses by theme, and identifying patterns across the data set.

Aligned with an identify-and-eliminate model, preliminary findings show that early elementary school students often experience negative emotions when they make mathematical mistakes. These negative sentiments are commonly directed at themselves–for not understanding the material, not completing the work accurately, or for facing consequences like receiving a low grade–and are similar to what is seen in the literature on adolescents' feelings towards errors. However, unlike adolescents, the young students in this study also acknowledged their potential to change and grow, often expressing that they had the opportunity to try again after making a mistake. These sentiments are more aligned with an error-for-inquiry model. If teachers leverage these sentiments and emphasize the opportunity to learn from mistakes, we can support students to build resilience, perseverance, and positive identities as mathematical sense-makers.

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MATHEMATICS AND SCIENCE SENSEMAKING IN LEARNING THE NONLINEAR DYNAMICS OF CLIMATE MODELS

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Keywords: Modeling; Sustainability; Undergraduate Education; Integrated STEM/STEAM

Global models of climate often consist of computer simulations that represent climate using mathematical equations carrying scientific meaning (Skovsmose, 2021). The mathematical and scientific foundations of climate models can provide opportunities to integrate mathematics and science instruction, fostering a deeper understanding of both subjects (Barwell, 2013). The North Atlantic circulation system, for example, has been modeled using systems of nonlinear differential equations, which are based on the salinity and temperature differences across the North Atlantic (e.g., Stommel, 1961). This model shows how ocean currents can slow due to rising temperatures and freshwater influxes from melting ice, limiting the flow of warmer water traveling to higher latitudes. With less warm water traveling to polar regions, equatorial regions remain warm as the polar regions become colder. To understand how students conceptualize such models, we examine undergraduate students' mathematics and science sensemaking processes while working on a model of the North Atlantic current.

We employ the *Sci-Math framework* (Zhao & Schuchardt, 2021) to understand how students engage in mathematical and scientific sensemaking during a climate modeling task. Scientific sensemaking can encompass how students interpret the meaning of variables and equations, as well as understanding the trends or patterns among such variables. Scientific sensemaking also includes how students point to mechanisms that explain how or why a scientific phenomenon occurs. Mathematics sensemaking can include how students complete mathematical procedures and identify mathematical rules to guide their calculations. It also includes how students make sense of the structure of an equation, the quantitative relationship between variables, as well as the underlying mathematical concepts behind an equation.

We examine how undergraduate students enrolled in a biology course make sense of a model of Earth's climate. During the climate model task, students adjust the initial values for temperature and salinity differences across the two regions of the North Atlantic, which affect the rate of ocean flow. Through discussing the stark contrast between initial conditions that create high and low water flow between two regions of the North Atlantic, students can make sense of the nonlinear behavior of climate systems by recognizing that the increasing global average annual temperatures do not always lead to higher temperatures everywhere on the planet.

We collected data from students working in small groups on the climate modeling task. Data include written student materials and video-recordings of their group work. To analyze the data, we use a priori coding schemes from the Sci-Math framework (Zhao & Schuchardt, 2021).

Preliminary results indicate that individuals notice patterns both qualitatively (i.e., general understanding of science concepts) and quantitatively (i.e., make sense of the rates of change and covarying variables). These results provide a springboard into understanding how the nonlinear dynamics of climate change can be taught in mathematics and science classrooms.

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EFFECT OF IBL VS LECTURE-BASED INSTRUCTION ON CALCULUS I STUDENTS' MATH ANXIETY

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Math-anxious students exhibit physical, mental, and emotional symptoms. These symptoms often have short-term and long-term impacts on students' mathematics learning. This research sought to understand the effect of inquiry-based learning versus lecture-based instruction on Calculus I students' math anxiety. Findings revealed that the activities, such as collaborative work, optional and ungraded work, and the instructor's caring nature decreased IBL students' anxiety, whereas the instructor's readiness to explain the material in class when students asked decreased lecture-based students' anxiety. However, the tests and exams and responding to the instructor's questions in front of their peer increased groups anxiety.

Keywords: Math anxiety, inquiry-based learning, collaboration.

Math anxiety significantly affects students' mathematics learning and academic performance. Students who are highly anxious exhibit physical, mental, and emotional symptoms. Physical symptoms include nausea, sweaty palms, and increased cardiovascular activity (Ashcraft, 2002; Chang & Beilock, 2016). Mental symptoms include an inability to concentrate and mind blanking (Plaisance, 2009; Ruffins, 2007). Emotional symptoms include extreme nervousness and apprehension (Mattarella-Micke et al., 2011). As a short-term consequence of these symptoms, students may dislike mathematics and take fewer mathematics courses, and in the long term, they tend to avoid mathematics and mathematics-related courses (Godbey, 1997; Hembree, 1990). Due to such impacts of math anxiety on students' mathematics learning and performance, this research investigated the root causes of math anxiety and explore the potential interventions to alleviate it. As such, this study explored students' perceptions and experiences of learning Calculus I via inquiry-based learning (IBL) versus lecture-based instruction and the impact of these instructional approaches on respective groups' math anxiety.

Literature Review

Math anxiety has been a part of the human experience for a long time. The verse, "Multiplication is vexation ... and practice drives me mad," can be traced back to at least the 16th century (Dowker et al., 2016). However, it wasn't until 1957, when Dreger and Aiken introduced the concept of "number anxiety," that math anxiety began to receive increased attention. Subsequently, Richardson and Suinn (1972) conducted the first formal study of math anxiety, defining it as "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide range of everyday life and academic situations" (p. 551). Since then, research on math anxiety has been extensively investigated.

Traditional lecturing, which is a predominant mode of instruction in college mathematics courses across the United States (Stains et. al., 2018) and is ineffective in helping students learn mathematics (Boaler, 2008), could be one of the possible reasons for evoking math anxiety among students. The lecture-based does not offer substantial opportunities for students to share each other's ideas and experiences with their teachers and peers. On the other hand, IBL, which is an active learning pedagogy, provides extensive opportunities for students where they can work in pairs or groups to make conjectures, gather information for problem-solving, and present their work to groups and to the whole class (Kogan & Laursen, 2014). Through a comparative study, Laursen et al. (2014) reported that students in IBL math-track courses achieved greater learning gains than their non-IBL peers in cognitive, affective, and collaborative areas. Similarly, Laursen et al. (2011) found that the IBL students were involved more in interacting with each other and with the instructor, and they were more involved in setting the course pace and direction. It is also reported that IBL enhances students' conceptual understanding (Jensen, 2006), communication skills, confidence, and self-efficacy (Laursen et al., 2011). Considering such benefits of IBL, this study examined the relative changes in the anxiety scores of Calculus I students' using an abbreviated version of the Mathematics Anxiety Rating Scale (MARS-S).

Methodology

Research Background and Participants

The sample of this study comprised of students who were enrolled in the Calculus I course and were taught using either IBL or lecture-based instruction during Spring 2021 at a university located in the Midwestern United States. Students who received IBL instruction were in the IBL group, and those who received lectures were in a lecture-based group.

The IBL instructor usually began the class by welcoming students and briefing them on the activities for the day. When he had to introduce a new lesson, he would begin the class with an interactive lecture; otherwise, he directed students to work collaboratively in Teams breakout rooms, where they usually shared each other's ideas and solved assigned problems. The instructor would visit each group and monitor their work. At the end, the students would return to the main room, where the instructor facilitated a whole class discussion.

On the other hand, the lecture-based instructor began the class by asking if the students had any questions from the previous class. If they had questions, the instructor explained the concepts as needed. Then, the instructor usually began the lecture by solving preselected examples using the Notability app from his iPad. Occasionally, the instructor paused during the lecture and asked some questions to the entire class. Students were never sent to breakout rooms and were never provided opportunities for group discussions.

Data Collection and Analysis

This study was conducted in two phases. First, data were collected from 15 students in the IBL group and 23 students in the lecture-based group using the MARS-S as a pre- and posttest. Based on the change in anxiety scores from pre to posttest, 9 students from the IBL group (3 greatly increased, 3 greatly decreased, and 3 did not change much) and 3 students from the lecture-based group (1 student from each category) were interviewed using semi-structured questions such as "Could you please describe your reactions when you felt anxious (e.g., sweaty palms, inability to concentrate, increased heartbeat)?" The data thus collected were transcribed using NVivo, a qualitative data analysis software, and several open codes were generated. Then, the codes were combined to identify various emergent themes. The results were then organized based on these themes.

Results

In the following sections, I present the major findings from the analysis of the data obtained from the class observations and student interviews. In so doing, I compare the results from the cross-analysis of the IBL and lecture-based group's data.

The in-class group work and lecturing reduced IBL students' anxiety; out-of-class group chats reduced lecture-based students' anxiety. IBL students experienced about 20% lecturing and 80% collaborative work in each class, while lecture-based students were fully lectured in each class, but some students managed to communicate via the GroupMe chat application. An IBL student said, "I think doing the task in a group reduced it for sure, and ... watching the videos in class, like that does not stress me out or anything." Because of the collaborative learning opportunities, IBL students became closer to each other and were comfortable asking questions, responding to groupmates' questions, proposing their solution strategies, and listening to their peers' different perspectives. Such activities made them feel relaxed and also helped to reduce their anxiety. On the other hand, lecture-based students initiated out-of-class collaboration via GroupMe. A lecture-based student said, "We all try to help each other out [on GroupMe] so I would say it's very collaborative." They maintained this throughout the semester, where they felt comfortable asking questions and responding to each other. The cooperation and mutual support received by these students during the chat reduced their anxieties and frustrations while learning Calculus I.

Instructors' readiness to meet and help students at any time decreased IBL students' anxiety; the long wait for the instructor's email replies increased lecture-based students' anxiety. The IBL instructor's readiness to meet with students at any time and discuss their questions, concerns, or problems in and out of class decreased IBL students' anxiety. IBL students felt comfortable asking questions to the instructor at any time—during class, at the end of class, right after class, or by scheduling a virtual meeting—and discussing their problems. They could approach the instructor at "eleven o'clock at night or 7:00 am that morning;" he was always ready to meet and talk with them. Students could also send their questions or concerns via email, to which he replied promptly. This kind of flexibility from the instructor made students calm down and reduced their anxiety. Alternatively, although the lecture-based instructor encouraged students to reach out to him with their questions, students had to wait for long hours for his email responses. Students were disappointed and frustrated by not receiving a response to their emails for several days. Liz, for example, mentioned, "We have an assignment that's due on Friday, and I need help. And I can't afford to wait three days for you [the instructor] to email me back because this piece of information is necessary for me to do 60 percent of the assignment."

Optional tasks and ungraded assignments decreased IBL students' anxiety, and the overwhelming number of online homework problems, especially at the beginning of the semester, increased lecture-based students' anxiety. In addition to the midterms, the IBL instructor engaged students in both optional and mandatory work, which was rarely graded. His pretasks were optional and ungraded but recommended for students to complete before each class. A student stated, "I know they're just used for practice and prep for the exams and for class. So, I don't feel as stressed about them." However, the tasks, mock exams, and group competitions were mandatory and required to be completed during class but were rarely graded. Students did not feel stressed working with the optional and ungraded tasks and activities because they did not impact their grades; rather, these activities reduced IBL students' anxiety by developing confidence and problem-solving ability.

Lecture-based students, on the other hand, experienced stress, frustration, and anxiety when they confronted so many online problems as homework assignments during the first week. A student stated, “He gave us homework assignments that had eighty-seven problems. ... But that was really the only thing that made me feel anxious about the assignments.” Students were anxious, thinking that they might continue receiving problems in the same manner. However, their anxiety reduced as the online homework problems decreased as the semester progressed.

Both IBL and lecture-based students felt anxious about being called on during the lectures and whole-class discussions. Students in both groups experienced anxiety, specifically during whole-class discussions, thinking that the instructor might randomly call on them for a response at. An IBL student, Camila, stated, “The rest of the time, I am comfortable until he directly calls on me.” These students were worried about their colleagues’ judgments when they could not respond to the instructor's questions correctly and instantly. Likewise, lecture-based students were worried about being called on during the lecture for similar reasons. Unlike the IBL students, lecture-based students were usually busy “trying to write everything down” that the instructor wrote, and they experienced nervousness when they did not respond quickly to the instructor’s abrupt questions. One of the lecture-based students, Liz, stated, “Occasionally he would … call on someone, and if they wouldn't know, he'd be like, you should know this. I would be, like, woo, I should know this by now, which would freak me out a little bit.”

Tests and quizzes, online learning, and proctored-track exams increased both IBL and lecture-based students’ anxiety. Both IBL and lecture-based students were anxious about learning online virtually and taking proctored-track exams. Although midterms and final exams were cumulative for IBL students, they were more anxious about taking the final exam than the midterms. “I was very confident going into it [exam]. And then, I started the test, and I felt like my mind just went blank, and I couldn't think.” Likewise, the lecture-based students were anxious about taking the tests and the final exams because they weighed the largest proportion of the overall grades. A lecture-based student reported, “I get anxious, especially when I'm taking a test; my heart races, and I feel like my throat sort of closes a little bit.” Also, lecture-based students were anxious to see a few questions on the exams because they would lose huge points if they did even a single problem incorrectly.

Online learning was another aspect that induced anxiety among both groups of students. IBL and lecture-based students occasionally missed part of or the entire class due to poor Wi-Fi connections, which added extra pressure on students. Moreover, proctored-track exams caused anxiety among both groups of students. They were anxious about being watched by someone remotely and hearing a loud noise when someone asked clarifying questions to the instructor.

Instructors’ questions increased both IBL and lecture-based students’ anxiety. Instructor questions—either during lectures or whole-class discussions—increased both IBL and lecture-based students’ anxiety. Three out of nine IBL students said that they feared the instructor’s questions, specifically when they were unsure of the answers and were still processing the information. Maria said she is terrified of instructor questions, thinking of giving an incorrect answer in front of her colleagues who were not from her small collaborative group. A lecture-based student said, “I was a little stressed out because he would call on people, and I was scared that he would call on me.” Although lecture-based students experienced similar types of anxiety in responding to instructor questions, they were also anxious about the instructor’s verbal pressures, such as, “There's been a couple of times when he [the instructor] said, you should know this, and I did not know it, and that made me very anxious," Liz from the lecture-based class said.

Discussion

Results show that factors such as the instructor's amicable and cordial nature, extended help for students at all times, in-class group work, and optional and ungraded homework decreased IBL students' anxiety. However, factors such as responding to the instructor's questions and asking questions in front of their peers increased these students' anxiety. On the other hand, activities such as an overwhelming number of online homework problems that were due every week, thinking of being called on for a response in class, responding to the instructor's questions, and fast-paced teaching increased lecture-based students' anxiety. However, the instructor's readiness to explain the materials in class reduced the lecture-based students' anxiety.

Based on the findings from the present study, I recommend that although some factors of the instructional environment, student behaviors, and instructor behaviors have been found to increase anxiety among both groups of students, many other factors have lessened their stress, frustration, and anxiety. It is suggested to the instructors of Calculus I and other similar mathematics courses to implement IBL in their classes or at least transition toward student-centered approaches to instruction because of the benefits of such instructional practices on students' critical thinking, reasoning, and problem-solving abilities. It is also suggested to avoid the instructional activities that have been found to increase students' anxiety.

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EFFECTS OF STUDENTS' IMAGES, ANXIETIES, AND ATTITUDES TOWARD MATHEMATICS ON THEIR VALUE OF THE SCHOOL SUBJECT

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Affective state may play a significant role in shaping what students value in mathematics. This study examines how high school students' attitudes, anxieties, and images of mathematics may affect their values toward mathematics. A quantitative survey was conducted with 208 grade ten students from 6 randomly selected high schools in Kathmandu, Nepal. The survey data was analyzed with factor analysis and structural equation modeling. The results revealed that students' overall attitudes toward mathematics significantly affected their perceived values of mathematics. Students' anxieties and images of mathematics did not significantly affect their values of mathematics. The pedagogical implications of the findings have been discussed.

Keywords: Images of mathematics, Math Anxiety, Attitude toward Mathematics, Values of Mathematics.

Introduction

There is a growing interest in the affective constructs of mathematics learning, such as images, anxieties, and attitudes toward mathematics. Several past studies focused on students' decisions to learn and use mathematics in everyday life (Wilkins, 2003), students' images of mathematics (Jankvist, 2015; Sam, 1999), mathematics anxieties (Ma & Kishor, 1997), attitudes toward mathematics (Niepel et al., 2018), and value of mathematics (Österling, 2013; Wakhata et al., 2022). Past studies interrelated students' anxieties and attitudes toward mathematics to their achievement (Ma & Kishor, 1997) or highlighted the importance of these affective variables in shaping students' interest, motivation, self-esteem, and perseverance to learn mathematics. However, most of these studies focused on images, anxieties, confidence, and attitudes toward mathematics and their relation to students' performance, but not in an integrated way to examine how they affect each other (Belbase, 2013). In this context, there are a few studies on the values of mathematics concerning students' images, anxieties, and attitudes toward mathematics (e.g., Lamichhane & Belbase, 2017; Paudel, 2019; Thapa & Paudel, 2020). Students' images, anxieties, attitudes, and values in mathematics have been a general concern in Nepal because many students either fail or perform poorly at the school level and national exams (Education Review Office, 2015 & 2020). It is the interest of teachers, students, and educators to know how these four constructs interact and influence each other. In this context, the research question for this study was: How do the affective constructs of mathematics anxiety, attitude, image, and value predict each other?

Images, Anxieties, Attitudes, and Values of Mathematics

The term image of mathematics has been studied from different perspectives, for example, teachers' images of mathematics (Mura, 1993), public images of mathematics (Sam, 1999), and students' images of mathematics (Jankvist, 2015; Lamichhane & Belbase, 2017). This perspective views mathematics differently, demonstrating different images of mathematics, for example, symbolic images, utilitarian images, and absolute images (Sam, 1999). Students' images of mathematics can be depicted with different metaphors as a useful tool to understand students' images of mathematics (Osman et al., 2010). These metaphors were solving riddles, using a calculator, considering bank manager as a profession, and mathematics as numbers. On the other hand, mathematics anxiety can be considered one of the negative factors that may cause "feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (Richardson & Suinn, 1972, p. 551). Therefore, mathematics anxiety may also cause them to realize that they are "powerless, out of control, lacking in self-esteem" (Zaslavsky, 1994, p. 19). The attitudes result from positive, neutral, or negative experiences developed over a long period, seemingly directed to act and behave differently upon any object, event, problem, or human being (Elci, 2017; Utsumi & Mendes, 2000). Students' attitudes toward mathematics are associated with value, self-confidence, enjoyment, and motivation (Primi et al., 2020; Tapia & Marsh, 2004). In this context, students' success or failure in mathematics may depend upon their attitudes and beliefs toward the importance, worth, and usefulness of mathematics (Elci, 2017; Kunwar, 2020). In contrast, students' values of mathematics focus on what they consider as a part of the subject matter and learning it in social, historical, cultural, and technological contexts (Baba et al., 2012; Tang et al., 2020). These values also connect with relationship, power, and identity (Guitérez, 2010; Österling, 2013; Skovsmose, 2009), and demonstrate one's normative and historical viewpoints about mathematics and its processes (Baba et al., 2012; Ernest, 1991). Therefore, students' value of mathematics may also relate to their mental, social, and cultural wellbeing (Clarkson et al., 2010; Bishop, 2012), which may provide them with confidence and positive thought about it.

Method

A quantitative cross-section survey design was used in this study to explore the high school students' affective states in terms of images, anxieties, and attitudes toward mathematics and how these states may influence students' values of mathematics. The study design was based on a Likert-scale questionnaire distributed to a sample of students studying at grade ten in public and private schools in Kathmandu. The tool was based on the theoretical construct of images, anxieties, and attitudes toward mathematics (Belbase, 2013; Fennema & Sherman,1976) and methodological assumptions of survey design (Cohen et al., 2018). The study site was Kathmandu, Nepal. A sample of 208 grade-ten students from six secondary (three private and three public) schools in Kathmandu was randomly selected for the study. A questionnaire was administered in the sample schools by visiting the schools by two researchers. Informed consent was obtained from the schools, mathematics teachers, and students in grade ten classrooms in the respective schools. The data were transformed into Statistical Package for Social Sciences (IBM SPSS 28) and AMOS 28 for analyses and interpretations. The overall reliability of the instrument was measured with a sample of 208 students, and Cronbach's alpha was found to be 0.932 for 41 items (Cohen et al., 2018). Confirmatory factor analysis was performed in IBM SPSS 28, and it was validated with a structural equation model in IBM AMOS 28 considering image, anxiety, attitude, and value toward mathematics as exogenous (independent) and endogenous (dependent) variables turn by turn to discover the best predictor model (Kline, 2005).

Results

A path analysis was performed in the structural equation modeling (SEM) to determine if students' affective constructs of anxieties, attitudes, images, and values of mathematics influence each other. The iterative processes of path analysis were performed for anxieties, attitudes, images, and values to validate the items in models 1 to 3. Fit indices of CMIN/DF, GFI, CFI, and RMSEA were used to validate the confirmatory factor analysis and the direct impact of independent variables on independent variables. The analysis of fit indices showed that the model fit was robust in the third iteration with CMIN/DF = 1.477, GFI = 0.861, CFI = 0.916, and RMSEA = 0.048. These results were acceptable (West et al., 2023). The structural equation models were fitted for each of these variables, considering them dependent variables one by one and others as independent variables in different models (Table 1, Figures 1- 4).

Results showed that mathematical attitude was a significant predictor of students' values of mathematics, but anxieties and images were not significant at 0.05 level of significance (Table 1 and Figure 1). Likewise, attitude significantly predicted students' mathematics anxiety at a 0.01 significance level (Table 1, Figure 2). Anxiety was also a significant predictor of students' attitudes toward mathematics (Table 1, Figure 3). None of the three variables, anxieties, attitudes, and values, were significant predictors of students' images of mathematics (Table 1, Figure 4). The model to predict attitude was the strongest among the four models, with the largest R-squared value. Results of Models 4-7 are presented in the Table 1 and Figures 1-4.

Table 1: Maximum Likelihood Estimates Standard Regression Weights (SRW) and R^2

Model 4		Model 5		Model 6		Model 7	
Variables	SRW	Variables	SRW	Variables	SRW	Variables	SRW
VAL←AN	0.078	AN←IM	0.103	AT←AN	0.670**	IM←AN	0.280
VAL←AT	0.576*	AN←AT	0.792**	AT←VAL	0.270*	IM←VAL	0.328
VAL←IM	0.217	AN←VAL	0.043	AT←IM	0.038	IM←AT	0.123
R^2	0.642	R^2	0.801	R^2	0.832	R^2	0.459

Note: *Significant at 0.05 level of significance. ** Significant at 0.01 level of significance.

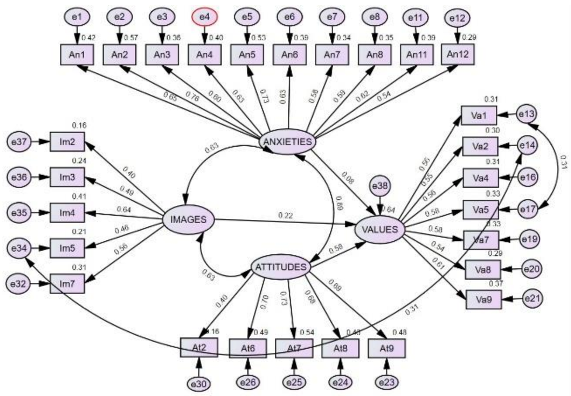

Figure 1: Model 4 for Impact of Images, Anxieties, and Attitudes on Values

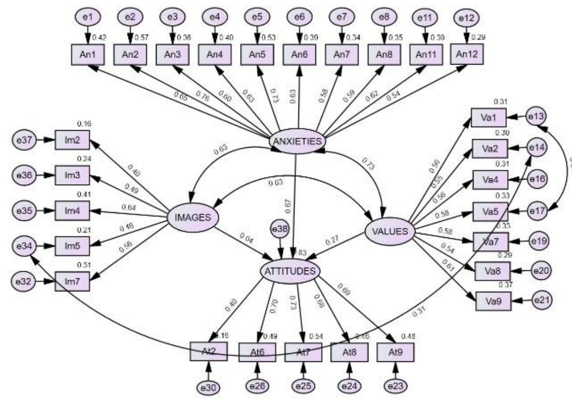

Figure 3: Model 6 for Impact of Images, Anxieties and Values on Attitudes

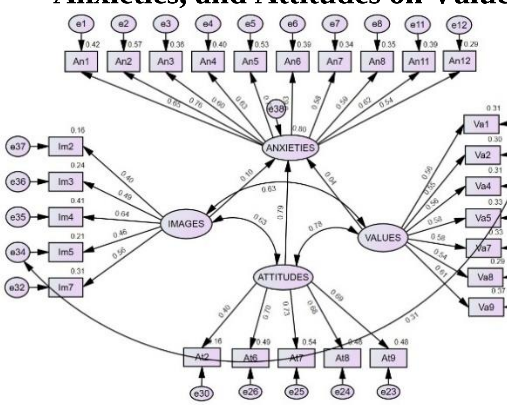

Figure 2: Model 5 for Impact of Images, Values, and Attitudes on Anxieties

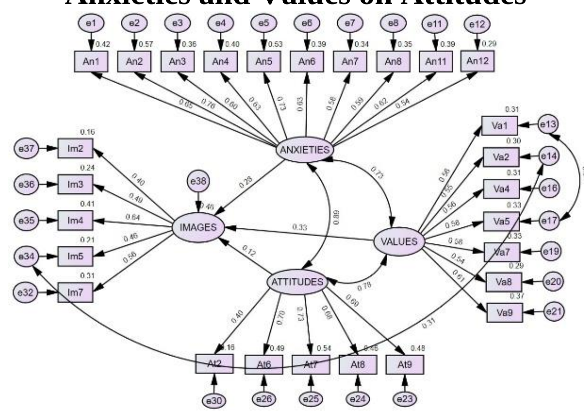

Figure 4: Model 7 for Impact of Values, Anxieties, and Attitudes on Images

Discussion, Conclusion, and Implication

The four components of affective states in students' mathematics were—anxiety, image, attitude, and value. The structural equation models demonstrated an impact of students' attitudes toward mathematics on values. These values are associated with self-efficacy, mathematical tasks, and its application in life and career has been studied and used to predict such mathematical behavior of learners (Gjicali & Lipnevich, 2021). Students' attitude toward mathematics significantly predicts their intention to study mathematics, their behavioral engagement as mediated through intention, and their mathematics achievement when mediated with intention and behavior (Gjicali & Lipnevich, 2021; Wang et al., 2020 & 2022). These studies and others related students' attitudes toward their value of mathematics regarding their intention, role, and action to perform or improve in mathematics. However, students' mathematics anxiety and image of mathematics were not good predictors of value. School and classroom context may significantly shape and interact among these psychological and affective constructs (Akey, 2006). The students may feel that mathematics is easy or difficult and that high or low anxiety should not affect their value toward mathematics (Lamichhane, 2020; Sam, 1999). Likewise, anxieties and attitudes toward mathematics seem to be strong predictors of each other, which is also supported by literature (Elci, 2017; Primi et al., 2020; Zaslavsky, 1994).

In conclusion, the results of this study showed that four constructs- images, anxieties, attitudes, and values may interact and influence each other. Students' attitude toward mathematics can predict their value of the subject positively. Attitude and anxiety can predict each other. However, the image of mathematics is a general construct that may not be predicted by attitude, anxiety, and value of mathematics. Mathematics teachers and educators may use students' mathematics attitude as a predictor of what they value and what is their anxiety level that may influence the quality of their mathematics learning. These psychological constructs may promote better mathematics classroom learning experiences (Everingham et al., 2017). Therefore, mathematics teachers should consider them while teaching students so that they are interested and motivated. Hence, mathematics teachers may focus on students' wellbeing through positive psychological development with images, anxieties, and attitudes for a greater value of mathematics learning (Hill & Seah, 2023). The study had some limitations due to the sample size of 208 students, which may influence the findings' generalizability in the larger student population in Nepal and elsewhere. Future studies should be conducted with a larger sample size and more sophisticated analyses to generalize results at national and international levels.

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EQUITABLE AND DYNAMIC APPROACHES TO ASSESSING EXECUTIVE FUNCTIONS IN THE MATHEMATICS CONTEXT

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Executive functions (EFs) are related to mathematics achievement, yet much is still unknown about how or why these relations exist. Improved measurement of EFs in students' math learning is needed to answer such questions equitably. Further, if research is conducted with the goal of informing teaching and learning, the EF assessments should focus on students' strengths and provide data that is useful and understandable to educators and students. In this paper, we explore how interdisciplinary teams of educators, developers, and researchers have assessed EFs in the mathematics context. We present strategies for assessing EFs more equitably and discuss implications for measuring EFs on various partners' activities within the research and development and implementation of mathematics curricula.

Keywords: Cognition; Equity, Inclusion, & Diversity; Assessment

Introduction

Executive functions (EFs) are a set of cognitive processes important for directing our thoughts and actions to what is necessary for achieving our goals. One popular model of EFs categorizes them as three separable, but overlapping functions known as working memory, inhibition, and cognitive flexibility (Miyake et al., 2000). Mathematics requires all three of these cognitive processes: thinking flexibly, holding and updating important information in working memory (e.g. Raghubar et al., 2010), and inhibiting misconceptions and irrelevant information or rules (e.g., Cragg et al., 2017). EFs are related to performance on mathematics tasks, and have been shown to predict mathematics achievement longitudinally (Cragg & Gilmore, 2014; Ribner, 2020). Teachers have also noticed that EFs are important for math learning based on their observations in the classroom (Gilmore & Cragg, 2014). However, there are few educationally-relevant causal studies of these relationships (Clements et al., 2016). We need better theory and measurement of executive functions in mathematics to understand why these relationships exist and whether executive functions should be intentionally targeted through math interventions (Medrano & Prather, 2023; Scerif et al., 2023; Wilkey, 2023; Younger et al., 2023).

EFs measured during a numerical task are more strongly related to math achievement than EFs measured during tasks that do not explicitly include math-related information (Gilmore et al., 2015; Wilkey & Price, 2019). This suggests that the mathematical learning context should be considered when investigating relationships between EFs and mathematics (Gilmore, 2023; Medrano & Prather, 2023; Niebaum & Munakata, 2023). Moreover, a more strengths-based approach to measuring executive functions is needed that reflects the diverse environments in which children are learning (Miller-Cotto et al., 2022). However, there is a large disconnect between research in math education and research in math cognition (Berch, 2016; Bruce et al., 2017). Math education researchers have long argued cognitive researchers should better consider the educational context in which math learning occurs (de Freitas & Sinclair, 2015; Verschaffel, Lehtinen, Van Dooren, 2016). Researchers in cognitive science and education often use different terminology to refer to similar constructs, which contributes to the divide across fields, and the gap between research and practice (Berch, 2016). For example, one study found that only 20% of teachers were familiar with the term executive functions (Cragg & Gilmore, 2014). EFs are typically assessed by having children complete cognitive behavioral tasks that measure accuracy and response time, or by having parents or teachers complete a rating scale of a child's everyday functioning, and these different measurement formats are not equivalent or interchangeable (Toplak et al., 2013). There is a building consensus that EF assessments need to account for individual differences and contexts; however, there is a dearth of practical solutions at this time. It therefore remains unclear exactly how or why teachers should assess and support executive functions in the mathematics classroom.

Context and Purpose of this Research

The EF + Math Program was developed to explore the core hypothesis that math intervention approaches which contain support for developing EFs can lead to improved mathematics achievement. EF + Math enlisted a portfolio approach, which entails multiple project teams designing and studying interventions which test this core hypothesis, along with other project teams focused on developing effective and equitable assessments and technologies to support measurement of EFs in mathematics. Each project team is interdisciplinary, with educators, researchers, and developers coming together to mitigate challenges that can arise in bridging research to practice (Uncapher, 2018).

The project teams established their approaches to answering the core hypothesis of the EF + Math Program through collaborative discussions, which allowed for the designed interventions and assessments, and their theories of action to be situated not only within the research base, but within the needs of real classroom contexts, educators, and diverse student populations. However, this inclusive approach to research and development raises tensions about the relationships between researchers' assessment of cognitive constructs and the ways that data informs educator practice and student learning (Uncapher, et al., 2022). We elevate one of those tensions in relation to our focus on executive functions in the mathematics contexts: to what extent is the assessment of EF in the mathematics contexts useful, and for which partners? Each project team encountered this tension in their work, and addressed it through different approaches, based on their conceptualizations of EFs in mathematics contexts and the ways that EF supports were incorporated into their interventions. In this paper, we ask, "how have project teams navigated the needs of researchers, developers, educators, and students in assessing EFs in math contexts? How can EFs be assessed in ways that inform both educational practice and mechanistic cognitive research questions?"

Conceptual Framework: Inclusive Research And Development

We examine this research question through the lens of inclusive research & development (iR&D; EF + Math, 2023). The IR&D model is part of the larger movement toward educational participatory research methodologies and partnerships (Amiel & Reeves, 2008; Vaugh & Jaquez, 2020). In these methodologies and partnership models, researchers, developers, and educators are positioned as collaborative partners and actively work to deconstruct hierarchies of power.

However, even in iR&D-driven projects, tensions can arise regarding the role and purpose of conducting research on interventions, especially given the relations different team members may have to the data. For example, researchers collect and analyze data to provide findings to developers and educators, as well as answering more generalizable research questions. Developers use data to inform future development of the interventions, including improving data collection mechanisms or structures to share data findings and implications with teachers and students within the intervention itself. Finally, teachers collect and use data to inform the supports they provide students, future adaptations of curricular materials, and other implications for practice; teachers also provide valuable context and interpretations of data to support researchers and developers in their work. Equity not only means that these different purposes and goals for engaging in research can cause project teams to take a particular approach to their assessment strategies, such as this paper's focus on EFs in math. Equitable approaches to assessment center the lived realities, needs, and assets of students, particularly students of color.

Data and Methods

The research questions and related findings presented in this paper are part of a set of larger, ongoing research studies. Each project team has conducted several cycles of inclusive R&D. Across these cycles, project teams created and updated study design documents, research plans, and measurement plans. For this analysis, our data sources included these documents, as well as notes taken during conversations about measurement and data collection with project teams throughout the inclusive R&D cycles. We have conducted initial analyses of the project team documents and conversation notes; as the teams' are actively engaged in inclusive R&D cycles, future analyses will include additional data sources and updated themes. To answer our research questions, we first identified sections of text that referred to the measurement or assessment of EFs. Within these sections, we coded for the "how" a project team assessed EFs and the "why" for their decisions. We looked for shifts across time for each project team to identify key moments for analysis. Finally, we looked to see how the methods for assessing or the justification for those methods were in response to researcher, teacher, or developer needs and uses of data. The themes presented in this paper represent two types of equitable EF assessments developed and refined through inclusive R&D.

Results: Equitable Approaches to Assessing EFs in Mathematics Contexts

Defining EFs within Mathematical Activity

Assessments do not always provide discipline-specific approaches to measuring EFs; given research on the role of context (Medrano & Prather, 2023; Niebaum & Munakata, 2023), it is essential for researchers to continue developing assessments that capture EFs in mathematics. Measuring EFs in the mathematics context can be approached by capturing in-the-moment data on student activity while actively doing mathematics. The Project teams worked to develop new assessments involving technological features and varied data sources to define EFs in the mathematical context more equitably and in ways that honor students' epistemic diversity.

One project team's intervention focused on the role of EFs in middle school students' collaborative problem solving practices (Kuchynka, et al., 2023). They developed a qualitative coding scheme that identifies what EFs look like in student talk or student actions during problem solving (Renninger, et al., 2023). This assessment strategy leverages qualitative data sources, such as transcripts or observations. Researchers are still able to conduct analyses on the relationships between EFs and math learning; such a coding scheme is also supportive of teacher use to develop an understanding of what EFs look like, noticing of when students are engaging their EFs, and cultivating awareness of how teaching practices or curricular materials can create opportunities for students to engage their EFs within a task.

Another team conceptualized EFs within the context of rational number learning, with an embodied cognition approach driving their intervention design and implementation. Their team developed a rational number knowledge assessment that requires students to engage their EFs to shift across strategies, filter information, and manipulate quantities flexibly. Administering this assessment allowed teachers to focus on assessing mathematical task performance and gathering rational number knowledge evidence, while still providing researchers with relevant and useful data on EFs.

Measuring EF Fluctuations and Communicating Findings Effectively

Traditional measures of EFs often provide static snapshots of student performance on EF tasks. In reality, students' engagement of EFs on a given day or within a task is dependent on a variety of factors, including social and cultural contexts, affective factors, and personal differences. Equitable assessments of EFs should be more flexible and dynamic to mirror this reality and disrupt static labeling practices common in schooling. Approaches to measuring student EFs can include more regular assessments, but also involve the communication of assessment data back to students and teachers to inform adaptations of teachers' practice, as well as supports for developing student agency.

One team has focused on the development and iteration of a machine-learning assessment tool that collects "in the moment" data on students' engagement of EFs and other metacognitive processes (Zhang, et al., 2022). This tool provides an asset-based approach to understanding the variability in students' EFs within problem solving tasks; the assessment aims to promote more accurate understandings of what students are able to do. This team's technology takes in large amounts of qualitative data and presents summaries of data trends to educators through a dashboard view. The teachers are able to view key information about students and classes, as well as implications for their pedagogy, use of particular materials, or scaffolds that may be high leverage supports for students based on their EFs needs that day.

Another project team is developing an app that can quickly and easily assess daily fluctuations in students' EFs (Ghil, et al., 2022). They are developing an adaptive technology that can optimize the EF assessment items students should complete each day; this capacity will allow researchers to model EF trends using minimal data and limiting teacher burden (Katsumba, et al., 2023). Additionally, this team has worked to develop a dashboard that displays these fluctuations to students and educators in positive, asset-based ways. The dashboard, which was co-designed with students themselves, presents data in understandable amounts, and includes recommendations for students to enact agency over their learning based on their data for the day.

Implications and Conclusions

The project teams have continued to iterate upon their approaches to assessing EFs as part of inclusive research and development processes. As the approaches are implemented in additional educational environments, new information is collected regarding their effectiveness and applicability. Further, researcher, educator, and developer goals for understanding EFs in math contexts will continue to evolve as part of the broader fields of math education and math cognition. The EF + MathProgram and its project teams will continue to explore the question, “to what extent is the assessment of EF in the mathematics context useful, and for whom?”

In collaborative partnerships, each party may require or prioritize different data sources or varied granularity to be able to achieve their own goals. Navigating these tensions and creating strategies for assessment in inclusive R&D cycles is non-trivial activities. By documenting the strategies developed by EF + Math’s project teams, we hope to encourage continued conversation and exploration of the role of assessment and measurement in iR&D activities. We envision a future for mathematics education where multiple partners are able to effectively collaborate to design and test solutions that address the challenges educators face in their mathematics classrooms in order to support all students in achieving their full potential as brilliant mathematicians.

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USING LINES OF PRACTICE IN IDENTITY ANALYSIS

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The literature using psychological constructs like "interest" has rarely been brought together with the robust and developing literature of sociocultural perspectives on math identity in K-12 classrooms. Similarly, work on hobbyist communities and other non-academic pursuits has yet to make consistent contact with the literature on developing STEM identities in school. In this study, we look at retrospective interviews following women who consider themselves lifelong knitters, and compare the processes of identification in both knitting and mathematics through the lens of lines of practice *as developed by Azevedo (2011). We find that while women describe consistent and flexible preferences across both domains, they are often met with two limiting factors that impede identification: binary thinking about math ability, and a highly circumscribed participation structure.*

Keywords: Gender, Informal Education, Integrated STEM / STEAM, Ethnomathematics

Objectives and Purpose of the study

While research on mathematics identity has been located almost exclusively in schools, clearly not all mathematics learning experiences are the same, and there is reason to question whether school is anywhere close to ideal. Ethnographic research on student learning has demonstrated that the practices of mathematics are locally constituted, and can transform the nature of mathematical activity. Early work conducted by anthropologists and cultural psychologists (Saxe, 1988; Lave, Murtaugh, & de la Rocha, 1984), enhanced our understanding of mathematical activity as a socially mediated process that connects to history, tools, and other people. This work demonstrated that school mathematics is a peculiar enterprise that has a profound influence on what people ultimately think about mathematics (Boaler & Greeno, 2000; Nasir, 2002; Schoenfeld, 1989). Other versions of mathematics could potentially transform the kinds of relationships to the discipline that we generally observe in school (Taylor 2009; Gonzalez et al 2001). The goal of this paper is to build on this literature in order to explore how identities are tied to local practices in different domains.

In this study, we use an interactionist perspective to explore the question of how school mathematics might be productively reorganized to invite broader participation among a more diverse group of students, particularly women and girls. In order to investigate what kinds of experiences might promote lifelong engagement with STEM, we look to textile crafting, which is a hobby often pursued by women across the lifespan, and which is frequently said to involve elements of mathematics. Our work builds on Azevedo's *lines of practice* (2011), combining psychological accounts of interest development with interactionist ideas about the inextricability of individual identities and the opportunities they have to develop in social practice.

Methods of inquiry

Participants were identified through participation in an online survey about mathematics in textile crafting. From over 1,500 responses, we interviewed roughly 40 participants. We conducted semi-structured, retrospective interviews to explore the connections and contrast between women's experiences with knitting and their experiences with K–12 mathematics. From those interviews, we chose eight women who represented an even balance across two categories—four who rejected a math identity in relation to school (though they considered themselves knitters, and agreed that knitting involves mathematics), and four who said that they enjoyed math in school and generally considered themselves "math people." These interviews were chosen for the depth of their answers to our questions.

Using a grounded theory approach, we build from data toward a theory of how long-term engagement is built and sustained that pulls together previous work using the constructs of interest and identity (e.g. Ames & Archer, 1988; Anderman et al., 2001; Gee, 2000; Hand & Gresalfi, 2015; Holland et al., 2001; Meece, Anderman, & Anderman, 2006; Murayama & Elliot, 2009; Wenger, 1998; Wigfield & Eccles, 2000). After our initial rounds of theory development, we consulted the relevant literature and found that our emerging theory fit well with Azevedo's observations of hobbyist communities. Thus, this preliminary analysis attempts to situate our grounded theory in relation to that framework.

Briefly, Azevedo's "lines of practice" framework describes how individual "preferences" and broader "conditions of practice" weave together to create "lines of practice" that support and sustain participation in a broader social practice over time. Any given individual may have one or many different lines of practice that constitute their participation in a community of practice, and two participants may have lines of practices that are similar or different from one another's. Thus, lines of practice are constitutive of broader social practices, but not wholly determinant of, or determined by, other definitions of those social practices.

Preliminary results

Reject Math Group

Although the knitters who shared a dislike of school math described a range of preferences for what they liked about knitting, they collectively described overlapping conditions of practice that nevertheless supported their ultimate identification with the craft. Rather than seeing knitting as a set of practices that are interesting only for some people, instead they described how practices were inviting and supportive of a range of ways of engaging. Indeed, most knitters identified more than one line of practice that supported their continued engagement. In contrast, the practices of school math, as described by the knitters, failed to support those very same preferences. Overall, what is most clear is that having access to assistance and an understanding of the "discipline," that is, why things work as they do, was widely supportive of a range of preferences for knitting. This same lack of access to understanding was uniformly what turned these women off to mathematics, even though for some of them, understanding mathematics was something that became more common and accessible for them later in life.

Identify with Math

Knitters in our sample who identified with math described very similar lines of practices to those who did not identify with math. These knitters widely had a preference for being able to figure things out for themselves, a preference that was met by conditions of practice that supported such independence, either through the fabric that knitters create or through the myriad resources available. These knitters also expressed preferences in mathematics that were consistent with their stated preferences in knitting, but here very different lines of practice emerged, which, for these women, ultimately led to identification. This largely appeared to be due to finding a different set of conditions of practice in their mathematics classes, either in the form of supportive or innovative teachers, support from home, or just a tendency to find that mathematics makes sense. Thus the difference between those who identified with math and those who did not identify with math did not seem to be based on a difference in preference, but rather based on a difference in conditions, with those who ultimately identified with math finding that the conditions of practice they encountered supported their preferences for engaging.

Participants express consistent preferences across domains

One possible explanation for the difference in identification between math and craft is that people who like both disciplines are simply more flexible, or that there are other factors that mediate their enjoyment, such as self-efficacy or confidence. Perhaps what people get from crafting is different from what they get from mathematics, and people who identify with math simply have a more favorable attitude toward what they are offered in that domain than others.

In contrast to this theory, we find remarkable consistency in the preferences people express in each domain. This finding suggests that people know what works for them—at least in retrospect—and that failing to provide opportunities to engage in their preferred way can lead to disidentification. Refusing to guide someone who prefers to defer to a trusted other is potentially just as damaging as telling someone who prefers to be her own locus of authority to memorize an algorithm for the sake of time. Just as importantly, however, when offered choices, people are often not limited to a single mode of engagement.

Across all eight interviews, while there was a fair amount of diversity in stated preferences throughout the group, we saw remarkable consistency within each interview across the two domains. In other words, we saw some people who preferred to be the locus of authority, and others who preferred to trust an expert, but anyone who stated a clear preference had the same preference in both knitting and math.

Rather than people being more flexible, our findings suggest that it is the disciplines themselves that appear to offer more or less flexibility to participants. This finding suggests that while people may have a primary preference, they are nevertheless willing to engage differently if the circumstances call for it. This observation also anticipates objections to the implications of our finding that there are "better" ways to engage in mathematics and people should be pushed to adopt these, despite their underlying preferences. Though it is beyond the scope of this paper to either agree or disagree with such a normative statement, other findings suggest that supporting people in their preferred method of engagement does not lock them into only that method, but rather supports a more diverse set of engagement patterns.

Stated preferences do not alone explain identification with a domain

Given that math was described by most participants as being relatively monolithic, one possible explanation for the diverging identification patterns is simply that while math and craft have some overlapping content, that overlap has little to do with the aspects of each practice that draw people to the disciplines. Were this the case, we might expect to see a profile of preferences that leads to identification in each discipline, or at least a subset that is required for liking math. Instead, we find multiple different "kinds of people" who found their way into each discipline. Thus, as many scholars have noted, the identification process must be a more complex interplay between individual preferences and the environment.

Of the four people who said they liked math in school, one said she preferred to focus on the experience, and on learning; one said she preferred to focus on learning, and preferred a different locus of authority depending on the task; one said she wanted to be the locus of authority and to focus on learning; and one said she preferred to trust either an expert or the discipline more generally. Of the four who said they did not like math in school, one said she preferred to be the locus of authority, but focused on either the experience or the outcome, depending on the project (Alicia); one said she preferred to be the locus of authority and focused on outcome or learning depending on the project (Carol); one said she preferred to trust an expert, and to focus on either the experience or the outcome, depending on the circumstances; and one said she preferred to be the locus of authority, though she never got that chance in math. In other words, there is no profile of preferences that would predict whether a person might come to love math or to hate it.

Identifying narratives on a longer timescale explains identification across lines of practice.

We found, to our surprise, that there were some people for whom the same conditions of practice in math and knitting seemed to elicit different interpretations of themselves, and ultimately, led to different patterns of identification. In looking more closely, it appeared not that our interviewees had different preferences in relation to these conditions, but rather, different interpretations of these conditions, which seemed to be attributable to a set of different cultural narratives about math and knitting. These included several well-known ideas, such as: math is an innate talent (that you either have or you don't have); speed is indicative of skill, and math involves creativity, but only once you've mastered the building blocks. Some of these narratives had parallels in knitting, though with noticeable amendments: crafters come in all types (and they're all valid); what is satisfying about knitting is open to personal judgement; crafting experience is learned/earned; and crafting is a learnable talent, but artistry is innate.

Discussion and Conclusions

Narratives about domains are hugely influential in how interactions are made meaningful in identity development. For example, struggle alone (either a lack of it, or some individual preference for it) does not explain how people come to view mathematics. Whether a person struggles or not interacts with prevailing cultural narratives about what that struggle means. In knitting, throughout our sample we hear reference to the idea that crafting expertise is learned/earned and that what is satisfying about knitting is open to personal interpretation. In math, however, the dominant cultural narrative is that math ability is a thing you either have or you don’t, though occasionally this is mediated or replaced by the idea that academic achievement takes effort. It is in the context of these narratives that personal struggle in math classes is seen as either evidence that you don’t belong, or a challenge that can be overcome with hard work.

As Azevdeo (2011) has observed, identity development in hobbyist communities often relies on a collection of lines of practice that sometimes overlap and occasionally conflict, rather than a single unifying experience. For our participants, this finding holds true in knitting, and appears to allow individuals to carve out an idiosyncratic path through the identification process that actually binds them to the community rather than splintering. In school math, however, binary thinking pervades—students are often told you’re either a math person or you aren’t—innately smart and talented, or not—and they are not given alternative ways to engage with the discipline that still “count” as math. The exception that proves the rule is often women who come to like math after they’ve left school, when they are afforded new and different opportunities to reclaim a math identity in a more flexible narrative environment.

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UNPACKING FRACTION UNDERSTANDING OF A NEURODIVERGENT STUDENT: CONNECTING AND QUESTIONING ACROSS FRAMEWORKS

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We unpack the fraction understanding of a neurodivergent student in the context of equal sharing tasks by considering three frameworks: equipartitioning (Confrey et al., 2009), mental actions in fraction schemes (Steffe & Olive, 2010), and persistent understandings (Lewis, 2014). During a teaching experiment guided by the equipartitioning learning trajectory (Confrey et al., 2014), the student, Macey, evidenced unanticipated fraction understandings across fraction representations (symbolic, discrete contexts, single and multiple continuous wholes). Each framework offers implications for Macey's thinking. We discuss how applying the frameworks to these data surfaces connections across them and numerous questions with implications for future research.

Keywords: Students with Disabilities, Cognition, Rational Numbers, Learning Trajectories and Progressions

The complexity of fraction understanding is evident in the numerous frameworks used to study it. It is also evident in the extensive literature describing challenges students face in developing rich conceptual understanding in this area. To date, a limited number of studies have investigated the conceptual understandings that neurodivergent students construct around meanings for fractions (e.g., Crawford, 2022; Hunt & Empson, 2015; Lewis, 2014). In the spirit of the theoretical and methodological shifts recognized by PME-NA 2024 conference organizers and expressed in the theme, "Envisioning the future of mathematics education in uncertain times," we center a neurodivergent student's thinking about fractions. We unpack this student's thinking by applying multiple conceptual frameworks, aiming to profit from the complexity and uncertainty that emerges. We claim that by embracing the potential of difference in centering neurodiversity and in interpretations across frameworks, we can move the field forward toward richer, deeper knowledge of student learning. Our goal is to engage in dialogue with the mathematics education community, bringing multiple perspectives together to collectively enrich understanding of the learning of fractions.

Conceptual Frameworks

We identified three frameworks which potentially provide insights into how a neurodivergent student, Macey, has constructed understandings of fractions: equipartitioning (Confrey et al., 2009), mental actions that make up fraction schemes (Steffe & Olive, 2010), and persistent understandings of fractions (Lewis, 2014).

The first framework we consider is equipartitioning. Equipartitioning is the foundational concept in a learning trajectory for developing understanding of fractions as ratios (Confrey et al., 2014). Equipartitioning combines the ideas of equivalence and partitioning in the creation of equal-sized groups through splitting, a cognitive operation grounded in awareness of the multiplicative relationship between partitions and the whole (e.g., given a ¼ share, the whole is four times larger; Confrey et al., 2009). The fraction meaning associated with splitting and equipartitioning is many-to-one, that is, understanding of fraction as ratio (Confrey, 1994). The idea of many-to-one is a relational idea applied to fractional shares of collections of discrete objects, single continuous wholes, and multiple continuous wholes (Confrey et al., 2014). Macey's development of relational thinking and multiplicative reasoning in the equipartitioning learning trajectory is described in depth in a previous publication (Crawford, 2022).

The second framework is mental actions within fraction schemes. This framework uses similar terms as the equipartitioning framework but there are subtle differences in meanings for fractions. This framework takes a contrasting many-as-one approach by focusing on partitioning and iterating a continuous whole unit, often represented as a bar model (Steffe & Olive, 2010). Attention is given to iterating and counting the shares that are part of a whole (e.g., a ¼ share would be iterated four times to create a one). The fractions schemes are cognitive structures used for operating with parts and wholes (Steffe & Olive, 2010), and there are hypothesized progressions in which these schemes are constructed (Hackenberg, 2007; Steffe & Olive, 2010; Wilkins & Norton, 2018). The schemes coordinate mental actions recognized by the student as operations for completing the mathematical task. These mental actions are: partitioning (creating equal-sized parts), iterating (repeating a unit of length to produce a connected whole), disembedding (taking out parts while maintaining awareness of the relationship to the whole), splitting (combining partitioning and iterating as inverse actions), and units coordination (maintaining relationships among levels of units). These mental actions can be applied in part-whole contexts (Olive & Vomvoridi, 2006; Steffe & Olive, 2010) or in measurement contexts (Wilkins & Norton, 2018).

The third framework we include is Lewis's (2014) "persistent understandings" of fractions. This framework attends explicitly to difference across individuals as they construct fraction understanding. Persistent understandings are theorized to have been constructed by neurodivergent individuals within socio-cultural interactions, recognizing the diverse ways learners interpret and construct meaning around representations (Lewis, 2014). Persistent understandings are those that are: a) likely qualitatively different from their peers, b) present challenges to the individual's ability to make sense of more complex fraction concepts, and c) not easily resolved through intervention. Some persistent understandings Lewis identified were: viewing fractions as "taking" the shaded part away (and attending to the fraction complement); applying a discrete set model to continuous models; viewing fraction representations as comprising two parts rather than part of a whole; understanding fractions as the number of pieces rather than the size of the piece; and comparing fractions based on the number of pieces. Lewis describes these understandings in symbolic representations and part-whole contexts.

Purpose

The goal of this analysis was to carefully examine Macey's fraction naming to characterize understandings that she had constructed. We present Macey's thinking for consideration in light of the three frameworks for describing fraction understanding: equipartitioning, mental actions in fraction schemes, and persistent understandings. We bring these frameworks into conversation with one another, and consider how, in combination, they might help to further our knowledge regarding the ways in which fraction understandings develop.

Method

Participant

At the time of this study Macey was in the summer between Grade 5 and 6 and was 11 years old. Macey attended a public school in the United States and received special education services in a pull-out program ("resource room") and private tutoring. Cognitive testing indicated Macey had strengths in visual-spatial, fluid, and deductive reasoning. She had difficulty with language processing, integrating information, abstract reasoning, and executive functions. All sessions took place at Macey's home with one of Macey's parents present.

Context

After two diagnostic meetings, a teaching experiment with Macey comprised 11 sessions, each ranging from 35–50 minutes. The teaching experiment was guided by the equipartitioning learning trajectory (Confrey et al., 2014). The first author, Angie, conducted all of the teaching experiment sessions. Angie has 18 years of experience teaching math in grades 1—8, including working in whole class, small group, and one-to-one settings with neurodiverse students. Angie designed a planning protocol with goals and tasks based on the learning trajectory, anticipated thinking and strategies that would emerge, and scripted teacher questions. Angie implemented tasks as described in research literature with supports planned to be enacted as needed. Angie also used a reflection protocol with prompts for summarizing Macey's activities, teacher responses, adjustments made to plans and rationale for these adjustments.

Data Sources and Analysis

Data sources were videos, transcripts, and Macey's written work from four tasks. The tasks elicited Macey's fraction understandings using different representations: symbolic only (i.e., ¾), fractional share of a collection (discrete representation), fractional share of a single whole (continuous representation), and fractional share of multiple continuous wholes (continuous representation).

Retrospective analysis of these data was completed using a three-level analysis (Simon, 2019). The first level of analysis remained close to the data, by descriptively coding student activity, representations used, features attended to, language used, etc. The second level of analysis involved making inferences across tasks. Codes across tasks were compared to look for patterns in fraction naming (including correctness of response) across forms of representation. The third level of analysis involved broad inferences to describe patterns not directly visible in the data (Simon, 2019). These are abductive inferences which, if true, can explain the data and speak generally to the topic under investigation. Thus, claims about Macey's understanding are abductive inferences supported by the patterns identified in the second level of analysis. These inferences led us to ask questions pertaining to the three frameworks.

Findings

We present Macey's responses to a series of tasks and provide our interpretations of her activity for each. In each of the tasks, Macey's strategy of distributing shares demonstrates the fundamental requirements of equipartitioning: creating a correct number of shares, ensuring shares are of equal size, and exhausting the whole. This leads her to creating "fair shares." As we progress across examples, we draw attention to aspects of her responses which are indicative of difference in her thinking. Finally, we offer the abductive inferences we make about Macey's understanding of fractions. Because Angie was the teacher-researcher in the interactions with Macey, Angie is referred to as "I" or by name in the findings.

Task 1: Meaning for a Symbolic Fraction

During our first meeting, I gave Macey an index card with ¾ written on it, and I asked, "What do you think of when you see this fraction?" Macey's response was, "I think like…I think like…three rows of four [draws a 3 x 4 array of dots]," (see Figure 1). I asked her what makes her think of four rows. She replied, "Like, say you have umm…13 cookies and they're in three rows of four. And you have three friends. And you give four cookies to each friend. And then you would have zero left over. So, three rows of four…would each…three rows… Yeah. Three rows of four makes… It's like 3 [gesturing over the rows] plus 4."

Macey represented the fraction with discrete objects within a collection. We note that Macey viewed the fraction as a sharing situation, in this case sharing cookies among friends, which is consistent with equipartitioning and mental actions within fraction schemes. Disregarding her statement of 13 cookies as a counting error, we note that shares were equally divided. The numerator was a count of the number of rows, or the number of sharers. The denominator was the number of items in the row, that is, the number of cookies. The fraction was not described as part in relation to a larger whole. Thus, Macey did not demonstrate many-to-one or many-as-one thinking. Rather, the fraction was two numeric and separate values in a multiplication context.

Figure 1. The image on the left shows Macey's drawing of three rows of four to illustrate the fraction ¾. The image on the right shows Macey's partitioning of 20 gold coins among five pirates.

Task 2: Fractional Shares of Collections

I presented Macey with the task of using 20 connecting cubes that represented gold coins in a treasure chest and asked her to share the coins evenly among five pirates. Macey distributed the cubes one by one into five piles (see Figure 1). I asked her how much each pirate received, and she answered, "Four." Then I asked what fraction of the whole treasure that was, and she replied, "1/4." I asked her to count out the fractional shares, and she counted "¼, 2/4, ¾, 4/4, 5/4."

Like the symbolic task, Macey appeared to have identified the numerator as the count of the shares (corresponding in total to the number of sharers). Also like the symbolic task, the denominator was not determined based on a referent whole. Instead the denominator reflected the size of the individual share, its quantity. This might be explained by the previous question directing her attention to how many each received or by the discrete nature of the cubes. However, in light of her response to the symbolic task, another interpretation of her thinking is that she understood the denominator as the quantity in the share (similar to the number of cookies in each row). That is, she may have been seeing five groups, each the "size" of four. This again indicates she may not be viewing the part in relation to the whole as is the focus of equipartitioning and or as disembedded from a whole as in the mental actions framework. We also note that her final count resulted in an improper fraction which did not seem problematic for Macey. If this fractions as counts is a persistent understanding similar to Lewis's (2014) fraction quantities as counts of partitions, it is not constrained to unit fractions or by the number of partitions in a whole.

Task 3: Fractional Shares of a Continuous Whole

I gave Macey a long strip of yellow construction paper and told her it was a giant french fry that friends could share (see Figure 2). Each time I asked her to create shares for a number of friends, she used her fingers to estimate a piece of the fry and iterated across the strip. When it was too big, she went back and made the distance between her fingers smaller and iterated again. Once she verified the size she was estimating would fit within the whole, she repeated the process again, marking the strip and then cutting it into parts. I asked her to name the unit fractions and to count the fractional amounts. She named and counted the partitions correctly (e.g., "⅓, ⅔, 3/3").

Macey demonstrated awareness that the number of pieces create a connected whole, an idea that is part of the mental action of iteration. As evidenced previously, Macey identified the numerator as counts of the shares. Unlike the cookies in an array and the pirates' treasure contexts, the value Macey identified for the denominator was based on the number of partitions to the whole, mapping to the number of sharers. In this context, of equipartitioning a single, continuous whole, Macey was able to give the mathematically correct answers.

We also worked with Cuisenaire rods and bar models without context. Told that a rod represented ¼, Macey iterated and traced each iteration, counting with fractions as she went. She continued beyond the whole, counting "5/4, 6/4" before stopping. I asked her if she could keep going, and she said, "Yes, but I will run out of paper."

Based on this activity, we infer Macey understood the numerator not just as a counting number but also as the number of iterations. This activity also indicates she had little trouble with the idea that one can iterate a partition beyond a whole. Considering this in light of the evidence that she was not attending to the referent whole with discrete representations, we suggest Macey was not disembedding the ¼ rod or viewing the rod as ¼ the size of the whole. Rather she may have considered this as a new quantity emerging from the partitioning and, as such, the original whole does not place limits on the number of iterations.

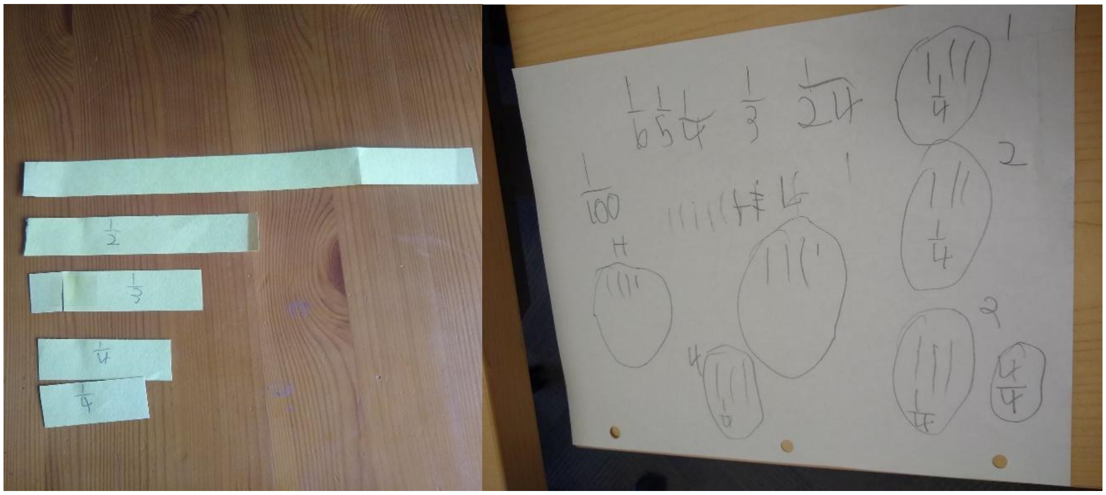

Figure 2. The image on the left shows the shares of a french fry created for a changing number of sharers. The image on the right shows Macey's solution to the task of four children sharing three licorice ropes. Circles representing each child are shown on the right and bottom center in four circles, labeled 1–4, and each containing three tally marks and the inscription, "¼".

Task 4: Sharing Multiple Continuous Wholes

I presented Macey with three long strips of red construction paper and told her they represented long licorice ropes. I then asked her to share the licorice among four children. Figure 2 shows four circles labeled 1-4, representing each of the four children (additional notations on the page are related to a previous task). The circled 4/4 in the bottom right of the picture is her final answer to how much licorice each child gets. Macey's response contains errors that are potentially informative about her understanding. Her solution process is presented in the following transcript:

> So, let's take and put them together [lining up one end of each strip].
> That would be about even. So [marks the middle of each strip to generate halves].
> So, one, two, three, four, five, six [counting halves].
> One, two, three, four, [pointing to each of the circles as she counts, pauses] five, six [returning to point at the first two circles].
> Each of these [unintelligible] again [marks halves of halves on the construction paper strips].
> So, one, two, three, four, [pointing at the circles on paper, looking back and forth from circles to construction paper strips] five, six, seven, eight [pointing again to each of the four circles] nine, ten, eleven, twelve [pointing again to each of the four circles]
> So, one two, three, four, five, six, seven, eight, nine, ten, eleven, twelve [putting a tally mark in each circle one-by-one as she counts].
> Each of them get a ¼, 2/4, ¾, 4/4 [pointing to the partitions in the first strip of construction paper].
> So, so ¼, ¼, [pointing to the first circle and second circle, then writing ¼ in each circle]. That equals 4/4.

When Macey found halves did not distribute evenly across sharers, she used her knowledge that halves of halves create fourths. She ensured the pieces would distribute equally across sharers and then marked tallies for each piece given. Her activity of guess-and-check with halves and then fourths does not indicate that she recognized she could partition each whole into fourths to match the number of sharers. We note this is in contrast to the case of a single continuous whole when she seemingly could map the number of sharers to the number of partitions needed.

Regarding naming of the fractional share, Macey said each gets "a ¼, 2/4, ¾, and 4/4" and then writes ¼ in each circle, and in doing this she appears to be repeating her process of distributing the four partitions of the first licorice rope. With the statement, "That equals 4/4," Macey pointed only to the first licorice rope and appeared to be referencing the 4/4 in a single rope. Macey disregarded or did not notice that she had three tally marks within each circle.

We noted some similarities with Macey's responses to previous tasks. Macey's gestures indicate the numerator was identified by counting partitions in one licorice rope, but she had distributed these pieces one by one to each sharer. In both tasks her numerator corresponds to the number of sharers. Regarding the denominator, there was no reference to the other two licorice ropes as she named the fractional shares, and there is no indication she understood one rope as the referent whole. Given responses in previous tasks did not show attention to a referent whole, we suggest the denominator in this case is the number of partitions in a group she perceives as salient, something consistent with her naming of fractional shares of cookies and gold coins. Again, there are similarities in Macey's response to the persistent understanding of fractions as numbers rather than indicators of size

Abductive Inferences

We infer Macey held these fraction understandings based on these equal sharing tasks:

- Awareness of equipartitioning (number of shares, equal size, exhaust the whole)
- The part represented a new, iterable quantity that was not seen in relation to the whole
- Numerators identified counts of shares, corresponding to iterations in the context of a single continuous whole
- Denominators were understood as "size" meaning quantity, as in the number of parts within a single whole or a single salient object/group

Discussion

We consider our abductive inferences through the three frameworks – equipartitioning, mental actions in fraction schemes, and persistent understandings. We are unpacking Macey's responses in the spirit of exploration from multiple perspectives and welcome the uncertainty we find. Our purpose is to engage conversation with the mathematics education research community about connections and questions across these frameworks about fraction understanding. Next, we identify our questions about the abductive inferences clustered around three topics: discrete representations of fractions, continuous representations of fractions, and numerators and denominators. Finally, we pose questions related to Macey's neurodivergence.

Questions about Discrete Representations of Fractions

When Macey was working with models of discrete objects (e.g., cookies or gold coins), she attended to clusters of these objects (e.g., dots or cubes) rather than to the larger whole.

- Does this indicate Macey has yet to construct a part-of-a-whole and/or a part-to-whole (Confrey, 1992) understanding of fractional shares of collections? What are the implications of this as it represents the first level of the equipartitioning learning trajectory (Confrey et al., 2014)?
- Persistent understandings are constructed by neurodivergent students in socio-cultural interactions (Lewis, 2014). Given the perceptual saliency of counts groups and partitions within groups (as opposed to counts of shares and number of shares in a whole), how might Macey's naming have arisen in the context of instruction? What does this suggest about tasks and instruction involving fair shares of collections?

Questions about Continuous Representations of Fractions

When Macey was working with a single continuous whole, she was able to iterate and re-create a whole as well as create improper fractions.

- Does Macey understand what a whole is? If asked to iterate a ¼ piece seven times, she would likely draw a bar of length 7/4, but would she indicate 4/4 as the whole?
- Macey seems to regard parts as new quantities that add up to the same amount as the original whole but not as part of the whole. Does this confirm that Macey does not yet disembed (Steffe & Olive, 2010)?
- Does her understanding of fractional shares facilitate an understanding of improper fractions without the mental actions of disembedding and units coordination (Hackenberg, 2007; Steffe & Olive, 2010; Wilkins & Norton, 2018), and if so, how might this be useful in supporting her subsequent learning?
- Given that a student with understandings such as Macey's may provide correct answers for fractional parts of a continuous whole, when are these persistent understandings reinforced by instruction and when do they become problematic (Lewis, 2014)? How can they be identified?

Questions about Numerators and Denominators

When we try to characterize Macey's thinking about numerators across tasks, we infer Macey understood numerators as the count of the number of shares or iterations. We infer Macey understood denominators as the quantity of partitions. These understandings worked for her in some situations and not in others.

- In what ways will these understandings impact her development at more advanced levels of equipartitioning and fraction schemes?
- This seems to be consistent with the "more pieces" understanding identified by Lewis (2014), one in which the understanding of the denominator is as the number of pieces rather than the size of the piece. The student in Lewis's (2014) study applied this only to unit fractions. Macey applied this thinking to non-unit fractions. Is this an understanding that other students construct?
- Is this understanding precipitated by instruction that focuses on counting the partitions and does not attend adequately to the relative size of the partitions, a lack of clarity about size as quantity and size as dimensions?

Questions about Neurodivergence

Macey was able to complete each of the tasks she was given by using additive reasoning, applying counting, partitioning, and iterating actions. She did not evidence activity indicative of understanding fractions as either part-to-whole or part-of-a-whole relationships.

- Given the perceptual saliency of counts groups and partitions within groups (as opposed to counts of shares and number of shares in a whole), how might Macey's naming have arisen in the context of perceptual processes?
- Macey reached solutions to each task presented to her, though she did not evidence using anticipatory thinking to do so. How did Macey's visual-spatial, fluid, and deductive reasoning strengths serve as assets when solving fraction tasks? How can these assets be used to support her in recognizing fractional parts in relation to a whole?
- How might a persistent understanding of fractions as counts–that is as uncoordinated numbers rather than a size–be related to Macey's issues with executive function or abstract reasoning? Might other cognitive processes be implicated?
- How might this persistent understanding be constructed based on perceptual saliency of representations and instruction which leaves ideas of multiplicative relationships or disembedding implicit (Lewis, 2014)?

Finally, we wonder more generally which of Macey's understandings might be qualitatively different from those more typical of students in the learning of fractions. If they are qualitatively different, but emerge in the socio-cultural use of representations, how many other students develop similar understandings? And if they persist and contribute to future difficulty with more sophisticated fraction operations and relational thinking, how might we design tasks and instruction to address these persistent understandings?

Conclusion

We center the thinking of a neurodivergent student, viewing this as an opportunity for our learning as researchers and educators. We consider the student's thinking from a diversity of perspectives–three conceptual frameworks for fraction understanding, and in doing so we find numerous areas of uncertainty. This represents a unique approach to exploring the fraction understandings of students. This is of benefit to the field by considering how frameworks can come into conversation with one another and surface questions which offer productive lines for research. Further, this exploration aims to provide evidence that explicitly positioning difference at the center of a study, in this case a neurodivergent student's thinking, can provide a valuable contribution to knowledge development around how fraction understandings emerge.

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BACKGROUND FACTORS AND MATHEMATICS BELIEFS THAT PREDICT 5TH - 12TH GRADE STUDENTS' MATHEMATICS IDENTITY

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We explored how background factors and mathematics-related beliefs were predictive of 5th through 12th grade students' mathematics identity through regression analysis. Results indicate both background factors (age, honors placement, tutoring received) and mathematics-related beliefs (mindset, anxiety, agency, nature of math, rules) were all predictive of students' mathematics identity. Student age, receiving tutoring, and mathematics anxiety were all negative predictors, while the rest of the variables in the model were positive predictors. This study highlights the variety of factors that play a part in a student's mathematics identity development and could provide insight into specific experiences or interventions that might be used to support students' identity development.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Middle School Education; High School Education

Research notes the important role that mathematics identity has on students' achievement (Bohrnstedt et al., 2021) and persistence, as shown through career choice (Cribbs et al., 2021a, 2021b). However, limited research explores how various background factors and mathematics-related beliefs influence students' mathematics identity. This is partly due to the limited quantitative research in the field (Graven et al., 2019) and also due to the small sample sizes that are often used to explore the complexities of identity development such as through discourse analysis and/or positioning (Bishop, 2012; Sfard & Prusak, 2005). It is the purpose of this study to fill a gap in the literature by exploring connections between various factors and students' mathematics identity to provide a picture of potential overarching patterns that could inform the field. This is particularly beneficial given the growing number of calls by the field for teachers to attend to students' mathematics identity development (NCTM, 2018, 2020a, 2020b).

Theoretical Framework

This study draws upon prior work in the field aligned with core identity (Cobb & Hodge, 2011; Gee, 2001). From this perspective, identity is viewed as thickening over time (Holland & Lave, 2001), allowing for a snapshot of students' identity to be taken at a point in time, such as through a survey. Initial work in the field provides validity evidence for measuring mathematics identity with undergraduate (Cribbs et al., 2015) and 5th-12th grade students (Cribbs & Utley, 2023). That work supports the inclusion of three sub-factors for measuring mathematics identity: Interest (a student's desire or curiosity to think about and learn mathematics), Recognition (how students view themselves and how they perceive others to view them in relation to mathematics), and Competence/Performance (students' beliefs about their ability to understand and perform mathematics). In combination, these factors capture students' mathematics identity.

Mathematics identity and background factors

Given that mathematics identity is informed by social and cultural norms and experiences (Holland & Lave, 2001), it is important to consider students' gender, racial, and ethnic identities when exploring their mathematics identity. A wealth of literature highlights the ways that gender (Cribbs et al., 2016; Damarin, 2000; Mendick, 2005), race (Berry, 2008; Martin, 2000), and ethnicity (Adams, 2018; Gonzalez et al., 2023) inform how students position themselves or are positioned by others as a *math person*. In addition, other background factors, such as grade level are important to consider when examining mathematics identity development. Research has consistently shown that as students progress through K-12 their dispositions regarding mathematics declines (Musu-Gillette et al., 2015). Other research found that Honor's placement positively influences changes in academic identity and school belonging (Legette & Kurtz-Costes, 2021). In addition to these background factors, specific family background/experiences have been shown to relate to students' identity. For example, research shows a connection between STEM identity and students' participation in informal STEM experiences (Dou et al., 2019). Additionally, research indicates receiving mathematics tutoring could have a negative effect on students' mathematics identity (Cribbs et al., 2021a). Another important connection to explore further is the number of student's family members who are in a STEM field. With literature noting the role that parents have on their child's beliefs and decisions (Turner et al., 2017). Other work notes that while the level of influence by peers may be less for individuals pursuing mathematics-related fields, career choice is still strongly influenced by family members (Sasson, 2019). However, much of this work focuses on females due to the potential stereotypes that might influence students' decisions. Given that little research explores mathematics identity quantitatively, little is known about how these factors might relate to students' mathematics identity for groups of students or cross-sectionally across K-12.

Mathematics identity and mathematics beliefs

Other work provides evidence of connections between mathematics-related beliefs and mathematics identity. For example, mathematical mindset and mathematics anxiety were both predictive of mathematics identity (Cribbs et al., 2021b). However, that study was with undergraduate students, which leaves a gap in our understanding of how these constructs might be correlated for K-12 students. Literature proposes a strong connection between mathematics identity and mathematical agency (Aguirre et al., 2013; Atabas et al., 2020; Turner, 2012). However, until recently a measure for mathematical agency was not available for exploring this relationship quantitatively (Cribbs & Utley, under review). Another important aspect of students' mathematics identity is their perceptions of mathematics as a subject. Boaler and Greeno (2000) make this clear stating that "It is possible that many able students who could become world-class mathematicians leave mathematics because they do not want to author their identities as passive receivers of knowledge" (p. 189). This statement highlights how students' beliefs about what mathematics is as a subject might influence their mathematics identity. By exploring beliefs about the nature of mathematics along with mathematics identity, this relationship can be further explored. In addition to these other beliefs, students' perceptions of the type of support they receive in the classroom are important to consider. Fredricks et al. (2018) found that both teacher and peer support positively associated with students' mathematics engagement. Given the social aspects of identity development, teacher and peer support is likely to have an influence on mathematics identity as well. In combination, exploring mathematics identity along with these mathematics-related beliefs can provide a better understanding of their influence on students' identity development.

Methods

A cross-sectional survey design was used to collect data from 5^{th}-12^{th} grade students to explore their mathematics identity and other mathematics-related beliefs.

Participants

Data were collected from two districts in a mid-western state in the United States. Students in grades 5^{th}-12^{th} grade were asked to participate, resulting in 1,655 surveys. After data was cleaned and responses with large portions of missing entries removed, the sample included 1,394 participants. With regard to gender, 46% identified as Male and 54% as Female. Participants were between the ages of 9 and 18. Twenty percent of the sample indicated they were of Hispanic origin. With regard to race, 56% indicated they were White, 24% Native American, 6% multicultural, 5% were uncertain, 3% Black or African American, 2% African American and White, 2% Asian, and 1% Other. In terms of grade level distribution, 18% were in 5^{th} grade, 13% 6^{th} grade, 10% 7^{th} grade, 10% 8^{th} grade, 12% 9^{th} grade, 20% 10^{th} grade, 11% 11^{th} grade, and 5% 12^{th} grade. In the state where the data was collected, a fourth year of mathematics is not required for high school graduation, which aligns with a smaller percentage of students represented from 12^{th} grade in the sample.

Data Collection and Analysis

A survey was administered to participants in late spring of 2022. In addition to background factors (e.g., gender, race, age), the survey included items for a variety of mathematics-related factors, which will be detailed in the subsequent sections.

Background Variables. Two types of background variables were collected for the study: demographic variables and family background variables. The demographic variables include gender (0=female; 1=male), age, Hispanic, race, grade level ($1=5^{th}$; $8=12^{th}$), and advanced math class. Family background variables include English is the primary language at home, family working in the STEM field (total number of male guardian, female guardian, and other family working in STEM), math and/or science as a family hobby, family help with math schoolwork, family arranged tutoring in math (0=no; 1=yes), and number of STEM camps or programs in which participants participated (scale from 0 to 5+).

Mathematics Identity. Mathematics identity was measured through a 16-item Likert-scale (1-Strongly disagree; 5-Strongly agree). The factor is comprised of three sub-factors (interest, recognition, and competence/performance; Cribbs & Utley, 2023). Although the instrument can be used to explore the three sub-factors separately, it can also be used to create an overall mean for mathematics identity, taking a snapshot of students' mathematics identity at the time the survey is administered. This scale included items such as "I see myself as a math person" and "I look forward to taking math."

Mathematical Agency. Mathematical agency was measured using three sub-scales: Discipline/Conceptual (7-items; e.g., "I can correct or fix my math errors when solving problems"), Collective (4-items; e.g., "In math class, we listen to each other's math ideas"), and Critical (7-items; e.g., "Math has helped me make sense of the world around me."). Given that the sub-factors for mathematical agency are theoretically and statistically distinct, they will be explored separately in this study (Cribbs & Utley, under review). Items were on a Likert-scale with 1-Strongly disagree and 5-Strongly agree.

Mathematics Mindset. Mathematical mindset drew on Dweck's (2008) work. However, only the incremental beliefs subscale was used (Blackwell et al., 2007; De Castella & Byrne, 2015) and adapted to be mathematics specific as in other research (Degol et al., 2018). The mean of the four items was used to measure the factor. This scale included items such as “No matter who you are, you can significantly change how smart you are in math” and “You can always change even the basic level of how smart you are in math considerably.” Items were on a Likert-scale with 1-Strongly disagree and 5-Strongly agree.

Mathematics Anxiety. Two different scales were used to measure mathematics anxiety as the scales measured different aspects of the factor. The first measure of anxiety used the Single-Item Math Anxiety Scale (SIMA) asking students to respond to the prompt “On a scale from 1-10, how math anxious are you?” Prior research provides validity and reliability evidence for the scale (Nunez-Pena et al., 2014), and the scale was a way for us to efficiently assess students’ overall mathematics anxiety. The second scale used was the modified Abbreviated Mathematics Anxiety Scale (mAMAS), which has validity and reliability evidence from prior work (Carey et al., 2017) and was used to assess two sub-factors of mathematics anxiety. The first sub-factor is Mathematics Evaluation Anxiety (MEA) and includes four items asking participants to rate their level of anxiety in response to items such as “Thinking about a math test that day before you take it” and “Taking a math test.” The second sub-factor is Learning Mathematics Anxiety (LMA) and includes five items such as “Having to complete a worksheet by yourself” and “Starting a new topic in math.” Items were on a Likert-scale with 1-Not anxious and 5-Extremely anxious.

Nature of Mathematics. Beliefs about the Nature of Mathematics was measured using items from the Teacher Education and Development Study in Mathematics (TEDS-M; Tatto et al., 2012). This factor included two sub-scales: Mathematics as a set of Rules and Procedures (6-items; e.g., “Mathematics is a collection of rules and procedures that prescribe how to solve a problem”) and Mathematics as a Process of Enquiry (6-items; e.g., “Mathematics involves creativity and new ideas”). These sub-factors are considered separately as “it is quite possible for them [respondents] to endorse both sets of propositions” (Tatto et al., 2012, p. 155). All items were on a Likert-scale with 1-Strongly disagree and 5-Strongly agree.

Teacher and Peer Support. Teacher and Peer Support was measured using four sub-scales: Teacher Academic Support (4-items; e.g., “My math teacher cares about how much I learn”), Teacher Person Support (4-items; e.g., “My math teacher thinks it is important to be my friend”), Peer Academic Support (5-items; e.g., “Other students in math want me to do my best schoolwork”), and Peer Personal Support (4-items; e.g., “In math class, other students like me the way I am”). Consistent with how the factor was used in literature (Odoy, 2018), the sub-factors were considered separately in analyses. Items were on a Likert-scale with 1-Never true and 5-Always true.

Regression analysis was used to create a model examining how demographic variables, family background variables, and mathematics beliefs were predictive of students’ mathematics identity. Prior to data analysis, simple random imputation was used to create a complete dataset, which is a better option than using listwise or mean imputation (Schlomer et al., 2010).

Results

A stepwise regression analysis was conducted in blocks, where background (demographic and family background) variables were regressed on mathematics identity first. After all non-significant items were removed from the model, mathematics-related beliefs were regressed on mathematics identity (also retaining the significant background variables). Items were removed from the model until only significant variables remained.

Table 1 shows that the only significant demographic variables were students' age and advanced/honors placement for mathematics class. Estimates indicate that students' age was a negative predictor, and honors placement was a positive predictor of mathematics identity. Only one family variable was a significant predictor for mathematics identity. Students indicating that their family arranged for them to get tutoring in mathematics negatively predicted their mathematics identity.

Table 1: Background Variables and Mathematics-Related Beliefs Predicting Mathematics Identity

N=1,394	Estimate	SE	t-statistic	Sig
Intercept	0.82	0.16	5.17	***
Student Age	-0.03	0.01	-3.52	***
Honors	0.35	0.05	7.50	***
Tutoring in math	-0.14	0.05	-2.56	*
Math Mindset	0.08	0.02	3.93	***
Overall Math Anxiety	-0.02	0.01	-3.26	**
Math Evaluation Anxiety	-0.08	0.02	-4.39	***
Learning Math Anxiety	-0.05	0.02	-2.03	*
Math Agency – D/C	0.54	0.03	21.21	***
Math Agency – CR	0.17	0.02	7.81	***
Nature of Math – Rules	0.11	0.02	5.48	***
Teacher Academic Support	0.04	0.02	2.03	*

*p<0.05 **p<0.01 ***p<0.001

In terms of mathematics-related beliefs, eight factors/sub-factors remained in the final model. Mathematics mindset, the discipline/conceptual and critical sub-factors for mathematical agency, rules sub-factor for nature of mathematics, and teacher academic support sub-factor for teacher and peer support were all positive predictors for mathematics identity. As one might expect, all three mathematics anxiety scales were negative predictors for mathematics identity. As a check for multicollinearity, the variance inflation factor (VIF) was assessed. The highest VIF was 2.3, well below the recommended value of 4 (Hair et al., 2010). Thus, all items were retained in the model.

Discussion

Results provide evidence for a variety of background and mathematics-related beliefs influencing students' mathematics identity.

Background Factors

Student age being a negative predictor for mathematics identity aligns with prior work noting a decline in students' beliefs and/or attitudes as they progress through school (Musu-Gillette et al., 2015). However, the estimate was relatively small ($\beta = -0.03$) indicating that this relationship might not be as strong as with other affective measures. This finding could be due to core identity being more stable over time (Cribbs et al., 2022) than other aspects of identity or a mediating relationship that was not tested in this study. Perhaps more targeted interventions for mathematics identity development are needed. Research found that changes in identity can occur when targeted practices are employed (Hima et al., 2019). However, little is known about interventions employed with large groups of students. Honors placement was a positive predictor for mathematics identity which aligns with prior work exploring academic identity (Legette & Kurtz-Costes, 2021). Students who are in honors courses might get reinforcing messages regarding their competency in mathematics as well as recognition by family, teachers, and peers. Additionally, the family background variable, tutoring in mathematics, had a similar result from prior research (Cribbs et al., 2021a), which might be indicative of students feeling they are not competent with mathematics because they are receiving tutoring. However, that study also found that students who tutored others positively predicted mathematics identity, so the act of tutoring supports identity development, but not vice versa. It is worth noting the non-significant variables, which were gender, race, ethnicity, English as the primary language spoken at home, number of family members working in a STEM field, math and/or science as a family hobby, family helped with schoolwork, and participation in STEM camps or programs. It is possible that exploring mediating relationships or interaction effects, not tested in the current study, could provide a more nuanced understanding of the role these variables might have on students' identity development.

Mathematics-Related Beliefs

Given that mathematics mindset was a significant predictor for mathematics identity for undergraduate students (Cribbs et al., 2021b), it was not surprising to see a significant relationship with 5^{th}-12^{th} grade students. Further work is needed to better understand the direction of this relationship. All three measures for mathematics anxiety were negative predictors for mathematics identity, similar to prior research with undergraduate students (Cribbs et al., 2021b). Although the estimates are relatively small, it is interesting to note that Mathematics Evaluation Anxiety had the largest estimate ($\beta = -0.08$), potentially highlighting the negative effects of assessment on students' anxiety and subsequent identity. This study hints at the potential negative ramifications student assessment has that goes beyond immediate performance as previous research highlights (Barroso et al., 2021) to mathematics identity, which impacts students' future choices (Boaler & Greeno, 2000; Cribbs et al., 2021a, 2021b). Two of the sub-factors of mathematical agency were significant predictors of mathematics identity. Discipline/Conceptual agency had the largest estimate in the model ($\beta = 0.54$), indicating a strong correlation between mathematics identity and the sub-factor. This could be partly due to the alignment between elements of the competence/performance sub-factor of mathematics identity. Critical agency was also a positive predictor for mathematics identity. Turner (2012) described critical mathematical agency as involving "students' capacity to: understand mathematics; identify themselves as powerful mathematical thinkers; [and] factor and use mathematics in personally and socially meaningful ways" (p. 55). Results from this study align with prior work (Aguirre et al., 2013) noting that agency plays an integral part in students' mathematics identity development. Interestingly, collective agency was not a positive predictor, particularly given the quorum effect that interest has on career choice in STEM (Hazari et al., 2017). The Rules sub-factor for nature of mathematics was a positive predictor for mathematics identity; the inquiry sub-factor, however, was not. This sparks some points for consideration. As many students are likely to experience direct instruction in their classroom as opposed to inquiry instruction, could this be indicative of students' experiences in their classrooms? Does this perception of mathematics potentially act as a barrier to students who might otherwise develop a mathematics identity? Finally, only one sub-factor of support was significant, teacher academic support. This finding highlights the important role that teachers have in students' identity development, as noted in multiple calls by the field (NCTM, 2018, 2020a, 2020b).

Connecting to the Conference Theme

In line with the conference theme, this line of research endeavors to use a different methodological approach to exploring mathematics identity than is commonly used (survey vs. positioning and/or narrative) by the field. This approach builds on theoretical perspectives from the field and provides a way to better understand how identity is being influenced by larger groups of students in varying contexts and backgrounds.

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STUDENT USE OF THREE REPRESENTATIONS OF SET RELATIONSHIPS TO REASON ABOUT LOGIC IN UNDERGRADUATE TRANSITION TO PROOF COURSES

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In transition to proof courses for undergraduates, we conducted teaching experiments supporting students to learn logic and proofs rooted in set-based meanings. We invited students to reason about sets using three representational systems: set notation (including symbolic expressions and set-builder notation), mathematical statements (largely in English), and Euler diagrams. In this report, we share evidence regarding how these three representations provided students with tools for reasoning and communicating about set relationships to explore the logic of statements. By analyzing student responses to tasks that asked them to translate between the representational systems, we gain insight into the accessibility and productivity of these tools for such instruction.

Keywords: Logic, multiple representations, Euler diagrams, undergraduate

Introduction

Using multiple representations to support student reasoning and problem solving has long been acknowledged as a cross-cutting theme in mathematics education (e.g., NCTM, 2000). Working within and across representations is often a productive means of supporting student reasoning and promoting communication in the classroom. In the realm of mathematical logic, there is a long tradition of developing various visual and symbolic representation systems (e.g., Venn diagrams, Euler diagrams, truth tables, logical calculus), which would suggest this is a ripe space for using visual and symbolic representations to support student learning. However, in our experience, the use of spatial representations such as Euler or Venn diagrams to teach undergraduate transition to proof (TTP) students is rare (see Dawkins et al., 2022). One explanation for this is that diagrammatic representations of logic generally rely on set relationships (Mineshima et al., 2012), but common approaches to teaching logic in undergraduate TTP courses generally base logical concepts on truth-values rather than sets (Dawkins et al., 2022). This is the case despite a body of evidence supporting the power of visual representations for student reasoning in logic (e.g., Stenning, 2002; Sato & Mineshima, 2015).

Based on a series of experiments (e.g., Dawkins & Cook, 2017; Dawkins & Roh, 2024; Dawkins et al., 2023; Eckman et al., 2023) involving a cycle of modeling student reasoning about logic and task design to support learning of logic, our team has developed a teaching sequence to foster learning of logic using set relationships. We use three primary representation systems (Goldin, 1998) to engage students in reasoning about set relationships: set theoretic symbols including set-builder notation, mathematical statements (rendered in English), and Euler diagrams. Figure 1 portrays how a subset relationship between two properties P and Q might be alternatively portrayed in the three representations. As Thompson (1994) explained regarding different representations of a function, we should not assume that students always see all of these as different representations of the same underlying object, even if that is our goal.

Set-theoretic symbolic representation
$A \subseteq B$
where $A = \{x \in S: x \text{ has property } P\}$
and $B = \{x \in S: x \text{ has property } Q\}$

Mathematical statement representation
Given any $x \in S$, if x has property P, then x has property Q.

Euler diagram representation

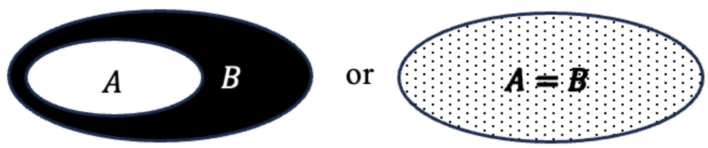


Figure 1: Expressions of a subset relationship in the three representations

As with much teaching using various representations, in our teaching experiments our research team did not want these representations to be the focus of instruction during most of the unit in which we taught logic. Rather, after they were introduced, we wanted them to be means by which students could reason and communicate about set relationships and the logic of statements. In two undergraduate TTP whole-class teaching experiments, we encouraged students to communicate within and across the three representational systems to learn set relationships, logic of statements, and proof techniques. We assigned a number of tasks in which students translated between the representations or generated new objects in one or more representations. In this report, we share our analysis of student work on such tasks to consider whether these three representations served as *accessible* and *mathematically productive* ways for students to reason about and communicate about set relationships. In particular, we share whether student use of these representations was normative – meaning the claims students made were mathematically accurate – and whether they were consistent – meaning a student's various claims for related tasks agreed, even if the interpretations or claims were non-normative. Our goals in this analysis are twofold: 1) to investigate the efficacy of teaching logic using these three representations for supporting student inquiry (as evidenced by the conditions in the previous sentence) and 2) pending positive evidence, to portray the potential of using these representations for instruction on logic and proof techniques (as it stands in contrast to common practice).

Relevant Literature

This section reviews literature relevant to our project before briefly reviewing our own line of research that informed the instructional approach employed in the teaching experiments.

Sets, diagrams, and the teaching and learning of logic

While Venn diagrams are perhaps the most well-known visual system for representing and reasoning about logic, a variety of such systems were developed, primarily in the 19th century. Research on how people reason with and learn from such systems is much more recent. Sato et al. (2010) compared how people solve syllogism tasks using a few different representation systems (see also Bronkhorst et al., 2022). In particular, they compared verbal solution methods (no diagrams), Venn diagrams (in which regions always overlap and regions are shaded), and Euler diagrams (in which possibilities are displayed by the overlap/non-overlap of regions, as in Figure 1), each with a short period of training in each diagram system. They found that the college student participants performed better with diagrams than with only verbal representations and performed better with Euler diagrams than with Venn diagrams. Those authors explain the value of diagrams by claiming, "we may plausibly assume that the semantic primitives of quantificational sentences in natural language are *relations* between sets, and that people's inferences with quantified constructions are sensitive to such a relational structure" (Sato et al., 2011, p. 2183). They assume that treating statements as relations among sets (as is portrayed in Venn and Euler diagrams) rather than relations quantified over ranges of individual objects (as is done in most standard treatments of logic, such as truth tables) is more consistent with natural language. Sato et al. (2010) further claims that Euler diagrams fostered better performance since they are in some sense "self-guiding" (p. 20) for minimally trained learners. We do not endorse such an interpretation of representational transparency as though learners must not engage in some constructive process of making meaning of the diagrams, but the evidence suggests that students find Euler diagrams easier to use with minimal training nevertheless.

Mathematics education studies of sets, diagrams, and logic

Deloustal-Jorrand (2002, 2004) provides a strong antecedent to the present work as she argued that student understanding of conditional statements should be built upon three viewpoints: formal logic (truth table definition and quantification), sets (represented by spatial diagrams), and implication (the conclusion can be inferred from the hypothesis). She hypothesized that "it is necessary to know and establish links between these three points of view on the implication for a good apprehension and a correct use of it" (Deloustal-Jorrand, 2002, p. 4). Similarly, Durand-Guerrier et al. (2012) emphasized that logic must be taught with attention to semantic and syntactic aspects. While they do not endorse spatial diagrams in particular, such diagrams are classically viewed as a representation of the semantics of statements which supplement the syntax of formal statements or symbolic expressions. These authors support the claim that students should reason about sets, likely expressed through spatial diagrams, to learn about the logic of statements and proofs – however, how students bridge between these representations has been explored less. In a forthcoming paper, Antonides et al. (in press) explore how students link spatial and logical structures, recognizing the challenge and opportunity posed by operating across these representational systems[4].

Our approach to teaching logic is consistent with these other studies, but only indirectly drew upon them. Our focus on sets arose from observations of productive student reasoning about mathematical statements (Dawkins, 2017). This led us to depart from the common truth conditions for statements defined by truth tables, and to adopt truth conditions based on sets. Specifically, the truth of a conditional corresponds to a subset relationship between the truth sets

[4] Both Sato et al. (2010) and Antonides et al. (in press) use Euler diagrams in which regions do not have existential significance. This means that in Figure 1, the left-hand diagram could represent both the case where $A \subset B$ and $A = B$. We do not adopt those conventions, but rather use the two diagrams in Figure 1 to separately express the two cases. This creates a two-to-one mapping between the diagrammatic representation and the symbolic and sentential representations in the way we teach, as portrayed in Figure 1.

of the two predicates, as portrayed in Figure 1. As we explored how to use sets to support students to reason about the logic of statements and proofs (e.g., Dawkins et al., 2023; Dawkins & Roh, 2024), we saw the need to more directly teach basic set theory beforehand (Eckman et al., 2023, provides examples motivating this need for instruction). The three-representation approach investigated in this report arose as a tool for enacting this teaching sequence. Two other aspects of our approach worth mentioning are that 1) we focus on sets defined by mathematical properties, not arbitrary sets such as $\{1, \pi, \text{Ford Taurus}\}$ and 2) we focus on mathematical statements rather than everyday, nonsense, or purely symbolic statements.

Methods

As part of a larger project investigating student abstraction of logic (NSF DUE #1954768 and #1954613), we conducted two whole-class teaching experiments (Steffe & Thompson, 2000) in undergraduate TTP courses. These courses occurred at two large Southwestern, public universities and were taught by the first and second authors. The data gathered consisted of videos of all class meetings in which sets, logic, and proof techniques were covered, student homework and exams, group interactions in target small groups, task-based interviews on logic with members of the target groups both before and after instruction, and pre- and post- logic assessments delivered online. Consistent with teaching experiment methodology, outside observers were present at all class meetings and the research teams at the two sites met weekly to discuss and conduct iterative analysis and planning.

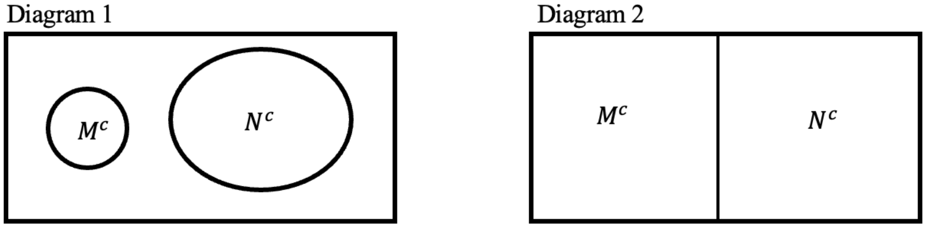
All students were invited to provide informed consent to participate in the study. The data analyzed in this report is limited to homework and exam work from students who opted into the study (13 students in each class), which represents 87% and 100% of the students in the two classes. This portion of the data was deemed most appropriate for analyzing all participating students' use of the three representations. The homework and exam tasks were not the same across the two classes, though some tasks were shared. This creates an asymmetry in the available data at the two sites. While this might be a problem if we tried to make claims about student learning over the course of the teaching sequence, our goals in this report are more modest. We want to consider the extent to which the three representations provided accessible and mathematically productive ways for students to reason and communicate about set relationships. We provide an example to illustrate what we mean below.

To answer this question, the third and fourth authors analyzed all homework and exam tasks from the portions of the courses on sets, logic, and proof techniques. We looked for all the tasks that invited students to operate between representations, often providing input in one and asking students to respond in one or both of the others (see Fig 2 for an example). The research team then selected a subset of these tasks for student response analysis, giving preference to those tasks used in both classes. The third and fourth authors then analyzed all consenting student responses to these tasks. Responses were coded for whether the response 1) made normative claims in the representation, 2) was internally consistent, and 3) exhibited any recurring feature observed in other responses and salient to translation between representations. As allowed by the structure of each task, we attended to whether students were internally consistent in the claims they made about sets across the representations. In other words, we sought to discern whether they used different representations as ways to express the same underlying relationship. Though many of the tasks in the courses were in a particular mathematical context (e.g., quadrilaterals), many of the tasks we analyzed dealt with arbitrary sets and properties. Such tasks were assigned to support student abstraction. These tasks are useful for this study to see how students related the three representations without underlying reference to the specifics of some underlying mathematical context.

Consider the task in Figure 2 to illustrate our coding. The normative responses for the first question were 11a – true-true, 11b – false-true, and 11c – false-true. To us, the three statements in question 12 corresponded to the set relationships in question 11: 11a~12b, 11b~12c, and 11c~12a. Since we know one of the two diagrams is the case, but we are not sure what is the precise state of affairs between these properties, the normative answers to question 12 are that statement b must be true while statements a and c may be true or may be false. Even if students did not give those normative answers to these questions, their answers to question 11 may be consistent with their answers to question 12. We interpret this to mean that they linked the statement and the set relationship normatively, but they might have read the diagram non-normatively. This is an example of what we would have called consistent, though not normative responses. These codes were then tallied to provide descriptive summaries that allowed us to survey student use of the three representations, as we shall share in the following sections. Since it is not our goal to compare the two classes, we aggregate all of the codes across the two classes for tasks used at both sites.

Assume that for the sets M and N, we know that one of the following two set diagrams is the case, but we are not sure which.

Diagram 1 Diagram 2



11) Determine whether the following set relationships is true or false in each of the two diagrams (in other words, give two answers, one for Diagram 1 and one for Diagram 2).
    a. $M^c \subseteq N$
    b. $M^c = N$
    c. $N \cap M = \emptyset$

12) For each of the following statements, circle the best explanation of its truth value based on the diagrams above.
    a. There exists some $x \in S$ such that x is in M and x is in N.
       i. Must be true. ii. May be true or may be false. iii. Must be false.
    b. Given any $x \in S$, if x is in M^c, then x is in N.
       i. Must be true. ii. May be true or may be false. iii. Must be false.
    c. Given any $x \in S$, x is in M^c if and only if x is in N.
       i. Must be true. ii. May be true or may be false. iii. Must be false.

Figure 2: Translation between representations task from a midterm exam

Results

In this section, we present our findings from analyzing student responses to three tasks (though each is multi-part). The first two tasks were given in both classes. The third (in Figure 2) was only given at one site. While we identified other tasks relevant to our investigation, space does not permit us to report more in this conference paper. When possible, we share instances of interesting student thinking to support the claim that the three representations were productive for student inquiry into logic.

Task 1: Building sets containing the given set

Figure 3 presents Task 1 in which students were asked to build sets containing the given set. This was both an opportunity to use set-builder notation and to think about how properties influence the membership of a set. While this task did not include either statements or Euler diagrams, we consider the coordination of properties and sets of objects another key aspect of constructing set relationships. Table 1 presents the results of the coding analysis. Most students used set builder notation as intended. Furthermore, more than 81.7% of the responses identified a superset of the given set (even if equal to the given set). Set D proved to be the most challenging for many students because there are no familiar sets of quadrilaterals that contain all trapezoids. Student responses tended to lean heavily on familiar conditions (i.e., those taught in school) to construct their supersets. One interesting pattern in some of the non-normative responses is illustrated by the student whose response for set D was the set of quadrilaterals in which all four sides are parallel. This produces a subset of D, but the student was likely thinking about how having two parallel sides is "contained in" having four parallel sides. This way of reasoning arises on other tasks, such as when students think the set of equilateral triangles contains the set of isosceles (defined inclusively) since three equal sides contains two.

Let $\mathbb{T}$ denote the set of all triangles. $\mathbb{Q}u$ is the set of all quadrilaterals.
$\mathbb{Z}$ is the set of integers. $\mathbb{N}$ is the set of natural numbers, which is $\{1, 2, 3, 4, \ldots\}$.

For each of the following sets, use set-builder notation to construct another set that contains the given set. In other words, the set given should be a subset of the set you define. (My answers are that $A, B \subseteq \mathbb{Z}$, $C \subseteq \mathbb{T}$, and $D \subseteq \mathbb{Q}u$, so you may not use those answers.) Each task is worth ½ point.

$A = \{x \in \mathbb{Z}: x\ is\ a\ multiple\ of\ 2\ and\ a\ multiple\ of\ 11\}$.
$B = \{x \in \mathbb{Z}: when\ x\ is\ divided\ by\ 4, it\ has\ a\ remainder\ of\ 2\}$.
$C = \{\Delta XYZ \in \mathbb{T}: \Delta XYZ\ is\ obtuse\}$.
$D = \{LMNO \in \mathbb{Q}u: at\ least\ two\ sides\ of\ LMNO\ are\ parallel\}$.

Figure 3: Task 1, which invites students to use set-builder notation

Table 1: Features of student responses to Task 1 (n=26)

Response Feature	Set A	Set B	Set C	Set D	Total
Used set builder notation	73%	81%	88%	85%	**81.7%**
Did not use set builder notation	27%	19%	12%	12%	**17.3%**
Superset	62%	81%	70%	23%	**58.7%**
Equivalent Set	31%	8%	27%	27%	**23.1%**
Subset or Incorrect Set	8%	12%	4%	50%	**18.3%**

Task 2: Constructing diagrams and new set relationships from them

Task 2 (see Figure 4) invited students to translate a set expression into Euler diagrams and then to use those diagrams to produce new set expressions. We had noted in our previous experiments that students often struggled to think about complement sets as the inside of regions in Euler diagrams. For this reason, we purposefully asked them to draw diagrams where the given information contained a complement and chose to provide them a diagram where the complement was inside of an oval region. There are two normative diagrams for this arrangement similar to those in Figure 1, but a range of other diagrams may be drawn if students represent S as the inside of a region or if they imagine any sets to be empty or universal. Accordingly, as displayed in Table 2, some students produced three diagrams. About one quarter of students could only produce one normative diagram.

Task 2 (2 pts): Assume for the sets S, T, which are in the universal set Ω, that we know $S^c \subseteq T$

a) Draw as many different diagrams as you can portraying the possible relationships between S and T.
b) Identify 2 other set relationships that must also be true among the sets S, T, S^c, T^c.
c) Suppose another student drew their diagram as shown to the right. Are your answers to part b still true in this diagram? How could you shade the diagram to show how your set relationships are true in this diagram?

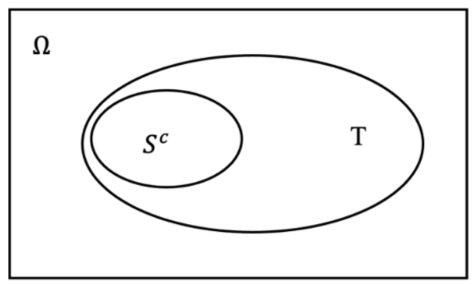


Figure 4: Task 2, which asks students to connect set notation and Euler diagrams

Table 2: Features of student responses to Task 2

Number of Normative Diagrams Produced		Set Relation Normativity		Internal Consistency	
Three	11.5%	Normative set relations	53.8%	Fully internally consistent	57.7%
Two	57.7%	Non-normative set relations	46.2%	At least one consistent	88.5%
One	26.9%				

The intended answers to part b (based on what was discussed in the classes) were that $T^c \subseteq S$ and that $S^c \cap T^c = \emptyset$ since these are both logically equivalent to the given condition, though other statements were possible. As the third column of Table 2 displays, 88.5% of students generated a set relation that was consistent with at least one of their diagrams. We interpret this as evidence of some fluency between the representations. The lower performance indicated in other cells points to the challenge of this task caused both by the presence of the complement and the many-to-one pairing between diagrams and set relationships. Less than 60% of student responses presented a set relation that matched all of their diagrams (according to our interpretation thereof). Since student diagrams were either non-normative or did not capture all of the possibilities, this meant that almost half of their given set relationships were non-normative (not necessarily true given $S^c \subseteq T$).

We want to highlight two types of responses reflected in this data. One student only drew a diagram in which S and T are complements of one another. This is consistent with the given information but does not show all possibilities. As a result, her first set relation, $S \cap T = \emptyset$, is consistent with her diagram, but is not normative since it is untrue when $S^c \subset T$. Her other set relation, $S \cup S^c = \Omega$, is a tautology. It is consistent with the given information, her diagram, and is normative, but it would be true of any set and does not depend upon the given information. A second type of response we noticed was when students generated other types of equivalent conditions, such as $S^c \cap T = S^c$. We had not taught these other conditions equivalent to a subset relation, so we infer that the Euler diagrams supported students in noticing new, normative relationships. Both these equivalent conditions and the tautologies suggest that the Euler diagrams were productive for students' ability to identify new set relationships.

Task 3: Evaluating set relationships and statements from given Euler diagrams

Task 3 appeared in Figure 2. As with Task 2, we purposefully introduced complement sets into this task to see if students could reason about them as sets much like any other set. Table 3 below presents how frequently student responses were normative and consistent with corresponding other responses. The codes group the parts of question 11 and question 12 that correspond according to normative logic. The data show that students frequently gave normative answers to most tasks, though responses to 12c were often neither normative nor consistent. We interpret this to suggest that the language of "if and only if" was still not being coordinated with the other representations in the manner intended. On the other tasks, students were consistent with their diagrams on nearly 4 out of 5 responses, suggesting relatively strong fluency between the representations.

Table 3: Student responses to Task 3

Problem Pair 1				Problem Pair 2				Problem Pair 3			
	D1	D2	Claim		D1	D2	Claim		D1	D2	Claim
Normative	100%	79%	86%	Normative	93%	93%	50%	Normative	71%	93%	79%
Non-Normative	0%	21%	14%	Non-Normative	7%	7%	50%	Non-Normative	29%	7%	21%
Internally Consistent		79%		Internally Consistent		36%		Internally Consistent		79%	

Discussion

This paper analyzed how students in two undergraduate TTP courses, which were designed to foster set-based meanings for logic and proof, used the three representations of set relationships to reason about and communicate about logic. This was done by analyzing student responses to tasks that particularly asked them to operate within and across the representations. Student responses were coded both in terms of whether they were, first, normatively correct and, second, internally consistent with students' responses to other parts of the task. Third, we also noted when students gave responses that were mathematically accurate even if logically less interesting, such as tautologies. While we are pleased by normative responses, we take consistent and tautological responses as supporting evidence that the three representations were productive for student reasoning and communicating.

Much of the data shows that a majority, but not all, student responses were normative and/or consistent. We draw two implications from this. We claim that operating in the representations was accessible and productive for students since most of them were able to demonstrate productive reasoning about set relationships on these tasks. By reasoning within and across the representations, we hoped students could construct and abstract set relationships both between particular properties and arbitrary properties. The second implication we draw is that reasoning between representations is non-trivial. By contrast, Sato et al. (2010) claim that Euler diagrams are self-guiding and certain inferences can be read directly from diagrams. We agree that they are facilitating, but we claim learning to operate in these representations is a meaningful accomplishment. Further, our tasks reveal how task features such as complement sets, coordinating different cases, and needing to generate non-familiar sets all increase the challenge of student reasoning about logic. We hope that future work will continue to explore student learning in this arena and future instruction will seek to make use of these three representations to support student progress.

Acknowledgements

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TRANSFORMATIVE IMPACT: SPECIFICATIONS GRADING IN CALCULUS AND ITS INFLUENCE ON STUDENTS' MATH IDENTITY

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For many college students, mathematics poses a significant academic challenge, particularly in courses like Calculus, which can become gatekeepers hindering their pursuit of STEM-oriented degrees. To address this issue, we advocate for innovative pedagogies such as Specifications Grading that we posit promote Latina/o students' mathematics identity development. In this study, we examine the mathematics identity development of 350 Latina/o students enrolled in Specifications Grading Calculus I compared to traditional grading. Using a validated survey and Repeated Measures ANOVA, we explore changes in recognition, and overall mathematics identity. The study highlights the impact of alternative grading systems, such as Specifications Grading, on Latina students' mathematics identity .

Keywords: Calculus, Instructional Activities and Practices, Undergraduate Education, Assessment

Purposes of the Study

The primary objective of this study is to investigate the influence of Specifications Grading on the development of Latina/o students' mathematical identity and contribute to the field of mathematics education by offering informed perspectives on pedagogical practices that positively shape students' mathematical identities. By comparing Calculus 1 students in Specifications Grading and traditionally graded classes, we aim to provide insights into the relationship between grading methods and their impact on self-concept, confidence, and motivation in mathematics. Specifications Grading, an alternative grading method, is characterized by transparent learning outcomes and assessment criteria (Nilson, 2015). The following research question guided our study: What impact, if any, does a specifications-graded Calculus 1 course have on students' mathematics identity development in comparison to students enrolled in a traditionally graded Calculus 1 course?

Theoretical Framework

The study adopts a theoretical framework emphasizing the importance of mathematics identity in students' academic journeys. Drawing on Cribbs et al.'s (2015) model, we focus on two of the four key constructs: *recognition* and *mathematics identity* (See Figure 1). In our study, Specifications Grading is posited as a pedagogical tool fostering stronger mathematical identities. For a qualitative study of the development of these four key constructs on a student taking Specifications Grading and a student in a traditionally-graded Calculus course, see the work by Fernandez, et al. (2023).

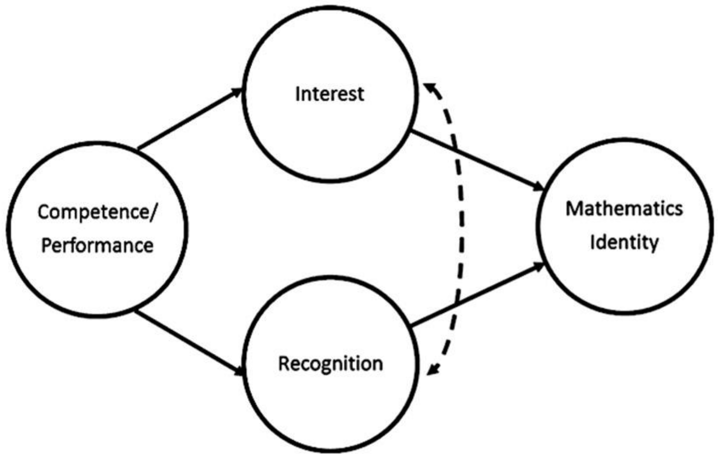

Figure 1. Cribbs et al. (2015) Establishing an Explanatory Model for Mathematics Identity

Methods

During two semesters, we implemented Specifications Grading to a subset of Calculus 1 classes consisting of 8/13 (or 62%) in-person sections in Fall 2022 and 11/14 (or 79%) in-person sections in Spring 2023; the remaining sections were taught in-person and considered traditionally-graded classes. We identified 27 learning outcomes/targets and grouped into 10 core, 11 supplementary, and 6 non-testing learning targets. Assessments consisted of weekly worksheets, four exams, and online homework associated with the learning targets. Worksheets and exams were graded on a pass/no-pass scale. Full credit for a learning target was earned by obtaining credit on the associated worksheet and exam problem. Worksheets could be resubmitted for full credit. If students failed to pass a learning target on an exam, retesting periods were available weekly on Fridays if they attended tutoring. Students from the Specifications Grading classes and traditionally graded classes who completed both the pre-survey during the second week of classes and the post-survey during the last week of the semester served as the participants of this study. In essence there were a total of 350 participants with 43.1% identifying as sophomores and 24%-25% identifying each as freshmen or junior students. For this study we adopted a comprehensive statistical methodology that included Structural Equation Modelling (SEM), specifically Confirmatory Factor Analysis (CFA) and Repeated Measures Analysis of Variance (ANOVA) to analyze the data collected from the participants. In this study, CFA was used to verify the mathematical identity construct, which was based on the work of Cribbs et al. (2015). The comparative fit index (CFI) was used as the metric to measure the accuracy of the construct, and it had a value of 0.941, indicating the goodness of the construct. Repeated Measures ANOVA was used to compare the means of the same group of subjects under different conditions at different times (Girden, 1992, as cited by Beins, 2017). Repeated Measures ANOVA was used to analyze the Mathematical Identity pre- and post-survey scores of students to determine whether there were statistically significant differences in the development of mathematical identity before and after taking the Calculus I course. ANOVA also helps to evaluate whether the differences observed are related to the intervention (Specifications Grading) or if they could be attributed to chance.

Results

Recognition: Specifications Grading vs Traditionally Graded Groups

Preliminary findings show that even when the Specifications Grading and traditionally graded groups started at different estimated marginal means in the pre-timeframe, participants from the Specifications Grading group had a slightly higher measure of ***recognition*** (that is, students' perception of being seen as a math person by others) in comparison to those in the traditionally graded group (See Figure 2). Although the observed differences between the two groups are not statistically significant as indicated by a *p*-value of 0.922, further investigation is needed to determine the factors that influenced these changes.

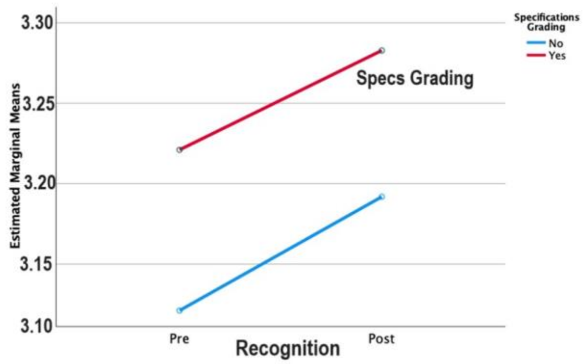

Figure 2. Comparison of Recognition among Specifications Grading and Traditionally Graded Groups

Mathematical Identity: Specifications Grading vs Traditionally Graded Groups

Preliminary findings indicate that even during the pre-timeframe, both groups have distinct estimated marginal means, with the traditionally graded group showing a slightly higher sense of ***math identity*** (that is, students identifying themselves as a math person). However, during the post period this group showed a notable decline in Math Identity, while the Specifications Grading group exhibited a slight increase (See Figure 3). Although the changes are not statistically significant, further investigation is needed to determine the factors influencing those changes. Studying math identity is important as it is correlated with student mathematics achievement (e.g., Fernández et al., 2022; Gonzalez et al., 2022; Matthews, 2020).

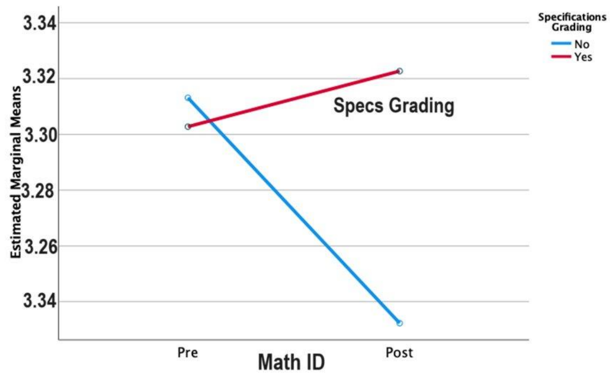

Figure 3. Comparison of Math Identity among Specifications Grading and Traditionally Graded Groups

Recognition: Specifications Grading Male vs Female

Preliminary findings indicate that from the students enrolled in Specifications Grading classes, both male and female students show a slight increase in their ***recognition*** levels (See Figure 4) with female students showing more gains.

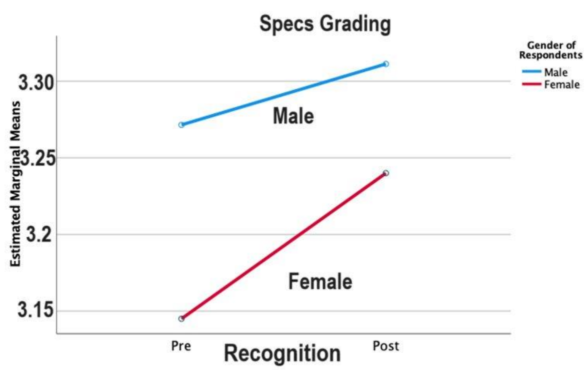

Figure 4. Comparing Recognition among Males and Females in the Specifications Grading Group

Mathematical Identity: Specifications Grading Male vs Female

Preliminary findings indicate that from the students enrolled in Specifications Grading classes, male participants showed a slight decrease in their ***math identity***. On the contrary, female participants showed a notable increase in math identity (See Figure 5).

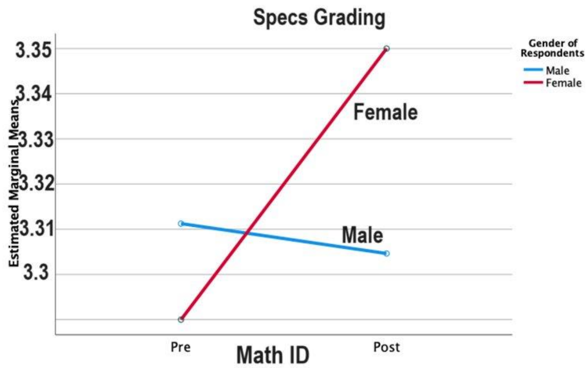

Figure 5. Comparing Math Identity of Males and Females in the Specifications Grading Group

Discussion and Conclusion

This research contributes to the ongoing dialogue on equitable mathematics education for Latina/o students, particularly in gatekeeper courses like Calculus I. The adoption of Specifications Grading emerges as a promising pedagogical strategy to positively influence mathematics identity development. The observed differences in*, recognition,* and overall *mathematics identity* between Specifications Grading and traditionally graded groups underscore the potential of innovative grading approaches in creating a more supportive learning environment.

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COLLEGE STUDENTS' USE OF LANGUAGE RESOURCES IN MATHEMATICS

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Language serves as a crucial tool for communication, comprehension, and problem-solving. People use language as a resource when they approach new problems, including in mathematics classes. In this study, we examined the language resources used by five college students as they solved a mathematics word problem in a language unknown to them. We found the most commonly used resources among the students were contextual (e.g., scaffolds provided by the teacher), cognitive (e.g., their background knowledge), and social (e.g., meaning co-constructed with their peers).

Keywords: Equity, Inclusion, and Diversity; Affect, Emotion, Beliefs, and Attitudes; Instructional Activities and Practices

Classrooms in North America are becoming increasingly multilingual (He & Yu, 2017). As mathematics education researchers envision a future for our field, we must continue to explore the ways in which language can be used as a resource in mathematics classrooms (Adler, 2000; Moschkovich, 2012; 2021). This is particularly crucial for multilingual learners (MLs) who may need to access additional language resources if they do not yet have access to the language of the classroom (Moschkovich, 2012). When asked to participate in a mathematics lesson in a language they do not speak, many teachers would assume their students would not be successful (Gallo et al., 2014). However, if language is viewed as a resource, and if the teacher supports comprehension appropriately (Krashen, 1992), then students may be able to be successful. The purpose of this study was to explore the language resources used by college students when solving a mathematics task in a language they do not know. The following question guided this study: *What language resources do college students use when solving a mathematics word problem in a language they do not know?*

Perspectives

Language and Mathematics

There is a robust body of research on the role of language in the mathematics classroom, especially with regard to MLs' mathematics learning (Barwell et al., 2016; Barwell et al., 2017; Halai & Clarkson, 2015; Moschkovich, 2010; Prediger & Schüler-Meyer, 2017). These studies take different views of language as a resource in the mathematics classroom, but overall have found that multilinguals, especially in classrooms where the language of instruction is not their home language, draw on what they know about language to interact with their peers and their teacher and make meaning (Moschkovich, 2008).

Many researchers have taken up the perspective of language as a resource in mathematics learning to counter deficit narratives about students whose home language is different than the language of instruction (e.g., Adler, 2000; Barwell, 2018; Planas & Civil, 2013). Here we take up Moschkovich's (2015) conceptualization of the multiple resources that MLs use in the mathematics classroom, which draws on a sociocultural perspective, including "not only oral and written text, but also multiple modes, representations (gestures, drawings, tables, graphs, symbols, etc.), and registers (school mathematical language, home languages, and the everyday register)” (p. 44).

In contrast to Moschkovich’s work, which was largely situated in English medium classrooms and focused on students who had at least some English language proficiency, our objective was to investigate the resources students would use to make sense of and solve a mathematics task in a language unknown to them. One critique of Moschkovich’s work is that her definition of language as a resource is all-encompassing to such an extent that it is challenging to discern the specific resources students accessed (Barwell, 2018).

Language Resources

Language resources can refer to external material and technologies, such as dictionaries, online translators, and writing resources (Oh, 2020) or the individual’s language-related resources (i.e., learners’ skills and strategies used to access and produce language). These latter resources include strategies that can be classified into cognitive, affective, social, and contextual resources (see Figure 1). Additionally, researchers have recently argued for the positive association between metacognitive (Hulstijn, 2015, 2019; Teng & Zhang, 2022) and metalinguistic awareness in students’ language learning. Thus, we consider these as resources too (see Figure 1).

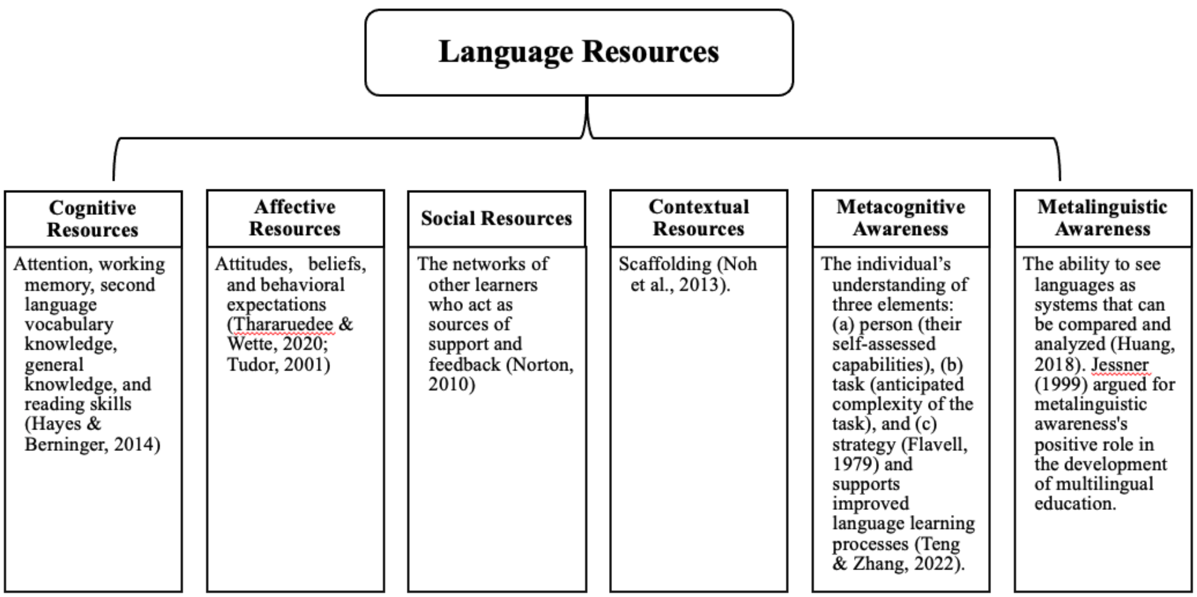

Figure 1: Individual Language-Related Resources as Reported in the Literature

The purpose of this study was to examine how the participants’ language resources allowed them access to a mathematics word problem in an unfamiliar language.

Methods

Participants

We invited undergraduate college students enrolled in a Southern urban university to participate in this study. We had five participants, four of whom identified as females, and one identified as a male. Two participants were sophomores, and three were juniors. Two participants identified as West African, two as Hispanic, and one as Asian. All five participants described having significant exposure to multiple languages and cultures in their lives. Participation included attending (a) an in-person mathematics session and (b) a focus group interview.

We had two in-person, video recorded, 30- minute mathematics sessions: three participants attended the first, and two attended the second. During the sessions, we presented a word problem in Arabic (a language none of them spoke) to the participants and asked them to try to solve it. Following this task, we led a 30-minute focus group interview for the participants to reflect on their feelings, thoughts, and strategies used during the task**.** The task– the population of Midtown is three times that of Karburg. The difference between the two populations is 2184. How many people live in each of the cities? (see Figure 2)– was presented in Arabic. The teacher spoke and wrote in Arabic throughout her presentation.

The participants were allowed to access any materials in the room (e.g., manipulatives, calculators, each other) apart from translation technology to solve the problem. Slowly, the teacher presented additional scaffolds, such as pictorial vocabulary instruction (see Figure 2), manipulatives, and a number translation sheet. We recognize that translanguaging– the ability to seamlessly transition between languages and a teaching approach where educators foster this skill (García & Kleifgen, 2020)– is a key practice to leverage students' language resources, and we allowed translanguaging among the students, but we chose not to have the teacher translanguage in order to emulate the situation in many classrooms where the teacher speaks the majority language and students' dominant languages are minority languages.

Figure 2: The word problem as presented to the students with math and vocabulary representations.

Data Analysis

We coded the transcripts of the videos for the language resources the participants either demonstrated as they worked on the problem or those that they told us they used during the focus group interview. We began with a priori codes derived from the literature review (see Figure 1). Together, both authors coded the entire dataset over multiple sessions. We maintained a code book and a data analysis log, which we referred to when making coding decisions. After applying these high-level a priori codes, we went through the excerpts within each code and open-coded them more descriptively. For example, some of the excerpts under social resources were coded “co-constructing a strategy” and “co-constructing understanding.” All coding at this level was done together so that 100% consensus was reached.

Findings

We observed the participants utilize *cognitive* resources such as “background knowledge” and “mathematical knowledge.” For example, Tiwa mentioned that because the second author introduced herself as an Arabic speaker at the beginning of the session, she was able to recognize that the task was in Arabic. That allowed her to deduce that since writing in Arabic goes from right to left, then perhaps numbers follow the same rules. Additionally, *metalinguistic* resources, such as “knowledge of the written form” and “metanumeric awareness,” were strongly present. For instance, Elena recognized the place value table that the researcher used during the presentation. Although the table showed the Arabic numbers, Elena made the connection and realized the similarities between the rules of place value as she knew them in English and in Arabic as they were presented. Making a connection between two different number systems demonstrates metanumeric awareness (Gallagher, 2021). These findings underscore the importance of language resources in providing access to mathematics tasks.

Additionally, we found that the participants relied most heavily on social resources. Indeed, “co-constructing understanding” was an ongoing activity between all team members. They did that by listening to each other’s thinking and discussing clues. For example, Salman said,

> I find that it's easier when you have a group to do it with, right? Because we build up on other people's ideas, right? So, she knew something that I didn't know. And when she told me, it started making sense to me. So, you know, it's good to work with the group.

Additionally, we observed the participants “co-construct strategy” together. The following exchange presents an example of this particular resource.

Tiwa: So, I think we set up a proportion to figure out how many people live in Midtown compared to the other places. So the first number would be 2184 over … because… I’m not good at proportions. Do you guys know how to set up the proportion?
Valeria: So, are you … are you thinking like one-third equals 2184?
Tiwa: Yes, but also no, but yes, something along that line.
Valeria: Like that structure?
Tiwa: Yeah, because the two is … 2184. So, we need to figure out that one to get the total and then compare it to the other city.
Valeria: [pointing at the problem in the sheet] So, if that number is 2184, then wouldn't that 2184 be 2/3?

Despite none of the participants speaking Arabic or having experienced the language in an academic setting before, they supported each other’s learning by co-constructing their understanding and strategies. Through their use of these multiple strategies, especially *contextual, cognitive*, and *social* resources, the participants were all able to make sense of the task and solve the problem, both arriving at a correct mathematical answer and writing that answer using Arabic numerals.

Discussion and Limitations

The purpose of this study was to examine the language resources that college students use to help them access mathematics tasks in a language unknown to them. Our findings echoed previous research on students' use of cognitive resources (de Araujo et al., 2018; Peng et al., 2020); metacognitive resources (Jessner, 2018); metalinguistic awareness (Reder et al., 2013); contextual resources, such as scaffolding (Noh et al., 2013); and social and contextual resources (Klang et al., 2021). The small sample size poses a limitation for this study. Additionally, we recognize that college-age students may have had more experience needing to make sense of new situations and therefore may have more resources than younger students. We hope to expand the study in the future and encourage researchers to examine the language resources used by students in mathematics classrooms across schooling, K-16.

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APPLYING ENACTIVIST EPISTEMOLOGY TO MATHEMATICS CLASSROOMS: AN INTERACTIVE PARADIGM FOR RESEARCHING STUDENT COGNITION

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Keywords: Cognition, Learning Theory, Research Methods

Researchers in mathematics education and educational studies have argued from post-positivist, constructivist, and emergent epistemological vantage points that understanding students' interactions is central to understanding their cognitive experiences in mathematics classrooms (e.g., Cobb & Yackel, 1996; Cohen et al., 2003; Mok & Clarke, 2015; Nuthall, 2007). In this poster, I explore how another epistemological perspective, enactivism, can further unify this body of work with implications for the conceptualization and design of mathematics classroom research. Enactivists posit that cognition *is* an interaction between an individual and one's environment, as cognition is an enacted phenomenon consisting of perceptually guided actions (Maturana & Varela, 1992; Reid & Mgombelo, 2015; Varela et al., 1991). Although enactivism has been applied in mathematics education research previously (see Simmt & Kieren, 2015), I seek to distill concepts and premises for researching students' cognition in mathematics classrooms to inform an *interactive paradigm* for mathematics classroom research. Indeed, paradigms "provide frameworks that describe, interpret, analyze, and in some cases explain both the knowledge that is being produced as well as the processes that are used to produce it" (Collins, 2019, p. 52). Thus, the articulation of an interactive paradigm for mathematics classroom research has implications for both the content of what one can conceive of researching as well as the methodologies and approaches one applies in investigation.

In this theoretical poster, I explore the following research question: *What does enactivist epistemology imply for how we might conceive of researching students' cognition in mathematics classrooms?* I will present an interactive paradigm in response, including my definitions for three core concepts—students, environment, and cognition—united around two guiding premises:

1. Classroom environments are actively constructed by each student as they relate to other actors and features of their environments.
2. Students' cognition is both agentic and interdependent with their classroom environment.

To contextualize the concepts and premises described above, I will draw connections not only to the extant literature but also to the research methodology and preliminary results of a case study I conducted in alignment with this interactive paradigm. I designed the case study to focus on the cognition and mathematical interactions of three eighth-grade students with their environment (including one another) in an algebra classroom during an instructional unit about linear equations (Premise 1). Through an embedded case structure and targeted analysis of transcript data, I sought to produce findings about algebraic cognition reflective of students' agency and interdependence (Premise 2). Through this poster, I will discuss how an interactive paradigm in this case study contributed to results around student cognition in classrooms that differ from non-enactivist research centering interactions. Importantly, by highlighting how an interactive paradigm might be applied as well as what it might contribute, this poster will share affordances of incorporating enactivist epistemology explicitly into mathematics classroom research agendas.

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FACTORS INFLUENCING WOMEN'S SENSE OF BELONGING IN UNDERGRADUATE CALCULUS

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Keywords: gender; affect, emotion, beliefs, and attitudes; calculus; undergraduate education

Women continue to be underrepresented in undergraduate STEM majors, and this gender gap is due at least in part to women's decisions to major in STEM (Carmichael, 2017). Fewer women than men enter into STEM majors and women leave STEM majors at a higher rate than men, especially after taking Calculus I (Chen et al., 2013; Eagan et al., 2016; Ellis et al., 2016). One major reason for women's decisions to leave STEM is that they feel a low sense of belonging (Seymour & Hunter, 2019; Shapiro & Sax, 2011). Sense of belonging is the extent to which one feels like an accepted member of an academic community, whose presence and contributions are valued (Good et al., 2012). Scholars have identified characteristics of sense of belonging such as its malleability (Anderman, 2003; Griffin, 2023), as well as factors that influence one's sense of belonging, including perceived competence, social connectedness, and learning environment (Anderman, 2003; Rainey et al., 2018; Rainey et al., 2019). Perceived competence is the extent to which one *feels* like they understand the material. Social connectedness is the extent to which one has relationships with their classmates and/or instructor. Learning environments have features such as classroom climate and classroom activities.

This report aims to explore the roles that perceived competence, social connectedness, and learning environment play in influencing women's sense of belonging in Calculus. This study takes place at a mid-Atlantic research university during the Fall 2022 semester. The university offers a year-long Integrated Calculus course that covers both Pre-calculus and Calculus and was designed to incorporate frequent opportunities for students to engage in active learning. In the Fall 2022, two sections of the course were offered, with 63 and 64 students enrolled in each. The sample for analysis was narrowed to students who self-identified as women, N=41. Participants were surveyed twice during the semester. They were asked to rank the following constructs from most to least impactful on their sense of belonging in mathematics: social connectedness with classmates, social connectedness with instructor, perceived competence, classroom climate, and classroom activities. They were then asked to describe why they chose their top two most impactful constructs. Frequencies of rankings were calculated to explore which constructs were most impactful for women's sense of belonging. Descriptions were analyzed using open coding to explore how and why these constructs impacted women's sense of belonging.

Results show that women consistently included social connectedness with classmates and perceived competence as one of their top two most impactful constructs on their sense of belonging. While classroom activities was not the most popular construct, when women did include this in their top two, they described its impact on social connectedness and perceived competence rather than its direct impact on their sense of belonging, suggesting that perhaps participants viewed classroom activities as farther removed from sense of belonging than social connectedness and perceived competence. These findings suggest that while students' perceptions of their own social connectedness and perceived competence have much influence on their sense of belonging, instructors may be able to influence students' social connectedness and perceived competence, and in turn their sense of belonging, via their pedagogical choices.

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IMPROVING METACOGNITIVE MONITORING IN MATH USING CUETHINKEF+

MEJORANDO EL MONITOREO METACOGNITIVO EN MATHEMÁTICAS USANDO CUETHINKEF+

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Metacognition (MC) has been found to be a critical component of mathematical problem solving (PS; Rhodes et al., 2023). We report on the results of a study that employed an ANCOVA to examine the impact of a PS intervention that included MC supports (e.g., journaling) on students' objective MC within a sample of 276 middle school students. The results suggest that the intervention group significantly improved in their MC.

Keywords: Metacognition; Problem-Solving

Research has suggested that mathematical problem solving (PS) should be embedded within core instruction and should not be isolated from other factors such as content and metacognition (MC; Cai & Lester, 2010). Moreover, MC has been found to be a crucial factor influencing PS proficiency (Rhodes et al., 2023; Lester, 2013). Specifically, MC is thought to support problem solvers in nearly every aspect of the PS process, such as making sense of the problem, selecting and applying strategies, and reflecting and revising their thinking (Tan & Limjap, 2018).

Given these findings, it is unsurprising that interventions that include metacognitive training have been found to significantly improve students' PS performance (e.g., Kramarski et al., 2002). Metacognitive training is operationalized here-in as any support designed to help students utilize MC in the moment (e.g., being more intentional in heuristic selection, self-generating feedback based on their progress, etc.). Indeed, research has suggested that it is ineffective to attempt to isolate instruction on PS from the cognitive and metacognitive factors that influence it (e.g., Cai & Lester, 2010; Lesh & Zawojewski, 2007). However, few studies have explored whether interventions that interweave PS instruction with metacognitive supports result in improved MC. Thus, the purpose of the present study was to examine the effects of a web-based application, CueThinkEF+, that intentionally integrated scaffolds and supports that targeted MC, executive functions, and PS, on students' MC as operationalized by miscomprehension accuracy (the degree to which a person is able to accurately predict performance on a given task or measure).

Learning strategy use emerges from both contextual, cognitive, and personal factors (Panadero, 2017). Research differentiates between types of learning strategies that are hypothesized to improve metacomprehension accuracy, such as when judgments of comprehension are delayed (Shiu & Chen, 2013), individuals receive feedback (Brannick et al., 2005), and learners receive practice (Bol et al., 2005; Gutierrez & Schraw, 2015; Gutierrez & Price, 2017; Hacker et al., 2008; Thiede et al., 2012). Similarly, Nietfeld and Schraw (2002) and Gutierrez and Schraw (2015) found that students who received learning strategy instruction showed superior learning and more accurate monitoring. In the Nietfeld and Schraw (2002) study, which involved performance on probabilities, participants received an instructional sequence of five learning strategies (e.g., self-generated feedback, planning, self-questioning, etc.) discussed during instruction. Gutierrez and Schraw (2015) adapted these strategies and added others. The CueThinkEF+ intervention employs, among others, prompts that encourage self-questioning and intentional planning and reflection.

Additional work such as that of Nietfeld et al. (2006) found that distributed learning strategy instruction with feedback produced higher performance, confidence, and metacomprehension accuracy among college students, while Huff and Nietfeld (2009) found that strategy instruction with 5th grade students improved performance and metacomprehension accuracy. Bensley and Spero (2014) demonstrated that the direct infusion of learning strategies increased students' posttest metacomprehension accuracy while Schleinschok et al. (2017) reported that students who were taught to diagram as a learning strategy improved their metacomprehension accuracy of how well they thought they learned. However, students in the control condition also showed improved monitoring at posttest, although smaller than the treatment condition. This research supports the hypothesis that diagramming, as an example of a specific learning strategy, can improve metacomprehension accuracy. Furthermore, Miller and Geraci (2011) found that strategy instruction was only successful at improving metacomprehension accuracy for lower-performing students because higher-performing students already exhibited high metacomprehension accuracy. Taken together, these findings uniformly supported the hypothesis that metacomprehension accuracy was trainable and could be improved because of learning strategy instruction in various contexts and levels of specificity, including math.

Given the literature surveyed, the purpose of the present study was to answer the following research question: what is the effect of the CueThinkEF+ intervention on middle school students' metacognitive monitoring accuracy?

Conceptual Framework

Efklides (2011) devised the Metacognitive and Affective Model of Self-Regulated Learning (MASRL) in which metacognitive and motivational processes are key, centered on task, person, and task by person levels. The present study employs Efklides' (2011) MASRL as a theoretical guide because this model focuses on the central role metacognitive monitoring and control processes play in learning. More specifically, this model helps us better understand how person-level characteristics (e.g., motivation, engagement, interest, autonomy, task value, etc.) interact with the task (in this case, the CueThinkEF+ platform) and how metacognitive skills like monitoring and control help the moderate this interaction.

Method

Description of the Intervention

Over the course of one academic year, students in the control group continued with business-as-usual instruction while the intervention group used CueThinkEF+ approximately 5-7 times. Incorporating elements from the research noted above, CueThinkEF+ intentionally targeted MC during each phase of the PS process. Exemplifying this several of the supports, students were asked to: 1) consider what they noticed and wondered; 2) restate the question being asked in their own words; 3) create a journal of their plan for solving each problem which could then be updated or revised as they solved the problem; 4) record themselves explaining how they solved the problem, showing their work on a digital whiteboard including various tools and manipulatives; 5) watch the recordings of their peers and write annotations (e.g., "My strategy is like yours because…").

Participants

Three middle schools from a suburban school district participated in the study with one school serving as the intervention group and the other two schools serving as the control group. The district was located on the West Coast and all mathematics teachers at the three participating schools who taught grades 6-8 were invited to participate in the study. Ten teachers signed up to participate in the study and all students of participating teachers were then invited to participate in the study. Of the participating students, 278 students had complete data on the measures utilized within the present analyses. Given that only two of the students with complete data were 8th graders, the present analyses were confined to students in grades 6 and 7. Of the 276 remaining students, 188 students were in the intervention group and 88 students were in the control group. Within each group, the students were evenly split between 6 and 7th graders. Additional demographic data on the students is provided in Table 1 and Table 2 below.

Table 1: Number of Participants by Group and Gender

	n	Female	Male	Non-binary	Other/Did not Specify
Intervention	188	106	73	1	8
Control	88	36	39	2	11

Table 2: Participants by Group and Ethnicity

	Asian	Black/ African American	Hispanic/ Latin(x)	Middle Eastern	White	2 or More Races	Other/Did not Specify
Intervention	7.4%	4.3%	33.5%	24.5%	1.6%	16.5%	12.2%
Control	4.5%	4.5%	22.7%	36.4%	3.4%	17.0%	11.4%

Instruments and Materials

Problem Solving. PS items were compiled from items written by Illustrative Mathematics (IM) and aligned by the researchers to district pacing guides. For each grade level, three cognitively demanding items were selected with the only modification being that directions were added to require students to show or explain their thinking, when needed. Each item was scored for correctness using answer keys that were developed by IM. The interrater agreement on scoring was calculated using Fleiss' kappa and was .961 for 6th grade and .880 for 7th grade.

Metacognition. Prior to completing the PS measure, students were given a list of the mathematical topics that would be covered by the problems on the measure. Students were then asked to predict how well they would do by writing a numerical answer between 0 and 100, inclusive. Prediction accuracy scores were calculated by comparing participants' confidence in performance judgments before the performance assessment against their actual performance. Comparing prediction confidence in performance judgments against actual performance yielded continuous, absolute accuracy scores, as described by Schraw (2009). A score of "0" indicates perfect monitoring; conversely, the further a score is from "0," the greater the inaccuracy.

Data Analysis

Data were first tested for requisite statistical assumptions and tested for univariate outliers using box-and-whisker plots. The data met the assumptions of normality using skewness and kurtosis metrics, homogeneity of variance, and homogeneity of regression slopes. Further, the data did not contain any extreme outliers that would otherwise undermine the trustworthiness of the findings. Hence, data analysis proceeded without making any statistical adjustments.

To answer our research question, we employed a one-way analysis of covariance (ANCOVA), with condition (intervention, control) serving as our fixed factor, pretest prediction accuracy scores serving as the covariate, and posttest prediction accuracy scores serving as the outcome. The results of this analysis demonstrated that pretest prediction scores did not significantly influence posttest prediction scores, and hence, the analysis reverted to a one-way analysis of variance (ANOVA). Eta square (η^2) served as the metric for effect size estimate. Cohen (1988) was used for interpretative guidance on effect size within the present study.

Results

Results of the one-way ANOVA revealed a statistically significant main effect for condition on posttest prediction scores, $F(1,273) = 13.23$, $p < .001$, $\eta^2 = .045$, indicating a small-to-moderate effect size. Table 1 contains the descriptive statistics for the intervention and control groups, and it indicates that the intervention group manifested significantly higher posttest prediction monitoring accuracy compared to the control group.

Table 1: Descriptive Statistics of Posttest Prediction Monitoring Accuracy for the Intervention and Control Groups

	M	*SD*
Intervention (*n* = 188)	32.19	22.09
Control (*n* = 88)	41.98	22.77

Discussion and Limitations

The purpose of the present study was to investigate the effect of CueThinkEF+ on students' metacognitive monitoring accuracy in math. Results revealed a small-to-moderate effect of CueThinkEF+ on students' prediction accuracy when compared to business-as-usual instruction. These results extend the extant literature on metacognitive monitoring accuracy in a math context. They are also congruent with the body of literature that concludes that learning strategy interventions are successful at improving not only learners' performance, but also their monitoring accuracy using control processes to adjust their confidence in performance judgments (e.g., Gutierrez & Price, 2017; Nietfeld and Schraw, 2002; Thiede et al., 2012).

Moreover, the results of the present study provide additional evidence that MC and PS instruction should be intertwined in instruction (e.g. Lesh and Zawojewski, 2007). Specifically, the present study demonstrates that the intervention led to improvements in students metacognitive monitoring accuracy, extending prior analyses showing that the intervention also resulted in increases in student's PS proficiency (Rhodes et al., in preparation). Thus, it is possible that intentionally intertwining metacognitive training and supports with PS instruction may lead to meaningful gains in both areas.

Despite these promising results, the generalizability of the findings is limited by the fact that the study only included samples of students from grades 6 and 7 from a single district. In addition, the present study operationalized MC as metacomprehension monitoring accuracy. Future research should consider using additional measures of MC as well.

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THE RELATIONSHIP BETWEEN BILINGUALISM AND CHILDREN'S SELF-EFFICACY BELIEFS IN MATHEMATICS THROUGH A SITUATED-SOCIOCULTURAL PERSPECTIVE

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This paper explores the relationship between bilingualism and children's self-efficacy in mathematics from a sociocultural perspective. Utilizing an explanatory sequential mixed methods approach, the study examines how diverse cultural contexts and experiences, related to varying language uses and community affiliations, influence children's self-efficacy beliefs in mathematics. Quantitative analysis of TIMSS 2019 data indicates that bilingual children exhibit significantly higher self-efficacy in mathematics than the nonbilingual students, even when controlling for other factors such as academic achievement. Grounded theory analysis of four bilingual families in the U.S. suggests that a bilingual, multicultural identity fosters a positive learning identity across subjects, enhancing self-efficacy in mathematics. Bilingual children also appear to encounter more meaningful challenges and mastery experiences, which strengthen their learning beliefs. Additionally, their unique contexts provide more opportunities to connect and compare diverse cultural perspectives in mathematics, and they receive more interest, attention, and support from parents, teachers, and communities, aiding in overcoming obstacles and developing positive beliefs.

Keywords: Affect, Emotion, Beliefs, and Attitudes, Equity, Inclusion, and Diversity, Culturally Relevant Pedagogy, Early Childhood Education, Self-Efficacy in Mathematics

Children's perceptions of their abilities significantly influence their motivation, performance, and achievement across various domains (Bandura, 1997; Schunk & Pajares, 2002; Wigfield & Eccles, 2000; Barroso, Ganley McGraw, Geer, Hart, & Daucourt, 2021). The self-efficacy beliefs, "the belief in one's capabilities to organize and execute the courses of action required to produce given attainments" (Bandura, 1977, p. 3), drive the 'will' to succeed, motivating individuals to invest time and energy in areas where they believe they can succeed.

Inevitably, self-efficacy is crucial for maintaining motivation in mathematics, enabling students to persist in skill development until reaching their potential (Bandura, 1977, 2012; Pajares & Miller, 1994; Zimmerman & Schunk, 2001). Instead of focusing on its significance of further potential development and growth, however, a prominent trend in self-efficacy research is the investigation of its role as a predictor of students' future academic achievement, course choices, and career trajectories, especially in context of science, technology, engineering, and mathematics (STEM) education (Lee & Stankov, 2018; Muenks et al., 2017; Pajares & Miller, 1995; Wang, 2013; Zeldin & Pajares, 2000). In young children, particularly, self-efficacy's significance extends beyond predictions for future outcomes to understanding how positive beliefs can be cultivated and solidified to thrive their potential. These beliefs, characterized by a future-oriented and malleable nature, hold theoretical and operational importance, emphasizing perceived confidence over competence (Bong & Skaalvik, 2003; Zimmerman, 2000). Therefore, this study aims to deepen the understanding of children's self-efficacy development within their sociocultural contexts, particularly in the circumstances of bilingual children.

Theoretical Perspectives

Traditionally, self-efficacy beliefs are seen as internal mental representations (Gee, 2008), with psychological perspectives emphasizing universal human functioning and mental processes. This view posits that well-designed sources of self-efficacy generally enhance these beliefs. However, the assumption of uniform internalization contradicts observed variations in children's learning and thinking (Gee, 2008; Usher, Weidner, Liem, & McInerney, 2018). Self-efficacy beliefs emerge from the interplay between personal and environmental factors, where situational and cultural elements lead individuals to interpret information uniquely, shaping their self-perceptions. Therefore, developing self-efficacy involves integrating information from multiple sources (Oettingen, 1995; Usher, 2009; Usher et al., 2018).

Mastery experiences are the most powerful source of self-efficacy beliefs (Bandura, 1986; Butz & Usher, 2015; Zimmerman, Schunk, & DiBenedetto, 2017; Sheu et al., 2018; Joët, Usher & Bressoux, 2011). However, it is not merely the existence of mastery experiences but students' interpretations of these experiences within specific contexts that shape their self-efficacy. Additionally, the circumstances under which a student gains mastery experiences can influence the interpretation, evaluation, and weight of these experiences (Gao, 2020; Usher, 2009; Usher et al., 2018). Usher and Pajares (2008) called for further studies to "capture the personal, social, situational, and temporal conditions under which students cognitively process and appraise their beliefs and experiences" (p. 784). This study emphasizes the importance of sociocultural influences in developing self-efficacy beliefs in mathematics.

Focusing on language as a significant construct of children's sociocultural circumstances, this study examines bilingual children. Language is fundamental to cognitive growth and mediates higher-order thinking (John & Brader-Araje, 2002). Vygotsky's sociocultural theory posits that language significantly mediates sociocultural forces, influencing mediated activity and experiences (Fernyhough, 2013; Rubik, 2017). Thus, modifying the cognitive tools accessible to a child, such as language, can fundamentally reshape their mind (Vygotsky, 1978). This paper investigates bilingual children's self-efficacy beliefs in mathematics through a sociocultural perspective, emphasizing the role of language as a cognitive and sociocultural mediator.

Research on students' self-efficacy has largely focused on high school and college-aged individuals in predominantly White settings. However, some studies highlight the significant impact of contextual and demographic factors, such as gender, ethnic background, and learning domain, on self-efficacy outcomes (Usher, 2009; Britner & Pajares, 2006; Lent, Lopez, & Bieschke, 1991; Usher & Pajares, 2006). Limited research has explored these factors' influence on efficacy beliefs, emphasizing the need to consider cultural dimensions in diverse settings (Klanssen, 2004). Klanssen's (2004) study of 270 Grade 7 students (Indo Canadian and Anglo Canadian) found that Indo Canadian students displayed a more hierarchical orientation in mathematics efficacy beliefs, influenced by social comparison. The study did not explicitly address language but suggested it as a factor. Clifton-Sprigg's (2015) research on bilingualism's impact on early childhood performance found bilingual and monolingual children performed comparably, with parental background contributing to variations in cognitive and non-cognitive skills development.

Methodology

This research employs an explanatory sequential design, starting with a quantitative phase followed by a qualitative phase to delve deeper into notable results (Creswell et al., 2011).

In the quantitative study, data from American 4th-grade students in the 2019 TIMSS is analyzed, focusing on the United States. To address missing data, all variables, primary sampling unit, strata variables, and appropriate weights were included in the imputation models. The study uses the Students Confident in Mathematics (SCM) scale from TIMSS as the dependent variable. Bilingualism is determined by students' use of the test language at home, with 23% identified as bilingual. Control variables include gender, achievements, engagement, and socioeconomic status. The quantitative phase examines the relationship between bilingualism and children's self-efficacy beliefs in mathematics, comparing variables between bilingual and non-bilingual students, followed by OLS regression to predict math self-efficacy beliefs.

In the qualitative phase, data is collected through interviews with two Spanish-English bilingual families and two Korean-English bilingual families, involving one 45-minute parent interview and two 30-minute child interviews per family. Parents discuss their family's identities, values, beliefs about language, culture, math, and their child's experiences. Children are interviewed about their understanding of family values, language and mathematics, and self-efficacy. Semi-structured interviews and grounded theory analysis explore how children interpret sources of their self-efficacy beliefs and how their environment and experiences influence these beliefs. The study investigates why bilingualism positively relates to children's self-efficacy in mathematics and how bilingual children develop higher self-efficacy in mathematics.

Results

In the quantitative phase, bilingual students score lower than non-bilingual peers (Table 1), yet there is no significant difference in self-efficacy beliefs. Bilingual children, often from socioeconomically disadvantaged backgrounds, reflect this in school SES. This socioeconomic disadvantage potentially impacts their math self-efficacy as prior achievement influences self-efficacy development.

Table 1. Means and standard errors of the estimates, for all variables[5]

	All Children		Bilingual Children		Non-Bilingual Children		Difference
	Mean	Std.	Mean	Std.	Mean	Std.	
Achievement	534.49	(1.89)	515.13	(2.52)	539.77	(2.07)	−24.64***
Self-Efficacy Beliefs	9.96	(0.02)	9.95	(0.05)	9.97	(0.03)	−0.02
Female	0.49	(0.01)	0.53	(0.01)	0.48	(0.01)	0.052***
Number of Books at Home							
1 (0–10)	0.17	(0.01)	0.21	(0.01)	0.15	(0.01)	0.06***
2 (11–25)	0.24	(0)	0.3	(0.01)	0.22	(0.01)	0.09***
3 (26–100)	0.32	(0)	0.28	(0.01)	0.32	(0.01)	−0.04**
4 (More than 100)	0.28	(0.01)	0.2	(0.01)	0.3	(0.01)	−0.1***
School SES							
Disadvantage	0.55	(0.02)	0.66	(0.04)	0.53	(0.02)	0.13***
Middle	0.21	(0.02)	0.17	(0.03)	0.22	(0.02)	−0.05***

[5] Means and standard errors of the estimates, for all variables. * $p < 0.05$; ** $p < 0.01$; *** $p < 0.001$

Affluent	0.24	(0.02)	0.17	(0.02)	0.25	(0.02)	−0.08***
Interactions	3.99	(0.02)	3.97	(0.03)	3.99	(0.02)	−0.03
Observations	10115		2333		7782		

The OLS analyses (Table 2) confirm a statistically significant positive correlation between bilingualism and children's self-efficacy in mathematics. Model 2 reveals significant associations between gender, achievements, interactions with math teachers, and school SES with children's self-efficacy, aligning with existing research. Initially, the link between bilingualism and self-efficacy appears negative but not statistically significant. However, when controlling for achievement, the relationship shifts to a significant positive correlation, indicating that bilingualism positively affects self-efficacy when accounting for achievement.

Table 2. Results in Descriptive Analysis

	Model 1		Model 2	
	Coef.	Std.	Coef.	Std.
Bilingual	−0.02	(0.05)	0.29***	(0.04)
Female			− 0.31***	(0.05)
Achievement			0.01***	(0.00)
Interactions with Math Teachers			0.12***	(0.02)
Number of Books at Home			0.06***	(0.02)
School SES			−0.20***	(0.03)

From the qualitative study aiming to find how bilingual, multicultural students' experiences influence their self-efficacy beliefs in mathematics, the impact of bilingualism on children's self-efficacy in mathematics cannot be simply attributed to individual factors like prior proficiency, observational learning, social influences, or emotional attitudes towards math, typically seen as key determinants in Bandura's social cognitive theory. Instead, it must be understood in the broader context of their upbringing and environment within families, communities, schools, and societies as bilingual individuals. Essentially, the connection between bilingualism and self-efficacy in math goes beyond mere direct experiences with the subject; it is intertwined with their familial and self-identities, cultural values and expectations, and overall disposition as bilingual children in specific settings. One interesting theme that emerged among the bilingual, multicultural children in my cases is that they see their families and themselves as special and valuable because they are different from other monolingual families or friends. They appreciate the challenges they have encountered and the effort and resilience they have demonstrated in various situations. These values and appreciations are initially supported by their parents at home and solidified by teachers and peers. Based on these wholistic beliefs and values in their families and themselves, bilingual students' self-efficacy beliefs in mathematics are shaped. Cognitively, bilingual students tend to use both languages to understand mathematics, which might give them more flexibility. Culturally, they likely have access to more contexts and resources that come from different languages and cultures.

Conclusions and Educational Implications

While the qualitative analysis had a small sample size and may not fully represent the entire population of the quantitative phase, this study revealed a significant link between bilingualism and children's self-efficacy beliefs in mathematics. It provides insights into how and why bilingual children develop positive self-efficacy beliefs within specific sociocultural contexts. The study underscores the importance of understanding how children's environments and experiences influence their self-efficacy beliefs, not just focusing on achievements.

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EXAMINING COMPLEX NUMBERS WITH LEARNING THROUGH ACTIVITY

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Simon and his colleagues' (2010, 2018) development of Learning Through Activity (LTA) offered a theory for explaining a mechanism for mathematics conceptual learning and an approach to the instructional design to foster it. LTA was an empirically based framework that was developed studying mainly rational number concepts. Taking a step forward, in this paper, we elaborate on and exemplify LTA instructional design principles using an advanced mathematics topic. For this purpose, we share an articulation of the Cartesian form of complex numbers as a mathematical concept and a task sequence to learn this concept. Also, we share an example of the reflective abstraction of this concept with data from teaching experiments with a prospective secondary mathematics teacher. Providing ways to utilize the LTA framework for an advanced mathematical topic, we discuss implications for teaching and learning.

Keywords: Mathematics Learning, Learning Through Activity, Complex Numbers

Conceptual mathematics learning for all students is the main goal of mathematics instruction (Common Core School Mathematics, 2010). Planning effective instructional designs is at the heart of conceptual learning (Gravemejier, 2004; NCTM, 2000; Simon & Tzur, 2004). Over the last three decades, Simon postulated the construct of hypothetical learning trajectory (HLT) for describing key aspects of planning mathematics lessons for promoting conceptual learning from a constructivist perspective (Simon, 1995) and extended it by postulating a mechanism for mathematics learning and instructional design principles for students' learning of mathematics through their own activity (Simon & Tzur, 2004; Simon et al., 2010; Simon et al., 2018).

Learning Through Activity (LTA) design principles deepen and extend HLT steps by further explicating the reciprocal relationship between tasks and learning goals. LTA design principles allow the teacher, first, to delineate "..an activity that students have currently available that can be the basis for the abstraction specified in the learning goal (Simon et al., 2018, p. 104). The activity refers to students' two or more goal-directed mental actions (Simon et al., 2018). Secondly, the teachers design or choose a task sequence intended to foster the particular activity and abstraction on the part of students. The tasks based on LTA design theory are as such: "… the learner can already solve that causes the learner to coordinate actions corresponding to prior concepts in such a manner that the intended new concept is inherent in this coordination" (Dreyfus, 2018, p. 217). In addition, the tasks neither explain nor mention the intended learning goal; however, they have the underlying goal of leading the students to grasp the logical necessity of the learning goal on their own (Dreyfus, 2018). Therefore, for both research and teaching purposes, researchers took attention to two important aspects of the instructional design: 1) an awareness and articulation of the learning goals through students' own activity, and 2) the creation or choice of tasks that have the potential to foster such activity based on which mathematics conceptual learning might be promoted (e.g., Dreyfus, 2018; Simon et al., 2018; Tzur, 2018). Further research is suggested to investigate the applicability of LTA to other levels of mathematical learning (e.g., advanced mathematics) (Dreyfus, 2018; Simon et al., 2018). Taking this as a challenge, in this paper, we elaborate on and exemplify LTA instructional design principles to promote conceptual learning of an advanced mathematics concept, the Cartesian form of complex numbers. Thus, we attempted to answer the following theoretical and empirical questions: What might be a possible example of LTA design principles used with respect to learning goals for the Cartesian form of complex numbers? What might be a possible learning goal for a Cartesian form of complex numbers as a mathematical concept? What might be a possible sequence of tasks affording the learning goal through one's own activity for a Cartesian form of complex numbers?

Learning Through Activity and Complex Numbers

Learning Through Activity instructional design is composed of four steps: The first two steps for the generation of HLT are determining students' current knowledge and identifying a learning goal. The third step is specifying "…an activity that students have currently available that can be the basis for the abstraction specified in the learning goal… The fourth step is the design of the task sequence" (Simon et al, 2018, p. 104). In particular, the trajectory for complex numbers is based on the assumption that, at the outset, the students currently can define any quadratic function with real coefficients and graph them on the Cartesian coordinate systems and know that given any quadratic equation, $ax^2 + bx + c = 0$ where $a, b, c \epsilon R$, $a \neq 0$ the two roots of the quadratic equation are of the form $x_{1,2} = \frac{-b}{2a} \pm \frac{\sqrt{\Delta}}{2a}$, where $\Delta = b^2 - 4ac$ and is called the discriminant (Δ). They also know that x_1 and x_2 can be located on the Real number line, as $(\frac{-b}{2a} - \frac{\sqrt{\Delta}}{2a}, 0)$ and $(\frac{-b}{2a} + \frac{\sqrt{\Delta}}{2a}, 0)$ respectively. Here, the component $\frac{-b}{2a}$ indicates $(\frac{-b}{2a}, 0)$, the abscissa-of-the-vertex (as well as the symmetry axis) and has a distance to the origin and a distance to the roots, that is $\frac{\sqrt{\Delta}}{2a}$ (Hedden & Langbauer, 2003). Regarding the learning goal, we first elaborate on the following: Mathematically, any complex number $z = x+iy$ is an ordered pair (x,y) of real numbers x and y, with $i = \sqrt{-1}$. However, mathematical definitions or theorems are not necessarily the same as mathematical concepts (Vergnaud,1997). Simon (2017) defined a mathematical concept as " a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (p. 123). So, we concur that a researcher's articulation of such a learning goal might be as follows: "Complex numbers, located in the complex plane as the solutions of a quadratic equation f(x)=0 with a fixed apsis of the vertex, vary continuously with the coefficients of f(x) such that the continuous changes in directed distances of the solutions to the apsis of the vertex and to the origin result in the changes in the locations of the solutions in the complex plane". In the LTA design, the third step starts with asking the question, "What activity, currently available to the students, might be the basis for the intended learning?" (Simon & Tzur, 2004, p. 96). Thus, for the learning goal, we identify the specified activity as continuously varying the locations in the complex plane of the roots of any quadratic equation with the same apsis of the vertex. For the fourth step in LTA, Simon et al. (2018) stated, "The task sequence must both elicit the intended student activity and lead to the eventual coordination of actions on the part of the students" (p.104). Thus, we generated the following tasks: *The first set of tasks: 1)* How many parabolas are there with the same abscissa of the vertex? 2)Draw what you imagine in the first question. 3) What changes and remains invariant in the parabolas you have drawn given the algebraic form, $f(x) = ax^2 + bx + c$? Why? 3a) How are the distances of the roots to the-apsis-of-the-vertex changing or not changing? *The second set of tasks*: 1) Given a set of quadratic functions such as $x^2 + 2x$ -8, $x^2 + 2x$ -4, $x^2 + 2x$ -1, $x^2 + 2x$, and $x^2 + 2x$+1 and their graphs as parabolas on a) Desmos and b) colored copy, what changes and what remains invariant? Why? 2) How are the distances of the roots and the abscissa-of-the-vertex changing or not changing? 3) Locate the roots on the Real number line from #1. How do the changes in the distances of the roots to the abscissa of the vertex relate to the different forms of Δ? 3a) How do you relate the changes in the distances of the roots to the abscissa of the vertex when the increment is bigger than zero, is zero, and is smaller than zero? 4) How could you re-write and locate the roots when the value of $\Delta = b^2 - 4ac$ becomes negative?

Methods

We report results from a teaching experiment (Steffe and Thompson, 2000). The first author acted as the teacher-researcher in the teaching experiment, which consisted of three 75- to 120-minute sessions. Data sources included transcripts formed from the video data and written artifacts. The study's participant was one prospective secondary mathematics teacher, Esra, who was in the fourth year of her five-year undergraduate program. Following the teaching experiment methodology, analysis was both ongoing and retrospective (Steffe & Thompson, 2000). Ongoing analysis occurred during data collection and involved formulating hypotheses of student thinking between sessions and designing subsequent sessions to test those hypotheses. For the retrospective data analysis, we followed three-level approach to abductive process (Simon, 2019). For the first level, we read the transcripts line-by-line, focusing on sequences in which Esra's actions and utterances provided information about her thinking. Then, "*we use the results of the first level as the "data" for the second level*, making inferences for chunks of these new data" (Simon, 2019, p. 119) by answering the questions such as what understandings Esra seems to have and how we can characterize her thinking. The third level involved our use of explanatory constructs, such as learning as a reflection on the activity.

Results

In lieu of space, we describe Esra's evolvement of ideas on the second set of tasks. For the second set of tasks, 1 and 2, given the examples on Desmos, Esra stated that the values of 'a' and 'b' and so, $\frac{-b}{2a}$, remained invariant in the examples and the values of 'c' changed, decreasing the roots' distances to the abscissa of the vertex (from out to inwards). She also commented on the roots, stating, "…with respect to the symmetry axis, they [*the different roots*] have the same, umm, they [*the roots*] are symmetric, exactly". Also, she further commented on the roots' distances to the abscissa of the vertex as changing. For 3 and 3a in the second set of tasks, Esra drew the following (*See* Figure 1). She did not use the exact values of the roots, but she marked some of the roots on the real number line as (x_1,0), (x_2,0), ($x_{1'}$,0), ($x_{2'}$,0), ($x_{1''}$,0), ($x_{2''}$,0), ($x_{1,2'''}$,0).

Figure 1. Esra's showing the roots on real number line she drew

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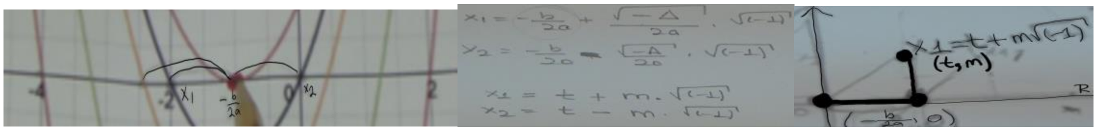

a *b* *c*

Figure 2. Esra's pointing to the real numbers as possible roots, re-writing the roots of the quadratic equations and plotting the roots of the quadratic equation on the plane

Furthermore, Esra re-wrote the roots for 4 in the second set of tasks (*See* Figure 2b) and stated: "Let's say there are infinitely many [*quadratic*] functions, and the abscissa-of-the-vertex of any quadratic function is t...and m is the distance from one of the roots to the abscissa-of-its-vertex…" Then, albeit with struggle, she positioned the roots on the plane(*See* Figure 2c) and commented on what (t,m) represented:

E: Umm, the root of the equation. The roots. They are the roots of the equation, quadratic equation. Umm, when they are not real, the delta is smaller than 0. Umm, when they are real, delta is greater than 0.

R: What do you call the algebraic expressions when delta is smaller than 0?

E: Umm complex numbers.

R: Okay. Where do you get those complex numbers?

E: From real numbers. All real numbers, on the real x-axis…Umm, I obtain them from the real roots of quadratic equations. If they are umm...okay, I obtain them from their real roots. Okay, I obtain from non-real ones as well […] Yes, they are complex numbers. The numbers obtained from the roots of all quadratic equations are complex numbers. Exactly. They give complex numbers.

Importantly, the data showed that Esra enacted the activity several times, such as in the first set of tasks, on the examples provided on Desmos and the colored-print copy (*See* Figure 2a), and when placing them as points on the real number. This way, her repeated mental runs through the activity of varying continuously the roots' locations on first the real number line and then on the plane allowed her images to become operative such that reflecting on the activity with the three cases of discriminant allowed her to anticipate that all the roots constituted the elements of a set of the roots of quadratic equations with real coefficients.

Conclusion and Discussion

We presented learning through activity design principles focusing on an advanced mathematical topic, the Cartesian form of complex numbers. Tasks designed with the principles of the LTA framework are distinguished from other task designs (e.g., Smith & Stein, 1998) in explaining the relationship to students' learning processes. Data indicated that engaging in the task sequence, Esra's enactment of the activity of varying continuously the locations of the roots of any quadratic equation allowed her to anticipate the logical necessity that any quadratic equation having both two real and two non-real number roots constitutes the elements of the set of complex numbers. Results suggest that the LTA design principles might be used to study advance mathematical concepts. We also argue that the learning goal shared in this paper might constitute a considerable change in students' conception of complex numbers. Previous research suggests that learners from different stages of schooling seem to struggle with the idea that any real number is a complex number (Nordlander &Nordlander, 2012). Thus, we argue that the aforementioned learning goal with the design of the tasks might allow to conceptualize real numbers as a subset of complex numbers.

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CHILDREN'S EXPOSURE TO MULTI-DIGIT MULTIPLICATION ALGORITHMS AND ITS EFFECT ON THEIR MULTIPLICATIVE REASONING

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Multi-digit multiplication is an important, yet understudied, topic in mathematics education. There are numerous algorithms that children may be exposed to, but it is commonly recommended that children transition from the area model to the box model to the partial products model before leaving the primary grades. The current study reports on pilot data exploring the effect of students' self-reported exposure to these algorithms on their multiplicative reasoning. Results reveal that participants in this sample were exposed to these algorithms in isolation of each other, and this led to negative effects of certain algorithms on their multiplicative reasoning.

Keywords: Elementary School Education; Number Concepts and Operations.

Students begin to learn multi-digit multiplication in the later primary grades (ages 9-11). There are numerous algorithms and approaches (Fuson, 2003; Hickendorff et al., 2019; Schulte, 2015)—many of which are largely unchanged from their development in India nearly 2000 years ago (Datta & Singh, 1938). Pedagogical innovations, such as the area model (using pictorial & concrete representations) and modifications to existing algorithms have emerged over the past century as means to facilitate meaningful understanding of the standard algorithm (Giles, 1975; Fuson, 2003). Despite the wide variety and history of algorithms and models for multi-digit multiplication, study of children's engagement with such algorithms is lacking (Amborse et al., 2003; Fuson, 2003; Izsàk, 2001). Rather, "there is little research into children's solution strategies use in multi-digit multiplication (Hickendorf et al., 2019, p. 553), as well as examination of how exposure to such algorithms affect children's multiplicative reasoning. Given the repeated calls for additional scholarship on multi-digit multiplication, the present paper reports on an exploratory study examining fifth-grade students' self-reported exposure to specific algorithms and this exposure's effect on their multiplicative reasoning.

Review of Literature

Multiplicative Reasoning via Scheme Theory

Multiplicative reasoning is a key mathematical concept for children's development of latter mathematics (Harel & Confrey, 1994; Norton et al., 2015). This study adopts Hackenberg's (2010) description of three multiplicative concepts to describe how multiplicative reasoning develops and may be characterized. Students operating at the first multiplicative concept (MC1) can anticipate one level of units while coordinating two levels of units in activity. Such students may solve 4×15 by repeatedly adding four 15s until they get to 60 (Steffe, 1994). The second multiplicative concept (MC2) involves anticipating two levels of units while coordinating three levels of units in activity (Hackenberg & Tillema, 2009). A student at MC2 may solve 8×15 by recognizing that 2×15 is 30 and 8×15 includes four sets of 2×15 (so it would be 4×30). The *third multiplicative concept* (MC3) involves coordinating three levels of units, but all levels of units are anticipated (Norton et al., 2015). Thus, a student at MC3 solving 8×15 is more likely to consider it by partial products (i.e., 8×10 + 8×5). This is because such children can consider multiple composite numbers simultaneously (Kosko, 2019).

Multi-Digit Multiplication

Children learn multi-digit multiplication in a variety of settings, using various algorithms. However, students’ algorithm use tends to follow what is emphasized by a particular curriculum or teacher (Hickendorff et al., 2018). While recall of basic multiplication facts is advocated as a prerequisite by some (Lin & Kubina, 2005), Hurst and Hurrell (2016) observed that despite “facility to recall multiplication facts was not an indicator that the students had a conceptual understanding of multiplication” (p. 37). Indeed, students need to develop an understanding of partial products to move beyond basic multiplicative reasoning (Ambrose et al., 2003; Hickendorff et al., 2019; Lampert, 1986).

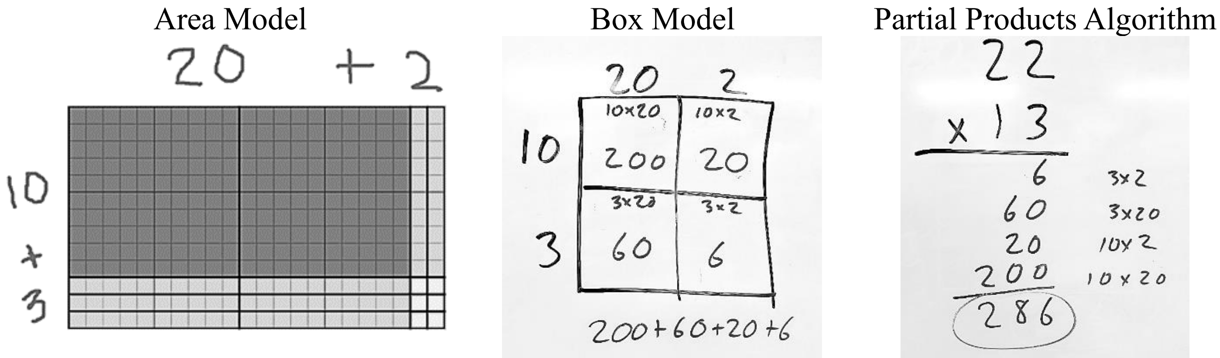

Figure 1: Three Algorithms for Multi-Digit Multiplication

Various scholars have advocated for the use of area and array algorithms, paired with the box and/or partial products algorithms for developing conceptual understanding of partial products (Fuson, 2003; Izsàk, 2001; Young-Loveridge & Mills, 2009). Working with a small group of fifth-grade students, Izsàk (2001) found that beginning with an area algorithm that included arrays in the visualization and then transitioning to a box model before using a partial products algorithm supported students’ understanding of partial products. Others have proposed the same pedagogical approach (Fuson, 2003; Kristen, 2021; Young-Loveridge & Mills, 2009). Although such scholarship is useful in characterizing a recommended pedagogical approach, there is little scholarship evaluating the effect of children’s exposure to these algorithms. The present paper reports on a pilot project assessing this phenomenon. Specifically, the purpose of this paper is to evaluate students’ self-reported exposure to the area model, box model, and partial products algorithm on their multiplicative reasoning.

Methods

Sample and Measures

Participants included 55 fifth-grade students (age 10-11 years) from two teachers’ classes. The sample included 47.3% self-identified females and 52.7% self-identified males. Participants completed the Multiplicative Reasoning Assessment (MRA) and a brief survey rating their familiarity and use of the three multi-digit multiplication algorithms of focus in the present study (area model, box model, partial products algorithm). Results indicated that all students had exposure to the box model. Interestingly, only 14.5% of participants had seen and used all three algorithms. Given this low percentage, exposure to these algorithms were examined as separate variables.

The MRA is a 21 item measure designed to assess children's multiplicative concepts at-scale. Validity evidence has been collected across multiple studies including, but not limited to, psychometric data for test content and internal structure, cognitive interviews on the items, and correlation of scores to other metrics for concurrent validity (Benjamin & Kosko, 2019; Kosko & Singh, 2018; Kosko, 2019; Kosko, 2020). MRA raw scores were transformed via a Rasch model to provide a continuous scale. Participants' scores in the current study were lower than average (*M*=-1.08, *SD*=2.45) indicating 50.9% operated with pre-multiplicative (count-by-1s) schemes, 16.4% at MC1, 20.0% at MC2, and 12.7% at MC3. A sub-sample of 16 students participated in clinical interviews to examine their strategy use for multi-digit multiplication, with findings planned to be reported elsewhere after analysis.

Table 1: Student-Reported Exposure to Multi-Digit Multiplication Algorithms

	Area	Box	Partial Products
Never Seen/Used	48.0%	0.0%	18.0%
Seen/Not Used	14.0%	3.8%	34.0%
Seen & Used	36.0%	96.2%	48.0%

Analysis & Results

Multiple regression was used to examine the effect of students' self-reported exposure to multi-digit multiplication algorithms on their multiplicative reasoning. The regression model is illustrated below with students' Rasch-based MRA scores as the outcome. Independent variables include a series of dummy-coded variables to model the statistical effect of having seen and used, or seen but not used, specific algorithms. Self-identification as male was included as a covariate. The illustrated model meets basic assumptions for regression: a predicted probability plot confirms normality of residuals and distribution of residuals meets criteria for homoscedasticity. Thus, linearity is assumed. Variance inflation factor (VIF) statistics were calculated to test for multicollinearity in the model. VIF statistics ranged from 1.123 to 2.091 across all independent variables. Thus, risk of collinearity was minimal and these variables were retained for analysis.

The final model is presented below. The intercept of the model, β_0, can be interpreted as the average MRA score for a female student who reported never having seen nor used the area model partial products algorithm, and who had seen but not used the box model. All other coefficients are interpretable as the effect of being male or having a certain form of exposure to these algorithms.

$$MRA\ Scores_i = \beta_0 + \beta_1 \cdot (dMale)_i + \beta_2 \cdot (Area_{Seen\ \&\ Used})_i + \beta_3 \cdot (Area_{Seen\ NOT\ Used})_i + \beta_4 \cdot (Box_{Seen\ \&\ Used})_i + \beta_5 \cdot (Partial\ Products_{Seen\ \&\ Used})_i + \beta_6 \cdot (Partial\ Prodcuts_{Seen\ NOT\ Used})_i + r$$

Results from the regression model were statistically significant (R^2=.293, F[5, 41]= 2.768, p=.024), and are presented in Table 2. Three independent variables were found to be statistically significant at the .05 level and one at the .10 level. Specifically, seeing, but not using, the area model (*B*=-2.271, *p*=.019) and using the box model (*B*=-3.936, *p*=.025) were both found to have a statistically significant and negative effect on children's multiplicative reasoning. By contrast, use of the partial products model had a positive effect on multiplicative reasoning (*B*=1.868, *p*=.052). To put these results in context, these Beta coefficients are unstandardized and represent changes in the logit-based scores on the MRA. A change of 1.00 logits is roughly equivalent to the difference between two adjacent multiplicative concepts. Thus, results here suggest that use of the box model, without exposure to either the area model or partial products algorithm, has a very large and negative effect on children's multiplicative reasoning. Similarly, exposure to the area model, but without actual use, also has a large and negative effect. Use of the partial products algorithm was the only algorithm found to have a positive effect associated with children's multiplicative reasoning. One surprising finding was that self-identified male participants demonstrated higher multiplicative reasoning in this sample than their peers (*B*=1.592, *p*=.017). This finding has not been observed in other administrations of the MRA and no data was available to explain the finding in the present study.

Table 2: Results from Multiple Regression

		B	Std. Error	*p*
Intercept		1.333	1.627	.417
d_Male		1.592	.638	.017
Area Model	Seen & Used	-.848	.683	.222
	Seen *not* Used	-2.271	.928	.019
Box Model	Seen & Used	-3.936	1.689	.025
Partial Products Algorithm	Seen & Used	1.868	.932	.052
	Seen *not* Used	1.338	.949	.166

Discussion

"Multi-digit multiplication is an important, though underrepresented, area of research" (Izsàk, 2001, p. 187). One key aspect of facilitating children's multiplicative reasoning in multi-digit multiplication is supporting their learning of partial products (Fuson, 2003; Hickendorff et al., 2019). The most commonly advocated approach is to introduce multi-digit multiplication with array-based area models before transitioning to a box model and then a partial products algorithm (Fuson, 2003; Izsàk, 2001; Young-Loveridge & Mills, 2009). To be clear, the present paper does not evaluate this approach. Rather, results presented here evaluate the isolated use of particular algorithms that can be used to facilitate the recommended pedagogy. Only eight of 55 participants surveyed reported having seen and used all three algorithms (area model, box model, partial products algorithm), with 32.0% of participants having only used the box model of the three. Thus, isolated exposure to these algorithms was the norm and there was no evidence of a sequenced progression from concrete to abstract. Results should be interpreted with this feature of the sample in mind. Thus, implications of these results suggest that isolated use of the box model, as well as demonstration of the area model without actual use, have a negative effect on children's multiplicative reasoning. By contrast, the partial products algorithm was found to be beneficial in the current sample. These results suggest further research is needed to better understand not only the appropriate pedagogy for teaching multi-digit multiplication, but the pitfalls of teaching particular algorithms without connections to other mathematical representations. Additionally, the present paper reports on an exploratory analysis of a pilot project and is limited due to sample size both regarding number of students and the classrooms they are sampled.

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EXPLORING STUDENTS' FLEXIBILITY IN SOLVING RATIO PROBLEMS FROM THE PERSPECTIVE OF SELF-REGULATED LEARNING

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Keywords: Flexibility, Strategies, Self-regulated learning, Ratio problem solving

Research Background

Flexibility has garnered significant attention in educational practices as a crucial element of students' higher-order thinking skills and creativity (Star et al., 2022). To explore and explain flexibility, researchers have examined the correlation between flexibility and various factors from different perspectives. In particular, evidence has shown that flexibility is related to a variety of factors, including task factors, personal factors, and environmental factors (Hong et al., 2023; Verschaffel, 2024). This paper aims to explore the relationship between learners' self-regulated learning and flexibility from a psychological perspective to better understand flexibility.

Research Questions and Procedures

Research Questions: What is the relationship between self-regulated learning and flexibility in solving ratio tasks? What are the self-regulated learning characteristics of problem solvers with different levels of flexibility?

Research Procedures: (1) Methodology: A mixed-method research design was employed. Learners' flexibility level was measured across three dimensions: multiple strategies generation, appropriate strategies identification and appropriate strategies switching. Learners' self-regulated learning characteristics were measured using the Motivated Strategies for Learning Questionnaire (Wang et al., 2023). (2) Convenience sampling was used, in a middle school in eastern China, with students in grades 7-9. (3) Math content: ratio problem solving.

Data Analysis and Conclusions

Data analysis examined the self-regulated learning characteristics of students across different flexibility levels. Structural equation modeling was used to explore the relationship between flexibility and self-regulated learning. The research results indicate that the relationship between flexibility and self-regulated learning is complex, intriguing, and worthy of continued investigation.

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REVISITING REORGANIZATION LEARNING THEORIES TO INFORM EARLY CHILDHOOD MATHEMATICS EDUCATION

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In this presentation, we examine the theoretical underpinnings of how young children reorganize their whole number understandings to construct fraction understandings (Steffe, 2001). By examining these reorganizations, we aim to inform instructional approaches in early childhood education, as well as build early forms of collaborative efforts between scholars with expertise in upper elementary mathematics education and early childhood mathematics education. Discussions focus on the conceptual resources that children carry into their mathematics activity, and, how, by taking up an asset-based frame, early childhood educators are better positioned to design equitable instruction for their children.

Keywords: Early Childhood Education, Learning Trajectories and Progressions, Learning Theory.

Often, early childhood educators must differentiate their instruction to support wide degrees of children's early number development. These efforts are often guided by curricula materials, locally developed resources, or national research-to-practice initiatives (e.g., U.S. Mathematics Recovery Council). In response to these efforts early elementary aged children may experience different instructional approaches in mathematics, some focusing on narrow ranges of mathematical content (Ginsburg et al., 2008). Many posit that learning differences are not only a reflection of children's cognitive differences, but the societal constraints we have in our school system to differentiate our instruction effectively for all young children (Baroody & Purpura, 2017). In this theoretical contribution, we push back on these societal constraints to discuss the significance of examining the underpinnings of children's whole number knowledge in terms of their construction of fraction understandings. Our objective is to emphasize and inform this developmental focus to be at the center of the early childhood classroom planning and instructional practices.

Theoretical Foci

To examine students' conceptual resources and the developmental process, we situate our work under Steffe's (2001) reorganization hypothesis which draws from a recognition template described as scheme theory; explaining how the conceptual resources within a child's mathematical reality are utilized to develop fraction understandings. Steffe and Olive (2010) describe scheme theory as drawing from a four-part model: an experiential situation, an overarching goal, an anticipated action(s), and an expected result(s). When the actual result(s) and expected results(s) do not align, a child will accommodate their scheme to change one of the four components within their model. An accommodation of a scheme explains learning, as an individual reorganizes their scheme to make sense of this misalignment. Scheme theory explains that as a student reorganizes their scheme, they also revisit earlier developed components of their schemes with the implicit goal to extend these components in new experiential situations. For instance, when introduced to equal sharing tasks (i.e., How much of this chocolate bar would each person receive if shared by three people?), children have the potential to use their activity from their whole number schemes to develop fraction schemes (Steffe, 2001). To better understand how scheme theory can frame children's reorganization of their conceptual resources, we examine the literature focusing on children's units construction and coordination.

Theoretical Framework: Units Construction and Coordination

We frame our discussion with constructs embedded in the units construction and coordination literature with regard to whole number schemes, fraction schemes, and relationships between them both. Hackenberg and Sevinc (2024) define units as, discrete 1s or lengths. Essentially, young children first construct units by isolating sensory-motor activity (i.e., motor activity, verbal utterances), providing them discrete units to carry into their earliest whole number activity. Children then construct length units through a similar unitizing process for fraction development, where experiential units are developed with length models (i.e., string, lines in sand). Children use partitioning to construct units and then iterate or copy units to create new units (Hackenberg & Sevinc). Children's organization of units provides constant relationships which can be coordinated for operation development, better positioning themselves with assimilatory structures for whole number to develop fraction schemes. Steffe (1992) also describes units coordination as the distribution of elements from one set of units across the elements of a second set of units. This is explained in more detail in the next section. As children's number understanding becomes more sophisticated, children progressively develop abstract units to facilitate coordination forming assimilatory structures described as a "units of units of units" (Steffe, 1992; Ulrich, 2016). Elementary-age children reorganize their assimilatory structures to develop fraction schemes and revisit their earliest activity informing educators of children's underlying activity developed in their the early childhood years.

Whole Number Scheme Development

Framing counting and number sequence development in the units construction and coordination literature, Steffe and Cobb (1988), describe counting as a scheme, whereby a student will approach a situation (e.g., perceiving items to count), develop an overarching goal (e.g., determine "how many altogether"), decide on an action (e.g., pointing at items in correspondence to number words stated in sequence), and decide upon an expected result (e.g., the final number in the counting sequence). This describes the process through which young children may construct, engage with, and reflect upon pre-numerical units (e.g., perceptual units, figurative units, motor units, and verbal units) before developing abstract units, marking an understanding of number as a quantity. Olive (2000) explains that once children strip away the contextual features of an operation, they interiorize number and develop abstract units. Once children interiorize their whole number units, they re-interiorize their number schemes by producing operations (e.g., partitioning, iterating, unitizing, and disembedding), with which to inform the results of their actions associated with their number schemes. For instance, children counting items can take the counting action and apply it to the result by understanding these items are "something to be counted" (Olive, 2000, p 5). Through a process of interiorization and re-interiorization, children develop assimilatory structures whereby "units of units of units" are understood. These three-level structures of number explain children's coordination of three units, such as, six twos relative to 12 (Olive, 2000, p. 7). Here children may be coordinating relationships between six, twos, and 12. By coordinating three levels of units, children anticipate the relations between three arithmetic units prior to constructing a solution of a mathematical task (Hackenberg & Sevinc, 2024). Additionally, when developing assimilatory structures, children rely primarily on their partitioning or iterating development. These operations can inform educators of meaningful conceptual resources to elicit for future mathematics instruction (see Figure 1).

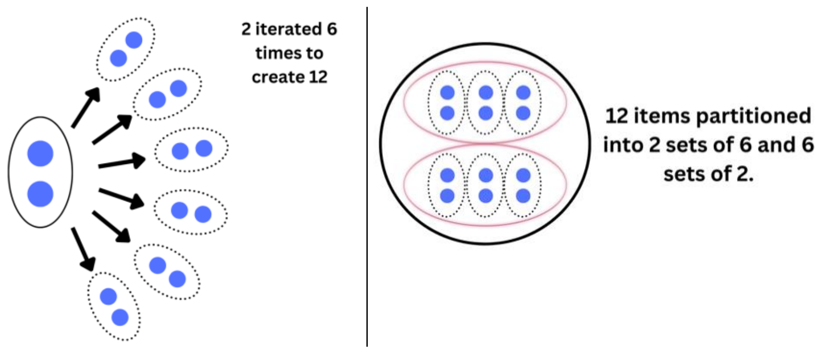

Figure 1: On the left, how children may iterate discrete units. On the right, how children may partition sets of discrete units.

While some children may have a "units of units of units" structures, other children may not yet have developed three levels of units, evidencing a reliance on pre-numerical units through a counting all strategy, a reliance on internalized number sequences through a counting on strategy, or a reliance on skip counting by keeping track of the third unit in activity (Olive, 2000; Ulrich, 2016). Examining these wide degree of variances found in early childhood classrooms allow educators insight into children's developing actions, which can be used for future instruction.

Reorganization for Fraction Scheme Development

When children develop three levels of units, they are much more capable of accommodating their whole number schemes to develop fraction schemes. Due to the inverse relations between whole number units and fractional units, children revisit some of their earliest actions when developing fractions (Olive, 2000; Steffe, 2001). In short this means children assimilating a three level units structure of *units of units of units* reorganizes their assimilatory structures and associated operations to develop *units in units* understandings. For instance, Olive (2000) explains that a student coordinating three levels of units with whole numbers is only able to carry in an assimilatory structure to solve a fraction of fraction activity (e.g., what is one-half of one-fourth? – see Figure 2). This is often because children developing fractional units coordination require inverse understandings of their whole number units coordination. For most children, this can be a rich and meaningful opportunity to examine the inverse relationships between whole number and fractional units. Children not yet coordinating three levels of units can oftentimes struggle understanding fractions. Yet, we posit children not yet coordinating three levels of units are still developing meaningful activity, which can inform their fraction development in a variety of ways. This activity can also inform the design of early childhood mathematics learning trajectories.

Contributions to Early Childhood Mathematics Learning Trajectories

Developmental learning trajectories, or representations of "learning that shed light on how children might engage with tasks, reflect on tasks, and develop knowledge through work on tasks" (Weber et al., 2015, p. 254) are effective tools when establishing early childhood curricula. We posit by examining children's reorganization of their assimilatory structures, early childhood educators gain insight into children's earliest operations. For example, when a student solves the task, how many fives does it take to make 30? They may first create a line of 30 counters before grouping them into random-sized groups before determining they need to create a group of five each time and then count the groups. This suggests the student relies on an internalized number sequence because they can produce 30 items, but their partitioning of these items is emergent, meaning they can reflect on their partitioning and the total number of items, but cannot do this along with their unitizing, evidenced in their unequal group development before reflecting on the size of the groups.

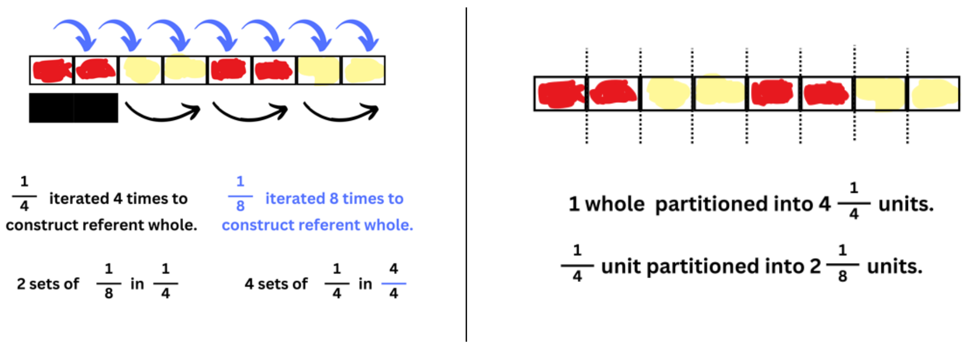

Figure 2: On the left, how children may iterate fractional units. On the right, how children may partition referent wholes and fractional units.

When reorganizing this to solve fraction tasks, educators would expect to see this emergent partitioning when the student solves the task, such as: "How much of this fry would each person receive if shared by three people?". We would also see the student resort to their pre-numerical units when applying their partitioning to the fry and the other materials and activity (e.g., auxiliary items, drawings, cutting, folding). As such, this can be used for future activity with fractions. By examining this activity with many students, educators are also given "typical" activity associated with children's earliest activities and operation development. For instance, if educators teaching eight and nine-year-olds see children partitioning or iterating but struggling with their unitizing, they may be able to discuss with their early childhood educator colleagues varying ways they can leverage more sophisticated grouping tasks, allowing children opportunities to use their partitioning or iterating to construct, engage, and reflect on their unitizing activity. By observing this reorganization, early childhood educators have the possibility to use this activity to design curricula materials and resources grounded in child-centered learning trajectories.

Discussion and Conclusion

Young children develop meaningful activity through their units construction in the early childhood classroom. By taking up scheme theory, educators and researchers are better positioned to revisit this activity by observing the children's conceptual resources that they use when solving whole number and fraction tasks. Observations of children's scheme reorganization, has the potential to design intentional discussions among educators and researchers in a variety of different fields (e.g., early childhood, upper elementary). Through these partnerships, educators can design trajectories that are student-centered and purposeful. By differentiating instruction in this way, early childhood educators can utilize an asset-based frame to differentiate and widen their classroom mathematics for all children.

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STRUCTURAL MODEL OF LEARNING RETENTION AND MOTIVATION OF UNDERGRADUATES IN FLIPPED AND NONFLIPPED COLLEGE MATHEMATICS

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Keywords: flipped instruction, direct instruction, college, precalculus

Background and Objectives

Previous studies have documented the benefits of learning in flipped instruction over the traditional lecture method (Akçayır & Akçayır, 2018; Nielsen, Bean, & Larsen, 2018). Some researchers attributed the successes to other factors rather than the pedagogical approach itself (Backlund & Hugo, 2018). For example, Kay and McDonald (2016) reasoned that the students learn more in flipped instruction because the model incorporates active and collaborative learning techniques. While Backlund and Hugo (2018) described the teaching method as multi-dimensional or a combination of multiple strategies. Thus, this study contrasted learning opportunities in flipped instruction (FI) with another non-passive method, identified as customized-direct instruction (CDI). This poster investigated whether students who received FI perform better than students taught using CDI on all assessments given in two introductory college mathematics courses after controlling for prior achievements and peculiarities.

Flipped instruction allowed learners to interact with all or part of the instructional activities independently outside the classroom. Then the grouped space is used to for cooperate and higher-order learning (Akçayır & Akçayır, 2018). While CDI encourages active participation (Voskoglou, 2019) by utilizing explicit instruction and group work for content presentation and learning. We focused on precalculus and calculus because of their gate-keeping roles for many college students (Rasmussen et al. 2019).

Methods and Data Analysis

We created a structural model (Figure 1) explaining how teaching methods, motivation beliefs, and past experiences contributed to learning retention and achievement. Model construction followed recommended standards by Schumacker and Lomax (2016). The participants were ninety undergraduates; 33 from FI and 57 who received CDI in precalculus at a four-year college. Students covered same topics, took same midterms and final exams, similar but different homework and quizzes, and had different instructors.

Two-steps structural equation modeling (SEM, Schumacker & Lomax, 2016) and MANCOVA were used to analyze and interpret collected data, which included students' precollege assessment reports, course grades, and self-reported end of semester course evaluation.

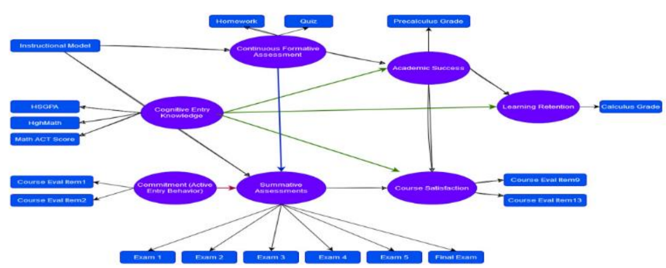

Figure 1: The Hypothetical Model

Results and Implications

Findings showed moderate to strong relationships between most of the variables considered. While a modified version of the theorized structural model indicated that no one instructional method is superior to the other for learning retention, the MANCOVA results show that the flipped model supports short-term learning achievement, while the direct instruction facilitates learning retention (see Tables 1 and 2).

Any active learning method and/or combinations of teaching approaches have the potential to increase learning retention and the achievement of desired goals.

Table 1: Direct, Indirect, and Total Effects of Model's Predictors on Retention.

Variable	Direct Effect	Indirect Effect	Total Effect
Method	0.145 ns	-0.339***	-0.194 *
Cognitive Entry Knowledge	-	0.031 ns	0.031 ns
Affective Entry Behavior	-	-0.572***	-0.572***
Formative Assessment	- 0.450*	0.305***	-0.146 ns
Summative Assessment	1.015***	-0.147 ns	0.867***
Academic Success	-0.157*	-	-0.157*

Table 2: Pairwise Comparison for Teaching Method

Dependent Variable	Adjusted Mean Difference by Teaching Method	
	Direct Instruction	Flipped Instruction
Formative Assessment		**0.565***
Summative Assessment	**0.667***	
Academic Success	0.065 ns	
Course Satisfaction	0.425 ns	
Learning Retention	0.161 ns	

Note. Bonferroni adjustment used for multiple comparison. * $p < 0.05$, ns = not significant.

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QUALITATIVE REVIEW OF INTERVENTIONS USED TO MITIGATE MATH ANXIETY

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Math anxiety is pervasive in our society, and it is causing problems for students who want to enter a science, technology, engineering and mathematics (STEM) pathway. Recent research has suggested that math anxiety, and not math ability, is a better predictor of performance and perseverance in STEM. Interventions to mitigate this anxiety must get to the root of the problem and provide students tools to help ease these feelings when they come to disrupt performance. Our approach uses techniques from mindfulness and self-compassion, which have been linked to reducing anxiety. We have developed modules that can be used in a math (or science) courses to help students understand the anxiety they face. In this paper we examine one of the interventions used, the pre-exam writing, and how it showed a 75 percent increase in statements of confidence and a doubling of statements of positive feelings over the semester.

Keywords: math anxiety, self-compassion, mindfulness

Tobias (1976) coined the phrase “math anxiety” to describe the feelings of panic, anxiety, paralysis, and mental confusion that occur when people face computational challenges. Math anxiety is a major barrier to broadening participation in STEM. First year college mathematics (math) courses are a gateway to STEM majors (Schleicher, 2018) and nearly one third of STEM-intending students in the U. S. enroll in remedial math courses at the college level (Chen & Simone, 2016). Math is a major source of stress and anxiety for many college students (Ramirez et al., 2018). Many students have the cognitive ability to carry out mathematical tasks successfully, but their fear and anxiety regarding math gets in the way (Tobias, 1976; Beilock & Maloney, 2015: Brunyé et al., 2013; Henslee & Klein, 2017; Samuel & Warner, 2019). Neuroscience research has established that the worries and anxiety that arise in math-anxious students usurp cognitive resources needed for thinking, reasoning, working memory, and maintaining focused attention (Beilock & Maloney, 2015; Brunyé et al., 2013). This causes students to perform below their actual ability and affects their motivation and interest in math (Brunyé et al., 2013).

Researchers have been examining ways to address math anxiety through non-academic interventions (Beilock & Maloney, 2015; Brunyé et al., 2013). Factors such as grit, mindfulness, self-compassion, and self-efficacy have been studied and are associated with anxiety, resilience, and academic achievement (Jarukasemthawee et al., 2021; Neff et al., 2005; Neff & Germer, 2013; Tubbs et al., 2019). In addition, research findings have indicated an inverse relationship between grit and math anxiety in college students (Holtby, 2018; Darrah et al., 2023) suggesting that increasing grit may foster resilience and perseverance by reducing math anxiety.

Our research team, including mathematics professors and a counselor who specializes in mindfulness and self-compassion, has developed and implemented modules aimed at reducing math anxiety in college students enrolled in entry level mathematics courses. The seven student modules focused on three main topics (math anxiety, mindfulness, and self-compassion) and were titled: Math Anxiety Introduction, Introduction to Mindfulness, Mindfulness as a

Superpower, Math Anxiety Writing, Introduction to Self-Compassion, Self-Compassion (Part 2), and Reflection. This research is part of a larger study (Leppma & Darrah 2022; Darrah et al., 2023) and in this paper, we focus on the Math Anxiety Writing assignments and the last module Reflection. We were interested in how student engagement with these modules influenced how students felt about their ability to learn mathematics. Our research questions for this paper were as follows:

1. How did student perceptions of themselves as learners of mathematics change over the course of the semester?
2. What was the students' reaction to the Math Anxiety Modules?

Background

Math anxiety not only interferes with academic performance, but may contribute to STEM attrition (Ahmed et al., 2017; Ramirez et al., 2018). Students who earn lower grades in STEM courses in their first year of college are more likely to switch to non-STEM majors or drop out of college altogether (Chen & Soldner, 2013). Recent research (Daker et al., 2021) suggests that math anxiety is a better predictor of students' participation and perseverance in STEM majors, as well as performance in math classes, than actual math ability. Results indicated that higher math anxiety was associated with avoidance of STEM courses and lower grades in STEM courses when tracking students over a four-year period.

Research demonstrates that mindfulness-related practices are effective in cultivating internal resources as protective factors against the distress associated with math anxiety, which may help attract more students to pursue and persist in STEM-related majors (Ahmed et al., 2017). Mindfulness – the intentional and nonjudgmental awareness and observation of the present moment, including thoughts, feelings, and physical sensations one is experiencing – is positively associated with improved emotional regulation (Meyer, et al., 2019) and inversely related to depression, anxiety, and severity of anxiety symptoms (Tubbs et al., 2019).

A related construct, self-compassion, is the ability to show kindness and caring toward oneself in the face of discomfort, failure, or suffering. Self-compassion encompasses three components: self-kindness, common humanity, and mindfulness. Rather than being critical and judgmental toward oneself when making mistakes, self-compassionate people recognize that personal failures are part of the human experience and recognize their negative internal states without judgment or overidentification (Neff, 2003).

Students who develop internal resources, such as perseverance, grit, hope, emotion regulation, and motivation can overcome math anxiety and persist in math classes (Beilock & Maloney, 2015; Duckworth, et al., 2007; Snyder et al., 2002). Mindfulness and related self-compassion practices have been shown to alleviate anxiety, including math anxiety, and improve resilience, grit, hope, well-being, and emotional and cognitive functioning (Beilock & Maloney, 2015; Doorley et al., 2022; Leppma & Darrah, 2022; Weed et al., 2021). Mindfulness and self-compassion practices help to develop valued skills in academic achievement, such as concentration, memory, focus, and test performance. They also cultivate skills associated with occupational success and wellbeing, such as resilience, emotion regulation, interpersonal skills, grit, and hope for goal attainment (Chiesi et al., 2022; Leland, 2015; Shapiro et al., 2015; Strohmaier et al., 2022). Moreover, mindfulness diminishes the experience of stress and fear

(Tubbs et al., 2019), and self-compassion increases intrinsic motivation to learn and improve (Akin, 2008; Manavipour & Saeedian, 2016; Neff et al., 2005). Thus, higher levels of mindfulness and self-compassion are associated with higher levels of grit and hope, and lower levels of math anxiety.

Methods

The research took place at a large public research university upon institutional review board approval. The participants in the project were a convenience sample of 59 students who were enrolled in an entry-level college algebra course and a college algebra co-requisite support course in the spring semester of 2022. Student placement into college algebra courses (with or without co-requisite support) is based on their SAT/ACT Math scores or how they performed on a Math Placement Exam (ALEKS). The co-requisite course focused on prerequisite skills including operations on real numbers and simplifying algebraic expressions, and metacognitive skills such as mindfulness, coping with math and test anxiety, and self-compassion. The co-requisite course was mainly taught by graduate assistants under the supervision of the course coordinator. The coordinator provided training on how to implement the modules mentioned in the Introduction.

For this paper, we focused on the Math Anxiety/Mindfulness pre-exam writings and the last student module titled Reflection. The purpose of the Math Anxiety/Mindfulness pre-exam writings was to help students acknowledge their thoughts as they prepared to take an exam. These writings were designed to help students practice mindful writing to bring awareness and clarity to their thoughts prior to taking an exam. These writing assignments took place before a practice exam given in the co-requisite support course but coincided with three exams taken in their 3-credit algebra course. Data collected were student responses on these three writing assignments. The prompt students were given was "Please take the next 7 minutes to write as openly as possible about your thoughts and feelings regarding the math problems you are about to perform." These writings were completed in February, March and April (before algebra exams 2, 3, and 4).

The researchers used the Content Analysis method to look for common ideas and themes in the student response data. Analyses began with the preliminary reading of the responses, followed by the development of preliminary codes based on ideas and themes that emerged within the responses to the writing prompt. The development of codes was also guided by the overarching research ideas of the student' perceptions of their ability in math and feelings of anxiety. The codes were grouped into three major categories: (1) confidence or lack of confidence in mathematical knowledge or ability to succeed on the pre-assessment, (2) specific feelings of anxiety or stress, and (3) positive feelings. The writing was not mandatory, so 97 students completed at least one of the writings, with 45 students completing all three writings. These 45 students were used as the analysis group, since we could compare across all three writings for changes.

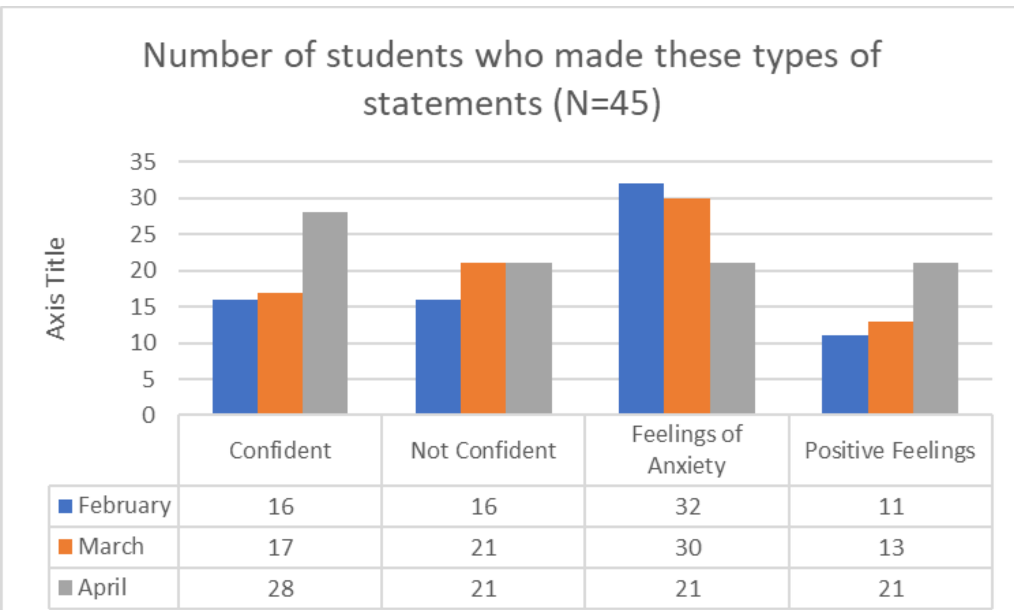
Additionally, to determine students' perceptions of the modules themselves, we presented the student responses from the last module, Reflection. This module asked students to reflect on the experience of learning about these ideas and practices (mindfulness and self-compassion) throughout the semester. Forty-seven students completed the final Reflection module at the end of the semester. Questions included "Will you continue to utilize some of the practices discussed in the modules in the future? If not, why not?", "What is your reaction to the series of modules?" The researchers used descriptive statistics to determine the percentage of students who found the modules useful and considered the open-ended responses for information about student perceptions and to extract ways to improve the modules or make them more useful to the students.

Discussion

We summarize the findings based on our two research questions.

Research Question 1: How did student perceptions of themselves as learners of mathematics change over the course of the semester?

For the 45 students who completed all three Math Anxiety/Mindfulness pre-exam writings we looked for change from February to April. Figure 1 below shows a summary of the analysis of the coding. We found that the number of students who made statements of confidence in their ability had a 75 percent increase. The number of students who made statements about positive feelings more than doubled, showing a 110 percent increase over the semester. On the flipside, the number of students expressing specific feelings of anxiety had a 34 percent decrease. For these 45 students, by April, they were making as many positive statements as negative statements about their feelings and more of them had statements of confidence than had statements of lack of confidence.



	Confident	Not Confident	Feelings of Anxiety	Positive Feelings
February	16	16	32	11
March	17	21	30	13
April	28	21	21	21

Figure 1: Summary of Coding of Math Anxiety/Mindfulness Pre-Exam Writing

Table 1 below is a sample of a few student responses that show the changes in the students' perceptions of themselves as learners. From the matched responses in the table, it can be seen that students took these few minutes before the exam to really think through what they were feeling. They talked about what was going on in their lives, sometimes mentioning other situations not related to math that were troubling them. This activity gave them a mechanism to let go of these thoughts and then to focus on the task at hand: taking the math test. Note the "VL" in the comments in the table stands for Video Lectures.

Table 1: Example Student Responses to Math Anxiety/Mindfulness Pre-Exam Writing

PROMPT: "Please take the next 7 minutes to write as openly as possible about your thoughts and feelings regarding the math problems you are about to perform."		
February	March	April
I feel like I am struggling because I don't really know what is going on. The word problems confuse me. And the VL are not anything like what the problems are, they do not help me when I do homework. They are completely different. Most of it does make sense to me at all and I am really not good at math or understand it.	**I am struggling so bad in math**. I do not like it because it is hard. I don't understand anything we are doing I literally do not comprehend it. The VL do not help me at all and I feel like it is not the same as the stuff we learn in class.	**I feel confident** because when I did the VL I actually understood what I was doing. I thought I understood though and then in math 126 I was so confused because he taught it a different way. I am not nervous because I felt comfortable and now I am not really sure.
I do not feel as prepared as I would like to for this upcoming assessment. I don't feel very confident in my knowledge on the current problems and material we are working on in Math 126. Though I am beginning to pick up how to do partial aspects of current material, I don't think I know enough to perform as well as I could on this diagnostic.	**I definitely feel more confident** going into this next diagnostic test than the previous ones. Before others I felt certain I would not do well at all and was very worried about how it would affect my grade. I am hopeful that this one will have a better outcome and relieve some of that stress regarding my grade.	**I feel prepared and good about the math diagnostic** that I am about to complete. Other times I have been quite anxious for the diagnostics because I didn't feel that I would score well on them and was worried about the effect that it would have on my grade in the class.
I am kind of confident but also nervous at the same time to do the math problems because there are parts that I am super familiar	**I am a bit nervous but also a bit confident** as I kinda know how to do the problems but I also am just am unsure of them. I think	**I feel okay about it** as I am going to be able to know the majority of what is on this.
I honestly have no idea what is going on, the second you get one second behind with a specific topic regarding the material we have went over thus far, you are just SOL. You just get behind so fast, if you struggle with one little thing it is honestly	**I barely understand what's going** on with anything we are going over in 126 right now, and honestly 106 doesn't help whatsoever. 106 is repetitive and we get lil to no help in either class. Feels	**I feel better about this module than I have any other**, it still scary because were getting close to the end of the semester. I feel like I should have felt better about this math a lot sooner.

impossible to continue with anything else. …	very pointless

Research Question 2: What was the students' reaction to the Math Anxiety Modules?

Forty-seven students completed the final reflection module at the end of the semester. When students were asked if they will continue to utilize some of the practices discussed in the modules, 81% of students said "yes". When asked, "What was your reaction to the modules?" students elaborated on their thoughts and feelings regarding the modules. Some example quotes included:

- "I felt like I learned a lot from these modules this year whether it was my least favorite or my absolute favorite. I felt like learning all these unique ideas and lessons, I can use them to make myself a better student, etc."
- "These modules made me improve my math scores throughout the semester. I kept improving as a student, and I will continue to do these practices."
- "I found the information in the modules more useful than expected. Learning not only about anxiety, mindfulness, and self-compassion, but how to apply them to real life made a huge difference. While one would not think the modules would not have a huge impact when coming to learning math, it created a difference mentally, and how one begins to think."
- "I loved them, I actually tried to apply what they were teaching me into my life."

Student responses from those who said that they would not continue to utilize these practices included:

"Probably not as I'm confident in my ability to work out math problems and I'm confident in my abilities already and don't need self-help."

"They all felt the same as they seem to just be about how I feel about math even though I enjoy math."

"Self-compassion felt silly, but I enjoyed the ted talk like videos."

Student feedback will be used to redesign modules or better explain their purpose to students. This feedback will also help inform professional development for instructors who will be implementing the modules.

Conclusion

Math anxiety is a problem for many students in our college courses (Tobias, 1976; Beilock & Maloney, 2015: Brunyé et al., 2013; Henslee & Klein, 2017; Samuel & Warner, 2019) and is causing a problem in our STEM pipeline (Ahmed et al., 2017; Ramirez et al., 2018). While the authors understand that these modules are not a silver bullet that will eradicate math anxiety in all students, we were pleasantly surprised by the number of students who took them seriously and who said they benefited from them. Over the years, we have not seen many resources that address the idea of anxiety in the math classroom and feel we have developed something to offer instructors who are faced with the student who suffers from math or test anxiety. We have also found through this research and reviewing the comments from the writing modules that the overwhelming majority of students in the co-requisite course started out with negative feelings, even though it may not have shown on their faces. We also note that these feelings, if addressed, can change throughout the course of the semester. The introduction of mindfulness techniques and self-compassion can make a difference in their perceptions of themselves as a learner.

We are continuing to redesign and improve the modules based on student feedback. Another aspect of the work is to create and provide professional development for faculty who would like to use the modules as part of their course. These modules can help instructors learn more about math anxiety and equip them with tools they can use to help their math anxious students persist in their studies. Lastly, we will continue the process of testing all the modules with students to determine their effectiveness at reducing math anxiety and increasing student persistence and performance. Future research will explore mechanisms, interventions, and effectiveness in a broader population.

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AN EMERGING THEORETICAL FRAMEWORK FOR USING COMICS STORYTELLING IN K–8 MATHEMATICS EDUCATION

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In this research report, the author draws on her diverse experiences in research and teaching to present an emerging theoretical framework for integrating comics storytelling as a pedagogical tool in K-8 mathematics education. The framework, deeply rooted in mathematics identity and funds of knowledge, serves as a bridge between students' mathematical experiences outside and inside the classroom. This framework carries significant potential to cultivate a supportive environment that fosters collaborative problem-solving, encourages students to share their unique stories, and enhance mathematics identity development. Through sharing insights from her learning experiences and her vision for comics storytelling, this research report aims to contribute to the ongoing discourse on creating inclusive learning spaces in mathematics education.

Keywords: Elementary School Education; Middle School Education; Integrated STEM/STEAM; Classroom Discourse.

Drawing from my experiences in research and teaching, I propose a theoretical framework that leverages comics storytelling to bridge the gap between in-school and out-of-school mathematics learning and to shape students' mathematics identities. This emerging framework has the potential to create a supportive environment that encourages students to share their unique stories, promotes collaborative problem-solving, and enhances the development of mathematics identity. Additionally, this report seeks to contribute to the ongoing discourse on fostering inclusive learning spaces in mathematics education.

Researcher's Positionality

I observed a recurring challenge among my students as a mathematics tutor in Nigeria – the challenge of connecting mathematical problems with their real-life experiences. To address this, I turned to the power of storytelling, specifically utilizing comics stories as a medium to intertwine mathematical concepts with indigenous lived experiences. Incorporating indigenous terms and characters in these comics stemmed from recognizing that many available mathematics resources were entrenched in western perspectives, featuring characters, language, and settings unfamiliar to the average child in my community. This led to the initiative "*Learning Mathematics Through Storytelling* (LEMATS)" (Oliwe & Chao, 2022). The foundation of this report rests upon my experiences with LEMATS and my ongoing theorizing of comics storytelling in K-8 mathematics education.

Conceptual Frameworks

Mathematics Identity

Researchers have indicated the significance of students' identities, especially in mathematics (Martin, 2000). Mathematics identity is defined as "the dispositions and deeply beliefs that students develop about their ability to participate and perform effectively in mathematical contexts and use mathematics in powerful ways across the contexts of their lives" (Aguirre et al., 2013). In mathematics education, the focus should extend beyond the development of essential skills and conceptual understanding to encompass the support for students in perceiving themselves as legitimate and capable participants in mathematics (Aguirre et al., 2013).

Recognizing the identities that students bring into the learning environment provides educators with valuable insights into the reasons behind students' positive or negative connections with mathematics. This understanding enables educators to implement necessary improvements, offering support and reinforcing a student's mathematical learning and their role as a mathematics learner. Also, students' experiences with mathematics in their classrooms significantly influence their perceptions of mathematics and their self-identification as mathematics learners and doers (Aguirre et al., 2013).

As mathematics identities can expressed through storytelling (Aguirre et al., 2013), I argue that the way students encounter mathematics in the classroom and connect these experiences with their home and community environments plays an even more substantial role in shaping their perspectives on mathematics and their self-perception as mathematics learners and practitioners. In fact, Sfard & Prusak (2005) proposed a correlation between mathematics and storytelling, asserting that identity formation is influenced by impactful narratives that provide understanding into one's experiences in learning mathematics. These identities comprise intricate and adaptable narratives individuals construct about themselves, rooted in diverse life experiences such as education, family, and media.

Funds of Knowledge

The concept of funds of knowledge in teaching recognizes the significance of "historically accumulated and culturally developed bodies of knowledge and skills essential for household or individual functioning and wellbeing" (Moll et al., 1992, p. 133). It acknowledges that students consistently bring these bodies of knowledge into the classroom and honors students' families, communities, and the knowledge they contribute to classroom mathematics. In the context of my work, these bodies of knowledge are mathematically enriched resources that can address the challenge of bridging out-of-school mathematics with in-school mathematics. Students' motivation to learn mathematics increases as they bring in their identities into the learning space and establish connections between the mathematics evident in their families and communities and the mathematics encountered in the classroom (Aguirre et al., 2013; Oliwe & Chao, 2022). However, the question arises: What kinds of opportunities and channels are available for students to share this wealth of knowledge they possess or to make meaning of it in relation to mathematics during their lessons?

Rehumanizing Mathematics

Gutiérrez (2018) outlines eight key dimensions that can contribute to the rehumanizing of mathematics education. These dimensions encompass (1) fostering inclusive participation and positioning, (2) valuing diverse cultures and histories, (3) providing windows into unfamiliar experiences and mirrors reflecting personal identities, (4) recognizing mathematics as a living practice embedded in everyday life, (5) encouraging creative expression and innovation, (6) broadening the scope of mathematical topics and perspectives, (7) acknowledging the role of the body and emotions in mathematical learning, and (8) empowering individuals to take ownership of their mathematical journey.

Connecting the above conceptual frameworks to inform the theoretical framing of comics storytelling in K-8 mathematics education, there is the emphasis that educators ought and should be encouraged to actively contribute to shaping positive mathematics identities among students by recognizing and acknowledging the various identities students develop, the wealth of knowledge they bring into a learning space, and creating opportunities for students to take an active role and ownership of the mathematics they are learning as they share their stories, engage in rich mathematical thinking and strategies dialogues with one another, and eventually create their own math comics creations.

Key Elements in the Emerging Theoretical Framework for Using Comics Storytelling in K–8 Mathematics Education

Definition

According to Eisner (1992), comics are a form of sequential art. McCloud (1993) elaborates on this definition by describing comics as images juxtaposed in a deliberate sequence to communicate an idea and/or evoke an aesthetic response, providing clarity to the expansive term "art." Notably, Abel & Madden (2008), addressing the absence of the term "text" in previous definitions, incorporate text in their working definition of comics.

My current working definition of comics storytelling in mathematics education is – a form of storytelling that involves using images and text to create stories in a juxtaposed sequential layout that not only give educators the leverage to create opportunities for students to explore, interpret, and solve mathematical problems but also invites the students to connect and share stories from their homes and communities wherein their mathematics thinking is situated, engage in rich problem-solving conversations, and visually represent their mathematical thinking through comics. These stories may be lived experiences, fictional, or a mix of both. The representation of these stories may be aesthetically pleasing or in simple format.

The Comics Structure

In the scope of my research, I find Eisner's (1995) narrative structure, encompassing the introduction/setting, problem, addressing the problem, solution, and conclusion, along with Chao et al.'s (2021) "*Math in My World*" model rooted in the Funds of Knowledge Theory (González et al., 2001), to be valuable structures for introducing both students and teachers to the art of storytelling and its connection to mathematics. Using this model not only positions students as learners and doers of mathematics but also gives them a sense of ownership of the mathematics they are learning. Furthermore, taking into consideration McCloud (2006) storytelling goals, educators and students should aim to (1) develop stories rooted in lived experiences and perspectives of oneself or the reader's (2) ensure authenticity in the creation of characters and their roles within the story's world (3) captivate readers by introducing them to unfamiliar places, cultures, and strategies (4) potentially evoke emotions by tapping into shared heritage and experiences, and (5) instill a sense of care and engagement in the readers toward their narrative.

Reading Between the Gutter

Reading between the gutter in comics storytelling involves two techniques – closure and transitions. McCloud (1993) defines "closure" as the phenomenon of perceiving the whole by connecting and making sense of individual parts. It occurs when a reader seamlessly links two or more comics panels into a continuous story (Abel & Madden, 2008). According to McCloud (1993), readers actively engage in the storytelling process by making decisions and filling in the gaps between depicted moments and perspectives chosen by the author (Figure 1). Closure, as described by McCloud (1993), can be a powerful force within individual panels and as well as between panels – known as the gutter, an empty area between panels (Abel & Madden, 2008). Embracing McCloud's (1993) concept of closure in comics storytelling in mathematics education, I argue that this presents an opportunity to invite students to contribute their perspectives and insights to the ongoing conversations by filling in the narrative gaps. Educators play a pivotal role in this closure process. To encourage students to actively participate by sharing their stories or strategies, educators must determine the extent of information expected from students to fill in and the types of mental jumps the students will need to make between panels. These jumps are referred to as "transitions" (Abel & Madden, 2008).

Looking at Figure 1, we can envision potential approaches an educator might tailor additional prompts or classroom activities to align with the intended lesson objectives, elicit students' funds of knowledge and perspectives, and facilitate mathematical dialogues.

Figure 1: Example of a 4-panel Math Comics

Story Circles

In my previous teaching and research endeavors, students primarily engaged in independent work or collaborated with their immediate seat partners. However, this approach had its limitations, particularly considering my aim to cultivate collaborative conversations within the learning space. Reflecting on these experiences, the story circle is central piece to my theoretical framing of comics storytelling as a pedagogical tool in K-8 mathematics education.

Story circles, characterized by small groups that emphasize active listening, provide a safe platform for individuals to share their evolving stories and offer feedback to one another (Lambert, 2013). Through this process, feedback from fellow participants aids in shaping and deepening the narratives until the central focus of their stories becomes evident. The goal of story circles is to empower all participants to embrace their roles as storytellers and cultivate collaborative conversations within the learning space. Additionally, story circles in mathematics education inspire educators and students to perceive and weave their mathematics stories uniquely, expressing their mathematical stories in their own voices (Chao, 2023).

Data Interpretation

In examining the students' comics stories, the data analysis concentrates on these elements: (1) the story plots, focusing on the central theme and setting; (2) characterization, which includes the names, personalities, physical traits, and expressions of the characters; (3) mathematics, identifying the types of mathematical problems and concepts present in the comics and analyzing how students use storytelling to showcase their understanding of mathematics; (4) the nature of dialogues; and (5) the relationships depicted in the comics. This approach enables one to identify and articulate significant patterns and themes that emerge from the analysis.

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STRETCHY MINDS: BUILDING FOUNDATIONS FOR DEEP CREATIVITY THROUGH EARLY EXPERIENCES WITH QUALITATIVE GEOMETRY

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Simply stated, *STEM needs creativity*. From equity to the economy to the environment, addressing major global issues demands both "deep creativity" and STEM knowledge and skills. Innovative solutions to address these large-scale problems can be derived through creative processes fundamental to scientific thinking (Cropley & Cropley, 2010). However, creativity is almost absent from mainstream approaches to STEM learning. If we do not resolve the disconnect between creativity, innovation, and the sciences, STEM graduates will be ill-equipped to tackle the most critical and persistent global issues. The "Stretchy Minds" project brings together elementary math teachers and researchers working in mathematics education, creativity, and embodied and emergent design to resolve this disconnect. *Conceived with an eye toward some imagined future,* this poster unveils this novel project and shares its findings.

Deep creativity yields novel forms of change, which have both quantitative and qualitative dimensions. Change-in-degree is an incremental *quantitative* change. Change-in-kind is a wholly and genuinely novel *qualitative* change. Configurations of Speks magnetic toys in Figure 1 demonstrate these forms of change using "number of loops" and as the criteria for equivalence (i.e., sameness in kind) and "size of loops" as the criteria for difference in degree. Shapes A-C are different in kind, since they each contain a different number of loops. Shapes B and D are different in degree in that the loop in D is larger than the loop in B, and size is a matter of degree.

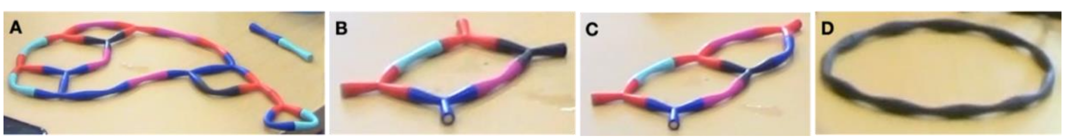

Figure 1: Configurations of the Speks toys express differences in kind and degree.

The Stretchy Minds project is undertaking design-based research (Brown, 1992; Collins, 1992) to produce responsive (Jacobs et al., 2011) curricular experiences (Dewey, 1925; Pinar, 2012) that engage and develop learners' thinking about deep creativity. We have conducted two pilot studies with elementary-age children to explore whether we could support the development of deep creativity through qualitative difference. The findings of these pilot studies have 1) established that qualitative geometry (Greenstein, 2014, 2018) is a viable context for research into children's thinking about qualitative difference; 2) determined that the Speks toys can be effective for mediating children's thinking towards more sophisticated understandings of differences in degree and in kind; and 3) revised initial tasks into game-based learning activities (Nguyen, 2020) that support even the youngest children's agentive and creative explorations. These findings have convinced us of the viability of a game-based approach to nurturing children's deep creativity for novel innovation. They will also enable us to contribute to the expansion of the theoretical understanding and instructional practice of creativity in the space of elementary mathematics education, and in STEM education more broadly.

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AN ANALYSIS OF INITIAL PERCEPTIONS OF MATHEMATICS IN A FIRST-YEAR COLLEGE COURSE

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First-year college mathematics courses serve a wide population of students with various mathematical backgrounds and experiences. Students' experiences shape the way they view themselves as mathematical learners and their mindsets toward mathematics. In this study, we analyzed reflections written in Fall 2023 by students in a college algebra course. We report our findings on students' mathematical perspectives and past mathematical experiences. Our work sheds light on students' dispositions towards mathematics at the start of a first-year mathematics course and provides valuable insights for practitioners and researchers.

Keywords: Affect, Emotion, Beliefs, and Attitudes

First-year college mathematics courses serve a wide population of students with various mathematical backgrounds and experiences. These experiences shape the way students view themselves as mathematical learners and their mindsets toward mathematics. The research presented here is part of a larger study that examines students' experiences in a first-year college mathematics course. As part of this work, we first sought to understand students' mathematical experiences and their perceptions of their mathematical abilities at the beginning of the semester. The research question we address is: *What prior experiences and perceptions of mathematics do students have at the beginning of a first-year college mathematics course?*

The primary data in this proposal come from written student reflections, which we call "mathographies." The prompts used in the mathographies were based on work done by Drake (2006) and offer insights into how students view themselves as individuals in the classroom. As described by Sawyer and Buckmeyer (2024), mathographies offer a human-centered perspective on students and their past encounters with learning and practicing mathematics that can contribute to the creation of a learning setting that is more inclusive. Moreover, writing a mathography often evokes emotions in students, particularly those students who identify as having struggled in the past (Sawyer and Buckmeyer, 2024). This research is critical to better understand how students' perceptions about mathematics have been shaped by recent challenges such as COVID-19 and mental health issues. By assigning and analyzing mathographies, instructors can better understand the circumstances under which students walk into a first-year college mathematics course and use this information to inform how they interact and support students in the classroom.

Theoretical Framing

Our work draws on the theoretical framework of students' mathematics-related beliefs set forth by Op't Eynde et al. (2002). Their framework describes how students' beliefs about mathematics fall into three categories: beliefs about the discipline of mathematics and mathematics education, beliefs about themselves, and beliefs about the social context of learning. These beliefs are present in the student mathographies we discuss in this proposal and underlie their past experiences and perceptions of mathematics.

Methods

Data for this study come from a written mathography reflection that was assigned to 611 students enrolled in 15 sections of a corequisite college algebra course at a large, Midwestern, metropolitan university during Fall 2023. The assignment was graded based on completion and included math-specific prompts such as “What do you think it takes for a student to be good at math?” as well as more general questions such as “What are your strengths as a student or as a person?” In this proposal, we report on 55 of those mathographies, which were selected based on responses from a Qualtrics survey with items from Cribbs et al. (2021) relating to mathematics mindset and self-efficacy. Mathographies were analyzed using MAXQDA 2022 (VERBI Software, 2021). To establish a preliminary set of codes, deductive coding was conducted with an initial set of codes based on previous work (Uhing & Bennett, 2023; Uhing et al., 2021; Wright & Uhing, 2023). At least two researchers tagged each response to prompts in the mathographies. Researchers then met to discuss and reconcile until agreement was reached. Excerpts were then sorted by initial codes and open, data driven coding (Saldaña, 2016) was used to create sets of themes for each of the codes that illustrate the mathematical mindsets and beliefs students have entering a first-year college mathematics course.

Findings

Mathematical Perspectives

In their mathographies, many students wrote about their beliefs about themselves and their abilities to do mathematics, relating to self-efficacy and control beliefs (Op’t Eynde et al., 2002). By the time students arrive in a first-year college mathematics course, these beliefs have been shaped by several years of past education. As one student wrote, “In general, my experience with math has been pretty poor. My intelligence favors the English [or] Reading side of education by a vast margin.” Another student expressed a similar viewpoint remarking, “I do not enjoy math it makes me frustrated how it doesn’t come easy to me and how I can’t understand it.” Other students described a common belief about the discipline of mathematics: “I believe that there are people who understand math more easily than others.” Similarly, another student said, “I know that some people are just ‘naturally’ good at math.” While many student mathographies that we analyzed had a somewhat negative outlook on the discipline of mathematics, there were a few students who had more positive dispositions. One student wrote, “I will use a positive attitude toward math and keep an open mind when I don’t understand something.” Another student commented, “I try my best in math courses, simply because math has always been a more difficult subject for me rather than others.” These quotes illustrate a growth-oriented mindset towards mathematics and students' learning abilities.

While analyzing students’ mathematical perspectives, we noticed that some students used passive language to describe what it takes to excel in math, while others used more active, first-person language. For example, one student discussed what makes someone successful at math: “I think anyone can be successful at math, I just really think it depends on the student. Are they patient? Do they understand it? If they are not, are they asking for help? Are they going to tutoring?" The passive language in this example shifts the focus away from the student and emphasizes beliefs and actions indirectly, referencing a third party (i.e. "anyone" and "they"). While many students used this passive language, there were a few who spoke in first-person terms. One student wrote, "especially in math, it has taught me that you're going to make mistakes. It's taught me how to accept these mistakes, learn from them, move on, and ask for help next time." Another student reflected,

> For me being good or decent at math has always required me to ask questions, understand every step to a problem, and practice. Because I am the type of person who is not the best at math but has applied themselves.

Overall, it was much less common for students to use this active, first-person language. The students who did, however, seemed to have growth-oriented mindsets towards their abilities to learn mathematics.

When discussing their mathematical perspectives, students also often discussed limitations of mathematical abilities. For example, one student wrote, "I think that anyone can be successful at math to an extent, but I do feel like if you enjoy math, you are more likely to succeed." Another student stated,

> I used to think that some are born with the ability to do math and some aren't. Although I feel that still applies to a certain extent, it's not that black and white. It's true that for some people, it just clicks and for others it doesn't.

Both students in these examples used the word "extent" to describe the ability to succeed and learn mathematics. This language may suggest a deeper belief about potential limits to a person's ability to grow in their mathematical understanding.

Past Mathematical Experiences

While sharing their perspectives on mathematics, students also often described their past experiences with learning mathematics. Many students wrote about their previous assessment experiences, specifically describing high stakes testing as a source of anxiety. For instance, one student discussed their disappointment after taking the ACT, "My low point in my math career was taking the ACT. I had always done good in math and gotten all A's. I took the ACT only one time because of COVID." This student went on to say, "I was very disappointed in myself after I got my score back and I did not perform to the level that I thought I would in math." Other students expressed similar sentiments, highlighting how standardized testing often has negative effects on students entering first-year college mathematics courses.

In addition to assessment, some students discussed how their mathematical experiences had been characterized by rigid, formulaic approaches to learning. For example, one student wrote, "public school has a very rigid, one-way-to-get-an-answer of doing things which caused some struggle for not only myself but a few classmates I've had throughout school." This student went on to compare this experience to their current course saying, "I do very much enjoy how in this Algebra class it is encouraged to seek different ways of getting an answer and to inquire to others as to how they got an answer especially if that process differs from yours." Thus, as illustrated by this example, past mathematical experiences shape students' beliefs about mathematics education and the social context (Op't Eynde et al., 2002), which can affect their expectations for college mathematics courses.

Another reoccurring theme in many students' reflections about their past mathematical experiences was the disruptions in their education due to the COVID-19 pandemic. One student described how they were turned off from math because of COVID:

> It was during Covid-19, and we were at home doing online class. I remember that I was taking algebra and to me it was so hard […] I did not understand but I didn't want to tell my teacher so for the first time even I got a F on my report card. When I looked at that F and looked at what class it was, I was so sad because I have always been good at math. I was thinking that math has changed so much and now I don't feel the joy about math anymore.

Unfortunately, this student's experience was not unique, with several students citing COVID as a low point in their mathematical trajectories.

Discussion

Students' beliefs about their abilities in mathematics are shaped through their myriad experiences with learning mathematics. These beliefs can be categorized in various ways, including beliefs about mathematics, beliefs about themselves as learners, and beliefs about the social context (Op't Eynde et al., 2002). As it relates to students' beliefs about themselves, we found that students in a first-year mathematics course often described self-perceived limitations using language such as "...can be successful at math to an extent..." and compared themselves to "people who are just 'naturally' good at math." This comparison of themselves to other, more gifted, mathematicians suggests that students may view themselves as being behind others when beginning a first-year college mathematics course. Moreover, when asked, "What do you think it takes for a student to be good at math?" some students responded in a passive second- or third-person voice rather than a more active first-person. While the way the question was phrased may have contributed to this effect, it is interesting to note that some students did use first-person language. This phenomenon warrants more investigation and further analysis is ongoing.

Experiences in previous mathematics courses lay a foundation for students' beliefs about mathematics education. Often students talked about how their assessment experiences were negative, causing many of them to have test anxiety. Students also discussed how meaningless formulas and rigid "one-way-to-get-an-answer" methods plagued their previous mathematical experiences. For some students, these inflexible ways of thinking and expectations caused them to lose interest and struggle with learning the material. Students' beliefs about mathematics education have been further shaped by the COVID 19 pandemic. Multiple students talked about how the shift to virtual learning was detrimental for them. Students described how they felt disengaged and how COVID-19 affected their enjoyment of mathematics. As one student said, "...I don't feel joy about math anymore." Students felt isolated and were often left to teach themselves the complexities of the mathematical content they were learning. Overall, these experiences shed light on the negative consequences that the COVID-19 pandemic had for many students in their beliefs and perspectives towards mathematics. These findings elicit questions regarding the widespread, continued impact of COVID-19 on student learning.

Students arrive in their first-year college mathematics courses with a variety of past experiences and perceptions of mathematics. These educational and emotional experiences affect their views about mathematics, themselves and the social context of learning (Op't Eynde et al., 2002). Assigning mathographies at the beginning of the semester can help instructors better understand the students in their classroom and help foster a more comfortable, creative, and supportive environment. Indeed, one of the goals of the corequisite college algebra course that these mathographies were collected from was to help students develop into confident, flexible problem solvers. Future research aims to assess how students' mathematical perspectives and beliefs may have changed over the semester using these initial perceptions as a reference point for comparison.

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SUPPORTING LEARNING THROUGH INTERPRETING OTHERS' SOLUTIONS FROM A RADICAL CONSTRUCTIVIST PERSPECTIVE: A THEORETICAL REPORT

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In this theoretical report, we leverage a radical constructivist perspective to explain how designing learning environments in which students work towards making sense of others' mathematical solutions may support learning. We elaborate on radical constructivist constructs—social goals, cognitive perturbations, and reflective abstraction—and use these constructs to model how engagement with others' mathematical solutions may engender learning. We illustrate our model with a task we designed to promote students' meanings for spatial coordinate systems. We conclude with implications for research and teaching.

Keywords: Learning Theory; Cognition; Problem-based Learning

Students working to make sense of worked examples (e.g., Barbieri et al., 2023) or classmates' solutions to mathematical problems (e.g., Webb et al., 2014) have been positively associated with mathematics achievement. Although researchers have described reasons why such activity may translate to achievement gains (Brown et al., 1992; Webb et al., 2023), they have not provided explanatory mechanisms for how such learning occurs for an individual. As a theory of learning focused on ways individuals develop knowledge, radical constructivism can provide such explanations (von Glasersfeld, 1995). In this report, we consider how students' working to make sense of others' mathematical solutions may support learning from a radical constructivist perspective. Additionally, we consider implications of our analysis for task design.

Learning in Radical Constructivism and Connections to Social Interactions

In this section, we present radical constructivist constructs and coordinate them to yield explanatory mechanisms through which a student may learn from others' solutions. First, we conceptualize that in a classroom, a student prompted to interpret a solution can experience a disturbance to their settled cognitive state (i.e., a perturbation). Second, as the student works to understand a solution, they can enact schemes relevant to their understanding of, and goals for, interpreting the solution. If the student experiences a cognitive perturbation, they may modify or reorganize their schemes to neutralize the disturbance. Such reorganizations can result in learning at a higher cognitive level (i.e., reflective abstraction). Next, we offer more detail about each construct and how they relate in the context of students examining others' solutions.

Schemes, Goals, and Perturbations

To begin, as students engage with others' solutions, we posit they would draw upon *schemes*, which entail "a situation, an activity triggered by how the person perceives the situation, and a result of the activity that a person assimilates to her or his expectations" (Hackenberg, 2014, p.

87). Any interaction prompting the student to draw on one or more schemes is a *disturbance* to the student's settled (equilibrated) state. A student uses their schemes as part of their goal-directed activity. Moreover, in classrooms where sharing solutions is prioritized, the student may conceive they are working toward a *social goal* with others (see Steffe & Thompson, 2000, for criteria to determine if a student is working towards a social goal). We note that even if other students are working toward a different goal (i.e., only intending to obtain a correct solution), a student's perception of a social goal can drive their goal-directed activity.

If a student works to interpret classmates' solutions, they have experienced a disturbance to their equilibrium; von Glasersfeld (1980) broadly defined a *perturbation* as any input that creates a disturbance in a student's equilibrium. Not all perturbations are *cognitive perturbations*. Students might be able to neutralize some perturbations with their current schemes without experiencing any discrepancies as they activate and anticipate the results; such perturbations are not cognitive perturbations. Neutralizing other perturbations may involve a student experiencing discrepancies in their use of a scheme (Steffe & Olive, 2009; von Glasersfeld, 1995). Such perturbations are cognitive perturbations. Cognitive perturbations are important because they can lead to a student reorganizing or modifying their existing schemes to achieve an equilibrated state (Steffe, 1991a, 1991b; Tillema & Gatza, 2024; von Glasersfeld, 1995).

When a student experiences a cognitive perturbation through engagement with others' solutions, they may experience a minor or major cognitive perturbation. Many researchers have equated perturbations with major cognitive conflict or the individual experiencing a 'problem' (e.g., Booker, 1996; Lerman, 1996; Simon et al., 2010). Although cognitive conflict is one type of cognitive perturbation, students can also experience minor cognitive perturbations without (consciously) experiencing cognitive conflict or a problem (Steffe, 2011; Steffe & Olive, 2002, 2009). To exemplify this distinction, we again turn to a student working to make sense of others' solutions. A student might experience a minor cognitive perturbation when a solution has some feature or way of reasoning that is novel for the student and the student is able to neutralize the perturbation with minor modifications to their current schemes. If an observer infers that the interpreting student undertakes a major modification or reorganization of their schemes, then the observer could characterize the perturbation as major. Finally, the student may experience a non-neutralizable cognitive perturbation if the student's current schemes do not support them in satisfactorily interpreting (from the student's perspective) the solution.

Reorganization of Schemes and Reflective Abstraction

To further describe the reorganization of schemes that may occur after a perturbation, we use Piaget's (2001) notion of abstraction. Abstraction is a mechanism explaining an individual's modification of their schemes toward greater cohesion and generality. In this report, we use the concept of *reflective abstraction*. In broad strokes, reflective abstraction entails two processes: a projection of actions or schemes to a higher level of thought and a reorganization that occurs at this higher level (Ellis et al., 2024; Piaget, 2001; Steffe, 2024; Tallman & O'Bryan, 2024; Tallman & Uscanga, 2020; von Glasersfeld, 1995). The reorganization can involve the creation of a coherent relationship or network of relationships between existing schemes as well as with new schemes (Piaget, 2001; Tallman & Uscanga, 2020). Such a reorganization involves taking prior meanings as input for further operating and thus can be considered a "higher" level. We note the cognizing subject need not be consciously aware of any reorganization.

Other researchers have argued for the importance of supporting reflective abstraction and have provided suggestions for doing just that. First, offering students repeated opportunities to develop schemes relevant to particular meanings can support their connecting their schemes and reasoning across similar (and different) contexts (Tallman & O'Bryan, 2024; Thompson, 2013). Second, offering explicit occasions for students to compare activity across tasks can support reflective abstraction (Ellis et al., 2024; Piaget, 1976, 2001; Tallman & O'Bryan, 2024). Taken together, we conjecture students' repeated opportunities to create solutions, consider others' solutions, and to explicitly reflect on solutions can also create opportunities for reflective abstraction. We illustrate these considerations with the following task design.

Exemplifying the Constructs: X-marks the Spot

We designed the *X-Marks the Spot Task* leveraging the above radical constructivist constructs to support students' work with spatial coordinate systems (described below). Our conjecture was that multiple rounds of describing (to classmates) and interpreting descriptions (written by classmates) of locations in space could occasion major or minor cognitive perturbations. Further, we offered deliberate opportunities for students to reflect on location descriptions at a higher level of thought. We intended these experiences to support students in engaging in reflective abstraction as they reorganize their meanings for organizing space.

Task Background: Spatial Coordinate Systems and Conventions

In this report, we focus on a task designed to support students' developing meanings for spatial coordinate systems (Lee, 2017; Lee & Hardison, 2016; Lee et al., 2020; Paoletti et al., 2022). A *spatial coordinate system* is a coordinate system (CS) that entails either mentally overlaying a CS onto some perceived space or overlaying a space onto an already established CS. In either case, objects within the space can be located via coordinates. Radar on a ship and GPS are different examples of spatial CSs (i.e., polar and Cartesian CSs, respectively).

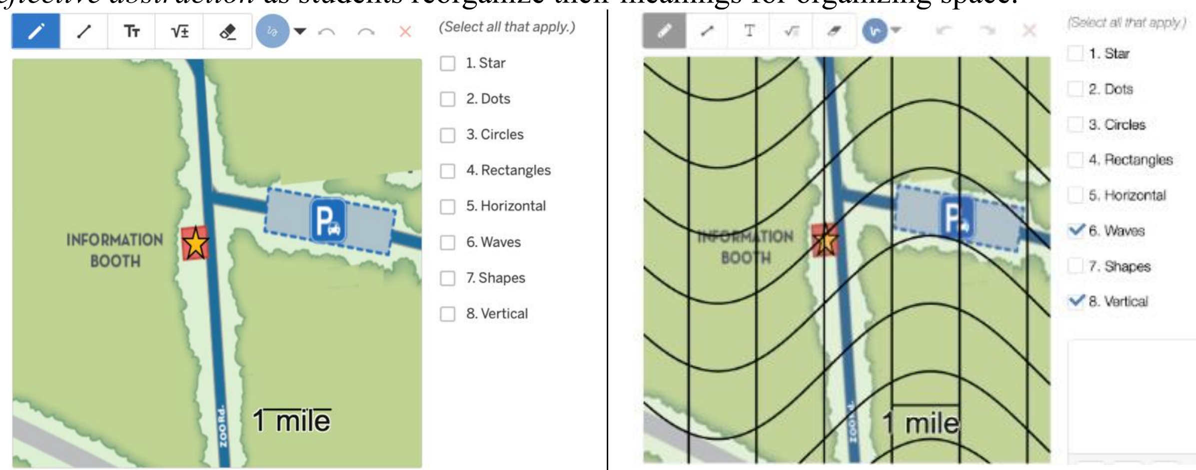
We note conventional coordinate systems involve choices often developed or adopted for the purposes of efficiency and communication (Moore et al., 2019; Zazkis, 2008). Given the communicative value of such conventions, we conjectured that we could support students in developing a *social goal* by offering them repeated prompts to describe locations in space, with the anticipation of classmates' interpreting it, and to interpret classmates' descriptions of locations. This goal could lead to activity that supported students in reorganizing their schemes for organizing space towards more clear and efficient strategies.

The Design of the X Marks the Spot Task

In the *X Marks the Spot Task* (Figure 1), we provide students with the map and buttons shown in Figure 1a with the prompt, "Play with the different overlays. In the next slides you will use the overlays either (a) to describe the location of an X or (b) interpret classmates' descriptions." Students can try each button and observe different overlays.

After exploring the different overlays, we task students (individually or in groups) with marking an X on the map. To support the creation of a *social goal*, we prompt students to use the overlays to provide a written description for the location of their marked X that classmates will interpret to determine the location of the X. In particular, a student may assume their classmates have the same shared goal of marking two Xs in the same location. To achieve this *social goal*, a student may try to write a description that is clear enough for their classmates to follow while also being descriptive enough to mark a precise location for the X. The student assumes their classmates will try to interpret the description and mark an X in the described location.

The act of interpreting real and hypothetical classmates' responses may result in a student experiencing a *cognitive perturbation*. We conjecture students would be able to interpret many descriptions using their current schemes (i.e., *without cognitive perturbation*). We also conjecture interpreting vague descriptions or observing discrepancies in the location of marked Xs for the same description could create *cognitive perturbations* for students. Further, we task students to provide feedback to the author of each description as we conjecture such activity can engender *reflective abstraction* as students reorganize their meanings for organizing space.



(a) (b)

Figure 1: (a) The initial map and (b) the map with Wave and Vertical overlays in the *X Marks the Spot Task.*

Contribution, Implications, and Areas for Future Research

In this theoretical report, we described how radical constructivist constructs can explain how a student's engagement with others' solutions can support their learning. We elaborated on our understanding of social goals, cognitive perturbations, and reflective abstraction. Given the emphasis on collaborative group work in mathematics education, such learning is likely to occur in classrooms where students interpret others' responses positively and reorganize their own meanings as a result of these interpretations.

We described how we designed the *X Marks the Spot Tas*k to support students in creating a social goal that could occasion cognitive perturbations. We conjecture that the use of classmates' descriptions to potentially provoke cognitive perturbations can be productive in this task due to the communicative nature of the mathematics at hand. We conjecture offering students repeated opportunities to first generate their own descriptions and then interpret (real and hypothetical) descriptions that communicate more or less effectively and efficiently increases the chances students would experience cognitive perturbations that could result in reorganizations of their schemes for organizing space. We conjecture there are other mathematical concepts that rely heavily on communicative goals such as conventions, in which students could be supported in learning via the use of (real or hypothetical) student solutions. We call for additional research exploring this possibility. This and other research could build on prior work showing how examining others' solutions to mathematical problems (Barbieri et al., 2023; Webb et al., 2014) can support learning. Further, such research could leverage the constructs outlined in this theoretical report to provide explanations for *how* examining others' solutions can lead to learning.

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A TOULMIN COMPARISON OF TEACHER'S AND STUDENT'S PROOFS

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Abstract: This paper takes data from the Proofs Project which was designed to address epistemological obstacles (Sierpińska (1987), Brousseau (1997)) related to mathematical induction and logical implication and uses Toulmin's model for argumentation to compare the structures of a proof presented in class from the instructor and the students. Under the assumption that math instructors set norms for and model ideal mathematical behavior, it is reasonable to look at how closely students adhere to the implicit or explicit norms set by the instructor.

Keywords: Undergraduate Education, Reasoning and Proof, Advanced Mathematical Thinking

In this report, we draw on the framework developed by Fukawa-Connelly (2014) and Inglis et al. (2007) to analyze the structure of verbal proofs toward answering the following research question: How are the proofs of math students similar or different from the proofs of their instructor? This report is also meant to contribute to an emerging interest in the role of mimicry in mathematical learning.

The data comes from a class part of the second cycle of a design-based research (DBR) study called the *Proofs Project* (Kokushkin et al. 2023). From the first cycle, the researchers replicated findings from existing literature (Brown, 2008; Dubinsky,1986; Dubinsky, 1990) emphasizing the difficulty of mathematical induction specifically from the induction hypothesis. A graded assignment for each student was the presentation of their proofs. Presentations consisted of a formal presentation and subsequent feedback from the instructor. So our use of the phrase "verbal proofs" refers to both the spoken and digitally displayed components of their presentations. Our unit of analysis is the proofs formally presented to the whole class represented by Toulmin diagrams, as it is hypothesized that the norms the instructor models for presenting proofs and his prior feedback are what the students emulate and the norms the students develop for their group work are more distinct from what the instructor models.

While Fukawa-Connelly (2014) used the Toulmin model to analyze only a teacher's arguments in an abstract algebra course, this paper uses the model to compare both the teacher's arguments and his students' arguments. For consistency's sake in both the wider study the data comes from and comparisons of proof, we look at their arguments regarding mathematical induction.

Literature Review & Methods

Toulmin's model was specifically adapted to a math education context around the turn of the century. While it was originally developed by Toulmin (1969), Inglis et. al. (2007) attribute Krummheuer (1995) for adapting it for a math education context. It was specifically adapted for the presentation of mathematical proofs by Fukawa-Connelly (2014) and additionally adapted for an evaluative context by Hancock (2019).

The paper from Fukawa-Connelly (2014) showcases an application of the model in understanding an instructor's proof while looking at both verbal and written forms of proof representation. Fukawa proposes that Toulmin's model of argumentation could be used to improve undergraduate proof teaching by detailing the constructions of the instructor's proofs. From his data, particularly the observations that the instructor implemented an "inconsistent use of backing and warrants" (p. 86), he distinguishes between proofs that convince and proofs that explain. Unlike Fukawa-Connelly's (2014) data, the episodes I analyzed consist of consciously distinguished proofs, whereas Fukawa-Connelly's (2014) study considered the presentation of proof as a more spontaneous affair.

We report on the similarities and differences between the following statements, with pseudonyms in parenthesis:

1. If the kth, k+1th, and k+2th terms of the Fibonacci sequence are odd, odd, and even (respectively), then so are the k+3th, k+4th, and k+5th terms. (F)
2. If k lines in the plane (none parallel and with no three intersecting at a point) form 1+(k(k+1))/2 regions, then k+1 such lines form 1+((k+1)(k+2))/2 regions. (P)
3. If the sum of the first k odd numbers is k^2, then the sum of the first k+1 odd numbers is (k+1)^2. (I)
4. If k!>2k, then (k+1)!>2k+1. (S)
5. A valid formula for the sum of the first n integers is n(n+1)/2. (Instructor, M)

Methods

I adopt and modify the methodology employed by Fukawa-Connelly (2014) to determine which aspects of a given proof correspond to the Toulmin components. As I am comparing the proofs of both instructor and student, we then compare the corresponding components of their resulting Toulmin diagrams or look for what common arguments arise as different components.

Although the proofs were written, because they were presented verbally, I also infer qualities of their modal qualification from implicit factors (Alcock & Weber, 2005) such as if they use hedging language in their delivery, including phrases such as "I think" or "It should follow that…". As the Toulmin diagram includes transcribed speech, but not the nuances of the speech, aspects such as inflection will be ignored. I also took their counterarguments to feedback from the instructor as rebuttals, but not the feedback itself.

Analysis

The instructor started his proof by commenting on a prior student's board work as data for his argument, pictured below in Figure 1. In linking the data to his final claim, he pointed out "the only thing you're adding is this k+1th term." Although his data was detailed algebra, the instructor did not explicitly break down every step. His modal qualification contained little information in terms of mathematical statements, but might be understood better in terms of his role as the instructor when he ended his commentary with: "That's how these arguments work. This is the whole creative part of mathematical induction." Finally, his argument was not followed by questions and thus had no verbalized rebuttals.

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \rightarrow \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)}{2} + k + 1$$

$$\frac{k^2}{2} + \frac{k}{2} + k + 1$$

$$\frac{k^2}{2} + \frac{3k}{2} + \frac{2}{2} \rightarrow \frac{(k+1)(k+2)}{2}$$

Figure 1: Writing Used in Instructor's Proof

Students L, M, and S presented similarly to the instructor in that they spoke with absolute mathematical certainty and were not pressed to provide a rebuttal. The students spoke with absolute authority in that they were reading directly from their presentation in contrast with the instructor who was speaking evaluatively. As they were questioned by neither their peers nor the instructor, their backing was nonexistent. Since their presentation was written as formal proof, we also assign them maximal certainty in terms of modal qualification. On the other hand, the other students, M and P, had backings or warrants in terms of entire alternate proofs, and thus required second Toulmin models to describe their proofs in full.

P's second proof functioned as a backing since it provided an additional interpretation to his warrant. Although M also uses a second proof to her claim, it is not related to the warrant of the first proof and is closer to a warrant to her claim than a backing, nor was it presented to the class. Their backings and warrants also used illustrations, while L had neither despite his statement admitting a geometric proof. The proof from P was also notable as it was the only one that included a backing in the form of a non-algebraic illustration, shown below. This is despite the statements from M and I also admitting geometric proofs. It also meets the definition of a modal qualifier as it utilizes the base case(s) to understand the inductive step.

The proofs by L and F lacked backing in their written portion. There was a rebuttal in F's argument motivated by a question from a fellow student, but only to clarify the parity of the sum of successive Fibonacci numbers.

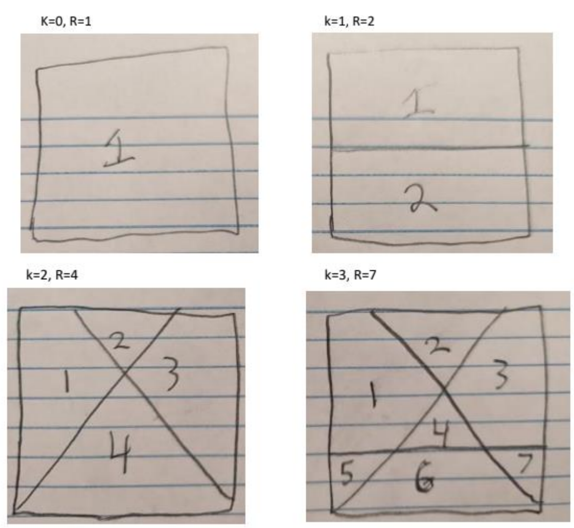

Figure 2: Illustrations from P's Backing

Finally, a common backing in all proofs was a claim that the warrant followed from that "you can kinda see how it works out through algebra." Verbal proofs use handwaving justifications via algebraic manipulations even if the written proof shows it explicitly (in the case of F) suggesting that the norms for speaking a proof, writing a proof, and reading off a proof may be subtly different. It is possible that a tendency to “handwave” algebraic manipulations was influenced by the instructor’s proof from the prior class session.

Regarding induction proofs specifically, it is notable that although the class and the wider project were about the proof schema of induction and the assigned proofs were only about the induction step, one student, S nevertheless presented a full induction argument of the base case, inductive case, and the relationship between them and still claimed to only prove the inductive step. Although he claimed that the property holds for integers greater than 4 and not all integers, this is still a claim about the set of integers, not about an implication. Generally, students would use the universal statement as backing for an inductive implication. During the presentations from S and P, they argued using the full universal statement as a backing for the induction step. However, this could also be understood in terms of both a mimicking strategy and a theme from Kokushkin et al. (2023) who found “students must develop an intellectual need to generalize the logic involved in building the quasi-inductive chain of inferences” (pg. 5) as the quasi-inductive arguments (Harel, 2002) in P’s proof aligned backing of a full induction argument and instructor’s illustration from the prior class session, recreated in Figure 3.

P’s proof gives another similarity between student and teacher. Compare the backing from P’s proof (Figure 2) to the class discussion from the prior class session, where the instructor drew Figure 3 in discussion a proof of the statement “If 3 lines in the plane (none parallel and with no three intersecting at a point) form 7 regions, then 4 such lines form 11 regions.”After drawing this diagram, he uttered “And this is the kind of thing you have to explain.” It is plausible that an emphasis on illustrations for geometric proofs from an instructional authority pushed P to provide illustrations as a backing even with a valid algebraic proof.

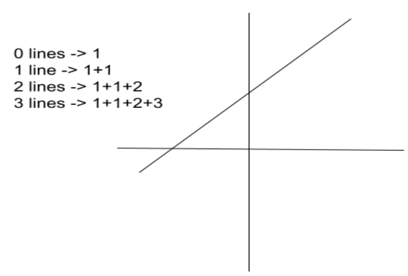

Figure 3

Conclusion

In his original study, Fukawa-Connelly (2014) suggests that Toulmin’s model and its application in his study could help instructors improve on proof teaching by examining their instructions concerning expected student outcomes. This report contributes to this suggestion by extending findings from Hazzan et al. (2003), who argued for the significance of proof mimicry by pointing out the abstract nature of many Linear Algebra concepts. As found by Zhou & Guo (2016), math students rely on the imitation of their teachers to support their learning more than students of other subjects. Zhou (2012) also argues that imitation is a cognitively sophisticated technique both students and teachers employ to support students' learning. Mimicry by both students and teachers can positively affect students' learning outcomes, as demonstrated by Zhou (2012), who found broad, positive, and statistically significant effects on student outcomes, especially among math and science students. Students were seen to mimic their instructor in a variety of ways, such as their language. But the Toulmin model also allowed us to identify that both parties refer to algebraic manipulations as a backing. The familiarities we observed in terms of similarities of backings, and modal qualifiers, support our main hypothesis that students mimic proofs to support their learning. Our work also shows a need to expand the theory from Zhou (2012) as mimicry seemed to occur not only between student and teacher in both directions but between students. And our methodology is not powerful enough to determine if mimicry was occurring, or if all the students were practicing the same sociomathematical norms. Generally, the boundary between norm adherence and mimicry should be investigated further.

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STUDENT THINKING AND ATTITUDES WHEN WORKING WITH DATA IN AN INTEGRATED, PLACE-BASED CURRICULUM

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Developing students' data literacy skills through an integrated place-based curriculum is important for creating informed citizenry as well as building student confidence around mathematical and statistical thinking. Initial results from attitude and assessment surveys from 74 sixth-grade students in a rural, coastal school district in the Northeastern United States reveal that students entered the school year with positive attitudes towards mathematics. While students hold more novice conceptualizations of what constitutes "data," their confidence in their math skills and performance offer an opportunity to sustain these positive attitudes as they encounter more challenging statistical thinking tasks across math and science classrooms.

Keywords: Middle School Education, Integrated STEM, Data Analysis & Statistics

Purpose of the Study

The oceans define our planet and are central to many challenges human populations face; addressing these challenges requires citizens who are ocean literate **and** data literate. We established a research-practice partnership (RPP) with teachers and community members from rural coastal communities in the Northeastern U.S. to promote integration of ocean science, data, and technology-related competencies into science curricula. This study reports initial findings regarding middle school students' conceptualizations of data, and characterizes their attitudes towards STEM (Science, Technology, Engineering, Math) subjects at the start of the school year.

Theoretical Framework

While many research studies indicate that integrated STEM instruction can benefit students and teachers alike, educators face significant challenges in implementing integrated instruction (Mayes, 2019; Ríordáin et al., 2016). Considerable research also supports the idea that STEM subjects should be taught in an integrated, authentic way that reflects the day-to-day practices and competencies of experts and scientists (Kelley & Knowles, 2016; NGSS Lead States, 2013; Roehrig et al., 2021). We draw upon the data science framework proposed by Lee and colleagues (2022) and the positioning of data literacy at the intersection of data science, authentic context, and quantitative reasoning proposed by Kjelvik & Schultheis (2019). These frameworks guide our investigation of students' reasoning when working with data and their conceptualizations about data, specifically regarding variability, measures of central tendency, and interpretation of data visualizations, aligning with the GAISE report principles (Bargagliotti, 2020).

Furthermore, we position our project within a marine context to make it relatable and accessible for students in a state with a substantial "blue economy." Linking learning to initiatives such as Powering the Blue Economy (PBE; Office of Energy Efficiency & Renewable Energy, n.d.) helps ground students' data literacy development in real-world applications that can focus on the innovative yet sustainable use of ocean resources to boost economic growth while preserving ocean ecosystems, which has direct impacts on their local communities.

Methods

This study took place in the context of a large grant-funded RPP exploring ways to integrate authentic research and technology into grade 6-12 classrooms, with a particular focus on marine science and data literacy. Key objectives include (a) providing professional learning for teachers to support the development of community-relevant, authentic student research, (b) studying the impacts of authentic research infused with technology on diverse students' knowledge of and engagement with the material, and (c) expanding students' career knowledge and awareness by establishing local community partnerships. Following the first series of professional learning sessions in Summer 2023, teachers developed lessons based on existing classroom instruction that incorporated these elements into instruction for the 2023 - 2024 academic year.

Participants

Student participants were recruited from grade 6-12 classrooms enrolled in the project; Institutional Review Board approval was obtained prior to any recruitment and data collection activities. This study focuses on pre-instruction data from participating 6th graders. Pre-instruction survey responses were collected from 74 6th grade students enrolled in Earth Science; post-instruction surveys will be deployed by the end of the 2023-2024 school year. Respondents include 38 females, 21 males, and 15 students who left the gender question blank or preferred not to answer.

Data Collection and Analysis

Data was collected using two pre-post surveys targeting both student reasoning about data and their attitudes towards STEM. Student attitudes were assessed using the Student Attitudes Towards STEM (S-STEM) survey (Friday Institute, 2012). The S-STEM survey conceptualizes student attitudes as representing both student self-efficacy and expectancy-value beliefs (Unfried et al., 2015). The research team created a Data Assessment survey that included a mix of Likert items and multiple-choice items drawn from previously published instruments (*e.g.*, Gormally et al., 2012; Zoellick et al., 2016) aimed at understanding students' views on what "counts" as data (Bargagliotti et al., 2020) and how they think about data in their lives.

Survey data was analyzed using SPSS statistical software (IBM, version 28.0.0.0). S-STEM analysis focused on the Math subscale and associated career/course question items. Likert answer choices to subscale items were scored from 1-5, with 5 representing the most positive views; averages and standard deviations from subscales are reported based on the recommendations by the survey developers (Friday Institute, 2012). For the survey item predicting class performance, simple contrast coding was used to run linear regression with categorical predictors to determine if responses were associated with Math subscale scores. Responses to the Data Assessment were summarized using descriptive statistics. Further analysis examining both classroom observation notes and artifacts for evidence of student thinking and reasoning is currently underway and will be complete by July 2024.

Results

Student Attitudes: Middle School S-STEM Results

Middle school students' attitudes were the most positive on the 21st Century Skills subscale, with a pre-instruction mean of 3.82 (SD = 0.50), followed by Math (3.47, SD = 0.69), Engineering (3.25, SD = 0.65), and then Science subscales (3.17, SD = 0.65). A one-way ANOVA revealed there was no significant difference between scores by gender ($F_{2,71} = 0.762$, $p = 0.514$). Subscale items around future plans for pursuing careers in math and/or doing advanced math had less positive student views than those about students' current mathematical self-efficacy. For example, only about 30% of students Agreed/Strongly Agreed with the statement "I would consider choosing a career that uses math," whereas "I am good at math" had over 70% of students Agreeing/Strongly Agreeing.

Students also predicted how well they expected to do in the current school year in their math, science, and ELA classes. Math had the highest percentage of students predicting they would do very well (31%), followed by ELA (28%), and Science (20%). Only 8% of students predicted "Not at all well" for both math and science; ELA only had 4% of students predicting such. Categorical linear regression analysis revealed that students' expectations for performance in math class was a significant predictor of their attitude score on the Math subscale (Adjusted $R^2 = 0.513$; $p < 0.001$). Students in the "Not at all well" prediction category scored an average of 1.75 points lower on the Math subscale than those who thought they would do "Very well," while those in the "OK/Pretty well" category scored an average of 0.84 points lower ($p < 0.001$).

Student Thinking: Middle School Data Assessment Results

Data Assessment responses ($n = 70$) to Likert statements indicated 70% of students agreed with statements characterizing data as useful for understanding the environment; 67% acknowledged descriptions/observations and 63% said stories from people with local/historical knowledge as useful data. Only 34% agreed data were part of their everyday life, and the remaining statements regarding data collection and analysis had large percentages (36–53%) of students selecting the neutral answer choice.

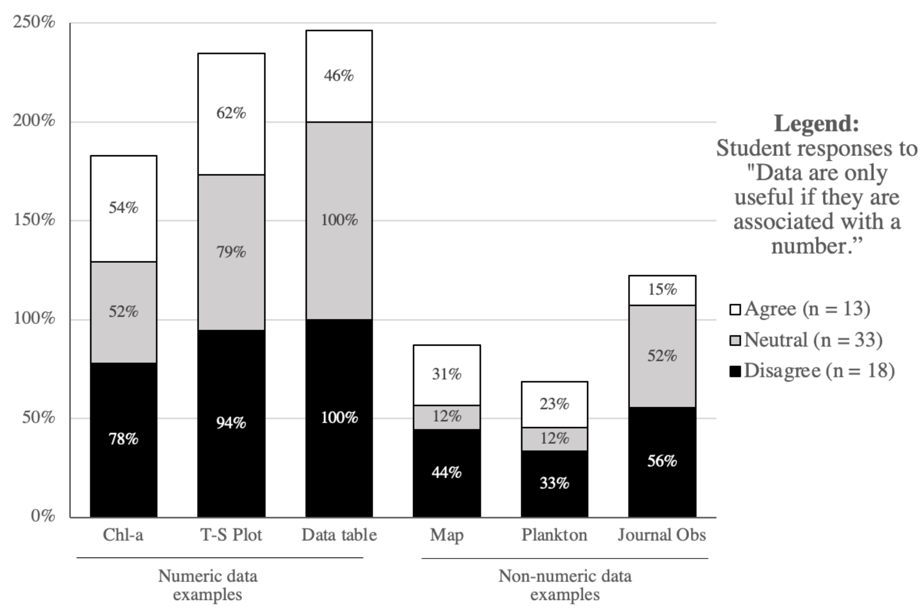

Figure 1: Percent of Students Within Each Response Category Who Identified Numeric vs. Non-Numeric Examples as "Data"

Students were shown six different images containing both numerical and non-numerical marine data sources and asked to choose all images they would classify as "data." Numerical examples included a satellite image of sea surface chlorophyll-a concentrations (Chl-*a*), a time-series plot of sea surface temperature and salinity (T-S Plot), and a table of temperature and salinity data at three locations (Data Table). Non-numerical examples included a map of a local bay (Map), a photo of a phytoplankton cell (Plankton), and a journal entry detailing intertidal observations (Journal Obs). The most popular images selected by students were the numerical sources: Data Table (81.4%), T-S Plot (72.9%), and Chl-*a* (55.7%). Less than half of the students (41.4%) chose Journal Obs as a kind of data; few chose Map (24.3%) or Plankton (18.6%).

Because of the small number of students who classified the non-numerical examples as representing types of data, we investigated their responses to this item in light of how they answered the Likert statement "Data are only useful if they are associated with a number" (Figure 1). 64 students responded to this survey question; 18 disagreed, 33 were neutral, and 13 agreed. Students who disagreed (i.e. held a more expert-like view of what "counts" as data) were more likely to choose non-numeric data examples over those who were neutral or agreed. For example, 44% of the Disagree students (black bars) identified the Map as a type of data, as compared to only 12% of Neutral students (gray bars) and 31% of Agree students (white bars). Because students could choose multiple options, percentages add up to more than 100%. In the final series of questions around handling hypothetical data dealing with outliers and variability, 54.3% of students correctly answered how to handle outlier data in a dot plot, but only 28.6% chose the correct description of variability in non-technical terms. 32.9% of students chose the distractor choice "The number of different values in the data set" for describing variability.

Discussion & Conclusions

We conclude that the average Math subscale score of 3.47 indicated more positive views towards math than science, when considering both subscale averages and prediction of performance in both classes in the coming year. Since participants were enrolled in their first standalone science class of their academic careers, uncertainty about what this course looks like could contribute to less confident attitudes towards science. The overwhelmingly positive responses to self-efficacy-type Math items were encouraging and suggest that younger middle school students may hold a more positive mathematical mindset than expected.

With the regression analysis indicating significant differences observed in overall math attitudes between the performance prediction groups, our results reinforce recommendations from previous work (e.g., Boaler, 2015) that building confidence and self-efficacy amongst the students who have negative performance predictions is critical to future success, especially as studies continue to demonstrate math anxiety having negative impacts on performance (Barroso et al., 2021). Efforts to build positive, growth-mindset oriented mathematical attitudes need to happen beyond math classrooms. More explicit opportunities for students to engage with mathematical thinking and reasoning in other subjects (like Earth Science) may help break down stigmas students have towards math and allow these practices to be more accessible.

Students' responses to the Data Assessment Likert questions had several statements with which students neither agreed nor disagreed; these may reflect more uncertainty than a true neutral stance. Modifying the survey to eliminate the neutral option and/or following up with an open response question or individual student interviews could be a way to better understand these responses. Despite students' agreement that data did not have to be numeric, Figure 1 shows that the majority of students still gravitated to data examples that had quantities attached to them, which may be a result of their familiarity with these types of representations.

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DESARROLLO Y REFINAMIENTO DE MODELOS MATEMÁTICOS POR ESTUDIANTES DE BACHILLERATO EN UN PROBLEMA DE DEFORESTACIÓN

DEVELOPMENT AND REFINEMENT OF MATHEMATICAL MODELS BY HIGH SCHOOL STUDENTS IN A DEFORESTATION PROBLEM

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Se presenta el análisis de los resultados de una investigación cualitativa relacionada con la modelación de una actividad cercana a la realidad por estudiantes de bachillerato. Se describen los modelos desarrollados por las estudiantes al abordar la actividad titulada "Deforestación en Jalisco". El marco teórico fue la Perspectiva de Modelos y Modelación, por lo que la actividad fue diseñada con base en los seis principios de diseño de una MEA (Model Eliciting Activity). Participaron dos estudiantes de sexto semestre de bachillerato quienes abordaron la MEA mediante la plataforma Zoom. Como resultado se observó que las estudiantes tuvieron oportunidad de conectar su conocimiento matemático con la situación de deforestación y reflexionaron sobre la importancia de los recursos forestales en términos de las implicaciones de su pérdida.

Palabras clave: Modeling, High School Education, Algebra and Algebraic Thinking, Online and Distance Education.

Introducción

Uno de los objetivos del acuerdo de París, adoptado dentro de los Objetivos de Desarrollo Sostenible (ODS) es contrarrestar la deforestación debido al impacto que la superficie forestal terrestre tiene como regulador climático natural (Organización de las Naciones Unidas [ONU], 2023). La Organización de las Naciones Unidas para la Educación, la Ciencia y la Cultura (UNESCO, 2021) señala que es necesario tomar medidas al respecto y el aula es un espacio donde se puede iniciar la sensibilización. Los ambientes de aprendizaje de matemáticas basados en la modelación promueven no sólo que los estudiantes desarrollen conocimiento matemático y habilidades para modelar (Corum & Garofalo, 2019; Lesh, 2010; Garfunkel & Montgomery, 2019), sino que también promueven la reflexión sobre la problemática del contexto (Vargas Alejo & Montero Moguel, 2023). En este estudio se describen los modelos generados por estudiantes de bachillerato al resolver una actividad cercana a la vida real diseñada en el contexto de la deforestación. La pregunta de investigación fue ¿qué conocimiento matemático sobre la función lineal, habilidades y reflexiones exhiben las estudiantes de bachillerato al enfrentarse a la MEA "Deforestación en Jalisco"?

Marco teórico

De acuerdo con la Perspectiva de Modelos y Modelación [MMP, por sus siglas en inglés Models and Modeling Perspective] los estudiantes deben aprender matemáticas de manera que su aprendizaje no se reduzca a su utilidad al aula (Lesh, 2010). Las matemáticas que aprenden deben apoyarlos a reflexionar sobre situaciones que trasciendan al salón de clases para entender su entorno (Lesh & Doerr, 2003). En lugar de enfocarse en la memorización o la mecanicazación de algoritmos, se debe priorizar el desarrollo de conocimientos y habilidades para usar ese conocimiento en la interpretación, descripción, explicación y predicción de fenómenos. Se considera que el proceso de aprendizaje es más relevante que el producto en sí, es decir, el modelo mismo. Lo importante es propiciar el proceso de refinamiento de conocimiento por los estudiantes al diferenciar, integrar, modificar y reorganizar conceptos (Lesh & Yoon, 2004) a través de ciclos de modelación donde los estudiantes usualmente matematizan.

En este sentido, la MMP propone el uso de actividades basadas en situaciones cercanas a la vida real, conocidas como MEAs, las cuales se diseñan con base en seis principios (Lesh et al., 2000; Sevinc, 2021; Sevinc & Lesh, 2018): el principio de realidad, el principio de construcción de modelos, el principio de documentación, el principio de autoevaluación, el principio de generalización de modelos y el principio de prototipos. Las MEAs permiten que los estudiantes desarrollen sus propias ideas matemáticas, teorías, procesos y habilidades para dar una respuesta ante un problema cercano a la realidad (Doerr, 2016), por lo tanto, el contexto es relevante, dado que influye significativamente en la matemática que se emplea, y por ende forma parte de los modelos que los estudiantes construyen.

Metodología

La metodología empleada en esta actividad fue un estudio de caso, participaron dos estudiantes de 18 años aproximadamente, quienes estaban cursando sexto semestre de bachillerato en una escuela privada.

La MEA "Deforestación en Jalisco" se diseñó con base en los seis principios de diseño (Sevinc & Lesh, 2018). Se implementó en un ambiente virtual mediante la plataforma Zoom. Las fases de implementación fueron las siguientes: a) Sesión virtual síncrona de una hora, para la lectura de la nota periodística de la MEA "Deforestación en Jalisco", b) Resolución de la MEA "Deforestación en Jalisco" mediante una carta, como tarea extraclase, c) Sesión virtual síncrona de una hora para discutir la carta y d) Refinamiento de la carta como tarea extraclase.

La MEA solicitaba a los estudiantes apoyar a la Comisión Nacional Forestal (CONAFOR) con la elaboración de una carta que incluyera una descripción del cambio del territorio forestal en Jalisco en los últimos años y para ello se les proporcionaba una tasa de deforestación constante curiosamente tomada de fuentes gubernamentales. También se pedía que la explicación sirviera a la CONAFOR para describir en el futuro cualquier otro territorio forestal con condiciones similares.

Los instrumentos de recolección de datos fueron videograbaciones de las sesiones, las cartas elaboradas por los estudiantes y la bitácora del docente. El análisis se realizó tomando en cuenta los modelos construidos por las estudiantes. Se codificó la información tomando en cuenta el conocimiento matemático exhibido, habilidades de modelación y reflexiones de las estudiantes sobre la problemática. La primera autora del artículo implementó la actividad e hizo un primer análisis el cual fue discutido con las dos siguientes autoras, lo cual permitió refinarlo.

Resultados

La evolución del conocimiento matemático, habilidades y reflexiones que exhibieron las estudiantes al enfrentarse a la MEA "Deforestación en Jalisco", se describe a través de las cuatro fases de la implementación.

Fase a

Durante la primera sesión virtual síncrona las estudiantes leyeron la MEA en voz alta. Se percibió gran interés y conocimiento en las alumnas respecto a la importancia de los territorios forestales para la conservación de la biodiversidad y, también, sobre acciones que se pueden tomar para evitar que se extienda el daño. En los siguientes extractos se observan algunas de sus reflexiones sobre la problemática.

E1: Se debería hacer un poco más de conciencia a la sociedad, bueno a los mexicanos, para saber el daño que se hace a los árboles por algunas empresas para su beneficio propio

E2: Se deberían hacer campañas para reforestar las áreas afectadas … iniciativas como por ejemplo con La Primavera, de tal manera que el espacio quemado no se pueda fincar en 10 años … para restringir el daño con estas reglas

En esta sesión las alumnas señalaron la necesidad de buscar más información sobre el territorio forestal en Jalisco en los años 2019, 2020 y 2021.En esta fase se observa cómo las estudiantes reflexionaron con base en su conocimiento del bosque cercano La Primavera. Esto concuerda con los hallazgos de Lesh y Doerr (2003) quienes señalan que cuando los estudiantes empiezan a resolver las MEAs, generalmente, hacen referencia a experiencias personales. También se relaciona con el principio de realidad (Sevinc & Lesh, 2018).

Fase b

Las estudiantes elaboraron su carta, solicitada en la MEA, para apoyar a la CONAFOR. En ella utilizaron el territorio forestal de Jalisco de $226{,}581\ ha$, $174{,}190\ ha$ y $167{,}811 ha$ correspondiente a los años 2019, 2020 y 2021, respectivamente. Realizaron cálculos comparativos para obtener el decrecimiento en porcentaje. Por ejemplo, para obtener el porcentaje de deforestación en el año 2020 con respecto al año 2019 realizaron la siguiente operación:

$$Porcentaje\ de\ deforestación = \frac{174{,}190ha * 100\%}{226{,}581ha} = 76.87\%$$

Con este dato calcularon la disminución porcentual de deforestación.

$$Disminución\ porcentual\ de\ deforestación = 100\% - 76.87\% = 23.12\%$$

De manera análoga, procedieron para obtener el porcentaje de deforestación para el año 2021 (74.06%) y la disminución porcentual de deforestación (25.93%). Enseguida obtuvieron la tasa de deforestación de $17{,}104.42\ ha/año$ correspondiente al periodo 2001-2021 y la compararon con la tasa de deforestación de $15{,}997\ ha/año$ indicada en la nota periodística, señalando que afortunadamente la deforestación había disminuido en 0.01%.

El conocimiento matemático utilizado fue razones, proporciones y porcentajes. Exhibieron habilidad para relacionar datos de manera multiplicativa y tomar decisiones. La MEA permitió a las estudiantes analizar matemáticamente la situación lo cual se relaciona con los principios de construcción y documentación del modelo (Sevinc & Lesh, 2018).

Fase c

Durante la fase c las estudiantes leyeron su carta a la profesora. Se concentraron en describir de manera cuantitativa la problemática de la deforestación en Jalisco y en la necesidad de difundirla, mediante una campaña. Reflexionaron sobre la importancia de incluir en su campaña las cifras obtenidas, ya que habían identificado un alarmante decrecimiento en la cantidad de hectáreas de bosque. Decidieron comparar estas cifras con el área de estadios de fútbol, de manera que fueran más significativas para la población.

E2: Aunque sí bajó, está bajando muy poco, y aunque es una diferencia que se escucha como 0.01%, en territorio son 2,000 ha, que son no sé qué tantos estadios de fútbol.

Adicionalmente, las estudiantes consideraron que para que la información generara un mayor impacto, las personas debían relacionar las cifras con acontecimientos negativos. Por ejemplo, debían conocer el tiempo restante para que los recursos forestales se agoten.

E2: Podríamos sacar estimaciones… [para mostrar que] podríamos perder en tanto tiempo toda nuestra diversidad forestal. Como el reloj de NatGeo, creo, o GreenPeace. Ellos tienen un reloj biológico de que en tantos años llegaremos al punto de no retorno… Que sean datos más fáciles de digerir para el público.

Fase d

Después de la sesión virtual síncrona las estudiantes refinaron su carta como tarea extraclase. En ella se observaron las dos reflexiones siguientes.

- Una comparativa de las hectáreas perdidas en promedio en el 2021 en relación con el área de una cancha de futbol soccer $(100m * 50m)$, que corresponde a $34{,}208$ canchas $(171{,}040{,}000m^2 / 5000m^2)$
- El comunicado del reloj climático en Nueva York para recordar el tiempo que nos queda para actuar, antes de que los cambios climáticos nos afecten irreversiblemente

El conocimiento matemático utilizado en las fases c y d siguió siendo razones, proporciones y porcentajes, conocimientos asociados a la función lineal; también realizaron estimaciones para describir el decrecimiento del territorio forestal en términos de estadios de fútbol. Las estudiantes en cada paso autoevaluaron sus procedimientos, lo cual se relaciona con el principios de autoevaluación (Sevinc & Lesh, 2018). Las alumnas tenían claro qué era lo que estaba cambiando (la cantidad de hectáreas de bosque), cómo estaba cambiando (constante y de manera decreciente) y cuánto estaba cambiando (0.01%). De acuerdo con investigadores como Carlson et al. (2002) saber qué cambia, cómo cambia y cuánto cambia son elementos fundamentales que permiten a los estudiantes profundizar en el concepto de variación y de función.

Discusión y conclusiones

Respecto a la pregunta de investigación ¿qué conocimiento matemático sobre la función lineal, habilidades y reflexiones exhiben los estudiantes al enfrentarse a la MEA "Deforestación en Jalisco"? En el primer modelo se observó el uso de razones, proporciones y porcentajes así como habilidades para buscar información en internet, identificar datos y relacionarlos mediante comparaciones multiplicativas para describir la deforestación. En el segundo modelo, el conocimiento matemático utilizado fue de nuevo razones, proporciones, y estimación. Las estudiantes identificaron una variación decreciente. Las reflexiones sobre la problemática y la necesidad de hacer una campaña de difusión fueron determinantes para abordar la situación y que surgiera en el segundo modelo la necesidad de realizar estimaciones para describir el decrecimiento del bosque en términos de estadios de fútbol.

Tal como menciona Lesh y Doerr (2003) las estudiantes durante el proceso de descripción de la situación lograron diferenciar e integrar información y datos, así como desarrollar un modelo el cual no solo contempló datos, sino también información sobre la problemática. La MEA "Deforestación en Jalisco" posibilitó que las estudiantes matematizaran de manera simultánea a su reflexión sobre la problemática, lo cual es importante ante el llamado de la UNESCO (2021) para apoyar la sensibilización sobre los problemas ambientales. El análisis cuantitativo permitió a las estudiantes corroborar la existencia de la problemática y pensar en acciones para afrontarla.

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MATH ANXIETY AND EMOTIONAL SKILLS: A PILOT STUDY IN MEXICO

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This research project aims to investigate the relationship between specific emotional skills and a student's ability to solve mathematical tasks in the presence of math anxiety. The study concentrates on primary school students, particularly those in the 3rd and 4th grades, as research indicates that math anxiety could appear in these age groups. Math anxiety hinders academic and professional development and has long-term implications for decision-making and daily activities. A pilot study using a mixed quantitative-qualitative methodology design was used to test various instruments to measure the involved variables. The sample consisted of 103 children from a school in Mexico City. This initial study is the first step towards conducting further research on a larger scale to determine the strength and direction of the relationships between emotional skills, the construct of math anxiety, and its effects on performance.

Keywords: Affect, Emotion, Beliefs, and Attitudes.

The construct of math anxiety and its dimensions

Learning mathematics can be challenging and stressful for many individuals, often leading them to avoid subjects related to numbers and problem-solving. This aversion can adversely affect professional development and the ability to manage everyday tasks (Ashcraft, 2002).

Math anxiety is a construct that can account for these blockages and negative attitudes toward mathematics. It is "a feeling of tension and anxiety that interferes with manipulating numbers and solving mathematical problems in everyday and academic situations" (Richardson & Suinn, 1972). The difficulties associated with learning mathematics are not solely due to learning problems but also negative emotional factors (Dowker et al., 2016; Mammarella et al., 2019). It has been observed that adults experience mathematical anxiety while performing simple numerical tasks like counting and estimating magnitudes. This suggests that the anxiety may have developed before the sixth grade (Maloney et al., 2015).

I conducted a systematic literature review on mathematical anxiety and its two defining dimensions to develop the theoretical framework. The cognitive dimension directly impacts executive functioning and is linked with learning mathematical concepts or topics that require greater utilization of working memory and attentional control (Ashcraft & Kirk, 2001). The emotional dimension, on the other hand, considers socio-emotional skills, such as self-regulation and perceived self-efficacy (Kaskens et al., 2020; Luttenberger et al., 2018), as well as intrinsic motivation, which can interact with mathematical anxiety (Karamarkovich & Rutherford, 2021).

There is a strong connection between the cognitive and emotional factors involved in learning mathematics and the generation of anxiety, which can affect mathematical performance (Barroso et al., 2021; Hembree, 1990; Namkung et al., 2019). This situation can result in poor academic performance (Rossnan, 2006), impacting students' interest levels, values, self-concept, and self-esteem (Gabriel et al., 2020). Therefore, developing specific emotional skills can provide children with the tools to manage anxiety while working on math tasks. Regarding the emotional dimension of math anxiety, a cyclical behavior has been observed wherein students experiencing continuous failures develop significant insecurity, which subsequently generates anxiety. Pekrun's (2006) reciprocal model explains this behavior, suggesting that control and value appraisals towards learning can predict academic anxiety, thereby affecting performance. The literature provides extensive support for using the Cognitive-Motivational Model of Achievement Emotions (Pekrun et al., 2018) and the Control-Value Theory (Pekrun et al., 2007) as theoretical frameworks.

Research Objective

The pilot study aims to evaluate the tools and techniques used and obtain initial results to address the research question: "How do specific emotional and cognitive abilities influence the relationship between math anxiety and mathematics performance?" Additionally, the study seeks to confirm the hypothesis that low emotional skills increase math anxiety and negatively affect math performance.

Method

An exploratory sequential design with a mixed methodology approach was implemented for the project, starting with a pilot study conducted at a private school in Mexico City. It included 103 students from third and fourth grades, comprising 71 girls and 32 boys, with an average age of 10.3 years and a standard deviation (sd) of ±0.65 years. The parents or tutors provided written consent for their participation in the study. Prior to the program, cognitive focus group assessed the duration and structure of the instruments, confirming their sustainability in terms of length and clarity. The program consisted of two sessions, conducted in separate days. The students were not informed about math anxiety to avoid bias.

Session one: Emotional Dimension. An instrument comprising five different tools to measure the abilities was designed using a Likert scale from 1 to 5 with emojis to answer. It was applied simultaneously to all students per group with a duration of 35-40 minutes:

1. Perceived Self-Efficacy: The Self-Efficacy for Self-Regulated Learning Scale, developed by Zimmerman, Bandura, and Martínez-Pons (1992).
2. Beliefs and Expectations Related to Mathematical Competence: The questionnaire developed by Wigfield and Eccles (2000).
3. Intrinsic Motivation: measured with the Spanish adaptation of the "Academic Self-Regulation Questionnaire" (SQR-A) developed by Conesa and Duñabeitia (2022).
4. Emotional Intelligence: measured with the Brief Emotional Intelligence Scale (BEIS-10) developed by Davies et al. (2010).
5. Mathematical anxiety was measured using the SEMA scale by Wu et al. (2012), translated into Spanish by Sánchez-Pérez et al. (2021), where responses ranged from no nervous to very nervous.

Additionally, general anxiety was measured to exclude students with trait anxiety using the Child Anxiety Scale (Gillis, 2003). Consequently, nine students were excluded from the study, and the sample size was reduced to 94.

Session Two: Cognitive Dimension. Children were individually assessed in a private room. The session included an interview with eight questions about their attitudes towards math, which took 5 minutes, and a performance test with ten items, which lasted 15 minutes. The test was developed in-house using items from the ENLACE standardized test (SEP, 2013). The sample size was reduced to 91 due to the absence of three students.

Interviews were transcribed and analyzed qualitatively. Responses were structured in tables to compare themes and students. Open and axial coding were conducted, and a comparative analysis observed differences and similarities. The emotional skills five-section instrument was scored and analyzed descriptively and statistically.

Results

I administered all tests to 91 students over the course of three weeks. Below is Table 1, which displays the main statistical data of the instruments and their results.

Table 1: Variables and Measurement Instruments Data

Variable	Mean (n=91)	Standard deviation	% students above average	Likert Scale Level	Cronbach's **α**
Self-efficacy	38.4	±6.58	52%	3.8	0.833
Perceptions and beliefs	46.3	±5.37	54.8%	4.2	0.823
Motivation	43.0	±6.81	52%	3.9	0.755
Emotional intelligence	44.1	±6.10	54%	4.0	0.788
Math anxiety	33.7	±10.94	40%	1.7	0.896

Based on the collected data, the participants obtained high emotional skills, with an average score of 4 on the Likert scale. On the other hand, the respondents reported a level of math anxiety, 1.7 on the Likert scale. In addition, the average score obtained on the math test was 7.4 out of 10, with a sd of ±1.52. It is important to notice that, in the case of math anxiety, the mean obtained is similar to the one achieved by Wu et al. (2012) in designing the SEMA scale, equal to 33.79 points (sd = ± 10.22) and by Sánchez-Pérez et al. (2021) equal to 27.92 points (sd = ±8.08) for 3rd-grade and 29.15 points (sd = ±8.54) for 4th-grade children.

A correlation matrix analysis using Spearman's correlation coefficient identified significant associations between factors related to math anxiety. Self-efficacy, perceptions and beliefs, motivation, and emotional intelligence were negatively correlated with math anxiety, indicating that higher emotional skills are associated with lower math anxiety. Specifically, self-efficacy ($\rho = -0.527$, $p < 0.001$), perceptions and beliefs ($\rho = -0.392$, $p < 0.001$), motivation ($\rho = -0.268$, $p < 0.05$), and emotional intelligence ($\rho = -0.244$, $p < 0.05$) showed significant negative correlations with math anxiety.

In addition, it was possible to identify certain concepts that posed challenges for students when solving problems. Third graders struggled with fractions and division reasoning, while fourth graders found problems involving fractions most difficult.

In the interviews, it was found that most students like mathematics but also feel nervous about it. The questionnaire revealed that only 9% of the students reported feeling nervous about math. However, during the individual interviews, students could differentiate between fear, nerves, and stress related to mathematics. Out of the 91 children who were evaluated, 33% reported that they did not feel any fear or nervousness. 43% of the children acknowledged feeling nervous, 17% indicated they felt stressed in different situations, and 7% admitted feeling fear. These results indicate that 67% of the children experience an emotion that generates nervousness or fear, whether termed nerves, stress, or fear.

The children perceive themselves as competent in performing mathematical tasks. Even when they found tasks difficult or their grades were not very good, their initial response was that they were good at math. Also, it was rare to find children who valued math for its real-life applicability or intrinsic pleasure, although there were exceptions as a 4th grader expresses: "...if you need a building, you can use them (math), for a video game as well." A pattern of disconnection with the practical relevance of mathematics also emerged, with children expressing that math problems seemed distant and unrelated to their personal experiences. A 3rd grader describes: "Reasoning is boring; they are not my problems."

Discussion

The pilot study collected quantitative and qualitative data about children's attitudes and perceptions towards math. Researchers spoke with the children before the study, creating a trusting environment that made them feel at ease in expressing their true feelings. The children showed interest and enthusiasm, sometimes admitting to not liking math.

Specifically, the situations that cause nerves in students are primarily related to exams, new class topics, difficulty understanding or knowing how to solve a problem or operation, time pressure or the required response speed, and the fear of failure. These findings align with what Hunt et al. (2014) report about the situations that can generate math anxiety. Some children have trouble paying attention in math class, especially if there is noise or their peers speak loudly. As Eysenck et al. (2007) note, attentional control is affected by the presence of math anxiety due to intrusive thoughts or external stimuli that cause worry and distract working memory.

According to the results, children find two-digit division, fractions, word problems, and reasoning the most challenging concepts in mathematics. This coincides with the literature suggesting that these concepts require more working memory and may increase math anxiety (Ashcraft & Krause, 2007). Also, it was observed that children tend to focus on basic mathematical operations and mechanization. This suggests that the mathematical topics covered in class may be limited to the basics, resulting in boredom and frustration. (Boaler, 2016). However, according to the beliefs and values appreciation results, students gave mathematics an average rating of 4.5 points. This indicates that they believe that mathematics is both useful and interesting, and they also consider it to be an important subject. So, one way to increase their interest in mathematics could be to link it with real-life problems that are relatable and relevant to them. This approach might help reduce their perception of the subject as difficult.

Upon investigation, emotional competencies such as perceived self-efficacy, intrinsic motivation, and emotional intelligence were found to be high among the children of the focal school. Scores averaged around four, indicating that the children are emotionally well-adjusted. This study's findings also show a negative correlation between mathematical performance and math anxiety, consistent with previous research (Ashcraft & Moore, 2009). This pattern suggests that a strong emotional foundation can alleviate math anxiety, improving students' mathematical performance. Therefore, the research hypothesis can be supported.

Limitations

Participants might have minimized their true math anxiety to fit social norms, possibly underestimating their actual anxiety levels due to desirability bias. Exam conditions were not included to avoid pressure biases. Although informal interviews were conducted to ease tension, they may not fully reflect real evaluation settings. Therefore, the findings should be interpreted cautiously, considering the differences from actual performance situations.

Conclusion

The pilot test was successfully conducted, allowing for the trial of all instruments with numerous students. Gathering their feedback was crucial for refining the final version, enhancing the quality of the content and the clarity of instructions and questions. The results from this pilot have strengthened the motivation to proceed to a scaling phase, aiming to broaden the research to include a more diverse sample with varying socioeconomic and academic backgrounds. This next phase will help verify whether a lack of emotional skill development is linked to a higher prevalence of math anxiety.

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EMERGENT BILINGUAL PRESCHOOLERS' SPATIAL AND SHAPE LANGUAGE IN STORYTELLING VERSUS SHAPE CREATION TASKS

LENGUAJE ESPACIAL Y DE FORMAS EN NIÑES BILINGÜES EMERGENTES DURANTE NARRACIÓN Y CREACIÓN DE FORMAS

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Past studies on children's spatial language have focused on block play or guided play with a focus on monolingual English speakers. In our study, we engaged preschoolers in shape composition guided play tasks with shape blocks. In particular, we present results for five emergent bilingual preschoolers who told stories using tangram puzzle pieces and played a tangram copycat game with a puppet. Results indicate that preschoolers' spatial language focused on different elements between the two tasks, and they code-switched their language mostly when naming shapes or when describing locations or the spatial features of their design.

Keywords: Geometry and Spatial Reasoning; Communication; Early Childhood Education.

As efforts to increase the diversity of STEM professionals continue, attention has turned to the importance of spatial reasoning, particularly with younger students (Taylor & Hutton, 2013). Much of our knowledge about preschooler's spatial reasoning comes from studies that focus on their engagement with blocks and puzzles, demonstrating a correlation between these experiences and their spatial development (e.g., Jirout & Newcombe, 2015; Levine et al., 2012; Verdine et al., 2014). Block and puzzle play naturally encourages the use of spatial language and terminology and gives students tangible experiences with spatial concepts (Ferrara et al., 2011; Levine et al., 2012). Further, these experiences facilitate the formation of mental representations and transformations of spatial relationships among objects in the world (Levine et al., 2012; Reifel, 1984). The use of pattern blocks and tangrams (blocks that emphasize mathematical shapes and are used in shape puzzles) offers additional opportunities for building and supporting spatial reasoning, while also promoting the composition of shapes and the application of shape language, underscoring the importance of language in these educational interactions (Clements & Sarama, 2009; Hallowell, 2020). This research engaged five preschoolers (emergent bilinguals), in two types of shape composition and guided play tasks with tangrams. Our study takes an initial step toward deeper exploration into language and shape composition and seeks to offer unique insights into the co-development of language and spatial thinking in emergent bilingual preschoolers.

Theoretical Framework

Composing Shapes and Spatial Language

In their analysis of spatial language across tasks with caregivers and children, Cannon et al. (2007) developed a system for analyzing spatial language across a series of dimensions, including spatial dimensions or sizes, shape names, locations and directions, deictics (e.g., this, that), orientations and transformations, amounts, spatial features or properties, and pattern words. Ferrara et al. (2011) compared caregivers' and preschoolers' spatial language depending on if they freely played with blocks, followed a set of directions to make a specific block structure, or played with a model made out of blocks. Caregivers and preschoolers used significantly more spatial language when building based on directions than when freely playing. Further, the preschoolers tended to use more location and deictic words across all conditions, while shape and orientation words were used the least (Ferrara et al., 2011).

Instead of having students work with caregivers, Cohen and Emmons (2017) investigated monolingual English students' (ages 4-5 and 8-12) language while they built predetermined types of block structures. For example, they might have to build something a certain number of inches wide and tall with a specific number of blocks. Similar to results found by Ferrara et al. (2011), students used location words the most; however, their second most references were to continuous amounts. Further, students rarely used shape and orientation words, and they also did not refer to spatial features much.

Use of Storytelling in Mathematics

In her analysis of storytelling in mathematics, Rodríguez (2007) asserts that storytelling in mathematics initiates early in childhood, influenced by cognitive development, language modeling, and expressive opportunities. She suggests that storytelling concurrently enhances literacy and mathematical skills by providing structured narratives with recognizable characters and language patterns, fostering crucial imagination and abstraction. This method in math education offers various benefits including contextualizing concepts, fostering connections, developing competencies, and motivating young learners aged 3 to 6.

Furthermore, Julca Fernández (2019) utilized storytelling alongside tangram puzzles to assess geometric skill development in Peruvian kindergarten students. Twenty participants were equally divided into experimental and control groups. Following the story presentation, experimental group students employed tangram shapes to recreate story scenarios. While pre-test results indicated moderate geometric skills in the experimental group, post-test outcomes demonstrated significant improvement, particularly in recognizing, representing, and distinguishing two-dimensional shapes, along with problem-solving related to object positions and movements. Notably, advancements were observed in identifying and graphically representing object positions relative to others.

Code Switching and Translanguaging

English immersion programs often adhere to the "two solitudes" assumption, favoring exclusive instruction in the target language (L2) while overlooking students' first language (L1) (Cummins, 2008). However, extensive research by Baker (2011), Creese & Blackledge (2010), Cummins (2008), and Lindholm-Leary (2006) underscores the transferability of literacy-related skills across languages, particularly affecting literacy development and math education. Recognizing emergent bilingual students' oral language development in STEM subjects is pivotal for favorable academic outcomes, yet their language practices are frequently marginalized, stifling their authentic voices (Flores, 2013). To counteract this, Flores advocates for embracing the fluidity of language to mitigate "linguistic Othering." This project adopts an ecological perspective on multilingualism (van Lier, 2004; Hornberger, 2002) and the Dynamic Model of Multilingualism (Herdina & Jessner, 2002), viewing emergent bilingualism as a cohesive dynamic system rather than distinct populations. Furthermore, García's (2009) notion of languages as dynamic processes informs the exploration of how translanguaging strategies correspond to mental imagery, spatial analysis, and verbal explanations in shape composition and embedding among emergent bilingual students.

Translanguaging acknowledges the utilization of various modalities and semiotic resources beyond linguistic signs, facilitating the "resemiotization" process (Iedema, 2003), where actions are reconceptualized across different semiotic modes, perpetually generating new meanings. Adopting a transmodal approach, which involves the movement of ideas across different communication modes (Murphy, 2012), challenges traditional cognitive-centric views of learning by recognizing the holistic nature of language acquisition, incorporating physical, emotional, linguistic, and artistic dimensions. This perspective promotes a dynamic construction of meanings through simultaneous engagement with multiple modalities, allowing learners to flexibly navigate diverse modes and enhance adaptability and expressiveness in constructing and conveying meanings (Newfield, 2014).

Current Study

We build on prior research by focusing on preschoolers' spatial language when working with two different types of shape composition tasks. In particular, we focus on emergent bilinguals whose home language is Spanish, Mandarin, or other languages. Our research questions include the following: (1) How do preschoolers' use spatial and shape language in a storytelling versus shape creation task? (a) In what ways does background knowledge support their explanations? (b) How does code switching play a part in emergent bilingual use of language in the tasks?

Method

Participants and Setting

We recruited 18 preschoolers between the ages of 3 and 5 from a midwestern city, targeting preschoolers who speak home languages known by our researchers: English, Spanish, or Mandarin. We chose 5 of these preschoolers to focus on to represent those whose home language is Spanish (2 preschoolers), Mandarin (2 preschoolers), and Hindi (1 preschooler). Notably, of the six researchers, two speak Spanish, two speak Mandarin, but none speak Hindi.

Preschoolers whose home language was Mandarin and Hindi were part of a daycare program that offered an English-only curriculum. Preschoolers whose home language was Spanish were part of a daycare that centered on free-play. Because the Spanish-speaking preschoolers were not yet part of an English-only curriculum, they resorted to speaking Spanish. To investigate a wide range of language situations, the researchers used English and the preschoolers' home language in different ways. For instance, our Spanish speaker researchers would begin implementing the tasks in Spanish but would switch to English if the preschooler would begin using English. They also would use English strategically if they suspected the preschooler did not know the Spanish vocabulary. In addition, both researchers speak a different variety of Spanish (Argentinian and Mexican) and adjusted the language to the one used by the preschooler. Our Mandarin researcher also used languages strategically. She used both English and Mandarin, often switching to Mandarin if the preschooler was not verbally responding.

Design and Materials

Overall, we worked with each preschooler across four sessions. In Session 1, we worked with them individually on tasks to help us understand their spatial reasoning and initial strategies and language around composing shapes and solving tangram puzzles (Figure 1). In Session 2, they worked with a partner to make different shapes from the tangram pieces and play a copycat game where one child would make a design and explain to the other child how to make it. In Session 3, preschoolers listened to the story *Three Pigs, One Wolf, and Seven Magic Shapes* and created their own version of the story by making additional objects from the tangram pieces to protect the pigs from the wolf. In Session 4, preschoolers did similar activities as in Session 1.

Figure 1: Pieces of a Tangram Puzzle

The focus of this paper is on the storytelling activity in session 3 and puppet task from session 4. For the storytelling task, preschoolers listened to the story *Three Pigs, One Wolf, and Seven Magic Shapes.* The Mandarin and Hindi speaking preschoolers heard the story in English, as was typical at their daycare. However, the Spanish speaking preschoolers wanted to hear it in Spanish, so the Spanish researchers translated it while reading. In this story, three pigs receive tangram pieces from a "magical creature" that instructed them to create an object to protect themselves from the wolf. The first and second pigs create a "cat" and a "candle," respectively, and are not successful at protecting themselves. The third pig, however, defends himself from the wolf by constructing a house and later sails away with his wife by creating a "sailboat" using the magical tangram pieces. At the end of the reading, the preschoolers used the tangram pieces to create more objects to protect more pigs from the wolf.

For the "puppet task," preschoolers created a design using the tangram pieces and then explain to a hand puppet (controlled by a researcher) how to copy their design. The puppet asked them prompting questions in order to do its design. In both tasks preschoolers were able to use whichever language they felt most comfortable, which resulted in researchers sometimes switching between languages to accommodate the preschoolers' linguistic choice. At times, some of our English-only researchers worked with emergent bilinguals, in these situations preschoolers were only using English during their explanations.

Analysis

To analyze the language the children used during the tasks, we used the framework developed by Cannon et al. (2007) but added additional codes. During our analysis, we nested their *Spatial dimension* and *Spatial features* codes into one *Spatial Features/Properties* code to better capture any reference in our particular tasks that would be intrinsic to the properties of the shapes we gave the preschoolers. To capture names that they gave to either individual shapes or composites of shapes that did not align with standard 2D or 3D shape names, we created a *background knowledge* code. This code also captured additional details preschoolers provided about the shapes that were descriptive (e.g., it shoots fire) but not standard *shape features*. We also added subcodes within each category so that we could capture their gestures, corrections, and references to the materials we used (e.g., storybook, paper with colored blocks). Because we were working with emergent bilinguals, we also included *code switching* and *translanguaging* codes to indicate when they switched between languages and/or used a dynamic and transmodal use of language. Finally, we included *leading* and *misleading* codes to help us locate points where the researchers were prompting for more information either by asking directly or purposely saying or doing the wrong thing in order to get preschoolers to correct them with additional language. See Figure 2 for descriptions of the codes.

We coded the first transcript together, then transcribed and coded all other transcripts, and went back through them together to address any questions. We coded the transcripts using a computer assisted qualitative data analysis software program (Dedoose). We included a coding scheme drawn from Cannon et al. (2007) as well as themes emerging from the data in the codebook. The codebook included 15 major categories and 41 subcodes.

Code	Description and Examples
Background knowledge	Participant uses background knowledge (experiences) to retell or explain. Makes connections with their everyday world / self
Code-switch	Participant alternates between two or more languages or varieties of language in conversation
Translanguaging	Participant draws upon different linguistic, cognitive, and semiotic resources to make meaning and make sense
Continuous amount	Words that are used to describe amount (including relative amount) of continuous quantities (including extent of an object, space, liquid, etc.).
Deictics	Words that are place deictics/ pro-forms (i.e., these words rely on context to understand their referent).
Location/direction	Words that describe the relative position of objects, people, and points in space.
Shapes	Words that describe the standard or universally recognized form of enclosed two- and three-dimensional objects and spaces.
Spatial features/ properties	Words that describe the size of objects, people, and spaces (e.g. small, large, big) *AND* the features and properties of 2D and 3D objects, spaces, people, and the properties of their features (e.g., round, bump, strong).
Spatial orientations/ transformations	Words that describe the relative orientation or transformation of objects and people in space.
Pattern	Words that indicate a person may be talking about a spatial pattern (e.g., big, little, big, little, etc. or small circle, bigger circle, even bigger circle, etc.).
Gesture	Participant moves of part of the body, especially a hand or the head, to express an idea or meaning (e.g., pointing)
Leading	Investigators prompted for more information by asking directly in order to get students to respond using additional language
Misleading	Investigators prompted for more information by purposely saying or doing the wrong thing in order to get students to correct them with additional language
Self-correct	Participant gives wrong answer and self-corrects with correct answer
Use of book/mat	Participant uses the storybook, or the 4-color mat provided in order to complete or add more information to his/her explanation

Figure 2: Language Codes (based on Cannon et al., 2007)

Results

The preschoolers' use of spatial language varied drastically between the two tasks (Storytelling and Puppet) we analyzed. First, we present results for preschoolers' language use in the two tasks, highlighting specific examples of their language use. Finally, we discuss results related to their code-switching and translanguaging.

Storytelling Task

In line with the descriptive nature of the storytelling task, the preschoolers mainly focused on using names for the shapes that were not standard shape names but reflected their *background knowledge* of objects (Rodríguez, 2007). For example, Mister used the shapes to make a house to protect the pig. She then pointed to the parallelogram and said, “This is a chimney.” Next, she pointed to the medium triangle and explained, “This is the roof” (See Figure 3A).

Preschoolers also discussed *spatial features* of their shapes more in this task, often to explain the objects that they created. Rosa did this when she decided to create “*manos fuertes*” (strong hands) to defend the pig from the wolf, later adding that she wanted to make “una mano larga” (a long hand) and then “una mano gigante” (a gigantic hand) to her spatial features descriptions. When asked by one of the researchers to describe how her design looked like a hand (prompting her use of *background knowledge*), Rosa replied by stating that the shapes were “*los huesos*” (the bones) while the invisible/transparent part of her design (“*transparente*”) was the actual hand, a feature that would allow the pig to trick the wolf who would be unable to see the hand approaching (See Figure 3B).

A. B. C.


Figure 3: Mister’s (A) and Rosa’s (B) Storytelling Pictures and Snow’s (C) Fox for Puppet

Puppet Task

In the puppet task, compared to storytelling, preschoolers more frequently utilized specific shape names, location words, and language related to spatial orientations to guide the puppet in replicating their design. They complemented their spatial language with gestures for clearer expression. They recognized all shape names from session 1, except for the more complex parallelogram. For instance, when prompted, "What’s this?" Snow responded, "The big one, the big triangle" while pointing to it.

They expressed abundant spatial feature words to explain their pieces and designs. For example, they used dimension words like small, medium, big to distinguish different sizes of triangles as well as color words. For example, Snow said, “This orange one goes here.” It's noteworthy that the researchers’ leading language included more deictics in this task too, which may have encouraged the students to speak more deictically. In fact, while they used fewer spatial orientation words, many of their orientation instructions combined deictics with gestures.

Some preschoolers used numerous location words. For example, Mister utilized a 4-color mat to clarify locations to the puppet, stating, "You need to put it at the middle" (glancing at the mat), then correcting to "At the bottom…no…at the top…" finally deciding on "No, at the middle top," and "put next to…" followed by "you make closer to these."

Their use of deictics was rich in spatial meaning, aiding the puppet's understanding of their references, going beyond just spatial orientation and location. They were also more likely to draw on deictics as they tried to get the puppet to move shapes “here” or “there” or “like this” or “that way.” When referencing specific shapes such as "parallelogram," they used "this" along with gestures to enhance their instructions. For example, Snow made a “fox” design for the puppet to copy. She placed the parallelogram and two big triangles and then said, “Then the two small triangles, one goes here, and one goes like this” (she used her finger to draw the outline of the triangle's orientation). She used "small" to distinguish which triangles to use but used deictics to help explain their location and orientation (along with hand gestures, Figure 3C).

Preschoolers' exploration of spatial descriptions was not confined only to the manipulation of the tangram shapes. Rather, they demonstrated a heightened awareness of positioning in reference to themselves and others. For instance, Rosa guided the puppet (handled by R3) on where to stand even before beginning the task. Employing deictic expressions such as "*párate aquí*" (stand here), Rosa showed her proficiency in conveying spatial instructions to have the puppet move next to her, thus sharing her vantage point. Additionally, she displayed her knowledge of location and direction by instructing the puppet to move "*más cerca*" (closer) when dissatisfied with the position chosen by R3 for the puppet.

Language Features

Code switching. We decided to identify as *code-switching* every time preschoolers alternated between two or more languages or varieties of language in conversation. For instance, Figure 4 shows part of the storytelling task when Mister is describing the houses she created to protect the pigs from the wolf. Here (Lines 4-6), Mister describes her houses as a strong house and a stick house. R4 (Lines 7-10) wants Mister to be more specific about what she means by "stick house" and prompts her to use and think in Chinese. Mister responds "wooden house" which provides more detail as to the way the house is built and adds to the *Spatial features/ Properties* of her design.

Line	Speaker	Original Transcript	English Translation
L1	Mister	I make three houses	
L2	R4	You want to make three houses to	
L3		protect them?	
L4	Mister	This is the strong house [she	
L5		pointed to the orange one], and this	
L6		is the stick house, and this is...	
L7	R4	What is the stick house? Try use	
L8		Chinese, do you know what the	
L9		Chinese name of that is?	
L10		What is stick?	
L11	Mister	木房	Wooden House
L12	R4	木房子	Wooden House
L13		What about this one? [R4 points to	
L14		the third set of the tangram]	
L15		What's your plan?	
L16	Mister	Brick house	

Figure 4: Wooden House

Translanguaging. Just like we coded for instances where speakers alternated between languages, we also identified instances where translanguaging instances were observed, where preschoolers drew upon different linguistic, cognitive, and semiotic resources to make meaning and make sense of the world around them.

Line	Speaker	Spanish Transcript	English Translation
L1	Olivia	*Aquí en donde están* ***los triangles***	Here where the triangles are
L2	R3	*¿Quieres decir abajo? ¿Así?*	You mean down here? Like this?
L3		*¿O así?*	Or like this?
L4	Olivia	*Así. Vuéltalo [sic]*	Like this. Turn it.
L5	R3	*Vuéltalo [sic]*	Turn it.
L6		*¿Así?*	Like this?
L7	Olivia	*Y... y...* ***los triangles***	And... and... the triangles
L8		*El... el* ***medium triangle*** *aquí*	The... the medium triangle here
L9		*y los pequeños aquí*	and the small ones here
L10	R3	*El* ***medium triangle****, ¿aquí?*	The medium triangle, here?
L11		*¿Así?*	Like this?
L12	Olivia	*[She gestures "no" with her head]*	*[She gestures "no" with her head]*
L13	R3	*¿Cómo hago?*	How do I do it?
L14	Olivia	*Así* *[She uses the triangle from her design and moves it below R3's design]*	Like this *[She uses the triangle from her design and moves it below R3's design]*

Figure 5: "Los Triangles"

Figure 5 shows how Olivia used both *code-switching* and *translanguaging* in order to explain to the puppet (handled by R3) how to copy her design. We observed that Olivia *code-switched* between Spanish and English mostly when referring to *Shapes* (Lines 1, 7, 8) and *Spatial Features/Properties* (Line 8). Since Olivia does not receive any formal Math instruction in her daycare, we are uncertain as to how she has acquired her English vocabulary, and we hope to be able to explore this further by performing parent interviews regarding home practices. Nevertheless, Olivia would display *translanguaging* elements by making use of semiotic resources, such as specific hand gestures and movements [Lines 12 & 14], in order to make meaning and be able to better express her answers. Therefore, to comprehend the task and communicate effectively, Olivia would not only utilize her linguistic/bilingual abilities but would also draw upon her complete range of semiotic resources.

Discussion and Implications

In examining preschoolers' engagement with storytelling and puppet tasks, we observed distinct uses of spatial language. The storytelling task saw the creative labeling of shapes based on background knowledge, enriching narratives with imagination. In contrast, the puppet task saw a more precise use of spatial language, with children employing specific shape names and location words, often accompanied by gestures, to guide a puppet in replicating a design. This contrast underscores the adaptability of young learners in applying spatial reasoning and language differently across contexts, moving from imaginative storytelling to detailed, instruction-based communication.

Overall, preschoolers' use of spatial language was closely aligned to the perceived needs of the tasks. As found by Ferrara et al., (2011), our preschoolers did not use shape names as much when storytelling because they were more focused on describing the resulting object (coded as *background knowledge*) that they composed with the shapes, which falls in line with the descriptive nature of the storytelling task (Rodríguez, 2007). Further, they named *parts* of their objects using this background knowledge rather than the individual shape names (e.g., chimney instead of parallelogram). This suggests that investigations into spatial language need to account for these contextual differences. At the same time, the preschoolers were very excited to share about their objects, leading them to provide more context for the object and description of the spatial features. Therefore, their background knowledge supported additional language use, even if the language did not focus on specific spatial terms. Future storytelling tasks could encourage more shape and location language in two ways. First, we could modify the story to include more language about which shapes the pigs used and how they composed them to make the objects to defend themselves from the wolf. Second, we could have preschoolers narrate a manual for creating their object so that other pigs could make the same thing. Similar to the puppet task, creating a manual could encourage preschoolers to use more location and shape names.

In this study, we also adopted a dynamic and fluid perspective on bilingualism, framing translanguaging as a bilingual approach focused on observable practices where linguistic features from different languages intertwine (Garcia, 2009). Translanguaging encompasses various discursive practices through which bilingual individuals navigate their linguistic realities, surpassing mere code-switching and translation, and involving the performance of bilingualism across diverse modalities (Garcia, 2009). Consequently, we categorized instances of *translanguaging* as any occurrences where preschoolers utilized different linguistic, cognitive, and semiotic resources to construct meaning and interpret their environment. Throughout our project, we observed numerous instances of *code-switching* and *translanguaging* practices among emergent bilingual preschoolers, particularly when paired with researchers who shared their native language. Preschoolers would often switch between languages when naming or describing shapes or designs, with researchers following their lead by continuing in the chosen language. Alternatively, researchers sometimes encouraged the use of the preschoolers' native language if they encountered difficulty in naming or describing something specific. This approach is significant, especially in language immersion settings like ours, where preschoolers are typically encouraged to respond only in English. By adopting a *translanguaging pedagogical lens* (Cenoz & Gorter, 2022), we aimed to soften the boundaries between languages and support the development of all languages utilized by the preschoolers.

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MATH IDENTITY AND MATH APPS: WHAT IN COMMON?

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In recent years, mathematics app use (e.g., DreamBox, Prodigy, Reflex) has continued to become more prevalent in K-12 and most math app research has examined achievement-related outcomes. The present study is part of a larger exploratory multiple-case study examining students' mathematical identity as it relates to math app use. Utilizing Cribbs et al. (2015) framework for math identity, I analyzed one student's math identity separated from technology (math identity) and related to technology (math technology identity). This paper introduces the case of Sarah, a third grader whose math identity, particularly her view of math, is impacted by her weekly use of math apps.

Keywords: Technology; Online and Distance Education; Affect, Emotion, Beliefs, and Attitudes.

Introduction

In recent decades, the adoption of blended learning (BL) has increased both nationally and globally (Barbour, 2018). BL has evolved into a comprehensive term encompassing programs that incorporate online learning (Hrastinski, 2019). Among the technologies commonly employed in BL programs, math apps have become particularly prevalent (Cleveland-Innes, 2018) and have been acknowledged as an influential part of learning mathematics (Griffith et al., 2020; Laato et al., 2020). Math apps have gained substantial popularity worldwide with over 14 million students using IXL worldwide (IXL, n.d.), one in four elementary students in the U.S. using Zearn (Zearn, n.d.), and six million students in the U.S. engaging with DreamBox (DreamBox Learning, 2022). A 2022 national survey of U.S. teachers revealed that more than a third of the additional instructional materials used for teaching math were comprised of math apps (Doan et al., 2022). Existing literature indicates that identity is an important aspect of students' learning experiences as it impacts their success and well-being in the classroom, and identity "play[s] a fundamental role in enhancing (or detracting from) [students'] attitudes, dispositions" (Bishop, 2012, pp. 34-35; National Research Council, 2001; McCarthey & Moje, 2002). Presently, there exists only one research paper on the relationship between math identity and math apps. Crossley et al.'s (2020) yearlong investigation of upper-elementary students' use of the math app *Reasoning Mind* revealed no substantial evidence indicating a change in math identity throughout the study period. This suggests that (a) little is known about the ways widely used math apps such as Reflex and Prodigy may influence identity, and (b) consequently, my proposed study can fill an important gap in the literature. The following research question guided this study: What is the relationship between math apps and a student's mathematical identity? I now describe my theoretical framework and the literature that guided my study.

Literature Review and Framing

The focus of the current study is to examine the relationship between math app technology and students' mathematical identities. My working hypothesis is that technology, a key part of students' learning environment, impacts the way students relate to mathematics and others in their class. Thus, technology impacts students' mathematical identity development and formation. It follows that I view identity as situated—that is, identity is a product of being, learning, and interacting within one's environment (Holland et al., 2001; Lave & Wenger, 1991; Wenger, 1998). This view of identity is also related to one's membership in a community (Lave and Wenger, 1991; Wenger, 1998) and the shared experiences of this group. Thus, I utilize Cribbs et al.'s (2015) framework for mathematics identity and view of mathematics identity as composed of the three interrelated factors of interest, recognition, and introspection (Figure 1). I chose Cribbs et al.'s identity framework because a key component of their framework focuses on how students perceive others to view them in relation to mathematics (Recognition). Their framework not only places an emphasis on recognition as a key factor of mathematics identity, but it also explicitly links Recognition to students' views of themselves related to mathematics—what is labeled Introspection in Figure 1—and students' view of math (what is labeled Interest in the figure).

In utilizing Cribbs et al. (2015) framework for mathematics identity, I use their factors of interest and recognition without any alterations or modifications to how they define these constructs (see Figure 1). However, I combine their factors of competence and performance, into one factor that I rename Introspection. Cribbs and colleagues define competence as "students' beliefs about their ability to understand mathematics" (p. 1051) and performance as students' "beliefs about their ability to perform in mathematics" (pp. 1051-1052). I combine Cribbs et al.'s factors of competence and performance into the single term Introspection for simplicity and ease of operationalization and broadly define this term as a student's view of themself related to math.

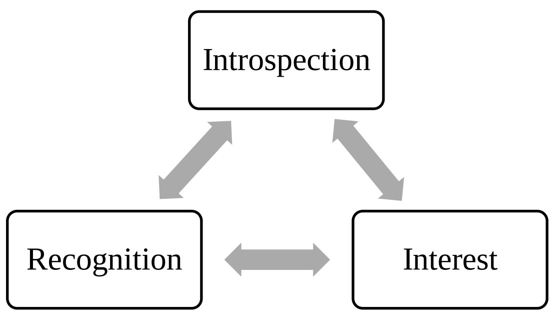

Figure 1. Cribbs et al. (2015) mathematics identity framework with Introspection in place of competence/performance.

Methods

This preliminary report was part of a larger exploratory multiple-case study that examined third graders' math identity and motivation related to their weekly use of math apps. The multiple-case study consisted of eight cases that were carefully chosen from a pool of participants in a classroom, each representing diverse mathematical identities and motivations. The focus of this preliminary investigation was on Sarah's case due to her insightful reflections and the revelatory nature of this case (Yin, 2016). Particularly, Sarah clearly articulated how the different utilized math apps related to her view of math, view of self, and how others viewed her while other third graders had a harder time expressing their math identity as it related to math app use. The two math apps utilized every week in Sarah's class were Reflex and Prodigy. Sarah's classroom is a part of X Elementary School, a large public elementary school located in the southern United States. X Elementary School has an economically disadvantaged student enrollment of 16% and over 70% of the school's students scored at or above the proficient level on the state math test. Drawing on the three components of math identity outlined by Cribbs et al. (2015), I used interest, introspection, and recognition as a priori codes (Miles et al., 2014) to analyze Sarah's responses to two sixty-minute semi-structured interview protocols (Rubin & Rubin, 2011). The first interview took place in the fall of semester 2023 while the second interview was conducted in the spring semester of 2024. Using interest, introspection, and recognition to serve as column heads of the math identity profile I created for Sarah, I analyzed her two interviews by creating a meta-matrix (Miles & Huberman, 1994) to keep track of her math identity separated from technology (math identity) and related to technology (math technology identity).

Findings

I start by summarizing Sarah's math identity and math technology identity (see Table 1). Several features of the math apps Reflex and Prodigy afforded varying opportunities for self-recognition, recognition, and shaping Sarah's view of math. In the sections that follow, I further explore Sarah's math identity and her math identity related to math apps.

Table 1. Overview of Sarah's Math Identity and Math Technology Identity

	Sarah's Math Identity		
	Interest	*Introspection*	*Recognition*
Interview 1	Sarah felt math was frustrating when she struggled, but experienced joy from getting correct answers.	Sarah viewed herself as good at math and a mathematician, but "not the best person at math."	Sarah believed her teacher, friends, and parents view her as good at math, but was clear she would not be voted as the best at math.
Interview 2	Sarah believed doing math meant solving problems you may have not seen before.	Sarah expressed confidence in her ability to do math, but no longer viewed herself as a mathematician.	While Sarah felt many people in her class are better than her at math, her peers would still view her as good at math.
	Sarah's Math Technology Identity		
	Interest	*Introspection*	*Recognition*
Interview 1	Reflex and Prodigy made doing math more fun, but Sarah acknowledged there is little math on Prodigy.	Prodigy and Reflex did not affect how Sarah saw herself as a math student.	Prodigy has a multiplayer game feature, but Sarah felt seeing avatars made it harder to understand her classmates' emotions.
	Sarah felt frustrated that math	Sarah felt good about her math ability	Prodigy is an app Sarah does not play when

Interview 2	was too much of a game when playing the math app.	when playing Prodigy but felt stressed by the timer on Reflex.	given the change and feels math apps have no bearing over how she is recognized.

Sarah's Math Identity

Sarah's view of math, or interest, was tied strongly to her performance and success. When asked what it meant to do mathematics, she responded, "to be fluent on most of your addition and subtraction facts." She also described the "feeling when I finish, um, a math question of get it correct" as the moment that mathematics is most enjoyable. During the second interview, Sarah's view of math had shifted to reflect an algorithmic and computational view of mathematics similar to that of the math played on Reflex. Sarah said, "I do not like showing my work when I could do the algorithm easily." Her enjoyment was still rooted in her success, and she said, "I like solving math problems because I'm good at it." Sarah's view of herself, or introspection, remained largely unchanged from the first to the second interview. In both interviews, she expressed viewing herself as good at math but was quick to point out there were several students better than her. However, during the first interview, Sarah considered herself a mathematician, but by the second interview, this was no longer the case. While her grades indicated she was good at mathematics and gave her confidence in her ability, she no longer felt she was a mathematician. Sarah was recognized by her classmates, teacher, and parents as good at mathematics in both interviews. She noted that while her peers all recognized her as good at mathematics, they also wouldn't vote her as the class mathematician at the end of the year.

Sarah's Math Technology Identity

During the first interview, Sarah expressed the math apps as having a significant role in how she viewed mathematics. Prodigy and Reflex made doing math more enjoyable, but by the second interview, she seemed to no longer enjoy the math apps. Further, she was frustrated by how much they had transformed math into games, saying "Math apps just kind of make it too fun when math is supposed to work. I kind of like, I like work." Sarah's view of herself remained unchanged by doing math on Prodigy and Reflex. This was in part due to the math apps not "really having that much of an effect on learning," but also because of Sarah's unchanging confidence in her ability to do math. However, certain attributes of the math apps such as the problem timer on Reflex stressed Sarah out and caused her to feel more anxious about doing mathematics. A multiplayer game feature of Prodigy allowed students to recognize each other but for Sarah, the avatars made it harder to understand the emotions of her classmates. She said, "Some of the avatars, they're just frowny faces the whole time, and then like mine just has the same expression." Because of these virtual characters, Sarah felt Prodigy allowed for fewer opportunities to genuinely interact and be recognized by classmates.

Discussion and Conclusion

While these results are preliminary, they suggest features of math apps relate differently to components of a student's math identity. For Sarah, the gamified nature of Prodigy and Reflex was at odds with how she viewed mathematics. This caused her distress as she experienced dissonance between how she was experiencing mathematics and what she believed mathematics was. Sarah also detailed features of Reflex and Prodigy that related to her views of herself and how others viewed her, but she felt neither app impacted her learning. One limitation of this study is its singular focus on the math identity of a single student, which precluded a comparative analysis of math identity across multiple participants. Future investigations and reports might expand their scope to include a broader range of participants, such as additional elementary students or individuals from diverse grade levels. I view research concerning elementary students' math identity in the context of math app usage as a crucial yet insufficiently explored domain within K-12 mathematics education. Given the growing prevalence of such technology, it is essential to persist in exploring various facets of students' math identity.

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USING DESIGN-BASED RESEARCH TO EXPLORE HIGH SCHOOL STUDENTS' INFORMAL LEARNING OF MATHEMATICS

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As the demand for STEM jobs increases, central to the success of STEM education and careers is a strong foundation in mathematics. However, students' interest in mathematics is often very low. Thus, it is imperative to cultivate interest in mathematics among high school students. To promote students' interests and positive attitudes in mathematics, we implemented informal learning using design-based research (DBR). We show that DBR is a compelling and suitable methodology for our research aims. Then we report how DBR can extend from previous studies in using informal learning for mathematics and foster motivating learning ecology in a school setting. Our DBR project has completed four iterations.

Keywords: Informal Education; Design Experiments; Affect, Emotion, Beliefs, and Attitudes; High School Education

It is well known that global job growth will be mostly concentrated in the high-skilled areas of healthcare and STEM (McKinsey & Company, 2023). For example, the US Bureau of Labor Statistics (2023) projected a 15% overall growth of computer and mathematical jobs in the next eight years, with jobs in data science and statistics experiencing 35.8% and 32.7% increases, respectively. However, education statistics imply that the supply of mathematicians and scientists entering those fields may soon be insufficient to satisfy the demand. The awarded mathematics and statistics bachelor's degrees growth rate is significantly lower than other STEM fields, despite an increasing trend of the overall STEM fields. According to Digest of Education Statistics (2023, Table 322.10), over the past decade, the annual growth rate of awarded bachelor's degrees in computer and information sciences was 22 times higher than that of mathematics and statistics, engineering growth 12 times higher, and biological and biomedical sciences growth 9 times higher. The number of high school students completing advanced mathematics courses (i.e., calculus) declined in the decade of 2009-2019 (NCES, 2022). Thus, it is imperative to cultivate interest in mathematics among high school students, which will eventually align the number of college students pursuing STEM degrees with workforce needs.

Informal learning, a type of *less classroom-bound, free-choice education* (Falk, 2001), has recently gained traction for improving STEM learning and for improving engagement in mathematics (Denson et al., 2015; Pattison, Rubin, & Write, 2017; Waldock et al., 2016). The "informal" and "free-choice" characteristics of informal learning make it an ideal medium for delivering education in uncertain times, offering a "free-choice" approach to engaging with information and knowledge. Cultivating positive mathematics or STEM identities is often a central focus for designers of informal learning experiences (Bell & Bevan, 2015; Feder et al., 2009; Zimmerman & Bell, 2012). We suggest that design-based research (DBR) from the learning sciences is a compelling and suitable methodology for exploration of both informal learning of mathematics and its outcomes measurement among high school students. In this paper, we report how DBR can extend previous studies in using informal learning for mathematics and foster motivating learning ecology in school settings. To exemplify, we briefly report on an NSF-sponsored DBR project that completed four iterations.

Theoretical Framework and Research Aims

Drawing on theories of experiential learning (Kolb, 1984) and related to active learning programs that have been shown to increase performance and motivation in STEM (Freeman et al., 2014; Weinberg et al., 2011), informal learning, as a type of less classroom-bound and more free-choice education (Falk, 2001), includes a wide array of experiential learning instances that happen when students actively engage in learning opportunities outside of the traditional context of teacher and classroom. Much of the research on informal learning emphasizes identity, seeking to influence identity as well as to understand identity development (Bell et al., 2009; Pattison et al., 2017). Math identity is believed to be an important component of students' achieving success in mathematics (Allen & Schnell, 2016; Bohrnstedt et al., 2020; Gonzalez et al., 2020). Identity work can be conceptualized as a process of alignment, drawing upon Anderson's (2007) four-dimensional model of mathematical identity as well as Wenger's (1998) three modes of being – alignment, imagination, and engagement. Furthermore, studies have shown that peer and near-peer led activities have a strongly positive impact on students (Brownell & Swaner, 2010; Carrell & Sacerdote, 2013; Cracolice & Deming, 2001; Quitadamo et al., 2009; Trujillo et al, 2015; Williams, 2009). In our project, we combined the processes of mathematical identity alignment with the supporting structure of near-peer mentoring.

Differing from many existing educational studies on informal learning that focused on activities held mainly in certain out-of-school or after-school settings, we seek a design scheme or solution for the infusion of near-peer, informal learning of mathematics for high school students in the school setting.

DBR – Literature Review and Why

Brown (1992) defined DBR in her seminal paper, followed by many literature references to DBR, including earlier ones focusing on the "what" and more recent papers shifting to the "how" of DBR (Puntambekar, 2018). Extending Anderson and Shattuck's (2012) review of the potential of DBR, of the characteristics of good DBR studies, and of the growing popularity of DBR approaches in educational research, Fowler et al. (2022) reviewed DBR studies completed in the decade up to 2011. Beyond being a specific research method, DBR is an approach that centers a series of iterative (often educational) designs as the unit of investigation, and frequently employs mixed research methods and tools. Two recent studies (Hoadley & Campos, 2022; Scott et al., 2020) demonstrated DBR's implementations in online learning and biology education. Scott et al. (2020) summarized what DBR is and pointed out four differences between DBR and experimental approaches, which deserves readers' special attention because most researchers and scientists are well trained for experimental approaches rather than DBR method.

Why We Chose DBR?

Design-based research (DBR) from the learning sciences, although considered a relatively young (about three decades old) educational research methodology, is compelling and suitable for our research aims. DBR has no solid requirements of instructional intervention form or evaluation measurements. Rather, the intervention design as well as outcome measurements can be developed or employed during the design process (Anderson & Shattuck, 2012; Sandoval, 2014). DBR focuses on investigating what the design process is and how it can be generalized (Cobb et al., 2003). Due to these features of DBR, it is suitable for innovative research in certain learning scenarios, such as ours. Tailoring the intervention and implementation process needs multiple iterations instead of a one-shot deal. Furthermore, the DBR approach enables a flexible methodology that accommodates specific situations, proving resilient and robust even in highly uncertain times. Notably, our project commenced amidst the COVID pandemic, and the DBR approach facilitated the customization of each iteration to suit the unique circumstances of each time period, as well as the progressing of our research agenda.

Our DBR Project

Following preliminary explorations and a smaller scale pilot study (Wilson and Grigorian, 2018) showing that near peer interventions have the potential to positively affect attitudes to mathematics, we carried out an NSF-funded project on informal learning of mathematics. Over the course of three years, this project involved 1,258 students from four high schools in two majority-Hispanic school districts in South Texas.

DBR Iterative Redesign Process

The DBR iterative redesign process is visually represented in Figure 1. In each iteration, the evaluation of both the delivery of the experiences and the data collection processes provided insights that informed the subsequent iteration's design. This resulted in a continuous cycle of innovation, evaluation, and refinement that ensured the experiences and the associated research methodologies remained responsive and adaptable to the unique learning contexts and challenges encountered throughout the project duration. To illustrate our design process for informal learning of mathematics in a school setting, we summarize the iterations.

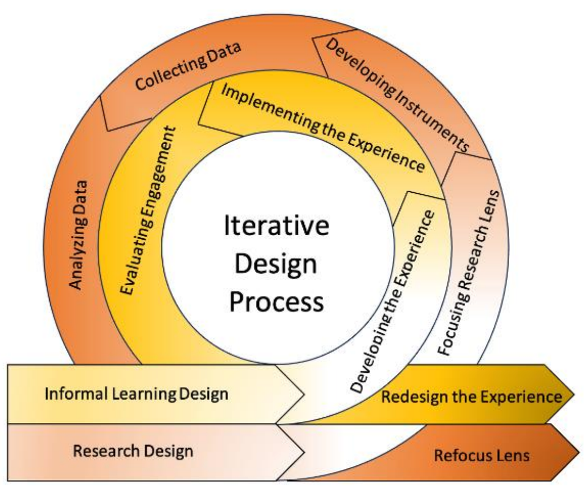

Figure 1: Iterative Informal Learning and Research Design Processes

The 1st iteration. We started in spring 2021 with fully online, synchronous MathShows presented to groups of classes via Google classrooms. The online modality was precipitated by COVID-19 pandemic restrictions. Viewing of brief pre-recorded video scenes produced by college student near-peers followed by live interactions with same near-peers via video and chat.

The 2nd iteration. An easing of COVID restrictions and resumption of in-person schooling required changes in the program design in Fall 2021. For this iteration, live, in-person MathShows were performed in a large school auditorium, with multiple classrooms in attendance simultaneously and combined viewing of pre-recorded video scenes and interactive activities with near-peer mentors.

The 3rd iteration. Live, in-person MathShows were performed in a large school auditorium, with multiple classrooms in attendance simultaneously. There were no pre-recorded video scenes, with more audience interaction, more prize opportunities for students, and more scripted acting by near-peers.

The 4th iteration. Live, in-person MathShows were performed by a smaller cast of near-peer mentors in individual classrooms, not in an auditorium. This allowed much more direct interaction between students and near-peer mentors, but each MathShow was shorter.

DBR-Iterative Instrument Design

DBR experiments are resource intensive (Scott et al. 2020). For the research aims of our DBR project, we collected large amount of qualitative and quantitative data via mixed methods. We hereby spotlight one instrument item for its iterative design process. During the 1st and 2nd iterations, as one of the main quantitative measures, this study used a mathematics identity survey item that was adapted from well-established attitude surveys. Students were asked to choose from a Venn diagram to describe how much they align with being a mathematician. In the focus group studies during the 1st and 2nd iterations, high school students shared their various perceptions of a mathematician. We followed up by asking them the reason for their response to the math alignment question. These qualitative studies revealed to us that when respondents saw the Venn diagram, the circle of "Mathematician" may have different meanings to them and also students have different reasons for making their choice. To capture these differences in perception, based on students' focus group input and using some of their exact words, we developed two novel items for surveys for subsequent DBR iterations to collect students' understanding of mathematician and reasons for their alignment choices.

Discussion and Conclusion

Our project shows that by employing DBR for designing and studying learning interventions, mathematics educators can develop both theory and practices for the informal learning of mathematics. For instance, the identity-measurement instruments developed in our DBR process exemplify how DBR invites utilization of mixed methods synergically. An example in this study of qualitative research informing quantitative research is that focus group interviews (qualitative research) captured students' perceptions of who a mathematician is. We then developed two more survey items (quantitative research) with choices written based on those high school students' words. On the other hand, as an example of quantitative research informing qualitative research, in later iterations, focus group studies consisted of participants pseudo-randomly recruited with a stratified sampling method based on certain quantitative data to ensure the inclusiveness of different types of students in the focus group. In addition, the design scheme developed in our DBR project is generalizable to broader learning settings. Middle and elementary schools are potential places for informal learning of mathematics. Moreover, math teachers may also be able to incorporate short and attention-catching informal learning components in their classrooms that nurture students' positive academic emotions. Similar expansion can be made to colleges as well.

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RELEVANCE FOR WHOM?: STUDENT REFLECTIONS ON RELEVANCE IN MATHEMATICS EDUCATION

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Keywords: Curriculum, Data Analysis and Statistics, Culturally Relevant Pedagogy

Choice of Problem

There have been calls for decades in mathematics education to make curriculum relevant to students (Civil, 2007; Gutstein, 2006; Ladson-Billings, 1995). In this study we focus on a case of written instructional materials created with the mission of providing "free, nonpartisan, relevant math lessons that prepare all students to think critically as citizens" (Skew the Script, 2024). The following research questions guide this investigation: 1) How do AP statistics students view the relevance of the issues investigated in Skew the Script? 2) How do AP statistics students view the partisanship of the topics investigated in Skew the Script lessons?

Theoretical Framework and Modes of Inquiry

Priniski et al. (2018), drawing from a wide range of theories including culturally relevant pedagogy, defines relevance as, "a personally meaningful connection to the individual" (p. 12) and exists on a continuum from personal association to personal usefulness to identification. We used this framing to understand how students view the relevance of lessons. 15 secondary AP statistics teachers and their respective statistics students were recruited with data collection conducted post-AP exam administration with 70 students responding. To gauge students' views of lessons, the survey also included items asking students to rate the level of relevance of each lesson with a 5-point Likert scale. To collect data on partisanship we asked students, "on a scale of 1 to 5 how would you rate the overall partisanship of the topics in the curriculum," and also "of the topics you covered in class which topics, if any, did you find right leaning, …left leaning?" We used exploratory data analysis and qualitative coding to analyses the data.

Findings and Discussion

The issues students found most interesting where those related to issues of race/racism and sports. There were followed closely by gun violence/control, then policing and politics. Several also explicitly mentioned anything relevant or related to the real world. When asked to rate the relevance of each lesson they experienced, a majority of students rated most lessons 'relevant' or 'very relevant.' No lessons were rated as not relevant by students. Only 12 of 70 students referenced a topic in response to being asked which ones they would consider right leaning, all others could not think of one. Of those found right leaning the only common topics were polling examples. In contrast, 35 out of 70 stated at least one topic to be left leaning. 51.4% of students rated the partisanship of the lesson overall as neutral (51.4% of respondents) with 40% reporting a left lean. The students in this study felt the wide scope of lessons, ranging from politics to the environment, were insightful and significant to their lives. In many cases, the topics were not directly related to them, but students still rated lessons such as these as relevant. This finding suggests that these students responded favorably to lessons that were both "windows" and "mirrors," to better understand both the world and themselves (Gutiérrez, 2012).

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EXPLORING DIALOGIC MATHEMATICS VIDEOS AS AN INSTRUCTIONAL TOOL FOR SUPPORTING SECONDARY SCHOOL STUDENTS' ALGEBRAIC REASONING

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Keywords: Algebra and Algebraic Thinking, Design Experiments, Instructional Videos

Researchers have positioned introductory algebra as a determinant of future academic success, and thus, it serves as a gatekeeper to more advanced courses in secondary mathematics (Stein et al., 2011). There is a general consensus among mathematics education researchers that algebra should move beyond rote symbol manipulation to a focus on concepts, and yet students still revert to procedural thinking (e.g., Christou et al., 2022; Liang & Moore, 2021; Tondorf & Prediger, 2022; Wilkie, 2022). This procedural emphasis is reflected in most instructional videos available for K-12 learning, which are similarly procedural in nature and lack student voices (Bowers et al., 2012; Klinger & Walter, 2022). One solution is to turn to an alternative video tool— the dialogic mathematics video—-that is conceptually oriented and features student-student interactions.

Empirical studies indicate that students evidence higher learning gains when viewing dialogic videos compared to expository videos (e.g., Chi et al., 2017). These studies demonstrate the potential for dialogic videos to enhance learning but have predominantly been situated in science contexts and in laboratory settings. In mathematics contexts, studies have utilized a qualitative approach but have focused on investigating the experience of students' engagement (Lobato et al., 2019, Lobato et al., 2023). There is a need to explore the role of dialogic videos as an instructional tool for learning in a classroom setting. For these reasons, this study aims to investigate how dialogic videos as an instructional tool influences students' reasoning in an algebra classroom context. Specifically, this study uses videos from Project MathTalk.

This study draws on the theoretical construct of instrumental genesis. Broadly, instrumental genesis (Artigue, 2002) is the process by which an artifact moves from being a human-made object to something that is meaningful to the learner. Within a mathematical context, a mathematical artifact becomes a mathematical instrument (i.e., the artifact becomes valuable and useful) as the learner develops the skill to express mathematical ideas with it. In this study, I interpret instrumental genesis to mean the mediation of an artifact into a meaningful instrument (à la Alqhatani & Powell, 2017). Hence, the guiding research question is: in what ways does the artifact of dialogic mathematics videos mediate secondary school students' ways of reasoning about algebraic expressions and equations?

The study employed a classroom teaching experiment methodology (Cobb, 2000) with thirteen 9th grade students. To capture the ways in which the video artifact was influencing student reasoning in the classroom, whole-class video-taped observations, whole-class inscription, and participants' written work was collected. Data was analyzed using a Thematic Analysis methodology (Braun & Clarke, 2006). The results of this analysis will report on themes that relate to specific ways that the video mediated students' reasoning during key moments in the teaching experiment. This study contributes to the literature on learning from dialogic videos by documenting the mediational role of dialogic mathematics videos in shaping students' reasoning. Understanding how these videos shape students' learning trajectories can inform instructional practices and curriculum development in secondary mathematics education.

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UNDERSTANDING FUNCTION TRANSFORMATIONS THRU TACTILE REPRESENTATIONS AND GESTURING WITH VISUALLY IMPAIRED AND SIGHTED STUDENTS

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This case study uses embodied cognition and universal design for learning to help three students, one with a visual impairment, understand the concept of function transformations. To accommodate students' learning, a group activity used tactile representational tools to give meaning to function translations on a coordinate plane. Results show that the learners constructed a conceptual and integrated notion of function transformations. In this paper, we illustrate how mathematical knowledge was understood and communicated through verbal discourse and gesturing and raise theoretical questions about embodied ways of knowing for both blind and sighted students in learning pre-calculus concepts. We propose these pedagogies be used with all students reflective of universal design principles.

Keywords: Algebra and Algebraic Thinking, Cognition, Students with Disabilities.

The use of interactive algebra and geometric systems to graph functions is well-researched in mathematics education to foster an integrated understanding of relationships between algebra concepts, graphs, and symbolic notation. While there has been progress in making this technology accessible to students with visual impairments (VI), learning experiences for students with VI are not equitable and often inadequate compared to opportunities for sighted students (Stone & Brown, 2023). There is a need to critique and rethink how to make mathematics accessible to all students using principles of Universal Design for Learning (UDL) in planning and implementing mathematics lessons (Abrahamson, et al., 2019; CAST, 2018). Mathematics educators have begun to understand mathematical experience as embodied; that is, mathematical ideas, physical objects, the environment, and our bodies are entangled in learning mathematics (de Freitas & Sinclair, 2014; Yu & Oslund, 2023) with gesturing as an enactment of embodied learning experience (Healy & Fernandes, 2011; Hostetter & Alibali, 2008). This case study looks at a lesson on algebraic transformations using tactile representational tools conducted with one student with VI and two sighted students in a collegiate precalculus course. In this paper, we report episodes from the lesson, describe students' emerging and integrated understanding of algebraic transformations on functions, describe the role of UDL, and connect these to the students' embodied learning experiences through tactile interactions and gesturing.

Theoretical Framework

This research study is grounded in Universal Design for Learning (UDL) (Gully, 2021) and embodied cognition (Alibali & Nathan, 2012; Healy & Fernandes, 2011; Lakoff & Nuñez, 2000). UDL principles include providing multiple means of representation, action, and expression (CAST, 2018); not all bodies have access to all representations or actions, therefore, multiple representations provide access to more students. The goal of UDL in mathematics classrooms is not to accommodate students with disabilities but to create learning environments benefitting all students (Gulley, 2021; Abrahamson et al., 2019). This can be particularly challenging for postsecondary instructors with limited exposure to UDL principles (Stone & Brown, 2023). In the case of students with VI learning precalculus mathematics, increasing use of computer algebra systems and interactive graphing technology poses a challenge given the visual acuity necessary. Abrahamson et al. (2019) suggest a reconceptualization of what it means to visualize that encompasses nonvisual learner experiences and provides necessary representational tools for students with VI. UDL considers how representational tools, visual, tactile, or technological, may benefit sighted students beyond the status quo curriculum and pedagogy.

Research has started to understand the mathematical learning of students with VI. Gerofsky and Zebehazy (2022) used motion to help elementary students with VI notice mathematically important features of graphs. Drawing from cognitive science research using fMRI techniques, Lakoff and Nunez (2000) suggest that visual systems in the brain are not restricted to visual input, stating, "...congenitally blind people, most of whom have the visual system of the brain intact, can perform visual imagery experiments perfectly well, with basically the same results as sighted subjects, though a bit slower," (34). They explicitly connect motor control and mathematical ideas; an *aspect schema* is a neural structure that controls both complex body movements and rational inferences about events and actions (Lakoff & Nunez, 2000). Of interest to this study are continuative, iterative, and imperfect aspect schemas (Lakoff & Nunez, 2000). In linguistics, aspect refers to expressions that denote temporal actions that may be perfect as in 'yesterday I rode my bike,' or imperfect as in 'when I was a kid, I rode my bike every day' (Smith, 1997). Specifically, we wondered how tactile representations of functions and students' associated gestures activate and mediate the aspectual nature of mathematics in a student with VI and sighted students. In keeping with the notion that bodies and their environments are not separate but influence one another (Chomney et al., 2019), we use *gestures* to refer to motions made by students as they learn, including those that involve manipulation of tactile materials. Research on gestures in mathematics education has sought to understand the relationship between gestures and mathematical understanding (e.g. Healy & Fernandes, 2011; Hostetter & Alibali, 2008). Hostetter and Alibali (2008) sought to understand mechanisms that give rise to gestures. They posit that gestures are based on mental images and actions. Nathan (2021) points out that gestures both reveal and influence the speaker's cognition. Lakoff and Nuñez (2000) described mathematics as arising from bodily experience. Although not specifically about mathematics, Johnson (2007) has posited that through movement we construct conceptual understandings, and the qualities of movement matter to the development of those concepts. The questions this study addresses are: (1) How does a UDL environment with tactile and other representational tools used in a novel way support mathematics learning for students with VI and sighted students? (2) How can the embodied notion of aspectual systems describe connections between physical action and conceptualization of mathematical ideas? (3) What role does gesture have in the communication and construction of mathematics in students with VI and sighted students?

Methodology

A naturalistic (Moschkovich & Brenner, 2000), multitiered teaching experiment (Lesh & Kelly, 2000) was conducted in an undergraduate precalculus course (n = 30) at a medium-sized Midwestern university. The professor of the precalculus course collaborated with one of the researchers, meeting once a week to discuss ways to provide instruction to a student with VI and sighted students using technology and manipulatives. The data consist of field notes and videos of whole class sessions, videos of individual student work sessions, and videos of selected small group sessions. The individual and small group sessions met once a week outside of regular class meetings and were led by the collaborating researcher, who assumed the role of teacher-researcher (Ball, 2000) and is identified as the instructor in this paper. There were nine video-recorded sessions, each session lasting approximately one hour.

This paper looks at one of the small group sessions, which was chosen for analysis because of the collaborative nature of the learning due to the use of the physical manipulatives and Instructor moderation of classroom discourse. This session included three STEM majors in the pre-calculus course (pseudonyms): Ian, Lexi, and Madelyn. Ian was a legally blind, male international student from Central Asia in his second year as a computer science major at the university. He was totally blind in one eye and had very limited vision in the other. Lexi was in her first year as an engineering major. Madelyn was in her first year as a secondary mathematics education major. The video was analyzed using iterative refinement cycles for video analysis (Lesh & Lehrer, 2000). The first iterative cycle was the transcription process. The second iterative cycle was a review of the video data with the transcripts by the first author to establish initial themes and time code those sections of video data for deeper analysis. The third iterative cycle was a review of the video and transcript data, by at least two researchers at once. The use of transcriptions with repeated viewing of the video, by multiple researchers, provided a means to obtain an increasingly reliable narrative (Gulley, 2021). This allowed for a deeper analysis of emergent themes, including the qualities of the participants' gestures, interactions with tactile materials, and the relationships between participants' movements and the group's discourse. For example, we noticed whether a student traced a tool with their fingers or tapped it into a certain position on their paper, the tentativeness or certainty of these motions, and how gestures either stood alone or were accompanied by verbal descriptions. The third iterative cycle was a review of video data by individual researchers to provide a deeper analysis of identified themes, including developing and refining written descriptions of the motions under study.

The Focal Episodes

In this section, we present two connected episodes from a small group lesson on exponential functions and transformations (translations). Before this small group session, in their precalculus class, students investigated function transformations. They used Desmos (Desmos, n.d.), a computer-based interactive graphical-algebraic system, to investigate how $y = f(x) + c$ would vertically shift the graph of $f(x)$ up or down by c units. Similarly, they used Desmos to explore function reflections in which $y = -f(x)$ and $y = f(-x)$ reflects $f(x)$ over the y-axis and x-axis respectively (Boelkins, 2019). Ian, being blind, was not able to do the Desmos-based lesson, which required the ability to visually perceive the movement of a thin-lined graph as different numerical values were input into the function using a 'slider' feature, which changes numerical values by dragging the onscreen slider left and right. At this point, Ian had limited understanding of the concept of function transformations. While Madelyn and Lexi had completed the Desmos activity, they had an emerging notion of the concept of function transformations.

This small group lesson used tactile graphs to explore how exponential functions would be affected by algebraic transformations. Each student was given large sheets of embossed graph paper (16" x 22") with Wikki Stix, which are made from yarn coated in wax, making them flexible and easily manipulated (Wikki Stix, n.d.). They can be made into sculptural elements or stuck onto paper to ‘draw’ 2D images and are typically used by children as a creative tool. Using principles of UDL, Wikki Stix provided a tactile representation to make graphing accessible to Ian while providing Lexi and Madelyn with an alternate representational tool to explore graphing and function transformations. Google Jamboard, a 55-inch digital touch screen and an online whiteboarding application, that allows for simultaneous group collaboration, was also used to record students’ mathematical ideas. Ian had enough sight though the outside corner of one eye, allowing him to write and read large block letters written on the 55-inch Jamboard digital touch screen. Lexi and Madelyn could also access, write, and read the Jamboard on digital tablets using the Jamboard application.

Episode 1 – Initial graph and finding the equation of $Y = A\,B^X$

When the instructor initially took out the large sheets of embossed graph paper and introduced the Wikki Stix, Lexi said, “These are my favorite when I was a kid… this is so exciting” indicating affective familiarity. Madelyn had also used Wikki Stix in elementary school art classes, and Ian had used Wikki Stix in a previous mathematics course. However, this was the first-time students had used Wikki Stix this semester so there was a brief acclimation period as they made sense of the tools. During this time, Ian ran his fingertips over the embossed graph paper feeling the physical boundary of the paper and locating the physical center.

The instructor put a Wikki dot at the physical center of Ian’s paper saying, “So the point right there is (0,0), OK?” Seeing where the dot was placed on Ian’s paper, Lexi and Madelyn drew their origin points at a similar spot on their graph papers. The instructor began by giving directions to the group, “Vertically on the Y-axis, pick a number either two or three [for the Y-intercept]. It's your choice, OK? [Then] go over to where X is equal to 10. And then you pick a Y value. The coordinate is going to be 10 comma, and your Y value can be anywhere from 6 to 10. So, you could do (10,6), (10,7), (10,5) ... then take one of the Wikki Stix and I want you to put what you think is an exponential growth graph through those two points.” The design of this activity allowed for each student to pick their own Y-intercept and point thru (10, y) so that each of their graphs were particular to their choice of points. The intent was to have three different, but similar exponential functions. Using his fingertips, Ian carefully placed the Wikki Stix through his points (0, 3) and (10, 6) (see Figure 1). The tactile nature of Wikki Stix made it possible for Ian to work at the same rate as Madelyn and Lexi. Ian and Lexi’s Wikki Stix graphs spanned both the first and second quadrant while Madelyn’s graph was only in the first quadrant.

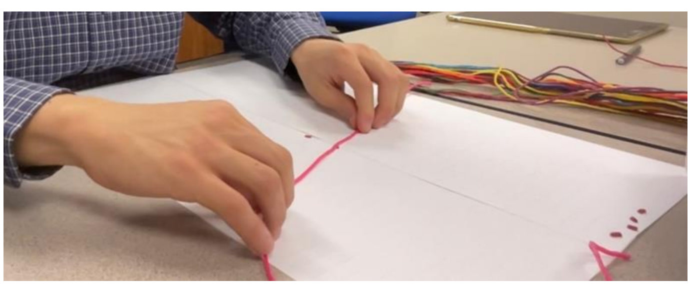

Figure 1: Ian’s placement of his graph on embossed graph paper with Wikki Stix

As they were simultaneously completing their graphs, the students were then instructed to find the exponential function that would go through their respective y-intercept and point (10, y), with the instructor saying, "[All three of you] get to your Jamboard [file]. Ian's on [page] 2, Madelyn, you can go to page 3. Lexi, you can go to page 4, and find the exponential function that goes through [your two] points. Get the formula for [your respective exponential functions.]" Each student spent 10 minutes working individually on algebraic calculations on the Jamboard to find the equation of their respective exponential functions. During this time, they were also instructed to use their exponential function to evaluate more points on their graph. The instructor said, "So now what you should be able to do is [to] use those [calculate coordinate] points to plot a few more of points on your graph…so you can really kind of dial [your Wikki graph] in…. Now we're done with the calculation, right? But try and make your graph as smooth exponential looking as possible." As Madelyn completed her calculations to find additional points on the graph, she reconfigured her Wikki graph so that it spanned both first and second quadrants. All three students used the coordinates from their algebraic calculations to create refined graphs for their exponential functions.

In this episode, we see the benefits of UDL in many ways (Gulley, 2021; Abrahamson et al., 2019). First, was the unintended, but positive affective response by Lexi and Madelyn in their recollection of using Wikki Stix as young children in creative and artistic contexts. For Ian, his familiarity with embossed graph paper and Wikki Stix provided a means for him to graph the functions at the same rate as the sighted students and with access to the same salient features of the representational tools as the sighted students. As a result, Ian's ability to reason mathematically was not hindered by the visual nature of graphing functions using traditional pencil and paper or digital technology. For all three students, the Jamboard provided a collaborative space to do mathematical calculations. In interviews, all three students have shared the benefits of Jamboard. Ian commented on how Jamboard was 'game changing' in providing a thinking space for his calculations. Lexi and Madelyn commented on how they like that all work done by their professor, small group instructor, and peers can be accessed via the Jamboard. Furthermore, this Jamboard became a mutual thinking space for all the students, which contrasts with working on scratch paper with pencil which would be inaccessible to a legally blind student like Ian. By thinking space, we mean a physical or cognitive space to wonder, postulate, calculate, or conceptualize. Thus, the Wikki Stix and Jamboard are reconceptualizations of ways to visualize emerging mathematical ideas for all learners (Abramson et. al., 2019).

Episode 2 – Moving the graph up two units to find $Y = A B^x + C$

For the next mathematical task, the instructor said, "translate the whole [exponential function] vertically 2 units up. But I want it to be as precise as possible." Ian took a second Wikki Stix, and starting at x = -14, and began to lay it adjacent to the original Wikki Stix function graph, but two units up. In doing so, his gestural motions were done in an artistic or sculptural way as he used his fingertips to 'feel' the two-unit difference, or space, between the two Wikki Stix. He carefully moved his hands from left to right along x = -14 to x = 14. By sculptural, we mean a wholistic and careful gestural motion, but mathematically intentional in preserving the two-unit distance. As he worked in the first quadrant, Ian was intentional about checking the two-unit distance at discrete points between the two Wikki Stix graphs. Lexi used a similar gestural motion, but more iterative. Starting from right to left, she tracked the two-unit difference at different points along the function with her fingers while looking at the graph paper, thus setting the second Wikki Stix a constant two-units vertically shifted up. After initially setting the second Wikki Stix, Lexi double-checked and refined her translated graph to be as precise as possible making incremental adjustments to the Wikki Stix graphs. Madelyn began her translated graph by starting at the y-intercept and setting the second Wikki Stix two units up from the original function at that point. Stretching the Wikki Stix, she then visually 'jumped' to a point on the first graph around x = 6 and translated that point up two-units. She then 'jumped' to a point around x = 10 and translated that point up two units as she stretched and placed the Wikki Stix to make her graph. She was very intentional about making sure that the point-to-point distance was two -units as she twice 'pulled up' the Wikki Stix along sections and reset it on the graph paper. In contrast, Lexi's revisions were more incremental, and Ian did not revise his translated graph due to his more deliberate gestural process.

In a few minutes, the instructor asked students to describe their process and thinking behind it. Lexi began, "I just took every unit and moved it two upward from wherever it was...so since 2 units is 1 box, first I just mirrored it. Not mirrored, but put it in the same place [two units up]."

Madelyn said, "I looked at the Y value [of the original function] and then added two and then put [the Wikki Stix] on the same X line." When asked how many points she checked, she replied, "Oh I looked at all of them, I only had seven points that I calculated but I looked at all of them."

When asked how he translated the graph, Ian said, "I looked at it, how wide is [it] with my fingers, and just go through, shoosh." Using his index and middle fingers as a measuring tool, he ran them together in the space between the two graphs in a smooth sweeping motion (See Figure 2).

The instructor commented to Ian, "OK. Yeah. You kind of did it almost [like] Desmos, almost like dynamically, right?"

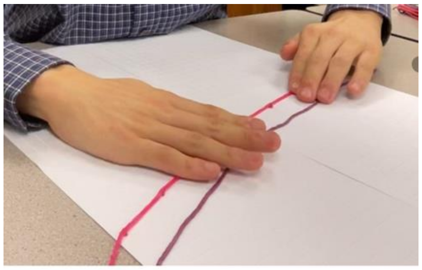

Figure 2: Ian's two fingered 'shoosh' motion

In this section, gesturing provided the means for all three students to both explore and explain their mathematical ideas. The students' mathematical reasoning related to translating the exponential function vertically two units can be explained using the aspectual structure framework (i.e., the structure of events) as described by Lakoff and Nunez (2000). An aspectual action is temporal and may be continuous or iterative. If the aspectual action is unending, or implies an infinite nature, it is considered an imperfect aspect. An example of an imperfect continuous aspect in this episode was when Ian described his approach of graphing the translated function, "I looked at it, how wide it is with my fingers, and just go through 'shoosh'," as he verbalized his gesture. His conceptualization of translating the function was through motor movements (gesturing and replacing the Wikki Stix) and being satisfied with the completion of the task. His gesture also had an imperfective aspect in that it was a continuous, unbounded, motion in which the 'shoosh' implied space between the Wikki Stix functions continuing to infinity. Mathematically, this imperfective aspect schema suggests that Ian had developed a notion of vertical translations to be holistic, smooth, continuous, and infinite. Lexi also used a gestural motion but was more incremental as if moving point by point along the x-axis from left to right on her Wikki graphs, which suggests an incremental imperfect aspectual structure. In her mathematics she was aware of a 'plus 2' relationship. This suggests Lexi had developed a notion of vertical translations to be discrete, incremental, and infinite. However, Madelyn's gestural motion as she placed the vertically translated Wikki Stix seemed more intentional at specific points along the graph, as she seemed to be intentionally translating specific points of the function. This indicates she had developed a notion of vertical translations to be structurally related to the values of the function. As suggested by Lakoff and Nunez (2000), students' motor movements reflected their mathematical conception of function translations, an imperfect aspectual structure for Ian and Lexi. Madelyn's gestures seemed to reflect the algebra of adding two units to each Y-value.

Then to the group, the instructor asked, "So that idea of how you added two to each one, right? What is the equation of that [second translated] function right there?"

Ian began by saying, "Whatever the Y is double +2? There is no formula because...."

Lexi added, "Well the rate of change [of both functions] is the same, so B is the same as what it was before, and A would be... plus two [to] whatever…A…so add two to you're A-value." Both Lexi and Ian concluded that the A value in their exponential equation would be 5.

The instructor then asked students to produce full formulas for their translated functions. Ian immediately said, "5 times 1.07 to the X power." Lexi affirmed that, saying her function would be 5 times 1.1 to the X power.

Madelyn then shared her idea saying, "I feel like you would just add [the 2] because it's a vertical shift and if you're adding...If you're multiplying by a large number, isn't your exponential going to be like? Messed up? Don't you just add 2 to the value that you get for Y?" As she said this, she gestured with her right hand tracing an exponential graph in the air.

Ian suggested, "What about if we do the reverse?" Placing his hands on his Wikki graph, Ian touched the Y-intercept of the translated graph, "It's already 5. Here in this point, it's already 5, not 3. We have nothing to do with the three, let's say, and we start from the five. What do we get?"

Realizing that the Y-intercept was a special case that afforded varied mathematical possibilities, the instructor said, "OK, let's go out to some arbitrary point…other than the Y intercept. There's two ideas, right? You could either multiply by 5 or we could add 2 to the whole thing." He then spoke to Madelyn saying, "and you said add 2 to which one?"

She replied, "Just add 2 to like the whole thing, because then your Y value would be up by two and then to be vertically translated up by two… So, I think you just add 2 to the end of the equation."

Touching the original graph at (14, 7.74), Ian said, "So you mean like the result of mine [the y-value of the original function at x = 14 is] 7.74, so [the y-value of the translated function would be] 9.74."

Madelyn confirmed, "Yeah, that's what I'm thinking."

Lexi added, "So mine would be 3 times 1.1 to the X power + 2."

The instructor said, “Now you can check values if you want, you can take a minute and like plug in values and verify it. But let's think about it conceptually… and seeing the function as the object itself… So, every single one of those points is representing your functional value and what you just physically did is you just kind of shifted [all of] them up [2 units] …You added two to each value.”

Madelyn concluded, “So it's function plus two,” with the instructor affirming her conclusion.

An important element of UDL in this episode is the discourse between students (Gulley, 2021; Abrahamson et al., 2019). All students could fully participate in the group’s construction of functional transformation. When students were asked to define the equation for the translated function, Ian offered, “Whatever the Y is of them, +2. Minus. there is no formula because…” At that moment in time, his graphical reasoning has been completed and he could not conceptualize a different approach (e.g., symbolic) of representing the translation. Lexi’s explanation focused on various aspects of the exponential function, like rate of change, initial value ‘A’, and growth factor ‘B.’ Lexi also had the concept of adding two to some part of the function and settled on adding two to A. Madelyn’s concept of the exponential function equation centered on adding two to ‘the whole thing.’ Through discourse, elements of Ian’s explanation of his motor movements and graphical explanation seemed to transfer to his conceptual development of symbolic representation. For example, later during group discussion, he used the word “reverse” indicating elements of his aspectual structural thinking-- taking the concept he already formulated and reversing it as a counterexample to explain his approach to translating the function. Towards the end of the discussion, in response to Madelyn’s suggestion to add two to the end of the equation, Ian concluded, “So, you mean the like result of, like mine got 7.74. so, it's 9.74...So then we don't have to touch the A then, right?” His use of “result” (to mean the solution for Y) and “touch” (to mean not changing the function’s initial structure) seemed to support him to structure his conceptual understanding of the symbolic representation of the function translation. Similarly, Lexi and Madelyn progressed in their symbolic formalization as they both reached a point of mathematical certainty, that the translated function was the original function plus two.

Conclusion and Implications

The purpose of this study was to investigate a UDL lesson with tactile representational tools (Abrahamson, et al., 2019) to understand embodied ways of learning mathematics. The questions we considered are: (1) How does a UDL environment with tactile representational tools support mathematics learning for students with VI and sighted students? (2) How can the embodied notion of aspectual systems describe connections between physical action and conceptualization of mathematical ideas? (3) What role does gesture have in the communication and construction of mathematics in students with VI and sighted students?

The UDL environment with novel tactile and technological representational tools supported mathematics learning for students with VI and sighted students. By the lesson’s end, all students were engaged in meaningful discourse to generalize their varied embodied ways of understanding function transformations into a symbolic representation. Interestingly, throughout the lesson, there was no mention or indication that one student had a visual impairment, nor did it appear to limit their participation. Activities focused on the mathematics task, interaction with tools, discourse with group members, and constructing mathematical knowledge.

The embodied notion of aspectual system (Lakoff & Nunez, 2000) was used to describe connections between physical actions with Wikki Stix and conceptualization of the mathematical concept of function transformations. Some aspect schemas of function transformation observed in this lesson were the before and after function transformation graphs, movement to get from one function graph to the translated function graph, and the mathematical conclusion that the resulting translated function is ‘function plus two.’ Ian and Lexi’s initial imperfect aspectual structures did not initially lead to the correct algebraic function formulas but did provide an experiential context that mediated the correct algebraic function with the help of Madelyn’s mathematical observation of adding two to the whole function.

The role of gesture in communication and construction of mathematics with VI and sighted students may be seen in the manipulation of the Wikki graphs and how those manipulations informed the student’s mathematical language (Alibali & Nathan, 2012) and mathematical understanding (e.g. Healy & Fernandes, 2011; Hostetter & Alibali, 2008). Moreover, the students’ gestural interactions with Wikki Stix gave insight into and influenced the students’ mathematical thinking (Nathan, 2021) in a manner we suggest differs from what might be elicited with interactive digital technology or pencil and paper graph construction alone.

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UNDERSTANDING GESTURES IN A CONCRETENESS FADING VECTOR ADDITION LEARNING INTERVENTION

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Abstract: In this study, we designed a concreteness fading intervention that taught middle schoolers vector addition. Our intervention allowed the participants to experience informal vector addition, solve the same problem in a simulation game, work on similar issues on a worksheet, and present their learning in a story creation activity. We examined their use of gestures during the intervention and discovered pointing and representational gestures mainly were used. The participants used pointing gestures to index objects related to the task and indicate the following representations, and their representational gestures simulated their actions, reflected their mathematical thoughts, replaced their speech, and performed abstract pointing. Our study provided a scrutinized way to inspect learning in a concreteness fading intervention.

Keywords: Cognition, Informal Education, Middle School Education

Introduction

Secondary mathematics educators often introduce mathematical concepts through formal notations (a "formalisms first" approach; Nathan, 2012) that often impede students' grounded meaning-making and transfer. Among various mathematical topics, Gubrud and Novak (1973) discover that vector addition causes learning difficulties for middle schoolers as they fail to develop a clear and stable concept of it. If students only learned those symbols of vector arithmetic from their math class, they would probably struggle with college-level vector arithmetic (see Knight, 1995). Therefore, developing an intervention that empowers students to gain a grounded understanding of vector addition is critical to secondary-level education. Another point worth mentioning is that most mathematical learning studies observe the participants' learning by comparing the performance of their pretest and posttest. Since mathematical understanding is usually embodied (Lakoff & Núñez, 2000), we think observing gestures in a vector addition learning intervention can enable us to understand better how students develop their mathematical thoughts.

Theoretical Framework

A vector is a highly conceptual idea and inaccessible in daily life. Bruner (1966) proposes three stages –enactive, iconic, and abstract– for learners to understand new concepts. Inspired by these three stages, there is an instructional approach called concreteness fading (CF) that refers to a learning process in which students start learning a new concept or skill with concrete learning materials related to their previous knowledge that gradually transitions to more symbolic learning materials (Fyfe et al., 2014). In a CF intervention, students will first encounter something that they are relatively familiar with, and then the same learning content will be presented to them in a way that gradually becomes more idealized so that it "involves the least effort of an agent to infer the invariant relation as part of generalization and transfer" (Fyfe & Nathan, 2019, p.6). Finally, formal notations can be introduced to students without intimidating them. With a CF learning process, students are expected to develop their understanding of the intervention. Previous studies explore the efficacy of the CF technique in teaching the equal sign (McNeil & Fyfe, 2012), basic circuits (Jaakkola & Veermans, 2018), and complex systems thinking (Goldstone & Son, 2005). Studies applying CF to vector addition among secondary students may have potential though it has not been researched before.

Regarding estimating students' understanding during a CF intervention, observing their use of gestures can be a good approach since people's developing ideas can be expressed by gestures (e.g., Hostetter & Alibali, 2019). Furthermore, focusing on gestures in an instructional intervention can reveal learners' mental representations of a concept (Nemirovsky & Ferrara, 2009) and indicate how learners ground the meaning of unfamiliar concepts and formalisms (Koedinger et al., 2008). For systematic analysis, McNeill (1992) categorized gestures into four types: 1) pointing, 2) iconic, 3) metaphoric, and 4) beat. Except for beat gestures, all the other three types of gestures convey semantic information relevant to learning contexts. Based on McNeill's categories, Alibali and Nathan (2012) distinguished between pointing gestures, which reflect grounded cognition in a physical environment; representational gestures, which demonstrate mental simulations of actions and perceptions; and metaphoric gestures, as a subcategory of representational gesture, which indicate body-based conceptual metaphors.

Previous research has discovered that iconic gestures can better help students embody mathematical concepts than pointing gestures (Swart et al., 2017) in a CF intervention. However, how different types of gestures allow students to construct mathematical thinking remains further examined. In this paper, we will use Alibali and Nathan's taxonomy to analyze the gestures in a CF intervention. Thus, our research question is how students use different gestures during a "concreteness fading" intervention that teaches vector addition.

Methods

Participants

We recruited six volunteer 8th graders (pseudonyms: Alice, Kevin, Jason, Emily, Anne, and Chris) from a charter school in a large Midwestern city in the U.S. Students' math performance in this school was below the state average, according to U.S. News & World Report. They had learned the coordinate plane in 7th grade and had never learned the concept of vectors before. According to our pre-study screening, none of them could conduct vector addition. They were randomly divided into two groups of three for this study.

Study Design

In this study, we designed an intervention to teach students vector addition using the vector components method. Reflecting Bruner's (1966) three stages of perceiving new concepts, the CF intervention has three stages (Enactive, Iconic, and Abstract), with one task per stage. After the three stages, we added an extra task (Constructionist Problem Design) to examine students' knowledge application in another setting.

Enactive stage. The task in the Enactive stage is called Physical task. The researcher will first ask the participants to tile the floor with cardboard tiles in two colors. Then, two students will step onto the tiles and randomly select two points to stand on and discuss how to describe their positions. After they come up with an approach to describe positions on the tiles, the researcher will give one participant a foam football and ask the participant to walk along the tiles to deliver the football to the other participant. Next, the researcher will ask the third student to use red sticks to measure the tiles that students walked leftwards or rightwards and blue sticks to measure the tiles that walked frontwards and backward. Then the researcher will ask the student who delivers the football to directly pass the football to the other student and ask the third student to measure the distance with yellow sticks (see Figure 1 left).

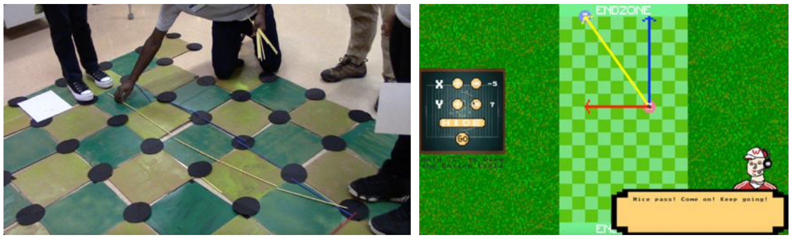

Figure 1: Physical task in Enactive stage (left) and Depictive task in Iconic stage (right)

Iconic stage. The task in the Iconic stage is called Depictive task. Students will play a football passing simulation game developed by the researcher (see Figure 1 right). In this game, students need to decide the units to add along the x and y axes (red and blue arrows) to make the quarterback pass the football to the wide receiver on a grid football field. If the ball is successfully delivered to the wide receiver, the wide receiver will move to a random place toward the end zone, and the quarterback will move to the place where the wide receiver stays before moving. Otherwise, the ball will be returned to the quarterback. Also, there is a button by clicking which students can see the projected trajectory (yellow arrow). The winning condition of this simulation is to have a touchdown pass. Each group will play two to three rounds.

Abstract stage. The task in Abstract stage is called Symbolic task. In this stage, students will have a worksheet that includes questions in a football context (see Figure 2). In addition, they will encounter formal vector notations on the Cartesian coordinate plane. Students will work separately on the questions first and then have a group discussion to share their answers and thoughts.

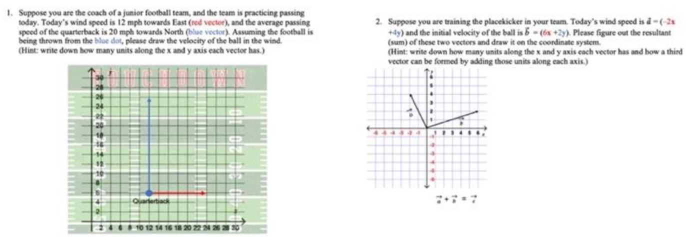

Figure 2: Symbolic task in Abstract stage

Task constructionist problem design. After three CF learning stages, there is an extra 30-minute task called "constructionist problem design" (CPD), in which participants built a story together to demonstrate their learning. The participants start with rough ideas by considering one of the learning objectives and selecting one of the contexts in the conceptual story design sheet we provided (see Table 1). They then use the materials–crafting sticks, IKEA artist's figures, color markers, and blank paper–to build a story related to the learning objective. When the participants make a rough story, the researcher will ask some prompt questions (e.g., how to describe the positions of your characters?) to let the participants iterate their story by adding more mathematical concepts. After they complete their iterative story building, the researcher asks every participant to retell the story and inform the mathematical concepts in their story.

Table 1: Conceptual Design Tool

Item	Option
Learning objectives	a. How to describe the movement (resultant) in terms of x and y? b. What are the most important things about vectors? c. What does your friend need to know about the coordinate system?
Contexts	a. Robotics b. Fashion c. Bicycling d. Soccer e. Basketball

Data Collection and Analysis

The researchers went to the middle school and conducted two study sessions in spring 2022. Each study session lasted approximately 2 hours after regular school hours. Alice, Kevin, and Jason were in the first study session, and Emily, Anne, and Chris took part in the other study session. Before each study session, we obtained consent from the participants' parents and the participants. During each study session, two video recorders were set up from different views to capture video and audio data.

We conducted a video analysis to analyze students' speech and gestures. We first captured every gesture in the videos and then segmented the videos into "clips" at the boundaries of verbal topic changes. If the participants had a couple of gestures when talking about one topic, these gestures were grouped but would be analyzed separately. Based on Alibali and Nathan's (2012) taxonomy, we developed a coding scheme (see Table 2) that included gestures. We first determined whether a video clip contained a pointing or representational gesture. For pointing gestures, we documented the object the participant was pointing to. For representational gestures, we examined whether it was metaphoric or iconic. If no clear metaphor was involved, we identified what the gesture depicted. If there was a metaphor, we identified the body-based concept this gesture represented.

Table 2: Gesture Coding Scheme

Gesture Type	**Pointing Gesture Target**	**Representation**	**Metaphoric**	**Metaphor**
Pointing or Representational	Specific target	Specific representations	Yes/No	Specific metaphors

Results

Pointing Gestures

This study analyzed 262 gestures from 248 segments. In response to our first research question, we discovered that students' pointing gestures were tightly associated with the surrounding physical environment and the undergoing learning materials. Most pointing gestures were used to link students' speech to referents. Concerning the concreteness level of referents, Fyfe and Nathan (2019) suggest examining the referent from three dimensions: physicality (whether an object is three-dimensional or two-dimensional), perceptual richness (how an object appears), and narrative context. During the Enactive stage (see Figure 3 left), participants pointed to the items of high physicality and perceptual richness (e.g., tiles and sticks they used) and real-world space (e.g., one side of the tile) highly contextualized in their conversation. When they were playing the football simulation game during the Iconic stage (see Figure 3 right), they still pointed at objects of high perceptual richness, such as buttons on the screen, but they also pointed at the arrows or the parameters of X and Y on the screen, which they realized had a relationship with the ball movement in the game. The targets the participants pointed to in the Abstract stage (see Figure 4) were mostly formalisms such as lines, axes, and dots on the coordinate plane. These gesture referents reveal that participants got more comfortable talking about idealized symbols on a coordinate plane across the CF stages. The similar contexts and problems across stages helped them transition from tangible physical learning to conceptual abstract reasoning.

Figure 3: Pointing Gesture in Enactive Stage (left) and Pointing Gesture in Iconic Stage (right)

Another noticeable use of pointing gestures is to help the participants explain mathematical thoughts in the Abstract stage. It might be because they lack accurate mathematical language and tend to use pointing gestures to assist their explanations. For instance, Figure 5 (left) shows the participant first used her pen to point at the red arrow and move to the location on the coordinate plane where she wanted the red arrow to move to and did the same thing to the blue arrow. In this excerpt, the participant had the idea of how the two arrows would impact the resultant, yet she did not have the concept of parallel vectors in mind, so she used pointing gestures to explain how to "copy-paste" arrows here.

The other function of pointing gestures is to index representations. This function of pointing gestures often takes place in the Abstract stage. For example, in Figure 5 (right), the participant pointed at 6 on the x-axis and then traced the grid line to point (6,2). In this example, his pointing at 6 was not referring to the number on the x-axis but helping him locate the coordinate point

(6,2) as he knew the number on the x-axis is a component of the coordinate space (6,2). Pointing at the axis and tracing the grid line helped him locate a position on the coordinate plane more easily.

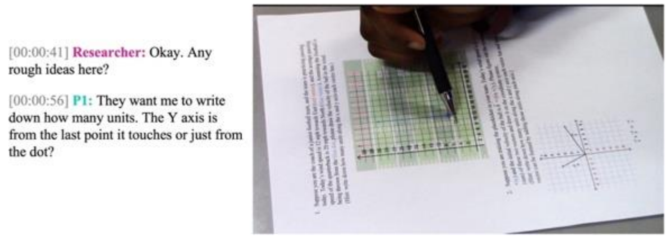

Figure 4: Pointing Gesture in Abstract Stage

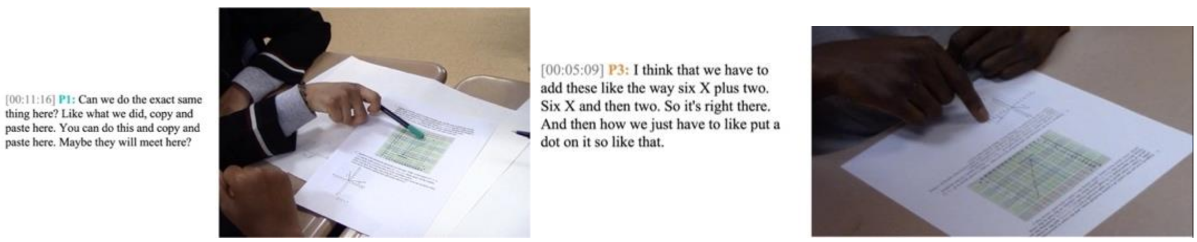

Figure 5: Pointing Gesture - Assisting Explanation (left) and Indexing Representations (right)

Representational Gestures

For representational gestures, we discovered four main functions. The first was to simulate actions. Hostetter and Alibali (2019) state that people automatically use gestures to reflect the motor activity that they are thinking about and speaking about, which is defined as Gesture as Simulated Action (GSA). GSA was also observed in this study. For instance, when Emily, Anne, and Chris were designing their own story to describe the movement in terms of x and y on a coordinate plane, they had a lot of details about how to pass a bouncing ball on a basketball court. When Chris said he needed a perfect angle and strength to pass the ball, he raised his hand and arm and gestured to throw a ball (see Figure 6).

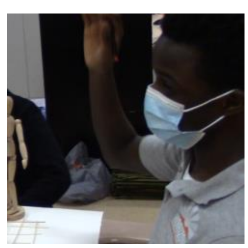

Figure 6: Throwing Gesture

The second was to simulate the behavior implied by a mathematical object. This type of representational gesture appears in every stage of the intervention. For example, in Figure 7, the participant was asked about the relationship between the yellow arrow and the red and blue arrows in the game. He waved his hand and said it was the diagonal line. In this example, the participant's waving his forearm to represent the yellow arrow meant he knew the direction of the yellow arrow was impacted by the blue and red arrows and it was like a diagonal line.

Figure 7: Representational Gesture-Mathematical Objects

The third was to replace their speech. Sometimes, when the participants would omit their words but had a gesture, this gesture delivered a clear message to others about what this participant intended to say. For example, in Figure 8, when asked about the relationship between the two vectors on the worksheet, Kevin used his pen to trace the right-angle symbol he drew between the two vectors and said, "it looks like..." His tracing of the right-angle symbol implied he wanted to say there was a right angle between the two vectors.

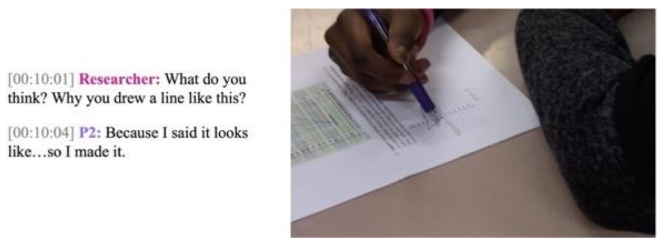

Figure 8: Representational Gesture–Replace Speech

The fourth was to have abstract pointing. Since the intervention in this study had three stages and an additional task, the participants sometimes would refer to some items that appeared before. They tended to point at the place where the item used to be. This was not a pointing gesture but a representational gesture, as their pointing represented the item they had encountered before. For example, in Figure 9, the participant was pointing at the table and said the question on the worksheet was like the game they did. He was paralleling the question and the simulation game, and this representational gesture indicated students made mutual referents among tasks.

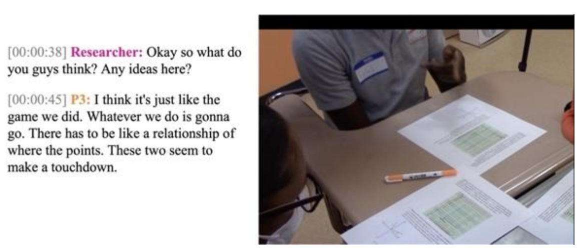

Figure 9: Representational Gesture-Abstract Pointing

Discussion

As mentioned above, the participants' use of pointing gestures is associated with the learning environment and the undergoing learning materials. This association leads the participants to a common ground. Common ground refers to knowledge, beliefs, and assumptions mutually shared by a group of people (Clark & Schaefer, 1989). When the participants worked on the tasks together, their brains established "a shared understanding" by temporally constructing a conjoint space in their social interaction (Gallese, 2003). In a learning setting, gestures can maintain the common ground among a group of people by referring to objects that are both physically present/non-present and introduced through language (Nathan et al., 2017). Thus, in every task of this intervention, the participants attempted to use pointing gestures to refer to the objects in the current learning context and created a common ground in their group to ensure everyone was on the same page.

The other two functions of pointing gestures are related to distributed cognition in their learning environment. Since pointing gestures for assisting mathematical explanation and indexing representations mostly emerged in the Abstract stage, the participants leveraged the learning environment and the elements in it (e.g., the coordinate plane grid on the worksheet) to help them develop their ideas and express their thoughts. Students' development of mathematical thinking is likely to be enhanced when they participate in mathematical activities that involve reasoning with tools and allow them to interact with the environment (Cobb, 2011). Moreover, by interacting with the learning materials and environment, they started to set up the prerequisite of developing a grounded understanding of the meaning and utility of generic symbolic representations.

The affordance of representational gestures was to communicate their ideas and demonstrate their mathematical thoughts precisely. Plausibly, the simulated action and abstract pointing gestures were for better communication during the intervention. In the CPD task, the participants had more representational gestures as simulated actions, and they had abstract pointing gestures in every task other than the first one. It was because they needed a clear image to let their teammates know what they were talking about by either acting it out or connecting to previously encountered elements. In addition, when the participants started to have gestures that implied mathematical objects, this could be the moment when they showed their grounded understanding of the concept they learned from the intervention. Because the topic in this study was fresh new to 8th graders, they lacked sufficient mathematical language to express thoughts and needed gestures as auxiliaries to help them precisely deliver their mathematical thoughts, which also led to their use of gestures to replace their speech. Sometimes, the learner cannot produce the exact mathematical word, so s/he uses a representation gesture to replace the speech, defined as direct consequence (Alibali & Nathan, 2012).

For the theoretical contribution of this study, we inspect CF learning through an innovative lens. There is no previous CF study incorporating gesture analysis to understand students' learning, so this study implies that examining learning in a CF study can be at a micro level besides assessing learning through a knowledge test (e.g., the content knowledge questionnaire in Jaakkola & Veermans, 2018). After identifying the way of using gestures in a CF intervention, the next step is to trace the train of reasoning of every learner and understand what representations are built during and after the course of a CF intervention. As for the practical contribution, we realize that students are uncomfortable with unfamiliar mathematical symbols. By asking them to explain their thoughts and observing their use of representational gestures, we can capture their trouble spots (see Alibali et al., 2013) and provide necessary scaffolding to help them get used to those symbols.

This study has several limitations. First, it is hard to capture every gesture when videotaping the learning process. Second, there is significant individual variability of gestural production. Third, due to the limited resources, this study only has two groups and six participants, making it hard to generalize a quantitative pattern to determine the relationship between the use of gestures and the different stages of a CF intervention.

Conclusion

Concreteness fading can be seen as a powerful and flexible instructional approach for teaching about vectors, which is an essential conceptual domain for mathematics and science. Gestures provide a rich window into children's intellectual development as they move through the stages of concreteness fading. By inspecting pointing and representational gestures in a concreteness fading intervention, we can better understand how students interact with the learning materials and environment and how they develop their mathematical thoughts. Future studies may consider conducting similar micro-level analyses to scrutinize learning in CF interventions.

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EXPLORING STUDENT ENGAGEMENT WITH CHATGPT IN CALCULUS LEARNING: A CASE STUDY ON CONTRASTING PERCEPTIONS

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This study explores students' perceptions of ChatGPT in a university-level calculus course and how their perceptions influence their actual use of the tool for calculus learning. Based on the Technology Acceptance Model (TAM), this study focused on how two students with contrasting perceived usefulness (PU) and perceived ease of use (PEU) engaged with ChatGPT. This study showed that with higher PU and PEU, the students are more likely to use ChatGPT in their calculus learning. We suggested that fostering positive perceptions of students is important to promote ChatGPT's potential as a learning aid.

Keywords: Calculus; Technology; Instructional Activities and Practices; Undergraduate Education.

The integration of artificial intelligence (AI) into education marks a significant turning point in the academic landscape. ChatGPT, a sophisticated AI chatbot launched by OpenAI in late November 2022, has demonstrated its innovative utility as an educational resource by answering academic queries, providing explanations, and generating both text and images (Lo, 2023). To date, research on the potential of ChatGPT in mathematics education is limited. Available studies suggested that ChatGPT's performance in solving mathematical problems varies based on the complexity of the questions and the specific subject area, with a tendency towards higher accuracy in simpler inquiries (Dao & Le, 2023; Wardat et al., 2023). In addition, ChatGPT was found to struggle with mathematical word problems, particularly when required to show its solving process (Shakarian et al., 2023). Research also found that ChatGPT's performance in mathematics is unsatisfactory compared to other subjects (Lo, 2023) and significantly below the level of a graduate student (Frieder et al., 2023). However, studies by Patero (2023) and Zafrullah et al. (2023) emphasized ChatGPT's positive impact on student attitudes and engagement in learning mathematics.

Previous studies have mainly focused on ChatGPT's performance in mathematics and there is a lack of research on students' perceptions regarding the use of ChatGPT in mathematics learning. Researchers (e.g., Chan & Hu, 2023; Kostka & Toncelli, 2023) have called for further exploration into students' perceptions and experiences with ChatGPT. This study explored how students' perceptions of ChatGPT influence their actual use of the tool in mathematical learning. The guiding research question for this study is: How do two college students with contrasting perceptions of ChatGPT engage with the tool for calculus learning?

Theoretical Framework

The Technology Acceptance Model (TAM) (Davis et al., 1989) has become a widely accepted framework for investigating the acceptance of learning innovations by various stakeholders in educational contexts (Abuhassna et al., 2023; Granić & Marangunić, 2019). TAM posits two primary factors that predict an individual's behavior towards accepting or rejecting technology: *perceived usefulness* (PU) and *perceived ease of use* (PEU). PU is defined as the degree to which a person believes that using a certain technology will enhance their performance (Davis, 1989). PEU denotes the degree to which a person believes that the benefits of technology outweigh the effort required to use or learn it (Davis, 1989). According to TAM, both PU and PEU are significant predictors of an individual's attitude towards, intended use of, and actual usage of a specific learning technology (Davis, 1989). Empirical studies in the educational domain have shown that PU and PEU can effectively predict the actual usage of learning technologies across various contexts (Hsu, 2012; Luan & Teo, 2009; Nagy, 2018; Teo et al., 2008; Ubaidillah et al., 2020), including in the study of mathematics (Zogheib et al., 2015). Researchers have reported that PU and PEU can significantly affect learners' acceptance of ChatGPT in education (Iqbal et al., 2022; Tiwari et al., 2023). However, there's a gap in the literature regarding how PU or PEU impacts students' actual usage of and interactions with ChatGPT, especially within mathematics learning contexts. In our study, we utilized the TAM model to explore how contrasting perceptions of PU and PEU among two college students influenced their actual use of ChatGPT for learning calculus.

Methods

This study was conducted in a university-level calculus course, which serves as the introductory class in a three-course calculus series offered by the mathematics department. During the semester, students were required to complete six lab assignments with topics such as derivative rules and related rates. These labs were designed to integrate ChatGPT 3.5 to support college students' conceptual understanding of calculus. Students were also required to use a validation framework (for more details, see Zhuang, 2020, 2023; Zhuang & Conner, 2022) to assess the correctness of ChatGPT's solutions to foster a rational and critical thinking approach when using AI tools for learning.

This critical case study (Yin, 2018) followed two undergraduates, Amy and Sue (pseudonyms), who had contrasting perceptions of ChatGPT as revealed by their attitude surveys. Both participants were White females majoring in STEM, with Amy receiving an A and Sue receiving a C at the end of the course. The data in this study included three attitude surveys, six lab assignments, ChatGPT chat history, and participants' assessment data. To fully understand participants' perceptions of ChatGPT's effectiveness and usability in calculus learning, we first reviewed survey results to assess students' PU and PEU of ChatGPT. Then, we conducted a constant comparative analysis (Corbin & Strauss, 2014) to analyze participants' lab assignments and chat history with ChatGPT. To examine participants' actual interactions with ChatGPT, we focused on the following three aspects: (1) the total number of interactions they had with ChatGPT, offering a quantitative measure of their engagement level; (2) their responses and evaluations when ChatGPT provided correct solutions to calculus problems; and (3) their responses and evaluations when ChatGPT generated incomplete or partially incorrect problem-solving solutions.

Results

This section presents the contrasting perspectives of the two participants on using ChatGPT for calculus learning and details of their actual use of the tool.

Case 1 – Amy

Amy's engagement with ChatGPT throughout the semester demonstrated a consistently positive attitude toward ChatGPT. According to the survey, she rated the usefulness of ChatGPT in learning calculus (PU) as 5 out of 5 at the beginning of the semester and 4 out of 5 at the end of the semester. As for the perceived ease of using ChatGPT (PEU), Amy rated 4 out of 5 at the beginning of the semester and 5 out of 5 at the end of the semester. For instance, after the first interaction with ChatGPT in first lab, she was surprised by the capability of ChatGPT in calculus (PU): "I think this thing has more power than I thought" (Survey 1). At the end of the semester, when we asked her about the biggest takeaway from using ChatGPT to learn Calculus, she stated in the survey, "I have been very impressed. I think overall, ChatGPT can help a lot of people with their calculus if you have a certain topic you are struggling with." (PU). Her lab reflections further revealed her intention to use ChatGPT for future calculus problems: "This is crazy…I am very surprised this [ChatGPT] worked so well. I might have to use it on some problems I get stuck on in the future!" (Lab 1 assignment).

Throughout the semester, Amy had 54 interactions with ChatGPT across six lab assignments. When ChatGPT provided a correct solution, Amy often followed up with requests beyond the solution generated by ChatGPT for additional supportive knowledge to deepen her understanding of calculus concepts and theorems. For instance, after ChatGPT used the intermediate value theorem to demonstrate that a function has a root, Amy asked, "Could you draw the graph of the function?" (Lab 1 chat history) to help her visualize the solution. After GPT3.5 indicated its limitation on graphic drawing and suggested she use a graphic calculator, Amy adapted her approach and attempted to achieve the same goal with a follow-up inquiry: "How can I put the interval in my graphing calculator?" (Lab 1 chat history). Moreover, Amy often asked ChatGPT to explain and demonstrate the additional problem-solving process to help her understand the application of certain calculus theorems. For example, she asked ChatGPT, "Can you show a more detailed version of step 4?" (Lab 5 chat history) to gain a clearer explanation of using the second derivative test in an optimization real world problem. When Amy noticed minor calculation errors in ChatGPT's solutions, she typically acknowledged the correct logic behind the problem-solving process without overly criticizing the errors in her evaluations. As she wrote in her evaluation, "The calculation was incorrect in finding the derivative using the chain rule. This was the only inaccurate part. All other steps were correctly applied…" (Lab 3 assignment). These comments showed that even though ChatGPT made mistakes in some instances, Amy was still willing to use ChatGPT to scaffold her learning in calculus.

Case 2 – Sue

Sue's engagement with ChatGPT throughout the semester showed a consistently conservative stance as she rated the usefulness of ChatGPT in learning calculus (PU) as 3 out of 5 at the beginning of the semester and 2 out of 5 at the end of the semester. In terms of the perceived ease of using ChatGPT (PEU), Sue consistently rated 1 out of 5 throughout the semester. Sue questioned the efficacy of using ChatGPT for calculus learning from the start: "I do not think this is a good tool for people trying to learn calculus." (Survey 1). She emphasized the limitation of using ChatGPT for calculus learning, as she commented, "ChatGPT understands the steps of how to do the math problem, but the actual algebra and easy math is where ChatGPT struggles." (Survey 1). Her conservative attitude towards ChatGPT did not change throughout the semester even though she gained more experience of using the tool for calculus learning through lab sessions. In her conclusion, Sue acknowledged that ChatGPT might be useful (PU) "for someone who knows the basics and needs more practice problems, this [ChatGPT] could be a helpful resource, but you must remember that the answer is often incorrect, and the theorems and definitions might also be incorrect." (Lab 6 assignment). Overall, Sue was reluctant to use ChatGPT for calculus learning: "I honestly did not enjoy these [ChatGPT] labs, but maybe they were helpful to others." (Survey 3).

Throughout the semester, Sue had a total of 38 interactions with ChatGPT. Sue often used minimal prompts that were provided by the instructor and asked limited follow-up questions when ChatGPT provided a correct solution. Her inquiries typically sought clarification on specific solution steps, such as: "Can you show and explain your work for step 5?" (Lab 5 chat history), rather than exploring broader conceptual understandings or the application of theorems. In addition, Sue's tolerance for calculation errors in ChatGPT's solutions was relatively low. Sue often provided critical feedback for simple calculation mistakes by ChatGPT, rating GPT-generated responses as partially correct even though the ultimate output showed the correct answer. For example, she commented "ChatGPT had the correct steps to solve the problem, however, they missed a simple math problem on the 3rd step. After I asked them to recheck this step, they replied with another incorrect answer. I told them the answer that I had gotten for step three, and then they finally agreed." (Lab 1 assignment). Sometimes when ChatGPT's overall problem-solving process was correct, excluding minor calculation errors or omissions in the final step, Sue expressed dissatisfaction with the tool's performance: "Although most of the work was done correctly, ChatGPT did not give a final answer. Although volume was given in the question, it left volume as V…" (Lab 5 assignment).

Discussion and Conclusion

We found that Amy, with higher PU and PEU, was more accepting of using ChatGPT for calculus, whereas Sue, with lower PU and PEU, showed less acceptance. These results are in coherence with the previous studies using the TAM model, which indicated that PU and PEU are valid predictors of actual usage (Poellhuber et al., 2018), learner satisfaction (Nagy, 2018), and learning motivation (Hsu, 2012). Our findings also extend previous research on the acceptance and utilization of AI-assisted tools in education to the context of mathematical learning. Learners with higher PU and PEU of ChatGPT engaged more frequently with the tool, tolerated its errors better, and showed greater motivation to use it for further support.

Our study revealed how students' perceptions towards ChatGPT affect their approaches to using it as an educational tool. Sue, who showed a conservative perception, posed fewer follow-up questions for understanding concepts or theorems and rarely inquired beyond the provided prompts for extra knowledge. In contrast, Amy, who holds a positive perception of ChatGPT, demonstrated more active uses of ChatGPT to support her learning of calculus. She often asked for further guidance from ChatGPT, including requests for geometric representations of solutions and definitions of specific mathematical concepts (e.g., critical points). In addition, Amy achieved a higher final course grade than Sue, indicating that Sue lacked a thorough understanding of certain calculus concepts and theorems. She strongly preferred to seek help from the instructor during office hours compared to asking ChatGPT for guidance despite the tool offering more convenience and flexibility regarding time and location. The cases of Amy and Sue suggest that fostering positive PU and PEU is important to help students maximize the potential of ChatGPT. Future research should explore teaching strategies to cultivate a productive disposition among students towards ChatGPT, thereby fully leveraging its power as an educational tool.

This study explored how two college students' approaches to using ChatGPT in a calculus class were influenced by their contrasting perceptions of the tool. When students possess a higher PU and PEU of ChatGPT, they are more inclined to use it for learning support. The contrasting case of Amy and Sue underscored the necessity for students to have a comprehensive understanding of both the capabilities and limitations of ChatGPT. The study also highlighted the importance of supporting students in developing an open-minded perception of ChatGPT as they integrate the tool into their mathematics learning.

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THE EFFECTIVENESS OF AN INNOVATIVE MANIPULATIVE IN FACILITATING CHILDREN'S DEVELOPMENT IN CIRCLE PARTITIONING

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The purpose of this study is to investigate the effect of using a play-dough fraction circle cutter manipulative in helping children partition circles into eight or fewer equal parts. Recognizing the challenges children face in partitioning circles, especially with odd numbers, this study leverages innovative tools to enhance understanding of partitioning. The study used a quasi-experimental, pretest-posttest design with children aged 5 to 6 years. Findings show significant improvement in partitioning skills post-intervention, particularly in dividing circles into odd numbers. The implications of this study for both teaching and learning partitioning are discussed.

Keywords: Rational Numbers; Geometry and Spatial Reasoning; Elementary School Education; Technology

When it comes to learning about fractions, elementary school students experience difficulties (Barbosa & Vale, 2021; Fazio et al., 2016; Sidney et al., 2019; Tunç-Pekkan, 2015). The ability to understand and work with fractions is not just a requisite for mathematical literacy at this stage but is also seen as predictive of success in higher mathematical pursuits, including algebra and calculus (Siegler et al., 2013; Soni & Okamoto, 2020). A lack of proficiency in fractions can significantly hinder a student's ability to grasp more complex mathematical concepts such as algebra and calculus, especially as they progress through higher grades (Flores et al., 2020; Wilkie et al., 2022).

In this context, partitioning – the ability to divide a whole into equal parts – is fundamental to learning fractions. However, many children encounter difficulty with this concept, particularly when they cannot grasp equity-related issues from the earliest years of primary education (Castro-Rodríguez et al., 2022). Recognizing this challenge, studies have established that innovative manipulatives are important to teaching fractions and they support children with learning mathematical procedures while also being more enjoyable in learning fractions (Fazio et al., 2016; Furman, 2017; Moyer, 2001).

This paper aims to leverage these insights, exploring how children used a 3D-printed play-dough fraction circle cutter that scaffolds children's use of radii as partitioning tools. Rather, the study investigated how such a manipulative can facilitate children's learning to partition in the context of circles. Through this exploration, the study seeks to contribute to a broader understanding of effective methods for teaching and learning fractions and partitioning in early mathematics education.

Review of the literature

Children develop an intuitive understanding of fair sharing at an early age (Empson, 1995). This understanding often emerges through practical activities like one-to-one correspondence and "dealing out" real-life objects among friends (Empson, 1995; Hunting and Sharpley, 1988). These experiences with fair sharing play an important role in introducing children to the basic principles of fractions, particularly the concept of partitioning. In the context of fractions, partitioning generally is defined as "the ability to divide an object or sets of objects into equal parts" (Pothier & Sawada, 1990, p.12). However, partitioning (also known as equal portioning) is not just about part-within-the-whole reasoning, but the ability to see the equal parts out of the whole and also recreating the whole by iterating the parts (Steffe and Olives, 2010; Hackenberg et al, 2016). Partitioning is also seen as a recursive process (Confrey et al., 2009), It is about breaking down a whole into equal parts and the process of reassembling these parts back into the whole. In the context of early fraction development, partitioning is concerned with two elements, number theory and geometric shapes (Pothier & Sawada, 1990). Several studies have been conducted on how children develop partitioning reasoning, either focusing on one of these aspects or on their combination (Hackenberg et al., 2016; Maloney &Confrey,2010; Pothier & Sawada, 1990; Steffe & Olives, 2010)

In the context of partitioning reasoning and number theory, research has identified different learning patterns among children. A number of studies have suggested that the process of learning partitioning is not necessarily numerically sequential. These studies highlighted that children typically find partitioning into halves and quarters more intuitive than partitioning into other fractions (Zolfaghari, 2023; Confrey et al., 2009; Gabriel et al., 2013; Pothier & Sawada, 1990). Pothier and Sawada's five-level theory illustrates a progression from basic sharing and halving concepts to more complex ideas of evenness and oddness, and eventually, to a multiplicative algorithm. Similarly, Maloney and Confrey's (2010) equipartitioning learning trajectory outlines a similar path, starting with simple 2-splits and evolving through more complex stages, including powers of two, even, and odd splits.

On the other hand, other studies contribute to a different perspective, suggesting that children's initial efforts at partitioning lengths can be sequential (Hackenberg et al., 2016; Steffe & Olives, 2010). These scholars have noted that children partition length into two or three parts, which might not be of equal size. As they progress, children develop the ability to create equal parts and exhaust the whole with a greater numbers of parts, eventually advancing to handling any number of parts and multiple wholes.

Children's partitioning development is also associated with the geometric shapes partitioned. A vast majority of research highlights the significant role of geometric shapes in teaching early fractions and partitioning (Cramer et al.,2008; Larson, 1980; Lesh et al.,1987; Ni, 2001). This approach is also prominently used in many elementary mathematics textbooks using different geometric shapes, but predominantly pre-partitioned (Pothier, 1991). According to Ni (2001), the most frequently used shapes in early fractions education are circles, rectangles, and length models. Of these, the circle is found to be more challenging for children to partition (Confrey & Maloney, 2010), despite being the easiest to recognize (Fisher, 2009). This paradox might stem from what Pothier (1991) observed in analyzing children's partitioning behaviors. Pothier (1991) found that children, when faced with various shapes tend to adopt specific partitioning techniques such as halving, drawing vertical or horizontal slicing, rather than focusing on the shape itself to determine the number of partitions. “The counting algorithm quickly focuses children on the parts, making the geometry of the whole ostensibly irrelevant” (Pothier, 1991, p.96). The issue of not attending to the whole geometric shape seems to be more evident with circles. Circles, in contrast to polygons (particularly rectangles), do not offer clear starting and ending points that might facilitate successful partitioning. This absence of clear linear guides in circles necessitates a greater emphasis on understanding the holistic shape, thus posing a greater challenge in partitioning tasks.

Given the challenges highlighted, especially those associated with partitioning circles, as well as the complexities involved in partitioning using different types of numbers (larger numbers or odd numbers), it becomes essential to investigate how children approach the task of partitioning circles and explore effective ways to support their learning process in partitioning circles. One of the important tools to scaffold children in partitioning development or mathematics, in general, is manipulatives. While models of manipulatives are diverse, physical and pictorial visual models have been argued to be crucial for scaffolding children's understanding of fractions (Martin et al., 2012; Wilkie et al., 2022). Moyer-Packenham et al. (2012) compared the use of static pictorial representations and dynamic virtual fraction manipulatives with 19 third graders with lower mathematics achievements scores, with evidence suggesting students using both manipulatives demonstrating growth. In contrast, Martin et al. (2012) observed that, when compared to picture models, the use of concrete manipulatives was more beneficial to students’ mathematics learning since the children could directly engage with how to arrange the materials. Similarly, physical manipulatives help children to have a deeper understanding of mathematical concepts including fractions while also being more enjoyable (Furman, 2017; Laski et al. 2015). Stigberg (2022) conducted a study on digital fabrication in mathematics education and claimed that manipulatives, primarily made of 3D printing, are a means of giving abstract mathematical concepts or operations in geometry, algebra, and fractions a more real experience.

Figure 1: The Play-Dough Fraction Circle Cutter

Despite extensive research on the effectiveness of manipulatives and the best practices for their use, there remains a gap in developing specific manipulatives for defined instructional purposes (Zolfaghari, 2023). To address this gap, we designed a manipulative that can be 3D printed and used with play-dough (Kosko et al., 2024). The play-dough fraction circle cutter is a 3D-printed circle with one fixed radius and several radii that may be inserted and then moved along the circle's edge. An additional circle cutter is included that cuts a circle template with the center marked by an indention. Thus, there are two primary ways children can employ this tool: they can press the radius-based cutter onto play-dough to create imprints of the partitions directly, or they can use the center-of-circle cutter to create a circle with only the center marked and then replicate these divisions (from the radius-based cutter). This latter approach is illustrated by a child in the right-hand image in Figure 1. Two arrangements with the radius-based cutter are illustrated in Figure 1. The radii on the circle can be adjusted by children to divide the circle into given equal parts.

With the design of this fraction circle cutter, we aimed to explore how children interact with and learn the concept of partitioning circles. Rather, the manipulative was designed with manipulatable radii to help scaffold children's use of the radius as a partitioning tool, and help young children intuitively understand the need to partition between the center and edge of the circle, instead of all the way across the shape. Specifically, the study aims to ask the following research question:

> How does the use of manipulatable radii, in a fraction circle cutter, affect children's ability to partition circles into different numbers of parts?

Method

The study involved 15 children, aged 5 to 6 years, enrolled in kindergarten in a childcare learning center within the mid-western United States. Among the participants, 7 self-identified as boys and 8 as girls. These children were divided into two groups for this study: an experimental group and a control group. The experimental group participated in teaching sessions focused on partitioning circles, utilizing innovative manipulative tools. These sessions were designed to enhance their understanding and skills in this area. Meanwhile, the control group students continued with their regular kindergarten curriculum and classroom activities during this period.

Pre-Experiment

Initially, students were asked to complete a survey with 7 questions about partitioning a circle into different parts (See Figure 2). The purpose of these open-ended questions was to assess whether students could equally partition circles into 2 to 8 parts. The survey items were designed based on previous studies and literature that explored children's understanding of partitioning (Confrey et al., 2014; Hackenberg et al., 2016; Hunting & Sharpley, 1991).

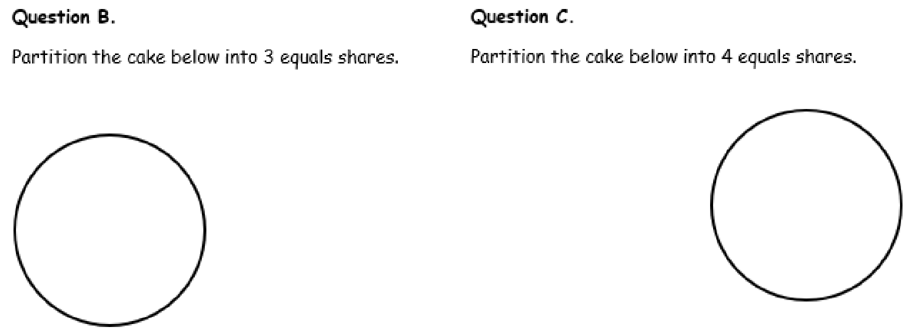

Figure 2: Example of Designed Items for Circle Partitioning Survey

To code students' responses to the survey, we utilized a transparent template that represents the acceptable range for partitioning circles into equal parts. This template was developed by Zolfaghari (2023) and helped determine the acceptable range to the extent to which students' partitioning could be considered equal. Utilizing this template, we assessed children's partitioning skills and assigned scores based on their accuracy: scores of 2 represented circles *equally partitioned* into the requested number of parts; scores of 1 represented circles *partitioned* into the correct number of parts but not equally sized; scores of 0 represented circles *incorrectly partitioned* either into incorrect number of parts.

Based on their survey scores, the first two authors selected two groups of four children each for the teaching sessions. The selection criteria for these students were based on their survey responses and encompassed the wide range of ability to partition, from unsuccessful to successful partitioning into various parts.

During the Experiment (Teaching Sessions)

In examining how children are involved with partitioning circles into different parts, we conducted a series of teaching sessions using an innovative manipulative (play-dough circle cutters), as shown in Figure 1. Eight sequential teaching sessions (Steffe & Thompson, 2000) were conducted over four weeks (two sessions per week). The intervention methodology is detailed in another study, as the current study's focus is on assessing the effectiveness of the manipulative and the emerging patterns in students' partitioning skills by looking only on their responses to the survey.

Post-Experiment

Upon completion of the teaching sessions, the same assessment was administered to the students. Of the children who participated in the teaching sessions, 5 were present and able to complete the post-survey. All 7 students from the control group were present and responded to the questions. Students' responses were scored using the same method explained above.

Analysis and Results

A Mann-Whitney U test statistic was computed to compare pre- and post-test scores between children who participated in teaching sessions using manipulatives and those in the control group. A Mann-Whitney test provided an appropriate statistical approach due to the small sample size (n=12) in this study as well as the ordinal nature of data for students' responses (Siegel & Castellan, 1988). Results indicated a statistically significant difference between the two groups

(U = 5.00, p = .039), indicating that the experimental group (Median = 11) was more likely to equally/successfully partition the circle into different parts than the control group (Median = 7). Results yielded a medium to large effect size (r = 0.586), suggesting a notable difference in partitioning between the two groups. However, it is important to note that these findings should be interpreted with caution due to the small sample size.

To further investigate how much change occurred within the experimental group due to having teaching sessions with the fraction circle cutter, a Wilcoxon signed-rank test was used to compare pre- and post-test scores for each group. For the experimental group, results indicated that there was a statistically significant improvement in the ability to partition circles post-intervention (Z = -2.201, p = .028), reflecting a significant positive effect of the scaffolding technique on the children's development of this skill. Such improvement was not observed in the control group in post-intervention (Z = -1.16, p = .246). We were also curious to know about the variation across different partitioning tasks and whether there was a specific pattern in students' responses to these tasks with and without the treatment. Figure 3 illustrates pre- and post-test scores for students in the control and experimental groups, but disaggregated by task (i.e., number of partitions requested for a paper-based circle). Recall that a score of 2 represents equally partitioned, a score of 1 was partitioned and 0 was incorrect number of parts or not partitioned.

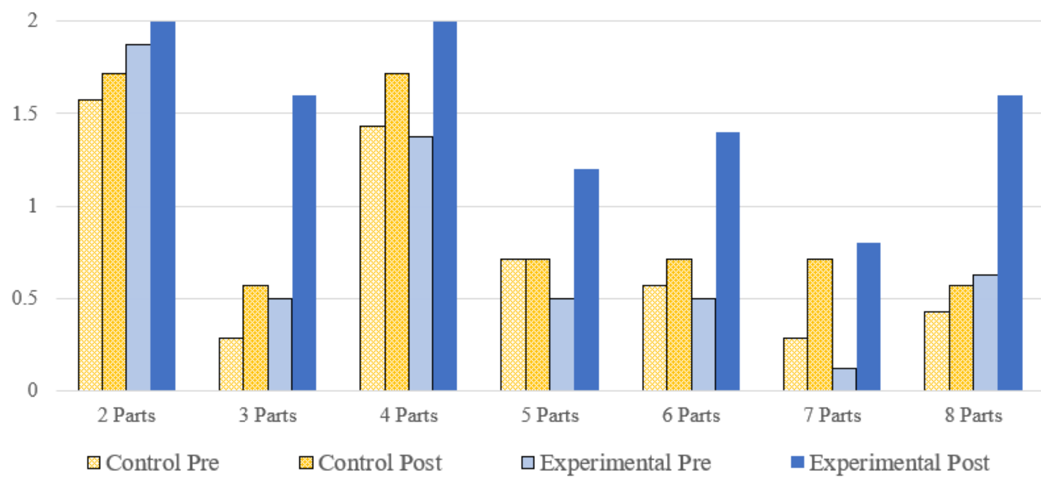

Figure 3: Changes in Task Partitioning Performance Before and After an Intervention for Control and Experimental Groups

Visual analysis of Figure 3 regarding responses from both experimental and control groups of students indicates a trend in their ability to partition circles. Tasks involving partitioning into 2 and 4 parts were the most successfully completed, as reflected by the highest mean scores. This suggests an initial comfort or familiarity with simpler partitioning tasks. Following these, tasks involving partitioning into 5, 6, and 8 parts showed moderate success. This is characterized by mean scores that range from .42 to 1.6. Lastly, partitioning into 3 and 7 parts proved to be the most challenging for students, as evidenced by the lowest mean scores (see Table 1).

The post-intervention data from the experimental group reveals a significant shift in trends from pre-intervention. The tasks of partitioning into 2 and 4 parts, which were already high, saw growth, with all children in this group demonstrating equal partitioning (*M* = 2.00, *SD* = .00). Interestingly, the task of partitioning into 3 parts, which was previously one of the least successful tasks for Kindergarteners, improved to become the second highest scoring task for students in the intervention (*M* = 1.6, *SD* = .54). This demonstrates a significant positive impact of the intervention on this specific skill. Tasks involving partitioning into 6 and 8 parts also showed noticeable growth. Tasks of partitioning into 5 and 7 parts, while still showing improvements, remained relatively more challenging for the students, as seen by the lower mean scores and higher standard deviations compared to other tasks.

Data for the control group on the post-intervention data did not exhibit growth in the way that the experimental group did. Specifically, in tasks involving partitioning into 2 (*M* = 1.71, *SD* = .75) and 4 parts (*M* = 1.71, *SD* = .75), the control group's mean scores did increase (from 1.57 to 1.71 and from 1.42 to 1.71, respectively), but not to the level of the experimental group. In other words, many Kindergarteners in the control group did improve in partitioning into 2 and 4 parts, but there were still students who could not partition into equal parts or even into the requisite number of parts requested on such tasks. The performance on tasks involving partitioning into 3 parts also improved (mean score from 0.28 to 0.57), but this was significantly lower than that of the experimental group (mean score from 0.5 to 1.6). This pattern was consistent across tasks, with the control group showing some improvement post-intervention but not to the extent of the experimental group.

Overall, results indicate that most Kindergarteners in both groups were capable of either partitioning or equally partitioning circles into two and four parts. However, when students were exposed to using the fraction circle cutter, they were much more likely to partition a circle into other requested number of parts. This was particularly the case with partitioning into three parts, with all students in the experimental group either partitioning or equally partitioning a circle into three parts, and very few of students in the control group doing so (see Table 1). In examining students' written work on the post-test, children in the experimental group focused more on using the center of the circle in attempting their partitioning whereas students in the control group did not use this property of the circle. This comparison highlights the effectiveness of using manipulatable radii on a play-dough fraction circle cutter in enhancing students' skills in partitioning circles.

Table 1: Mean Scores of Students on Partitioning Tasks Before and After Intervention

	Pre-intervention				Post-intervention			
	Experimental group		Control group		Experimental group		Control group	
	M	SD	M	SD	M	SD	M	SD
Partitioning into #2	1.87	0.33	1.57	0.72	2	0	1.71	.75
Partitioning into #3	0.5	0.5	0.28	0.45	1.6	.54	.57	.53
Partitioning into #4	1.37	0.85	1.42	0.72	2	0	1.71	.75

Partitioning into #5	0.5	0.5	0.71	0.45	1.2	.7	.71	.48
Partitioning into #6	0.5	0.5	0.57	0.49	1.4	.54	.71	.75
Partitioning into #7	0.12	0.33	0.28	0.45	.8	.44	.71	.48
Partitioning into #8	0.62	0.48	0.42	0.49	1.6	.89	.57	.53

Discussion

The purpose of this study was to explore the effectiveness of using a fraction circle cutter with manipulable radii in scaffolding children to partition circles into equal parts and to explore emergent patterns in their partitioning skills. Prior research has indicated that while circles are one of the first and easiest shapes for children to recognize (Ni, 2001), partitioning them into equal parts, particularly with odd numbers, is extremely challenging (Malony & Confrey; 2010). In this study, we used an innovative manipulative to help students focus on the task of partitioning and to examine the development of their skills in this area. The results showed a significant improvement in the experimental group's ability to partition circles into equal parts, aligning with prior research on the efficacy of manipulatives in education (Martin et al., 2012; Stigberg, 2022).

What sets this manipulative apart is its targeted design, which actively engages children in attending to the shape as they adjust the radii to create equal parts. This addresses a critical aspect identified by Pothier (1991), where students often neglect the geometry of the whole shape itself in determining the number of partitions. We believe that this tool helps students to focus simultaneously on both the counting algorithm (number of parts) and the geometric aspect of the circle, thereby integrating both aspects of the task. This integration suggests that types of representations that press students to interact with the geometry of the circle may provide a resource for teachers and students as they learn to partition shapes before formal instruction on fractions.

We observed distinct patterns in students' abilities to partition circles in both the experimental and control groups, before and after the intervention. Before the intervention, our observations revealed specific patterns in students' difficulties with partitioning circles into eight or fewer parts. Initially, students found it easiest to partition shapes into halves and quarters. This was followed by the challenge of partitioning into numbers that are powers of 2 (such as 6 and 8), and subsequently, they faced greater difficulty with odd numbers (3, 7). These observed patterns in students' abilities to some degree align with the patterns suggested by previous researchers (Pothier & Sawada, 1990; Maloney& Confrey, 2010). However, after the introduction of the fraction circle cutter, with manipulatable radii, this pattern changed significantly. While partitioning into halves and quarters remained the easiest tasks, dividing into thirds became the second easiest task for students in the experimental group. After this, tasks involving powers of 2 (such as 6 and 8) came next, and finally, the more challenging tasks were partitioning into the larger prime numbers 5 and 7. We conjecture that the changes with partitioning, especially with partitioning the circle into three parts, highlight the effectiveness of manipulatable radii in guiding students to focus not only on counting parts but also on engaging with the whole geometric shape of the circle. We believe this approach fosters out-of-the-box thinking, enabling children to conceptualize the circle in new ways. It encourages them to understand the limitations of common partitioning methods (Pothier & Sawada, 1990) and to explore alternative starting points (i.e. the center of the circle). This is a practice not typically emphasized in conventional school instruction for teaching fractions or partitioning. By focusing on such scaffolds—especially given that circles are commonly used to teach fractions (Ni, 2001)—we believe teachers can establish a strong foundation for students' partitioning in the early grades and better prepare them for learning fractions more effectively.

Despite these promising findings, we are still in the exploratory phase of examining this manipulative and pedagogical approach. Recognizing limitations, such as the small number of students involved, further investigation is needed to explore the broader effectiveness of this tool, especially when used by different teachers. Additionally, more data is needed to confirm the observed patterns. Ensuring the consistency of these patterns across a wider student population will provide a more comprehensive understanding of the tool's effectiveness in teaching and learning partitioning.

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Chapter 14:
Teaching Practice and Classroom Activity

NAVIGATING PARTICIPATION THROUGH COMMUNICATION: INSIGHTS FROM GROUP ROLES IN A GRAPHING TASK

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Group work is widely employed in mathematics classrooms to foster communication and enhance students' mathematical thinking. Drawing on principles of participatory equity, this study explores the role of group roles in promoting equitable participation. The study highlights the benefits and challenges of group work, foregrounded by the importance of culturally responsive teaching and open-ended tasks. Specifically, we attended to how group roles influence participation in a graphing scenario task conducted at a summer math camp. Data analysis reveals insights into students' communication patterns and the impact of group roles on participation. The findings suggest that group roles offer opportunities for students to engage and express their ideas, contributing to more inclusive mathematics classrooms.

Keywords: Middle School Education, Communication, Culturally Relevant Pedagogy, Equity, Inclusion, and Diversity

Group work is a common strategy to utilize in mathematics classrooms for students to communicate their ideas. Communication is one of the process standards in the National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* as a means for students to reflect on and modify their mathematical thinking (NCTM, 2000). However, a path forward to envision equitable participation in group work is a complex goal to achieve due to the nature of the social aspect that plays out. Yet, the National Council of Teachers of Mathematics' *Principles to Actions: Ensuring Mathematical Success for All* advocates that mathematics programs should aim to have access and equity in their classrooms (NCTM, 2014). That is, all students need to have the necessary support so that they reach their full learning potential (NCTM, 2014). However, the real question becomes how to translate such access and equity into a groupwork task where human interactions vary depending on the social dynamics of the group.

Shah and Lewis (2019) noted that inequity in collaborative learning tends to increase or decrease depending on the circumstances, but they recommended striving for participatory equity, when all students have an equal opportunity to participate in a learning interaction. That is, the authors emphasized that when two or more people enter a joint endeavor, factors, such as group norms, task structure, and the distribution of authority, can either promote equitable participation or reinforce inequitable opportunities (Shah & Lewis, 2019). One way to target productive groupwork and equal opportunities to participate among students is to develop different group roles that value different skills and interdependence (Cohen & Lotan, 2014).

Literature Review & Theoretical Framing

Many researchers have investigated the implications for learning and equity in using small groups in mathematics classes. Cohen et al. (1999) recorded gains in achievement for students who participated in small group classrooms, specifically on higher-order questions where students were asked to demonstrate more abstract knowledge beyond recalling facts. Cohen (1994) observed that students in groups who often take on the task of explaining to their peers have especially benefited in terms of achievement. Additionally, researchers have studied participation patterns and opportunities for students to be positioned as mathematically competent within their groups (e.g., Campbell & Hodges, 2020; Langer-Osuna, 2011; Esmonde & Langer-Osuna, 2013). For example, Esmonde and Langer-Osuna (2013) observed a student, Dawn, positioning herself with more power than she previously had opportunities to, by inserting her social talk to shift the figured world of the classroom and engaging in productive mathematical practices in the process.

While the literature has seen promise in group work, it is not a guarantee of positive outcomes. Cohen (1994) examined the complications of reaching productivity in group work as obtained "only under certain conditions" (p. 2); Esmonde (2009) also highlighted complexities such as classroom context, status indicators, task design, and the teacher's role. Even when group work seems to be leading to gains in participation, rich learning opportunities can be sacrificed. Campbell and Hodges (2020) identified "co-construction" group structures that allowed all group members to insert valued ideas, but meanwhile, in favor of agreement, students were not able take up productive mathematical debates with each other. Other researchers have also noted that students need to be guided in group work to work together as intended and these skills are not automatic, that is, "something must be done to provoke the desired behaviors" (Cohen, 1994, p. 7), and benefits will never be straightforward when students are "left to their own devices" (Heck et al., 2019, p. 437).

Moreover, researchers have studied the assignment of group roles as a provided structure in students' participation in groups. Heck et al. (2019) illustrated how reinforcement of group roles aided students in "enhancing the group's mathematical discourse" and "protecting" their contributions when others in the group attempted to be more dominant (pp. 439 – 440). While group roles are not enough to ensure equity, as several factors are at play (e.g., Langer-Osuna, 2011), they can be a start in guiding students to engage in group work in a purposeful way, as teachers continue to monitor and address group dynamics (Esmonde, 2009).

Throughout our design and implementation of our graphing task, we adopted the theoretical ideas of "culturally responsive teaching" to guide us (Gay, 2010; Ladson-Billings, 1994). Culturally responsive teaching asserts that to open opportunities for rich learning in diverse students, teaching should draw on students' own strengths by leveraging their cultural knowledge, prior experiences, and performance styles. In creating our task, we strived to include access for students to insert their personal experiences into the activity by creating our task to be open-ended and designing purposeful group roles (see Methods for the explanation of the task). Researchers have emphasized that open-ended tasks that have properties of uncertainty with no single right answer are beneficial for group work (e.g., Lotan, 2014; Heck et al., 2019), as they "increase the need for interaction since they force students to draw upon each other's expertise and repertoire of problem-solving strategies" (Cohen et al., 1999, p. 83). However, we acknowledge that even with culturally responsive teaching at the forefront of our design, there are many factors that could still impact student participation and ability to express their personal ideas into the task, including authority structures within the group (Cobb et al., 2009).

In this report, we share an analytical perspective on how group roles intertwine with the participation of group members' ideas and contributions in a graphing scenario task, where students alternated group roles three times but stayed in the same group. We are guided by the research question, "How do designated group roles influence the types and frequency of participation among middle school students during a collaborative graphing task?"

As the aforementioned literature has addressed many complexities to "participation" while students work in small groups, we kept this in mind and operationalized participation by considering students' engagement throughout the task in two main ways. We attended to students' patterns of contributions of idea proposals within each group role, but we also investigated participation by attending to the types of responses group members provided to these proposals. To do so, in our analytical choices, we adapted the coding frameworks for proposals and group member responses Barron (2003), Campbell and Hodges (2020), and Langer-Osuna (2011) developed and applied for investigating these participation dynamics within small groups.

Method of Study

The data for this study was gathered from a 2023 summer math camp in the southern United States, which provides a distinctive mathematical learning community for elementary and middle school students. The camp places students into one of five levels depending on their age and mathematical knowledge for a 2-week, half-day program. That is, the first level consists of students from grades third through fourth, while the fifth level consists of middle school students in grades seventh and eighth. Depending on the level, the mathematical content ranges from beginning concepts in algebra to advanced problem solving and discrete math.

Data Collection

To explore how students may participate within their groups, we designed a graphing scenario task with group roles (Table 1), in which students had to create an original story from suggested scenarios (e.g., pet following you to school), along with a sketch of the graph that aligns with the story. One thing to note is that students were building a distance versus time graph with no implied numerical values on the axes.

Table 1: Group Roles

Storyteller	Creates their own story to build the graph from suggested scenarios.
Screen Writer	Writes notes in their index card or sheet of paper on what the group notices about distance and time from the story and graph, which can include rate of change, constant speed/velocity, acceleration, or stationary conditions.
Grapher	Creates the graph on the poster from the storyteller and can ask the group if the graph aligns with the story.
Director	Presents the graph along with the storyline to the class.

This report focuses on one group of students from level five (seventh and eighth graders) working on the graphing scenario task. On the day of the activity, students were first introduced to standard nonnumerical graphs by having a class discussion on certain differences between the graphs. Namely, there was a discussion highlighting concepts like rate of change, constant speed or velocity, acceleration, and when the object is at a standstill. This took about 20 to 30 minutes before beginning the graphing scenario task. After the class discussion, students were placed into groups of four that were preplanned. Each group received written instructions for each group role, a large sticky sheet, a small white board, and markers. The Grapher used a large sticky sheet to sketch their graph, while the Screen Writer used a small white board to take notes. During the group activity, there were three rounds in which the group rotated different group roles. In other words, each group member had the opportunity to participate in three different group roles (Table 1) in the graphing activity. We video and audio recorded the groups as they worked on the graphing scenario task. The group of students focused on in this report consisted of three female students (Taylor, Katy, and Natalie) and one male (Chris). The dialogue during the group activity for the three rounds was transcribed to analyze.

Data Analysis

Each of the four authors acted as a coder. Three of the authors initially watched the video of the first round of the group activity to gain familiarity with the interactions. As a foundation, we started with the coding scheme proposed by Barron (2003), where codes recognize when a group member proposes an idea or contribution. However, we expanded the coding scheme with more codes and distinctions to reflect on specific types of student responses to a group member's idea or suggestion from our data; for example, we added a "resistant uptake" once noting properties of it in the videos and referring to Langer-Osuna (2011)'s codes of this. Additionally, completely new codes emerged such as distinctions in the type of "discussion" responses. We later brought a fourth author to code with us on the second round of video interactions to discuss some commonality in the code book and make minor modifications to be more concise. This resulted in a total of 9 codes (Table 2). Although the fourth author came into the coding of the second round of the group activity, we coded the first round of the group activity again to stay consistent with the modifications and to include the fourth coder.

Table 2: Coding Scheme for Student Responses & Examples

Code	Definition	Example
(1) Idea/ Contribution	A student verbally proposes a suggestion to include in the task.	Taylor: We should do murder mystery.
(2) Successful Uptake	A group member accepts the idea without any complaint.	Chris: Forgets how to get home. [Chris acknowledges the idea while writing down story.]
(3) Resistant Uptake	A group member accepts the idea with a complaint or hesitation.	Natalie: I'm not gonna do the line. Actually, I'll do it with a pencil. This is great.
(4) In Task Discussion	A group member who did not propose the new idea but initiates a discussion with the intention of	Katy: Yeah. Oh my God. That's what we should do. Draw an airplane and stick it to

	advancing the idea.	a pencil. Just have it go [hand motion].
(5) Clarification Discussion	A group member who did not propose the new idea but requests further information about the new idea or inquires about the rationale for considering it.	Natalie: He's late for school and his school is far away from home, right?
(6) Rejection	A group member verbally refuses the idea or requests another suggestion.	Taylor: No, normal stories are boring.
(7) Ignore	A group member was verbally (or non-verbal) switching the topic or not adding to the idea.	Katy: You should draw a thunderstorm. That's why turbulence happens. (code 1) Natalie: I've never been to an airport either. Actually, I have. (code 7)
(8) Individual Role-Reference	A group member comments or asks for directions about the task with explicit reference to their own group role.	Natalie: I know but I have to write it down. Do I make my graph already?
(9) Peer Role-Reference	A group member explicitly directs or reminds another peer to partake into the task in respect of the peer's group role.	Chris: You're the Grapher. You're the one that has to design the whole graph. It's none of our job.

In the coding phase, initially, each coder read the three rounds of the small group transcripts and independently assigned an appropriate code(s) on certain talk turns. Moreover, we did not code turn talks from the teacher or when conversations were off topic. Once every coder was finished coding one round of transcripts, we gathered to discuss everyone's codes to arrive at a consensus. As a group, we had a discussion to lay out the reasoning for everyone's code and to convince all coders of what the appropriate code was. During the deliberation, we either rewatched the small video segment as a group, reread the previous lines of dialogue to gain more context, and/or revisited the codebook to emphasize what the code is targeting. Once all four coders agreed on a single code, we moved on to the next coded turn talk. The unit of analysis was when a student introduced a new idea or contribution to the graphing activity (code 1). Codes 2 through 7 acted as consequential responses, following that idea code from other group members who did not initiate a contribution. This means the turn talks focused on the proposed idea or contribution (code 1) from the one who initiated the conversation were not coded, mainly because we were interested in how the other group members verbally responded to such suggestions. For example, the following conversation occurred during the third round of small group work:

Natalie: We should write Kobe with signs and then
Chris: Why are we doing basketball players. We did it. Michael Jordan Now on...Kobe
Katy: Stop

Natalie proposed a contribution to the group activity by suggesting a character in the storyline. This turn talk was coded as 1 by all coders, so there was already a consensus, and no further discussion was needed. Chris then inquired for justification for the continuation of a basketball player to be included in the story. Two coders coded that as 5, while the other two coders coded the turn talk as 3. We then discussed our reasoning and eventually came to a consensus for code 5. For Chris's turn talk, we agreed on a code of 5. As for Katy's turn talk, two coders had it as 6, while the other two coders had it blank. We then deliberated and agreed that Katy's turn talk was reflected as a code of 6. One thing to note is that we did not code turn talks from Natalie centered on her inclusion of Kobe with codes 2 through 7. We wanted to see how everyone else responded when certain group members initiated a contribution, given their group roles. As for codes 8 and 9, we were interested in parts of the conversation that were motivated by the students' group roles.

Findings

We organize our results first by reporting on findings for each of the three rounds in terms of how group roles intertwined with the participation of group members' ideas and contributions to the graphing scenario task. We then examine all three rounds and observe similarities or differences in the rounds as students changed their roles. We examined 93 total ideas proposed during the three rounds with 50 ideas from Round 1, 25 from Round 2, and 18 from Round 3.

Round 1

In Round 1, the story that evolved was about a cat that traveled from home to school at a certain distance away, taking breaks then running along and having incidents such as the loss of a wallet along the way. Figure 1 provides a visual representation of the participation for each role in Round 1 in terms of ideas suggested and responses contributed to others' ideas. Among the four roles of Storyteller (ST), Screen Writer (SC), Grapher (GR), and Director (DI), ST proposed the most ideas with code 1, followed by the SC, then the GR, and lastly DI. On the other hand, responses were those turn talks coded as 2 through 7. As indicated in Figure 1, SC contributed the most responses. The responses most frequently occurring for SC included agreeing with the idea (code 2) six times and seeking clarification (code 5) seven times. The GR followed with the most rejection (code 6) six times and seeking clarification (code 5) six times with only one agreement to an idea. DI's responses included seeking clarification (code 5) six times followed by rejection (code 6) four times. Lastly, the responses from ST were the least frequent with mainly agreement (code 2) four times and resistant uptake (code 3) twice.

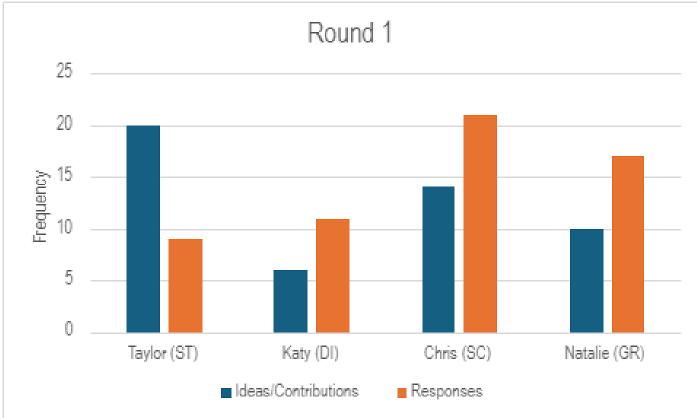

Figure 1: Round 1 Ideas and Responses by Roles

Round 2

For Round 2, the story was developed around Spiderman saving MJ, Michael Jordan, from a burning building. The story involved Spiderman going up to get MJ to bring him back down, and the time associated with those tasks. Figure 2 breaks down the number of new ideas suggested and responses to others' ideas from Round 2. Like Round 1, the ST contributed the most ideas in Round 2. However, in this round, the GR offered the next most new ideas followed by the DI, and then the SC. As responses to ideas went in this round, the DI led the number of responses with five of those being coded 2 and three codes of 5 and 6 each. Next was the ST largely coming from three successful uptakes and two rejections. Finally, the SC and GR responded to ideas the same number of times where all five of SC's where coded 5 and GR had two coded 2, two coded 5, and one coded 4.

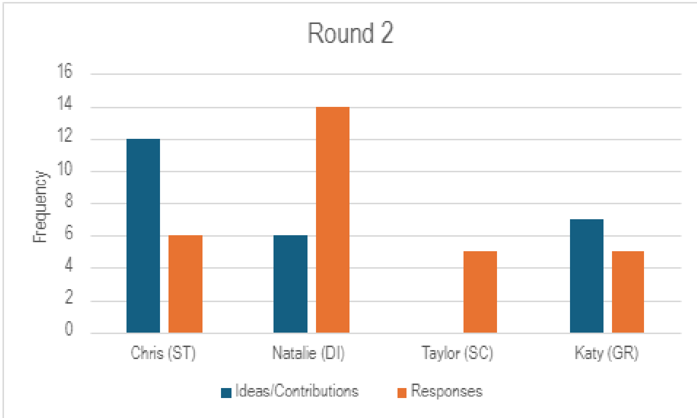

Figure 2: Round 2 Ideas and Responses by Roles

Round 3

The third and final round's story revolved around an airplane crashing due to turbulence and a thunderstorm. Figure 3 displays the number of new ideas and suggestions from each group member and their role. Again, in this round, the ST suggested the largest number of new ideas, followed by the SC and GR. The ST shared the lead in the responses category, primarily coming from three coded 2 and two ignore codes. GR had the same number of responses as ST where three clarifying questions were coded as the largest portion of responses. Lastly, DI responded the least with only one coded 5.

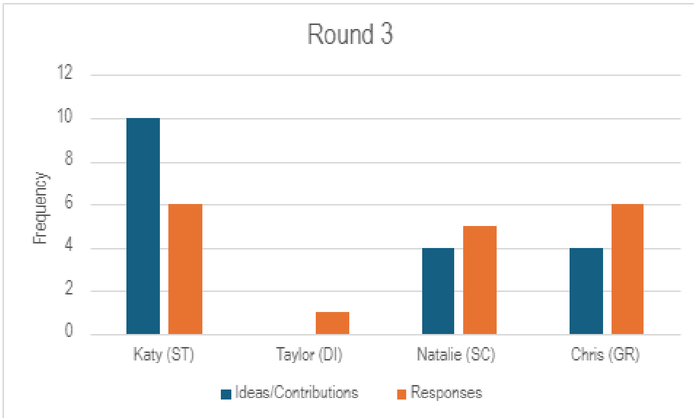

Figure 3: Round 3 Ideas and Responses by Roles

Role Acknowledgement

We noticed that acknowledging one's role (code 8) or another's role (code 9) were more prominent in round 1 and 2 with little being referenced in round 3. We did not consider this an explicit response to the idea but more to the group participation. However, there seemed to be an awareness that everyone's contribution was important in the ways defined by the roles. In round 1, there were 13 mentions of roles, two being code 8 and 11 code 9. Meanwhile, round 2 had eight mentions of other's responsibility and role (code 9) and ten mentions of one's own role (code 8). In the last round there was just one instance of code 8 and one of code 9. We conjecture this reduction in the final round is due to the development and understanding of each role from each group member as they progress through the rounds.

Discussion

The goal of the study was to gain insights on how middle school students communicate in a designed task, given group roles. Furthermore, we addressed the type of responses students engaged in and in what ways the group roles played a part in their participation. For instance, the Storyteller had the obligation to create a scenario for the group to sketch a graph, which contributed to the Storyteller having the highest number of proposed ideas/contributions on each of the three rounds. Although the other three group roles had distinct duties, there was not a specific group role that consistently had the highest number of responses (codes 2 through 7) to group members' proposed ideas/contributions. Round 1 had SC with the most, round 2 had DI with the most, and in round 3, there was a tie between ST and GR.

We also note that while we observed patterns discussed in the previous section, we maintain that these patterns interact with more than simply group roles. For instance, we hypothesize that group dynamics throughout the rounds and broader social factors could have impacted participation moving forward, such as in Langer-Osuna's (2011) work. We encourage future research for this complexity. Although there are some limitations this study holds, such as the focus on one group and not investigating the deeper social dynamics that also can influence participation, there are certain implications that the findings can inform educators to pursue equitable participation. Since literature suggests that at times mathematics classrooms create an atmosphere influenced by gender and race (Battey & Leyva, 2016; Leyva, 2021), our data indicates that a group role that positions students to constantly propose ideas can give those students who come from marginalized communities a chance to express their voice and "protect" their contributions (e.g., Heck et al., 2019). The role of the ST pushed students to take on the responsibility of sharing ideas with the group while the other members responded to them. Nevertheless, our study suggests that group roles can provide unique affordances for participation, which can pave the way to integrating students who typically do not express their ideas in a mathematics classroom. Such affordances for participation could be related to the design of the task being grounded in culturally responsive teaching via allowing opportunities for open-ended participation through the activity and given group roles. For example, stories in this group consisted of references to popular athletes, while created graphs incorporated students' own pictorial additions to the ST's created story. Our limited knowledge of the individuals' backgrounds prevents us from making strong arguments, but we anticipate that task design and group role choices could impact students' opportunities to insert their own personal experiences into the mathematical activities at hand.

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COMPARATIVE ANALYSIS OF MATHEMATICAL TASKS: UNVEILING DISPARITIES BETWEEN DEVELOPED AND DEVELOPING COUNTRIES

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This study explores how inclusive and differentiated instructional practices, specifically focused on designing and implementing mathematical tasks, influence student learning outcomes in mathematics education across Nigeria, Kenya, Ghana, and the USA. By integrating insights from prominent research in each context, the research aims to uncover the transformative potential inherent in mathematical tasks. Drawing from established concepts like higher-order thinking, problem-solving, scaffolding, and differentiated instruction, the study navigates diverse educational contexts, emphasizing the impact of socioeconomic contexts, inclusivity, and collaborative learning environments. The findings aim to offer practical guidance for educators and policymakers, contributing valuable insights to enhance mathematics learning environments.

Keywords: Instructional Activities and Practices, Problem-Solving, Mathematical Representations

Previous studies, such as those by Gravemeijer (1994) and Masingila (1993), emphasize the importance of aligning mathematical tasks with students' backgrounds and surroundings. This principle is not universally applied, yet research consistently shows significant differences in mathematics instructional practices between developed and developing countries. Educators in developing countries often face resource constraints that limit opportunities for scaffolding, problem-solving, and student-centered learning, leading to more teacher-centered approaches (Amoah, 2018; Ampadu & Danso, 2018; Fletcher, 2010). In contrast, developed countries offer various resources, such as computational tools, curated textbooks, and extracurricular facilities, enabling enriched mathematical tasks that foster creativity.

Mathematical tasks are essential for promoting problem-solving, critical thinking, and mathematical reasoning. However, Bature et al. (2016) observed that many developing countries prioritize rote memorization and procedural fluency over conceptual understanding. This underscores the need for interventions to ensure all students, regardless of location or socioeconomic status, receive high-quality mathematics education that encourages deep conceptual understanding and critical thinking skills.

In developing countries, mathematical tasks often emphasize algorithmic procedures and standardized test preparation. For example, in Kenya, educators design tasks based on standardized test items and assess students through high-stakes exams (Nandwa et al., 2015). Similarly, in Nigeria, traditional practices significantly impact student performance (Falebiba & Olofin, 2020; Ojonubah, 2016). Addressing diverse learning needs requires adopting differentiated, student-centered instruction that promotes conceptual understanding and problem-solving (Njagi, 2014; Amoah, 2018; Ampadu & Danso, 2018).

Conversely, developed countries prioritize student-centered approaches in mathematics instruction. These approaches actively engage students through hands-on activities, collaborative problem-solving, structured curricula, and real-world applications (Stecher et al., 2016; McGaffrey et al., 2001). Numerous studies highlight the significant influence of instructional practices on mathematics achievement (McDonald et al., 2013). Teachers in these contexts employ diverse mathematical tasks, from open-ended investigations to structured problem-solving, to deepen students' understanding of mathematical reasoning and concepts (Anderson, 2003; Henningsen & Stein, 1997).

Our study conducts a comparative analysis of mathematics instructional practices across Nigeria, Kenya, Ghana, and the USA, focusing on inclusive and differentiated approaches. The investigation aims to uncover how socioeconomic contexts create disparities that impact student learning outcomes. By examining these differences, this research seeks to inform educators, policymakers, and researchers about the contextual factors shaping instructional practices and student outcomes in various national settings. Our guiding question is: How do inclusive and differentiated instructional practices, particularly in the design and implementation of mathematical tasks, impact student learning outcomes in mathematics education across Nigeria, Kenya, Ghana, and the USA?

Mathematical Instructional Practices

Mathematical fluency and conceptual development are essential for students to navigate complex problems and real-world applications confidently. A comparative analysis of instructional practices in Nigeria, Kenya, Ghana, and the USA highlights the diverse approaches to mathematics education. In Nigeria, Bature et al. (2016) emphasize the importance of inclusive instruction, showing its positive impact on student outcomes through tailored tasks. Similarly, Nandwa et al. (2015) highlight structured instruction and collaborative activities in Kenya, fostering a dynamic learning community.

Teachers' attitudes toward differentiated instruction play a crucial role in shaping instructional practices. In Kenya, teachers' favorable attitudes toward differentiated instruction, evidenced by varied tasks catering to different readiness levels and learning styles, contribute to a supportive learning environment (Njagi, 2014). Teachers who tailor instruction to accommodate diverse abilities create an inclusive environment where all learners feel valued and supported, this aligns with Mayer (1999) emphasizes intentional task selection to enhance students' mathematical experiences. Effective mathematics instructional practices in the United States involve differentiated and individualized approaches, particularly benefiting students with difficulties (Morgan, Farkas, & Maczuga, 2015; McKinney et al., 2009). Emphasizing conceptual understanding over rote memorization is linked to higher student achievement, with innovative practices showing positive outcomes (O'Dwyer, Wang, & Shields, 2015; Osborne, 2021; Herbst & Chazan, 2020).

Challenges such as language barriers and individual differences necessitate a spatial justice lens in mathematics education (Larnell & Bullock, 2018). Khalo et al. (2022) and Gee (1999) discuss how language difficulties can hinder problem-solving, highlighting the need for practices that accommodate linguistic diversity. Mereku & Cofie (2008) and Rubel & Nicol (2020) advocate for spatial justice to address individual differences and ensure equitable access to learning opportunities. Recent research, such as Amoah (2018) in Ghana, focuses on practical tools like the AKIOLA Core Mathematics Series and Enriched Mathematics textbook to enhance instructional practices. Studies by Shultz (2022) and Villanueva & Prudente (2022) further contribute to the conversation on effective mathematics education by exploring teachers' knowledge, attitudes, and practices. These findings highlight the importance of evidence-based practices and continuous professional development to improve instructional quality and student outcomes globally.

Methods

The study utilizes a comprehensive comparative analysis approach to examine mathematics instructional practices in the USA, Nigeria, Ghana, and Kenya. Content analysis, as outlined by Bowen (2009), serves as the primary method for reviewing documents to extract meaningful passages. Leveraging the researcher's firsthand experiences in these countries adds a distinctive perspective to understanding instructional challenges and the influence of socioeconomic factors on mathematics education. Merriam (2009) and Krippendorff (2004) emphasizes the methodological rigor of employing content analysis for making replicable and valid inferences from textual data to their contexts.

Thematic coding and analysis were conducted on a curated selection of peer-reviewed publications, standards/practice documents, and curriculum frameworks. This process involved coding raw data and categorizing them into themes based on the content characteristics. Major themes identified included higher-order thinking (HOT), problem-solving activities (PSA), and open-ended assessment techniques (OEAT), emphasizing their significance in shaping student mathematical learning. This approach captures diverse perspectives on instructional practices and provides valuable insights into the challenges and contextual factors influencing mathematics education. The methodological approach ensures a comprehensive exploration of the subject matter, facilitating a deeper understanding of the diverse landscapes of mathematics instructional practices across the selected countries.

Findings and Discussion

The findings from an in-depth analysis of a curated selection of peer-reviewed publications, standards/practice documents, and curriculum frameworks from Nigeria, Kenya, Ghana, and the USA underscore the pivotal role of mathematical tasks in shaping student learning outcomes in mathematics education (Bature & Jubrin, 2015; McGaffrey et al., 2001; Spillane & Zeuli, 1999). While the curriculum standard documents are well-articulated, with clear content and learning objectives, there is a discrepancy observed in the curriculum objectives enactment and development and use of enriched mathematical tasks among teachers in Ghana, Nigeria, and Kenya compared to their counterparts in the USA. This difference may be attributed to varying access to necessary resources and ongoing professional development opportunities for instructional improvement.

Thematic coding reveals recurring themes such as higher-order thinking (HOT), problem-solving activities (PSA), and open-ended assessment techniques (OEAT), highlighting their influence on student mathematical learning. These findings resonate with Bature & Jubrin's (2015) emphasis on the importance of tasks stimulating higher-order thinking skills in Nigeria, facilitating meaningful connections between various mathematical concepts. Similarly, McGaffrey et al. (2001) in the USA advocate for activities of varying difficulty levels, fostering students' understanding and enhancing their ability to generate diverse solutions, aligning with the universal pedagogical goal of promoting critical thinking and creativity in mathematics education.

Furthermore, the significance of problem-solving activities, recognized across diverse educational contexts (McGaffrey et al., 2001; Spillane & Zeuli, 1999), plays a crucial role in developing students' mathematical proficiency. This aligns with the observations of McGaffrey et al. (2001) in the USA, highlighting the positive impact of presenting students with activities that involve varying levels of difficulty. The emphasis on problem-solving is consistent with Spillane & and Zeuli's (1999) exploration of patterns of practice in the context of national and state mathematics reforms.

Open-ended assessment techniques, implemented in Kenya and the USA (Nandwa et al., 2015; McGraffrey et al., 2001), encourage personalized, critical responses, fostering a comprehensive understanding of mathematical concepts. Nandwa et al.'s (2015) research in Kenya aligns with McGraffrey et al.'s (2001) insights, both emphasizing the importance of open-ended assessments in promoting a deeper understanding of mathematical concepts. These themes collectively underscore the global importance of adopting inclusive and differentiated instructional practices, particularly in the design and implementation of mathematical tasks, to enhance student learning outcomes.

Conclusion

Through an exploration of various studies conducted across diverse countries, including Nigeria, Kenya, Ghana, and the USA, this paper seeks to unpack the impact of such practices on student learning outcomes. The studies across Nigeria, Kenya, Ghana, and the USA collectively stress the importance of designing and implementing diverse mathematical tasks that accommodate diverse learning needs, foster collaboration, and create engaging student-focused environments. The design and execution of mathematical tasks are highlighted, emphasizing their function in meeting a range of learning goals, encouraging teamwork, and creating stimulating learning environments for students. the transformative role of scaffolding emerges as a critical factor in shaping students' understanding of moral mathematical knowledge. Through scaffolded support, educators can guide students towards mastery, providing the necessary assistance while gradually fading it as students become more proficient. This gradual release of responsibility empowers students to take ownership of their learning, ultimately leading to improved learning outcomes.

Implication

The insights gleaned from these studies have significant implications for mathematics educators worldwide. It emphasizes how crucial it is for teachers to intentionally construct tasks, taking into account the varied requirements and backgrounds of their students. Teachers may establish inclusive learning environments where all students feel valued and encouraged by including a variety of mathematical exercises that accommodate varied learning styles and abilities. Moreover, teachers are urged to use scaffolding as a teaching tactic to aid in their students' mathematical progress. Teachers can scaffold students' learning journeys and give them the confidence to tackle increasingly difficult mathematical concepts by offering focused support and coaching.

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STUDENT PERCEPTIONS OF GROUP WORTHY TASKS IN PROOF-BASED COURSES

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This report presents initial findings from a project aimed at enhancing equitable group work in undergraduate proof classes. The study explored student perceptions of these tasks compared to traditional group work, addressing questions about engagement and collaboration. Quantitative analysis, utilizing the Assessing Student Perspective of Engagement in Class Tools (ASPECT) instrument, indicated overall positive perceptions of the tasks. However, qualitative analysis signaled that structured task designs acted as a key factor in supporting collaboration and understanding. However, varying attitudes towards assigned roles suggest the need for further investigation into their impact on participation. This research underscores the importance of intentional task design in creating equitable learning environments in proof-based courses.

Keywords: Undergraduate Education, Reasoning and Proof, Equity, Inclusion, and Diversity

Overview & Purpose

In this report, we share some initial findings from the project Structuring Equitable Participation in Undergraduate Proof (STEP UP) aimed at supporting more equitable groupwork in undergraduate proof classes. Groupwork is becoming an increasingly common part of undergraduate proof classes with professional organizations (Saxe & Braddy 2015; the MAA Instructional Practices Guide, 2018) advocating for more student-centered approaches in undergraduate instruction. In fact, a recent survey of abstract algebra instructors found that over 90% used groupwork at least once in their course (T. Fukawa-Connelly, personal communication; Johnson et al., 2019). While groupwork can support richer student engagement, it is also a space where participation can be very imbalanced and students perceived as having higher status may dominate (e.g., Cohen et al., 1999; Esmonde, 2009). Proof-based courses have high potential to amplify status differences as students are enculturated into a new language and form argumentation (Weber & Melhuish, 2022) and where competence may be misperceived as unidimensional: ability to produce a formal proof (Hanna, 1991). Thus, there is a need to think about not just the quality of mathematics in tasks, but the nature of the activities and how the tasks may be designed to better support equitable participation.

During the first year of the project, we supported 10 mathematics instructors in designing tasks using principles of Complex Instruction (Cohen et al., 1999; Featherstone et al., 2011). For the scope of this report, we focus primarily on group worthy features of interdependence and individual responsibility. During the fall, four mathematicians implemented between 1 and 3 of these tasks designed for Topology, Linear Algebra, Analysis, and Introduction to Proof, respectively. To explore how students experienced these tasks, we take a mixed methods approach. The students took a brief Likert-scale survey, the Assessing Student Perspective of

Engagement in Class Tools (ASPECT) (Wiggins et al., 2017), designed to capture group work experiences. We then interviewed 11 students to both better understand their survey responses and get more in-depth information about their groupwork experiences. For the scope of this paper, we focus on the following research questions:

- RQ 1 In general, how do students perceive the STEP UP tasks in their proof classes?
- RQ 2 How do students perceive differences between their typical group work and STEP UP task days?

Background Literature & Theoretical Perspectives

There is a lot of potential for active learning and groupwork to support students in developing rich understanding and engaging in mathematical practices. However, the literature is mixed on the relationship between inquiry and equity in proof-based courses. For example, Laursen et al. (2014) showed more affective gains for women in inquiry-based classes; Johnson et al. (2020) found that inquiry-oriented abstract algebra was associated with men, but not women, outscoring a national sample on a conceptual assessment. Johnson et al. (2020) conjectured that groupwork may lead to a gendered hierarchy where men engage in a disproportionate amount of the mathematics. Brown (2018) further illustrated the ways that group work may serve to marginalize certain students in an inquiry class in which two women were "excluded" from participating in the group work. From our preliminary work, we have found that men may hold more authority during group work tasks (Hicks et al., 2020; Melhuish, Dawkins et al., 2022). Ernest et al. (2019) identified explicit instances in which student discourse was overtly sexist as well as implicitly aggressive towards women during small-group interactions in an inquiry setting. These scholars problematize the notion that group work necessarily creates equitable learning spaces, when in fact "small-group work can provide fertile ground for inequities to emerge" (p. 168). These results are consistent with K-12 literature establishing the presence of group work in classrooms as insufficient for fostering more equitable learning environments (Cohen et al., 1999; Esmonde, 2009a; Langer-Osuna, 2016; Shah & Lewis, 2019). When students discuss mathematics in small groups, status hierarchies may form, positioning some students as more expert helpers and others as novices in need of help (Esmonde, 2009b).

With these results in mind, we take the position that status, which is influenced by societal factors such as race and gender and comprised of both academic status (perceived mathematical ability) and peer status (social status and popularity) impacts opportunities to engage and learn in the classroom (Cohen & Lotan, 2014). If groupwork does not include features that may disrupt a status hierarchy, then it is likely that high status students will participate the most and thus learn the most. However, specific structures built into group work tasks have the potential to mitigate problematic status hierarchies from forming (Cohen & Lotan, 2014; Dunleavy, 2015; Esmonde, 2009a), which can reduce (rather than amplify) inequities in inquiry settings (Shah & Lewis, 2019).

In our work, we have emphasized a series of principles to support tasks being group worthy in proof classes stemming from complex instruction (e.g., Cohen et al., 1999; Featherstone et al., 2011) and an expansive view on proof activity (Melhuish, Vroom, et al., 2022; Weber & Melhuish, 2022). We consider a task to be group worthy if it allows for *multiple access points* and strategies, foster a sense of positive *interdependence* among group members, and have *structures to hold the group responsible for the participation and learning of each team member* (Cohen & Lotan, 2014). In the context of a proof-based class these structures need to be paired with opportunities for competency to expand beyond proof construction to include activities such as comprehending proofs, building and reasoning from examples, and comparing and modifying proofs (Melhuish, Dawkins, et al., 2022). That is, not only should tasks include social structures (such as group roles or different students being provided with different information), but the nature of the tasks needs to allow for students to engage in mathematics in different ways to support the recognition of different strengths at play. We conjectured that these features could support more positive group experiences that are less dominated by pre-existing status perceptions.

Methods

This data comes from a larger project STEP UP supporting proof-based instructors in designing and implementing more equitable group tasks. In general, the project borrows heavily from complex instruction and notions of group worthy tasks (e.g., Cohen et al., 1999; Featherstone et al., 2011). During summer workshops, instructors who teach different courses collaborated to design tasks where students engaged in theorem and proof comprehension, theorem and proof comparison and analysis, and proof construction (via conjecturing a major theorem and developing lemmas from visual representations.) The tasks were designed to elicit an array of mathematical strengths. They were all embedded with specific roles (e.g., definition manager) and/or responsibilities (e.g., become an expert on proof A, lead discussion about the focal question on your index card). The tasks were designed so each student had mathematical responsibility for components of the activity and that different needed knowledge was distributed throughout the group.

Sample and Procedure

The central research design for this study is an exploratory mixed method (Creswell & Plano Clark, 2011). The purpose for this method is for the qualitative data to explain the quantitative results. We include a quantitative instrument: Assessing Student Perspective of Engagement in Class Tools (ASPECT; Wiggins et al., 2017) and follow-up interviews to better understand and explain the student responses to the ASPECT instrument.

Students were recruited to participate in this study after their instructor (Fall 2023 and beginning of Spring 2024) agreed to run these group work activities in their proof-based courses. In all, we had 76 students consent to participate. Students who consented were recorded (all but one class) in their group work and asked to complete the ASPECT survey either directly after their groupwork task or the following class day. One class, Topology, completed 3 groupwork tasks, the Fall Intro to Proof course completed two tasks, and the Linear Algebra and Real Analysis completed one task. Thus, some students took the ASPECT survey three times. For this analysis, only their first ASPECT scores were analyzed.

Quantitative Measure and Validity Evidence. ASPECT is a 16-item construct with three factors: 1) value of activity (9 items), personal effort (3 items), and instructor contribution (4 items). The measure was designed to assess a student's perception of engagement in an active-learning classroom on a Likert scale (1 = *Strongly Disagree* to 6 = *Strongly Agree*). In this case, ASPECT was used to measure a student's perception of engagement regarding the STEP UP tasks. Only the first factor (value of activity) was utilized. The instrument has been previously validated in an introductory biology course. We used Rasch modeling to assure the validity of the tool in our context focusing on the value of the activity subdomain. An initial item analysis from the first survey responses (n = 77) indicated a slightly less than acceptable item reliability of .72 (above .80 is considered a high reliability index) and an item separation of 1.62 (less than 2.00 suggests a lack of breadth in item difficulty) for the 9-item sub-construct. However, as the score is over 1.5, it is not demeaning (Ariffin at al., 2010) even though it may not distinguish to the desired degree. Despite this, the psychometric results indicate that the data fits the model from the average infit ($MNSQ = 1.00, Z = -.20$) and average outfit ($MNSQ = 1.03, Z = .00$). Our person reliability was slightly above the ideal threshold (above a .80) with a .85, but an ideal person separation (greater than 2 suggests a range in abilities of the students) of 2.34 (Linacre, nd). The Wright Map (Figure 1) displays the spread of the participants (top) on the horizontal scale illustrating the variability in responses.

Follow-Up Interviews. Towards the end of the semester, we sent a survey to all three courses asking students who would volunteer to participate in individual interviews about their interactions with groupwork. We conducted 11 semi-structured interviews with all who volunteered: 5 students from the Introduction to Proof course, 4 students from the Topology course, and 2 students from the Linear Algebra course. We conducted interviews after their finals and interviewed students online via Zoom. During the interview session, there was one researcher leading the interview while a second researcher was taking notes.

The interviews lasted close to an hour, and we had an interview protocol composed of three major parts. The first part asked students to describe their experiences with typical group work, such as their description of the group work that happened in class and their interactions with their group members. The second set of questions in the interview were almost identical to the first part but were focused on the group work from the STEP UP tasks (e.g., thinking back to the tasks on the video recorded days, how do you think this group work was the same or different compared to a typical day?). The last part of the interview centered on asking the participants to elaborate on their reasoning for their score on certain survey items (e.g., could you elaborate on what you meant by this score to the statement: I made a valuable contribution to my group during the Proof Activity?). We shall note that all 11 interviewees were present on at least one day when the STEP UP tasks occurred.

Analysis methods

We report on the results of the Rasch analysis and present some descriptive statistics from the survey. We situated our interview participants based on the scores. For the qualitative portion, the first stage of analysis involved using the interview notes and going back to the video-recorded interviews. For each interview, one member of the research team identified all the instances that a student described a typical workday and instances of explaining the groupwork on the project day. For each individual, a set of key quotes were selected from the transcripts that provided insight into how group work was perceived and how project groupwork days were seen as similar or different. The next stage involved condensing themes (Braun & Clarke, 2006) based on whether the students were discussing either cognitive (learning) or social (participation) features. Additionally, we attended to whether the sentiment was positive or negative based on linguistic cues used to signal appraisals (Eggins & Slade, 2004).

Analysis and Findings

Quantitative Results

Rasch modeling was chosen over a classical approach due to Rasch transforming raw data into continuous data via a logistic transformation (Bond & Fox, 2015). The Rasch model utilizes the response patterns from the item and participants to create a logistic model to transform the data into logits (Bond & Fox, 2015). The visual spread of the logits can be seen in Figure 1 on the Wright Map. To note, typically a Wright Map is displayed as a vertical scale, here we report the map on a horizontal scale. The students on the right of the map (top squares and triangles) are interpreted as more favorable or seeing more value towards the STEP UP tasks. As seen on the Wright Map, the logit score of the students (*M* = 1.21, *SD* = 1.38) are higher, on average, than the items (*M* = 0.0, *SD* = 0.29). This suggests that most of the students are scoring these items highly (Likert scale value 4 or above) and agreeing to the value of the STEP UP tasks. As for the items, we see that the items are clustered together towards the left end of the horizontal scale. In Rasch, this means that these items are easier to endorse or agree with by the participants. In other words, all items are interpreted as agreeable. Since there are no items toward the right of the horizontal scale, this suggests that there aren't any items, as a whole, that are predicted by the model to illicit a disagreeable response. While ideally the mean Logit scores of the items and the participants are meant to be near each other, this suggests that the students are favorable of the [BLIND] groupwork tasks created and implemented in these courses.

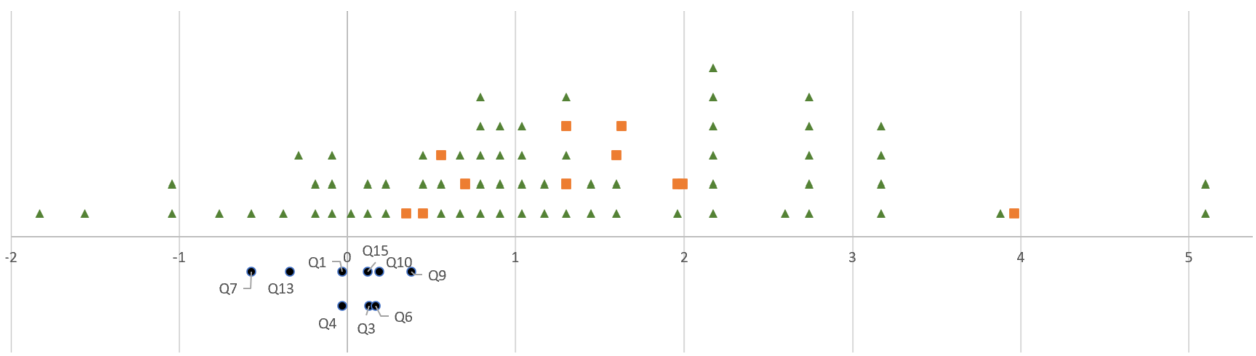

Figure 1. Wright Map

For this subscale, there were two items we focused on which were Q4 (group discussion during [the activity name] contributed to my understanding of the material) and Q9 (I would prefer to take a class that includes [group activity] over one that does not include this group activity). As seen from the Wright Map (Figure 1), item Q4 lies in the middle of the cluster which indicated that this item is agreeable (*M* = 4.59, *Mdn* = 5, *SD* = 1.38) but not as agreeable as item Q7 (the left most item on the horizontal scale). Item Q9 is the right most item on the horizontal scale. This means that item Q9 (*M* = 4.26, *Mdn* = 4, *SD* = 1.51) is not as agreeable or as likely to endorse as Q4. However given its position on the Wright Map, the Likert statement is still likely to be agreed with by the majority of the participants but less favorable than others with a median response of a 4 (neutral to agree).

Table 1: Interview Participants

Participant	Gender	Race/Ethnicity	Course	Score
Alexis	Woman	White	Topology	3.88
Lily	Woman	Hispanic	Topology	1.99
Lee	Man	Asian/Mongolian	Linear Algebra	1.96
George	Man	White	Intro to Proofs	1.60
Crocodile	Man	Hispanic/Latino	Intro to Proofs	1.60
Karli	Woman	Not Provided	Intro to Proofs	1.30
Gabi	Woman	Mixed/Latina	Intro to Proofs	1.30
Joy	Woman	Hispanic	Intro to Proofs	0.67
Julia	Woman	White	Topology	0.56
Joe	Man	Spanish	Topology	0.45
Marla	Woman	Not Provided	Linear Algebra	0.33

To understand some of the results from the ASPECT survey, we interviewed 11 students (Table 1). Our sample of students had a higher ASPECT logit average (M = 1.43, SD = 1.02) than the whole samples group (M =1.21, SD = 1.36), but still represented a true subset as seen by the Wright Map (Figure 1). On the Wright Map, the interviewed students are represented by the squares and the rest of the participants are represented by the triangles. More about the participants and their ASPECT scores can be found in Table 1.

Qualitative Results

Our goal with the qualitative analyses was to both validate the survey responses and provide more explanatory insight for students reported positive or negative experiences. We subdivide these results into two sections: cognitive-focused and participation-focused.

Cognitive-Focused: Understanding and Activity. None of the interviewed students reported any negative impact of the tasks on their cognitive understanding of the content. Five students identified overtly positive distinctions for the project tasks they felt resulted in differences in understanding the material. Students commented on the structured nature of the tasks with Lily further elaborating, “I honestly just really liked the activity. I felt like it really helped my understanding... that was one of the concepts in the class that I felt like, really, like confident about” with Gabi similarly commenting on structure and the role of having a goal: “So, like in a typical group work, you didn’t necessarily have a goal. It was just more like talking.” Two other students commented on the nature of the activity with Alexis noting that the activity, “made me look at proofs differently, and made me understand a little bit more about like what like a professor or other mathematician might be seeing when they are reading a proof,” and George explaining that their groups helped “explain things in a way that made sense” and supported visualizing.

We see these comments as focusing on three elements: the structured and goal-oriented nature of the tasks, the atypical type of activity (e.g., proof comprehension), and the role of peers. We highlight that the students explicitly noted the “structured” nature of the task in supporting understanding and contrasted it with “just talking.” While structuring the task was an initial design element, this is language students spontaneously introduced during the interview.

Finally, we note that not all students had positive reflections about supporting understanding. Several students still preferred lecture over any group work (although this did not impact their understanding of the material.) Joe indicated that he read the material in advance and suggested he did not benefit in the activity to the degree that students who had not prepared may have.

Participating-Focused: Collaboration and Contributions. Nearly all of the students mentioned differences in participation. Many of the students commented on roles and responsibilities supporting more equalized and collaborative involvement where Joy noted, "I think it [participation] was well spread out because we followed the jobs [group roles]. I think they helped with making sure [we were] learning the entire thing and making sure …we all stuck to contributing. I think if there weren't roles, one of us would have definitely wanted... to step up [meaning take over the work] after seeing how lost all of us were at the beginning." Other students commented that the group work was "more collaborative (Marla)" and that the roles led to "everyone getting involved (Karli)." Students again noted how the tasks were structured differently with Lily explaining how in typical group work they just have a set of exercises to finish, but the structure of the [blind tasks] meant they had to "share it with each other and compare, and all of that stuff that made, you know, made me have to share it-- made others have to listen to what I have to say as well."

A few commented on people who tended to share a lot on typical days, but who did not share as much on project task days. Marla described a "leader" who would tell others what to do, they would do it, and he would check their work. She noted that on the project task day, this student was given a non-leader role and when the group members traded out roles he did not want the leader role. It is possible he felt some relief from an unintended role he fell into and then didn't know how to navigate away from on his own.

Some students suggested they liked certain structures more than others. "I really loved the first and the third one a lot, a lot, a lot, a lot. The second one-- I think the structure was confusing, and so I felt like we had to slog through a little more. But I actually don't remember the activity very much. I remember the roles, and that they were confusing (Alexis)." Lily compared "free range" group work to "not as much freedom" in the project tasks due to the roles. She further elaborated that because they each had different information (via their roles) and they had to combine it to find an answer, she "didn't mind" and compared it to solving a mystery.

In contrast, there was one student who voiced a negative reaction to the task structure. George explained, "It was frustrating that I couldn't contribute, because I might have already known the answer." Other students noted that they stepped back during some role activities or wished they had a larger role or a different role (one student said her group switched up roles right away so everyone got what they wanted). However, George was the most direct about feeling the structured task roles held him back. In that interview he commented that he'd hear a groupmate say something wrong and would want to correct them, but didn't feel his role allowed for that. The same student also noted, "Yes, yes, I've never had an issue with respect in the class. I would hope that I properly respected everyone in the class as well. Again. It's hard for me sometimes to tell if I'm being disrespectful. It's not something I'm very good at." He did not elaborate and it's entirely possible he has some differences in interpreting social cues and nuance in how others perceive statements. It also may be helpful for this student to practice restraining from correcting others—however, it is not a task design intent to ban anyone, despite group role, from speaking up or commenting on others' thinking.

Discussion and Conclusion

This study provides initial evidence that structures from K-12 group work can be successfully integrated into undergraduate proof-based contexts. As group work becomes an increasingly common part of advanced mathematics, it is crucial that we consider how we implement it to not amplify status issues that are particularly prevalent in proof-based classes. To address this issue, we developed task templates (stemming from an earlier exploratory abstract algebra context, Melhuish, Dawkins et al., 2022) that disrupted what types of activities students are asked to engage in by moving away from traditional proof construction as the primary task, and integrated intentional structures to support interdependence and personal responsibility. The quantitative results from this study suggest that across four classes, students by-and-large reported positive experiences with the STEP UP tasks. The interviews then shed light on what students perceived as key differences between typical group work days and STEP UP tasks and the ways the STEP UP tasks did or did not support learning and collaboration. In this way, we are contributing to Adiredja and Andrews-Lasron's (2017) call to better understand student experiences in active learning to gain insights into what circumstances may support positive experiences.

All students noted at least one difference between project task groupwork and regular groupwork—almost all had something to do with the structured nature, which they contrasted with "just talking" in typical groupwork. Students who appreciated this structure suggested it led to more collaboration, better understanding, and appreciated having a clear goal in mind. Most students indicated they felt the structures did equalize participation in terms of different student contributions. However, we note that the roles were most divisive and most brought up in the interviews. The students who were explicit about usually being chatty, not having any problem jumping into conversations, or self-identified as strong mathematically commented either an ambivalent relationship to the roles (liked them here; not there) or did not like them (the one student who described them as frustrating). It may be that some students like roles because they have a harder time jumping into discussion and others feel less positive because they did not usually have a hard time. Our theoretical view on status may provide insight into this relationship. We conjecture that the "high status" students may feel more constrained when not left to dominate conversation.

Because the data in this study is all self-reported, we are limited in terms of making conclusions about actual participation rates and nature of collaboration. In future research, we plan to consider empirically the contribution rates in groups to examine the degree that status appears to predict or not predict contributions. Additionally, we are bringing positioning lenses in to explore not just participation rates, but how students are engaging with each other and the mathematics. This initial phase of the research points to several promising avenues for continued work in developing more equitable group work situations in proof-based classes. We suggest other researchers who engage in design of classroom tasks at this level consider not just how to support cognitive and activity goals, but also what structures may support increased involvement and collaboration of all students in small groups.

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PORTRAITS OF MATHEMATICAL AUTHORITY IN MIDDLE-GRADES CLASSROOMS

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In this paper, we consider who leads and who follows when a classroom community engages in collective mathematical activity—we describe this as mathematical authority. Building from theoretical writings on authority by Benne and Weber, we advance an activity-based perspective of mathematical authority which we exemplify through vignettes of middle-grades classrooms. Using data from 117 hours of video of 129 math lessons, we employ Loglinear models to identify relationships among the co-occurring authoritative activities of authoring, speaking, and representing. Results extend our understanding of the ways mathematical authority is negotiated during whole-class interaction and the importance of representing and speaking as high-leverage opportunities to foster students' mathematical authority.

Keywords: instructional activities and practices, classroom discourse, middle school education

Social groups and group activity are core to our experiences as learners and teachers, and, sociologists would argue, to making sense of lived experience in general. Social activity almost always involves some form of leading and following. Whether its playing peek-a-boo with a young child, listening to an orchestra perform, children playing a game at the park, or students learning in classrooms, someone is leading, directing, and planning these collective activities. In this paper, we consider who leads and who follows when a classroom community engages in collective mathematical activity—we describe this as mathematical authority. We believe that authority is an ever-present but overlooked feature of mathematics classrooms. In this paper we build from theoretical writings on authority by Benne (1970) and Weber (1947) to advance an activity-based perspective on mathematical authority which we exemplify using vignettes from middle-grades classrooms. Using data from over 117 hours of video of 129 math lessons, we employ Loglinear models to identify significant relationships among co-occurring authoritative activities. These results extend our understanding of the ways mathematical authority is negotiated during whole-class interaction and the importance of representing and speaking as high-leverage opportunities to foster students' mathematical authority.

Theoretical Perspectives & Related Research on Authority

Sociologist Max Weber and educational philosopher Kenneth Benne each framed authority as a feature of social groups that arose from the need for coordinated social action. As a macro-sociologist, Weber focused on broader institutional, historical, and societal structures. Thus, he alluded to social action in a general way as commanding and obeying in service of maintaining group order (the norms, values, behaviors, and roles of the group). For Weber (1947), authority was "the probability that a command with a given specific content will be obeyed by a given group of persons" (p.152), and the action of commanding established a hierarchical relationship between the head and followers. Benne, too, framed authority in terms of directing and obeying:

> [Authority] operates in situations in which a person or group [subjects], fulfilling some purpose, project, or need, requires guidance or direction from a source [bearer] outside himself or itself. The need demarcates a field of conduct or belief in which help is required. The individual or group grants obedience to another person or group … (1970, p. 392).

A key distinction here, one that Benne himself highlighted, was his attention to the field, or context, of authority relations. His explicit identification of the field as a constitutive feature of authority suggests, to us, a more dynamic and transient view of authority wherein the roles of bearer and subjects are fluid and determined by mutually-negotiated purposes that can shift over time and situations. We draw on both Benne's and Weber's work in our conceptualization of authority to further clarify the *field* within which mathematical authority exists. We define *authority* as a dynamic and negotiated relationship within a given field in which one party agrees to lead, while another party agrees to follow that lead. (The words "lead" and "follow" are from Pace & Hemmings, 2007). Further, we suggest that within any field, there is a collection of activities through which a member of the field can hold authority by leading one or more of those activities. *Mathematical authority*, then, is the authority relevant to the field of doers of mathematics that is accessible to all within a classroom and determined by who leads mathematical activities valued by the community.

Much of the authority research consists of classroom studies focused on aspects of instruction related to authority to understand how mathematical authority is constituted in those contexts. For example, in their collaborative body of research Herbel-Eisenmann and Wagner documented how authority was encoded implicitly in our often unconscious linguistic choices in math classrooms (Herbel-Eisenmann & Wagner, 2010; Wagner & Herbel-Eisenmann, 2014a, 2014b.) Their work is part of a growing body of authority research that studies how interactional and discursive routines can constrain or support students as mathematical authorities (Arnensen & Rø, 2022; Engle, et al., 2014; Hamm & Perry, 2002; Kinser-Traut & Turner, 2020; Langer-Osuna, 2016; Langer-Osuna et al., 2020; Solomon et al., 2021). Talk moves such as the types of questions posed, validating activities, wait time, and the appropriation of student ideas can support students to take responsibility, ownership, and authority for classroom mathematics.

In general, we can describe much of this research as linking mathematical authority to certain discursive forms or features of interaction. We agree that authority is indeed at work in the details of micro-level interaction as the extant literature clearly shows, but argue that these interactions are, in part, constituted by and given meaning from the larger activity or goal of a situation. Thus, in this paper we take a different methodological approach and operationalize authority through the lens of leading and directing mathematical activity, drawing from both Benne and Weber's focus on collective social action as a key driver of authority relations. This perspective allows us to describe *who* has authority for *what*; to document how authority changes from one situation to another as participants' roles in different mathematical activities change; and to do so in ways that clearly connect to existing theory.

To operationalize our activity-based perspective of authority, we reviewed the authority literature with a focus on mathematical activities. The majority of research emphasized a specific activity as the primary marker of mathematical authority: the authoring of mathematical ideas

(Arnensen & Rø, 2022; Gerson & Bateman, 2010; Langer-Osuna et al., 2020; Otten et al., 2017; see also Cobb et al.'s [2009] "making mathematical contributions"). That is, who is responsible for generating the mathematical ideas that are taken up in the classroom. We agree that authoring ideas is a core mathematical activity but propose the consideration of other valued mathematical activities. For example, authoring a mathematical idea does not always entail communicating that idea publicly (e.g., Goffman's [1981] distinction between authoring and animating). In fact, this distinction is seen in NCTM's (2000) process standards which differentiate the mathematical activities of communication and problem solving. Communication, for us, is the way in which mathematical ideas are publicly expressed so they are accessible to the group. In many cases authoring and communicating mathematical ideas coincide, but not always. Although the dominant mode of communication in K-12 classrooms is oral, we consider two types of public communication in this study: speaking and representing. *Speaking* is our term for oral communication whereas *representing* is the visual communication of a mathematical idea, including written and gestural forms. Representations include graphs, diagrams, written notation/symbols, as well as physical models that capture "a mathematical concept or relationship in some form" (NCTM, p. 67). We suggest that students' opportunities to produce and engage with various forms of written text, graphs, and models—what we term representing—is a key aspect of learning mathematics and developing proficiency with mathematical discourse in addition to being an important marker of mathematical authority (Arcavi, 1994; Pimm, 1987; Solomon & O'Neill, 1998; Staples, 2007). Thus, in this study, our focus is on who has authority for the mathematical activities of Authoring, Speaking, and Representing during whole-class instruction though we recognize other important mathematical activities are present (e.g., authority for justifying a claim). The research questions we sought to answer are: (1) How is mathematical authority negotiated among teachers and students in middle-grades classrooms?, and (2) How are the different authoritative activities of Authoring, Speaking, and Representing related?

Methods

Setting, Participants, and Data

This study is part of a larger research program investigating productive mathematics discourse in middle-grades classrooms. The participants were 11 grades 5-7 teachers and their students. Participating classrooms were located in four states across the US, and all teachers had at least 6 years of experience. Teachers were recommended by administrators, math coaches, and professional development trainers as good mathematics teachers, but they had different styles of teaching which led to varying instructional routines during whole-class instruction (e.g., some were more lecture-based and others more discussion-based). Data was comprised of video recordings and transcripts of 57 algebra and 72 fractions lessons, comprising a total of 117 hours of instruction. At least 9 lessons per classroom were filmed in a single academic year, with a minimum of 3 lessons filmed in each content area within a classroom. We chose the topics of algebra and fractions because these are common topics across the grade levels in our study and because of their significance in secondary curricula and overall academic success.

Analysis

Unit of analysis. In our definition of authority, the field is more than the setting, group, and group activities; it also incorporates the mathematical focus, participation format, and who has authority for leading each mathematical activity in a given interaction. We account for the field of authority relations in our unit of analysis—a segment. A *segment* is a series of turns of talk with a common focus (e.g., task, strategy) and a consistent form of participation (whole-class, independent work, etc.). Boundary markers for segments were indicated by a change in the problem, task, or topic often indicated by changes in intonation, resources or materials, physical orientation, participants, or the use of particular phrases or linguistic markers. We note that our analysis was restricted to segments of whole-class instruction only. The transcript below illustrating our coding framework is an example of a segment. There were a total of 1688 whole-class, mathematical segments coded across the 129 lessons (an average of 14 per lesson).

Coding. Within each whole-class segment, holistic decisions were made about who led each of the activities of Authoring, Speaking, and Representing which are defined in Table 1. One of the mutually exclusive codes of Teacher, Students, Both (the students and teacher co-lead an activity), or None (the activity did not occur) was assigned on the basis of who led, or had mathematical authority, for *each* activity in a segment. When coding, we considered students' collective mathematical activity rather than individual students' actions because individuals were often not possible to identify from video. We double-coded over half of the 129 lessons (44 algebra and 25 fractions lessons; 954 segments). Coding discrepancies were discussed until coders achieved consensus. Values for Cohen's kappa for Authoring, Speaking and Representing were $\kappa = .71$, .76, and .86, respectively, which indicate substantial agreement.

Table 1: Mathematical Authority Analytic Framework

Mathematical Activity	
Authoring	Generating the mathematical idea that is the focus of the segment.
Speaking	Orally communicating mathematics in a way that is publicly accessible.
Representing	Visually communicating mathematics within a segment in a way that is publicly accessible, visually observable, and mathematically meaningful. Includes writing, gesturing, and modeling.

Exemplifying the analytic framework. To exemplify our analytic framework for authority, we share a segment from a 6th-grade class and explain our coding. The class had been working on writing equations to represent generalized solutions. Students worked independently on the following task: As a contractor, you specialize in outdoor brick stairwells. How many bricks will you need to build a 10-brick-high stairwell? (See Figure 1 below.) Consider how you would code authority for each of the activities of Authoring, Speaking, and Representing.

Teacher: Anybody else have another strategy they want to share? Kelli.
Kelli: (Walks to doc camera.) Mine's kind of like Danica's. So, I found that the, um, tallest part of the brick stairwell is ten (points to the rightmost column), the shortest is one (points to the leftmost column). So I added those together and I got eleven. And the second shortest and the second tallest got eleven. And I kept doing that until I got that (gestures to the five 11s written diagonally in her work). I had five elevens, and I added them together and I got 55.
Student: That's a good one.
Kelli: Oh, you can't really see. I should make it darker.

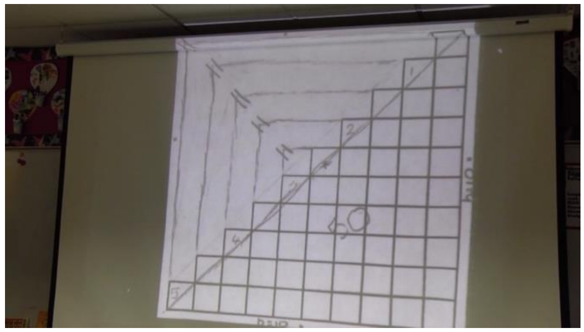

Figure 1: Kelli's written work for the staircase problem

Teacher: Yeah, it's hard to see. Let me zoom in. (Teacher helps with document camera.) Zoom just on your picture. … Alright, one more time now that everybody can see.

Kelli: So I added the shortest one (draws a square around the leftmost column) with the tallest one (draws a square around the rightmost column). And I got eleven (points to the "11" in top left corner). Then I added the second shortest (traces over the second column from the left) and the second tallest (points to the second column from the right) and I got eleven. And then I added the third tallest (points to column) and the third shortest (points to column). ... And I got elevens for all of them (points to the row of "11s" in drawing). Then I added the 11s and got 55.

Teacher: Aah, very cool.… Three claps on three for Kelli.

In this segment, students had authority for Authoring, Speaking, and Representing the mathematics because they led, or directed, these mathematical activities in this situation. Kelli, a student, publicly explained the strategy she authored both verbally and by referring to her written work. She repeatedly gestured to specific columns (i.e., stairs) to connect her written symbols to the problem context and pictorial representation. Her gestures emphasized the pattern she noticed in the representation—that the sum of pairs of 'outer' columns was constant—which she then connected to symbolic notation. Kelli's Representing activity in this vignette, which was primarily gestural but also included some writing, was critical to the clear communication of her ideas. You may wonder why we did not assign the teacher as having authority for any of the mathematical activities despite her verbal contributions: she opened (and closed) the interaction by giving Kelli the conversational floor, acknowledged Kelli's strategy, and supported clarity in Kelli's communication. However, the teacher did not directly communicate or author any mathematics. Thus, the authority Kelli's teacher held was *pedagogical authority*, which we differentiate from *mathematical authority*. *Pedagogical authority* is the authority for directing instructional activities such as selecting topics or tasks, opening and closing tasks, and nominating speakers (Wilson & Lloyd, 2000; see also Oyler's [1996] process authority). Just because a teacher has pedagogical authority does not mean she also has mathematical authority.

Quantitative analyses and loglinear models. While coding segments, we noticed patterns in some codes. For example, when students had authority for Representing in a segment, they also tended to have authority for Authoring. Because we hypothesized that who had authority for different mathematical activities was related, we used a combination of contingency tables, conditional probabilities, and Loglinear models to systematically explore relationships among the mathematical activities in our analytic framework. A Loglinear model is a regression model for non-normal outcomes that represents counts of categories and is appropriate for categorical data. In the Loglinear model, we assume that counts (i.e., cells in a contingency table) follow a Poisson distribution and depend on the categories of variables (i.e., row effects, column effects, and interaction effects for a two-way, or two-variable, model) (Agresti, 2007). We used the Generalized Linear Framework to analyze the multi-factor contingency table created by counting co-occurring codes across the mathematical activities in our authority framework.

Findings

Negotiating Mathematical Authority: Authority Structures

To answer our first research question we share vignettes that illustrate the diverse ways classroom members negotiated mathematical authority for the mathematical activities of Authoring, Speaking, and Representing during whole-class interaction. We introduce the term *authority structure* to describe, in a more general sense, patterns we observed in how authority was distributed across mathematical activities within segments of whole-class interaction. We identified four authority structures in our data—Teacher as Primary Mathematical Authority, Students as Primary Mathematical Authority, Sharing Authority Across Different Mathematical Activities, and Sharing Authority Within the Same Mathematical Activity. These authority structures exemplify common, yet qualitatively different ways authority was negotiated during whole-class interaction. We begin with segments in which one party has authority for the majority of mathematical activities (i.e., two or more activities). We describe the authority structure in these types of segments as a primary authority structure, the difference being *who* is the primary authority—the teacher or students. Consider the segment shared in the methods. This segment exemplifies the authority structure of Students as Primary Mathematical Authority because students had authority for Authoring, Speaking, and Representing (the majority of math activities). Segments in which students were the primary mathematical authority were not the norm but did comprise about 20% of our data. In this authority structure, students are positioned as capable doers of mathematics with the authority to clearly communicate their mathematical thinking. In contrast, consider segments in which the teacher drove the content; we characterize that authority structure as Teacher as Primary Mathematical Authority. Instead of illustrating this authority structure, we ask you to imagine an interaction in which a teacher authors mathematics content and represents it publicly, perhaps giving students opportunities to verbally communicate some parts of the content. In fact, segments in which the teacher had authority for Authoring and Representing, and both the teacher and students had authority for Speaking was the usual way in which Teacher as Primary Mathematical Authority occurred.

For the first two activity structures presented, one party (teacher or students) was the *primary* mathematical authority. Below we present a vignette to illustrate a different type of authority structure in which mathematical authority is *shared*. In this vignette, students in a 5th-grade class solved this problem: How much does each person get if 4 friends share 11 brownies evenly?

Teacher: James, what did you do?
James: I made eleven squares and I divided – and I, um, cut them into fourths.
Teacher: Okay. And those squares represented what?
James: A brownie. (Teacher draws eleven squares on the SMART board.)
Teacher: And then you said you did what to those brownies?
James: I cut them into fourths. (Teacher partitions each square into four equal parts.)

Teacher: Alright. James, after you split each brownie into fourths, then what did you do?
James: I put a number next to each [fourth] so I know that that's one kid.

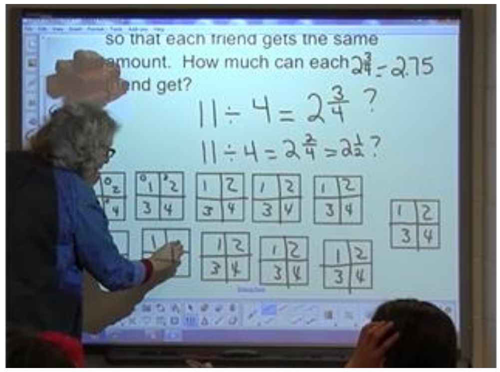

Figure 2. Recording James's solution on the SMART board

Teacher: Okay. So what you're meaning is that piece (points to a fourth) goes to child one? (James nods.) So how did you know from that picture how much each child got?
James: Well, I counted, um. Every child got a fourth of each brownie. So every four squares, that would equal a whole. Each kid gets two wholes and three fourths. ...

In this vignette, the mathematics under consideration was generated by James, but he is not the only person communicating his idea to the class. Because our operationalization of authority goes beyond Authoring to include Representing and Speaking, we can document how the teacher plays an important role in the oral, gestural, and written forms of communication. Although students had authority for Authoring the focal mathematical idea, the teacher had authority for Representing the idea by creating a public visual record of his thinking (and both had authority for Speaking). Thus, we describe the authority structure in this vignette as *Sharing Across Activities*. In this type of shared authority structure, students and teachers simultaneously claim authority for *different* mathematical activities. In other words, each party has sole authority for at least one mathematical activity in a segment. However, at times, teachers and students may share authority for the *same* activity. In fact, we see this in the previous vignette when the students and teacher shared authority for Speaking. But consider if the teacher had also explained how the 11 one-fourths each child received was reflected in the drawing and was equivalent to 2¾. Had this occurred, the students and teacher would have shared authority for Authoring the focal mathematical idea, resulting in codes of both for Authoring, both for Speaking, and teacher for Representing. This hypothetical situation exemplifies our final authority structure of Shared Authority Within the Same Mathematical Activity; this authority structure is characterized by segments in which two or more mathematical activities are coded as both. The Shared Authority Within authority structure was the most common in our data, accounting for 40% of segments.

The four authority structures we shared here provide common yet qualitatively different ways mathematical authority was distributed across the activities of Authoring, Speaking, and Representing in our study. Moreover, our distinctions between the authority structures of Shared Authority Across Different Activities and Shared Authority Within the Same Activity provide clarity to the construct of "shared authority" which is a consistent recommendation across research but with various meanings in the literature (Amit & Fried, 2005; Arnensen & Rø, 2022; Gerson & Bateman, 2010; Langer-Osuna et al., 2020).

Relationships Among Authoring, Speaking, & Representing: Loglinear Models

To answer our second research question, we systematically explored relationships among the activities in our framework using a Loglinear model. We started with a saturated model that included all categorical variables and interactions. Using a backwards selection process, we eliminated interactions in layers to identify the 'smallest' model that fit the data as indicated by model fit statistics (Agresti, 2007). Our final model was: $log(\mu_{ijk}) = \alpha + \beta_i + \gamma_j + \lambda_k + (\beta \times \gamma)_{ij} + (\beta \times \lambda)_{ik}$ where X_{ijk}, the count of the $(i, j, k)^{th}$ cell, is Poisson(μ_{ijk}); each category of Authoring was estimated by β_i, $i = 1, \ldots, 3$ (the three possible authority codes of Students, Both, and Teacher for Authoring); each category of Speaking was estimated by γ_j where $j = 1, \ldots, 4$ (the four possible codes of Students, Both, Teacher, None); and each category of Representing was estimated by λ_k where $k = 1, \ldots, 4$. The final model contains all main effects and pairwise interactions among Authoring and each of Speaking and Representing. Following Agresti (2007), we interpret significant results in terms of odds ratios that compare the relative likelihood of different categories of one authority variable occurring while holding other variables constant. (Due to space, we do not report all coefficients for the final model.)

Our goal with this analysis was to understand which combinations of authority codes co-occurred more frequently than expected—for example, what codes co-occurred when students had authority for Authoring or Representing? We found that when students had authority for Representing ideas, they were 3 times as likely to also have authority for Authoring compared to both or teacher Authoring. [Relevant odds ratio is $\log\left(\frac{\mu_{121}}{\mu_{321}}\right) = 0 - \left(\hat{\beta}_3 + \widehat{(\beta \times \gamma)}_{32}\right) = .72 + .41 = 1.13$, so $e^{1.13} = 3.1$. The 95% family-wise confidence interval for the odds ratio is (2.26, 4.24)]. In contrast, when students had authority for Authoring, the teacher was twice as likely as students to publicly Represent those ideas [$\log\left(\frac{\mu_{123}}{\mu_{121}}\right) = \hat{\lambda}_3 - \hat{\lambda}_1 = .67 - 0 = 0.67$, so $e^{0.67} = 1.95$; the 95% family-wise confidence interval for the odds ratio is (1.58, 2.44)]. Although the authoritative activities of Authoring and Representing were related, that relationship depended on *who* had authority. In general, Representing is a mathematical activity for which the teacher is most likely to have authority, regardless of who has authority for Authoring. But when students did have authority for Representing, it was most likely their own ideas they were communicating. We also found that when students had authority for Speaking, they were more likely to have authority for Authoring, regardless of who had authority for Representing (odds ratios comparing students Authoring to other authorship categories holding students as Speaking constant were all greater than 1). And perhaps not surprisingly, when teachers Authored ideas or shared authority for Authoring with students (both code), the teacher was more likely to be the one Representing those ideas. These findings indicate that when students have authority for the mathematical activities of Representing and Speaking, they are more likely to also have authority for Authoring. Thus, we suggest these two activities may be particularly important opportunities to foster students' mathematical authority.

Discussion & Implications

By incorporating multiple, simultaneous mathematical activities into our activity-based perspective of authority, we expand the field's understanding of how mathematical authority is negotiated in classrooms and provide a useful lens to reflect on practice. We found both consistency and variety in the ways mathematical authority was distributed across the activities of Authoring, Speaking, and Representing which we documented with four authority structures. We suggest that the authority structures of Sharing Within and Across Activities provide specific and productive ways to share mathematical authority in ways that amplify, clarify, and build on students' contributions. Yet despite the prevalence of shared mathematical authority in our data, students rarely had authority for Representing though they often had authority for Authoring. Moreover, authority for Authoring did not imply authority for Representing: the directionality of these relationships was complex. However, the activities of Representing and Speaking seem to be powerful indicators of students' overall mathematical authority and we encourage teachers to seek opportunities for students to communicate mathematical ideas.

In closing, we quote Benne who, over 50 years ago said, "We do not … know the shape of the future society and culture into which we as educators, along with those we are helping to educate, are now moving" (1970, p. 404). He wrote this as part of his argument for an authority that balanced freedom with responsibility in a self-renewing, interdependent community that adapted to the needs of the group, both now and in the future. As we look toward an uncertain future, we believe the activity-based perspective of authority we shared here provides a new and more positive conceptualization of authority that foregrounds the empowerment of students as doers and leaders of mathematical activities in ways that can adapt to accommodate the new activities future mathematics communities deem worthy of learning and learning to lead.

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INSTRUCTIONAL MOVES TO SUPPORT MIDDLE SCHOOL STUDENTS' ENGAGEMENT DURING A GEOMETRY CAMP

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Supporting engagement at the middle school level as well as providing access to high-quality mathematics is crucial. We explore instructional moves that contributed to engagement during an informal geometry summer camp for middle school students, centered around advanced geometry such as braids, symmetries, and platonic solids. We report variation in engagement and explore how instructional moves of open-ended questioning, real-time feedback, and prompting supported positive moments of engagement.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Geometry and Spatial Reasoning; Informal Education; Equity, Inclusion, and Diversity

Middle school is a pivotal time in students' development, and access to opportunities during this age can determine students' confidence in pursuing particular disciplines (Buffum et al., 2016; Shin, 2011). For example, access to high-quality, engaging mathematics during middle school can determine a student's confidence to pursue STEM avenues (Butler-Barnes et al., 2021; Furner, 2017). Unfortunately, many students lack opportunities for high-quality mathematics due to continued inequities in U.S. schools along racial/ethnic and socioeconomic lines (Morgan et al., 2016; Sirin, 2005). Experiential learning that engages students in collaborative and hands-on experiences may work against opportunity gaps (Kolb et al., 2014), while also aligning with community and socially-oriented approaches to learning that are especially important for students from historically marginalized racial/ethnic, cultural, and/or socioeconomic groups (Gray et al., 2020; Xu et al., 2018). We sought to understand middle school students' engagement experiences during a week-long summer geometry camp centered around experiential learning-based activities by addressing the following questions: (1) *What was the variation in students' engagement during the experiential geometry activities, by day and by student?* and (2) *What instructional moves during the experiential learning geometry activities contributed to moments of positive engagement in the camp?*

We focused on geometry to leverage students' daily experiences with space and movement. Up to and through middle school, public school students receive geometry instruction, however, tendencies to provide lecture-style instruction and "teaching to the test" remain prevalent (Close et al., 2020). We sought to provide rich experiential learning opportunities to students so that they may explore abstract geometric concepts that they may not experience in their traditional mathematics classroom: braids, knots, symmetry, etc. Our purpose is to understand how students experience engagement with experiential learning activities centered around geometry and how to support students in order to create positive learning experiences.

Theoretical Framework

Engagement is a multidimensional construct as conceptualized by various researchers that refers to how students connect to learning (Fredricks, 2011; Hospel et al., 2016; Middleton et al., 2017). We used Wang et al.'s (2016) five dimensions of engagement - behavioral, cognitive, affective, social, and agentic - to guide our observation of students during each activity. Our protocol (adapted from Ben-Eliyahu et al., 2018) included columns to document student actions (e.g., "Consider what students are doing: asking, describing…"), level of participation (e.g., "*Active*: takes initiative, *Passive* +: listening, attentive, *Passive* -: unfocused, not on task…"), and affect (e.g., "+: Amazed, joyful…, *Neutral*: calm, relaxed, - : Distressed, angry…").

This study is informed by Kolb et al.'s (2014) experiential learning cycle which consists of a) *concrete experiences* (engaging in an authentic situation), b) *reflective observation* (noticing what has occurred and relating to past experiences), c) *abstract conceptualization* (translating perceptions into abstract understandings), and d) *active experimentation* (honing skills in a new experience). The cycle encourages students to engage in both concrete and abstract thinking through experience and reflection, allowing them to interact with the content and discover connections between ideas. For example, within this study, students engaged in a series of activities centered around identifying the number of mathematical braids that meet a particular characteristic. Following an initial discussion of braids in which students shared their knowledge of what braids are and how they work, the students engaged in a *concrete experience* in which they created braids using pipe cleaners and then attempted to draw the braids using different color markers. Next, the students engaged in *reflective observation* when the instructor asked students how they might draw strings in their illustrations to clearly distinguish which strings were woven in the front versus behind. The students engaged in *abstract conceptualization* as they discovered that leaving a space between the background string and the forward string on both sides of the forward string represented the section of the background string that they could not see. Lastly, students entered into the *active experimentation* phase following this discussion when the instructor posed the question, "How many 3-string braids are there that have just one crossing?" The students used their new understanding to experiment with different possible cases. This cycle was used to engage students in opportunities to develop understanding of a variety of concepts from advanced-level geometry. Instructional moves in each of these phases refer to teacher actions that occurred prior to and during students' engagement in the experiential geometry activities (Bobis et al., 2021; Frey & Fisher, 2010).

Methods

The context for this study was a community-based geometry camp offered on the campus of a large urban university in the Mid-Atlantic. The public school division in which the university is situated is a high-needs school district, serving nearly 23,000 students (75% Black, 13% Latinx, 9% White, and 3% Other) from low-income backgrounds (in the 2017-2018 school year, 100% of students qualified for Free and Reduced Lunch). Priority was given to students from the inner-city school district and portions of surrounding districts known to be low-income areas.

Forty-one students attended the week-long camp; 36 students' data was included in analysis. The students included 18 girls (50%), 17 boys (47%), and one student who identified as non-binary (3%). Eighteen students identified as Black (50%), eight as two or more races (22%), eight as White (22%), and two as Asian (6%). Nineteen students (53%) had completed 6th grade, 10 (28%) students had completed 7th grade, and seven (19%) students had completed 8th grade.

Data sources analyzed were observation protocols, video-recordings of lessons, and surveys. Students took survey items measuring dimensions of student engagement at the end of every day. Video-recordings were transcribed as needed. For the first research question, the second author analyzed survey data by converting responses to a Likert scale of 1 (*strongly disagree*) through 5 (*strongly agree*) and computed averages. For each student, their average scores were calculated for each day of the camp. Using these scores, the students' total averages were calculated to analyze overall engagement per day. Open-ended questions at the end of each survey were analyzed for common key phrases students attributed to their engagement efforts.

For the second research question, the first author watched video to identify cases of "engaged moments" (Yin, 2009) based on the level of students' cognitive (e.g., asking questions, describing their actions) and behavioral engagement (e.g., showing a knot they made). These moments were analyzed using Wang et al.'s (2016) dimensions of engagement, for students' ideas (cognitive), actions (behavioral), participation level (social), and affect (emotional). Instructional moves were inductively coded and analyzed to generate themes in relation to the phase(s) of the experiential learning cycle that was/were demonstrated in the moments.

Results

Quantitative Results

Mean engagement across five days of camp were relatively high: 3.47 (Day 1), 3.53 (Day 2), 3.90 (Day 3), 3.71 (Day 4), and 3.79 (Day 5). Results showed that 0, 14, and 22 students fell into low (< 2.5), average (2.5-3.5), and high (>3.5) engagement groups, respectively. Open-ended survey responses showed that students attributed infrequent breaks to lower engagement, whereas they attributed hands-on activities to higher engagement ratings.

Qualitative Findings

Two cases in which various forms (e.g., cognitive, behavioral) of student engagement were visibly present during the geometry activities were analyzed in-depth to understand the instructional moves that supported students within the experiential learning cycle. One case involved one-on-one interaction between the instructor and a student, while the other case involved interaction between an instructor and three students. Both cases of interaction occurred when students were encouraged to explore a concept individually and/or with their group members, during which the primary and secondary instructors moved around the room to support students' exploration. Overall, findings showed that instructors played an important role in supporting students' engagement in experiential geometry learning. Three major teacher moves that emerged from the analyses are summarized below.

Open-ended questioning. Questions were categorized as open-ended (e.g., "If I were going to try to make yours out of string, what would I do?") or closed (e.g., "It is a type of braid but it is somehow different from this braid, right?"; Chin, 2007). The majority of questions were open-ended: nine out of 10 in the group engagement case and two out of three in the individual engagement case. Use of open-ended questions encouraged students to actively participate by explaining their thoughts regarding what was posed. For example, during the small group case, the instructor posed the question, "What kind of process happened to make this braid that was different from the process that was used to make this one?" Student A responded, "I was folding over and over and over, and he was twisting and twisting." Here, posing open-ended questions created opportunities for reflective observation. Specifically, the student was able to reflect on the differences in braiding techniques which resulted in the production of different braids.

Real-time feedback. Analysis of the engaged moments revealed the instructor's use of real-time feedback which supports students' engagement with and understanding of the concepts (Wisniewski et al., 2020). This was often through the use of positive commentary supporting the students' work, followed by guidance for the students to consider in moving forward. One example of this real-time feedback occurred during the individual engagement case. The student was working to identify the number of braid crossings. The instructor supported the student's thinking and approach by saying, "This is a great way to approach this counting problem. You've broken it into categories where you're going to count what are the possibilities [sic]." Following this positive feedback about the student's moves, the instructor identified an important component to consider within the process, for duplicates and exceptions: "but the things to look for now that are actually technically the same or don't count for some reason [sic]." She then gave an example using cases, which (she explained) is a process mathematicians use. The instructor then encouraged the student to go through his cases and identify whether there are more cases that are equivalent within the cases he had already identified.

Prompting deeper experimentation. Analyses revealed the engagement that occurred following the instructor's use of prompting encouraged students to elaborate on their thinking (Walshaw & Anthony, 2008). Prompting sometimes took the form of encouraging students to consider a particular idea, while other times encouraged physical engagement. For example, in the small group case, the instructor prompted student C to demonstrate how they completed a type of braid. The instructor responded, "Oh okay, you're going full over." As a result of this prompt, student A became involved as well, which led to sensemaking between the two students:

Student A: "Wait, there's an easier way to do that."
Student C: "Huh?"
Student A: "Wait, can I try something? You could have just done… that… just twisting it."
Student C: "Yeah, but that doesn't twist the entire thing like that."
Student A: "Yeah, that's true."

In this example, the instructor's prompting created an opportunity for the two students to engage in active experimentation, where they developed a deeper understanding - together - of a discrepant event: why two braiding processes did not result in the same outcome.

Conclusion

In conclusion, quantitative and qualitative analyses revealed students reported and demonstrated overall high engagement in the camp. The qualitative analysis of two cases involving student engagement during geometry activities provides valuable insights into effective instructional strategies within the experiential learning cycle. The varied instructional moves implemented during these activities support existing literature arguing for intentional instructor actions that support students in engaging with the content (e.g., Chin, 2007; Walshaw & Anthony, 2008). While the geometry camp presents unique contextual considerations that vary from traditional K12 classrooms, the instructional moves presented can be implemented by traditional K12 classroom teachers as a means of supporting students' deeper engagement with geometry concepts and beyond. Our analysis provides a fine-grained look at specific actions making up these instructional moves and student responses, which can provide guidance for instructors looking to implement supportive instructional moves in a variety of learning spaces.

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HOW BEGINNING SECONDARY MATHEMATICS TEACHERS RECONCILE COMPETING PROFESSIONAL OBLIGATIONS

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Building on the theory of practical rationality, we explore how three beginning secondary mathematics teachers reconcile competing professional obligations, namely: disciplinary, individual, and institutional obligations. As these teachers transitioned from supervised teaching to teaching their own classrooms, they reconciled competing obligations and developed their own ideas about mathematics teaching and learning. The analysis revealed that it was only institutional obligation that conflicted with either disciplinary, or individual obligation, or with teachers' own teaching preferences. No other two obligations appeared to clash. The conflict with institutional obligation was reconciled in favor of institutional obligation in less than 30% of instances. In the vast majority of cases, another obligation took precedence.

Keywords: Beginning Secondary Teachers, Teacher Obligations, Practical Rationality

"I thought that my job was to teach math. I was not emotionally prepared for all the other things I need to do in the classroom." These words of a first-year secondary mathematics teacher illustrate the complexity of classroom teaching and the many demands inherent in the profession. Raising the question of "How teachers manage to teach", Lampert (1985) asserts that the work of teaching requires constant management of practical dilemmas caused by competing responsibilities or commitments. For example, a commitment to attend to an individual student's understanding may clash with a commitment to "cover the curriculum;" or a commitment to advance academic achievement may conflict with providing a comfortable learning environment to students. Becoming a mathematics teacher involves, among many other things, learning to manage these types of dilemmas (Herbst & Chazan, 2003; Windschitl, 2002).

Beginning mathematics teachers transitioning from university-based teacher preparation programs to school teaching need support in learning how to recognize and deal with such dilemmas (Bieda et al., 2015). One way to support beginning teachers in this process is through an internship—supervised teaching experience, in which a teacher candidate is placed full-time in the classroom of a mentor-teacher. However, research shows that rather than being supported, interns experience additional competing commitments: toward the university supervisor advocating for ambitious teaching vs. the often-traditional practices of the mentor teacher (Bjerke & Nolan, 2023; Gainsburg, 2012). When entering their first teaching job, novice teachers assume additional classroom responsibilities, some of which have previously been managed by their mentor (e.g., communicating with parents, reporting to administration, coordinating instruction with other teachers). As a result, beginning teachers may feel overwhelmed, and enter a survival mode (Stokking et al., 2003) characterized by rigid, traditional teaching styles. Yet, some novice teachers hold on to the ambitious teaching practices learned in teacher preparation programs (Gomez Marchant et al., 2021; Thompson et al., 2013).

The implied interconnectedness of beginning teachers' early-field experiences and their emerging classroom practices indicates the importance of enhancing our understanding of how beginning teachers learn to reconcile and manage the multiple dilemmas of classroom teaching.

However, research examining this topic, especially longitudinally, has been limited (Cochran-Smith et al., 2008; Gainsburg, 2012). Our study aims to address this research gap.

The study reported herein is part of a larger NSF-funded project that explores the professional growth of beginning secondary mathematics teachers across four years, in multiple settings: from the senior year in their teacher preparation program, into the supervised internship, and their own classrooms. In this paper, we focus on three such beginning teachers: Nancy, Olive, and Diane (pseudonyms) who volunteered to participate in the study. All three teachers excelled in their academic studies as secondary mathematics education majors and demonstrated high buy-in for integrating ambitious teaching practices, as evidenced by the dispositions survey they completed as undergraduates. Moreover, the three participants had some experience with integrating ambitious practices in real classroom settings in their undergraduate preparation (Buchbinder & McCrone, 2023). We examine how these well-prepared beginning teachers coped with the challenges of transitioning from university to school teaching; and, how they reconciled competing commitments and teaching dilemmas.

Theoretical Perspectives

Teacher decision-making draws on many resources, such as teacher knowledge, personality traits, and beliefs. Herbst and Chazan's (2003, 2011) theory of *practical rationality* suggests that beyond individual characteristics, there are certain professional *obligations*, that are common to anyone who holds the position of teacher in the institution of schooling. The authors identify four broad types of professional obligations. The obligation to the *discipline* of mathematics involves authentically representing mathematical concepts, and engaging students with mathematical ideas, values (e.g., accuracy of vocabulary and notation), and practices (e.g., discovery, reasoning, and proving). The obligation to students as *individuals* involves attending to fairness, and consideration of individual student's needs, cognition, and emotions. *Interpersonal* obligation considers the class as a whole, requiring the teacher to manage social dynamics, intergroup relations, and ensure fair sharing of resources, time, and space. The *institutional* obligation requires the teacher to follow school, district, and state policies related to curriculum assessment and standards, and adhere to practices and guidelines shared by members of school mathematics departments.

Becoming a mathematics teacher involves adopting a decision-making framework for managing the work of day-to-day classroom teaching in the institution of schooling. The four professional obligations are an inherent part of this framework, whether explicitly acknowledged by teachers or not. The obligations do not prescribe teacher actions, but rather serve as sources of justification for those actions (Chazan et al., 2016). As pointed out by Bieda et al., (2015) the "obligations can be found in teacher talk as they warrant claims, either explicitly or implicitly, about what should or should not, or might or might not, be done in classroom interaction." Additionally, due to their often-implicit nature, obligations can be captured in situations where they come into conflict with one another. As teachers describe their classroom dilemmas their obligations come to the fore; and by examining the action taken following the decision-making process, we can learn about how the beginning teachers reconcile the competing obligations.

In this paper, we focus on negotiation, managing, and reconciling competing obligations by three beginning secondary mathematics teachers. We examine the following question: *In the discourse of beginning secondary mathematics teachers, what types of obligations surface as* *competing with one another, and how do teachers reconcile competing obligations?*

Methods

Data Collection and Analysis

Three beginning teachers: Nancy, Olive, and Diane volunteered to participate in the study as undergraduates and remained with the project for four years. In this paper, we focus on two time periods: (1) a supervised internship during which each intern taught in their mentor-teacher classroom, and (2) the first year of autonomous teaching. For Nancy, this supervised internship was a traditional length of one year with occasional stretches of autonomous teaching in the second semester. Olive and Diane were promoted to full-time autonomous teaching in their second semester due to the staffing needs of their schools and in light of their exceptional performance. For each participant, we collected multiple video observations, lesson artifacts, and interviews. The data for this paper comes from three interviews conducted with each participant during their first year in classrooms: one at the beginning of the internship and two in the second semester after lessons in which the participants taught autonomously. The fourth interview was conducted in March the following school year by which time they were all novice teachers. All interviews were conducted after one of the researchers observed a lesson taught by the participant. The interview questions probed the instructional decisions involved in the planning and enactment of the lesson.

The interview transcripts were split among three researchers (the authors of the paper), and each interview was coded individually by two researchers. The three researchers met weekly to discuss the coding and reconcile disagreements; such that each code was reviewed by at least two researchers. Teachers' discourse in these transcripts was examined at the utterance level for the presence of an action or decision made by the participant and the justification for that action/decision. These justifications were coded according to the four professional obligations described above (Chazan et al., 2016). In addition, some actions were justified on account of personal resources (i.e., knowledge, beliefs, preferences), when the participants described how they wanted their classrooms to look and feel. For example, consider the quote: "They [students] surprise me every day. I love that every time you do an activity, you're never doing the same thing twice. And every time I try to tweak it a little bit in the right direction." In this quote, Olive's action of "tweaking" an instructional activity is justified on account of her personal enjoyment of the teaching process and breaking the routine ("never doing the same thing twice.")

In this paper, we focus on those instances in our data where two obligations (or an obligation and a personal preference) appeared as conflicting with each other, and the teachers reconciled between them. Although these instances were relatively rare, they illuminate the dilemmas these beginning teachers faced and resolved early on in their professional journey. Hereafter, we refer to these instances as *reconciling obligations*.

Results

Of the total 492 obligation codes, only 35 codes (7%) involved reconciling obligations. For Diane, 8 of 132 obligation codes (6%) were reconciling; for Olive, it was 15 of 156 obligation codes (9%); and for Nancy 12 of 174 obligation codes (7%) involved reconciling. In all instances of reconciling two obligations, one of those obligations was *institutional*. Meaning that either disciplinary or individual obligations or personal teaching preferences conflicted with the institutional obligations. In 28% of all instances, the conflict was reconciled in favor of institutional obligation, but in most cases (72%) it was reconciled in favor of some other obligation (See details in Table 1, below).

Reconciling Institutional and Individual Obligations

In the participants' discourse, institutional obligation most often conflicted with an obligation to students as individuals. This occurred in 18 out of 35 instances (51%).

At the beginning of the internship, some institutional obligations were represented by the need to adhere to the teaching style and the guidance of the mentor teacher. For example, Diane described that her mentor teacher suggested she should show students two ways of solving problems of calculating percentages, but Diane felt it would confuse the students. She said:

> It's like one extra step that they'd [students] have to do that I think would be confusing for them. So, I don't know if I would've even mentioned it, but my cooperating teacher suggested mentioning it.

Here, Diane's desire to avoid confusing students (individual obligation) conflicted with the institutional obligation of following the suggestions of the mentor. Olive experienced a similar tension, although in her case, Olive felt she had to adhere to her mentor teacher's advice to instruct students to solve linear equations by collecting variables on the left side only. Despite her reservations, and because she "didn't want them [students] to do something that the other class didn't do," Olive upheld the institutional obligation and followed her mentor's advice.

As the teachers transitioned to autonomous teaching, the relationships with the mentor were no longer a concern. A different aspect of institutional obligation was now discussed by the teachers but remained in tension with the individual obligation. Specifically, the teachers talked about the tension between the need to "cover the curriculum" while at the same time attending to individual students' needs, prior knowledge, pace of learning, and even moods. Teachers mostly reconciled this tension in favor of the institutional obligation. For example, Diane described her context saying that there are "two students that are completely behind every time and I can't stop and constantly work with them, but I feel bad moving on." Similarly, Olive described the need to move along the curriculum while attending to individual students' learning pace. She shared:

> The reality of the situation is that I cannot give the same work to every kid in this class and expect them all to do well on it. That is just not the case. There are kids who need more than other kids. […] It's definitely taught me a lot about trying to differentiate and trying to make sure I have extra […] resources and things for them to do so that they're not bored.

In this quote, Olive acknowledged that students learn differently; therefore, her solution to managing this diversity was to differentiate her curriculum, ensuring some students were "not bored" while others receive different types of tasks in order "to do well."

Of the three teachers, Nancy was the one to express the tension most explicitly between institutional and individual obligations. She described this as follows:

> I feel like the standards and the things that they want us to teach are taking away from some of the fun stuff that we can do with it. And so, […] I get kind of frustrated because I wanna do fun stuff, but I also know I have other stuff I have to do and it's just kind of trying to find a happy medium, which I don't think I've gotten to yet.

In this quote, Nancy described the tension between the institutional obligation to address the content standards of her curriculum ("things they want us to teach") and her desire to create an engaging learning environment for the students ("fun stuff we want to do"). She described her frustration with the situation admitting that she has not yet found a "happy medium."

Despite the strong influence of institutional obligation, all three teachers described how they resolve, or strive to resolve the conflict in favor of individual obligation. For example, Nancy explained how she worked on revising the curriculum that was handed down to her to make it more engaging and "fun" for the students:

> That's something I've been working on since the beginning because in the beginning of the year it was basically just worksheet review, worksheet, review, worksheet, notes. Like there was no engagement happening. Um, and that's just because those were the resources I was given. But now it's a lot more interactive. There's a lot more fun activities built in.

Diane talked about reconciling the tension between institutional and individual obligations in terms of finding ways to "set them [students] up to be successful without making it too hard, but also without lowering expectations too much." Diane disagreed with some of her colleagues' advice of breaking the problems into isolated skills. She said:

> Talking with some other teachers, they would be like, 'oh, you'll never mix quotient rule with negative exponents.' […] I really want to push them [students] to be able to problem solve through a couple of steps. My goal is trying to find a manageable way to set them up to do that without making the work so difficult that they don't do well.

The institutional obligation in Diane's quote is represented through the community of teacher colleagues, and the advice given to her. Yet the obligation toward individual student thinking and their ability to problem solve takes precedence for Diane in this instance.

Reconciling institutional and individual obligations in favor of the latter sometimes took the form of developing greater sensitivity to students' feelings. Olive explained this as follows:

> I've got a pulse for how the kids change every day. Sometimes they're in a […] good mood. Sometimes […] it's not time to bug that student. […] that kid is in a place today that this math is not the biggest concern right now. […] I can work with that kid tomorrow and that's okay for today because that kid is not having a day.

In this quote, Olive described how she gradually developed "a pulse" for her students' moods, and their ability to act as students in her classroom. Using the word "kids" rather than "students" indicates her obligation to them as individuals, as Olive described prioritizing their well-being over moving on with the curriculum.

Of the 18 instances of conflicts between institutional and individual obligations, the teachers reconciled the dilemma in favor of institutional obligations 5 times and in favor of individual obligations 13 times. While these numbers are small and cannot be generalized in any way, we report on them to provide a sense of data trends.

Reconciling Institutional and Disciplinary Obligations

Institutional obligation competed with disciplinary obligation in 15 out of 35 instances, and in 10 of them the conflict was resolved in favor of disciplinary obligation. The disciplinary obligation was represented by the teachers' commitment to have students learn important mathematical concepts and procedures meaningfully and thoroughly. Additionally, this entails having students learn mathematics in ways aligned with disciplinary values and practices such as exploration, discovery, reasoning, proving, tinkering, and figuring out things for themselves.

Enacting classroom activities that uphold these disciplinary practices may be demanding for novice teachers and may compete with the institutional obligation. For example, as a novice teacher, Nancy received instructional materials from a more experienced colleague. Nancy expressed frustration with the traditional nature of these materials and with an implicit expectation to align her instruction with this mode of teaching. She said:

> We're doing translations and there are so many cool things you can do with translations. […] And I just feel like I need to do the guided notes and practice problems, even though, I don't know, I guess I don't necessarily have to, but it's just like, there are these notes and things that in the past it's been really great for her [another teacher] and it works for her. And I'm just like, I just don't wanna sit there and talk to them for 30 minutes at a time.

This quote shows Nancy's perceived obligation to uphold institutional expectations to coordinate instruction among teachers of the same grade level ("I feel like I need to do the guided notes"). This conflicted with her obligation to disciplinary practices of doing meaningful mathematics ("many cool things you can do with transformations"). Despite Nancy's dissatisfaction, the institutional obligation seems to take precedence in this instance.

Other examples of reconciling in favor of institutional over disciplinary obligation occurred when teachers were pressed for time or struggled to manage classroom discussions. As a result, they cut short an exploratory activity (e.g., "We didn't end up doing this [exploration] because of time. I had to get to the next stuff"); or lowered the conceptual depth of the discussion (e.g., "[If I] try to circulate that room and have a deep conceptual conversation with each of those 28 [students], I don't even think I'd have time in the block to do that").

Nevertheless, in most instances (10 out of 15) the teachers prioritized disciplinary obligation over the institutional. For example, as an intern, Olive modified her mentor's lesson plan about linear inequalities to introduce a short exploration for students to understand why multiplying or dividing an inequality by a negative number changes the sign of the inequality. Olive admitted she had to "push to do that just because in the original lesson plan […] the idea of flipping the inequality sign is not really explored at all." Olive explained that she was "worried that they [students] were going to ask why, and I didn't want to not have an answer to that question." This shows that Olive had to overcome the institutional authority of her mentor and of the prescribed curriculum, to provide a conceptual justification for a mathematical rule. As a novice teacher, she continued to modify her curriculum to make for a more conceptually rich practice. She said:

> If there [is] a worksheet with a ton of problems on it, I will try to deliberately choose three different ones that […] really throw you for a loop, so that they [the students] see different representations of problems where they're doing a similar process, but they're seeing there's something novel about each one.

Similarly, Diane described how she attempted to uphold the institutional goal of having students practice surface area and volume formulas through an exploratory activity where students calculated the surface area and volume of physical objects wrapped in aluminum foil:

> I didn't do much practice with them even using surface area formulas […] and volume formulas. So, they [the students] had the practice of figuring out what the formula means,

plugging the numbers in. […] It was different […] from every day because they [the students] were kind of drawing their own conclusions at the end.

In this quote, Diane almost apologetically admitted that she "didn't do much practice" with the students. This practice was achieved by breaking out from the everyday routine and by upholding the disciplinary obligation of having students "figure out what the formula means, plugging numbers" and "drawing their own conclusions."

The theme of breaking the routine, "doing something different," (Diane) introducing "fun stuff" (Nancy), and "just really hate[ing] the drill and kill idea" (Olive) was common for all three teachers. Sometimes, exploratory activities "did not fit in naturally," in Nancy's words, with the ongoing curriculum topics. For example, Nancy used the pretext of Pi-day (March 14th) to have students explore the value of Pi while teaching linear equations. Despite the tension with the institutional obligation, all three teachers found creative ways to uphold the disciplinary obligation.

Reconciling Institutional Obligation with Personal Preferences

Occasionally, teachers described tensions between institutional obligations and their personal beliefs and preferences about teaching; these tensions were resolved in favor of the latter. Olive discussed her choice to shorten homework assignments, saying that the shorter assignment was more "fair" for students and that this choice was "the first time I had done something that was my idea." Similarly, Nancy described going against her mentor teacher's typical grouping of students into teams of three, saying "I wanted to do groups of four because I feel like groups of four are better". In each case, the institutional obligation (giving homework, grouping students) was overridden by teachers' personal preferences for how they wanted to organize their classrooms (shortening homework and changing group size), which ultimately shaped their actions. Although such conflicts were rare, it was important to these teachers to follow their personal beliefs about teaching, rather than always strictly adhering to institutional norms.

Summary and Discussion

Table 1 summarizes the distribution of codes showing how the three participating teachers reconciled between competing obligations in favor of one of them. For each type of code, the percentage is calculated out of the total N=35 reconciling codes. For each teacher, the total percent of reconciling codes is calculated out of the total number of obligation codes per participant. The results of case studies are not meant to be generalized statistically (Yin, 2017); we report on these frequencies to provide a general sense of data trends.

Table 1: Distribution of Codes for Reconciling Obligations (N=35)

Teacher	**Institutional vs. Individual** Resolved in favor of		**Institutional vs. Disciplinary** Resolved in favor of		**Institutional vs. Personal preferences** Resolved in favor of the latter
	Institutional	**Individual**	**Institutional**	**Disciplinary**	
Diane	2	3	0	3	0
Olive	2	6	2	4	1
Nancy	1	4	3	3	1

Total	5 (14%)	13 (37%)	5 (14%)	10 (29%)	2 (6%)

As mentioned above, reconciling codes constitute only 7% of the total obligation codes; however, they are significant as illuminators of the tensions the beginning teachers encounter as they transition from university to school. Our data show that the main source of conflict for the participants along this journey was their *institutional* obligation. It appeared to clash primarily with either disciplinary or individual obligations and occasionally with participants' beliefs, in our case, their desire to enact ambitious teaching in their classrooms. Similar to the observations in the literature (e.g., Bieda et al., 2015; Smagorinsky et al., 2004), the institutional obligation first surfaced at the beginning of the internship when the participants needed to adhere to the teaching practices and styles of their mentors. This obligation became even more pronounced during autonomous teaching since teachers had to assume additional responsibilities in their classrooms and adhere to the practices of their schools (Lampert, 1985; Windschitl, 2002).

Teachers reconciled in favor of the institutional obligation when they felt pressed for time, either in specific lessons or more broadly, with respect to the pace of the school curriculum and content standards, for example, when there was an expectation to follow the shared curriculum and coordinate instruction amongst multiple teachers and classrooms. Additionally, the class size and the need to manage multiple students or groups inhibited teachers' perceived ability to go into conceptual depth on certain mathematical topics. In these instances, the teachers often had to compromise other obligations to ensure they were in line with the institutional expectations.

However, our data show that in the vast majority of situations (72% of reconciling codes) the conflict with institutional obligation was resolved in favor of some other obligation: disciplinary, individual, or personal preferences. On a side note, we did not encounter cases of institutional obligation clashing with interpersonal obligation. This may be due to the overall low frequency of reconciling codes within the data set; an observation that bears future exploration.

The beginning teachers reconciled in favor of *individual* obligation, in situations in which they felt the curriculum did not support students' classroom engagement. Nancy strived to include more "fun" and "interactive activities" breaking away from the "worksheets and review" routine. Diane devised tasks to make procedures "manageable" and "not too hard" for students, but without compromising problem-solving and "without lowering expectations too much." For Olive, reconciling institutional and individual obligations in favor of the latter was realized in developing heightened sensitivity, "a pulse" in her terms, to students' feelings. She talked empathically about how she can suspend the institutional obligation to the pace of the curriculum to accommodate a student who is stressed or maybe "just not having a day."

Additionally, all three teachers found ways to uphold the *disciplinary* obligation when it conflicted with the institutional obligation. This was apparent in the teachers' expressed desire to enact ambitious teaching practices. This took the form of integrating exploratory activities for discovering mathematical rules and relationships. Olive talked about doing "explorations" to justify the rule of 'flipping the inequality sign," while Nancy integrated an activity about discovering the value of Pi despite its loose connection to the ongoing curriculum topic. The obligation to engage students with disciplinary practices took the form of engaging students with disciplinary mathematical values and practices, like free exploration (Diane), reasoning and justifying (Olive), and allowing students to choose which solution method they want to pursue (Diane). Reconciling the tension in favor of disciplinary obligation did not come easy but took the form of *small* steps like "tweak[ing] things ever so slightly" or doing "little explorations."

Our findings illustrate the strong influence of institutional obligation on the day-to-day work of teaching and the pressure it imposes on beginning mathematics teachers. While the literature suggests that beginning teachers tend to gravitate toward traditional teaching practices in their schools (Gainsburg, 2012; Windschitl, 2002) we are encouraged by our results showing the ability of these beginning teachers to navigate institutional obligation without caving into it.

These findings shed light on the complex situations beginning teachers face as they transition from the idealized setting of their teacher preparation programs into the challenging realities of school teaching. The theory of practical rationality and the four professional obligations (Chazan et al., 2016) help to conceptualize these transition processes as the *socialization* of beginning teachers into the teaching profession during which teachers adopt a particular decision-making framework that makes their classroom practice manageable (Herbst and Chazan, 2011). Teacher educators can build on these conceptual tools, and on the results of the current study, to support future teachers in retaining ambitious teaching practices in the institutions of schooling.

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WORKING TOWARDS STRENGTH-BASED PEDAGOGIES IN MATHEMATICS EDUCATION

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The purpose of this brief research report is to provide insight into Kaja's ongoing doctoral work on storylines and student positionings in mathematics education. Based on a mentorship program in a Norwegian multicultural primary school, where 7th grade mentors collaborate with and mentor 3rd grade mentees during mathematics classes on a weekly basis, this research report explores how storylines can influence the positioning of the mentors. The report presents two storylines that indicates that the mentorship program could have a positive impact on the mentors positionings.

Keywords: Culturally Relevant Pedagogy, Affect, Emotion, Beliefs, and Attitudes

Context: A beautiful transformation of students' positioning

In February 2022, when the first author, Kaja Burt-Davies, started her PhD work, she visited the teachers and classes that had agreed to participate in a larger research project. She immediately understood that one class faced challenges. The teacher apologized for the behavior of the class and explained that it had been a challenging class since they started school. But with a twinkle in her eye, she also said something like, "Well, at least in this class, we have real challenges. It's not the kind of class you read about in textbooks". The classroom was chaotic. At one point, there were 7 adults in the room to manage 18 5th graders. Tired and frustrated after a lesson, the teacher declared: "The only thing that works with this class is to take a bus or be mentors." Kaja noted this in her notebook. A few months later, she saw mentorship in action. The mentees were 1st graders in a classroom across the corridor. At that time, to her, it seemed almost unthinkable that this class would mentor anyone; they couldn't even take care of themselves! However, she was astonished when she saw how the older students embraced the role of mentors. It was as if someone had cast a magic spell over the students. In groups of 2–6 students from both classes (carefully assigned by the teachers), the older students focused and quietly helped the younger ones with mathematics tasks. The older students read problems, counted on their fingers, and demonstrated with pen and paper to the younger students. When she heard one of the most challenging boys say to a younger boy, "You have to read the problem if you don't understand what to do; I'll read it out loud for you," she had to stop. What was happening here?

Objectives and research question

This brief report explores how storylines may impact students' opportunities to assume positions that are beneficial for learning. The aim of the report is to emphasize how mathematics teachers can utilize storylines to establish learning environments where students can position themselves as mathematics learners. To illustrate this research, two storylines are drawn from data collected in the previously mentioned classes, now consisting of 3rd and 7th graders in southern Norway. For three years, the two classes have been meeting weekly through an in-school mentorship program to engage in mathematical activities together. This leads to the following research question: How do mentees' storylines impact the positioning of the mentors?

Towards strength-based pedagogies through the lens of positioning theory and storylines

Kaja's doctoral work is affiliated with the research project *Mathematics Education in Indigenous and Migrational contexts: Storylines, Cultures and Strength-based Pedagogies* (MIM project), where position theory (Davies & Harré, 1990) and the concept of storylines are central. The MIM project explores strength-based pedagogies in mathematics education. We aim to explore how students' strengths and resources (both personal and academic) can be used as a starting point to create learning contexts where students perceive themselves as mathematics learners.

Positioning theory

Herbel-Eisenmann et al. (2015) define positioning as a discursive process involving action and communication to create social structures. In contrast to the static nature of the concept of "role", Davies and Harré (1990) propose that "positioning" directs attention to dynamic aspects, emphasizing how we understand ourselves in dialogues through the terms "positioning" and "subject position" based on existing narratives. Through the MIM project, we aim to promote positioning theory as it brings new perspectives to understanding learners. As most mathematics teachers probably recognize, it is common in mathematics classrooms to have students who desire to learn mathematics as well as those who do not (Andersson et al., 2015).

Through the lens of positioning theory, it is possible to say that students accept or reject positions as mathematics learners. By adopting a positioning theory lens, we assert that teachers can create learning contexts that offer opportunities for students to pivot (Gerbrandt & Wagner, 2023) or change their positions to enhance their engagement when learning mathematics.

Storylines

Based on our experiences as mathematics teachers, we have observed that students' positions often align with one or more narratives that support their views on their position as learners or non-learners. Positioning theory (Davies & Harré, 1990) refers to these types of narratives as storylines. Herbel-Eisenmann et al. (2015) define a storyline as a culturally shared narrative derived from an individual's lived experiences. Interactions among participants contribute to the creation of storylines, which serve as prerequisites for different positions in people's lives. Storylines play a significant role in shaping and influencing the positions individuals are assigned or have access to (Herbel-Eisenmann et al., 2016). During an interview about mathematics, their view on their futures and dedication in mathematics, Farrokh, a 6th grade student, shares one of his dreams:

Farrokh: I want to become a doctor.
Kaja: A doctor?
Farrokh: Yeah, something like that.
Kaja: Why is that?
Farrokh: Because... well, I don't really know what I want to be myself, so my mom wants me to become a doctor. Since I don't know myself, I want to listen to her. And being a doctor is actually a good job.
Kaja: Mmm, why is it a good job?
Farrokh: I don't know much, but I've heard... my mom says it's because, firstly, it has a pretty good salary. And being a doctor or a physician is something you need everywhere you go. So, it's easy to find a job if you're a doctor and have a medical education because

> doctors are needed everywhere. That's why it's easy to get a new job if you move and stuff. That's why it's a good job.

This example illustrates how Farrokh relies on narratives and information he receives at home to construct a narrative for his school life. He positions himself in relation to the future because of the conversations he has with his mother at home. As a result, a storyline such as "becoming a doctor is a good career choice" is supporting Farrokh in his decision to learn mathematics because the effort seems crucial to achieve his goal of becoming a doctor.

Methodology

To understand how the mentor-mentee relationship influenced mentors change of positions, semi-structured group interviews were carried out with both 3rd and 7th grade students, in total 10 students from each class. The 3rd graders were interviewed in pairs, and the 7th grade students individually, in pairs, or in groups of three. For practical reasons, interviews with 3rd graders were conducted first, followed by interviews with the 7th graders.

Thematic analysis

Even though the ongoing analytic process has an inductive framework, the analysis is not free from theoretical or epistemological commitments (Braun & Clarke, 2006). Committed to the [project name], and the context of positioning theory and strength-based pedagogies, the analysis is guided by the search for storylines that can explain "the magic process", described initially—the beautiful transformation of the students' positioning. The analysis process started with manually transcribing all audio recordings, followed by categorizing the text into various themes. These themes were partly derived from the interview questions, which primarily addressed practical aspects like student collaboration and mentor guidance. However, additional themes and sub-themes were also integrated. Notably, *admiration* emerged as a theme in mentee transcripts, while *being a little teacher* was a theme in mentor transcripts. These two themes, later developed into storylines, constitute the foundation of this research report.

Initial results

Storyline 1: "I admire you"

One storyline that emerged most prominently during the analysis of the mentee's interviews was *I admire you.* This storyline was formulated based on many student statements but is rooted in the fact that the mentees look up to the mentors. The reasons for their admiration are many and varied. For instance, Nora explains that *the mentees need the extra little teachers.* Besides acting as *little teachers*, Nora also states that mentors *can be of great help.* Not just with mathematics but also if they *get stuck or something like that*. Jakub og Fariah have other reasons. Jakub talks about the mentors' experiences: *so every day, they had math, that's why they are so good at it,* and Fariah says: *they are getting taller and taller and bigger, and then they almost become adults, and that's when they become better at math every single day.*

Storyline 2: "I like being your little teacher"

In response to storyline 1, "I admire you," storyline 2, "I like being your little teacher," is visible in all interviews. Mentors were asked what they thought about being called a little teacher or almost an adult. The reason behind this question was to find out if mentors' accepted positions made available through storyline 1. Despite facing some challenges, they agree that mentoring younger students positively impacts their self-esteem, with expressions of feeling proud, tough, and skilled. Due to space constraints, we've selected one example of mentor responses:

Kaja: But do you feel grown up? [when working with the mentees]
Madelen: Yes, very much.
Kaja: How is that?
Thea: It's kind of fun.
Madelen: You feel like you're so smart and strong!
Thea: You feel like you're the big, strong, smart, cool one.
Madelen: Not cool?
Thea: No? I feel...tough!
Kaja: But if we removed the whole mentor thing from school? Does it give you something to be mentor?
Madelen: Yes, I feel like I like it because I feel smart. And I'm actually smart in my class too, just saying.
Thea: Yes, you are.
Kaja: But how does it make you feel when others think you're smart?
Madelen: I feel [giggles, laughs] better than the others, I feel a bit egoistic but [giggles].
Kaja: Yeah, but there's something about it, your self-esteem grows. Do you think it helps that you have even better self-esteem because of the mentees?
Madelen: Yes, I feel that.

Discussion

The storylines presented in this report are based on interviews linked to a mentorship program. "The magic process" was chosen because the change of context from their ordinary classroom had a major impact on the mentors' behavior. When the mentors changed their role from student in a challenging learning environment and instead entered the role of mentors where they interacted with different, younger students, they accepted and acted out a position as little teachers. We believe this was because the mentors experienced admiration and as Madelen confirms in the dialogue above, working with the mentees helps her self-esteem. In the MIM project, the goal is to move towards strength-based pedagogies in mathematics education. Through the lens of positioning theory, we are trying to understand how students' strengths and resources (both personal and academic) can be used as a starting point to create learning contexts that offer students positions where they perceive themselves as mathematics learners. We believe that the example highlighted in this report demonstrates how contexts can have a significant impact on students' self-perceptions. This, in turn, may influence their view of themselves as mathematics learners, ultimately leading to a pivot or change in students' positions. In this way, a focus on storylines could potentially serve as a foundational support for transitioning to strength-based pedagogies.

The authors of this article encourage readers to actively participate in contributing their valuable insights and thoughts. Your perspectives are crucial in fostering a collaborative dialogue on the potential applications of positioning theory and storylines as effective tools for the development of strength-based pedagogies. Your contribution will not only enrich the ongoing discussion but also help us better understand how these frameworks can be used to create an empowering and inclusive educational environment. Feel free to share your experiences, reflections, and ideas, both with fellow readers and us.

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UNDERSTANDING TEACHERS' UPTAKE OF INSTRUCTIONAL NUDGES

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Keywords: Professional Development, Research Methods, Instructional Activities and Practices

One goal of professional development (PD) is to improve practice by having teachers incorporate what they learn into their existing instruction. We refer to teachers' incorporation of these practices as uptake. Our project seeks to incrementally impact (Star, 2016) teachers' practice through their uptake of instructional nudges, modest instructional suggestions, that are in concert with their existing instructional practices. We designed 16 instructional nudges that varied with regard to whether they were intended to impact curriculum materials (task nudges) or teachers' actions (teacher nudges). For example, Pivot is a teacher nudge that encourages teachers to change the instructional format during their class period (e.g., shifting from whole-class discourse to independent work time) with the goal of increasing student engagement by refocusing the classes' attention. Rate & Review, a task nudge, encourages teachers to provide students with worked examples to rate and review in terms of the quality of the solution in the hopes of growing students' conceptual understanding. The purpose of our study is twofold. First, we seek to understand which of our instructional nudges were high-uptake (Author, 2022. Second, we want to understand what features impact the rate of uptake.

To accomplish our aims, we piloted a PD experience in which we provided instructional nudges to seven algebra teachers. The teachers ranged from novice to veteran and varied in their local context (e.g., school racial and ethnic diversity, location, and socioeconomic status). The main data source for the present study were individual, semi-structured interviews during which each teacher interacted with a heat map activity. The heat map consists of two axes on a continuum of, Hate It to Love It, on the horizontal axis and, Number of Tries, on the vertical axis. Each participant placed each of the 16 instructional nudges on the continuum with respect to their affinity towards the nudge (e.g., hate or love) and the number of times they tried it in their classroom, asking them to think-aloud as they placed each one. Each teacher also completed a survey that aimed to understand their perspectives on their instructional practices, and we conducted three observations across the year to examine teachers' practices both before and after accessing the PD. We coded the interview data for each teacher individually with regard to the factors they considered in using particular nudges and the factors that impacted their view of the nudges. We then looked across the teachers for patterns regarding the factors impacting uptake. We used the survey and observation data to confirm our interview findings.

Our preliminary findings indicate teachers heavily consider the amount of preparation necessary in deciding to take up nudges. In addition, nudges that align with teachers' goals and current practices contribute to their uptake. We will share the results of the heat map interviews. Our findings provide important insights for the design and development of PD and suggest the need for further research into features impacting teachers' uptake of PD in their practice.

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TEACHERS' VIEWS OF MATHEMATICS IN DATA VISUALIZATION ACTIVITIES

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Keywords: Data Analysis and Statistics; Integrated STEM / STEAM; Informal Education

Data visualization has been proposed as a multidisciplinary topic to integrate mathematical thinking, statistical thinking, critical media literacy, and art or design in ways that highlight critical perspectives (Matuk et al., 2022; Rubel et al., 2021; Woods et al., 2024). While research has focused on students' engagement with data science tasks and curriculum (e.g., Calabrese Barton et al., 2021; Lee & Delaney, 2022; Matuk et al., 2023; Wilkerson & Laina, 2018), less research has focused on how teachers or other educators take up the role of supporting student learning about data. Building on theory from interdisciplinary learning about the importance of teachers' views about the disciplines and their connections (Cohen et al., 2022), we focus on how teachers view the discipline of mathematics in relation to data visualization.

This research aims to explore the following research question: how do novice mathematics teachers view mathematics or mathematical thinking as part of youth's engagement with data visualization activities? We present the preliminary analysis of episodes in an ongoing exploratory case study (Yin, 2009) of the sensemaking of instructors who co-led a 13-week data visualization informal education program for middle and high school grades youth. The team of instructors included an experienced art teacher, an experienced English teacher, and two novice mathematics teachers. The authors identified critical episodes in transcribed video-recorded planning sessions and reflective debriefs by searching for the words math, statistic(s)/stat(s), data, and art in the transcripts. We identified episodes where multiple participants used these words to discuss a similar topic that lasted longer than 1 minute. Both authors identified themes through collaborative analytic memoing and discussing episodes (Saldaña, 2015).

We found that instructors compared math, data, and art as distinct labels for, or types of, activities, youth's interests, and sensemaking. One example comes from an debrief episode about activities that stood out to instructors. Luis, a novice math teacher, commented that in contrast to youth showing interest in art, he wished he "would have seen that kind of spark when we did the math portion…at the very beginning they were, like, already negative…[about] the math part, the mean, median, mode, [and] range." Beth, a novice math teacher, shared that the "dear data" portion was "something so simple and making it math…how many times I took a picture?...It's saying, 'hey, math can be so small, it's what you do every day, you don't know it.'" Sarah, an experienced English teacher, contrasted doing math with making meaning, saying, "A lot of them were just stuck on tallying things and adding them up. 'I'm like. okay, what about as a percentage?'...And so, [our focus is] not even doing math…we're trying to get at like 'is mean meaningful?'" This example shows how the novice math teachers use the label of "math" for activities and youth's experiences in different ways, and suggest potential tensions in disciplinary definitions among teachers that may constrain or expand learning opportunities.

These findings suggest that explicit conversations and sensemaking about what "counts" as mathematics in connection to data science education and data visualization could be important for mathematics teachers, including novice mathematics teachers, to support expansive learning.

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AN EXPLORATION OF PRESERVICE TEACHERS' USE OF DISCRETIONARY SPACES AND IMPACT ON STUDENT POSITIONING

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As the field of mathematics education research has progressed, scholars and professional associations have contributed to and advocated for an increased focus on equitable teaching practices that are to be adopted and employed by mathematics teachers. This study analyzes how two preservice teachers (PSTs) use discretionary spaces during their first field experience for a secondary methods course. The data consisted of video recordings, transcripts of those recordings, and a reflection paper from both PSTs. To perform the analysis, we used an empirical discourse approach to identify existing themes. Our work suggests that choices made in discretionary spaces came from the preconceived expectations PSTs had of social roles in the classroom. This indicates that an intervention shifting the expectations of classroom roles can promote considerations of equity in the positioning of teachers and students.

Keywords: Instruction Activities and Practices, Classroom Discourse, Preservice Teacher Education, Teacher Beliefs.

Introduction

As the field of mathematics education research has progressed, scholars and professional associations have contributed to and advocated for the increased focus on equitable teaching practices that are to be adopted and employed by mathematics teachers (e.g., Aguirre et al., 2013; Leonard et al., 2010; Crespo et al., 2021; Goffney & Gutiérrez, 2018; National Council of Supervisors of Mathematics [NCSM] and TODOS, 2016; Association of Mathematics Teacher Educators [AMTE], 2017). In this same breath, we as a field, have also looked inward to assess the effectiveness of the ways in which we, as researchers and teacher educators, prepare teachers to effectively enact these practices (McDonald 2005; Hollins, 2011; McDonald et al., 2013; Lyiscott et al., 2018). One potential way of doing so is highlighted in Indicator C.4.2. of the AMTE (2017) *Standards for Preparing Teachers of Mathematics*, which states that "well-prepared beginning teachers of mathematics recognize that their roles are to cultivate positive mathematical identities with their students" (https://amte.net/node/2270). Our work explores the choices preservice teachers make during classroom discussion that either shift or reinforce the positioning of students. These positionings, in turn, prompt a negotiation of roles that have direct impacts on student mathematical identities (Ruef, 2020).

We decided to isolate moments of teachers cultivating mathematical identities within *discretionary spaces* (Ball, 2018; Berry, 2022; Berry, 2023) as these spaces provide focused, in-the-moment insight into teachers' use of discretion in the classroom that is not dictated by policy or curriculum. These "in-the-moment" decisions of facilitating classroom discourse and the corresponding impact on student *positioning* (Herbel-Eisenmann, et al., 2015; Bishop, 2012) contribute to the cultivation of student identities. Previous research provides examples of the impact of communication acts, both verbal and non-verbal, on the mathematical identity of students. For example, Wagner and Herbel-Eisenmann (2008) explain how "just" can be used as an imperative ("just focus on your work") or as an indicator that the answer should be easy ("just subtract five from both sides"). Both cases illustrate how communication acts affect the development of roles within a classroom. When "just" is used as an imperative the teacher takes an authoritative tone creating a sense of hierarchy within the classroom and indicating that the student should blindly follow. On the other hand, "just" as an indicator of simplicity can result in students who don't understand the concept feeling inadequate and as an outsider to the conversation. This assignment of roles and the results thereof has the ability to influence future student contributions and their perceived sense of legitimacy (Wagner and Herbel-Eisenmann, 2008).

To better understand these issues, we investigate this research question: *How do pre-service teachers navigate discretionary spaces in mathematics classrooms, and to what extent does their use of these spaces impact the positioning of students in the learning process?*

Theoretical Framework

We draw on positioning theory (van Langenhove, L., & Harré, R., 1999; Anderson, 2009; Dennen, 2006) as it explains how each participant in a communication act plays a part in negotiating social structure. Within a classroom, teachers often exercise authority as a means of classroom management and control. In this storyline, teachers assume the role of subject matter experts, which leaves students as recipients of information (Wagner & Herbel-Eisenmann, 2015). An alternative is a shared authority structure where students become co-constructors of knowledge, contributing to the general discourse with the teacher as a guide or possibly even on the peripheral as an observer (Harper & Kudaisi, 2023). We are interested in how these negotiated storylines are enacted by preservice mathematics teachers in their early field experiences. Due to the nature of our study, and the complexity and scope of positioning theory, we have narrowed our investigation to only focus on how positioning is used to identify and apply this theory to help better understand how teachers position students during classroom interactions.

Further, positioning is a dynamic action, changing from moment to moment based on moves that either reinforce the established position or shift how participants understand their role with others. For example, in a bulleted list, Bishop (2012) identified moves between two students based on the dialogue they shared, these include the use of an authoritative voice (being critical of the actions of others), face-saving moves (reducing the appearance of a lack of understanding) and the building of solidarity and provision of encouragement. Each of these moves shifted the identities enacted by the students. While we did not limit our analysis to these three movements, they appeared in discretionary spaces within the observation.

Methodology

To conduct this work, we used previously recorded video and audio data from a secondary methods course during the first round of field observations for a group of 14 preservice teachers (PSTs). The course focused on teaching at the middle school level, with an emphasis on analyzing and understanding student thinking and implementing instructional practices in middle school classrooms. The course-embedded field experiences took place in a midwestern public middle school with a majority African American population, located in the heart of a small city that was less diverse. Each PST completed a brief survey at the beginning of the course that was intended to gauge their experience with teaching as well as their beliefs and attitudes of "good teaching" prior to their first field experience. After the experience, they submitted a reflective paper that detailed their impression of the enactments based on an analysis of a video recording of their teaching and the student work associated with that enactment.

As part of the teaching experience, the PSTs were paired and given the same instructional routine of Contemplate then Calculate (Lucenta & Kelemanik, 2022) to teach to a group of 12-15 middle school students during their mathematics period. This allowed PSTs to elicit student thinking strategies by posing complex integer addition and subtraction number expressions that had underlying structures for the students to recognize and use to mentally evaluate the expression.

Of the 14 total PSTs taking the course, seven agreed to participate in the research study. Guided by our research question, we analyzed three video enactments where both PSTs agreed to participate in the study. The video enactment of the seventh PST was analyzed separately because their partner did not consent to participate. The two researchers independently reviewed the videos of the enactments with a focus on identifying episodes of PSTs using language in ways that influenced the positioning of students in the classroom. After this initial round of independent analysis, researchers reconvened to discuss their findings; both identified the same two PSTs for further analysis of the construct, Jaime and Alex (pseudonyms) because although they were presenting similar tasks, their individual approaches to teaching resulted in vastly different responses from students. It is important to note that Jaime and Alex did not perform their enactments as a pair, therefore, the analysis of their use of discretionary spaces was done independently from their partners. We operationally define discretionary spaces to analyze moments during instructional time where teachers make a choice that shifts or directs the learning trajectory of either individual students or the entire classroom. These spaces were identified when PSTs made an instructional move (choice) or used their discretion to facilitate the lesson and ended whenever PSTs shifted their focus away from the initial topic and began a new one.

Once the two PSTs were selected, we transcribed the recordings using an online resource (https://otter.ai) and verified the transcriptions, making minor adaptations as needed. We used an empirical discourse analysis approach (Hodges et al., 2008) that allowed us to identify broad conversational themes contained within discretionary spaces (as defined above) that had the potential to shift or influence student identities in the classroom. We performed a second, more targeted analysis that focused on the two selected PSTs. This second analysis was also performed individually then transitioned to a discussion between researchers. Both researchers identified the same discretionary spaces and labeled them as rich opportunities to examine various ways students are positioned based on the language (verbal and nonverbal) used by the respective PSTs. The identified moments were categorized into *statements* and *interactions*. Statements are uninterrupted utterances by the PSTs, while the interactions include the verbal and non-verbal responses of students. We acknowledge that the positionings discussed in the following sections may vary based on the perceptions of the researcher. Therefore, we do not claim that our positionings listed are the only possible interpretations; they are based on our history and familiarity with the current literature.

Findings

We addressed our research question first by focusing on various discretionary spaces. Initially we looked at individual statements uttered by the PSTs that gave meaningful insight to positioning structures, then at brief interactions. During several moments of the enactment, the two PSTs navigated similar discretionary spaces, which gave an opportunity for comparison

Statements

In Figure 1, similar discretionary spaces between the two PSTs were placed side-by-side for comparison. On either side of observation quotes are possible interpretations of the language used by Alex and Jamie. Notice that there may be alternative positioning interpretations, but based on the perspectives of the researchers, these interpretations seemed most plausible.

Alex		Jamie	
Positioning of Students	Utterance	Utterance	Positioning of Students
Student as an active contributor of knowledge	"Good morning you guys… So we're gonna be going over an instructional routine called contemplate then calculate, [it] will help you practice looking for shortcuts using your own mathematical knowledge…"	"Alright… so today we're going to be working with y'all on contemplate then calculate… we do this to try to get you think like mathematicians…"	Student as a passive receiver of knowledge
Student as voluntary participant	"[C]an I get any volunteers to share what they notice about the problem?"	"We're going to share our noticings out loud"	Student as involuntary participant
Student contribution as part of group	"We have a good observation back here"	"[Y]ou said something really interesting about that"	Student contribution as individual
Student as valuable contributor	"So now I'm going to hand out a piece of paper… you only have to answer one of these… to help you think about how can we use what we know about the problem… and how they relate to one another…"	"Alright, so now we'll move on to reflecting on learning. I'm gonna pass out a paper, if you have a pencil, get them out…just go through, fill in whatever blank you feel like filling in"	Student as neutral contributor
Students as capable	"…using your own mathematical knowledge. Y'all have brilliant brains that y'all can use."	"You don't have to get it right away. Honestly, who expects anyone to? … No one's expecting you to know how to solve it yet."	Students as incapable

Figure 1: Examples of Teacher Language and Impact on Student Positioning

Interactions

From the analysis, we identified three pairs of related episodes of teacher-student interactions that illustrate positioning based on teacher communication acts.

In the following transcript, Alex creates a sense of solidarity and collaboration with the students early in the class, which aligns with Alex's previously stated intention of focusing on the "teaching" aspect of mathematics education.

Alex: "Are y'all ready to think like mathematicians?"
Students: "No"
Alex: "No? Come on, I need some better energy. Come on!"
Students: [chuckling] "Yes"

Conversely, the following interaction in Jaime's class highlights the authoritative structure created by Jaime's desire to be the subject matter expert. During a large-group discussion in response to a question about noticing, a student mentioned there were no variables in the mathematical expression on the board.

Jaime: "Does everyone know what a variable is?"
Student: "ABCDEFG"
Jaime: "Not always. You can also use Greek symbols." [nods to other PST, turns their back to the students and immediately begins talking about another topic]

Here, we see Jaime enforcing this authoritative structure by self-identifying as the subject matter expert, and situating students as mere recipients of information. The physical move of turning their back and beginning to talk about a different topic closes any possibility of question or retort, indicating that Jaime did not expect to be challenged or questioned.

The second episode allowed us to compare similar moments in which we recognized a similar discretionary space treated very differently between the two PSTs. As shown in Figure 2, both PSTs set an expectation that students were to put writing implements away to focus on mental mathematics. However, the way the PST discussed these expectations with the students varied greatly, as did the student response. In the following interactions, Jaime and Alex both set clear expectations; however, the way they address the students in this space is significantly different.

	Alex	Jamie
Interaction 1: The teacher is explaining the task for the day. The teacher made it known to students what they will not be using writing utensils for the beginning of the task.	"The first three steps you do not need a pencil or paper. You'll not need anything to write down. Just use your mind and that's it. Okay? We can do it. All right. Let's put away our pencils. Put away our papers."	"Pens and pencils away. You don't need them for this at all until the very end. We prefer you not to have them out for this because it is supposed to be more mental than verbal."

Impact (Student Response)	Immediately after: Students start putting away things, you can hear shuffling and what sounds like a pencil case or a backpack zipper. Alex: "Y'all ready? Paying attention?"	Immediately after: Students start putting things away, you can hear some shuffling. One pencil remains on the desk (can be seen from camera angle) Jaime moves immediately into expectations for the activity.
Interaction 2: The teacher has completed the task and is moving to the reflective portion of the activity.	"So now I'm gonna hand out a piece of paper. I'm gonna hand out a piece of paper."	"Alright, so now we'll move on to reflecting on learning. I'm gonna pass out a paper, if you have a pencil, get them out"
Impact (Student Response)	[Teacher was walking around handing out paper and pencils after giving students instructions] Alex: [To Student 1] "Do you need a pencil?" Student 1: "Yes" Student 2: "I'd like one" Student 3: "Me too" [Inaudible conversation] Student 4: [In response to another student voicing frustration] "She's [Alex] giving out pencils"	Multiple Students: "Can I go grab my pencil?" Jaime: "We have pencils I believe" Student 1: "Is it mechanical?" Jaime: "No" Student 1: "Can I go grab *my* pencil?" (emphasized) Jaime: "No, we have another group next door also doing the activity." Student 2: "So I can't run and grab my pencil?" Student 1: "Right?" [The class gets a little louder, students start to fuss] Student 3: "Oh my *god*!" [audibly annoyed]

Figure 2: Examples of Teacher-Student Interactions and Impact on Student Positioning

We see here that Jaime set expectations as rules to be followed, which aligns with the previous pattern of Jaime as a rule-setter and subject matter expert, positioning students as rule-followers. Alex, on the other hand, sets the expectation, then follows with encouragement of the students' capability as learners. Alex self-identifies in solidarity with students with the statements "we can do it" and "let's put away our papers".

Within each discretionary space, language shifts the assumed role of teacher and students to fit the narrative storyline that is unintentionally designed by the PST. In Alex's class interaction, the negotiated role of teacher and students were as co-contributors working in solidarity. Jaime's language set expectations more as subject matter expert and receivers of information. Although these are negotiated positions, the inherent power imbalance gives students little influence over these enacted roles.

Discussion & Conclusion

Each PST acted through their previous experience and expressed focus to use discretionary spaces in what they deemed to be meaningful ways. Jamie expressed in the survey that in the phrase "mathematics teacher," the emphasis on mathematics was most important, while Alex emphasized the teacher. As a result, the words and phrases they chose to use reflected their individual focus. Because Jaime emphasized the mathematics, the role she played was as subject matter expert. Conversely, Alex's emphasis on the teacher placed her in the role of encourager and guide, creating a classroom structure of shared authority and established students as co-constructors of classroom knowledge (Ruef, 2021; Langer-Osuna, 2017).

Using the term guide as a metaphor for Alex elicits imagery of a tour guide, giving clear directions of what's to come while walking alongside the students with shared authority. We see this in the use of first-person pronouns of "we", "us", "our" and "ours". Putting herself in the same figurative space as the students builds a sense of solidarity. In stating that students should use their own mathematical knowledge, Alex adds, "Y'all have brilliant brains that y'all can use," identifying the capability of students, setting them as co-contributors to classroom knowledge. She further builds solidarity by asking for volunteers to share their knowledge rather than answers. In addition, with the shared solidarity of the class, students gladly participated in conversation. Each noticing was met with encouragement, and the students were able to build on each other's ideas. The students' participation in the conversation and the unrestrained sharing of ideas confirmed that students took up the position of co-contributors and acted out of this position.

Jaime's focus on being subject matter expert created a hierarchy in which the teachers are the experts and authority. In a pattern of teacher moves, Jaime created an expectation of providing rules for the students to follow, then changing the subject or physically moving in a manner that removed herself from the conversation. The use of first-person pronouns referring to the PST with second-person pronouns such as "you" and "yours" referring to the students creates a division between authoritative voice and submissive voice. In explaining the idea that students would learn to think like mathematician, she stated that "we do this to get you to think like mathematicians." In this statement, "we", Jaime and the other PST, are doing the action to create a result of "getting you to think like mathematicians". This sets the students as passive receivers of information with little agency, which appeared to be a role they were not ready to accept.

In the event of misaligned expectations, students may push back. In Figure 2, we highlight an interaction within a discretionary space where directions were given to students to use pencils for the final activity. Although Jaime acted out her expectations of being a subject matter expert with students as followers, the students challenged this positioning with crosstalk and lack of compliance. It can be noted that this pushback is a method of renegotiating roles in an environment where a power imbalance exists. To counter this imbalance, students band together, all speaking at once and advocating for a common goal. In this case, the goal became about going out of the classroom to get their own pencils, rather than using the pencils given to them. This interaction gives evidence of students' dissatisfaction with the role of passive recipient.

After the field experience, each PST wrote a reflection of their experience with the students. It is interesting to note that Alex mentioned several opportunities for improvement, citing additional ideas that could have helped students connect to strategies of interest. Jaime, on the other hand, used this reflection as an opportunity to synthesize and justify her perception of events that occurred, describing in detail whether the students found the correct answers as a measure of successful teaching.

The difference between these two PSTs is clear in their use of discretionary spaces and the subsequent classroom response. Alex established herself as a guide to the students and worked in solidarity with them, while Jaime attempted to be a subject matter expert in a hierarchical classroom which resulted in pushback from the students. What emerges then is a sense that the choices made in discretionary spaces came from the preconceived expectations PSTs had of social roles in the classroom. The resulting social roles manifested in the ways students were positioned during classroom engagement. When not agreed upon by the other actors (students), the process of positioning has the potential to be divisive, which stifles opportunities for collaboration. On the other hand, communication acts that create a sense of belonging or guide students have the potential to shift positions of the students into co-constructors with shared authority.

Our work suggests that without proper, intentional preparation, PSTs enforce classroom roles that are influenced by their prior experience with mathematics and the teaching of mathematics. Oftentimes, these roles conflict with those shared by the students, as seen in the findings. For mathematics education to shift towards a brighter, more inclusive future, PSTs must engage in explicit shifts of expectations where students' diverse identities, inclusive of cultural and linguistic backgrounds, are leveraged. Student roles must be negotiated rather than dictated and should reflect the brilliance they bring to the classroom. By bringing attention to classroom dynamics that occur within discretionary spaces, we can begin the work where we evolve from simply envisioning to physically creating a future of mathematics education where students are nurtured and positioned in ways that cultivate positive mathematical identities.

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COMET: COLLABORATING WITH MATHEMATICIANS TO ENHANCE TEACHING

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In this paper, we describe how mathematics education researchers and mathematicians collaborated to introduce active learning pedagogy into a proof-based linear algebra course. This description highlighted the goals, values, and obligations that mathematicians had for their pedagogical practice, and how challenging it can be to introduce active learning pedagogy into mathematics classrooms that is compatible with these. We also illustrate how mathematics educators' understanding of mathematicians' perspectives allowed mathematics educators to help create instructional techniques that mathematicians are willing to use in their practice.

Keywords: Professional Development, Curriculum, Instructional Activities and Practices, Reasoning and Proof

Introduction

The purpose of this paper is to discuss a recent collaboration project between mathematics education researchers and mathematicians to improve instruction in undergraduate mathematics. In this collaboration, mathematics education researchers and mathematicians worked together to introduce active learning into a proof-based linear algebra course in a manner consistent with the goals and values of the mathematicians who were teaching this course. We illustrate how this collaboration proceeded by describing the challenges and resolution of designing short questions that can be used during lectures that encourage student activity and elicit student thinking. As we describe what transpired, we will discuss mathematicians' values and goals and the importance of attending to them.

Literature review

Most university mathematics courses are taught by lecture (Artameva & Fox, 2011; Melhuish et al., 2022). There is a general consensus amongst researchers in undergraduate mathematics education that this situation is not ideal. Lecturing is largely viewed as an ineffective pedagogy; students who emerge from lecture-based classes in advanced mathematics typically have a poor understanding of central concepts and an inability to write proofs (e.g., Ko & Knuth, 2009; Rasmussen & Wawro, 2017). There is also evidence that students' understanding, performance, and affect improve when active-learning strategies are used (e.g., Freeman et al., 2014; Laursen et al., 2014; Rasmussen & Wawro, 2017). It follows that a key way to improve instruction in undergraduate mathematics is to introduce active learning pedagogy. However, lecture remains the dominate form of instruction across undergraduate mathematics (Johnson, 2019).

This leads to a natural question: Why aren't mathematicians using more student-centered forms of instruction when there is evidence that this pedagogy leads to better learning outcomes than lecturing? We believe that a major reason is because alternative instructional methods frequently do not address mathematicians' most pressing concerns. A primary goal of advanced mathematics courses is to enable students to successfully prove theorems (e.g., Alcock, 2010) and, as Stylianides and Stylianides (2017) observed, there are few research-based interventions that have been shown to improve students' abilities to write proofs that meet mathematicians' standards in undergraduate mathematics education research. Another often cited, but largely unaddressed concern is content coverage.

Braun et al. (2018) argued that the instructional innovations designed and promoted by mathematics educators have designed is not the only way to make university classrooms more active and improve teaching and instead emphasized that there are smaller steps that mathematicians can use to increase student activity in a lecture. For instance, think-pair-share questions and whole class discussions can increase student activity while still enabling lecturing. However, active-learning pedagogy that is compatible with lecturing has not been the subject of much mathematics education research. In this paper, we describe how we worked collaboratively with mathematicians to introduce exactly these types of active learning strategies with their lectures. In doing so, we respond to Artigue's call for projects that are "collaborative projects, building and negotiating, jointly with mathematicians and other university teachers, problématiques that make sense for all those involved, and meet their respective interests and needs" (2016, p. 12).

Theoretical perspective

We use three constructs to categorize teaching. A teacher's *values* correspond to the broad goals they want to achieve in their classroom. Values might include things such as enabling students to prove interesting theorems, preparing students to enter graduate school, of having students regularly engage in authentic mathematical debate. *Strategies* are common adaptable pedagogical techniques that teachers use to achieve their goals. These include things like modeling mathematical reasoning during lectures, asking open-ended questions with adequate wait time, or having students solve problems collaboratively and present their work. *Implementations* are the specific embodiment of a strategy. For instance, the implementation of the strategy of "use a think-pair-share questions" would involve the specific question that was chosen as well as how the question was introduced to the class. We argue that active-learning strategies are most typically strategies while the specific mathematical questions are implementations or tactics.

Broadly speaking, we believe that the lack of uptake of undergraduate mathematics education research is that these are based on strategies that do not align with many mathematicians' values and strategies that are difficult to implement. However, we believe that there are strategies that encourage active learning that are compatible with mathematicians' values. For instance, using think-pair-share questions can be done in a lecture format, or in more inquiry-based classrooms. We drew on Brownlee et al.'s (2017) conceptual description of teacher beliefs and practices as reflexively coevolving. They claimed that as teachers engage in new practices that their beliefs will change, and, as their beliefs change, their valuation of- and engage-in practices will also change.

Methods, Data, and Analysis

During the Fall 2023 semester we formed a collaborative group of mathematics educators and four mathematicians who were teaching different sections of the same proof-based linear algebra course. The research team's progressive goal was to support mathematicians in using more student-centered instructional techniques over time. To do so, we conducted weekly meetings (starting in October) with four mathematicians teaching the course and the five members of the research team. Our study followed a design research framework (Cobb et al., 2003) in which we are simultaneously trying to develop an effective collaborative framework for improving mathematicians' teaching as well as a theory for why our framework is effective and how mathematicians' beliefs and practices evolved.

After the semester, we engaged in retroactive analysis. In instances where mathematicians judged an active learning strategy to be infeasible (either dismissing it before using it, or deciding it did not work for them after using it), they were required to give a rationale for their decision. Using a thematic analysis (Braun & Clark, 2017), we analyzed the mathematicians' rationales to identify what obligations (in the sense of Chazan et al., 2016) that mathematicians had to their institution, discipline, and students that made the active learning strategy infeasible. We then developed a narrative of how our active learnings strategies were ultimately adapted by the mathematicians as feasible given their perceived obligations.

Results and Discussion

In our first meeting with the mathematicians, we initially invited them to use Exit Tickets. The aim here was showing the limited understanding that students had of lectures, as well as helping the mathematicians see the value of attending to student thinking. Our suggested was rejected on the grounds that the mathematicians already knew that the most students did not understand their lectures all that well. As Mathematician D put matters, "I think most of them would say they don't understand yet". To the mathematicians, understanding only came after students had the opportunity to reflect upon the lectures. Our next suggestion was for the mathematicians to give the Exit Tickets for homework problems. Our rationale was that this would give mathematicians the opportunity to see student thinking, but since they were homework, they would not require cutting into any lecture time. This was tried, but mathematicians felt that even these questions took too much time. From our field notes on the implementation of the Exit Tickets, the mathematicians would offer complete answers to the Exit Ticket questions that they asked. Apparently, mathematicians felt an obligation to give a complete and rigorous answer to every question that they proposed.

We proposed a think-pair-share structure for questions. The mathematicians rejected the structure for multiple reasons, including that they did not want to try to force students to talk to each other and it was too late in the semester to introduce the new practice. One noted that the students had established, spatially distanced, seating patterns and asking them to move would be too much.

In general, mathematicians valued the conceptual questions that our team initially generated, as well as the student engagement that they elicited. However open-ended questions, with the nuance and detail that the mathematicians requested, required more time in class than the mathematicians were regularly able to devote to their implementation. This was largely because these questions took students a long time to process and the mathematicians felt obligated to lecture the correct solutions. A structural solution that we found that worked for the mathematicians was to ask clicker-type or voting questions immediately after they introduced a new topic or concept on a topic that they intended to lecture about anyway where the questions had to be ‘understandable’ within about a minute of reading them. This required careful balance between precision in the formulation of the question and the importance of the ideas.

For example, one question that Mathematician B used was, “Do we get equality or inequality in the Cauchy-Schwartz inequality if one of the vectors is a multiple of the other?” This format alleviated the concerns about time in two respects, while maintaining the conceptual focus of the question. First, because the questions were introduced within the context of the lecture, students were already familiar with many of the core ideas of the question so it would take less time to process. Second, because the mathematicians had planned to cover the topic of the questions anyway, no time was “lost” going over the solution to the questions that were asked. We also agreed that students did not necessarily formulate complete answers to the questions that were posed. They could simply make predictions, and discuss with their peers, whether certain statements were likely to be true or not. This could engage the students with thinking about the key theorems that would be covered in the class before seeing their proofs. An innovation that the mathematicians suggested was having all students commit to a yes or no vote on whether statements were likely to be true, so all students had to engage in the task and the mathematicians were able to see the thinking of all students (one even created voting cards that students would hold up).

Overall, mathematicians were satisfied with the format that we created. They liked the conceptual questions that our research team generated—our research team had the time and expertise to generate questions that elicited students thinking about the key linear algebra topics in the course—and they liked the format by which they could use the questions. Mathematician C reflected, “So that’s actually, that’s just like what [Mathematician D] said. It does not take much time and it is actually effective because I can ask everyone someone who would never want to raise their hand. I sort of forced her to raise her hand so that it’s actually useful.” Mathematician D was also enthusiastic about the questions that were used and the collaborative meetings in general:

> Yeah, I thought the suggestions for the questions [generated by the research team] were great. And that's really helpful. Also, just to see the suggestions from other instructors.

Conclusion

In this paper, we illustrated how mathematics educators and mathematicians can collaborate to introduce active learning strategies into the mathematicians’ classrooms. We use our description to suggest three broader points. First, implementing active learning pedagogical techniques that are commonplace in K12 classrooms may not be immediately applicable to advanced mathematics classes. Second, mathematicians had goals and values that of which we were not initially aware that caused problems with implementing our strategies. For instance, mathematicians’ desire to have questions about abstract vector spaces beyond R^n made it challenging to generate conceptual questions that could be answered (or even understood) quickly by the students. Finally, we illustrate how mathematicians and mathematics educators working together is a promising model for generating active learning strategies that mathematicians are willing to use. This supports Artigue’s (2016) contention that changing university instruction may work best if we collaborate with mathematicians to solve problems that are meaningful to them with solutions that they believe are viable.

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A CASE OF DECENTERING TO SUPPORT RESPONSIVE TEACHING WITH MIDDLE SCHOOL STUDENTS

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A classroom study was conducted to understand how to engage in responsive teaching with 18 seventh grade students at three stages of units coordination during a unit on proportional reasoning co-taught by the first author and classroom teacher. We found that teacher-researcher decentering was a mechanism underlying the practice of inquiring responsively in small groups. Decentering involves adopting the perspective of another person by setting one's own perspective to the side and using the other's perspective as a basis for interaction. This paper shows a pattern of decentering actions and a type of question, leveraging questions, supported a student at stage 1 of units coordination to sustain challenges and learn.

Keywords: Instructional Activities and Practices, Communication, Proportional Reasoning

Being able to respond to students' diverse mathematical ways of thinking in a classroom is viewed as essential for supporting student learning—for teaching (e.g., Dyer & Sherin, 2016; Franke et al., 2015). Jacobs and Empson (2016) define responsive teaching as "a type of teaching in which teachers' instructional decisions about what to pursue and how to pursue it are continually adjusted during instruction in response to children's content-specific thinking, instead of being determined in advance" (p. 185). Not surprisingly, this kind of teaching requires considerable skill and expertise (Webel et al., 2021) and is not easy to enact (Empson, 2014).

In our classroom study that took place during a 26-day unit in a 7th grade mathematics class, we aimed to engage in responsive teaching with students who were learning proportional reasoning. We uncovered diversity in students' thinking through assessment of students' units coordination stages (explained later). These stages indicate the multiplicative reasoning that students engage in based on how they conceive of and organize units (Hackenberg & Sevinc, 2024). Since proportional reasoning involves multiplicative reasoning, we used a framework about multiplicative reasoning to track diversity in students' thinking.

In this paper, we look at one teaching practice, inquiring responsively in small groups (Hackenberg et al., 2021), because we found this practice useful in supporting student learning across units coordination stages. As we analyzed our data, we found that decentering, or "taking actions that adopt the perspective of another" (Bas-Ader & Carlson, 2021, p. 2), was a mechanism underlying our practice of inquiring responsively. Researchers who study decentering have argued that it provides a mechanism for understanding teachers' mental actions and behaviors that underlie responsive teaching (Bas-Ader & Carlson, 2021; Teuscher et al., 2016). Our study provides more evidence for this claim from a typical seventh grade classroom.

In our study, 18 students learned to make two cars travel the same speed during a classroom unit on proportional reasoning. Elsewhere we have articulated what three focus students in the class learned, where each focus student was operating at a different stage of units coordination (Hackenberg et al., 2023). In this paper we describe the decentering that supported that learning. Our research question is: How did teacher-researcher decentering in a classroom study with seventh grade students support them to learn to make two cars travel the same speed?

Literature Review: Teachers Interacting in the Moment with Students to Support Learning

How teachers inquire responsively in small groups is part of a larger domain: How teachers interact in the moment with students to support their learning. Researchers have studied this domain from the perspectives of responsive teaching (e.g., Jacobs & Empson, 2016), questioning (e.g., Stein et al., 2008), and decentering (e.g., Bas-Ader & Carlson, 2021). Jacobs and Empson developed a framework that articulated what an experienced elementary teacher did as she monitored—circulated in a class and interacted with groups of students who were working on a story problem. The case study is a valuable documentation of expertise. However, as Bas-Ader & Carlson have pointed out, it does not address what teacher thinking went into making these moves. In addition, Jacobs and Empson do not demonstrate how the teacher's enactment of particular categories of teaching moves supported student learning.

In contrast, Hunt and colleagues (Hunt et al., 2019; Hunt & Silva, 2020) have engaged in responsive teaching in one-on-one teaching experiments with students diagnosed with learning disabilities. In their studies they have linked carefully-crafted teaching moves to specific acts of student learning. However, this work has not been done in full classrooms.

The types of questions teachers ask certainly play a role in responsive teaching: Researchers have studied questions that probe students' responses (Boaler & Brodie, 2004; Stein et al., 2008) or support students to rethink or elaborate on their initial ideas (Franke et al., 2015). For example, Smith and Stein (2018) have proposed two types of questions to ask when teachers are monitoring: assessing questions and advancing questions. The purpose of assessing questions is to find out how students are thinking about a problem or idea. The purpose of advancing questions is to support students to make progress in their thinking about a problem or idea. Smith and Stein encourage teachers to stay with the group to listen to student responses to assessing questions in order to gather information about how students are thinking. In contrast, when the group works on an advancing question, teachers can walk away in order to visit another group.

Bas-Ader and Carlson (2021) used decentering as an approach to study the nature of in-the-moment interactions between graduate student instructors and college students in a precalculus program at a large university. They characterized instruction using five levels of decentering. At the first two levels, instructors primarily tried to get students to adopt teachers' ways of thinking. At the other three levels, instructors exhibited gradations of reflective action. Their study provides "insights into the rationale for the teacher's actions" (p. 17) as they worked to understand their students' thinking and use it in instruction.

Theoretical Frame

Knowledge and Interaction

To understand decentering requires distinguishing between first-order and second-order knowledge. First-order knowledge refers to what a person constructs "to organize, comprehend, and control his or her experience" (Steffe, 2010b, p. 16). Second-order knowledge refers to what a person constructs to describe and explain their observations of another person's experiences (Steffe, 2010b). In decentering, a person sets their own first-order knowledge to the side and tries to build second-order knowledge of the other person's first-order knowledge. Here, knowledge consists of the interactive constructions that people make to organize their experiential worlds.

Piaget and Inhelder (1969) viewed decentering as essential in the construction of knowledge and in socialization because it marks being able to separate one's own viewpoint from that of others, a task that young children cannot yet perform. Indeed, decentering is necessary for the development of mental actions, or operations, because it allows children to move from a state in which all of their operations are centered on themselves to a state in which their operations take their place within a larger world of objects, events, and other people. Other people are especially important in this process, because as children construct others who have their own views, children have to reconcile their views with the views of others, which involves decentering on a social level. Thus, the "decentering of cognitive constructions necessary for the development of operations is inseparable from the decentering of affective and social constructions" (p. 95).

Students' units coordination stages

One way to understand the multiplicative cognitive constructions of students is to use a framework called *units coordination* (Hackenberg & Sevinc, 2024). People create units from "abstracting out the 'one'-ness from some aspect of experience" (Ulrich, 2015, p. 3). With middle school students, a *unit* typically refers to a length or a discrete 1. A *composite unit* is a unit of units, such as conceiving of a package of four yogurt drinks as one item and as four items.

Units coordination refers to how students distribute the units of one composite unit across the units of another (Steffe, 1992). For example, to determine how many yogurt drinks in seven packages, a student might distribute four ones across each of the seven units (packages), for a total of seven fours. A *units coordination stage* is a researcher's generalized model of particular operations and schemes with units that students construct and use to structure problem situations (Hackenberg & Sevinc, 2024). In our study, we had students at all three stages of units coordination typical in middle school. However, in this paper we can only report on one group of students at stage 1. So, here we discuss the units coordination of only students at stage 1.

Students at stage 1 have constructed composite units, and they can track a sequence of these units recursively (Steffe, 1992). To determine the number of yogurt drinks in seven packages of four, students at stage 1 typically count on by 1s past known skip-counting patterns for 4s. These students can also construct *connected number sequences* (Steffe, 2010a). A connected number sequence is made from iterating a length unit to create a set of lengths that are two iterations of that unit, three iterations of that unit, etc. However, students at stage 1 do not construct an a priori multiplicative relationship between a length unit and lengths made from iterating that unit.

Method

Participants

The classroom teacher with whom we conducted the study, Ms. W, was one of 15 middle school teachers who had participated with us in a year-long study group to explore differentiating instruction. We observed Ms. W's only seventh grade mathematics class and invited students to participate. Eighteen out of 20 students submitted consent forms. Before the unit, we sought to develop an initial understanding of students' units coordination stages and to select six focus students, ideally two at each stage. So, we administered written assessments of students' units coordination stages (Norton et al., 2015) and conducted individual interviews prior to the unit.

In the interviews we developed a deeper understanding of students' thinking and sometimes revised our assessment of a student's units coordination stage. Following the interviews, we had five students at stage 1, nine students at stage 2, and four students at stage 3. We selected as focus students two students at stage 1, three students at stage 2, and one student at stage 3. For the analysis for this paper, we included all six focus students, but due to space constraints we present in detail our interactions with one focus student at stage 1, Emily.

Data Collection

All 26 class meetings of the unit were filmed with four cameras: a GoPro that captured the whole classroom, a stationary camera trained on the board, and two roaming cameras. Video from the latter three cameras was inset into the GoPro video for analysis. The classroom was organized into six tables for students to work in groups of three or four. Audio was captured with six microphones fed into a mixer and audio-recorders on each table.

During class meetings, Ms. W and the first author co-led inquiry-oriented instruction with the *Comparing and Scaling* unit of Connected Mathematics Project 3 (Lappan et al., 2014), which addresses proportional reasoning. Research team members operated roaming cameras and interacted with students. Between class meetings the team processed video data, scanned student work, discussed conjectures, and planned for the next class with Ms. W. The first author kept a research journal and wrote a summary of each class session. In the middle and at the end of the unit, we interviewed the six focus students to assess their understanding of topics.

Summary of Speed Investigation

The unit consisted of three investigations: quantifying orangeyness (Days 1-8), quantifying speed (Days 9-18), and understanding percents (Days 19-26). We followed *Comparing and Scaling* for the first and third and designed a replacement for the second based on Lobato's MathTalk project (https://mathtalk.sdsu.edu/wordpress/). We made this replacement because we conjectured that after investigating orangeyness, which is controlled by two continuous quantities, a speed context, also controlled by two continuous quantities, would support students' proportional reasoning. During the orangeyness investigation we found that students at stage 1 were not iterating two quantities as a composed unit to make other mixtures with the same flavor. A composed unit is a correspondence between two composite units (Nabors, 2003). Because we also observed differences among students at stages 2 and 3, we decided to group students by units coordination stage for the start of the speed investigation, Days 9-13.

On Day 10 and 11, students worked on introductory tasks about speed. On Day 12 students worked on the Same Speed Task (Figure 1). We differentiated instruction by giving different numbers to different groups (not relevant to this paper and explained in Hackenberg et al., 2023).

Figure 1. Same Speed Task.

Same Speed Task. The blue car travels ____ miles in ____ minutes.
Find a distance and time for the red car so that it travels the **same speed** as the blue car. Your distance and time must be different than the blue car's distance and time!

When you find a distance-time pair that works, **explain** why it does and **draw a picture**.

Data Analysis

For this paper, analysis occurred in two phases. First, we built second-order models of the focus students. A second-order model is a researcher's constellation of constructs to describe and account for another person's ways of operating (Steffe & Thompson, 2000). Second, we used an existing coding scheme for decentering (Bas-Ader & Carlson, 2021) to analyze individual-environment interactions that were involved in the accommodations. The result of the first phase analysis is reported in Hackenberg et al. (2023). This paper reports on the second phase.

In the second phase, we analyzed the decentering actions of the teachers in interacting with students in small groups during the speed investigation. To do so, we coded all interactions between a teacher and the six focus students during the three days of intensive small group work

(Days 10-12) during which students worked on speed tasks. We used the codes from Bas-Ader and Carlson (2021) to code all utterances of the teacher in each exchange, basing code selection on what the interchange with the student was about. So, even though we were coding the teacher's utterance, codes were contingent on what students said and did in relation to the teacher's comments and questions.

During the coding of data, we had 15 transcripts and videos to code, one for each focus student's small group interactions for each of Days 10-12 (two focus students were in a group together; the other four were in separate groups). Each transcript was coded by at least two authors. The three authors met bimonthly to discuss our code use and what coding revealed about decentering in relation to students' work on speed tasks (e.g., Figure 1). We ran intercoder investigations for each transcript, revising coding until we reached at least 90% agreement. Once we stabilized the coding, we looked for patterns in relation to accommodations students made. That is, we identified instances where specific acts of learning occurred, in 5 of the 15 transcripts, and we examined the nature of the decentering actions for those acts of learning. We compared these patterns to patterns of decentering actions across all 15 transcripts.

Findings

We found two patterns of decentering actions that supported specific acts of student learning. In the first, a challenge to the student (coded PS, *Perturbing the student in a way that extends the student's current way of thinking*) was followed by questions coded LST (*Leveraging the student's thinking to advance the student's thinking*), which we call *leveraging questions*. Then the teacher used follow-up questions and comments to understand students' thinking about their response to the leveraging question (coded FMST, *Following up on response to make sense of students' thinking*). In the second pattern, a challenge (PS) was followed by follow-up questions and comments (FMST), which were followed by a leveraging question (LST). In both patterns, chains of FMST and LST often continued while the students worked to address the challenge.

We found that these two patterns helped us support students to sustain challenges across stages of units coordination. In all cases, sustaining challenges yielded learning, although not always the learning that the teacher was aiming for. We show the first pattern with Emily.

Emily: Constructing Doubled Journeys as Connected Numbers

Student work and decentering analysis. Emily's group worked on making the red car travel the same speed as a blue car traveling 18 mi in 3 min. The group tried 9 mi in 6 min, 18 mi in 6 min, and 18 mi in 2 min. Then Emily's groupmate suggested 36 mi in 6 min. The group found that it worked, and Emily expressed excitement. The first author, known as Ms. H, asked them to explain why doubling produced the same speed and to draw a picture to justify.

Although most students were challenged by the request to draw a picture to show same speeds, Emily seemed especially stymied. Ms. H immediately modified the challenge, asking Emily and groupmates to draw only the 18 mi-3 min journey. Emily said, "I don't really know how to show it. I know how to tell, but I don't know how to show it." When asked how she would tell, Emily said that 18 times 2 was 36 and 3 times 2 was 6, so you could divide each number by 2 or "reduce" it. She did not draw. Thus, modifying the challenge to draw only one journey did not sufficiently support Emily to draw.

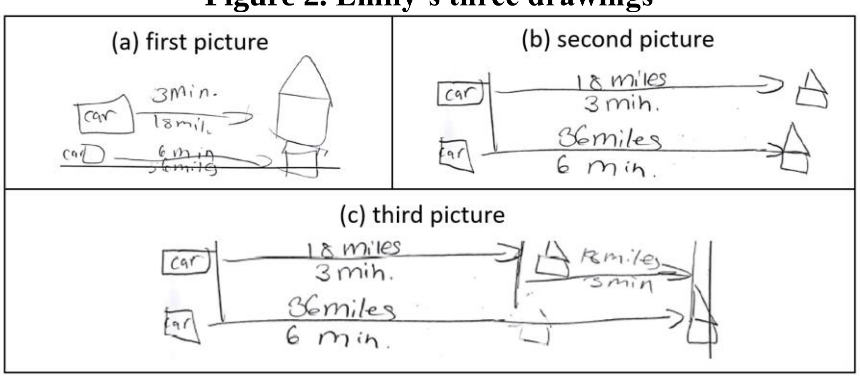
Ms. H then asked Emily to think about the quantities, not just the numbers, because the quantities would give her more power in understanding speed. Ms. H said, "You said something about multiplying by 2. Is there any way to show that in a picture, like... to show 36 miles in 6 minutes is twice 18 miles in 3 minutes?" This question is a leveraging question because Ms. H was asking Emily to think about how to represent her idea of multiplying by 2 in a picture. Emily wrote down numerals to show that 18 x 2 was 36 and 3 x 2 was 6. Ms. H acknowledged her work but said she wanted to know why doubling produced the same speed.

At this point, in a further attempt to support the group Ms. H repeated the modified challenge, asking Emily and groupmates to draw just the 18 mi-3 min journey (Table 1; turn 1). Following this modification, Emily began to draw (Table 1; turn 2).

Table 1. Emily Begins to Draw

Turn	Code	Data
1	Modified PS (repeated, as discussed above)	AH: How could you just draw—you don't have to draw a car—you don't have to draw anything other than something to show 18 miles in 3 minutes. What could show it? [7-s pause] What if you want to show, say, hey somebody, the car went 18 miles and it happened in 3 minutes, what would you draw to show the car's journey?
2		EMILY [draws a car]: That's a bad car. But you have a car. And it's going [draws a segment from the car to a house with "3 min" written over the segment, Figure 2a].
3	LST	AH: Okay, great. I noticed, Emily, you have 3 minutes there. It's kind of like you're saying this is this distance that got covered in 3 minutes. What's the distance again?
4		[Emily writes "18 mil" under the segment.]
5	LST	AH: All right, super. Right here you have this segment, or line, that would show that distance. Now, we also have the other car going 36 miles in 6 minutes. Do you think you could show that one?
6		[Emily draws another segment below the first, Figure 2a.]

Figure 2. Emily's three drawings



Emily's initial drawing (Figure 2a) may have indicated motion from one place to another, not a distance traveled in 3 min. The arrows at the end of her segments suggest this interpretation. However, Ms. H interpreted the segment as a distance covered in 3 min and asked Emily to write in the distance (Table 1; 3). This question is a leveraging question because Ms. H used what Emily had drawn and supported her to develop it so that it might show a distance traveled in a number of minutes. Following this exchange, Emily drew the other car's journey. Ms. H then posed follow-up questions (Table 2; 7, 9, 11, 13) to get a sense of how Emily was viewing her drawing and whether she was thinking about the length of her segments.

Table 2. Emily Revises her First Picture

Turn	Code	Data
7	FMST	AH: Now, would these [segments] be the same length?
8		EMILY: No.
9	FMST	AH: What's the relationship between the length of this one and that one?
10		EMILY: That one's [top] shorter than that one [bottom].
11	FMST	AH: It's shorter, okay. Can you tell me anything more? How much shorter—I mean right now yours is shorter. Do you think it shows well that this is 18 and that's 36?
12		EMILY: Yeah.
13	FMST	AH: How come?
14		EMILY: Because it's labeling them.
15	LST	AH: Do you have the idea of doubling in there though, that you were talking about?
16		[Emily writes "x 2" between the segments.]
17	PS	AH: Okay, well can you show it with the lengths? Emily, can you show the doubling of the lengths? Do you get what I'm asking about?
18		EMILY: It's two times larger.
19	LST	AH: Yeah. Can you draw it that way? Maybe draw it again over here. [Emily draws a second picture, Figure 2b.]

Based on Emily's response that the drawing showed the two journeys well (Table 2; 12), Ms. H posed another leveraging question, "Do you have the idea of doubling in there though, that you were talking about?" (Table 2; 15). Here, Ms. H used Emily's work so far—she had talked about the importance of doubling and drawn a picture where one segment was not double the other in length from our perspective—and asked her to continue to develop it. So, the question was a support for Emily to sustain the challenge of drawing without telling her what to draw.

Emily responded to this leveraging question by writing "x 2" on her picture (Table 2;16). Ms. H then repeated the original challenge about trying to show doubling with the lengths (Table 2; 17). When Emily said, "it's two times larger," Ms. H asked if she could draw it that way (Table 2; 19), a leveraging question that again built on Emily's expressed idea of one journey being two times the other, pressing to see whether she could draw that idea. Emily then drew a second picture (Figure 2b). From our perspective, Emily's second picture shows journeys that are the same size. Ms. H complimented the picture and said to Emily what she saw in it (Table 3; 20).

Table 3. Emily Revises her Second Picture

Turn	Code	Data

20	FMST	AH: That’s a really nice picture. To me, though, when I look at it, it looks like this journey and that journey are the same size. Are they the same size?
21		EMILY: No.
22	LST	AH: Something’s the same about them, the speed. Is there any way to show how the one journey is, as you said, twice as long, twice as big?
23		[Emily extends the 36 mi-6 min segment, but she reaches the edge of the paper and does not make it twice as big as the 18 mi-3 min segment, Figure 2c. AH asks if that shows it exactly. Emily spans the first 18 mi-3 min segment with her fingers, and then she moves her fingers over to make another one, drawing it in.]

When Emily agreed that the journeys in the picture were not the same size, Ms. H posed another leveraging question, “Something’s the same about them, the speed. Is there any way to show how the one journey is, as you said, twice as long, twice as big?” (Table 3; 22) Here Ms. H again used Emily’s idea of “two times larger” (Table 2; 18) and asked Emily to show that relationship in her picture. Following this question, Emily extended the 36 mi-6 min journey, using her hands to indicate fitting two 18 mi-3 min journeys into the 36 mi-6 min journey.

During this interaction, Emily constructed doubled journeys as connected numbers (Hackenberg et al., 2023). The teacher’s decentering actions did not cause Emily to do so, but they opened the possibility for it. We argue that the decentering actions supported Emily to view her segments as lengths and use iteration, an operation she had constructed, on the segment that represented the original journey to create a new journey that consisted of two of the original journeys. Since Emily then spoke of her picture (Figure 2c) as “not drawn to scale,” we have evidence that size was now relevant to her. The next day in the whole class discussion Emily presented to the class how the 18 mi-3 min journey fit two times into the 36 mi-6 min journey, and she explained that her drawing was not accurate, confirming the importance of size to her.

Patterns in Decentering Actions. The pattern we found in these decentering actions is that after a challenge (PS), a leveraging question (LST) often immediately followed. Then the teacher used follow-up questions and comments (FMST) to try to understand Emily’s response to the leveraging question. This understanding then allowed the teacher to pose another leveraging question (LST) based on Emily’s current ideas.

In particular, after Emily first started drawing, the leveraging questions were about helping Emily to expand her drawing or see it in a new way. After her first picture, the leveraging questions (Table 1; 3, 5) asked Emily to expand her drawing to show both journeys and to identify both a time value and distance value for each journey. These questions may have helped Emily to make a correspondence between 18 mi and 3 min, as well as between 36 mi and 6 min, viewing each pair as a composed unit. The follow-up questions and comments (Table 2; 7, 9, 11, 13) were posed to understand how she viewed the changes she had made.

Then the next leveraging questions (Table 2; 15, 19; Table 3; 22) asked Emily if she could follow through (from our perspective) on her ideas to show doubled journeys in her drawing. After Ms. H posed the third of these leveraging questions, Emily changed her drawing to show the larger journey as made of iterating two smaller journeys.

Discussion

In this paper we argue that decentering actions supported to Emily sustain engagement in the challenge of justifying same speeds. Like Jacobs and Empson (2016), we documented responsive teaching moves but with a theoretical grounding of decentering that bases these moves in second-order model building. Like Hunt and Silva (2020), we demonstrated how particular teaching moves supported specific acts of student learning but in a whole classroom.

Our leveraging questions were based on closely observing students' contributions and then asking students to build on that contribution a small amount, toward a large challenge such as drawing a picture or justifying. So, leveraging questions could be seen as a version of advancing questions (Smith & Stein, 2018). Yet they are a quest for a small advance, close to students' work. The judgment of the size of the advance is contingent on the second-order models of students' mathematics that the teacher is building. To determine the efficacy of a leveraging question, a teacher-researcher has to stay to hear the student's response. So, in contrast with advancing questions, leveraging questions require staying put.

We argue that leveraging questions are important for middle school students at stage 1 of units coordination. These students are quite challenged to meet the multiplicatively-heavy demands of topics in typical middle school mathematics curricula and instruction. Leveraging questions may support them to use their abilities to coordinate two levels of units in activity in order to make interpretations of at least some of these topics. In the larger study, we found leveraging questions to be useful with students at all three stages. We view them as a tool for both teacher-researchers and classroom teachers.

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PROPOSED FRAMEWORK FOR EXPLORING THE INTERPLAY OF TEACHER INTENT AND STUDENT PERCEPTION IN DISCOURSE

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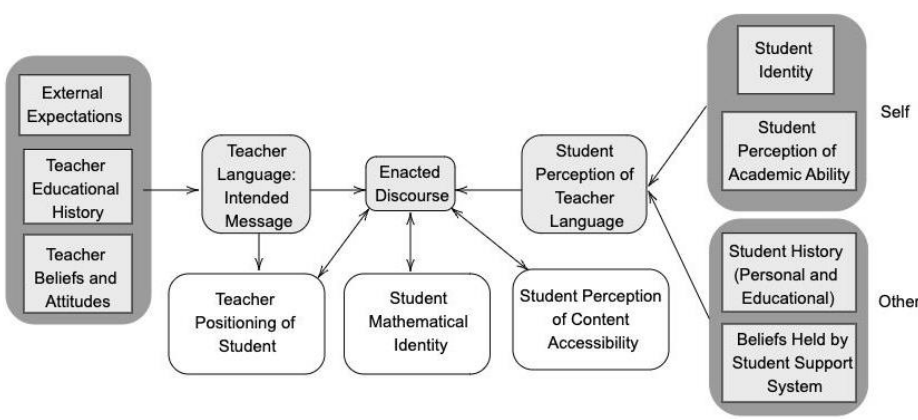
The purpose of this poster is to present a framework that uses discourse and positioning theory as a tool to define relationships in discourse between a teacher and their students and provide a connection between discourse and student perceptions of content accessibility, as well as the student's mathematical identity. Positioning theory evaluates conversations by examining verbal and non-verbal communication patterns (Bishop, 2012), common phrases (Herbel-Eisenmann et al., 2010), or by looking at individual words (Wagner & Herbel-Eisenmann, 2008) for meaning. We use positioning theory as a lens through which we view discourse from classroom observations to describe the apparent negotiation of social roles in that classroom.

Researchers tend to identify roles and positions through their understanding of the social norms of the dominant culture. Students and teachers come from various cultures, shifting how they view social constructs. An aspect missing in positioning theory is identifying teacher intent, driven by their prior beliefs and attitudes, and student perceptions, shaped by the student's previous experiences. Bishop (2012) described a scenario with students where their pre-existing self-identity is enacted through utterances and conversation moves such as interrupting and question-and-answer patterns.

The proposed framework illustrates the interplay between teacher and student, with their outside influences, attitudes, and beliefs. The outcome of the interaction between teacher and student is that students internalize the messages they receive which, in turn, shapes their perceptions of how accessible the content is and their self-identity within the mathematics community. In our future research, we will utilize the framework to combine student perception, teacher intent, and positioning theory to understand how teacher language impacts student beliefs about their position in the classroom and as "doers" of mathematics.



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INTRODUCING EULER DIAGRAMS IN AN INTRODUCTORY PROOFS CLASS: A SEMIOTIC ANALYSIS

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Despite calls to make classroom mathematics more accessible to students, research on instructional moves that support accessibility is scarce, particularly at the undergraduate level. We present one teacher's use of Euler diagrams as a pedagogical tool to support students' understanding of logical implication. Through semiotic analysis, we describe how her instructional task sequence may support students' understanding of the semantic, syntactic, and pragmatic features of Euler diagrams and their connections to mathematical logic. Further, we discuss how addressing all three of these features may aid students in forging vital connections between their own intuitive understandings and formal mathematical reasoning. Our framework considers interactions between semiotic features, not just the features themselves—a divergence from the current semantic-syntactic dichotomies used in mathematics education research.

Keywords: Instructional Activities and Practices; Reasoning and Proof; Undergraduate Education; Semiotics; Euler Diagrams

Undergraduate mathematics students' reasoning in proof and proving has become a focal point in recent research. Weber and Alcock (2004) suggested the importance of understanding the semantic and syntactic features of proof production, and Dawkins (2012) highlighted the significance of understanding the underlying structure of students' mathematical activity in proof and proving. However, their use of these semiotic features to investigate teaching and learning of proof does not attend to the ways these features might interact. Furthermore, the pragmatics of a representational system are not addressed in these works, a divergence from the analytic framings in other disciplines, namely applied linguists. This paper explores whether utilizing applied linguists' analytic framing allows us additional insight into student understanding of the biconditional statements within the logical representation system of Euler Diagrams. We analyze the semiotic features (semantics, syntax, and pragmatics) explored in a small group discussion in an introductory proofs class.

Semantics and Syntax in Mathematics Education Research

Semiotic analysis in mathematics education has predominantly attended to semantics and syntactics, focusing attention on the continuum between students' informal and formal understandings which Vinner (1991) called concept images and formal definitions, respectively. Weber and Alcock (2004) described these understandings by identifying features that differentiate the proof production of senior undergraduate students and more seasoned mathematicians. They found the former tended to use "rules" in proof activity, while the latter considered the more intuitive mathematical features when proving. Weber and Alcock (2004) called these two approaches to proof production *semantic* and *syntactic*, respectively.

Hiebert (1985), explains that we understand when we connect new knowledge to our existing knowledge. To develop robust understanding, he argues, we need to connect symbols to their underlying meaning (semantics), connect procedures to their underlying rationale (syntax), and check the reasonableness of the procedures and solutions in the light of other mathematics we know (pragmatics).

Semiotic Features in Linguistics

Hiebert's (1985) approach aligns with applied linguists' analytic framing. According to the American Speech-Hearing Association (ASHA, 2023), *syntax* is the form or structure of language, *semantics* is the content or meaning of language, and *pragmatics* is the use of language in context. For linguists, semantics describes the meaning associated with words and sentences. Words are symbols associated with referent objects or ideas–they have lexical meanings. In discourse, these lexical meanings are semantics but may be subsumed by their pragmatic, or applied situational meanings (Stotts, 2020). Furthermore, the meaning of a word is dynamic, adapting with individual experience and applied context (Assimakopoulos, 2012; Stotts, 2020).

Theoretical Framework

Applied linguistics defines the features of semiotic systems to both assess and remediate language development, considering the interaction of semiotic features. Interactions between semiotic features are seldom considered in mathematics education literature. This paper aims to illuminate these interactions through analysis of students' work using Euler diagrams, a simple representational system: Euler diagrams. We adapt ASHA's semiotic definitions to fit the mathematical context of Euler diagrams in Table 1. Symbols (e.g., words and numbers) are objects used to encode meaning (i.e., mathematical logic). In Euler diagrams, the truth set of a statement is depicted by a region enclosed by a circle, while the universal set is depicted by a rectangle. The spatial relationships between circles within the universal set encode the logical relationships between truth sets (syntax). It is from the interaction of these semantic and syntactic features that students ultimately abstract appropriate logical meaning based on the mathematical context (pragmatics).

Table 1: Semantics, Syntax, and Pragmatics of Mathematical Representation Systems

Semantics	Syntax	Pragmatics
The intuitive meaning of symbols and what they are understood to mean in terms of experience.	Meaning associated with the placement of symbols in relation to one another.	Combinations of semantics and syntax that are mathematically functional or appropriate.

Methods

We were interested in students' interaction with Euler diagrams because they have recently been leveraged in research-based instruction on logical implication (Dawkins & Roh, 2022). Antonides et al. (in press) provide evidence of the spatial elements of this representational system aiding students' understanding of logical structures. The data were video, audio, and students' written work (on iPads) that was collected throughout one instructor's semester-long undergraduate introduction to proofs course. Motivated by the findings of Hub and Dawkins (2018), Euler diagrams are used during instruction to support students' understanding of logical implication. The class included group activities and whole class discussions which refined their understanding. There were many cases where students utilized Euler diagrams spontaneously in groups to support their arguments. The whole classroom dialogue was transcribed, and pertinent portions of the instructor's notes were identified. Screenshots of these are included as figures in the results section. We present one vignette to illustrate how students leveraged their syntactic understanding of Euler diagrams to make sense of the logic underlying their task.

Results

We present an excerpt of a small group discussion demonstrating how students can leverage the syntactic aspect of Euler diagrams to understand biconditional statements. This task arose at the end of a sequence of activities which we conjecture supported students to build an understanding of the semantic and syntactic features of Euler diagrams by utilizing students' pragmatic understandings. Figure 1 depicts an Euler diagram of the true statement ``P implies Q" (P→Q).

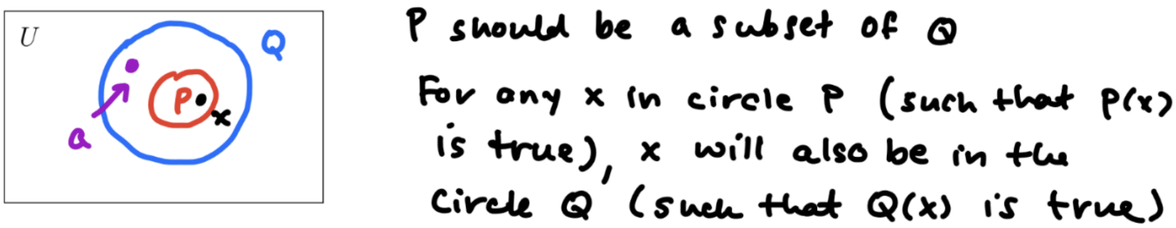

Figure 1: The prototypical Euler diagram for the true implication P→Q

Using the Syntax of Euler Diagrams to Make Sense of Biconditional Statements

Before this vignette, the whole class had discussed whether the converse of the statement, Q→P, must also be true when P→Q is true. They agreed that it was possible to populate an area in Q which was not in P. Thus, Q→P was false. The discussion led one student to wonder what the Euler diagram would be like for the biconditional statement "P if and only if Q" (P↔Q). The instructor encouraged the class to explore this idea in groups.

While not all groups were able to interpret the relationship between the truth sets or successfully generate an Euler diagram representing this case, one group had the following discussion:

Student E: P implies Q. That would just be Q, a giant circle with P in it. So like, something is a multiple of 6, then it's a multiple of three. [Drawing the rectangle]
Student F: You can just draw one circle.
Student G: It's the same circle. [In agreement]
Student E: [Pauses to consider, then drawing a single circle] They have to be the same circle. It's a really direct relationship. They're always both true. (Labeling the circle P, Q)
Student F: It has to go both ways.

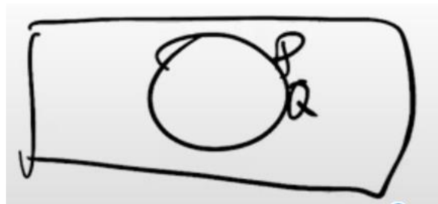

Figure 2: Students' representation of P↔Q

In this interaction, Student E considered the conjecture and the supporting statements of her group. She agreed with her peers, offering a justification for the resulting Euler diagram as she generated it. By attending to the syntactic relationship of the two statements in the Euler diagram, she refined her understanding of biconditional statements.

Student E referenced the initial exploratory task to ground the chosen syntactic structure. This indicated that the syntactic feature itself had become a symbol for her, its own (pragmatic) referent. These more nuanced semantic and syntactic features were used to support the understanding of mathematical inferences that can be made from the diagrammatic interaction of the truth sets. Using features of Euler diagrams to make logical conclusions in this task was mathematically appropriate and functional. The syntactic and semantic features made the implicit logical relationship between the truth sets explicit. This allowed a more flexible and connected use of the representational system, a pragmatic feature of Euler diagrams.

Discussion and Conclusion

This analysis showed how students' understanding of not only the semantic and syntactic features of a mathematical representation but also their interaction empowers them to utilize these features pragmatically. In considering P→Q, Student E supported her abstraction of the meaning of syntactic features by spontaneously connecting the task to a past class example ("So, like, something is a multiple of 6, then it's a multiple of three"). She was able to utilize the relationships between symbols pragmatically. In this way, the syntactic features forged connections between intuitive knowledge and formal mathematical concepts.

Researchers have recognized the importance of connecting procedures, or syntactic processes, to their underlying meaning (see for example Hiebert, 1985). Existing frameworks have described syntax as synonymous with procedural understanding (Bayaga & Bossé, 2018) or instrumental reasoning (Skemp, 1976; Weber & Alcock, 2004). However, these views alone do not capture the importance of the conceptual features encoded in the syntax of the mathematical representation. Adapting ASHA's definitions of semantics, syntax, and pragmatics in our analysis allowed a clearer picture of how students develop competence with mathematical systems of representations. Our reframing of the categories of semiotics is a starting point, elucidating the interaction of semantics and syntax in mathematical sense-making.

While our analysis suggested the utility of Euler diagrams in making explicit the features that are implicit in formal mathematical language and informal conversation, we did not explore this. Future studies should consider an in-depth analysis of students' spontaneous use of Euler diagrams to support reasoning in more complex tasks as a means of developing an intuitive understanding of mathematical statements. In addition, careful analysis of how students develop the semantic and syntactic understandings of these representational systems is needed. The proposed framework could ultimately yield productive insight into which tasks instructors might use to target specific semiotic features of mathematical representations. This insight could then be used as a starting point for unpacking students' difficulties with more complex mathematical representations, including how the order of quantifiers impacts the logical meaning of formal mathematical statements.

Acknowledgments

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ESCAPE ROOMS AS A TOOL FOR ENGAGMENT AND CONFIDENCE-BUILDING IN INTROCUTORY MATH COURSES

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Keywords: Instructional Activities and Practices, Classroom Discourse

Issues Facing Students in Introductory Math Courses

This poster attempts to address certain issues faced by students in introductory math courses. We piloted this activity in a course which serves as a prerequisite for *College Algebra*. Many students in the course are first-time first-year and/or first-generation students. Students in such courses frequently struggle with confidence and doubt their own mathematical ability. Many students seem hesitant to seek help from their instructor or tutors on campus when they struggle with the material. The final issue facing students in these courses that we identified is student engagement. Many students see these courses as a requirement they must fulfill but have no interest in the topic of mathematics. Some of these students see the course as a repeat of material already learned and regularly miss class. This leads to poor attendance, so the students miss material that is new or complicated to them.

Escape Room Icebreaker Activity

We implemented an icebreaker activity where students work in groups and apply their mathematical and logical abilities. The activity was an escape room game with a math theme. An escape room is a series of puzzles that participants must complete to "escape" the room/scenario. We used several sets of "Escape Room: The Game ©" by Identity Games. The scenario involves a group of prisoners working together to break out of prison using clues left by a former inmate who was a mathematician and close to escaping. This scenario was perfect for this course since many puzzles were mathematical but only required order of operations, which was an early topic in the course.

To encourage attendance, students were warned that there would be a special activity during the next class period and that successful participants would earn extra credit on the next exam. We worked with the Student Success Center [SSC], which offers free tutoring on campus, to have tutors attend class and join in the escape room activity. Each team included 4-5 students and a tutor. Tutors worked through a similar escape room scenario earlier in the week, so they knew how the game worked and could help guide their teams. Note that this activity does not fit well into common theoretical frameworks, since the goal is to encourage confidence and comfort with tutors, rather than having tutors guide students through course content.

Results and Data

The goal of this activity was to have students use math in a fun way so that they were more engaged with the topic and the course. Tutors were brought in so that students could meet and interact with them in an informal setting and destigmatize seeking help. The class enjoyed the activity, and each group finished the 1-hour activity within a 50-minute class period. The number of SSC visits for students in this course increased from 6 in Fall 2023 to 19 in Spring 2024, despite the SSC seeing fewer visits in the same period. We plan to expand this activity to more introductory math courses next semester.

BEYOND TEACHER NOTICING: DEFINING SECOND ORDER MODELS FOR TEACHERS

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In this paper we examine Second Order Models (SOMs) in contrast to teacher noticing, focusing on how SOMs go beyond attending by inferring students' underlying mathematical mental activity, specifically their construction and use of units. Through empirical examples from practicing teachers, we illustrate how SOMs attribute students' mathematical actions to inferred mental activities. By distinguishing between teacher noticing and SOMs, we highlight the importance of developing SOMs for effective mathematics instruction.

Keywords: teacher noticing, elementary school education, number concepts and operations

To imagine the future of mathematics education, it is crucial to draw on past foundational research, using it as a basis to enhance future learning beyond current knowledge. In this paper, we build on recent research (Hodkowski, 2018; Hodkowski et al., under review; Smith & Hodkowski; in press) on teachers' shifts towards instruction that uses Second Order Models (SOMs; Steffe, 2000) to examine how SOMs may go beyond teacher noticing (Jacobs et al., 2010; Mason 1998, 2008), providing empirical examples from existing teachers to present evidence for those differences. Specifically, we asked: What distinguishes SOMs from teacher noticing?

Teacher noticing is an active, proven, pedagogical process where teachers attend to and make sense of their students' work, words, and actions (Mason, 2008; Sherin et al., 2011). Teacher noticing includes teacher attention to students' mathematical strategies and problem-solving, which is more than just noting the type of strategy (e.g., algorithm versus partial product). Rather, the teacher uses their observations to get "a window into children's understanding" (p. 172, Jacobs et al., 2010). The teacher interprets their students' mathematical understandings using evidence from the students' work as the basis for their judgements. Failing to seek understanding of a student's actions and, instead, saying "they just didn't try" would demonstrate unproductive noticing by the teacher (Jacobs et al., 2010). Finally, in teacher noticing, the teacher uses their noticings of student work to guide *potential* instructional decisions. Jacobs and colleagues (2010) noted this as an anticipated intention rather than an enacted one, where the reflection on student reasoning precedes any actual teaching acts. The practice of teacher noticing is considered critical for intentional, effective instruction (Schoenfeld, 2011).

SOM, also referred to as a model of someone else's mathematical reality (Steffe, 2000) is inferences made of a students' thinking (mental operations; Steffe & Thompson, 2000; Thompson, 2000; Ulrich et al., 2014). SOM differs markedly from the observer's (researcher/teacher) own mathematics, which is referred to as one's First Order Model (FOM). When thinking about how the concept of SOM can become a practice useful for teachers, it is important for teachers to make a distinction between their FOM and a student's thinking (SOM). Often, teachers conceive of their students' mathematics through the lens of their own FOM

(Tzur, 2010); that is, the teacher understands a mathematical concept in a particular way and interprets students' mathematics by using their own FOM. Such a teacher likely assumes that, through instruction, students will come to understand mathematical concepts the same way the teacher does (Simon et al., 2000). For example, a teacher may use the "I do, we do, you do" instructional model as a way to teach mathematics however this model teaches mathematics the way the teacher ("I do") understands it. Thus, the teacher uses instructional techniques driven by their FOM understanding of mathematics. The issue is that FOM, while making complete sense for the teacher, may be insufficient for fostering student learning of the intended mathematics (Steffe, 2000).

Developing an SOM has an explicit focus on recognizing not only what the student did, but more importantly, why they did it based on the mental operations inferred from the observer/teacher. This focus on the "why" provides insight into the student's existing conceptions as the starting point for their future learning. In contrast, during noticing, the teacher may work to interpret the student's math but not understand why this math was available to the student. For example, teacher noticing would be considered when a teacher explained that the "student answered the problem 12+5 fairly quickly, however drew 12 circles and then drew 5 more and then counted all the circles. The student understood to add, but needed to draw pictures to solve it." This statement attends to specific mathematical details, but not necessarily why the student made the mathematical decisions they did. On the other hand, SOM for the same student, would be considered if the teacher explained, "The student answered the problem by drawing out all the circles, 12 and then 5, and then counting each one. They understood to add but operated on units of one instead of counting on from the 12 or 5. This indicates the student does not yet recognize and operate with a unit composed of ones just the ones." Teachers with an SOM can describe an intention behind the math action as opposed to just naming the math action (as in noticing) or judging it against their own (FOM).

Instructional shifts from just teacher noticing toward SOM is an essential pedagogical realignment away from the teacher's math as the driver for learning and instead toward student conceptions. Focus on the students' conceptions for learning can pivot instructional shifts from "I do, we do, you do" towards instruction more centered on the students' specific individual needs for advancing conceptual understanding (Hackenberg et al., 2023).

Second Order Models in Research and Instructional Practice

Second order models have been described to be made by researchers and teachers. At the researcher level, SOMs are reliant on a theoretical perspective of learning that requires the core notion of assimilation (Piaget, 1971). Assimilation is a mental process by which a learner's existing understanding organizes and shapes their ongoing learning experience. It is the lens by which we "see" the world around us. Literature has described the different levels that *researchers* make SOMs: Emerging SOM, Developed SOM, and Elaborated SOM (Ulrich et al., 2014). At the Emerging level, researchers have insight into participants'/students' mathematical thinking as separated from their own mathematical understanding, but are not yet able to make instructional adaptations as a result of the SOM – as the model is still being constructed. At the Developed level, a researcher can anticipate and plan interactions with participants/students based on the SOM. Finally, at the Elaborated level, the researcher is able to determine a viable SOM of the participant/student and situate this with other SOMs in a class to create several "like" models of participants/students that drive design and enactment of mathematical instruction for groups.

Teachers’ shifts toward SOM are somewhat parallel to that of researchers', however, at the instructional level the focus for SOM is to create a model of students' existing concepts and acknowledge those concepts’ influence on solutions presented by the student. Specifically, two stages for teachers’ shift toward SOM are cogitation and distinction (Hodkowski, 2018). Cogitation refers to teachers’ increased ability to contemplate and think deeply about what their students’ mathematical mental activity might be. Shifts towards cogitation allow teachers to further separate students’ thinking from their own (FOM), and think more deeply about how students make sense of the mathematics. It is most closely related to the Emerging level of SOM used by researchers and similarly does not necessarily result in instructional adaptations. Distinction is a stage of SOM development where teachers’ ability to distinguish a students’ mathematical reasoning from other students while maintaining a separation of student reasoning from their (the teachers’) own reasoning (FOM). Distinction is likely what researchers may incorporate regularly at all three levels (Emerging, Developed and Elaborated). Particularly in the Elaborated level, researchers use one student’s understanding to attribute similarities to a group of students and organize instruction based on inferences into all students’ available conceptions.

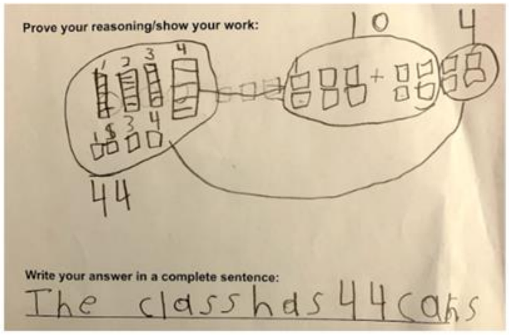
To shift towards cogitation and distinction requires teachers to infer into a student's existing conceptions (what the student can do) and separate these from her (teacher’s) own mathematical knowing (SOM; Hodkowski, 2018; Smith & Hodkowski; in press; Hodkowski et al., under review). Essentially, SOMs enable teachers to infer how students' existing mathematical knowledge shapes their reasoning and mathematical decisions. This shift allows teachers to gain a deeper understanding of why students focus on specific aspects of mathematical problems (Hodkowski, 2018), or make certain errors in reasoning when solving problems (Smith & Hodkowski, in press), or operate on units in ways (Hodkowski et al., under review).

Recognizing what students are attending to (as in teacher noticing) is a crucial initial step for teachers, but transitioning toward SOMs (cogitation and distinction) requires more than mere observation. An SOM encompasses the teacher’s capacity to infer how the student’s existing mathematical knowledge influenced the reasoning presented from the student. Incorporating SOMs involves not only noticing students' abilities but also attributing these abilities to their underlying conceptions. With SOMs, teachers can both observe students' mathematical strategies and delve into the conceptual roots behind them. This pedagogical shift expands the concept of teacher noticing by framing it as a foundation for inferring into students' mental mathematical activities and understanding the conceptual basis of their reasoning through units and how students operate on them. Following we present empirical evidence illustrating that SOM is more than just teacher noticing.

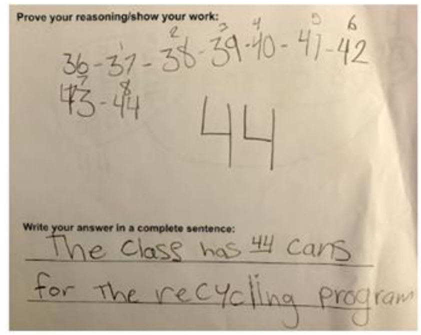
Differentiating Between Teacher Noticing and Second Order Models

We build on recent research (Hodkowski, 2018; Hodkowski et al., under review; Smith & Hodkowski, in press) examining potential indicators toward teachers’ shifts in SOM. Specifically, we present evidence and comparisons between practicing teachers’ noticing as distinct from but related to developing SOMs. To identify evidence of distinctions between teacher noticing and SOM, we asked practicing teachers to reflect on the following scenario and solutions from two students who were not part of the teachers’ current classrooms:

Student A and Student B (work pictured below) were asked to solve the following question: Ms. Smith's class collects 36 cans for the recycling program. Then, Camilla brings in 8 more cans. How many cans does the class have now?



Student A



Student B

Six teachers were sampled from a large, public school district in the Southern United States. Teachers ranged from early career (first year) to veteran (30+ years) across the K-6 spectrum and were anonymously asked five questions based on the pictured work using Google Forms. The questions were: 1) Looking at the work of Student A and Student B, what do you notice? 2) Looking at the work of Student A and Student B, what do you wonder? 3) Looking only at Student A, what can you describe about their thinking and how they solved the problem? 4) Looking only at Student B, what can you describe about their thinking and how they solved the problem? 5) When comparing and contrasting the work of Student A and Student B, what differences can you describe in their reasoning (with evidence). Is this comparison important for their next steps in learning? Why/not?

We used constant comparison analysis (Glaser & Strauss, 1967) to examine teacher responses and distinguish between instances of teacher observation (teacher noticing) and instances of more complex student reasoning descriptions (SOMs). Initially, responses were individually coded by each author for evidence of teacher noticing or indications of cogitation and/or distinction (SOM). The authors then compared their individual coding with each other through multiple conversations to reach consensus between teacher noticing statements versus SOM statements. This collaborative process allowed us to better characterize each teacher's statement and establish how SOM differed from teacher noticing.

Following, we present a sample of teacher responses through our classification of teacher noticing versus indications of SOMs. Our aim is to illustrate that while teacher noticing involves recognizing the students' solutions and strategies, SOMs, particularly cogitation and distinction, describe student thinking based on the construction and operation of units each student might have been using to think about and solve the problem.

Examples of Teacher Noticing

Teacher noticing involves teachers' attention to observing and analyzing how students solved the problems through mathematical strategies and problem-solving approaches. This is exampled from the teacher descriptions seen in Table 1.

Table 1
Examples of Teacher Noticing

Teacher 1	[Student A] found the sum by counting on their fingers. Then proceeded to break the problem up to show the sum of 44 using tens & ones. Then they sort of used arrows to show 4 tens & 4 ones equals 44.
	[Student B] simply counted on.
	Student A is on their way to understanding multiplication.
Teacher 2	[Student A] only have the answer drawn out, so I believe they have a better understanding of the numbers within the problem.
	[Student B] They counted on from the first number, so their thinking shows they have an understanding of the numbers within the problem but may still need some concrete support while working.
	They used different strategies in solving, I think [Student] A may have a better understanding because they did not really have to work out the problem, they just showed the answer. I believe [Student] B has an understanding of the work but may not be quite on the same level of abstract thinking.
Teacher 3	[Student A] likes to group items and numbers to make it easy for them
	[Student B] likes to subtract and can understand the regrouping process.
	Well comparing them they understand regrouping carrying a number but they just go about it in a different way neither of them are wrong. It is not important because they have shown their work and how they got the answer. Next step is give them another strategy.
Teacher 4	I think [Student A] has a solid understanding of how numbers work and has a good understanding of number partners.
	[Student B] was able to count on and also numbered them to show her thinking.
	Student B has a solid understanding of the procedure for counting on, but I am not sure they really understand quantities. Student A seems like they are ready for bigger numbers. It is important because on is ready to fly and the other one still needs support and perhaps more hands on practice.

Teachers’ noticing statements acknowledge that Student A may be on the path to a deeper comprehension of mathematical concepts but conceptual reasons as to why or inferences into how the student operated to arrive at their answer (e.g. counting by ones versus breaking apart to make ten) is not yet accounted for. For example, Teacher 1’s description of Student A's use of finger counting, breaking down the problem into tens and ones, and employing arrows to represent the sum of 44 demonstrated robust noticing and attending to the mathematical details presented. However, in Teacher 1’s description, inferences into the mental operations for how the student thought out and operated on the tens and ones to arrive at 44 seems absent. Similarly, for the teacher noticing group (Table 1), Student B's reliance on counting on is noted which, as suggested by the teachers, is “an understanding of a procedure” (Teacher 4) and “a potential need for additional concrete support” (Teacher 2) but how the student counted on (tracking by ones with each additional count) is not mentioned by the teachers.

Teachers' noticing descriptions recognize the diversity of strategies each student brought, such as Teacher 3's noticing that Student A "used grouping" and that Student B had an "understanding of regrouping". However, why each student would reason this way (inferences into how they operated on tens and/or ones to arrive at the grouping or regrouping) or use such strategies seemed lacking in these teachers' descriptions. From the examples above, teacher noticing statements acknowledge that both students' strategies were valid but do not yet categorize either strategy as being more (or less) conceptually advanced.

Examples of Second Order Models

Teachers' statements which included SOMs, unlike teacher noticing statements, describe inferences made by teachers into students' mental mathematical activities as the conceptual basis of their reasoning through units and how students operated on them, as illustrated in Table 2. These statements suggest the teachers were cogitating and making distinctions about students' mathematical thinking in terms of the units and operations students used; instead of referring to strategies students used to arrive at the correct answers, as seen in the noticing statements (Table 1).

Table 2

Examples of SOM

Teacher 5	First [Student A] broke up the number 36 by tens and ones. Then they add [sic] 8 ones, then had another group of tens. [Student A] then took the left over ones and left them, this is how they got 44.
	[Student B] had to count on and keep track that they counted up 8.
	Student A has a deeper understand[ing] of numbers value and how numbers are formed. Student B does not see numbers the same and sees this more as counting on and not looking at the value of the number. Yes, student A has an understanding of value while student B is still at the surface level of understanding numbers.
Teacher 6	Student A decomposed the 8 more cans into 4 and 4 and decomposed 36 into 3 groups of 10 and 6 ones. She then composed a group of 10 using the 6 and 4. That gave her 4 groups of 10 and 4 extra ones for a total of 44.
	Student B solved the problem by starting with the first quantity and counting on 8 more numbers to add the second quantity.
	I believe Student A has cardinality and is able to subitize with larger numbers. She understands the quantities that are being added and that making a 10 is an efficient way to solve a joining problem. Student B understands the pattern for counting and can count on from a given number. He understands the process for solving a problem by counting on but there is no evidence that he understands the quantities that have been added. (He may understand them, but it is not indicated in his answer.) Both students have used a strategy that works for them to solve a joining problem with result unknown. I believe that Student A's explanation shows a more advanced understanding of the math behind the solution and therefore Student A is ready for other problem types or joining problems with larger

	quantities and that Student B perhaps is not. However, I will admit that I am not really sure about what Student B would need at this point.

There is a distinguishable difference between Teachers 1-4 statements (Table 1) and the statements of Teachers 5 and 6 (Table 2). We attribute this difference to teachers developing SOM as exampled through the cumulation of Teacher 5 and 6's descriptions. For example, when describing Student A's work, both Teacher 5 and 6 noted an approach where the student decomposed 36 into tens and ones (3 groups of ten and 6 ones) and 8 into a group of 4 and a group of 4 and composed numbers, using the 4 to make a ten and then determining 44 with 4 more ones. Teachers 5 and 6's statements explicitly describe distinct units which the student operated, whereas Teachers 1-4 did not provide evidence of this.

Later, when comparing the two students' work, both Teacher 5 and 6 worked to identify why Student A was able to reason as they did (Teacher 5 describing Student A's understanding of "value" while Teacher 6 described it as "cardinality and subitizing"). We attribute this to a developing SOM regardless of the mathematically/ conceptually accurate thinking that a researcher might attribute to these students. The teachers' attention to the "why" Student A was mentally able to work with various, distinct units demonstrated more than teacher noticing. We see Teachers 5 and 6's and statements as attempts to understand what the students were thinking as a result of their solution as well as the "why" (units and how students operated on them). Said differently, for statements in Table 2, teachers seemed to make attempts to infer into what students did know as a way to describe the work that was presented.

Discussion

This paper examined the distinctions between teacher noticing and Second Order Models (SOMs) and presented empirical evidence to illustrate these differences. Teacher noticing, as described in the literature, involves teachers' attention to and interpretation of students' mathematical strategies and problem-solving approaches. In contrast, SOMs involve recognizing not only what the student did but also why they did it based on the mental operations inferred from the observer. This focus on the "why" provides insight into the student's existing conceptions as the starting point for their future learning. While teacher noticing is a crucial initial step, SOMs enable teachers to infer how students' existing mathematical knowledge shapes their reasoning and mathematical decisions.

The empirical evidence presented in this paper illustrates the differences between teacher noticing and SOMs through a comparison of responses from six practicing teachers. The examples of teacher statements provided demonstrate that while teacher noticing involves recognizing the students' solutions and strategies, SOMs, particularly cogitation and distinction, describe student thinking based on the construction and operation of units each student might have been using to think about and solve the problem. An implication of SOMs in instruction is teachers' attention to differences in the way students solve problems based on the units involved. While more research is needed on teachers' development of SOMs and their use of them in instruction, teachers' attention to differences on how students operate can result in more individualized instruction to students who can arrive at the same answer albeit different thinking.

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TEACHERS' ENACTMENT OF SOCIAL JUSTICE MATHEMATICS LESSONS

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Mathematics class can support students to develop civic dispositions that are foundational to productive citizenship (Education Commission of the States, 2014) and prepare students to see mathematics as relevant and meaningful. Social justice mathematics lessons (SJML; Gutstein, 2003) may provide opportunities for students to consider both mathematics and social issues (Kokka, 2022), yet balancing attention to both is challenging (Bartell, 2013). Citizen Math, one set of SJML, has been documented to engage students in social issues while improving middle school students' mathematics achievement (Jackson & Makarin, 2018). We investigate teachers' enactment of Citizen Math lessons to inform professional learning and further research of SJML.

We utilize Cazden's (2001) two dimensions of interactional routines, *sequential* and *selectional dimensions*. S*equential dimension* refers to routines that are consistent across classrooms and/or lessons. *Selectional dimension* refers to the ways teachers exercise their agency to enact segments of a lesson. Classroom routines are influenced by published materials (Banilower et al., 2018) such as off-the-shelf SJML (e.g., Conway et al., 2023). Yet, even with common resources teachers may enact components differently which is visible in the *selectional dimension*. One difference is in how teachers position mathematics and students, which has implications for teachers' and students' mathematical authority (Herbel-Eisenmann, 2007).

We share our analysis of 22 observed SJML in middle school classrooms to answer the research question: *How do middle school teachers learning to use SJML enact such lessons?* Findings in the *sequential* dimension indicate that teachers enact segments of Citizen Math lessons in a predictable manner consistent with teachers' guides. The timing of each segment was the most variable aspect. In the *selectional* dimension, we found variance among which mathematical concepts were emphasized and how the lesson and mathematics were positioned. For instance, in a 7th grade lesson, students considered the fairness of wages in different professions by analyzing unit rates. Mrs. Amber emphasized that students should consider fairness from their perspectives (e.g., "You're going to determine whether you all think it's fair, whether you think they should be able to make that much…") and throughout the lesson students used mathematics to make sense of the concept of fairness. On the other hand, Mr. Brown's perspective suggested the task was to find a correct answer (e.g., "Is it more valuable to be a football player?") and regularly directed students to use particular procedures. Where Mrs. Amber allowed students to make assumptions, Mr. Brown funneled students' reasoning so that all students performed the same computations. Multiple examples from a variety of these SJMLs illustrate differences across lessons, particularly in the *selectional* dimension. By identifying the various ways that teachers exercise their agency, for example through questioning patterns and constraining authority, we hope to contribute to the discourse about professional development efforts related to SJML. This analysis can support teachers to skillfully implement SJML that advance mathematical goals while also cultivating positive mathematics and civic dispositions.

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"I'M STUCK": NAVIGATING CRITICAL JUNCTURES WHEN TEACHING MATHEMATICS THAT IS RESPONSIVE TO STUDENT THINKING

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In teaching a lesson that is responsive to students' mathematical thinking, teachers urge students to solve problems using whichever strategy they prefer. Prior work has described the difficulties teachers may face in teaching responsively, yet little has sought to characterize the particularly high-stakes moments that may derail a responsive lesson, what we refer to as "critical junctures." We examined 27 prospective elementary teachers' reflections on the challenges they encountered in teaching a responsive lesson in a rehearsal, as well as video of the most common critical junctures. We find that critical junctures involve confusion on the part of the teacher, the students, or both. We also find that many of the challenges teachers described in their reflections did not rise to the level of a critical juncture, suggesting that these may need to be introduced during rehearsals if prospective teachers are to have opportunities to navigate them.

Keywords: instructional activities and practices, preservice teacher education, teacher educators

Imagine the following. A teacher educator (TE) is modeling a Number Talk for a group of prospective elementary teachers (PTs) (Humphreys & Parker, 2015). The TE asks PTs to solve 75 – 29 however they prefer. After a few minutes, a whole-class discussion begins, in which several PTs share their strategies. The first PT describes a strategy the TE anticipated: 29 ➔ 25 and 4; 75 − 25 = 50; 50 – 4 = 46. The second PT describes a compensation strategy the TE had also anticipated: 75 – 30 = 45; 45 + 1 = 46. A third PT then shares: 80 – 30 = 50; 50 – 5 = 45; 45 – 1 = 44. And the TE is stuck, unsure what the PT did, why it did not work, or what to do next.

This scenario took place in a mathematics methods course taught by the first author as he sought to model instruction responsive to students' mathematical thinking (Dyer & Sherin, 2016; Jacobs & Empson, 2016). While difficult to convey in writing, this moment engendered a sense of panic. Such moments, what we call *critical junctures*, can derail a responsive lesson, and even lead a teacher to abandon efforts to teach mathematics in a way that is responsive to students' thinking (Borko et al., 1992). While some work has described what makes responsive teaching challenging (e.g., Ghousseini, 2015; Gibbons et al., 2017), little has characterized particularly high-stakes moments like the one described here. Work characterizing such moments, including what they have in common and how they can be navigated may support TEs in preparing PTs to teach responsively. It may also result in more students experiencing the merits of this approach to instruction for their learning and dispositions (Boaler & Greeno, 2000; Carpenter et al., 1989).

We examined the critical junctures 27 PTs described facing in teaching a responsive lesson in a rehearsal. We answered the following question: What is the nature of the critical junctures PTs encounter when teaching a lesson designed to be responsive to students' mathematical thinking?

Conceptual Framework

Responsive Mathematics Teaching

In teaching responsively, teachers urge students to solve problems using their own preferred strategies rather than a single predetermined strategy shown to them by the teacher (Carpenter et al., 1998). Several of these strategies may then be compared and contrasted in a whole-class discussion devoted to surfacing and developing students' understanding of a key mathematical idea (Sleep, 2012; Stein et al., 2008). Calls to teach responsively are rooted in research showing that such teaching is associated with the development of rich mathematical understandings and productive dispositions towards the subject (Boaler & Greeno, 2000; Fennema et al., 1996).

What Makes Responsive Mathematics Teaching So Challenging?

Despite these benefits, responsive teaching remains somewhat rare (Bishop, 2021; Cobb & Jackson, 2021; Stigler & Hiebert, 1997), due in large part to the many challenges of teaching in this way. In teaching responsively, teachers must necessarily improvise (Borko & Livingston, 1989). Even with substantial prior planning (Stein et al., 2008), teachers are likely to encounter strategies they did not expect or have never seen before, which requires them to comprehend the strategy on-the-spot and decide how to respond to the student who used it (Jacobs et al., 2010). This challenge may be especially salient for prospective teachers, as they tend to be less familiar with the nonstandard strategies students use (Jacobs et al., 2011; Shaughnessy & Boerst, 2018). Even if they follow every strategy, teachers may encounter a challenge in making these strategies the basis of a whole-class discussion (Singer-Gabella et al., 2016) or leveraging them in an effort to surface the lesson's key mathematical point (Ghousseini, 2015; Sleep, 2012).

Such challenges are integral to responsive teaching, yet may not derail a lesson. Critical junctures – the focus of this study – are not only challenging, but threaten to derail a lesson. And while they may share much in common with the challenges described here, they may well differ.

Methods

Participants and Context

Twenty-seven PTs were recruited from an elementary math-methods course. Most presented as white and female, but a range of gender, linguistic, and racial identities were represented. PTs experienced live representations of a Number Talk, Quick Images, Choral Count, and Number String taught by the first author, who also facilitated an unpacking of each representation with the PTs (Lampert et al., 2013). PTs then co-planned a similar lesson to rehearse with a partner.

Data Collection and Analysis

Data consisted of written reflections in which PTs described three challenging moments from their rehearsal, including what occurred, what made it challenging, how they responded, and how things proceeded. Video-recordings of the rehearsals were also gathered with an iPad on a Swivl robot. Analysis began with the creation of a codebook comprised of a priori codes for the types of challenges teachers may face in teaching a responsive mathematics lesson (Ghousseini, 2015; Singer-Gabella et al., 2016; Sleep, 2012). We then read a sample of PTs' reflections to assess the applicability of the codes, which resulted in several emergent codes being added. However, we soon recognized that some codes were unrelated to responsive teaching or described challenges not high stakes enough to comprise a critical juncture. We thus revisited the codes, asking if each challenge: a) was relevant to responsive teaching and b) did or could derail a lesson. Only codes meeting these criteria were kept. We made these decisions by considering our own teaching experience and asking if a given challenge had indeed halted lessons when it occurred. Next, we independently coded a sample of challenges, with those that did not meet our criteria coded NA. We then compared and revised codes as needed. Once codes were stable, we coded additional samples until achieving adequate interrater agreement. The first author then coded all challenges, also applying a code for the lesson phase in which the challenge occurred: a) launch, b) whole-class discussion, or c) lesson closure. We then wrote analytic memos (Maxwell, 2013) in which we described patterns in the codes. To capture what critical junctures entail, we identified instances representative of the most common critical junctures, viewed video of them, and wrote memos describing what occurred therein. We present our analysis of one such instance below.

Results

Table 1 portrays the distribution of codes applied to PTs' descriptions of challenges faced.

Table 1: Critical Junctures

Code	n	Lesson Phase
Unanticipated solution strategy	4	Whole-class discussion
Following a student's thought process	8	Whole-class discussion
Explaining why	1	Whole-class discussion, Closure
Limited range of strategies	4	Whole-class discussion
Partially correct strategy	4	Whole-class discussion
Steering toward the mathematical point	0	Whole-class discussion, Closure
Closing lesson by foregrounding key point	4	Closure
Adjusting in-the-moment	11	Whole-class discussion
NA	45	

Adjusting in-the-moment and following a student's thought process were common critical junctures. Less common junctures were responding to unanticipated strategies, a limited range of strategies, and partially correct strategies, as well as closing a lesson by foregrounding its key point. We had a code for "steering toward the mathematical point," but PTs did not mention this challenge, suggesting that it was not salient for PTs or they perhaps experienced it, but did not write about it as it is less of a "challenging moment" and more of a recurrent challenge woven throughout a lesson. We applied NA codes to 45 challenges. For example, consider the following challenge shared by one PT: "When students answered the questions that we had planned, it was hard to come up with more questions to push them to think deeper." While relevant to responsive teaching, this challenge is unlikely to halt a lesson and thus fell short of being a critical juncture.

A Critical Juncture

We now describe an instance of one of the most common critical junctures PTs faced in their rehearsals: following a student's thought process. For this instance, PTs were teaching a Number String and had already asked students to solve 4×7 and 8×7, which PTs chose as the answer to 8×7 is double that to 4×7. After posing a third problem, 20×7, the following exchange occurred:

Student 1	One-forty [140].
PT 1	And how did you get that?
Student 1	I did it two different ways, actually.
PT 1	Okay.
Student 1	And the first one was, I actually multiplied 28 by five.
PT 1	Perfect. Twenty-eight by five?
Student 1	Yep. Because I knew that four times five is 20, so I just multiplied that by five.
PT 1	Okay, so four times five equals 20 and then you just multiplied that by five?

Student 1	I multiplied the, I knew that the first number string, the four times seven, was 28.
PT 1	Right.
Student 1	And I knew that four times five is 20, so I did the 28 times five, and I got 140.
Student 2	Girl, what?
Student 3	Oh, okay, so she did seven times four to get 28 and then times that by five, to get, I get, I get what she's saying.
PT 1	Okay. So, you did this right here?
Student 3	So, she broke up, she broke…
PT 1	The seven times four to get the 28.
Student 1	Exactly.
PT 1	And then?
Student 1	And I saw that the new equation was 20 times seven, I knew that 20 was four times five, so I just multiplied the 28 by five.
PT 1	Times four, times five? I'm stuck.

The TE then asked Student 1 if she solved 28×5 in one step or solved 20×5 and 8×5, then added the products. Student 1 said that she did the latter and the TE wrote the following on the board:

$$20\times7=$$
$$\wedge$$
$$(4\times5)7=$$
$$4\times7=28$$
$$28\times5=$$
$$\wedge$$
$$(20+8)5=$$

The TE then turned to PT 1 to ask if things were making sense, to which PT 1 responded, "a little bit." PT 1 then asked the class if anybody had a different way of solving the problem, 20×7.

In reflecting on this moment, PT 1 shared: "Student 1 shared a strategy that not only us 'teachers' were confused by, but fellow 'students' were confused by, too. Even after eliciting, I struggled to understand what she did and was unable to revoice or represent her strategy." Thus, PT 1 seemed to concur that this was a critical moment related to following a student's thinking.

Discussion

We contribute to a growing literature seeking to characterize the challenges teachers face in teaching responsively (Ghousseini, 2015; Gibbons et al., 2017; Singer-Gabella et al., 2016). Unlike prior work, we focus not on "challenges," but those particularly high-stakes moments encountered in teaching responsively that either do, or have the potential to, bring a lesson to a halt, what we refer to as *critical junctures*. Critical junctures involve confusion on the part of the teacher, students, or both, which may be expressed overtly (e.g., "I'm stuck," "Girl, what?") or in more subtle ways recognized by a TE, though perhaps not by PTs. Following a student's thought process (Jacobs et al., 2011; Shaughnessy & Boerst, 2018) was a common critical juncture, as was adjusting on-the-spot. At times, the latter was in response to something the TE suggested.

About half of PTs' challenging moments did not rise to the level of a critical juncture. This implies that such junctures may be rare in rehearsals, perhaps because students in rehearsals are adults who are forthcoming with their thinking, thereby limiting confusion. TEs may thus need to introduce critical junctures in rehearsals (Baldinger et al., 2021) if PTs are to have opportunities to navigate them. The lack of critical junctures may also have been due to PTs not recognizing junctures when they arose or junctures being resolved by a TE before becoming critical for PTs.

An objective of ours moving forward is to develop even greater clarity regarding what makes a critical juncture a critical juncture. As we describe above, the critical junctures in this study either derailed, or had the potential to derail, a responsive mathematics lesson. They were also characterized by notable confusion. However, in our analysis, we recognized that some of the critical junctures that fulfilled these criteria (e.g., following a student's thought process) seemed more critical than others (e.g., limited range of strategies). In future work, we plan to specify such distinctions and may even create tiers in terms of a critical juncture's criticality.

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AN EXPLORATORY STUDY INVESTIGATING RESEARCHERS' PRODUCTIVE STRUGGLE CONCEPTUALIZATIONS IN TWO DISTINCT CONTEXTS

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Keywords: Instructional Activities and Practices, Problem-Solving.

Identifying key features of *productive struggle* helps build a common language and shared understanding of this term across conducting research and enacting the National Council of Teachers of Mathematics' (NCTM, 2014) teaching practice "[supporting] productive struggle in learning mathematics" (p. 10). However, productive struggle is a complex phenomenon. One way to better understand this phenomenon is by investigating how different researchers conceptualize productive struggle, such as by depicting it through the figures in their published papers. Thus I investigated the research question: *What do researchers' conceptualizations of productive struggle in distinct contexts reveal about key features of productive struggle?*

This exploratory study emerged from a larger ongoing study that investigates the definitions of productive struggle (e.g., Kamlue & Van Zoest, in press). As I analyzed studies that investigated productive struggle, I noticed that Warshauer's (2015) study [W15] and Granberg's (2016) study [G16] had several key differences, including interaction opportunities, mathematics content, and study foci. For example, in W15, students had opportunities to work individually, in a small group, or as a whole class, and the students received support from their teachers. In contrast, students in G16 worked in pairs and received automatic feedback from GeoGebra software. Moreover, while G16 used linear function problems for upper secondary school students in her study, W15 used proportional reasoning for middle school students. I also noticed that while G16 focused on the problem-solving process (e.g., correcting prior knowledge), W15 focused on classroom interactions (e.g., teacher-student interactions).

Since these two authors investigated the productive struggle construct through different lenses (e.g., different study foci), I analyzed these two articles, particularly focusing on the depictions in their figures as a proxy of their conceptualizations, to identify commonalities in their approaches. I analyzed the two articles using the four dimensions (*tasks*, *student struggles*, *teacher responses*, and *outcomes*) that were identified in W15's *productive struggle framework* (p. 391) as a starting point to investigate how G16 aligned with or differed on those four dimensions. Then, I discussed the initial findings with other researchers to confirm or disconfirm the findings.

The initial results indicated at least three commonalities between how W15 and G16 conceptualized productive struggle in learning mathematics that can inform future work on productive struggle. First, tasks that are unfamiliar to the students promote productive struggle. Second, struggles that tended to be productive concerned important mathematics (e.g., *error concerning prior knowledge* (Granberg, 2016, p. 39), not *error due to carelessness* (Warshauer, 2015, p. 386)). Finally, student responses to the feedback they receive emerged as important, regardless of the form of the feedback (e.g., teacher responses, automatic feedback from GeoGebra, peer responses). This poster will illustrate my analysis of these studies and elaborate on how my results can inform future research into productive struggle.

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'YOU DO NOT LEARN MATHEMATICS; YOU GET USED TO IT': AN INVESTIGATION INTO GRADUATE MATHEMATICS

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Keywords: Tertiary Mathematics, Teaching Practice and Classroom Activity, Systemic Change

Efforts to make mathematics more equitable and inclusive for anyone who wishes to pursue it have been happening for the past 40+ years in K-12 grades. More recently, efforts to reform undergraduate mathematics -- in particular, the calculus sequence (Smith et al., 2022) -- have been underway as well. Mathematics education in proof-based courses, like those taught in graduate classes, and teacher pedagogy and practice in college mathematics are both understudied fields in education (Weber, 2012; Melhuish et al., 2022; Speer et al., 2010). Conversations regarding graduate-level mathematics instruction could open up communication between researchers in mathematics education and research mathematicians, which could benefit students enrolled in graduate courses, professors who teach these courses, and students who aspire to pursue mathematics at any level.

This project aims to bridge the mathematics and education departments within a university in the Rocky Mountains in an ethnographic endeavor to study teaching and learning practices in a graduate mathematics course. Graduate courses are prime candidates for studying learning based on the motivation of the students that participate in them and given the similarity between modern mathematics teaching methods and the goals and methods of mathematics research. The type of work accomplished in graduate mathematics courses includes developing intuition about how ideas are connected and the validity behind statements, developed through exploring patterns, formulating conjecture, and seeking solutions to them (Schoenfeld, 1992). While performing mathematics research, one must enact these practices and understand the material that forms the foundation, which is typically what graduate courses are for. So, understanding how mathematics is taught in graduate courses and relating it back to the work of a mathematics researcher can open up possibilities for creating classrooms where learning mirrors research more closely.

This study inquires into how graduate mathematics content is learned and taught in university classrooms. To gain understanding about the teaching and learning process, the researcher has conducted interviews with 15 students enrolled in graduate mathematics courses (graduate, undergraduate and from schools outside of math), six instructors of graduate mathematics courses and one postdoctoral scholar. Classroom observation data was also collected using a modified P2C2 protocol from the SEMINAL project (Smith et al 2022). These observations focused on student interactions and teacher moves.

I perceive teaching and learning as social activities. Prior research has shown enhanced student understanding of mathematical concepts and desire to pursue mathematics further (Boaler and Greeno 2000). Looking at what strategies students are using to learn content and what professors do in class to motivate student learning can begin conversations about what strategies are currently happening as well as envision futures where more students are drawn to learning mathematics and have enhanced opportunities to do so at this level.

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STUDENTS' EXPERIENCES OF DIFFERENT GROUP FORMATION METHODS

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We report on undergraduate students' experiences of different group formation methods used by eight instructors of an introduction to proof course. We offer themes of student experience of three group formation methods: random selection, student selection, and instructor selection.

Keywords: instructional activities and practices, undergraduate education, reasoning and proof

Group work implementations vary across many dimension, such as the frequency of the group work, the content worked on, and the instructor assistance offered (Smith et al., 2020). In this report, we focus on one such dimension: *group formation*, that is, how groups are formed. In particular, we are interested in undergraduate students' experiences of different group formation methods and seek to learn:

- How do undergraduate students experience different group formation methods?

To this end, we report on the experiences of 29 undergraduates from eight sections of an introduction to proof course where about two-thirds of class time was dedicated to group work. Although group work has been extensively studied in the K–12 context, it has received less attention at the undergraduate level. Due to the contextual differences, research results from the K–12 context may not map directly onto the collegiate context.

Literature Review

Below, we briefly review existing literature on group formation using Hagelgans et al.'s (2001) four-part classification of group formation methods, which will also frame our results.

Random Selection

Groups can be organized by random assignment, a method argued for by Liljedahl (2021). He noted that groups should be randomized every hour or so to avoid students settling into active or passive positions, and he observed that groups needed to be *visibly* randomized, for otherwise students doubted that groups were random and returned to fixed positions. Yet, random selection can also lead to unfortunate groupings, such as minoritized students being isolated in their groups (Reinholz, 2023). For example, Hwang et al. (2022) documented two undergraduate students' experiences of being linguistically isolated in randomly formed groups.

Pseudo-Random Selection

Hagelgans et al. (2001) used the term pseudo-random selection to refer to "[t]he instructor mak[ing] a few adjustments to randomly selected groups" (p. 19). They suggested that any such adjustments should be made before the class so that students do not see the adjustments. This stands in contrast to Cohen and Lotan's (2014) suggestion to make adjustments openly. Reasons for making these adjustments could be to pair newly arrived students with translators or to separate unproductive friends/enemies. Further, paying attention to race, gender, and individual needs and making adjustments accordingly can serve to counter known issues such as white men dominating participation (Reinholz, 2018, 2023).

Instructor Selection

By determining groups themselves, instructors can ensure heterogeneous groups, paying attention to race, gender, and individual needs (Reinholz, 2018). Instructors can also utilize student input. A concern raised by Cohen and Lotan (2014) is that some instructor choices, for example with respect to gender and race, can be easily noticeable and lead students to viewing other students as stereotypical representatives rather than as individuals.

Student Selection

Instructors can also permit students to form groups on their own. Cohen and Lotan (2014) raised several concerns about this method: (a) friends will typically choose to work together and be focused on play rather than work, (b) socially isolated students may be isolated even further by not being selected, and (c) some students who need extra support may be less likely to receive it. Reinholz (2018) added that student selection may also lead to excessively homogeneous groups, which can also lead to the exclusion of students.

Methods

This study's data set is a subset of a larger data set that was collected to understand students' experiences in university mathematics classes across multiple years. Data collection began when students took a group work-based introduction to proof (ITP) course at a large public university in the Midwestern United States and continued into subsequent proof-based courses.

ITP Context

The ITP course was a multi-section course, where each section was typically taught by two graduate students, one serving as the instructor, the other as a teaching assistant. The course was coordinated by a faculty member, who asked instructors to use group work with groups of three to four students. No further group work implementation instructions were provided to the instructors. Around two-thirds of all class time was dedicated to group work.

Data Sources

We drew on two data sources in this study: (a) classroom observations, and (b) semi-structured interviews. The classroom observations were of eight sections, split across two semesters, each observed by one or two researchers. We tried to observe classes at least once a week and thus have a total of 116 observations, of which 83 included group work.

In each of these eight sections, we recruited between two to five participants for a total of 29 participants. In our recruitment, we attended to gender, race, ethnicity, and major in the hope of accurately representing the diverse set of experiences of the ITP student body. All but one of the 29 participants took part in two semi-structured interviews: an early-semester interview (ESI) several weeks into the semester, and a late-semester interview (LSI) towards the end of the semesters. (The 29th participant only completed the ESI.) In the interviews, we sought to learn about students' course experiences more broadly, and although group work was an important experiential component addressed by all participants, it was not the sole focus of the interviews.

Data Analysis

First, we used a bottom-up approach to classify the group work formation methods we witnessed instructional teams implement. This approach resulted in seven group work profiles. Comparing our profiles to Hagelgans et al.'s (2001) taxonomy, we were able to recognize our group work profiles as subcategories of the taxonomy. For example, we identified two different types of random selection: one in which the groups were randomized in public (i.e., in front of the class) and one in which the groups were privately randomized by the instructional team. Yet, our refinement of Hagelgans et al.'s (2001) taxonomy is not the focus of this report: We view the classifying of the classroom observations as context for students' reported experiences.

To learn about students' experiences, we used a series of coding methods to identify themes. The first author began by using two rounds of structural coding (Saldaña, 2009): first, to identify interview segments that addressed group work, and, second, to narrow the group work segments to those subsegments pertaining to students' experiences of the group formation method. The group formation subsegments were then themed (Saldaña, 2009) by the first author, that is, each subsegment was summarized by a several sentence-long thematic statement. Last, the first two authors used pattern coding (Saldaña, 2009) on the thematic statements to identify central themes of student experience. For each theme, we wrote an extended description. We include an example thereof in the Results.

Results

In the 83 observations that included group work, we observed student selection methods 42 times, random selection methods 26 times, instructor selection methods 12 times, and pseudo-random selection 3 times. Accordingly, participants' comments about their experience mainly focused on random selection and student selection, with a smaller number of comments about instructor selection. Furthermore, mirroring participants' experience in the classroom, participants' discourse on random selection was entirely about *weekly* random selection and their discourse on instructor selection was almost entirely about instructor selection *with student input*. In Table 1, we present the themes identified for the three broad types of group formation methods discussed by the participants. Thereafter, we offer our description of Random Selection Theme #2 as an example of one of our theme descriptions.

Table 1: Students' Experiences of Group Formation Methods Themes

Group Formation Method	Student Experience Themes
Random Selection (Each Week)	1. Participants' overall experience in randomized groups was positive or neutral 2. By working in multiple randomly formed groups, participants met many of their peers, which had several positive consequences 3. Participants also identified downsides of random selection 4. Participants identified advantages and disadvantages of randomizing groups *each week* 5. New interactions and positions in new randomized groups
Student Selection	1. Participants offered a mixed picture of group work experiences in student-selected groups 2. Student selection typically led to participants working "with" their neighbors for the semester (subject to minor adjustments) 3. When given the option, some participants chose to work alone

	4. Fixed interactions and positions in student-selected groups
Instructor Selection (With Student Input)	1. Not all participants submitted group preferences, but those who did asked to work with peers they were comfortable with 2. Being in a permanent group created by the instructor with student input made many, but not all, participants happy

Example Theme Description of Random Selection Theme #2: By Working in Multiple Randomly Formed Groups, Participants Met Many of Their Peers, Which Had Several Positive Consequences

Since instructor 1 randomized groups every week of the semester and instructors 6 and 7 randomized groups for a large part of the semester, students who experienced random selection worked with many of their peers. The participants reported multiple positive outcomes, including *getting to know their classmates*. Even though this may seem natural or even trivial, five participants (O, M, G, F2, E2) shared that they had not gotten to know their peers in past classes—even in small classes and classes with group work. Consider G's story:

G:	[...] I work with my friend. [...]
Interviewer:	Did you meet when you came to class, or did you know each other before?
G:	Not known [*sic*] very well, but I was seeing him in my Calculus 2. First semester I remember him, and Calculus 3 I think I've seen him in the class, but I haven't talked to him until I came to this class. And I saw him in class, and I remember him, I remember his face, but I never know [*sic*] him before. (ESI)

Aside from G, multiple other participants (K, C2, F2, N2, E2) reported *being able to make friends* in the class. More broadly, M shared in her ESI that "[in] high school I felt like I definitely had a bond with my other math kids that I feel a similar bond to the math kids in [this course]." Aside from getting to know their peers and forming friendships, three participants (M, H2, N2) described *benefiting from hearing different perspectives*.

Interviewer:	Do you feel like you've learned directly from anybody in class?
M:	Yes [...] I know that [NAME 1] usually uses contradiction work and I use direct proofs. His way of doing it is always different. Then [NAME 2], like I said, just has a completely different way where he brings in other variables if he needs them [...] (LSI)

Last, participants highlighted that changing groups provided them with two valuable insights: (a) *understanding one's speed in relation to one's peers'* (i.e., "know[ing] how fast, or how slow I am compared to a part of the class" [H2, ESI]), and (b) *determining whom one works with productively* (i.e., "who I work with better or who I found to be more productive" [N2, LSI]).

Discussion

In this study, we explored students' experiences of group formation methods by identifying 11 themes across three group formation (sub)methods. The themes paint a generally positive picture of weekly random selection and raise concerns about student selection. From the two themes about instructor selection (with student input), we learned that letting students provide instructors with their group preferences may be useful but not a cure-all.

We acknowledge the importance of the ITP context for this study. Several participants identified key aspects of the ITP context that made their group work experience (positively) different from past group work: (a) the utility of group work in an ITP course, unlike calculation-heavy courses like Calculus, (b) their peers' motivation and seriousness about the work, and (c) the emphasis on understanding and not receiving a grade for group work. Particularly the first two appear to us a dividing line of typical early and later undergraduate students' experience. We are left wondering to what extent the line can be (re)moved via active learning implementations of early undergraduate courses, like Calculus.

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BE EQUITABLE!: WHAT DOES THAT EVEN MEAN?

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In this theoretical paper, we present the results of an analysis of literature on equitable mathematics teaching. We bound our review to research published in Educational Studies in Mathematics *between 2010 and 2023. In our review, we considered how authors conceptualized equity, the recommendations they made to teachers, and how they positioned students.*

Keywords: Equity, Inclusion, and Diversity; Systemic Change; Teacher Educators

This year's theme asks that we envision the future of mathematics education in uncertain times. In order to envision the future, we first turn to the past. In particular we trace the recommendations made as to how classroom teachers should create more equitable mathematics classrooms. Given the large body of work now addressing equity in mathematics education, we are interested in what it is that mathematics education researchers are saying about equity and how the work of addressing equity is transferable to actions that teachers could take in K12 classrooms. We conducted a literature review of *Educational Studies in Mathematics* (*ESM*). *ESM* was chosen for three reasons: (1) It is considered "top-tier" journal in mathematics education (Nivens & Otten, 2017; Williams & Leatham, 2017); (2) In our initial search which included *Journal for Research in Mathematics Education* and *For the Learning of Mathematics*, *ESM* returned the highest number of articles that met our criteria; (3) the journal aims indicate that it is, "open to all research approaches and research foci, including cognitive, socio-cultural, socio-political, and language-related aspects of mathematics education" (*ESM,* 2024). Given this aim of the journal, we expected that *ESM* might provide the most diverse representation of views on equity. The research questions that guide our review are:

1. What is the scope of research on equitable mathematics education?
2. How are authors framing equity and who do they position as *needing* equity-focused pedagogies?
3. What actions has the research on equity in mathematics education suggested for classroom teachers?

Phrases such as 'equitable mathematics teaching', 'equity-focused pedagogies', and 'equity in mathematics education' convey distinct nuances, our purpose is to engage with the way mathematics education researchers have employed their use in explorations of equity-related issues in mathematics education.

Perspectives and Theoretical Framework

There is no single set of directions or procedures that can be followed to create an equitable classroom environment, and we have found in our research that enacting equitable pedagogies takes substantial time (Castanheira et al., 2024). We begin by stating that equitable teaching is an ongoing, elusive, never finished endeavor. Due to the disciplinary and societal positioning of mathematics in K12 environments, we, and others (Martin, 2015, 2019; Stinson, 2004), argue that creating equitable *mathematics* classrooms requires working against the grain. All change in schools requires recognizing and then challenging the taken-for-granted (Britzman, 2003), in this paper, we analyze the ways the mathematics education researchers are framing equitable practice and the actions necessary to create equitable mathematics classrooms. An underlying assumption to the calls for more equitable teaching practices is that the inequities present in schools can be addressed and remedied by teacher actions. Some question whether teachers can effect substantial and sufficient change within the current structures (Bullock, 2023; Martin et al., 2019), we honor these perspectives; however, in this project, we are curious about the scope of recommendations and how they position teachers. Therefore, we traced how equity was conceptualized within a subset of Mathematics Education Research (MER). and further consider how teachers are positioned in terms of their agency to address inequities.

Modes of Inquiry

In 2010, Katherine, Heid, asked "Where's the math (in mathematics education research)?" Given this push to recenter and prioritize mathematics content, we were curious as to patterns we might find in the MER related to equity from that point until the present. Thus, we searched articles from 2010 to 2023 in *ESM*. We identified 190 total records. Of the initial search we excluded 98 duplicates, book reviews, and editorials. Then, the team reviewed the abstracts of the remaining 92 identified articles retaining any article that met the following criteria:

1. Audience is K12 mathematics education researchers or mathematics teacher educators
2. Equity is a central focus of the article
3. Manuscript suggested recommendation(s) to support equitable mathematics teaching and learning

After these criteria were applied, there were 22 manuscripts to review (see Figure 1). We reviewed each article and considered the following. What, if stated, was the author's stated purpose? To whom were equitable teaching recommendations or interventions directed?; How, if at all, did the authors conceptualize or describe equity?; What recommendations were made for actions that K12 teachers should take to teach mathematics more equitably? How do the stated recommendations position teachers in terms of their agency within the classroom? And, how, if at all, did the authors acknowledge structural and systemic influences on classroom instruction?

Figure 1: Literature Review Process

Results

In this report, we share our analysis of three areas, conceptualizations of equity, recommendations for and positioning of teachers, and positioning of students. We begin by considering the *conceptualizations of equity*. Across the articles 11 of 22 articles did not directly define equity, or a form of equity. In these cases, we acknowledge that all authors did not omit a definition, but that some focused heavily on social justice or equity adjacent ideas. Of the articles that did provide conceptualizations of equity, there were ten which were borrowed from other authors. There were two main themes. Equity as *taking up space or making room*. Lui (2022), Hand (2012), and Ruef et al. (2020) take the view of equity as taking up space or allowing student ownership of classroom space or mathematics. This definition within a classroom space aligns with NCTM's Access and Equity Position Statement (NCTM, 2014), which is drawn upon by Yilmaz et al. (2021). Matthews et al. (2021) and Xie et al. (2021) use a similar conception of taking up space, but with the space being within larger society. For example: "the creation of liberatory spaces that disrupt dehumanizing conditions and provide avenues for hope" (Matthews et al., 2021, p. 336). Equity as *societal fairness*. Valoves-Chavez (2018) and Roos (2023) conceptualize equity as fair chances, or fair distribution of opportunities, for advancement of the self or mathematics knowledge. Neither identify where, or with whom, the responsibility of distribution of fairness rests. Whereas Bollock (2023) and Chen (2023) explicitly and critically situate fairness and opportunity within societal structures and acknowledge that teachers do not have control nor individual responsibility to overturn systems of oppression.

Next, we analyze the *recommendations for action and positioning of teachers.* We analyzed the way in which definitions of equity position the teacher, students, classroom space, and society and who has the ability to enact change. Within equity definitions, some position the choices of the teacher as the creator/driver of equitable space (specifically Boylan, 2016). With Hand (2012), Ruef et al. (2020), Yilmaz et al. (2021) picking up definitions which see the teacher's choices as providing agency to students. Chen (2023) acknowledges that while teachers make equitable instructional actions, these actions do not solely promote necessary change to dismantle inequitable systems.

Discussion

Across suggestions for teacher practice, we found a continuum from practical and observable actions to overarching ways of being in mathematics classrooms. Several authors described planned recommendations for teacher practice that could lead to more equitable mathematics teaching and learning. Hand (2012) suggested ways of noticing and responding to classroom life that teachers could enact to create a trusting environment. Bonner (2014) recommended ways for teachers to develop knowledge of students and their community to build relationships and trust. They recommend relationship building as providing a "more practical place from which to work" (p. 397), thus allowing classroom change responsive to a student's community. Conversely, Yilmaz et al. (2021) suggested overarching professional development on implementation and utilization of mechanisms to support student needs. These recommendations are non-specific but identify teachers as the decision-makers and enactors of change within the classroom space.

A subset of articles included conceptualizations of equitable teaching practice that are made-in-the-moment. These conceptualizations primarily draw on post-structural and new materialist theoretical frameworks. They present equitable or ethical classrooms as being made by teachers in ongoing and inter-acting relations between students, curriculum, and discursive elements. Boylan (2016) upholds this perspective in their discussion of ethical practice drawing on Levinas. Liu and Takeuchi (2023), drawing on feminist new materialist theory, acknowledged "spatial and temporal configurations of pedagogy that enable or hinder racialized multilingual learners" (p. 270). And further discussed student agency as being fostered through the use of space, discourse, and the decentralization of the teachers. While many of these authors also acknowledge the impact of societal and school structures, these statements and guidance imply agency within the teacher to fix inequities within classroom spaces. We found few examples of explicit acknowledgement of systemic inequities (e.g., Bulluck, 2023; Chen, 2023; Louie, 2017)

Some studies focused on making changes to the classroom environment to better serve specific student populations. For instance, Hand (2012), Liu and Takeuchi (2023), and Bonner (2014) directed their research towards reconceptualizing mathematics pedagogical spaces to better include and serve traditionally minoritized students. These studies acknowledged the inherent deficiencies in the teaching and learning of mathematics and aimed to reposition "learners from nondominant ethnic, racial, and linguistic backgrounds" (Hand, 2012, p. 233) and "racially and linguistically minoritized students" (Liu & Takeuchi, 2023, p. 268) from being seen as learners of deficiency to being recognized for their potentials and what they can contribute.

Across other studies, there seems to be a preference for an asset-based approach to describing students by focusing on the strengths, potentials, and rich cultural values they bring into the classroom. This is evident in the way these populations are framed as not merely recipients of mathematics education but as contributors who can be active in the learning process. For instance, Matthews et al. (2021) emphasized reimagining work highlighting the need for a shift towards recognizing and harnessing the culture and excellence of black learners. Also, Takeuchi (2017) focused on the mathematical resources students of immigrant parents access at home, which suggests that we should recognize the valuable and often overlooked assets these students come into their various mathematics classrooms with as learners. Despite the intentions of some research studies to position students from an asset-based lens (e.g. Hand, 2012; Liu & Takeuchi, 2023; Matthews et al., 2021; Takeuchi, 2017), which emphasizes the strengths and potentials, some findings from our review suggested a contrasting perspective at the classroom level. Notably, Darraugh and Valoyes-Chávez (2019) drew attention to how their data showed that "students in general were viewed as being a problem." and that "teachers spoke more about the students than any other factor as being one of these challenges" (p. 431).

Conclusion

Given the history of reform practices, we analyzed the recommendations by the level of control they attributed to the teacher in enacting the recommendations and whether they accounted for any external factors. We identified two continuums of conceptualizations of the teacher in relation to equitable practice. The first continuum moves from conceptions of the teachers as an independent actor able to perform new practices to create a more equitable learning environment to a conception of teacher as in ongoing relation to the students, discourses, material environment with some level of agency. The second continuum moves from articles that largely ignored systemic and structural conditions to those that explicitly acknowledged and named systemic and structural influences and interactions. By considering the ways that MER conceptualizes equity, who it is for, and how it might be enacted, we are left to wrestle with the ambiguities within this body of research. Leaning too far toward a conception that positions teachers as in complete control creates the possibility for teachers to serve as scapegoats. Likewise, giving too much power to systemic influences can create apathy. As mathematics teacher educators envisioning a different future for students and teachers in K12 environments, questioning the taken-for-granted has propelled us to consider the multiple levels of work, and we reiterate for ourselves and the field that "equity" is a target that is always on the move in relation to our communities, commitments, and expressed and unexpressed values.

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RESEARCHERS LEARNING FROM TEACHER NOTICING: THE CASE OF MR. THOMPSON

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In this exploratory study, we analyzed one mathematics teacher's annotations of a transcript of their teaching. The teacher was prompted to annotate the transcript for actions that contributed to or hindered their enactment of a complex teaching practice. We analyzed these noticings to explore what we could learn about the teacher's understanding of the practice, and then what these understandings revealed about our own conceptualization and communication of the practice. Our approach to analyzing teacher noticing illustrates how the study of noticing can contribute to advancing researchers' understanding not just of teachers' noticing but also of the phenomena they are noticing.

Keywords: Classroom Discourse, Instructional Activities and Practices, Research Methods, Teacher Noticing

Research on teacher noticing has typically centered around how to best support teachers in learning to notice salient features of the classroom. The majority of this work has focused on understanding the effectiveness of specific noticing interventions (Jacobs et al., 2010; McDuffie et al., 2014; Santagata et al., 2021; Schack et al., 2013; Stockero et al., 2017), but studies have also focused on comparing different methods of documenting noticing (e.g., Lee, 2021) and understanding how noticing develops (e.g., Bragelman et al., 2021; Simpson & Haltiwanger, 2017). Other studies have focused on understanding how teacher noticing interacts with factors such as teaching experience (Yang et al., 2021), mathematical knowledge, and affect (Jong et al., 2021), which allows the field to consider how such factors might need to be accounted for when supporting teachers in learning to notice. Still other research has examined the long-term outcomes of noticing interventions by focusing on the extent to which noticing skills developed in a teacher education intervention transfer to classroom teaching practice (Sherin & van Es, 2009; Stockero, 2021). In all of this work, the intent was to examine teachers' noticing in order to understand how to better support that noticing.

We encourage the field to consider, however, how the study of teacher noticing can contribute to advancing researchers' understanding not just of teachers' noticing itself but also of the phenomena they are noticing. One example of this approach is Louie et al.'s (2021) work focused on understanding anti-deficit noticing. In their work, they used a noticing interview to analyze how one teacher's noticing aligned with and differed from deficit and equitable frames from the literature. The outcome of their analysis was a conceptualization of anti-deficit noticing and what it might look like in practice. We contend that the study of teacher noticing could also help us understand other aspects of teaching. In this exploratory study, we examine how studying a teacher's noticing around his enactment of a complex teaching practice can help us better understand our own conceptualization and communication of that practice.

Theoretical Framework

Broadly speaking, our work aligns with the cognitive-psychological perspective on noticing in that we "recogni[ze] that human perception is limited and that teachers must learn to pay attention to certain instructional aspects while disregarding other aspects" (König et al., 2022, p. 3). In particular, our research has focused on supporting teachers to notice student contributions that have high potential to advance students' learning of mathematics if taken up during a lesson. We have also focused on understanding how teachers might subsequently keep students focused on jointly making sense of those contributions during a whole-class discussion by noticing and taking up ideas that support such sense-making and putting aside those that do not.

Mathematical Opportunities in Student Thinking (MOSTs) (Leatham et al., 2015) are high-leverage student contributions that are important for teachers to notice because they create an opportunity to engage the class in making sense of the significant mathematics embedded in the instance. Building on a MOST (hereafter referred to as building) is the teaching practice that takes advantage of the opportunity a MOST provides by engaging the whole class in a joint sense-making discussion focused on understanding the mathematics at the heart of a MOST. Building is "comprised of four elements: (1) Establish the student mathematics of the MOST so that the object to be discussed is clear; (2) Grapple Toss that object in a way that positions the class to make sense of it; (3) Conduct a whole-class discussion that supports the students in making sense of the student mathematics of the MOST; and (4) Make Explicit the important mathematical idea from the discussion" (Leatham et al., 2021, p. 1393).

In this exploratory study, rather than considering how to support teacher noticing, we consider what we as researchers can learn from a teacher's noticing as they reflect on their enactment of building. That is, we theorized that examining the instances a teacher identified as important in their enactment of building and how they discussed those instances in relation to a conceptualization of building that had been shared with them could advance our own understanding—both about the practice of building and how we communicate that practice to teachers. In a sense, we extend van Es and Sherin's (2002) notion that noticing entails "making connections between the specific of classroom interactions and the broader principles of teaching and learning they represent" (p. 573) to consider how researchers can use the connections a teacher makes in their noticing as a venue for advancing their own understanding.

Methods

As part of a larger research project, twelve teacher-researchers (TRs) worked with the authors as they conceptualized the teaching practice of building by enacting the practice in their classrooms and then allowing the authors to analyze those enactments. Mr. Thompson, the subject of this case study, was one of the twelve TRs. At the time of the study, he had been teaching junior high school mathematics for about 20 years, had been recognized for his excellent teaching, and worked part-time as an instructional coach for his school district. Following the final two enactments of the mini-tasks, Mr. Thompson was provided a transcript of each enactment and asked to annotate instances where his actions either contributed to or got in the way of enacting an element of building.

Mr. Thompson provided 24 annotations across two different enactments of building. Given the prompts for the annotations, we viewed them as instances of noticing with respect to the practice of building and used them as the unit of analysis to answer the research question: *How can teachers' noticing help researchers refine both their theorization of a teaching practice and how the practice is communicated to teachers?* Looking at each annotation in the context of what happened during the enactment, the researchers determined whether what Mr. Thompson noticed provided evidence of understanding, partial understanding, or lack of understanding of building. After deciding what a given “noticing” told us about Mr. Thompson’s level of understanding, we identified what contributed to this level of understanding.

Results

There were two primary categories of what we as researchers gained from our analysis of Mr. Thompson’s noticings: (a) an expanded conceptualization of building, and (b) an expanded understanding of how to develop teachers’ understanding of building. We use examples of Mr. Thompson’s noticings to illustrate each of these categories.

Expanded Conceptualization of Building

At times our analysis of Mr. Thompson’s noticings with respect to building prompted us to consider aspects of the practice that we had never considered. At other times our analysis helped us to better understand aspects of the practice of which we were aware, but about which we still had uncertainties. In such situations, the TRs were using their classroom expertise to try to enact a broad vision of the practice and their attempts allowed us to further conceptualize what it would take to enact moves that would align with our own vision of the practice. For example, at one point during Make Explicit Mr. Thompson asked the class, “Now we’ve talked about this year, what we call the value that you plug in for x, it has a name. Do you remember, what’s it called? What do we—what do we call the possible things you can plug in for x?” He attached the following comment to his transcript: “Not contributing very well (Make Global): As soon as I attempted to have students consider the understanding about this specific problem that has now been shared things went awry.” We still had many questions at this point about the kinds of moves a teacher might make when trying to scaffold the class toward articulating the mathematical point that could be gleaned from their prior sense making about the MOST. Mr. Thompson’s attempt to provide such scaffolding, the difficulties that ensued, and his own noticing of those difficulties, allowed us to better understand the nature of the teachers’ scaffolding at this point in building. In particular, we realized the complexity of determining the extent to which students themselves might be positioned to articulate the mathematics and the extent to which the teacher might need to do some of that work.

Expanded Understanding of How to Develop Teachers’ Understanding of Building

At times our analysis of Mr. Thompson’s noticings with respect to building confirmed our conceptualization of the practice. In such instances we often gained insights into our own efforts to develop Mr. Thompson and his fellow TRs’ understanding of building in preparation for their enactments. There were times when we had been explicit in our work with the TRs about a given aspect of building and yet Mr. Thompson’s noticing seemed to provide evidence of incomplete understanding of that aspect. There were other times when his incomplete understanding was likely due to a critical oversight on our part. That is, while the aspect of building was mutually understood by the research team, it had remained implicit in our work with the TRs. Finally, several of Mr. Thompson’s noticings with respect to building revealed how some important routines from his typical task-based teaching hindered his ability to enact our conceptualization of the practice. For example, at the beginning of the Conduct element of one enactment, Mr. Thompson elicited a contribution that was clearly related to making sense of the established MOST. The protocol in these situations was to invite the class to consider how this contribution might help them to further make sense of the MOST. Instead, Mr. Thompson set aside the contribution and collected contributions from five additional students. He attached the following comment to the section of the transcript where he elicited this collection of contributions:

> I'm not sure that this is a real nice fit with the framework for MOSTs. The closest fit might be Conduct? Anyway, I do think it is helpful for there to be a bit of sharing like this to occur….a chance for some initial thoughts and rough draft thinking to get out there.

We see here an awareness that this collecting rather than connecting a contribution to the MOST at the beginning of Conduct likely did not fit the conceptualization of building Mr. Thompson was enacting. Nevertheless, he felt it was important to deviate in this way. We realized in analyzing this noticing (and others that were similar) that we needed to pay more attention to teachers' typical teaching routines, considering which aspects of those routines might be valuable to activate during building and which might need to be "deactivated" for the sake of building. In this case, Mr. Thompson seems to have assimilated building into his prior practice rather than accommodating that practice to create new space for the building practice.

Discussion and Conclusion

In this exploratory study, we focused on analyzing the noticings from a single teacher. Data from the larger project will allow us to analyze noticings from a dozen teachers. We suspect that analysis of the larger data set will provide additional ways teacher noticings can be used to develop researchers' conceptualization and communication of a teaching practice. In addition, we might learn which of these ways are more common or more useful for these purposes.

The noticing literature has tended to focus on whether teachers are noticing what we, as researchers or teacher educators, want teachers to notice. And for good reason. It is important that teachers develop the skills for noticing certain things over others (e.g., the substance of students' mathematical ideas over students abilities to articulate those ideas). The goal of such research is to better understand how to help teachers develop those noticing skills. The results of this exploratory study demonstrate how teachers' noticings can provide insight into their understanding of the phenomenon they are noticing which, in turn, can provide insights into the phenomenon itself. Mr. Thompson is an experienced teacher with substantial noticing skills. As researchers we were able to leverage that noticing to help further our understanding of building.

Although it is certainly important that the field of mathematics teacher education continue to seek ways to help teachers improve their noticing skills, it also seems important, particularly when working with inservice teachers, to position them as important resources from whom we as researchers and teacher educators can learn (Freeburn et al., 2024). Mr. Thompson is a skilled and accomplished teacher who engaged enthusiastically in our research and demonstrated a desire to learn from the experience as well as help us to learn. In fact, he expressed on numerous occasions that his work with us was one of the best professional development experiences he had had. Thus, when his noticings revealed a mismatch in understandings of the practice of building it in no way reflected poorly on him as a teacher or learner. When his noticings revealed a difference, we were able to hone in on that difference and consider whether and to what extent the difference might matter. His noticings also provided ample opportunities for us to reconsider how we communicate the building practice to others. Perhaps there is a lesson in this work for all of us as mathematics teacher educators and mathematics education researchers: Let us use the expertise of the students and teachers with whom we work to shine the light on ways we can improve, to evaluate our own work more so than theirs.

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UNDERSTANDING ONE MIDDLE SCHOOL'S MATHEMATICS INTERVENTION CONTEXT

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Keywords: Curriculum, Middle School Education, Teacher Beliefs

Mathematics intervention is an often under-investigated aspect of mathematics instruction. Mathematics intervention is generally understood to be supplemental instruction for students who require "increasingly intensive instruction matched to [students'] needs" (RTI Action Network, n.d.) and is often conceptualized through the lens of special education (e.g., National Center on Intensive Intervention, n.d.). When general education teachers are responsible for mathematics intervention instruction, they may especially lack guidance about how to support a range of learners in supplemental instructional contexts (e.g., Perry et al., 2015). The purpose of this study was to understand how general education teachers at one middle school thought about and enacted mathematics intervention curriculum and instruction.

Context, Participations, and Data Collection

Unity Middle School (a pseudonym) is in a small Midwestern city. In 2021–2022, the state designated Unity as an "on watch" school since it was in the lowest-performing 5% of all schools in the state. The school has almost 900 students, the majority of whom are Black or Hispanic (59.9%); 19% of Unity students receive special education services.

This research team has been co-designing research activities with Unity since the 2021–2022 school year. In spring 2022, the mathematics instructional coach and the two authors of this paper, co-designed this investigation. Driven by the instructional coach's interest in reimagining the intervention space, we decided to first collect data about the current landscape of intervention instruction—what was happening in classrooms and how teachers talked about mathematics intervention itself and in relation to core instruction.

There were six voluntary participants—four mathematics teachers and two science teachers; five participants taught one section of mathematics intervention and one participant was the school's mathematics interventionist. Using a co-designed observation tool, each team member conducted three, 50-min classroom observations in each teacher's classroom. Each teacher also participated in one, 60-min semi-structured audio-recorded interview. The instructional coach collected one-third of the classroom observation data; the first author conducted all interviews.

Preliminary Findings and Possible Implications

As part of our research team, the instructional coach has been coding and analyzing data with the paper authors. Preliminary findings reveal that the majority of teachers in this study viewed remediating basic skills as the purpose of intervention and believed that core instruction and intervention should be more different than similar. These articulated beliefs were consistent with observational data, which revealed intervention curriculum and instruction aimed at below-grade-level content and focused on "procedures without connections" (Smith & Stein, 1998, p. 346). Our co-analysis will conclude in the summer of 2024. Findings from this project can inform how schools and districts conceptualize mathematics intervention and at what efforts they might aim for resources.

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DEPRIVITIZING INSTRUCTIONAL PRACTICE USING INSTRUCTIONAL CIRCLES

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Keywords: Professional Development, Mathematical Knowledge for Teaching, Teacher Knowledge.

Background

The effectiveness of professional learning opportunities depends on their content and how the content is facilitated, *what* teachers' study, and *how* they study it (Van Es et al., 2014; Kazemi et al., 2008; Kennedy, 2016). In my study, I focus on the way in which professional learning opportunities are facilitated, on how teachers are engaged in studying teaching.

Instructional circles are somewhat informal gatherings of small groups of teachers to discuss instructional practices (Melville, 2022). These circles seem oriented to analyzing specific practices and sharing advice rather than collectively designing a lesson or learning about a specific instructional method. Some local teachers and educators voluntarily attend instructional circle sessions that usually meet once every few months. Japanese teachers have described them as important professional learning activities that serve an essential function quite different from lesson study. This session will look at how instructional circles support the development of teachers' ability to critically discuss improvements to instructional practices enabled through the deprivitization of their practice.

U.S. teachers often have a difficult time deprivitizing their practice among other professionals (de Jong et al., 2019). Often when teachers do deprivitize their practice, three main types of interactions occur: giving praise, giving advice, and (uncommonly) engaging as challenging colleagues (Males, 2009). Engaging as challenging colleagues is less about critiquing the professionalism of the teacher but working together to improve the instructional practices found within a lesson. Thus, taking the focus from the teacher and put onto *teaching* (Hiebert & Grouws, 2007). This session will present how instructional circles are able to help teachers deprivitize their practice and engage as challenging colleagues to improve their instructional approaches to teaching.

Methods and Results

An ethnographic approach was taken to learn about how instructional circles engage U.S. teachers in deprivitizing their practice in a community of learners. We focused specifically on what teachers brought to the instructional circle, what they taught, and then how they felt the process beneficial for their learning. We did participant observations, interviews, artifact collection, and journaling.

So far, we have found that instructional circles addressed the need for a safe place for teachers to share their practice with others. In other words, teachers felt comfortable deprivitizing their practice to allow for high-quality discussions about the improvement of instructional practices as identified by the teachers themselves. Teachers were able to develop a community of learners through their focus on improving a specific aspect of the lesson plan. For example, one teacher was trying to increase student engagement and asked the other teachers for their input. Teachers offered advice, but the focus was more detailed through the lens of trying to accomplish the learning goals.

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CONFERRING TO ELICIT AND ADDRESS EPISTEMOLOGICAL OBSTACLES

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Epistemological obstacles (EOs) are cognitive challenges students experience that persist despite research-based instruction (Brousseau, 2002; Sierpińska, 1987). Informed by prior research, The Proofs Project (NSF DUE-2141626) designed and implemented instructional materials that elicit and address these obstacles in transition to proof (TTP) courses through cycles of small group exploration and whole-class discussion. We investigate the discursive routines of one teacher as she elicited and addressed EOs around transforming conditional statements and we discuss implications for teaching with this theory.

Keywords: Instructional Activities and Practices; Classroom Discourse; Reasoning and Proof

Teaching in ways that elicit meaningful participation in classroom mathematical discourse has become a primary objective of modern mathematics education (NCTM, 2014; Speer & Wagner, 2009), but developing these practices is known to be challenging for teachers (Andrews-Larson et al. 2019, Bishop et al., 2020, Henderson et al., 2011). In K-12 education, researchers who have investigated these practices in whole-class discussions have found that features like responsiveness and intellectual demand can support students' learning (Bishop, 2021) as well as their mathematical attitudes and identities (Boaler & Greeno, 2000; Wagner & Herbel-Eisenmann, 2008). At the undergraduate level, Mesa & Cheng (2010) illuminated some of the ways instructors' discursive moves shaped students' agency in calculus and introductory courses like college algebra. Building on this work, we wondered how teachers in transition-to-proof (TTP) courses might use student-thinking to drive mathematical discussions, given the "didactical gap" between the mathematics of K-12 education and proof-based mathematics (Balachef, 2010).

To bridge this gap, students must take up the rigorous logical reasoning and communication practices of mathematicians in a context with distinct epistemic and heuristic commitments from computation-based courses, like formality and abstraction (Melhuish et al., 2022; Lew & Mejía-Ramos, 2019; Sfard, 2014). They must also adapt to higher demands for conceptual understanding and autonomous problem-solving as compared to prior math courses (Selden, 2012; Tallman et al., 2021). In addition to these difficult shifts in mathematical practice, there are unique cognitive challenges students encounter in TTP (Dubinsky, 1990; Harel, 2002; Stylianides et al., 2007). Working with logical implications (LIs), or conditional statements, is one such challenge (Arnold et al., in press, Antonides et al., in press). Students may erroneously interpret conditional statements as biconditional (Girotto, 1991) and conflate implications with their converse (Duran-Guerrier, 2003) or inverse (Knuth, 2002). Further, students struggle to utilize the equivalence of a LI with its contrapositive (Dawkins & Hub, 2017) and its disjunction (Hawthorne & Rasmussen, 2015). Brousseau and Sierpińska (1987) conceptualized these necessary challenges as *epistemological obstacles* (EOs): "cognitive challenges that persist even in response to research-based instruction," perhaps due to teachers circumventing them, rather than addressing them (Kokushkin et al., 2023, p. 1).

Considering the challenges students face in TTP from both a discursive and cognitive perspective, it would be edifying to understand how teachers might elicit students' thinking around these EOs and address them during whole-class discussion. We investigate one teacher's routines in the context of eliciting and addressing EOs around logical implication in an undergraduate TTP class.

Theory and Methods

The theory of epistemological obstacles emphasizes the importance of eliciting and exploring students' thinking through questioning, probing, and challenging tactics aimed to bridge students' informal ideas with formal reasoning, or, as Bishop (2021) framed it, teaching in ways that are *responsive* to students' thinking and require *intellectual work* from both teacher and students. These discursive routines enacted in classroom interactions are related to Munson's (2019) *conferring* interactions in which the teacher elicits and probes student thinking and then responds to what has been uncovered (Munson, 2019, p. 2). We investigate the ways these discursive modes arise in a TTP class with attention to specific EOs entailed in the lesson, especially because the research-based instruction we study aims to "elicit and address students' experience of epistemological obstacles head-on" (Arnold & Norton, 2022). Reasoning about hidden quantification (Qh) is one EO from prior investigations particularly relevant to this study (Norton et al., 2023).

The data in this study were collected as part of a larger project focused on addressing cognitive and instructional challenges in TTP. The data consist of one 50-minute class video of a TTP course taught at a large, land-grant university in the eastern United States by the third author, Dr. A. To begin data analysis, the first author created an initial transcript from the class video using Otter.ai, then edited the transcript by hand for accuracy. Episodes of whole-class mathematical discussion were identified for analysis; discussions that were not mathematical in nature (i.e., about assignment due dates) were not analyzed. Within each episode, several related verbal exchanges between teacher and students occurred. Here, an exchange is comprised of a corresponding initiation, response, and follow-up. We coded each of the teacher's follow-ups for intellectual work and responsiveness, using the following coding scheme from Bishop (2021). Intellectual work determined whether the teacher's move requested information from students (demand) or provided information (give). For both types, low-level and high-level moves are distinguished. Several additions to categories were developed to accommodate the mathematical activity in TTP classes. For instance, Bishop's (2021) framework did not account for requests to evaluate logical statements at a particular value in the universal set because the subjects of her study were not TTP students. However, students at this point in the course could be expected to make this calculation routinely, so responses of this type were coded as Low demand. Similarly, connecting an idea to formal notation or logical structures entails high intellectual work for students, so responses of this type were coded as either High give or High demand depending on the response type.

Bishop's (2021) responsiveness moved beyond conceptualizations of responsiveness as binary—i.e., teacher interactions as either responsive or not (see Munson, 2019)—to investigate responsiveness as a continuum describing "the *extent* to which student ideas are elicited, incorporated, and built on" in class conversation (p. 12), like Jacobs and Empson (2016). Responsiveness has three levels: Low or No responsiveness, Medium, and High, depending on whose idea became the focus of the response. Low or No responsiveness follow-ups include evaluating, revoicing, acknowledging, or making a related statement or question to a student's idea. Medium-level follow-ups involve corrective moves, co-opting student responses to make a point, or incorporating basic information provided by students to supplement the teacher's response. The last level is subdivided into levels High I and High II depending on whose reasoning was displayed in the follow-up: the teacher's or the students', respectively (Bishop, 2021, p. 478-79).

Because our curiosity lies at the nexus of Dr. A's eliciting and addressing moves around EOs and their function as responsive or intellectually demanding speech, we attended to shifts in responsiveness and intellectual demand within Dr. A's follow-ups. We expected Dr. A's responses to shift while eliciting and addressing an EO because eliciting students' thinking around EOs necessarily makes demands of students whereas addressing EOs requires both responsiveness to students' thinking and intellectual work from the teacher.

To identify these shifts, we collected the codes for all exchanges in class discussion and looked for when one or more codes changed across exchanges. For instance, we noted shifts from Medium responsiveness to High I or from High demand to Low demand. We also identified, and member-checked with Dr. A, the EOs underlying the mathematical content discussed within each segment to note their possible connections to these shifts. A shift across exchanges became a routine if a) it was repeatedly enacted across the lesson and b) a feature of an EO (e.g., quantification, or Qh) was being discussed. Note that we do not view these shifts as entailing a change in the quality of a response; a decrease in responsiveness or intellectual work does not necessarily imply a lowered quality of teaching. In contrast, a shift from High II to High I responsiveness meant the follow-up was (neutrally) less responsive because the teacher's thinking, rather than the students', drove the response—an essential move for refining and formalizing students' thinking.

Results

We present a discursive routine found in Dr. A's class that is characterized by tandem shifts in the intellectual give and demand of Dr. A's responses to students: as intellectual give increases, intellectual demand decreases. We refer to this routine as C*onferring to Elicit and Address*, adopting the terminology of Munson, (2019). This routine arises in 12 out of the 25 exchange pairs we analyzed. The remaining exchange pairs either did not exhibit shifts that were repeated multiple times (failing our first criterion) or did not address an EO (failing our second criterion). We present a vignette displaying this routine and discuss Dr. A's moves below.

Vignette: Truth sets are subsets

Prior to this vignette, students had been working together in small groups to create an Euler diagram depicting the true statement P implies Q, denoted P → Q.

Dr. A: Alright, let's come back together. What should the relationship between the truth sets P and Q be, in the event that the implication P implies Q is actually true? [calls on Alex] Yeah?

Alex: P should be the smallest circle within a larger circle, Q.

Dr. A: P should be inside - the circle for P should be inside the circle for Q, what do you all think? What's another way that we could describe that relationship? [calls on Joe] Yeah?

Joe: P is a subset of Q.

Dr. A: Yeah, P is a subset of Q, so we want to make sure that we're seeing that in our picture. [drawing the Euler diagram] So this one should be Q and then the one on the inside should be P. And we said that means that P should be a subset of Q.

There was a notable shift in Dr. A's intellectual work across her responses to Alex and Joe. After Alex described her group's diagram, Dr. A revoiced the contribution and asked, "what do you all think? What's another way that we could describe that relationship?" This was a High intellectual demand response, requesting that the class make sense of Alex's statement and then probing to elicit a more formal version of it. This was also High II responsiveness because it placed Alex's contribution at the center of discussion.

When Joe articulated the correct subset relationship, Dr. A's response shifted from High to No demand, instead giving information to students in the form of a summary of the discussion. The shift in intellectual work occurred precisely when a student had attended to the subset relationship between the truth set of P and the truth set of Q. This centered on the Qh EO because it attended to the sets of values for which each statement is true and the relations between them. Rather than pressing Joe for further explanation or requesting other students to make sense of his thinking, Dr. A connected the two contributions herself to summarize the main ideas and drive the discussion toward the next topic. This move also contained a shift in responsiveness, from High II to Medium because Dr. A was revoicing Joe and Alex's statements.

Discussion and Conclusion

In analyzing Dr. A's implementation of curriculum materials designed to elicit and address EOs in her TTP class, we characterized the ways instructors might elicit and support high quality student thinking in classroom discourse while maintaining mathematical rigor. By attending to Dr. A's intellectual work and responsiveness, we identified a clear discursive pattern, called Conferring to Elicit and Address, in which Dr. A used high-demand responses to elicit students' thinking about a particular EO and then address it using high-give moves. This routine aligns with the theory of EOs that informed the larger project as it ostensibly elicited students' initial understandings and ways of reasoning about the task, creating the need to address the EOs they experienced. In turn, Dr. A's intellectually giving moves appeared to refine students' ideas into more formal mathematical notation and reasoning. By hinging on key aspects of students' experience with EOs, this routine emphasizes the production of intellectual need (Harel, 2013) in discourse, so that the teacher's giving moves organically impart mathematical disciplinary practices as useful tools that empower students to formalize their own ideas.

EOs are not limited to the context of TTP; practitioners are aware of a wealth of cognitive challenges in mathematics that persist despite best practices. To orchestrate mathematically rich class discussions on all grade levels, teachers can design in advance their Conferring to Elicit and Address routines by drawing explicit connections between features of the task, possible student contributions, and the EO(s) they've identified in their lesson. This supports the effective enactment of the intellectually giving and demanding responses necessary to address EOs when Conferring to Elicit and Address. Finally, because responsiveness and intellectual work are about whose thinking is on display and whose ideas drive mathematical discussions, great care must be taken in utilizing these discursive structures in ways that support equitable participation for students who have been historically marginalized in mathematics. Although we have identified one discursive routine to address cognitive challenges, more work is needed to understand the ways such discursive routines might support students in terms of equitable participation, access, and affect. Additionally, future research should investigate the discourse of a variety of teachers in TTP courses to further explore the consistency of this routine across contexts.

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ELEMENTARY MATHEMATICS SPECIALIST COACHES' STRATEGIES FOR GROUPING STUDENTS FOR INSTRUCTION BASED ON EVIDENCE IN ARTIFACTS

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When elementary mathematics specialist coaches observed written artifacts of student thinking, they organically created groupings and pairings of students. Their interpretations of student work reveal two broad categories for meaningfully grouping students for instruction. One grouping strategy subjectified students, rendering a global evaluation of their performance. Mathematizing strategies were based on the evidence of mathematical thinking shown in students' work. Both strategies have the potential to inform instruction, but the mathematizing grouping strategies may have more potential to support mathematical instruction, while subjectifying strategies may pose threats to student identities as learners of mathematics. The noticings and spontaneous groupings described in this study illustrate different approaches to forming working groups that may serve a broader variety of instructional goals.

Keywords: Teacher Knowledge, Assessment and Evaluation, Instructional activities and practice, Elementary Mathematics Specialists

The formation of student groups is essential for promoting productive classroom discourse and allowing students opportunities to learn from each other's mathematical thinking. This report focuses on coaches' observations, specifically related to their strategies for grouping students for instructional purposes. The results suggest that a binary choice (success or non-success) for grouping students is simplistic.

What coaches noticed in student work (Sherin, Jacobs, & Philipps, 2011), how they interpreted it, and then responded raised issues of mathematical identity, reflected in the participants' utterances (Haye, et al., 2011) as they described students' work. In order to make sense of the utterances participants used to talk about students and their work, the identity perspective shared by Heyd-Metzuyanim and Sfard (2012) was invoked. *Mathematizing* describes the utterances about objects of mathematics– the symbols, the representations, etc. Utterances that address the doer of mathematics are referred to as *subjectifying* language and threaten to fix students' identity in mathematics (Aguirre, Mayfield-Ingram, & Martin, 2013).

Four licensed teachers serving as coaches were invited to observe and reflect on 13 written student work samples on a task. The elementary mathematics specialist coaches all independently and spontaneously used their investigation of the student work samples to plan for instructional groups. Using Heyd-Metzuyanim and Sfard's (2012) framework, their utterances were coded as *mathematizing* or *subjectifying.* Two broad categories emerged. In one category, the coaches generalized student performance and formed groups based on global performance. For the second category, the coaches' utterances reflected the mathematizing category, referring to the mathematics present in the student work. Coach utterances that focused attention on details of mathematical thinking fore-fronted student thinking to steer instructional group creation.

A coach can leverage language cues to draw attention to grouping strategies, either by attending to their own language or noting the language of their client teachers. The coach can then use this information to constructively guide the teacher client to create working groups, remedial groups, or other groups designed to maximize students' opportunities to learn. Future study might examine the noticings of novice and experienced teachers

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REIMAGINING MATHEMATICS CLASSROOM DYNAMICS: EXPLORING PLACE AND SPACE AS CONTEXTUAL VARIABLES THROUGH THE LENS OF MATHEMATICS TEACHER EDUCATION (MTES)

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This study explores how place and space influence mathematics classroom interactions, focusing on insights from Mathematics Teacher Educators (MTEs). Through geospatial lenses, we investigate how teachers' and students' accessible identities and positioning shape teaching practices, student mathematics learning, and specific aspects of knowledge generation and flow within the mathematics classroom. Employing semi-structured interviews with MTEs, we unveil the influence of geographic and non-geographic factors on mathematics classroom dynamics. Our findings offer practical insights for educators striving to create inclusive learning environments. Aligned with the conference theme of envisioning the future of mathematics education, this research provides a fresh perspective on the crucial role of place and space in shaping the dynamics of mathematics education.

Keywords: Place, Space, Student Mathematics Learning, Mathematics Teacher Educators

In this research, we employ dialogic lenses to illuminate the complex dynamics shaping interactions within mathematics classrooms. Our focus is unraveling the positionalities and identities available to teachers and students during mathematical lessons. Specifically, we delve into the influence of place (the origins of students and teachers) and space (the characteristics of the learning environment) on the identities accessible to individuals in their mathematics classes (Butler & Sinclair, 2020; Tate, 2008). Drawing on the works of Tate (2008), Masingila (1993), and Rubin (2007), we acknowledge the significant influence of students' out-of-school experiences and discourses on their prior knowledge, engagement, and experiences with the conventional mathematics classroom. Furthermore, Leyva (2021) underscores how institutional ideology, culture, and beliefs shape students' mathematical experiences within diverse classrooms, varying across institutions, school districts, and geographical contexts. (Anderson, 2014; Hogrebe & Tate, 2012).

Central to this argument is the assertion that a teacher's proficiency in teaching practices is linked to the identities they embody and the identities accessible within the classroom setting (Darragh, 2006; Omoze et al., 2024; Simon, 2012). Simultaneously, student learning, classroom interactions, and the generation and sharing of knowledge are profoundly influenced by student identities, positioning, and discourses permitted or restricted by the teacher during lesson exploration (Andersson & Wagner, 2019; Robin, 2007). However, these dynamics operate within the constraints and influences of place and space factors, delineating the geographic and non-geographic realms of the learning institution, as well as the diverse backgrounds of teachers and students (Leyva, 2021).

The primary aim of this study is to reposition Mathematics Teacher Educators (MTEs) within the context of mathematics classroom interactions. By eliciting reflections on their past experiences, MTEs offer valuable insights into how place and space as contextual variables can be effectively addressed to enrich students' mathematical conceptualization. This study focuses on the strategies and approaches that can mediate the influence of place and space, ultimately enhancing students' learning experiences. Through in-depth interviews and leveraging the expertise of MTEs, this research endeavors to illuminate the significance of space and place in the mathematics classroom, contributing to the existing body of knowledge in mathematics education and offering practical implications for educators striving to create inclusive and supportive learning environments. The central research question guiding this investigation is: How do Mathematics Teacher Educators (MTEs) perceive place and space as contextual variables in mathematics classroom interactions?

Conceptual Framework

In recent decades, there has been a notable shift in research toward employing geospatial lenses to analyze and critique mathematical learning outcomes, diversity, equity, and inclusion in mathematics education (Larnell & Bullock, 2018; Rubel & Nicol, 2020). Many researchers in the field of geospatial analysis assert that place and space significantly influence students' mathematics classroom experiences and, consequently, their learning outcomes, highlighting the growing recognition of their impact on educational dynamics (Hogrebe & Tate, 2013; Holland et al., 1998; Tate, 2008). Understanding the complex relationship between geospatial factors and students' identities is paramount for fostering a holistic educational approach (Andersson & Wagner, 2019; Bishop, 2012; NCTM, 2000).

Socio-spatial Framework

Larnell and Bullock's (2018) Socio-spatial Framework (SSF) provides a comprehensive perspective that outlines the geographical, social, and temporal dimensions of the mathematics learning environment. Although Larnell and Bullock originally intended to develop theoretical lenses for understanding urban mathematics education, we focus on the human and critical geography aspects of their framework to examine how mathematics teacher educators (MTEs) perceive place and space in their practices. Specifically, we are using the SSF to interpret learning environments, including the planning and organization of mathematics classrooms. For example, they explain that "by socio-spatial dialectic, we mean that the social significations and spatial considerations necessarily interact to determine meaning for urban contexts" (p. 48). Suggesting that the geographical characteristics of a small town or city, combined with a particular community's cultural beliefs and practices, create a meaningful place. This meaningful place then contributes to shaping and defining local discourses, language, out-of-school mathematical experiences, and embodied identities for students and educators.

The SSF highlights the complex relationship between geographical factors, social influences, and temporal dynamics within mathematics classroom settings. The spatial dimension underscores the importance of understanding the spatial context of mathematics learning, including geographical contexts, classroom arrangement, resource accessibility, and organizational layout (Hogrebe & Tate, 2013; Tate, 2008). The social dimension acknowledges the profound impact of sociocultural contexts on students' engagement with mathematical concepts, emphasizing culturally responsive teaching strategies and diverse perspectives within the learning environment (Ladson-Billings, 1995; Gay, 2018; Wachira & Mburu, 2019). Finally, the temporal dimension delves into the dynamic nature of learning experiences over time, highlighting the continuous process of knowledge construction and the enduring influence of past experiences on student learning outcomes (Larnell & Bullock, 2018).

Place as a Contextual Variable

Place, as a contextual variable in education, refers to the geographical location endowed with unique non-geographical characteristics that shape individuals' experiences, perspectives, and interactions within a specific setting (Butler & Sinclair, 2020). It encompasses the physical aspects of a particular location, including its geographical coordinates, geopolitical demarcations, and imaginary boundaries (Angweh, 1987; Massy, 2009; Tate, 2008). The geopolitical and imaginary boundaries can include school districts, urban, suburban, rural areas, states, countries, or even continents, each with distinct landmarks, physical features, and cultural attributes.

The meaning attached to a place goes beyond its geographical coordinates and is derived from factors such as the historical, cultural, or social attributes associated with that place (Angweh, 1987). The place is not solely defined by its physical characteristics, but also by the unique characteristics that shape individuals' perceptions and understanding of the world (Butler & Sinclair, 2020; Holland et al., 1998; Weiland & Poling, 2022). For example, a person living in Greenland, where snow is prevalent, may have developed specific ways of interacting with and understanding the world in terms of snow, while someone living in the Sahara Desert, experiencing consistently high temperatures, may perceive the world in terms of sand dunes. These different environmental contexts significantly influence individuals' experiences, perspectives, and knowledge construction.

Space as a Contextual Variable

Space extends beyond the physical boundaries of a classroom to encompass the broader educational landscape. The spatial distribution of schools, educational resources, and infrastructure within a community or district can significantly influence students' access to quality education (Hogrebe & Tate, 2012). Disparities in spatial distribution and resource allocation can create inequalities in educational opportunities, affecting students' learning outcomes and academic achievement (Lee & Ready, 2009). Therefore, understanding the spatial dimensions of educational environments is crucial for addressing educational inequalities and promoting equitable access to education.

Place and space contextual variables play distinct roles in shaping and influencing students' mathematics learning experiences (Butler & Sinclair, 2020). Understanding these differences is essential, as they highlight unique contextual variables within the educational discourse. Recognizing these distinctions helps educators navigate the specific variables of the mathematics classroom, thereby enhancing their teaching practices and fostering inclusive learning environments

Methods

Context and Setting

MTEs' extensive knowledge and insights, gained from teaching mathematics across diverse educational environments, provide valuable context for understanding the spatial and situational dimensions of mathematics education. Their varied experiences navigating different classroom settings and encountering diverse student populations enrich the exploration of how space and place shape instructional practices and student learning outcomes. Moreover, MTEs are uniquely positioned to interact with both preservice and in-service educators, contributing to the study's depth. The research captures the diverse range of MTEs' conceptualizations of space and place through semi-structured interviews, shedding light on the factors influencing instructional decisions and student engagement (Jacobs et al., 2010).

Data Collection and Analysis

The study involved semi-structured Zoom interviews with Mathematics Teacher Educators (MTEs) from a mid-western US university. Purposeful sampling ensured diverse faculty and graduate student perspectives. Interviews explored MTEs' views on place and space in mathematics classrooms and their impact on teaching. Audio recordings were transcribed, deidentified, and securely stored. Data analysis used Saldaña's (2014) dual-engagement strategy and Braun and Clark's (2008) thematic analysis, identifying themes through narrative inquiry. Descriptive coding highlighted key elements, addressing challenges in interpreting distinctions between "place" and "space" with insights from Poland and Pederson (1998).

Findings and Discussion

In response to our research question, MTEs perceive place and space as integral components shaping mathematics classroom interactions, and their beliefs underscore the transformative influence of these perceptions on instructional practices (Rubel & Nicol, 2020).

For instance, one MTE participant illuminates the distinction between "place" and "space," defining "place" as the physical environment encompassing geographical factors, student backgrounds, and the educational setting. In contrast, "space" is viewed as the classroom's physical layout, including seating arrangements, technology, and teaching tools. This definition aligns with Larnell and Bullock's (2018) socio-spatial framework's spatial dimension, focusing on physical location and background, a notion supported by Gravemeijar (1994) and Masingila (1993). It also resonates with Hogrebe and Tate's (2012) concept of geographical locations. However, their vision of space is confined to classroom design, including desk arrangements and teaching tools (Rubin, 2007).

> "Place to me is all about where you're located, where you're teaching, where your kids are coming from, and where they're at. Space, on the other hand, is the actual layout of the classroom, how you set up your desks or tables, how you use technology and various teaching tools."

Another perspective offered by participants defines space as the environment, which could be sociocultural, sociopolitical, or intellectual. According to one participant, "Space is more about kind of the social and intellectual environment that you're in." This broader conceptualization emphasizes the influence of the sociocultural and intellectual atmosphere on the learning environment, aligning with Larnell and Bullock's (2018) social signification dimension (Leyva, 2021; Rubin, 2007).

> "Space is more about kind of the social and intellectual environment that you're in."

The significance of cultivating an environment that allows students to personally connect with mathematics is underscored by MTEs, echoing Gravemeijer's (1994), Masingila's (1993), and Omoze et al. (2024) observations that student backgrounds impact how they interpret mathematical problems. They stress the importance of allocating time for individual or group work, enabling students to share insights with their peers (Yackel & Cobb, 1996).

Conclusion and Implication

MTEs play a pivotal role in advocating for pedagogical practices that foster inclusive and effective learning environments grounded in the understanding of place and space variables. Their insights highlight the need to address the challenges posed by fixed classroom layouts and promote flexibility to enhance student engagement and collaboration. As MTEs continue to navigate the complexities of mathematics classroom interactions, their contributions serve as guiding beacons for creating dynamic and equitable learning spaces. This study highlights the critical role of MTEs in shaping diverse, equitable, inclusive, and effective learning environments. The findings underscore the need to address challenges posed by fixed classroom layouts, promoting flexibility to enhance student engagement and collaboration.

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EXPLORING HOW EARLY-CAREER MATHEMATICS TEACHERS USE AN INTERACTIVE/DIALOGIC APPROACH TO COMMUNICATION

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Inviting students to participate in mathematical dialogues could help them develop a better conceptual understanding. For classroom dialogues to be effective, teachers' communicative approaches must be consistent with the activity's purpose and create support for students to participate in joint meaning making. This paper gives examples of how two early-career teachers use an interactive/dialogic communicative approach to invite and engage students in dialogues. We describe how teachers invite students to participate in meaning-making processes of mathematical content in three different situations over a series of mathematics lessons—to activate previous knowledge, to contrast students' solutions and to work on problem-solving tasks. In the ongoing analysis, we ask how these types of situations can be related to teaching purposes and the progression of students' understanding of the mathematical content.

Keywords: Communication, Classroom Discourse, Instructional Activities and Practices

Research show that participating in conversations about mathematics promotes students' understanding of mathematical concepts and the development of a formal mathematical language (e.g. Barwell, 2016; Schleppegrell, 2007). For such conversations to be successful, the teacher must balance between focusing on the process of students sharing their ideas and focusing on the learning of the specific mathematical content (Sherin, 2002). In practice, classroom talk varies in the level of student participation and the possibilities for them to share ideas or perspectives (Mercer & Howe, 2012; Mortimer & Scott, 2003; Warwick & Cook, 2019). To activate the learning potential in classroom talk, the teacher must not only invite students to participate in dialogues but also support their meaning-making processes toward a scientific view of the subject (Scott et al., 2011; Walshaw & Anthony, 2008). How the mathematical content is presented will also influence students' engagement and understanding (Dietiker et al., 2023).

Although the teacher's approach to communication can set frames for meaning-making processes in the classroom (Scott et al., 2006), students' reactions, questions and expectations all influence classroom interactions. The exchange of initiatives and responses in interactions between teachers and students adds a dynamic aspect to teaching, requiring teachers and students to be prepared to respond to one another in meaning-making processes. Then, teachers' planned actions in each moment may change, resulting in improvisation (Bishop, 2008). The teacher must make new decisions for how to respond or what of the students' input to include further in teaching, maintaining the conversation's mathematical productivity. However, inexperienced teachers may struggle with dynamic aspects such as instantaneously enacting strategies that encourage students to share their ideas or building on students' contributions in a whole class setting (Alexander, 2020; Lewis, 2014).

In this paper, we focus on how early-career teachers implement an interactive/dialogic communicative approach to engage students in classroom talk and negotiate the meaning of mathematics. Our guiding research questions are: How are students invited to participate in the negotiation of the meaning of the mathematical content within an interactive/dialogic communicative approach? How are the teachers building on the students' contributions?

Theoretical approach

We take a dialogical approach to this research on classroom interactions to highlight "the role of interaction and contexts, as well as language and the contribution of 'the other'" (Linell, 2009, p. 7). Dialogic theories describe meaning making as multi-voiced and interactive with others or oneself. The meaning of actions and knowledge is created in dialogue with others when trying to make sense of the world in a specific context. The others are not only the physical people we communicate with but also people present in traditions, expectations, previous knowledge, or ideas brought into a conversation by the participants. When using a dialogical research approach to explore interactions in mathematics classrooms, all the participants are positioned as contributors to the meaning of mathematics (Barwell, 2016). Such meaning-making processes could be about understanding representations of a mathematical concept or what counts as an acceptable mathematical explanation/justification in the classroom (Cobb, 1999).

Mortimer and Scott (2003) have identified four classes of communicative approaches used by science teachers to support meaning making in classroom talks. These approaches address aspects of classroom talk that are also relevant to mathematics teaching and learning.

- *Interactive/authoritative approach*: One perspective/idea is explored (authoritative). Students are invited to contribute (interactive), but the teacher evaluates their ideas to support the scientific view.
- *Interactive/dialogic approach*: All participants are invited to take part in the talk (interactive) and to contribute with perspectives/ideas on the subject (dialogic).
- *Non-interactive/authoritative approach:* The teacher alone stands for the content and perspective/ideas (authoritative), and students are not invited to contribute with responses or new initiatives (non-interactive).
- *Non-interactive/dialogic approach*: The teacher makes different perspectives/ideas accessible to the students (dialogic), but the teacher presents them rather than the students (non-interactive).

The approaches provide different opportunities to participate in classroom talk and contrast everyday views on a subject with the scientific view (Scott et al., 2006). So, depending on the content and the teaching purpose, the teacher can use shifts between these approaches to support students' understanding of mathematical content. Scott and Ametller (2007) associated a dialogic communicative approach with a teacher opening up classroom talks by, for example, inviting students to compare several ideas. In contrast, an authoritative communicative approach, such as the teacher reviewing presented ideas, was used to close down classroom talks. The study by Mortimer and Machado (2000) demonstrated a correlation between different approaches and talk patterns. For example, an interactive/authoritative approach was characterized by talk patterns of teacher initiation, student response and teacher evaluation (IRE-patterns). In an interactive/dialogic approach, the third turn (teacher evaluation) in such patterns instead consisted of teacher feedback or a new invitation for others to participate.

This paper focuses on teaching episodes in which the teacher shifts to and from an interactive/dialogic communicative approach to explore how teachers balance incorporating students' contributions in teaching and supporting students in making sense of mathematical content. We note that although teachers may choose or indicate a specific communicative approach to interactions, other participants may respond differently to what the teacher expects. Moving forward, we use dialogues to describe classroom talks where several participants contribute to the content.

Methodology

This classroom study was conducted in two Scandinavian upper secondary schools. Two early-career mathematics teachers were observed during a sequence of lessons covering a mathematical curricular unit (quadratic equations and vectors, respectively). The teachers themselves planned all the teaching. Both participating teachers had a master's degree in engineering and retrained as mathematics teachers for upper secondary school. At the time of classroom observations, Teacher 1 had taught for less than two years, and Teacher 2 had around five years of teaching experience. All observations were video recorded (approximately 14 hours per teacher) by the first author. Microphones worn by the teachers captured dialogues during group and whole-class activities. The students were 16–18 years old and **took a mathematics course for further science studies**. All participation in this research was voluntary, and the researchers had no prior connections to the participants. The participants were informed of the fundamental ethical principles of integrity, confidentiality, and anonymity (Clark et al., 2021; Forskningsetiske komiteene, 2018). Students who did not consent to video recording were seated outside the camera view, and their conversations and interactions with the teacher were not included in the data material.

In the analysis, we concentrated on episodes that involved dialogues with the whole class or with students during group or peer discussions. Teacher's communicative approaches were identified based on the level of interaction with students (interactive to non-interactive) and the extent to which students' ideas were allowed to influence the progression of a lesson (dialogic to authoritative) (Mortimer & Scott, 2003). Patterns in the teachers' and students' turn taking were explored through the type of questions, amount of talk time, and presence of evaluation and feedback in teacher responses. The approaches were first identified by the first author. In a validity process, the coding was then discussed collectively in a research group. Consensus on the features behind the coding was reached after reviewing video material and transcripts from various episodes from both teachers. At the PME-NA conference, we will share examples of transcripts to explain how the approaches were identified, teachers' prompts for inviting students to participate and their responses to students' contributions. Here, we focus on results from the analysis of when the teachers used an interactive/dialogic communicative approach.

Results

Three types of situations in which the teachers were more likely to apply a dialogic communicative approach were identified in both teachers' classrooms: 1) to activate or make visible students' previous knowledge, 2) to contrast students' solutions, and 3) to guide students' group discussions during problem-solving tasks. The situations could be linked to specific teaching purposes expressed by the teacher either in the introduction of an activity or as part of an evaluation at the end of a lesson. The first two types of situations occurred as part of teaching episodes, which started with a group activity and were followed by a whole class discussion. The third type occurred during students' group work with problem-solving tasks as the teacher moved between group dialogues.

The first type of situation was used by the teachers to make visible students' previous knowledge of mathematical content. The teachers used them as transitions between activities during a lesson or as a start of a lesson, often before introducing a new mathematical concept or using a familiar concept in a new setting. In these situations, the teacher invited students to engage in collaborative discussions in pairs or groups, prompting them to recall prior knowledge. The purpose was to activate the students' knowledge and to find ideas to build on or address during upcoming activities. For example, Teacher 1 asked the students to remember the characteristics of a quadratic function before introducing them to a zero-product procedure. Teacher 2 asked the students, before presenting the mathematical definition of a vector, to think about when it could be helpful to know both the size and direction of a physical quantity. In the whole-class discussion that followed the peer discussion, the teacher invited multiple students to share their ideas and asked them to explain or develop their answers. This first type of situation often ended with the teacher shifting to a more non-interactive approach, summarizing contributions, or providing examples of correct procedures. In the follow-up phase, the teacher stayed in an interactive/dialogic approach if the students agreed with the invitation to share their ideas. However, if there were no responses or just a single student answered, the teacher shifted to a non-interactive approach rather than attempting a new strategy to encourage participation.

The second type of situation occurred when students were given the opportunity to contrast different solutions to mathematical tasks. As in the first situation, these situations involved shifts between work in groups or pairs and whole-class discussions. The teaching purposes were to support students' conceptual understanding of rules/procedures and to provide opportunities to reflect on correct notations and mathematical language in their communication. For example, both teachers created discussion tasks around students' errors in their solutions on a previous test. Teacher 1 handed out premade solutions for the students to correct, and Teacher 2 presented solutions on the whiteboard and asked the students to evaluate them. In the follow-up phase, different groups were asked to share reflections, show their solutions, or elaborate on someone's explanation. The teachers asked questions to clarify mathematical language use or highlight what mathematical rules were being used (or misunderstood). These situations ended with the teacher repeating the correct solution or reminding the students of the correct mathematical rule in a non-interactive/authoritative communicative approach. If there were aspects of the mathematical content that the students did not bring up, the teacher presented them at the end.

The third type of situation where the teacher used an interactive-dialogic approach occurred during more extended group activities with problem-solving tasks. Positive feedback such as "You are well on the way! Nice opening! Clearly described!" was a typical start to these dialogues. To include students in the dialogue, the teacher could ask questions such as "How have you started? What strategies have you tried? Do you see any patterns?" These questions aimed to get ahold of students' starting points and guide (or challenge) them toward a solution or understanding without showing them how to do it.

Discussion

In all three types of situations, students were invited to share their ideas without the teacher knowing precisely what contribution the students would have. However, how students' ideas were incorporated further in teaching varied in our material. In exploring how students are invited to negotiate mathematical content within an interactive/dialogic communicative approach, we will continue our analysis by comparing the results with previous research (e.g., purposes identified by Mortimer and Scott, 2003). Additionally, we will compare the three types of situations to search for additional characteristics. In this work, we ask a couple of questions to distinguish or find similarities between the situations: What is the mathematical focus of the meaning-making processes? What positions are made available for the students when the teacher invites them to think and share their ideas? How are the students responding to the invitations to participate in meaning making? In what ways can an invitation to participate be linked to where in the sequence of lessons/activities the situation appears? We intend to share some of our answers and insights into these questions at the PME-NA conference to contribute to research on how early-career teachers handle dynamic aspects of teaching during classroom dialogues.

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OBSERVATIONS OF MATHEMATICS CLASSROOM DISCUSSIONS: EXAMINING RACIALIZED PARTICIPATION

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Classroom discussions are important for students' mathematical learning, but often, who and how students participate illuminates persistent issues of gendered and racialized bias. Teachers can disrupt potential issues of inequity by providing varied opportunities for students to participate in discussions. In this paper, we consider an observation tool to make sense of the nuanced features of interactions between a teacher and students during mathematics classroom discussions. When examining the lesson with attention to student race, new understandings are illuminated in the interactions between teacher and students.

Keywords: Classroom Discourse; Equity, Inclusion, and Diversity; Instructional Activities and Practices

Classroom discussion has long been shown to support students' broad learning and engagement across content areas (Khong et al., 2019; Mercer et al., 1999; Reznitskaya et al., 2009) and specifically in mathematics (Kazemi & Stipek, 2001; O'Connor & Snow, 2017). Teachers, then, are asked to create meaningful opportunities for discussion in their instruction (O'Connor & Michaels, 2019; van der Veen et al., 2017). However, research has found that some students from historically marginalized groups can be further marginalized within classroom discussions (Chen & Horn, 2022), which in turn influences these students' opportunities to participate (e.g., Grøver Aukrust, 2008; Reinholz & Shah, 2018).

The benefits of classroom discussion for students' mathematical learning combined with the challenges of making such discussions inclusive spaces for diverse groups of students require more attention to how mathematics classroom discussions unfold. This includes attending to what teachers and students do. To address this concern, we build on our prior work in which we analyzed how the teacher and students participated as well as differences between how boys and girls participated for a focal subset of students. In this analysis, we ask: How does fine-grained coding of teacher and student participation, combined with student demographic data, reveal differences in patterns of participation by race? Our intent in examining this question is not to evaluate a teacher on her practice in one lesson but to focus on what our analysis tool may reveal about patterns of participation with respect to race that may then be used to support future teacher professional development.

Theoretical Framing

Our conceptualization of classroom discussion is rooted in sociocultural views of learning (Mercer, 2000; Vygotsky, 1978) and Bakhtin's (1981) theories of the dialogic. As such, we see classroom discussions as spaces where interactive (i.e., dialogic) student talk supports understanding. This talk supports the co-construction of knowledge as it influences learning in classrooms (John-Steiner & Mahn, 1996). In other words, how people think and talk with one another supports shared learning.

Such conceptions of learning include the interaction of talk between people and the context of their lived experiences (Bakhtin, 1981; Wells & Arauz, 2006). Given the interplay between talk and context, we analyze classroom discussion, attending to the teacher's and students' contributions, with particular attention to patterns of student participation for different demographic subgroups. How these three elements interact with one another creates a clearer picture of the events of a classroom discussion, where teachers solicit student ideas, students share their own thinking or add to the thinking of others, and teachers take up or extend those students' ideas in some way (Bishop, 2021; Michaels & O'Connor, 2015; Webb et al., 2014).

Student participation plays a central role within the combined frameworks of dialogic talk and sociocultural perspectives of learning. Research has examined the influence of social demands in classrooms on how students may be enabled or constrained to participate (Kovalainen & Kumpulainen, 2007; Ng et al., 2021). Dialogic and sociocultural perspectives provide lenses for making sense of what happens in classroom discussions. Using this perspective, we attend to teachers' actions and the participation of the students in relation to the teacher's actions within a dialogic conversation. We also examine the role that race may play in student participation during classroom discussions as an aspect of the sociocultural nature of their learning environment, particularly given the research on how whiteness has influenced who has power and is valued in education spaces (Battey, 2013; Haviland, 2008).

Methods

This analysis takes place in the context of a partnership with elementary schools aimed at supporting teachers to engage in dialogic discussion across content areas. To better understand teachers' ongoing learning and facilitation choices, we observed whole-class discussions in mathematics, ELA, and science for each teacher in each year of the study. This analysis focuses on video-recorded observations of mathematics classroom discussions for nine teachers (grades K - 5) at Rivers Elementary, focusing on one teacher (Allison). First, we explain the analytical approach, the results from our prior analysis, and then explain the subsequent layer of analysis.

Analytical Approach

To best make sense of the dialogic aspects of classroom discussions and who gets to participate in this discussion, we combined aspects of two previously used observation tools: (1) the Equity Quantified in Participation (EQUIP) framework (Reinholz & Shah, 2018) and, (2) the Low Inference Discourse Observation (LIDO) instrument (LaRusso et al., 2023; O'Connor et al., 2016). The EQUIP framework was designed to focus on student participation, while the LIDO instrument was designed to assess the dialogic nature of teachers' and students' talk. We coded classroom observation videos by individual student contributions. For each contribution, we documented the teachers' original question (teacher solicitation), how a student was asked to share (solicitation method), what the student said (student contribution), and how the teacher took up what the student said (uptake type). Table 1 shows these four dimensions coded and the different coding options.

Table 1: Coding Schemes and Descriptions

Teacher Solicitation	Solicitation Method	Student Contribution	Teacher Uptake Type
(S2S) Encouraging students to respond to students	(Called On - by teacher) Teacher selects a student to share	(S2S) Student to student talk	(Follow-up) Teacher asks a question with a connection to the student contribution
(Explain) Teacher asks students to explain, clarify, give evidence	(Called On - by student) Student selects another student to share	(S ref S) Student referring to another student's ideas	(Elaborated) Teachers adds detail or clarity to a student idea
(Continue) Encourages students to continue	(Student Called Out) Student shares without being called on	(Because) Student provides reasoning	(Revoiced) Teacher repeats or paraphrases what a student said
(Open) Open question	(Choral Response) Multiple students respond together	(Q4T) Student asks Teacher a question	(Acknowledged) Teacher acknowledges student idea
(Semi-open) Between closed and open	(Turn and Talk) Students are directed to discuss the question with a partner or small group	(Extended) Student response longer than a simple clause	(Redirected) Teacher redirects student idea
(Closed) One expected answer		(Simple) Minimal student turn, brief	(Evaluated) Teacher makes a judgment about student contribution
(None) No solicitation		(Unrelated) Not connected to discussion	(None) No uptake by the teacher

Videos were coded with four raters, who were trained on the coding scheme and engaged in practice and consensus coding across multiple rounds. Inter-rater reliability was assessed by comparing codes from all raters after training and by comparing codes from pairs of raters in an ongoing manner. Our Krippendorf's alpha scores for training ranged from .68 to .96 and ranged from .57 to .81 for independent coding. Since the independent reliability tests were conducted in pairs, there were fewer comparisons than the training reliability across four coders, which likely contributed to the reduced alpha statistics.

Our prior analysis (Wilhelm et al., 2024) examined the nature of the mathematics discussion and how student participation differed by gender for a focal subset of students (n=8) in one teacher, Allison's, classroom. That subset consisted of all students in Allison's class for whom we had parental consent to participate in the study. This sample of students consisted of four girls[1] and four boys (Alma, Samika, Lucy, Casey, Jaden, Kai, Derrick, and Robert, all pseudonyms).

[1]While we recognize that gender is more expansive than the categories of girl and boy for students, we report on only these categories as the listed options in the school's demographic data.

The girls within the sample identified racially as white[2] (n=2), Hispanic (n=1), and Asian (n=1). The boys within the sample identified racially as white (n=2), Black (n=1), and Black and white (n=1). Given our attention to participation by race and the role of whiteness (Haviland, 2008; Taylor-Heine et al., 2022), we focus on the six students from this set who identified as white or Black. Students who identify as only white are categorized as white, and students who identify all or in part as Black are categorized as Black for this analysis. We share the results and statistical comparisons of Allison's lesson. Given that the focal sample is small, with just one lesson's worth of information about their participation, we also aggregated the participation patterns of Black and white students in the observations of the teachers across Rivers Elementary in order to see if some of the emergent patterns from Allison's discussion were true across a larger population of classrooms and students. We provide descriptive information about the four categories described in Table 1: Teacher Solicitation, Solicitation Method, Student Contribution, and Uptake Type. For the aggregated sample, we use Chi-squared tests to compare the distributions and report statistically significant differences.

Prior Analyses

Before answering our research question about how patterns of participation varied by student race, we describe Allison's lesson and summarize results from our prior analysis of Allison's mathematics lesson (see Wilhelm et al., 2024 for more detail). The classroom discussion we analyzed focused on supporting students in representing three-digit numbers (e.g., expanded form, word form, standard form). During the discussion, Allison specifically introduced students to the expanded form with input from students into the definitions of key terms like sum, value, and digit. Looking at the data in the aggregate, we saw evidence of a lesson where students were encouraged to share their ideas, often in response to questions with one expected answer, one after the other. Frequently, this looked like Allison listening to what a student said and then calling on a new student to support the flow of discussion without asking a new question. While it was most common for Allison to invite students to participate by calling on a new student to share their idea (49.3%), we noticed her intentional use of choral response (16.0%) and turn and talk (9.3%) that supported more students talking and getting students to talk directly with one another as these moves allow everyone to share an idea. Likely tied to the prevalence of Allison posing questions with one expected answer, the most common type of student response in the discussion was simple, one-word or brief answers (64.8%). Finally, the most common approach Allison had in responding to a student's idea was to allow or encourage other students to engage (no uptake, 37.3%), offering no further questions or rephrasing of her own.

Results

When looking at the subset of students in Allison's classroom by their racial demographics (see Table 2 below), some other differences emerge in how their ideas were solicited, shared, and taken up, particularly between the Black and white students. The patterns of questions (teacher solicitation) students received were fairly similar. However, white students tended to receive more questions asking them to explain their thinking (5% Black, 30% white). For example, after stating, "sum is the value of an addition problem," Allison asks one of the white students to

[2]We make an intentional choice to leave white uncapitalized and to capitalize instances of Black to describe race where other scholars have documented white as a social construct and Black a "self-determined name" (Dumas, 2016, p. 12).

explain what value means in that statement. Related to what students said in the discussion (student contribution), 80% of the responses from Black students and 60% of responses from white students were classified as simple. Additionally, most responses from white students included longer answers (extended; 30%). Finally, concerning how Allison responded to student contributions (teacher uptake type), Black students were most likely to be asked follow-up questions from the teacher following their responses (40% vs. 30%), and white students were more likely to have their responses repeated (revoiced) by the teacher (30% vs. 10% revoiced).

Table 2: Classroom Discussion Participation of White Students (N=4) vs. Black Students (N=2)

Teacher Solicitation	White	Black	Solicitation Method	White	Black
S2S	0.0%	0.0%	*Called On (by teacher)*$^{+}$	*100.0%*	*85.0%*
Explain^{+}	*30.0%*	*5.0%*	Called On (by student)	0.0%	0.0%
Continue^{+}	*10.0%*	*20.0%*	S Called Out	0.0%	15.0%
Open	10.0%	15.0%	Choral Response	0.0%	0.0%
Semi-Open	10.0%	15.0%	Turn and Talk	0.0%	0.0%
Closed	20.0%	25.0%			
None	20.0%	20.0%			
Student Contribution	**White**	**Black**	**Teacher Uptake Type**	**White**	**Black**
S2S	0.0%	0.0%	*Follow-up*$^{+}$	*30.0%*	*40.0%*
S ref S	0.0%	5.0%	*Revoiced*$^{+}$	*30.0%*	*10.0%*
Because	10.0%	15.0%	Recorded	0.0%	0.0%
Q4T	0.0%	0.0%	Elaborated	20.0%	15.0%
Extended^{+}	*30.0%*	*0.0%*	Acknowledged	10.0%	15.0%
Simple^{+}	*60.0%*	*80.0%*	Evaluated	0.0%	0.0%
Unrelated	0.0%	0.0%	Redirected	10.0%	0.0%
			None^{+}	*10.0%*	*20.0%*

Note. This table compares discourse elements for a mathematics lesson in Allison's 2nd-grade classroom, comparing the participation of 6 focal students by race. The number of students here is too small to determine statistically significant differences. Italicized values point out a difference but not at a significant level ($^{+}$ qualitative difference visible).

The differences in teacher questions and student responses between Black and white students are illuminated in an excerpt from the lesson below. Allison asks the class if the equation 156 = 50 + 100 + 6 is true or false. In what follows, Jaden (the only Black student in the exchange) is prompted with multiple closed-ended questions with simple responses compared to the other peers (Alma and Casey) who share here:

Allison — Alma, can you tell us why you think it's false? Then we'll come to you [Jaden] if it's different. [To Alma] what made you say it was false?

Alma — There's 50, then 100, then 6. It's the order

Allison — Because there's 50, then 100, and then 6? Ok. Their order? 5 1 6? Ok, she's saying it's because of the order…Jaden, tell me what you think.

Jaden	What I think is, so when there's 50 and 100 that gets 100 and then we have the 6 and I think that's 106.
Allison	So you think it's 106. Someone who thinks true, Casey, you said true. Now I want Jaden to listen because if you want to revise your thinking you can. Casey, why did you think it was true?
Casey	I said it was true because I saw that there was the same numbers up on the board but they were just flipped flopped. So I still saw a 100, I still saw 50, and I still saw 6.
Allison	So you saw all the components but you knew they were flipped flopped, or mixed up.

[Allison walks through recording the place value before turning back to Jaden]

Allison	Do you want to revise your thinking now?
Jaden	Yeah
Allison	Ok, on your board do you see a 50?
Jaden	Yeah
Allison	On your board do you see a 100?
Jaden	Yeah
Allison	On your board do you see a 6?
Jaden	Yeah
Allison	Ok, what is that number? What does 100 and 50 and 6 equal?
Jaden	One hundred fifty six.
Allison	One hundred fifty six.

In this exchange, we see some misconceptions from the students Alma and Jaden. Notice how Alma and Casey are prompted with questions to explain their thinking in this exchange (e.g., "why did you think it was true?"). However, Jaden (who identifies as Black) is posed a series of closed, test-like questions to attend to the revisions in his thinking.

This pattern in uptake type for white students was similar to the pattern in uptake for girls we saw in our previous analysis, with girls receiving more revoicing of their ideas (see Wilhelm et al., 2024). Additionally, the pattern in uptake type for Black students may be explained by one of the students in the subset, a Black boy, who had the majority of responses across the group that included a follow-up from the teacher. The intersectional identities of these students are important to this analysis of their discussion participation. For example, both Black students in our focal sample identify as boys. The teacher questioning frequencies by race showed more use of closed-ended questions by the teacher and simple answers from the students. However, these frequencies are more aligned with how girls participated in the discussion (i.e., asked more closed-ended questions and answered with more simple responses).

Table 3: Aggregated Classroom Discussion Participation of White Students (N=35) vs. Black Students (N=14)

Teacher Solicitation	White	Black	Solicitation Method	White	Black
S2S	2.80%	3.30%	Called On (by teacher)	93.80%	93.30%
Explain	17.2%	10.00%	Called On (by student)	0.70%	0.00%
Continue	12.40%	16.70%	S Called Out	5.50%	6.70%

Open	12.40%	13.30%	Choral Response	0.00%	0.00%
Semi-Open	15.20%	15.00%	Turn and Talk	0.00%	0.00%
Closed	25.50%	25.00%			
None	14.50%	16.70%			
Student Contribution	White	Black	Teacher Uptake Type	White	Black
S2S	0.00%	0.00%	Follow-up	51.00%	53.30%
S ref S	1.60%	1.70%	Revoiced	20.70%	18.30%
Because	10.50%	12.10%	Recorded	1.40%	1.70%
Q4T	0.80%	1.70%	Elaborated	14.50%	11.70%
Extended	**25.00%**	**1.70%**	Acknowledged	5.50%	6.70%
Simple	**60.50%**	**82.80%**	Evaluated	0.70%	1.70%
Unrelated	1.60%	0.00%	Redirected	2.80%	0.00%
			None	3.50%	6.70%

Note. This table compares discourse elements for all observed mathematics lessons of teachers in Y1 of the study with consented students who identified as Black or white. Values that are bolded point out a statistically significant difference.

We were curious if we would see similar participation patterns when we looked across more lessons. Table 3 shows the aggregated analysis of participation patterns of consented Black and white students at Rivers Elementary across grades K - 5 classrooms. Some of the differences in Allison's classroom discussion are not apparent in the aggregated set, such as how students were called on (solicitation method) and how teachers took up students' responses (uptake type). Some of the same differences we saw in Allison's lesson were present, but to a lesser degree, such as the types of questions the teachers ask (more explain your thinking questions for white students, more continue your thinking questions for Black students). What is interesting is the continued differences in student contributions, where white students were more likely to provide an extended answer than Black students (25% white, 1.7% Black), and Black students were more likely to provide a brief answer ("simple," 60.5% white, 82.8% Black). These differences in how students responded were statistically significant.

This fine-grained analysis of the mathematics lesson suggests differences in how students participated based on race, which exemplifies the potential of analyzing data at the individual level. With an even larger sample of student contributions, it would be worthwhile to analyze this data by intersectional subgroups as done in other literature (e.g., Reinholz & Wilhelm, 2022). The use of the observation tool helps make visible the moves of the teacher and students in a classroom discussion, with the opportunity to pay attention to participation related to student demographics.

Conclusions

We began this deeper analysis of Allison's mathematics classroom discussion to understand the details of a teacher's facilitation and student participation and what would be visible when specifically paying attention to student race. Our sociocultural and dialogic framework thus centers how these features of facilitation and participation interact with one another (Bakhtin, 1981; Wells & Arauz, 2006). Attending to issues of equity within dialogic talk is important because of the numerous ways that mathematics classroom discussion can constrain student participation (Black & Radovic, 2018; Ng et al., 2021). The intent of this analysis is not to draw generalizable conclusions about Allison's specific practice but to provide an image of what such an observation tool allows mathematics educators to see and track in future classroom discussions to understand if such findings become larger patterns pertaining to student participation, such as the emerging results of the aggregate data for Rivers Elementary. Analyzing the nuances of what happens in classroom discussions thus illuminates who is potentially marginalized in their participation (Chen & Horn, 2022).

In our focal set of students from Allison's classroom, we found that Black and white students participated differently. For example, we found that despite the fact that both Black students were boys, who generally tended to be asked more semi-open questions, the Black students were more likely to be asked closed questions, while white students were more frequently asked to explain their thinking. With just six focal students, it was not possible to meaningfully disaggregate the data by intersectional subgroups (e.g., Black boys), but that intersectionality is critical and likely contributes to explanations of participation patterns. Other studies have shown significant differences between intersectional subgroups, with a large body of qualitative evidence demonstrating differences in classroom experiences (e.g., Gholson & Martin, 2014; Joseph et al., 2019; Morris, 2007).

We want to reiterate that Allison's classroom discussion results are based on a single observation in one content area with a small sample of focal students. Although some emergent patterns were not present or not as prevalent in the larger set of Rivers' classroom discussion data, we do see patterns in how white vs. Black students are entitled to contribute their ideas to a discussion. We specifically consider how students may have been entitled to participate, recognizing the interrelated role of interactions between teachers and students that can inform how students are "entitled or constrained" to participate (Black & Radovic, 2018, p. 274). In our data, we see that white students were more likely to provide extended responses, whereas Black students were more likely to provide simple, brief responses. These differences suggest continued examination of why these students participated in this way. In drawing attention to these patterns of participation, we aim to demonstrate that by disaggregating the data by particular social demographics, it is possible to ask questions about whether participation is equitable (Lavigne & Good, 2021; Reinholz & Shah, 2018).

Details about the nuances of student participation in a mathematics classroom discussion were illuminated through the use of our observation tool. Originally, LIDO was designed to assess students' and teachers' dialogic talk, while EQUIP was designed to focus on the equity of student participation. By combining aspects of the LIDO instrument (LaRusso et al., 2023; O'Connor et al., 2016) and the EQUIP framework (Reinholz & Shah, 2018), we were able to attend more carefully to student participation through the interplay of a teacher's solicitation and student contribution of ideas during a classroom discussion.

Investigating mathematics classroom discussions through a combined tool to track dialogic talk and student participation supports the future of mathematics education in such uncertain times. The tool provides two ways to attend to the challenges of the educational and political landscape. Related to educational needs, teachers are being asked to demonstrate disciplinary expertise across content areas, including cross-content classroom discussions (Fitzgerald & Palincsar, 2019). Such a tool, then, could support analysis of classroom discussion across disciplines to leverage features of similarity for facilitating quality instructional practices. For example, from the current study, this observation data was collected at the beginning of our work together, where teachers participated in a 3-year long professional learning program to support and reflect on their classroom discussion practices. Related to the current political landscape, we see a voice of doubt that argues such issues of race and gender are even present, and thus, ways that such activity to support equitable classroom practices are limited (Lopez et al., 2021; Wuest, 2018). A tool that looks at teacher and student engagement in discussion examined by factors such as race and gender can provide empirical evidence for particular inequities that cannot be as easily dismissed. Mathematics education could thus benefit from purposeful attention to instructional practices to identify potential patterns of inequity. Analyzing how these practices unfold in classroom discussions provides a first step to articulating their presence empirically.

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GETTING COMFORTABLE: DELEGATING AUTHORITY WHILE POSITIONING STUDENTS WITH AGENCY

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This multiple case study explores how two elementary mathematics teachers made themselves and their students comfortable with the teacher delegating authority to students while also positioning students with agency. I conducted three Video Stimulated Recall (VSR) interviews and pre- and post-semi-structured interviews with each teacher to understand their practices and thoughts about moments from prior lessons related to authority and agency. The results indicate that, while one teacher expressed being quite comfortable delegating authority to students and positioning them with agency, the other expressed reservations. Results also indicate that teachers make students and themselves comfortable delegating authority and positioning students with agency by creating time to understand students' solutions, getting to know students' assets outside of the classroom, and employing Social Emotional Learning (SEL) strategies.

Keywords: elementary school education, teaching practice and classroom activity

Implementing an ambitious approach to teaching elementary mathematics is complex work that requires teachers to delegate authority to students while positioning them with agency (Aguirre et al., 2013; Amit & Fried, 2005; Dunleavy, 2015). Existing literature demonstrates the benefits of providing opportunities for students to engage in such a vision of ambitious mathematics instruction (Ball, 1993; Carpenter et al., 1989; Jackson et al., 2017). Teachers in the United States may experience discomfort and tension, however, when they shift from traditional mathematics instruction – in which students typically lack authority and agency (Stigler & Hiebert, 1997) – to an ambitious instructional approach, in which students have more say about the validity of mathematical ideas (i.e., authority) and freedom to solve problems as they choose (i.e., agency) (Ball, 1993; Langur-Osuna et al., 2020). To understand how teachers combat this tension, this multiple case study (Creswell & Poth, 2016) explored how teachers delegated authority while positioning students with agency (Hicks et al., 2023; Turner, 2013). I answered the following research questions: (1) What practices and activities do teachers engage in to help make themselves feel comfortable with teachers delegating authority to students while also positioning them with agency in elementary mathematics classrooms? (2) What practices and activities do teachers engage in to help make students feel comfortable with teachers delegating authority to students while also positioning them with agency in elementary mathematics classrooms?

Theoretical Framework

Authority

Authority in mathematics resides with whoever is regarded as mathematically legitimate (Amit & Fried, 2005; Dunleavy, 2015). An ambitious vision of mathematics instruction (Jackson et al., 2017) requires teachers to delegate authority to students so that they play a role in deciding which contributions are mathematically sound and valid, and why (Gresalfi & Cobb, 2006; Hicks et al., 2023), thereby contributing to students feeling mathematically legitimate.

Agency

Agency is having the freedom to solve problems as one sees fit (Dyer et al., 2023). In the context of the mathematics classroom, agency necessitates day-to-day attendance (Aguirre et al., 2013; Turner, 2003). Therefore, in an elementary mathematics classroom, teachers must routinely position students with agency for them to develop a sense of agency (Ball, 1993; Boaler, 2002; Lampert & Ghousseini, 2012).

Delegating Authority and Positioning Students with Agency is Difficult

Teachers may be uncomfortable delegating authority and positioning students with agency. (Dunleavy, 2015; Turner, 2003). Societal metanarratives and prior knowledge of what elementary mathematics classrooms should look like might impact teachers' beliefs of who holds mathematical authority (Aguirre et al., 2013; Bartell, 2016). Moreover, when teachers delegate authority to students while positioning them with agency, it can create an unpredictable learning environment where a lesson can take unanticipated turns (Kazemi et al., 2009; Stein et al., 2008). Students might be uncomfortable as well. All students come to school with a set of beliefs, cultural and community norms, and an idea of "what to do" in order to learn (Delpit, 1995; Lampert et al., 1998). These factors might be at odds with ambitious mathematics instruction (Lubienski, 2000); therefore, students might not be used to being delegated authority or positioned with agency (Aguirre et al., 2013). However, the teacher is responsible for helping students transition into this environment and has the power to develop students' comfort with being delegated authority and positioned with agency in the mathematics classroom (Jackson et al., 2017; Langer-Osuna et al., 2021).

Methods

This multiple case study (Creswell & Poth, 2016), inspired by Gutiérrez's statement, "I have always believed we learn best from understanding 'success' cases" (Gutiérrez, 2011, p. 21), explored how teachers made themselves and students comfortable with delegating students authority while positioning them with agency in elementary mathematics classrooms.

Positionality

My previous professional roles, such as an elementary classroom teacher, mathematics instructional coach in Title I schools, and teacher educator supporting pre-service teachers, contribute to my background knowledge, biases, and interpretations of data.

Participants and Context

This study, which took place at the beginning of the school year, included two participants, each teaching at elementary schools in the same district in the Pacific Northwest. Though unknown at the time of data collection, the district participates in a social-emotional program, *Everyday Speech*. Participant A, who was in their 16th year of teaching at the time of the study, teaches third grade at a school that has Title I status, participates in the Advancement Via Individual Determination (AVID) program (AVID, 2012), and serves a diverse population. Participant B, who was in their eighth year of teaching, teaches first grade at a school in the same city that does not qualify for Title I status. The participants in the study met the following criteria: (a) the participant was seeking to delegate authority to students and position them with agency during mathematics instruction, (b) the participant taught first, second, or third grade, and (c) the participant taught in a public elementary school setting.

Data Collection

I used Video-Stimulated Recall (VSR) to allow teachers to articulate their thoughts and feelings related to interactions in their teaching (Martinelle, 2020; Muir, 2010; Nguyen et al., 2013). The data collection protocol was as follows: (a) lessons were video-recorded using a SONY™ EV-10 camera and audio-recorded with a Lavalier microphone or ZOOM H1 exterior microphone, (b) after the lesson, I selected moments from the lessons that exhibited interactions related to mathematical authority or student agency, (c) occurring the same day, and as soon as possible, the teacher and I met for an interview, in which the teacher watched and reflected on selected moments (Calderhead, 1981; Nguyen et al., 2013).

Additionally, pre- and post-semi-structured interviews were conducted with each participant via Zoom (Creswell & Poth, 2016). As Calderhead (1981) states, VSR alone cannot completely capture teachers' thoughts. Therefore, interviews were conducted before and after the VSR process to add context, confirm or disconfirm findings from the VSR data, and access additional insight regarding teachers' practice with respect to delegating authority and positioning students with agency (Calderhead, 1981; Creswell & Poth, 2016; Lincoln & Gruba, 1985).

Data Analysis

Data was analyzed using the process of thematic analysis (Braun & Clarke, 2021). A codebook was developed that consisted of a priori codes connected to relevant literature (Carpenter et al., 1989; CASEL, 2022; Moll, 2019) while allowing for unanticipated codes to emerge (Saldaña, 2021). Multiple iterations of the codebook were developed throughout the data familiarization process (Braun & Clarke, 2006). An external audit process took place with another researcher familiar with the project to stabilize code descriptions and examples. Additionally, the audit process enhanced the accuracy of my interpretations (e.g., distinctions between authority and agency were discussed during the coding process) and conclusions for this research project. The themes were derived from the coded data (Table 1) by analyzing frequency, co-occurrences, and triangulation (Creswell & Poth, 2016; Lincoln & Guba, 1985).

Results

Through thematic analysis of the VSR interviews, I identified the following themes: (a) Tension, (b) Time to think for teachers and students, (c) Getting to know students' knowledge and culture, (d) Tending to students' emotions, and (e) Being or becoming comfortable. Below, I share examples of reflections from Participant B demonstrating the themes of *Time to think* and *Tending to students' emotions*.

Table 1: Topics Discussed by Teachers in VSR Reflections

Teacher Discusses		Teacher A	Teacher B	Total
Authority	Teacher Delegating Authority	13	7	20
	Student Initiating Authority	4	5	9
	Teacher Withholding Authority	0	0	0
Agency	Teacher Positioning Students with Agency	9	8	17
	Student Assuming Agency	5	3	8
	Teacher Withholding Agency	0	1	1
Comfort	Teacher Comfort	1	2	3
	Student Comfort	5	2	7
	Teacher's Lack of Comfort (Tension)	0	5	5
	Student's Lack of Comfort (Tension)	0	0	0

Time to Think	Time to Think to Make Themselves Comfortable	1	4	5
	Time for Student Thinking to Make Students Comfortable	3	2	5
Knowledge, Culture, and Language	Student Knowledge, Culture, and Language to Make Themselves Comfortable	0	0	0
	Student Knowledge, Culture, and Language to Make Students Comfortable	4	0	4
Emotions	Social Emotional Learning to Support Delegating Authority and Agency	7	8	15

Time to Think

Both teachers spoke of the importance of *time to think* in order to understand students' thinking, thus, fostering students' mathematical legitimacy (authority) and freedom to solve problems as they see fit (agency). As an example, Participant B reflected on a moment when a student posed a subtraction representation (whole-part=part; 9-1= 8) for a part-part-whole activity (the whole was 9), stating, "I felt frantic in the moment." When looking for the intended representation, the teacher asked the student "Oh, but what is our whole number?", then went on to discuss the following:

> …because I saw 8 at the end of her equation. and then she said, "But it is subtraction." And I said, "Oh, we just haven't talked about subtraction," I called out that she did not have the 9 that I was looking for. But then she, I don't want to say called me out, but said, "But it is subtraction, I did 9-1."

Participant B went on to share that they felt relieved that they had time to make sense of the student's representation while conferring one-on-one before the whole-group discussion, stating, "If I was teaching whole group, they [students] are all staring at you. It gave me some time, and I feel like that is what they need, too."

Tending to Students' Emotions

Participant B discussed *tending to students' emotions* to create a comfortable environment for students to exercise their mathematical authority. In the following example, Participant B discussed how they sought to delegate authority by promoting students' ability to give feedback on their peers' mathematical solutions while also receiving feedback from their peers ("glows and grows"):

> It's a process, I've only started the 'glows and grows' last year. It's nerve-wracking…we are also trying to teach them social-emotional [skills] and how to make their peers- I don't want to say better versions of themselves- but to help them, without hurting their feelings…I don't want to say back in the day, but when you call someone out, it's embarrassing or makes you feel super sad, or I am just not good at this. Before we even get here, we go over 'growth mindset' and how we can't do things yet, the power of yet.

Participant B continued to stress the need to *tend to students' emotions* to create a classroom environment in which students are not embarrassed or hurt by feedback (peers exercising mathematical authority), saying, "So, there is a whole social-emotional piece before we dive into academics, specifically math…It's social-emotional…It's a work in progress."

Discussion

This study contributes to the field by illuminating the ways teachers make themselves and students comfortable with the teacher delegating students authority while positioning them with agency. Results indicated that teachers make students and themselves comfortable by creating time to understand students' solutions (Carpenter et al., 1989), getting to know students' assets outside of the classroom (Moll, 2019), and employing Social-Emotional Learning (SEL) strategies (CASEL, 2022).

Time was limited for this study. The next steps might be to use similar methods to explore the longitudinal development of teachers' practice in this realm.

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JUXTAPOSITIONING CASES OF DELEGATED MATHEMATICAL AUTHORITY TO ELEMENTARY STUDENTS IN THE CONTEXT OF COORDINATE GRAPHING

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Keywords: Algebra and Algebraic Thinking; Classroom Discourse: Elementary School Education; Equity, Inclusion and Diversity

This poster focuses on how two 3rd grade teachers taught conventions associated with coordinate graphing with a goal of developing a method for capturing the degree a teacher delegates mathematical authority. This study came from a larger study of an analysis of a common set of early algebra lessons enacted in 78 classrooms guided by the question, "How are mathematical conventions introduced and treated by teachers in elementary classrooms?" (Ristroph, 2024). Those findings revealed those teachers rarely explicitly addressed conventions as such, most frequently introduced mathematical conventions in a manner of direct instruction, and when spoken of they were often attributed to external agents (e.g., "they", "people", "mathematicians"). The few episodes in which teachers delegated authority to their students to explore alternatives to conventions gave motivation for this work.

The ways in which elementary teachers of mathematics confer mathematical authority are explored by asking the following:

How is the power to form and justify mathematical ideas either withheld by the teacher or delegated to students? How does the teacher's talk moves, uptake of student's ideas, and participation structures constitute a delegation or withholding of mathematical authority?

Perspective

Mathematical authority is the view of another subject (i.e. person, community, object) as a legitimate source of mathematical knowledge or mathematical reasoning and, thus, able to make meaningful mathematical contributions (Gresalfi & Cobb, 2006; Hamilton, 2022). The delegation of mathematical authority in the classroom can be construed as the set of teacher moves that position students as having the power to rely upon mathematical authority situated internally or within the community of peers' "taken-as-shared" knowledge (Dunleavy, 2015; Wood et al., 1991).

Development of Methods

A lesson introducing coordinate graphing was taught and video recorded in 26 third-grade classrooms. Instances were flagged in which the power to form mathematical decisions was largely positioned either within the teacher or students'—this pass yielded the resulting two seemingly diametrically opposing episodes which became the focus of this poster. Teachers' talk moves, uptake of student ideas, and participation structures are currently being operationalize for qualitative analysis.

A coding framework and preliminary findings will be shared.

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TEACHERS' PERSPECTIVE ON USING STORYTELLING TO TEACH MATHEMATICS

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Storytelling is a versatile and inclusive pedagogical tool that accommodates for different ways of mathematical thinking and contexts including cultural and linguistic. This study sought to understand factors for realizing the power of storytelling as a pedagogical tool for mathematics namely teachers' perspectives of the role of storytelling, and how well the curriculum supports incorporating storytelling. Using qualitative research methodology, the results show the need for teacher training and curriculum materials that support story-based pedagogies.

Keywords: Teaching Practice and Classroom Activity

Research demonstrates that storytelling is an effective teaching tool shown to increase achievement and facilitate authentic learning through motivation, engagement, and contextualization of information (Doğan, 2021; Wilkerson & Laina, 2018). This aligns with the goals of the U.S. Department of Education for "creating authentic learning experiences that encourage and prepare learners" (National Science and Technology Council, 2018, p. 5). Storytelling has also been shown to be a versatile and inclusive teaching style that accommodates different ways of thinking and contexts including cultural, linguistic, identity, subject, and age (Doğan 2021; Posey & Lavik, 2021). The narrative paradigm (Fisher, 1984; Roberts, 2004) posits that through narratives, we develop a deeper understanding of our own and others' knowledge of culture. It can address the opportunity gap expressed by Harvard-sponsored Panelists as reported by The Harvard Gazette on the need for more inclusive STEM education (O'Rourke, 2021). It makes math topics accessible to a diverse group of students when the student identities are intentionally taken into consideration.

Storytelling is a practice that contextualizes math to create a more authentic learning experience rather than focusing on rote memorization (Lemonidis & Kaiafa, 2019; Doğan, 2021). Lemonidis and Kaiafa 's study with experimental and control groups showed that "the use of storytelling had a positive effect on students' achievement in fractions, as the experimental group performed significantly better than the control group" (p. 165). Gould and Schmidt (2010), describe how storytelling was able to increase motivation in trigonometry. Furthermore, it can be identified as a form of student-centered learning (Doğan, 2021; Büyükkarci & Müldür, 2022). Thus, storytelling in math is more than just a motivator, it's also an effective tool for increasing achievement.

Objective

Low performance and achievement gaps in math persist while improving the teaching and learning of mathematics continues to be of national importance. Empirical studies provide evidence that storytelling can be one of the effective tools for addressing challenges facing STEM education. Critical factors in the effectiveness of storytelling include teachers' pedagogical content knowledge, beliefs about the role of storytelling, teaching practices, and their perception of the place of storytelling in the curriculum. Studies on these critical factors are scarce in the literature. Therefore, this study sought to study storytelling in math through these research questions:

a. How do math curriculum materials support teachers' use of storytelling for teaching?
b. How do teachers incorporate storytelling as a tool for math teaching and learning?
c. What are teachers' perceptions about the role of storytelling in mathematics?

Methodology

This research utilizes a qualitative research design as it seeks to understand through descriptions and experiences. Fossey et al describes qualitative research as "a broad umbrella term for research methodologies that describe and explain persons' experiences, behaviors, interactions and social contexts without the use of statistical procedures or quantification." (Fossey et. al, 2002, p. 1). Eleven teachers participated in this study. All participants were elementary (10 teachers) or middle school teachers from schools in one of the states from the Midwest. The participants' teaching experience ranged between fifteen and 30 years. Ten participants self-identified as female and one as male. Data were collected through semi-structured interviews. All interviews were audio recorded.

Data were transcribed using computer software: Nvivo and Microsoft. The transcriptions were further checked by researchers to ensure accuracy. Using both software services and human transcription is important because "While AI may offer a cheaper and quicker alternative to human transcription, these transcripts will need to be meticulously checked by the researcher to ensure accuracy, fill in missing details or edit for context and readability" (McMullin, 2021, p.3). The steps for thematic data analysis begun with getting familiar with the data, then generating initial codes based on the literature and participants' own words, then lastly revising codes based on responses. Transcriptions were coded using line by line coding which involved naming each data which allowed the researchers to "remain open to the data and to see nuances in it" (Charmaz, 2006, p.50). Identifying themes involved identifying repeated patterns in the line-by-line codes to "summarize, highlight key features of, and interpret a wide range of data sets" (Kiger & Varpio, 2020, p. 8). Data analysis was conducted by multiple coders to "improve both the internal quality and external reception of qualitative studies" (O'Connor & Joffe, 2020). All the data were coded by two researchers.

Results

Math Curriculum Materials

Almost all teachers said they think their curriculum materials incorporated storytelling but with some sort of caveat. Excerpts 1 through 4 were responses when participants were asked how well their math curriculum included storytelling. As the excerpts show, the caveats included grade level as a factor, story readability, school context, and relatability of story problems. The participants wished the "curriculum was very intentional in including stories that went along with what was being taught in that unit" but did not perceive their curriculum as doing this well.

Interview excerpt 1.

"I think in the younger grades probably more so, but in the 6th grade we're moving at such a fast pace. I mean, we really don't have the number talks and we don't go through what some of the younger grades do. So, I really think at my grade level, it doesn't lend itself as much."

Interview excerpt 2.

I do think the curriculum incorporates math storytelling as a tool for teaching math. One of the only problems I am finding though is that the students who cannot read well have a big aversion to that section and are completely put off by it. This makes it very hard for me since I need to walk them through every problem and sometimes feel like if the curriculum had used simpler language this wouldn't be such a big deal. Being in a such a small, rural area also poses a problem with the students' comprehension of multicultural names or activities that cannot be sounded out to pronounce.

Interview excerpt 3.

Our curriculum already has many story problems throughout the lessons, but many are hard to relate to. At times I will change them up and use my own students as examples and insert their names and friends so they can understand it better.

Interview excerpt 4.

We have even just some books that we could get and read, but even the way that those are done, they're still so disconnected to what we're actually teaching, and it's not like they're even any have high interest stories that the kids really want to hear. It's just so disconnected, so our curriculum does a horrible job of having anything extra like that involved in it.

Storytelling in Math Instruction

Most of the participants said they always or almost always use storytelling in lesson delivery. The most common way storytelling is incorporated is for the application of math concepts such as for problem-based learning. Storytelling is also used for lesson introduction as one teacher explained "Starting a lesson, you always want to do some type of spark, so you are really striving to relate it to the current culture or what you're currently studying." Eight of the ten participating teachers said they do not have students tell stories in math learning. When teachers reported that their students tell stories in math, it was unintentional and was because students were telling personal stories. Only two participants identified having students tell stories intentionally. The justification for not intentionally using storytelling included that students "see stories as part of literacy, and not as part of math." Another teacher explained that it is because students "solve the problem and just focus on numbers that they forget how much language is involved in the math."

The Role of Storytelling in Math

Multiple themes emerged when teachers were questioned on how they perceive the role of storytelling in math classrooms. The themes included engagement, personal connection, application, retention, and perseverance.

Engagement. The most robust theme for the role of storytelling in math was engagement. Participants believed that storytelling is necessary "so that the math sticks, is interesting, alive and it's applicable." The engagement theme also focused on personal connections, as explained in Excerpt 5. They view personal connections as support for engagement, information retention, and application. One teacher said, "I think it is important that students relate math to the real world. I think if your students hear short stories or scenarios that relate to them, they will be more engaged in the learning." Other teachers thought many elementary students are not able to make personal connections with the math they learn without storytelling.

Interview excerpt 5.

I think it's a piece of engagement and sometimes with these stories that you share from your own experiences it gives students a better idea of who you are. . . I think in teaching you're not just the so-called sage on stage and if you want to do a good job you need to know a little bit about your students. But that also means at times to give a little bit of yourself, so they know a little bit more about you as well.

Application. As may be noted, the referral for real-world applications of math through storytelling is not suggested only in the sense of supporting content connections of how to apply mathematical content in real-world situations. The theme of application is also focused on how the mathematical content can be applied to the students and make those personal connections.

Information retention. The role of storytelling for information retention emerged from the data in the context of mathematics learning. A teacher explained, "I think it is very important because wherever you can insert real-world applications to what the students are learning, they tend to retain more information."

Perseverance. Storytelling in math was also perceived to aid in perseverance. Many responses echoed that teachers share their previous struggles with math and how they worked hard and sought support as students to be successful. The teachers identified value in using personal experiences with mathematics to engage students in the "if I can do it so can you" mentality to motivate students to persevere.

Intentionality and Readiness to Teach with Storytelling

Teachers explained that their storytelling is intentional if their curriculums have adopted it. These curriculums that teachers viewed as having adopted storytelling were Amplify, MyMath, and Big Ideas Math. When discussing students sharing stories, this was also done unintentionally and often not purposefully. Furthermore, 10 out of 11 participants reported that they have never had any professional development or training related to story-based pedagogies math. The one participant who had reported having professional development was an educator in a Montessori school. The participant's Montessori curriculums are designed with stories and base their curriculum on 5 main stories. This participant has three professional development days per year to learn about the story-based curriculum. Additionally, participant's Montessori School hold weekly meetings with spaces for learning about how to teach using storytelling.

Discussion and Conclusion

Some of the study's results align with the existing literature. For example, teachers perceive storytelling as a tool for engagement, motivation, and improving achievement as reported in the literature (Doğan, 2021; Wilkerson & Laina, 2018). Unlike literature that identifies storytelling as a tool for inclusion pedagogy (Piipponen & Karlsson, 2019; Mahmood et al., 2020), inclusion was not a robust theme in these data. The study also adds to the literature the finding that although storytelling is a valuable tool in math classrooms, lack of intentional incorporation and training are persistent problems.

The implications of this paper are that schools should provide more professional development around the intentional use of storytelling in the classroom and be more intentional about incorporating story-based pedagogies in teacher education. Schools and curriculum specialists should consider developing and selecting curriculums that motivate story-based pedagogies. Teachers themselves should take the storytelling they utilize in the classroom and make its incorporation more purposeful.

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NAVIGATING BELIEFS AND KNOWLEDGE: THE IMPACT OF DEFICIT THINKING ON TEACHING SLOPE

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This study examines the impact of teachers' beliefs on the implementation of Mathematical Knowledge for Teaching (MKT) slope, focusing on an in-depth case study of an in-service teacher, Ms. R. Through classroom observations and interviews, we explore how Ms. R's beliefs about her students' abilities and backgrounds influence her teaching of slope. Findings reveal that deficit beliefs significantly mediated the implementation of MKT slope, affecting instructional decisions and practices.

Keywords: Teacher Beliefs, Mathematical Knowledge for Teaching.

Teachers' beliefs affect their instructional strategies and shape their classroom practices, and one particularly influential aspect of teachers' beliefs is their asset or deficit thinking about students. This study narrows its focus to Mathematical Knowledge for Teaching slope (MKT_{slope}), investigating how one teacher's beliefs about students' mathematical abilities, behavior, and potential career paths affected the enactment of her MKT_{slope}. We conducted a series of classroom observations with an in-service teacher and then, in response to one classroom incident, which we introduce below, we collected a sequence of follow-up interviews to understand the dynamic between the teacher's MKT_{slope} and her beliefs about her students. In this paper we examine how a teacher's beliefs influenced her actions to constrain student engagement with the mathematical concept of slope. In doing so, we address the following research question: How does one teacher's beliefs about student groups affect the implementation of MKT_{slope} in a classroom setting?

Literature Review and Background

Teachers' Beliefs and Expectations in Mathematics Education

Teachers' beliefs significantly influence their instructional actions in mathematics education. Beliefs about mathematics, teaching, and learning are pivotal in shaping classroom practices (e.g. Conner et al., 2011; Liljedahl, 2009; Thompson, 1984). A number of researchers have also studied more specific or nuanced teacher beliefs. In particular, studies have shown that teachers' beliefs about their students, including their needs and backgrounds, play a critical role in determining their teaching approaches (Sztajn, 2003; Skott, 2001). This study addresses how one teacher's deficit beliefs about her students affected her instruction.

Deficit thinking involves attributing academic challenges to perceived deficiencies in students (Valencia, 2010), often linked to their racial or cultural backgrounds (e.g., Diamond et al., 2004; Bol & Berry, 2005), socio-economic status (e.g., Rubie-Davies, 2016), or language (e.g., De Araujo, 2017). This perspective tends to overlook students' existing skills, focusing instead on their weaknesses. Deficit thinking is not only a personal bias but is also entrenched in educational systems and societal structures (Parks, 2010). It leads to biased expectations and inequitable treatment in educational settings (Martin, 2009; Irvine & York, 1993; Townsend, 2000). For instance, academic failures among marginalized students are often linked to inherent deficiencies, overlooking the role of teaching methods and educational practices (Delpit, 1992).

More recently, researchers have increased the amount of attention given to deficit-based beliefs because these beliefs influence teachers' choice of tasks and instructional strategies, often limiting mathematical opportunities for those to whom they attribute deficiencies (Jackson et al., 2017; Peck, 2021). Marginalized and lower-achieving students frequently encounter tasks emphasizing procedural skills rather than conceptual understanding (Ferguson, 1998). These beliefs can result in simplified language and less challenging mathematical tasks (De Araujo, 2017), lower cognitive demands in activities (Jackson et al., 2017), and a diminished sense of responsibility for student learning, ultimately leading to lower expectations and fewer opportunities for students (Diamond et al., 2004; Flores, 2007). This creates a negative feedback cycle that only exasperates the issue. In attributing deficiencies to particular student groups, teachers may then lower the cognitive demand of the tasks they implement, which then limits students' reasoning opportunities and prevents them from building the competencies teachers would like to see, thus leading to a self-perpetuating cycle of lowered expectations.

This study explores how a teacher's deficit thinking about her students' mathematical abilities, behavior, and career paths affected her teaching of slope. Unlike previous research that has drawn on surveys, NAEP data, or simulations (e.g., Bol & Berry, 2005; Battey et al. 2021; Irvine & York, 1993; Lubienski, 2002), we examine a specific classroom-based incident in which a teacher's beliefs influenced her in-the-moment decision making in a manner that reduced opportunities to reason meaningfully about slope as a ratio of change.

Mathematical Knowledge for Teaching (MKT)

Teachers' MKT is crucial for effective teaching (e.g., Ball et al., 2008; Harel, 2008; Kahan et al., 2003.; Rowland et al., 2005). We use Silverman and Thompson's framework (2008), which is rooted in the concept of key developmental understandings (KDUs), which are essential for developing a teacher's MKT. Initially, a teacher must identify and develop their own KDU for a specific mathematical topic, which equips them with knowledge that has the potential for pedagogical application. This knowledge must then undergo a process of reflective abstraction to transform into pedagogically powerful MKT. The framework further requires a teacher to adopt students' perspectives ("decentering," p. 508), envision how students might grasp mathematical concepts similarly to themselves, and devise supportive activities and discussions.

Researchers have identified various meanings of slope among teachers (see e.g., Nagle & Moore-Russo, 2013; Byerley & Thompson, 2017). Three prevalent meanings include slope as steepness, in which slope is seen as a physical property with visual steepness; slope as rise over run, in which slope is understood as a geometric or algebraic ratio often focused on procedural movement (i.e., "rise" and "run") on a Cartesian plane but lacking in multiplicative reasoning (Byerley & Thompson, 2017); and slope as a ratio, in which one compares changes in two quantities to multiplicatively form a new quantity, which is an interpretation applicable across various contexts and related to the mathematical property of constant rate of change (Diamond, 2020; DeJarnette et al., 2020). This last interpretation, though less common, is essential for a comprehensive understanding of slope and its applications in mathematical concepts and real-world scenarios.

Diamond (2020) defined MKT_{slope} as teachers' personal understanding of slope, teachers' understanding of students' developed meanings for slope (e.g., slope as steepness, slope as formula, slope as ratio, etc.), and how the teachers use classroom activities and discussions to support the development of these meanings. We examine a teacher's MKT_{slope}, particularly her focus on the slope-as-formula meaning, and how it is mediated by her beliefs about students.

Interaction Between Teachers' Mathematical Knowledge and Beliefs

Research indicates a complex interplay between teachers' mathematical knowledge and their beliefs, which affects their instructional practices (Bray, 2011; Campbell et al., 2014; Wilkins, 2008). Teachers' beliefs mediate their instructional decisions, and these beliefs are, in turn, influenced by their mathematical knowledge (Fennema & Franke, 1992; Zhang & Wong, 2015). Studies have shown that although strong knowledge is essential, beliefs play a critical role in how teachers engage with instructional practices (Charalambous, 2015; Copur-Gencturk, 2012). However, the specifics of this interaction, especially instructional decisions in the moment, remain underexplored (Philipp, 2007; Wilkins, 2008; Yang et al., 2020). Most research addresses beliefs about mathematics and its teaching and learning, with limited studies on how teachers' beliefs about students might mediate their knowledge to influence instruction. Our study bridges this gap by examining how a teacher's deficit beliefs about students affected her use of MKT_{slope}, revealing how such beliefs constrain student engagement with complex mathematical concepts.

Methods

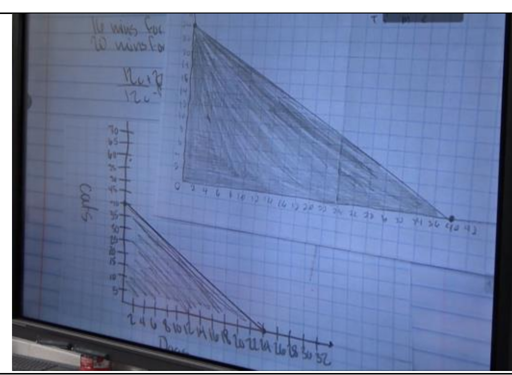
This study is part of a larger project focused on understanding how teachers support students engaging in mathematical generalization (see Ellis et al., 2024). We conducted a series of classroom observations and interviews with several teachers to observe what happened in practice and to understand their beliefs about generalization and their perspectives on the lessons. Within this broader project, we identified Ms. R, a sixth-year high-school algebra teacher from a rural district, for an in-depth case study (Merriam, 1998; Yin, 2009) due to her insights into the teaching and understanding of slope. While teaching her lesson on Systems of Equations and Inequalities, Ms. R used the Pet Sitter Task (Figure 1a). She asked students to collaborate and represent the constraints algebraically and graphically. As she facilitated the discussion, Ms. R make some choices about which she later expressed regret, and the details of the situation are discussed in the next section.

Carlos and Clarita have been worried about space and start-up costs for their pet sitters business, but they realize they also have a limit on the amount of time they have for taking care of animals they board. To keep things fair, they have agreed on the following time constraints.

<u>Feeding Time:</u> Cats will require 12 minutes to eat per day. Dogs will require 20 minutes to eat per day. Carlos can spend up to 8 hours each day to feed the animals.

<u>Playing Time:</u> Cats need 16 minutes each day to be brushed. Dogs will need 20 minutes each day playing with the ball. Clarita can spend up to 8 hours to play with the animals.

Write inequalities for each of these additional time constraints. Shade the solution set for each constraint on



separate coordinate grids.	

(a) (b)

Figure 1: (a) The Pet Sitter Task and (b) Two differently oriented graphs from two different students for the feeding time

Through a series of six semi-structured interviews (Roulston, 2022), which were conducted over a mixture of in-person and Zoom-based settings, we explored her MKT, her beliefs, and how these factors influenced her instructional actions. MKT interviews focused on Ms. R's understanding of slope and were adapted from Diamond (2020). The questions included tasks intended to explore how her knowledge might impact her teaching such as the Five Students problem (Figure 2) and follow-up questions that incorporated contexts from her classroom to explore the knowledge related to the classroom incident that is illustrated in the next section.

Five students are discussing the meaning of slope in a linear context. Student A says that slope is $\frac{y_2-y_1}{x_2-x_1}$. Student B says that slope is the steepness of the line. Student C says that slope is rise over run. Student D says that slope is the rate of change of the line. Student E says that slope is the number *m*.

Figure 2. Five Students problem (Adapted from Diamond, 2020).

Beliefs interviews focused on her beliefs about mathematical generalization and the capabilities of her students and allowed us to follow-up with her to confirm our conjectures about her beliefs. These interviews incorporated video clips from her classroom so that she could reason and provide context to the choices that she had previously made in teaching. This retrospective reasoning was not intended to understand her in-the-moment decision making, but it helped us to explore the beliefs that she holds.

We analyzed the interviews through a multi-phased qualitative process. Initial analysis focused on Ms. R's beliefs about her students' mathematical abilities and her MKT_{slope} and were coded using an open and axial coding approach (Strauss & Corbin, 1998). As a result, categories of her beliefs about students' knowledge and ability emerged as did her understanding of slope. Later interviews were modified based on previous data to continue to explore her beliefs and understanding. As a result, we created an account of explanations of how her beliefs and MKT influenced her teaching actions.

Background: Classroom Incident

Here we detail a classroom incident that sparked our curiosity about Ms. R's MKT_{slope} and beliefs. We share this incident as a separate section from the Results, as it is the event that initiated further data collection in the form of interviews to better understand both Ms. R's MKT_{slope} and her beliefs about her students. This incident occurred during the implementation of the Pet Sitter Task (Figure 1a). During the task's implementation, the students worked in groups to write inequalities. As they began to graph the solution sets on separate coordinate axes, the students asked Ms. R which quantity, feeding time or playing time, should go with which axis.

Ms. R decided to allow the students to choose how to orient their axes, stating, "Let's just see who comes up with what. I think that'll be better… that'll be cool to see." As the students continued to work on their graphs, Mr. R moved from one group to another and appeared to regret her decision to allow the students to choose their axes orientations. She said, "Man, I wish I had never said anything about y'all's axes." Despite this apparent regret, Ms. R nevertheless proceeded to foster a discussion about the situation, comparing two students' graphs with different orientations (Figure 1b).

Ms. R put the two graphs under the document camera, saying, "I thought it was interesting, um, how you guys graphed. So, the cats and the dogs, the axes were different and similar." She then asked the class, "Alright, so do you guys see the difference between these two graphs?" Several students responded that the dogs and cats were "flip-flopped" on the axes of the coordinate plane. Ms. R then highlighted the difference in the axes and drew the students' attention to the slopes of the two graphs. Some students claimed that both graphs had the same slope, whereas others believed that the slopes were different. In trying to navigate this disagreement, Ms. R drew the students' attention to the quantitative referents, claiming "they [both graphs] represent the feeding time… the one has cats as a *y*-axis, one has dogs as a *y*-axis." She then asked the students again, "So, would the slopes be the same or different?" One student pointed out that in comparing the two graphs, "the rise and the run would be, like, switched." Referring to the graph at the top (Figure 1b), another student said that the slope is "Negative 3 over 5 and, like, if you start with 40 cats [referring to the bottom graph in Figure 1b] and you go down 5 and over 3." In response to this argument, through employing the "rise over run" method, the majority of students concluded that the slopes were different, in fact, they were "flip-flopped," seeing the reciprocal values of the slopes as different.

Following this classroom incident, we were intrigued by Ms. R's expression of regret about allowing the students to choose their own axes orientations. We were also interested to understand why, despite this regret, Ms. R decided to still bring the two graphs into a whole-class discussion and why, during this discussion, she compared the slopes of these graphs.

Results

In this section, we initially focus on Ms. R's reflective interview concerning the classroom incident. Subsequent sections will dive deeper into her MKT_{slope} and beliefs about students, exploring how these factors may have influenced her instructional actions.

Ms. R's Reflection

We were curious about Ms. R's stated regret over her decision to let the students choose their axes orientations, particularly given that she then nevertheless used this as a learning opportunity by presenting two students' graphs to the class. We decided to conduct a reflection interview with Ms. R to gain insights into (i) her potential regret and its driving factors, and (ii) her decision to use the student graphs with different orientations. It was during this reflection interview that Ms. R's descriptions of the classroom incident suggested that her beliefs about students and her MKT_{slope} might impacted her actions.

Ms. R's beliefs regarding her "on-level" students' abilities may have influenced her perception of her instructional decision as a mistake, as she believed that discussing graphing in different orientations, or "axis flipping," was appropriate only for honors settings. She noted, "it [*referring to the axis flipping*] is only something that I think should be discussed in like honors setting," and added, "on level, I mean they already cannot figure out which like they are like which way does it go," indicating a preference to "let's just focus on the basics." This stance suggests she reserves more complex discussions for honors students, underlining a belief that on-level students are better served by focusing on basic graphing skills due to perceived limitations in handling advanced topics.

We also explored Ms. R's decision to use two student graphs with different orientations

(Figure 1b) in a class discussion despite her regret. During a video interview, Ms. R reflected on her decision watching a clip of herself. She noted her wish to have more clearly demonstrated the slope formula, citing concerns that some students (especially her "on-level" students) might misconstrue the graphs' similarity in decline and spacing, potentially assuming identical slopes. Ms. R remarked, "both of them [graphs] are decreasing and they look about the same spread," highlighting a potential misunderstanding outside an honors context where students might overlook differences, emphasizing, "I can see if that had not been in honors gifted class, it would have easily been oh yeah everything is the same." Her intention was to clarify that slope analysis goes beyond visual inspection to require formula application, aiming to show, "we are just looking at how this vertical distance is changing over this horizontal distance," to discern the distinct slope values. Our interpretation is that Ms. R's understanding of the slope concept and her MKT_{slope} may have significantly influenced her teaching approach, as she emphasized the importance of the formula over visual steepness in understanding the concept of slope.

The emphasis Ms. R placed on the slope formula and the numerical values of slopes while comparing the two graphs sparked a line of inquiry for the research team as there are other ways of thinking about slope including slope as rate of change[3]. Given the value of understanding slope as a rate of change, particularly for interpreting the two graphs in the pet sitter task, we wondered why Ms. R discussed the meaning of slope as a formula instead of slope as a rate of change. It could have been interesting to see how both Ms. R and the students could conclude whether the slopes of those lines were different or the same when considering the rate of change meaning that is connected to quantities in the context of the problem. We argue that the slopes of these two graphs would be the same as both graphs show the same quantitative relationship: "every time you are done feeding 3 dogs, you can feed 5 more cats." This understanding would require someone to think unconventionally (see Moore et al., 2014, for "breaking conventions," p. 151) about how we represent inputs and outputs in a Cartesian plane.

Conjecture #1: Ms. R's MKT_{slope}

Given Ms. R's emphasis on slope as formula in the initial interview, we hypothesized that her understanding of slope, as well as her MKT_{slope}, might not include the concept of slope as a rate of change. To verify this hypothesis, we conducted MKT_{slope} interviews, drawing on Diamond's methodologies, to explore her perspective. Contrary to our hypothesis, Ms. R demonstrated a multifaceted understanding of slope, primarily as a rate of change between two quantities, often using real-world examples like fuel costs to illustrate this relationship. She considered this interpretation applicable in various contexts beyond formulas or procedures.

We wanted to get further insight into Ms. R's views on students' understanding of slope. Presented with a task (see Figure 2), she highlighted her focus on slope as a rate of change, viewing it as a deeper, more meaningful understanding than just a memorized technique. She considered conceptualizing slope as a formula more substantial than seeing it as mere "rise over run" and deemed understanding slope as steepness, represented by "*m*," as the most basic level.

Ms. R used practical examples, such as "25 miles/hour," broken down into a relatable format ("for every 1 hour, the car goes 25 miles"), to teach slope as a rate of change. Her goal was to enable students to create similar phrases and apply this understanding across different mathematical representations, like graphs, equations, and tables.

[3] We adopt Ms. R's terminology, using "rate of change" to describe the "ratio" understanding of slope, despite our awareness of the conceptual differences between the two terms.

In order to get her insights into how she would compare the slopes of the two graphs oriented differently (Figure 1b) form the rate of change perspective, we designed and presented a task where a hypothetical student claims the following regarding the two graphs in Figure 1b: *Slope is the rate of change. So, both graphs show that "For every 3 dogs you are done feeding, you can feed 5 more cats". So, the rate of change in both graphs are the same. Therefore, the slopes are the same. The student also knows that inputs can be represented on the y-axis and outputs can be represented on the x-axis.* Our goal was to see how Ms. R would interpret the hypothetical student's understanding. Ms. R indicated that the student's emphasis lay in understanding the relationship between quantities and the meaning of slope, rather than fixating solely on the numerical value of the slope. According to her, "these numbers [*referring to the numbers in the student's phrase*] represent something and have meaning."

When directed to the aspect that the student also asserts that the slopes are equivalent, she explained, "The phrasing [*referring to the student's phrase*] to me is more about like a relationship. And they're saying, okay it's [*i.e., slope*] a rate of change. So, their rate of change is the same" because "these two graphs represent the same relationship although they just look different visually." Although she acknowledged that "the slopes are technically different"—using "technically" to denote "the literal exact value"—she argued that by conceptualizing the slope as a rate of change, the slopes are indeed identical. The interview data revealed that Ms. R emphasized slope as a rate of change connected to varying quantities. It was therefore more confusing that she did not adopt the rate of change perspective in her teaching, especially when comparing two graphs. This finding adds complexity to our understanding of her teaching approach regarding the concept of slope.

Conjecture #2: Ms. R's Beliefs about Students

Our initial interview with Ms. R suggested she held deficit beliefs about certain students. Therefore, we created an alternative hypothesis that perhaps her beliefs mediated the way she implemented her MKT_{slope}. To test our hypothesis, we investigated Ms. R's stance on the feasibility of engaging her students in a discussion about slopes as rate of change and, more importantly, a discussion about the equivalence of slopes—as exemplified by the hypothetical student's response. Results confirmed our conjecture. Ms. R's beliefs about student behavior issues, abilities, and their future paths played a mediating role in implementing her MKT_{slope}.

Ms. R recognized the value of teaching slope as a rate of change but emphasized its complexity, noting, "I think that's a good idea and a good thing to talk about, but I also recognize how big of an idea that is," and contrasting it with simpler, procedural methods like the slope formula. She observed that students struggle with abstract concepts, expressing, "like we are talking about a relationship and meaning versus the concrete smaller procedural like 'let's do slope formula.'" Concerned about lower-level students' reactions to difficult material, she mentioned, "when they're confused, it's like they're angry and they start misbehaving," leading her to prefer straightforward approaches to minimize disruptions. Ms. R pointed out that in a gifted setting, complex topics were more feasible due to fewer behavioral issues but anticipated "blank stares" and resistance from lower-level students. Additionally, she linked behavioral issues to external factors like home life, suggesting, "a lot of it [*referring to behavior issues*], I would say probably stems from something happening at home … if they [*referring to parents*] were more proactive, I think that would help a lot with kids being more engaged."

Ms. R's beliefs about her students' readiness to understand concepts like switching *x* and *y* axes on the Cartesian plane influenced her use of MKT_{slope} in the classroom, particularly hesitating to introduce such topics to lower-level students. She considered these concepts more accessible to gifted students and challenging for others, noting, "I think that is a big jump for those, that level of kids [*referring to lower-level students*] to handle the different orientations." This provided additional evidence that her beliefs about students' ability moderated the enactment of MKT_{slope} as she decided to not bring slope meaning as rate of change when comparing the two graphs. In other words, even though she had more powerful MKT_{slope} (slope as a rate of change), her deficit beliefs interfered with them leveraging that MKT. Moreover, Ms. R expressed doubts about teachers' understanding of slope as a rate of change, saying, "as for like teachers as a whole, like, honestly, I don't know that teachers even understand it." Crediting her own understanding of the slope as a rate of change to being labeled as gifted, she observed, "I don't want to sound cocky, but I think it's because I was labeled gifted as a kid in school," and noted a tendency among teachers towards rote learning. This stance implies she views the rate of change concept of slope as potentially too complex for standard classroom settings, fitting more for advanced learners and educators.

Moreover, Ms. R held varying expectations for her honors and lower-level students, influenced by her perceptions of their future career paths. For honors students, she emphasized challenging mathematics tasks relevant to their projected careers requiring advanced skills. She stated that the rigorous mathematics is important "for the ones that I'm thinking that want to go to, like a four-year college and potentially major in something where they're gonna have to use a good bit of math." In contrast, she expected less from lower-level students, tailoring instruction to simpler concepts she deemed more practical for their potential vocational paths like "nursing," and noted, "I think the advanced, the depth of math they need is just depending on where they want to go." Additionally, Ms. R highlighted how external factors such as social class and parental education, especially in lower socio-economic backgrounds, impact students' educational directions, observing, "I think it's because the parents don't know [about career options]... so it's harder for them to educate their own kid about it."

In summary, Ms. R's deficit beliefs about certain student groups and her perception of their abilities and future paths significantly influenced her implementation of MKT_{slope}, particularly in her approach to teaching slope concepts.

Conclusion and Discussion

Our findings indicate that Ms. R's beliefs about students mediated her MKT_{slope} to influence instructional actions and decisions in various ways. Initially, she regarded the decision to allow students to choose their own axis orientations as a mistake, driven by her beliefs about their abilities, leaning towards a "focus on the basics." However, once she recognized this perceived error, she used it as a learning opportunity. Guided by the interaction between her beliefs about students and her MKT_{slope}, Ms. R emphasized the formulaic meaning of slope while discouraging the steepness interpretation. This combination of beliefs and knowledge led Ms. R to lead a discussion about graph differences and their respective slopes, redirecting students' attention from visual appearances to the numerical aspects of slope as a formula. Furthermore, despite Ms. R's MKT encompassing the concept of slope as a rate of change, she chose not to introduce the rate of change interpretation when comparing the two graphs, influenced by her deficit beliefs about student behavior issues, her perception of students' abilities, and future career pathways.

This paper highlights the profound influence of teachers' beliefs on the implementation of MKT in the context of teaching slope. Through an in-depth case study of Ms. R, it provides nuanced insights into how a teacher's perceptions of students' abilities and potential can shape instructional practices, particularly in complex mathematical concepts like slope. Moreover, our findings extend the discourse on the interplay between teachers' MKT and their beliefs, offering a unique perspective on the dynamic interactions that occur in real classroom settings. By focusing on slope as rate of chance—a concept that is pivotal for students' deeper mathematical understanding—this study sheds light on the missed opportunities for enriching students' learning experiences due to the constraints imposed by deficit thinking. Thus, it calls for a reevaluation of teaching practices and belief systems in mathematics education, aiming to empower teachers with the knowledge and strategies to effectively nurture and leverage students' mathematical understanding, irrespective of their backgrounds. This research underscores the necessity of addressing and challenging deficit beliefs within teacher professional development programs, advocating for a more holistic approach that includes fostering an understanding of diverse student capabilities and promoting instructional strategies that are inclusive and supportive of all learners.

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HOW MENTORS PROVIDE FEEDBACK TO ELEMENTARY TEACHER CANDIDATES ON ELICITING STUDENT THINKING IN MATHEMATICS

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Mentor teachers may have the most immediate influence on teacher candidates' (TCs) practices, suggesting that teacher educators must continue to explore the messages conveyed about ambitious mathematics instruction in field-placements to bridge the gap. The purpose of this case study is to explore mentors' perceptions of eliciting student thinking in mathematics and to understand how they go about modeling and providing feedback to their TCs about such practices. Data were collected with four TC-mentor dyads via recorded mathematics lesson observations, coaching conversations and interviews. The findings focus on how the TCs and mentors made meaning of what it means to elicit student thinking, the structures of the feedback conversations, and the coaching moves used during feedback conversations.

Keywords: Instructional Activities and Practices, Preservice Teacher Education, Elementary School Education, & Teacher Educators

Problem Statement and Conceptual Framework

TCs learn across both the university setting and in K-12 classrooms (Grossman et al., 2012); however, for some time, scholars (e.g., Feiman-Nemser & Buchmann, 1985) have cautioned us to be cognizant of the lack of a shared vision across these two settings, especially in mathematics education. And still, researchers have emphasized (Smagorinsky et al., 2006; Ronfeldt et al., 2018a) how mentors are seen as having the most immediate and enduring influence on how TCs teach. Thus, if we as mathematics teacher educators within teacher education programs (TEPs) hope to truly impact TCs' teaching practices, we have to seek avenues by which we can cultivate a shared vision of ambitious mathematics teaching (Walkowiak et al., 2018) with mentors.

This notion of a shared vision and partnership not only incorporates communal ideals around best mathematics practices but also ideals surrounding the various roles that mentors play (Butler & Cuenca, 2012; Parker et al., 2021). Mentors' conceptualization of such roles is influenced both by their beliefs (Rozella & Wilson, 2012) and their perceptions (Leatham & Peterson, 2010). Matsko and colleagues (2020) reflect upon the various ways in which a mentor is both a model of teaching practices and a coach, as they intentionally target TCs' learning. Early work defined *coaching* as ongoing cycles in which a coach facilitates "observation and feedback" to improve implementation of new teaching strategies (Joyce & Showers, 1981, p. 170). Other work has moved beyond these earlier definitions by focusing on components of coaching conversations (e.g., Knight, 2007), alternative structures for coaching (e.g., Gibbons & Cobb, 2017), and the multitude of modalities for *how* one should provide coaching and within varying contexts (e.g., Dozier, 2006). Hoffman et al. (2015) review what we can learn across various studies surrounding what it means to *coach* a TC and what we can learn from those interactions.

Thus, we seek to understand not only how mentors are modeling mathematics teaching practices for their TCs as teacher educators but also the ways in which they go about communicating feedback (Lawley, 2014) and coaching their TCs about ambitious practices. Yet, in approaching this work, we are cognizant of, just as teaching is complex work, there are many differing contextual factors at play that may impact mentors' work (Roegman & Kolman, 2020) and TC learning (Grossman et al., 2012).

Eliciting Student Thinking in Mathematics Education

The *eight mathematics teaching practices* provide a framework for standards-based or ambitious mathematics instruction (NCTM, 2014). *To elicit and use evidence of student thinking* is defined as, "…us[ing] evidence of student thinking to assess progress toward mathematics understanding and to adjust instruction continually in ways that support and extend learning" (NCTM, 2014, p.10). This practice heavily overlaps with *posing purpose questions*. We are interested in five aspects of *eliciting*, including: 1) formulating questions to elicit and probe student thinking, 2) posing questions, 3) listening to and interpreting student thinking, 4) developing additional questions based on student responses, and 5) making sense of what students know and can do (Shaughnessy et al., 2020).

Purpose

A broad goal for this work is to enhance the quality of elementary mathematics instruction by understanding the skills of mentors with practices of eliciting student thinking in mathematics and providing feedback (Cohen et al., 2020; Ronfeldt et al., 2018b) to their TCs about such practices. This nested case study sought to understand how such practices were being implemented; while the larger research study has sought to explore how we can use these findings and materials to support and develop (Matsko et al., 2020) the skills of mentor teachers, who will, in turn, better support the next generation of TCs during such uncertain times.

Research Questions

There were three research questions for this work, but we have focused heavily on Research Question 3 within this proposal, as seen in the findings.

1. How are the TCs within this study eliciting student thinking? What are their perceptions of the practice of eliciting?
2. What are the mentors' perceptions of what it means to elicit student thinking? What are the mentors noticing about the TCs' practices with eliciting student thinking?
3. How are the mentors providing feedback to their TCs and coaching them about the practice of eliciting student thinking?

Methods

For this study, we followed a case study design (Yin, 2018), examining the four, teacher-TC dyads within our elementary TEP, who were all serving a diverse student population.

Sample

The participant group for this research consisted of four mentor-TC dyads. The mentors were each recommended as experienced mentors who had worked locally with our TEP for at least five years. The four TCs were all seeking degrees in our elementary undergraduate program and were student teaching at the time. We sought to have more diversity represented within our sample, but unfortunately, as reflective of the teacher population in our area and nationally, the majority of experienced teachers working within the TEP were white women.

In what follows, we outline the dyad information for pairing of mentors and TCs. Jade (white woman) a second-grade mentor teacher worked with Thea (white woman). Kali (white woman) a kindergarten mentor teacher worked with Raya (white woman). Maria (white woman) a first-grade mentor teacher worked with Imani (woman of color). And lastly, Phoebe (white woman) a third-grade mentor teacher worked with Alex (white man).

Data Collection

The four TCs were each asked to submit two recorded mathematics lessons (approximately 30 minutes each) taught within their internship placements. The TCs were asked to select lessons that specifically focused on eliciting student thinking. The respective mentor teachers were asked to engage with their TCs, following each mathematics lesson, in recorded feedback conversations. These conversations were approximately 20 minutes each. The mentors were not given a substantial amount of guidance, but were told that we wanted to learn more about how mentors provide feedback to TCs on eliciting student thinking in mathematics.

Later, the mentors and TCs were each individually interviewed (approximately 30 minutes) to discuss their experiences with these feedback conversations. The TC interview focused on perceptions of *when* and *how* they attended to eliciting student thinking as well as unpacking how they felt about the coaching received. Sample mentor interview questions included: 1) How do you provide feedback to your TC about his/her mathematics instruction; 2) how do you decide how to provide feedback to your TC about eliciting student thinking; and 3) are there specific strategies or coaching moves that you use to give feedback? If so, please explain.

Data Analysis

All eight of the mathematics lessons and the eight feedback conversations were recorded and transcribed. Additionally, in reviewing all of the video materials, we kept detailed fieldnotes as if we were in the elementary classrooms. Further, all eight of the interviews were also recorded and transcribed. Throughout the interview process, analytic memos were written intermittently, data were coded in Dedoose, and all data sources were triangulated. We further engaged in peer debriefs to ensure credibility throughout this process aligning with research.

Results

Eliciting Student Thinking as a Practice

In examining the first two research questions, a number of themes emerged. It became evident that while these mentors have a great amount of expertise, their conceptions of what it means to *elicit student thinking* in mathematics did not always align with mathematics education research. For example, at times, there was a lack of depth in the mentors' comments and missed opportunities to coach for probing questions. Perhaps, this is because the TCs also grappled with expressing fully formed ideas of what eliciting student thinking looks like. This also translated into practice where we noticed that the TCs made efforts to initially pose "how" questions but they missed the mark with further probing the students' thinking or with making assumptions. Additionally, the mentor interviews revealed that the mentors' perceptions of their feedback and coaching often did not align with what was observed within the videos. For instance, they often thought that they provided much more tangible coaching on eliciting than what was observed.

Structures of Feedback Conversations

The feedback conversations, structured by the mentors, varied in approach. For example, some mentors like Phoebe were very intentional in making sure that the conversation had a beginning (asking Alex how he thought it went), a middle (in which most of the feedback for future work was provided and there was some practicing), and an end (bringing closure to the conversation and looking ahead to future practice). However, mentors like Kali had a list of points that she wanted to discuss and she simply worked through the list in sequential order. While these two examples stand out for being extremes of one another, the coaching conversations of Maria and Jade did not consistently follow a specific structure, rather the structure was seen as being highly dependent upon what happened during a given TC lesson and what seemed pressing to discuss. Overall, Phoebe's structure was seen as being most influential.

Coaching Moves within Feedback Conversations

While the structure of the conversations varied, the coaching moves also differed. For instance, Jade had more "suggestions for future moves" and rarely asked Thea questions about her teaching nor had her make future projections about practices. Similarly, across both feedback conversations, Kali did not ask Raya a single question, rather Raya only spoke when she wanted to interject or defend her decision making. Kali gave very directive comments, telling Raya what she should do the next time; Kali would often sit with a small white board and actively model instructional strategies. Maria and Phoebe both sought more of a balance between soliciting their TCs' ideas and providing suggestions and directives for future practice. Specifically, we saw how Maria and Phoebe posed questions to have their TCs reflect on what they did, while immediately making moves to transition into phrases like, "next time, I would really like to see you to try…". These directives were usually followed by additional questions like, "why do you think that would be better?" or "how do you feel about that?". In the following excerpt, we see some of Phoebe's coaching moves while attending to the practice of eliciting student thinking.

> Phoebe: So, after they had explored [by using manipulatives], we turned to the brownies (with 1/6), how do you think that that went?
> [Alex talks about the scenario which asked the students to consider where they would want to purchase brownies from, given the shape of the equal parts.]
> Phoebe: Then, we did the discussion and a couple students answered, but what could you have done to make sure that everyone was understanding?
> Alex: I could have asked another question… to demonstrate that no matter how it was cut, it was still 1/6.

Implications & Conference Theme

The first finding asserted that mentors who work within our TEP could use professional development on the practice of eliciting student thinking as a teacher learner to ensure consistent understanding of these practices. In reflecting upon the latter two findings, we noted that even with a small sample size, the coaching that our TCs received varied substantially, having impacts on their practice and their perceptions of their performance. This too points to potential professional development opportunities on what it means to coach and engage in effective coaching moves within the role of a teacher educator. A particular implication for teaching would be to have mentor teachers record and re-watch their own feedback conversations with TCs for professional growth, just as we often suggest with self-analysis of mathematics lessons.

Our research team affirms the value in establishing partnerships between university and school stakeholders. Specifically, we point to the need for a shared vision that enables TCs to develop equitable mathematics practices such as eliciting student thinking as we seek to envision the future of mathematics education in uncertain times. In recent years, teachers have faced immense challenges and feel the rippling effects of a global pandemic and political shifts, and so, in the face of such adversity, we must focus on the importance of building partnerships to support both practicing teachers and novices, for the sake of best serving the needs of learners.

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EQUITY-DIRECTED MATHEMATICS INSTRUCTION THROUGH MATHEMATICAL MODELING PROBLEMS

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This article explores equity-based and culturally responsive mathematics instruction in elementary education, with a specific focus on mathematical modeling (MM). Guided by the Noticing for Equity Framework, this case study of a 5th grade classroom identified and explored three key approaches: (1) selecting and adapting tasks relevant to students' lives, (2) launching tasks connecting to out-of-school knowledge, and (3) creating opportunities to learn and value input from peers. The findings illustrate the practical application of these approaches. The study highlights the importance of contextually relevant tasks, emphasizing the need for purposeful task selection and adaptation to create an equitable and engaging learning environment. Implications for teachers include recommendations to integrate familiar scenarios into lessons to deepen student understanding and appreciation for mathematics.

Keywords: Culturally relevant pedagogy; Equity, inclusion, and diversity; Modeling; Teacher noticing.

In elementary mathematics education, a transformative shift is underway—one that recognizes the impact of lived experiences on students' learning journeys (Turner et al., 2021). In the United States, the Common Core State Standards for Mathematics (NGA Center & CCSSO, 2010) call for activities that allow students to "apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (p. 7). The process of mathematical modeling (MM) aligns with this standard. MM involves the application of mathematical concepts and procedures to solve real-world problems by formulating a mathematical question, selecting computational methods, interpreting results, and creating generalizable models to apply to other situations (Suh et al., 2021). While engaging in MM, students use their knowledge and experiences to identify quantities and mathematical relationships (Aguirre et al., 2020; Turner et al., 2019). Moreover, student-centered pedagogies that emphasize contextual relevance and culturally responsive mathematics instruction can be incorporated into teaching and learning with MM (Turner et al., 2019; Suh et al., 2021).

Researchers and policymakers advocate for engaging students in tasks that foster both mathematical reasoning and problem-solving, as well as equitable instructional practices that attend to students' backgrounds, experiences, and cultural perspectives and traditions (National Council of Teachers of Mathematics, 2014; NGA Center & CCSSO, 2010). Despite these calls, a persistent gap remains in understanding how to implement equitable instruction in ways that align with students' interests and experiences (Suh et al., 2021; Turner et al., 2024). We argue that presenting situations that are relatable and significant to students supports meaningful mathematics instruction and serves as a cornerstone for fostering equitable practices.

Purpose and Research Questions

We investigated the teaching and learning of MM with a focus on equitable and culturally responsive pedagogies. Our research focused on deepening understandings of teaching practices and classroom interactions that incorporates students' lived experiences in MM instruction. We explored the following research question: *How do elementary teachers notice equity in planning and enacting MM instruction?*

Research Informing the Study

This study centers on equity-directed instruction in the context of MM. MM problems provide opportunities to enact equitable practices as students experience real-world connections between mathematics and their own lives (Suh et al., 2017, 2021). Our theoretical framework for studying teaching and learning of MM centers on the *Noticing for Equity Framework* (van Es et al., 2022). "Noticing" in this body of literature represents ways teachers attend to elements within the classroom, make sense of them, and provide appropriate responses (Jacobs et al., 2010). van Es and colleagues (2022) applied noticing research to equitable teaching practices to develop the *Noticing for Equity Framework,* and they identified critical ways to notice and foster equity in mathematics education. This framework is built on previous work on equitable practices, including research from Gutiérrez (2007) and Rubel (2017). Correspondingly, their constructs for equity and equitable practices informed our research as we focused on culturally responsive and sustaining pedagogies, access, achievement, identity, and power. Through a lens of the *Noticing for Equity Framework,* we examined culturally sustaining practices, such as positioning students as capable through discourse, recognizing students' sociocultural selves, and promoting academic success (van Es et al., 2022).

Methodology

We conducted a case study involving classroom observations and analysis of mathematics lessons and lesson artifacts (e.g., instructional materials and worksheets) (Yin, 2015). Case studies provide rich examples for educators to consider when improving their own practices. Through purposeful selection, we identified and recruited two experienced 5th-grade teachers, Anna and Liz, along with their respective students at Sunset School, located in the northwestern United States (all names are pseudonyms to protect confidentiality). This public school serves K-5 students with diverse backgrounds, ensuring representation across different demographics, in line with equity-based instructional principles (Rubel, 2017). Data collection included teacher interviews before and after each lesson and lesson observations. Interviews focused on mathematics instruction, decision-making, equitable practices, and students' learning. We observed four mathematics units, with each unit featuring three mathematics lessons, resulting in 12 observed and video-recorded lessons.

Data analysis involved multiple coding cycles to identify themes, patterns, and differences (Saldaña, 2021). During the first cycle of coding, we used *in vivo* and descriptive coding to identify content relevant to the research question (Saldaña, 2021). For the second cycle of coding, we used Atlas TI (qualitative data analysis software) and employed axial and thematic techniques, integrating first-cycle codes, as well as codes from relevant research and theory (e.g., fund of knowledge, mathematics thinking, academic success). This stage involved deeper examination of the data to identify overarching themes and synthesize information (Saldaña, 2021). We wrote analytic memos throughout data collection and analysis. Finally, we ran reports and used functions in Atlas TI to test emerging themes and generate findings.

Findings

Guided by the *Noticing for Equity Framework* (van Es et al., 2022), our analysis revealed three primary approaches for planning and enacting MM instruction as teachers noticed for equity, including: (1) *selecting and adapting tasks* that are meaningful to their students' lives, (2) *launching tasks* to connect to students' out-of-school knowledge, and (3) creating opportunities for *learning from and valuing input from peers*. We illustrate these three ways in an example from a lesson Anna taught. For context, we first describe the MM task. Next, we discuss our findings for the three approaches Anna used in planning and enacting the lesson.

For this lesson, Anna selected the "Better Buy" task from her curriculum materials. For this task, students analyzed the cost-effectiveness of purchasing 8 granola bars for $10 compared to buying 20 bars for $23 (Matassa et al., 2017, p. 27). Anna *selected* this task because she thought it had strong potential for engaging students in MM. The Better Buy task prompted students to consider real-world prices and relationships, a crucial aspect of the modeling process (Turner et al., 2021). While this task did not involve generalization, it encompassed other key components of MM, such as formulating a mathematical question, selecting computational methods, and interpreting results, making it suitable for students in aspects of MM (Suh et al., 2021).

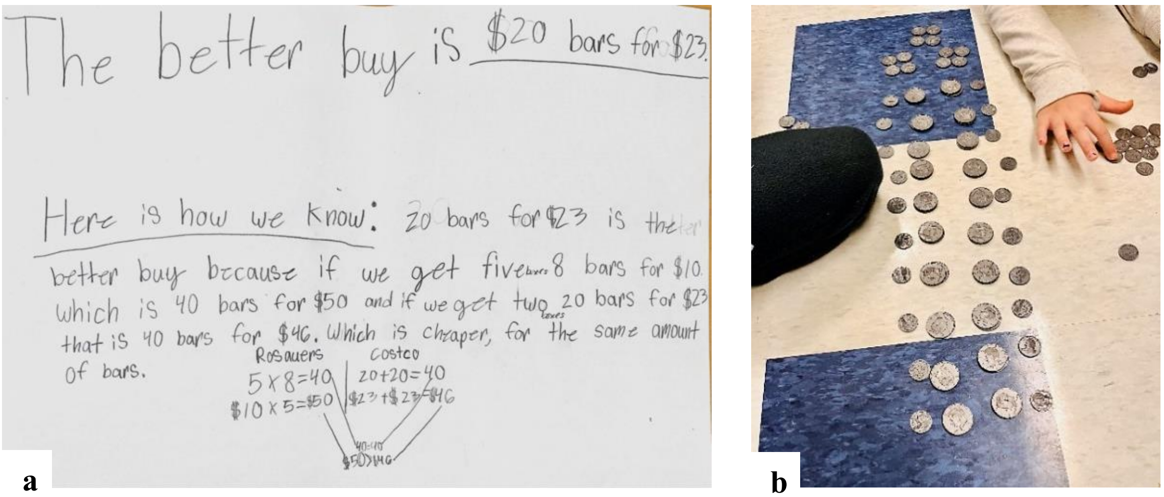
Anna evidenced noticing for equity when she stated that students would connect with the experience of buying granola bars. In the post-lesson interview, Anna said:

> I think that's what happened today. It [the Better Buy task] made sense for them [students], because that was completely a familiar context. ... "I will go buy granola bars for all of us", or "I eat granola bars every day at snack, and I need to find the cheapest ones around. So, I need to figure out what's the better deal."

To further support connections to students' out-of-school knowledge, Anna *adapted* the task by setting it in the context of grocery stores that were familiar to her students, using the names of "Rosauers" and "Costco." Anna's strategic framing of the task within the context of familiar grocery stores allowed students to draw upon their lived experiences and enabled students to apply mathematical reasoning in a real-world setting, forms of noticing for equity (van Es et al., 2022). During the lesson, Anna *launched the task* by prompting students to recall instances of their experiences with grocery shopping, deliberately facilitating connections to students' lives. Anna also asked students to share scenarios in which they might need granola bars (e.g., birthday parties, social gatherings). Anna facilitated group work and ensured that students *valued input from peers* while students shared different strategies to solve the task. At the end, students presented their strategies to the class and explained the rationale for their final decision on the best deal for granola bars. Anna viewed making posters and having a class discussion as a way for students to *learn from peers*. These three approaches (i.e., selecting and adapting tasks, launching the tasks, and valuing peer learning) aimed to ensure that all students could relate to and engage meaningfully with the MM activity.

Next, we present students' work on the task to demonstrate the effectiveness of the identified approaches in fostering meaningful engagement in the MM task. A substantial number of students drew on their personal experiences shopping at both grocery stores as they solved the task. For instance, one student confidently stated, "Of course, we all know Costco is cheaper than Rosauers." When another student asked about the reason for his assumption, he explained that he compares prices while accompanying his parents to both stores.

Students’ group work demonstrates the effectiveness of the identified approaches and illustrates additional strategies used by students while working on the MM tasks. In Figure 1 (a) students chose to equate both deals to 40 bars for comparison. Students identified 40 as a common multiple of 8 and 20, which facilitated the comparison by providing an equal basis for evaluating each deal. In Figure 1 (b), another group showed their work by comparing both prices using coin models as a visual representation to aid their understanding and decision-making process. By representing each price option with coins, they could visually compare the quantities and values to analyze the situation.



a b

Figure 1. Students’ Work

Anna reflected on the task and stressed the value of grounding mathematics in real-life contexts. In the post-lesson interview, Anna explained:

> It is good if you can give them [students] a context that makes sense for them.... There are random questions that don’t necessarily have that [meaningful context]. ... So, if [students] don’t have the background knowledge to even understand what a question is asking, then ... that’s where the hang up continues to be.

Anna’s perspective underscores the potential challenges students face when presented with abstract or disconnected mathematical problems, emphasizing the need for teaching contextually relevant tasks. She deliberately addressed this challenge by *selecting* the scenario of choosing the best buy for snacks and *adapting* it by using familiar store names. This scenario resonated with students, making MM relevant for decision-making in their daily experiences.

Conclusion and Implications

Our analysis revealed three effective approaches for elementary teachers in noticing for equity while planning and enacting MM instruction. Anna’s enactment of the “Better Buy” task exemplified how real-life scenarios can be integrated into MM, prompting students to draw from their personal experiences. The implications for research and practice stemming from this study underscore the need for equitable practices in mathematics education, particularly in elementary classrooms. For teachers, this implies a focus on purposeful task selection and adaptation, along with deliberately facilitating engaging learning environments. By integrating familiar scenarios with MM, teachers can foster a deeper understanding and appreciation for mathematics among students. At the same time, teaching MM with a focus on equity is complex work, and teachers might need support to engage in this work. Moving forward, it is essential to continue exploring and developing equity-directed approaches in mathematics education. Ongoing collaboration between researchers and practitioners can support efforts to prioritize equity and access in elementary mathematics classrooms through mathematical modeling tasks.

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DESIGNING A TOOL FOR TEACHER NOTICING FOR EQUITY IN MATHEMATICS INSTRUCTION

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In this paper, we propose a mathematics professional development tool designed to support teachers' noticing for equity and improve their ability to provide powerful mathematics and an inclusive discourse community for each and every student. Used within the context of coaching cycles, this tool serves as a reflection guide for teachers to consider the extent to which all students had opportunities and access to rigorous mathematics and a discourse community and were engaged as doers and communicators of mathematics during a lesson. Our theoretical basis for the tool and iterative design process are described, followed by an example of its utility. We conclude with a discussion of what we learned from our retrospective analysis of our process.

Keywords: Classroom Discourse, Teacher Noticing, Professional Development

The significance of teacher noticing to promote effective mathematics teaching and learning has been highlighted in mathematics education research. Definitions of noticing call for teachers to identify classroom events and act on them during instructional interactions (Jacobs et al., 2020). Teacher noticing includes teachers paying attention to student thinking and using their mathematical knowledge for teaching to make sense of students' mathematical ideas (Sherin et al., 2011). It is often conceptualized as comprising three interrelated skills: attending, interpreting, and deciding how to respond (Jacobs et al., 2010). Such definition emphasizes the dynamic nature of noticing, which goes beyond observing a moment during classroom interactions to asking that the observer (often teachers) use their knowledge to make sense of the moment and make decisions in responding (Jacobs et al., 2010; van Es & Sherin, 2021).

Noticing is not an isolated concept and has to be considered within a broader framework for analyzing instruction. Schoenfeld (2011), for example, stated "teachers' decision making is shaped by what teachers notice...But what teachers notice, and how they act on it, is a function of the teachers' knowledge, goals and orientations." Therefore, Schoenfeld claimed that noticing had to be "situated within the larger picture of teacher decision making" (p. 233). Tying noticing explicitly to equity, van Es et al. (2022) indicated that "teachers' noticing of classroom activity shapes who is invited to participate, who is valued, and whose forms of knowing are included in mathematics classrooms" (p. 114). They suggested that supporting mathematics teachers' noticing is critical for an equitable mathematics learning environment (van Es et al., 2022).

Although noticing is key for equitable instruction, not all teachers inherently engage in noticing and noticing skills do not necessarily develop solely through teaching experiences (Jacobs et al., 2020). Some teachers might learn to notice effectively in their own classrooms, but professional development (PD) can support teachers in being more attuned to this practice and cultivating understanding about what to notice in the moment and how to interpret and act on those significant moments. In their review of the literature, Jacobs and Spangler (2017) stated that PD initiatives focused on enhancing teacher noticing have shown that teachers can learn to gain in-depth insights into their students' mathematical thinking and learn to make productive instructional decisions, such as improving the quality of instructional tasks and increasing opportunities for students to engage with mathematical concepts.

This paper focuses on the question of how to support teachers' noticing for equity in mathematics instruction within the context of a PD program. In response, we share our development of a tool for teacher noticing for equity and its use in coaching cycles. We have designed the tool to support mathematics teachers and coaches' noticing for equity and improve teachers' ability to provide powerful mathematics, as well as an inclusive discourse community for each and every student. The tool also serves as a reflection guide for fostering productive coaching conversations on equitable practices in mathematics classrooms.

In what follows, we first present the PD context for which the proposed tool was developed. Then, we provide theoretical background that contributed to the design of the tool and describe the design process. We conclude by providing an illustrative example within the tool. Additionally, we suggest recommendations for and discuss the implications of the tool's utility, while also acknowledging its limitations.

Context

Beginning in 2010, Project AIM (All Included in Mathematics) received a series of grants from the National Science Foundation (NSF) to develop and test comprehensive educative PD materials (Davis & Krajcik, 2005) for K-2 mathematics. Originally, these materials comprised a core teacher PD program (AIM-PD) and a facilitator preparation program and were designed for school-team participation with local facilitators. These AIM materials were tested and revised across six implementation cycles that involved over 230 teachers and 14 facilitators. Additional components including a school-leader program (AIM-LP), a coach preparation program (AIM-CP), and materials for coaching cycles were recently developed to extend the set of PD materials.

Project AIM built on successful discourse techniques established in elementary literacy instruction to develop its 30-hour core teacher program. Throughout the AIM-PD, while teachers learn the AIM mathematics discourse techniques (Sztajn et al., 2021), they are asked to immediately integrate them into their classroom instruction. Teachers then use their experiences to examine what it takes to implement the techniques effectively and how to use them purposefully to improve high-quality mathematics discourse for their students.

By providing teachers with techniques that can be immediately implemented, the AIM-PD "flipped the script" (Guskey, 2020) on PD design to allow teachers to quickly change aspects of their practice before they reevaluate their knowledge and beliefs. The PD starts with lesson organization, moves into the implementation of purposeful AIM discourse techniques, and circles back to connect teachers' reflections on implementation to knowledge about high-quality mathematics discourse (Sztajn et al., 2021). The Project AIM team calls this quick change in aspects of teachers' practices "getting it going." Over the years, the project has demonstrated the impact of its flipped script on participants' knowledge, beliefs, and practice (Alnizami et al., 2019), reinforcing the understanding that teachers in the AIM-PD get changes in their practice going. To extend and enhance the impact of AIM-PD on teachers–or as Project AIM team refers to it, to move from "getting it going" to "getting it good"--Project AIM added the AIM-LP and the coaching cycles.

Coaching is considered a productive way to bring knowledge into practice (Joyce & Showers, 1981, 1982) and evidence of the positive impact of coaching on teaching is growing (Campbell & Makus, 2011; Russell et al., 2019). In the context of AIM, the coaching component particularly focuses on teacher noticing for equity (van Es et al., 2017) to promote transformative changes in practice. Leadership also plays a critical role in shaping teachers' practice (Cobb et al., 2003), influencing the way teachers interact with coaches and fostering the ways in which teachers incorporate PD experiences into their practices (Coburn & Russell, 2008). Thus, AIM-LP engages school leaders in establishing a shared vision for high-quality mathematics instruction and envisioning what productive mathematics classrooms look like.

Theoretical Background

The National Council of Teachers of Mathematics (NCTM) committed to approach mathematics education research through an equity lens, highlighting the growing importance of equitable mathematics learning for all students (NCTM, 2000; NCTM, 2014). This commitment significantly influenced the way mathematics educators explore equity and equitable practices in mathematics instruction. For instance, Gutiérrez (2007; 2012) provided a structured framework with four dimensions for equity in mathematics education: access, achievement, identity, and power. Access involves tangible resources for participation in high-quality mathematics; achievement focuses on tangible results; identity centers on whether mathematics is meaningful and relevant to students' cultural and linguistic background, and power analyzes voice in the classroom. NCTM's (2014) access and equity position statement highlights what equitable mathematics instruction entails:

> Being responsive to students' backgrounds, experiences, cultural perspectives, traditions, and knowledge when designing and implementing a mathematics program and assessing its effectiveness... Addressing equity and access includes both ensuring that all students attain mathematics proficiency and increasing the numbers of students from all racial, ethnic, linguistic, gender, and socioeconomic groups who attain the highest levels of mathematics achievement. (p. 1)

Moschkovich (2012) offered a synthesized understanding of equitable practices in mathematics classrooms by reviewing several works (e.g. Esmonde, 2010, Gutiérrez, 2012, Wagner & Borden, 2012) that highlighted:

> a) supporting mathematical reasoning and mathematical discourse (because we know these lead to conceptual understanding and learning);
> b) broadening participation for students from non-dominant communities (because we know that participation is connected to reasoning and learning) (p. 100).

As researchers explore what equitable mathematics instruction entails, teachers' noticing in their instruction appears essential for supporting teachers in forming equity-focused perspectives and implementing equitable practices (Hand, 2012; Turner et al., 2012). Thus, over the past decade, researchers initially explored the concept of teacher noticing, recognizing it as a fundamental construct in effective teaching (van Es et al., 2022). Research on high-quality mathematics instruction has provided evidence that noticing underpins teachers' in-the-moment decision-making as they attend to and make sense of important moments during instruction (Jacobs et al., 2010, Jacobs & Spangler, 2017; van Es & Sherin, 2021). Furthermore, researchers have provided evidence that teachers' noticing of students' mathematical thinking influences teachers' instructional practice and students' mathematics learning (Jacobs & Spangler, 2017; van Es et al., 2022).

However, noticing students' mathematical thinking and activities is insufficient for fostering equitable learning environments (Turner et al., 2012). It is imperative to extend noticing to equitable practices (Jacobs & Spangler, 2017). van Es et al. (2017) define "noticing for equity" as "how mathematics teachers observe classroom activity with consequences for whether particular groups of students feel more or less empowered to engage in mathematical reasoning" (p. 252). Others indicate noticing for equity calls for attention to aspects like classroom participation (Wager, 2014), access (Gutierrez, 2007), positionality (Esmonde, 2012; Hand, 2012), identities (Gutierrez, 2012), as well as culture and home knowledge (Turner et al., 2012).

Studies on equitable instructional practices also call for exploring teachers' noticing for equity (van Es et al., 2017). These studies indicate that equity-focused teachers notice key aspects of practice such as equitable discourse opportunities, access, how students position themselves and are positioned, the role of power, and the role of culture and language in mathematics classrooms (Jacobs & Spangler, 2017). Additionally, equity-focused teachers engage in pedagogical practices "that do not perpetuate deficit (or privileged) perspectives about groups of students" (van Es et al., 2017, p. 254).

Project AIM Perspective on Equity and Noticing for Equity

Several research studies highlight various aspects to be noticed with an equity lens, yet teachers often maintain a holistic perspective on classroom practices, leading to simplistic and general inferences based on their observations (Wager, 2014). They usually do not notice nuanced details related to equitable practices such as who engaged with the mathematics and how during the lesson (Hand, 2012; Wager, 2014). They tend to notice classroom mathematical activities as distinct from other forms of classroom participation (van Es et al., 2017). Addressing this issue, Hand (2012) suggested teachers should cultivate a lens of noticing for equity. This is essential for implementing equitable classroom practices and responding to noticing in equitable ways. To cultivate such a lens, it is critical to offer teachers both support and opportunities for "more frequent and nuanced noticing of classroom participation" as it contributes to a deeper understanding of equity issues (van Es et al., 2017).

Project AIM's coaching component supports teachers to notice equitable practices in mathematics instruction by creating opportunities for them to engage in nuanced noticing with an equity lens. We contend that noticing for equity needs to attend to both the dominant (access, achievement) and critical (identity, power) components of instruction (Gutiérrez, 2007). For us, "dominant noticing" regards teachers' use of their MKT as they learn to attend to, interpret, and decide how to respond to students' thinking (e.g., Jacobs et al., 2010). "Critical noticing," however, focuses on attending to, interpreting, and responding to equitable discourse practices. The project also focuses on including *each and every* student in mathematics discourse, attending particularly to emergent multilingual learners (Malzahn et al., 2019). It promotes instructional practices that have been shown to foster equity such as attending to the role of language and status in classroom discourse participation, positioning all students as competent, and welcoming students' home knowledge and language (Jacobs & Spangler, 2017; Wilson et al., 2019).

Design Process

The tool developed is Project AIM's reflection tool for use by teachers and coaches. Our primary focus is fostering growth and improvement in teachers' noticing for equity through the use of this tool in the coaching cycle. The design of the tool incorporated several key decisions, which emerged from the wealth of knowledge and insights derived from Project AIM, literature on noticing for equity and equitable mathematics instruction, and individual and collective reflections within the research team during three iterative design cycles.

First Cycle: Design Decisions

We began our design with Moschkovich (2012) emphasis on the pivotal role of mathematical reasoning and discourse in supporting conceptual understanding and learning alongside broadening participation for students from non-dominant communities. This resonates with Project AIM's emphasis on providing high-quality mathematics to each and every K-2 student, focusing on equitable discourse practices, and supporting teacher noticing with a focus on equity. Thus, we identified two critical aspects for the tool, adapting Moschkovich's (2012) nomenclature, and named them initially: 1) Powerful mathematics and 2) Communication/Inclusive learning environment.

The research team proposed a matrix format to unpack the actions of both teachers and students in the classroom, specifically homing in on how teachers notice these actions within each aspect. Recognizing the complexity of teaching, our tool design aims to support teachers in noticing for equity within significant moments, considering the challenge that they cannot attend to every event during classroom instruction (Jacob & Spangler, 2017). Thus, in the design process, our intentional focus is on leveraging the insights and experiences teachers acquire through the AIM-PD to identify and engage with these significant moments.

Our collective reflection meeting led us to focus on detailing the actions of both teachers and students, emphasizing noticing lenses of *opportunity* and *access* mainly in Teacher Actions, and *participation* and *power/voice* mainly in Student Actions (Column headings of the tool). These areas were chosen within the overarching context of Powerful Mathematics and Communication/Inclusive Learning Environment aspects (Row headings of the tool). Each intersection represented by a cell (See Table 1).

Subsequently, we decided to incorporate characteristics of equitable practices in a mathematics classroom as noticing actions within each cell of the tool. To identify these noticing for equity actions, the research team reviewed the related literature (e.g. Gutiérrez, 2012; Hand, 2012; Jacobs et al. 2010; Jacobs & Spangler, 2017; Munter, 2014; Moschkovich, 2012; Sherin et al., 2011; van Es, 2017) and conducted individual written reflections to address a key design question:

> If we want teachers to notice for equity as it relates to students' mathematical thinking/reasoning and student participation/communication, what specifically do we want them to pay attention to during the lesson and/or think about in planning?

Individual written responses to these questions were analyzed by a team member and organized within each cell of the tool. This process helped us pinpoint the initial noticing for equity actions within each cell. For instance, in cell of Communication or Inclusive classroom learning environment-opportunity& access (teacher actions), we included “Implementing techniques and moves to encourage and support student participation and draw on their assets” and in participation voice/power (student actions), we included “patterns of participation” as noticing actions.

Second Cycle: Design Decisions

In the tool’s second iteration, instead of presenting a mere list of noticing actions, we decided to delve deeper into each cell. This involved identifying focal actions within each cell, incorporating reflection questions, and reorganizing the placement of those actions identified in the first cycle under the new focal actions. The bolded noticing actions were purposefully selected and/or synthesized from the initial version of the tool to represent key practices that teachers pay attention to within each cell. We used Teach Math (McDuffie et al., 2014), and existing AIM-PD resources such as the Math Discourse Matrix (Sztajn et al., 2021) to identify the initial focal noticing actions.

Reflection questions were also purposefully added to help teachers reflect on or examine each focal noticing action. The reflection questions were not broad inquiries such as "what teachers notice about"; rather we created questions to promote reflection on specific qualities of equitable practices in mathematics classrooms. For instance, we unpacked the initial noticing action of "patterns of participation" by including questions like, "To what extent do the reciprocal interactions among students and teacher-student contribute to the development of shared understandings of mathematical concepts?” We placed this question under the focal noticing action of "Who participates? Where does the majority of the math 'work' take place in the classroom (e.g., front of the room, small group, individual desks) and how does this contribute to participation?" within Communication or Inclusive classroom learning environment-access & opportunity (teacher actions). These noticing actions and questions within the tool are not exhaustive, but represent key points of emphasis in Project AIM that address equitable instructional practices highlighted in the literature.

During collective reflection meetings, we decided to fine-tune the tool for better accessibility and user-friendliness for teachers and coaches.

Third Cycle: Design Decisions

In this cycle, we refined key elements of the tool, including the renaming of aspects and revising focal noticing actions, as well as revising the reflection questions. In the first cycle, we initially named the second aspect of the tool as "Communication/Inclusive learning environment." However, given Project AIM's commitment to equitable discourse practices, and the acknowledgment that all actions emphasized in the tool contribute to an inclusive learning environment, we decided to rename this aspect "Inclusive Discourse Community."

We decided to use language in the reflection questions to prompt thoughtful consideration rather than evaluation or yes/no answers. Specifically, we chose to frame questions with prompts such as "in what ways" and “how” rather than "to what extent" or “do/does.” For instance, instead of inquiring about “to what extent are students encouraged/expected to ask each other questions that press for mathematical reasoning, justification, and connections among ideas?”, we revised the question to "How are students encouraged/expected to ask each other questions that press for mathematical reasoning, justification, and connections among ideas?"

While "how" questions can effectively encourage teachers in noticing their actions, the same approach for student actions lacks the specificity needed for teachers to notice which students engaged in powerful mathematics and discourse. Consequently, we made the decision to rephrase the questions related to student actions, incorporating both "how" and "which/who." For example, in powerful mathematics-participation & power/voice cell, under the "Engage in math reasoning for sense-making of concepts " focal noticing action, we modified the question to be more targeted: "Which students are making meaningful connections between procedures and concepts? How?"

Table 1 presents the short version of the final tool (without the reflection questions) as a result of the three iterative design cycles.

Table 1: Tool for Teacher Noticing for Equity: Short Version

	Opportunity & Access (Teacher Actions) *Notice what the teacher does to create equitable opportunities and access to powerful mathematics and discourse.*	Participation & Power/Voice (Student actions) *Notice who is engaged and how in powerful mathematics and discourse*
Powerful Mathematics	● Selects a challenging, discourse promoting task ● Implements the task to maintain challenge while building on students' prior knowledge ● Promotes student thinking and sense-making Capitalizes on students' mathematical, cultural, and linguistic assets as resources for learning	● Engage in problem solving for conceptual development ● Engage in math reasoning for sense-making of concepts ● Utilize prior knowledge, resources, and assets to understand and engage with the math
Inclusive Discourse Community	● Attends to socio-mathematical norms for engaging in high-quality and equitable math discourse ● Values, encourages, and supports multiple modes of communication ● Creates structures to encourage and support broad participation ● Promotes academic language	● Use multiple modes of communication to share and justify their math thinking and reasoning ● Critique others' thinking and reasoning and form math connections among ideas ● Take ownership/responsibility for the math discussion ● Collaborate with and utilize peers as resources to make sense

	of each other's thinking and the math ideas

In this paper presentation, we also unpack the inclusive discourse community aspect of the tool. We share the reflection questions related to focal noticing actions in the cell and discuss the utility of the tool. For instance, for the focal action "Use multiple modes of communication to share and justify their mathematical thinking and reasoning," the reflection question was "Which students are representing a concept or solution using one or more modes in addition to language – gestures, writing/drawing, technology, concrete objects, mathematical symbols? How?"

Discussion and Conclusion

Our work as designers of Project AIM has repeatedly led us to appreciate the power of tools that bring concepts from research into teachers' professional learning and support aspects of their professional practice. That is, the tools form a foundation for learning and improvement by providing practical ways of engaging with new ideas in the PD setting and applying those ideas in teachers' school and classroom experiences. We consider these tools to be boundary objects (Sztajn et al., 2014) spanning this territory of research, professional learning, and teaching practice. We were intentional about which aspects and actions to highlight, knowing that a tool can't include everything and still be practical and useful. We spent considerable time deciding which actions to include in the tool and engaged multiple design cycles to avoid overwhelming users.

In our current work, we set out to create a new tool to support coaches and teachers who will engage in coaching cycles for a year following their experience in the existing AIM-PD. Undertaking this work allowed us to be attentive to and reflective on our process for creating the kind of boundary objects we have found supportive for bridging research, professional learning, and teaching practice. Although we did not set out with a defined process for this work, in retrospect, we discovered three identifiable cycles in our design work that mirrored the territories we intended for the tool to bridge. This intention includes (1) drawing on research generated concepts; (2) providing teachers access to these concepts in ways that meaningfully connect to their experiences; and (3) engaging teachers in applying these concepts to reflect individually and collectively on their learning and how it informs their teaching practice.

We found that the three design cycles we identified in our work largely align with these three intentions. The First Cycle aligned with the first intention. We began with powerful concepts from research literature to define the overall domain, Noticing for Equity (Jacobs & Spangler, 2017, van Es et al., 2017) and the space of the tool as a two-by-two matrix crossing aspects of equitable mathematics instruction (Moschovich, 2012) with actions to notice which tie to dimensions of equity (Gutierrez, 2007, 2012). Finally, we identified multiple actions in related research and located them in this space.

The Second Cycle centered on the second intention, providing teachers and coaches access to the many concepts incorporated in the tool. Two key steps made up this cycle: organizing actions into a smaller number of categories recognizable to teachers, and framing these actions within questions teachers and coaches could consider, answer, and discuss when examining specific instances of their own practice. We drew on tools with similar purposes from our own (Sztajn et al., 2021) and colleagues (McDuffie et al., 2014) work to inform these steps.

Finally, in the Third Cycle we addressed the third intention of promoting teachers' and coaches' application of the tool. Two main decisions arose in this cycle, renaming one aspect of equitable mathematics instruction (Inclusive Discourse Community) to more closely align it with what would be familiar to teachers when examining practice in AIM-PD, and rephrasing the reflection questions to promote thoughtful and critical reflection on instances of practice rather than evaluative or binary (yes/no) judgments. This design decision was crucial for the tool's accessibility and usability, reinforcing that it represented familiar ideas reorganized and unpacked to highlight aspects of equity.

This retrospective examination of our work uncovered a process for developing tools as boundary objects for professional learning to improve mathematics instructional practice, specifically for equitable practice. Reconsidering and recounting our experience allowed us to codify for ourselves and offer to other developers specific intentions and actions for engaging in and studying the design of tools for mathematics teachers' professional learning experiences.

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EXAMINING THE RELATIONSHIP BETWEEN CLASSROOM OBSERVATIONS AND NATIONAL BOARDS COMPONENT SCORES DURING INSERVICE PROFESSIONAL DEVELOPMENT

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Alabama's Practitioner Leaders for Underserved Schools in Mathematics (APLUS in Math), funded by the National Science Foundation, is a group of secondary (grades 6-12) mathematics teachers recruited from 20 schools across 10 school districts, creating two cohorts of Master Teacher Fellows to become masters of instruction and teacher leaders. Utilizing two measures with validity evidence, we examine the relationships between classroom observation data on teaching practices collected at multiple points of the school year and the National Board for Professional Teaching Standards (NBPTS) portfolio rubric scores obtained during the same year as the observation data. Significant results include strong relationships between classroom observations and specific portfolio results. We highlight validity evidence and measures with findings towards accomplishing the project's overarching goals to improve instructional quality, student experiences, and school change.

Keywords: Professional Development; Instructional Activities and Practices; Teacher Educators; Instructional Leadership; Assessment; Instrumentation, Validity Evidence

Since 2000 on a macro-scale, the teaching and learning of mathematics in the United States has been guided primarily by the National Council of Teachers of Mathematics (NCTM). The Standards for Mathematical Practices [SMPs] (NGA & CSSO, 2010) rooted in the NCTM mathematical processes (NCTM, 2000), along with the Mathematics Teaching Practices [MTPs] (NCTM, 2014) present opportunities for dynamic mathematics classrooms. Teacher professional development (PD), at the micro-level, has tended to reflect these practices as higher priorities for the last decade-plus. Classroom teachers that incorporate the Mathematics Teaching Practices and engage students in the Standards for Mathematical Practices in concert during instruction can, arguably, be described as providing an effective environment for the teaching and learning of mathematics for all students. Moreover, mathematics classrooms that regularly demonstrate the MTPs and SMPs in concert provide ideal clinical settings for field-based experiences of teacher candidates (TCs) as they work towards establishing productive dispositions and early career teaching practices. This brief research paper focuses on the early analysis and findings of the relationship between live observation ratings of secondary teachers using the Mathematics Classroom Observation Protocol for Practices [MCOP^2] and the National Boards' Component 2 (Differentiation of Instruction) and Component 3 (Teaching Practice & Learning Environment) portfolio scores. National Boards is much more related to affective domains of teacher beliefs, attitudes, and self-efficacy, whereas live observations of instruction are actual teacher behaviors. There has been conflicting evidence as to whether teacher espoused beliefs about their teaching of mathematics are their actual enacted practices and behaviors (Buehl & Beck; 2014; Polly et al., 2013). We wonder the degree to which National Boards results align to that of the MTPs and SMPs classroom enactment.

Foundation and Framework

Measurement Framework

Over the past decade, nearly 50 mathematics education researchers have assessed the validity of measures in mathematics education. A subset of this group analyzed the validity evidence of affective and behavior "teacher education instruments" published in 24 mathematics education field journals. Within the domains of affective and behavior measurement, it has been found that a significant portion of these instruments lacked strong validity evidence or in some cases no evidence (Gallagher et al., 2022).

The MCOP^2, developed by Gleason et al. (2017), evaluates student engagement (SE) with Standards for Mathematical Practice (SMPs) during classroom instruction, as well as teacher facilitation (TF) related implicitly to the NCTM Mathematical Teaching Practices (MTPs). Gleason et al. (2017) published validity evidence for content, reliability, response processes, and internal structure. Later, Zelkowski et al. (2016, 2024) have published addition evidence for the consequences of use, and relationships to other variables strengthening the validity argument for using the MCOP^2 with live observations of classroom mathematics teaching to capture the two factors of interest (i.e. SE and TF).

The National Professional Boards for Teaching Standards (NPBTS) has witnessed a significant rise in nationally board-certified teachers (NBCTs), with states offering salary incentives for certification and two to three times as much for teaching in high-need and/or hard-to-staff schools (Gooden et al., 2023; Darling-Hammond & Sykes, 2003). States, districts, and schools have developed support mechanisms, professional development, and writing peer mentorship groups as an effort to support teachers in their pursuit of such certification. The NPBTS assesses four domains, including affective and behavior dimensions (Components 2, 3, 4) and content knowledge (Component 1), through established rubrics with some demonstrated validity evidence in scoring (NBPTS, 2016) yet virtually no empirical published research connected to the degree to which NBCT mathematics teachers demonstrate the MTPs and their students engage in the SMPs.

Professional Development

A professional development [PD] program requires extensive planning for design, implementation, and assessment to make change as needed to maximize the program's goals (Penuel et al., 2007; Kennedy, 2016; Loucks-Horsley et al., 2009). Some of the five goals of the APLUS in Math project include teachers reaching "teacher leadership" status through ongoing PD to become masters of instruction, leading PD, mentoring other teachers and TCs. The project is a five-year, three-phase designed PD project. Phase-1 consisted of 15 months of foundation graduate coursework (18 credit hours) in content knowledge, pedagogy, and teacher leadership, whereas Phase-2, the target phase of this research, consisted of professional workshops and pursuit of National Boards over two academic years [AY] as a means to "master instruction" demonstrating regular use of the MTPs and student engagement in the SMPs. Phase-3, not the focus of this report, includes the implementation of teacher leadership projects.

This initial research focuses on understanding the relationship between observed instructional practices (SE, TF) with the MCOP^2 and National Boards' differentiation of instruction (Component 2) and teaching practices and classroom environment (Component 3). This early work aimed to inform adjustments for future professional development in the project, improve instructional practices, and guide national board workshops while looking for additional validity evidence related to the MCOP^2, National Boards, the MTPs, and the SMPs.

Our research question: What is the relationship between MCOP^2 SE and TF observation ratings and National Board Component 2 and 3 portfolio scores?

Methodology

To assess the impact of Phase-1 and Phase-2 PD, three MCOP^2 classroom observations are collected during each AY spread apart by about 2-3 months. According to Gleason et al. (2017), three observations of a teacher provide enough data for an annual measure of SE and TF (previously discussed). During year 1 of the project, baseline observations were conducted in the 2020-21 and again in year 2 (2021-22) when MTFs were working on Component 2 in the fall and Component 3 in the spring. In December of 2022, results of both components of the first cohort of MTFs were received which are reported in this paper.

We conjectured that baseline year MCOP^2 SE and TF scores would not be significantly related to Component 2 and 3 scores given they occurred in different school years, whereas we further conjectured that the SE factor score would be more likely related to Component 2 [differentiation] and that the TF factor score would be more likely related to Component 3 [teaching practices] given the foundation and framework discussion during the year of teachers working on national boards. We performed Spearman correlations given the interval level scoring of NB rubrics and then followed that with regression analyses to account for highly correlated independent variables with Component 2 and 3 as the dependent variables (interval scored, a caution to consider), with the MCOP^2 SE, TF, total score ratings, and then MCOP^2 item level ratings as the independent variables using a stepwise forward procedure to understand the variance explained by the observation measures on the national board's performance level. Eleven MTFs had Component 3 scores, and ten MTFs had Component 2 scores. All MTFs had year 1 and year 2 MCOP^2 observation ratings.

Results

Spearman Correlations two-tailed tests were significant, in order, as follows ($p<0.05$):

COMP2 ~ [Yr 2 MCOP^2 item 13, total score, SE score, TF score]

COMP3 ~ [Yr2 MCOP^2 item 7, 6, 4, 13, total score, TF score]

All MCOP^2 items were highly significant in relation to each other (rho>0.7). To address multicollinearity and to understand the better potential predictors, the secondary analyses found significant regression equations explaining 58.3% and 59.6% of the variance respectively of Component 2 and 3 based on MCOP^2 observation recorded data.

EQ1: COMP2 (rubric rating) = 1.653 (MCOP^2 SE Factor Score in year 2)

EQ2: COMP3 (rubric rating) = 1.485 (MCOP^2 TF Factor Score in year 2)

The constants were non-significant in the models whereas the MCOP^2 coefficients were significant ($p\leq 0.01$). Year 1 MCOP^2 factor scores were non-significant.

After recognizing the two factors that were statistically significant for components 2 and 3, we conducted additional analyses of the respective factor score items from the MCOP^2 in relation to the component scores. The results indicated one TF item and one SE item were statistically significant in predicting the associated component score. However, the results were surprising which we discuss. The relationships explained 51.2% and 62.0% respectively where:

EQ3: COMP2 (rubric rating) = 0.847 (MCOP^2 Item 13 rubric mean from year 2)

EQ4: COMP3 (rubric rating) = 1.429 (MCOP^2 Item 6 rubric mean from year 2)

The constants were again non-significant in the models whereas the MCOP^2 coefficients were significant ($p<0.02$). Year 1 MCOP^2 items on each respective factor were non-significant.

Lastly, we were curious as to whether there were any relationships to the content knowledge assessment (component 1) with respect to the MCOP^2 observational data. Again surprisingly, there was one MCOP^2 item with a significant relationship to the total component 1 score, item 7 was significant, the modeling focused item.

EQ5: COMP1 (rubric mean score) = 2.578 + 0.632 (Item 7 rubric mean from year 2)

None of the sub-scores (exercises 1, 2, or 3, and multiple response) on Component 1 had significant predictors in the data, but the total component score generated a significant model.

The Spearman Correlational analyses initially found the items that were statistically significant in relationship to NBCT component scores. Given the high correlations of multiple variables (rho>0.5), we then explored which variables are the greater predictors to understand which were most related in understanding NBCT related scores. The MCOP^2 scoring can be interpreted as follows. The mean SE and TF factor scores ranging 2.5-3.0 would be "excellent" with 2.0-2.49 as "very good". Further, 1.5-1.99 would be "good to average" and less than 1.5 is below average teaching practices. Based on EQs (1, 2, 3, 4, 5), we find that Component 2 is strongly related to the level of student engagement observed throughout the year's observations during initial NBCTS submission. The model explains nearly 60% of variation in the outcome scores. Important to note, a mean score of ~1.8 on the MCOP^2 SE factor is related to a Component 2 score of 3 (clear evidence of accomplished teaching). This is a lower score than anticipated. With the differentiation component mostly about how teachers plan to engage all students in their planning and enactment and a purely written component of portfolio artifacts, it is not so surprising in hindsight. Component 3 though tells a different story. A mean MCOP^2 TF factor score requires slightly more than a 2.0 to warrant clear evidence of accomplished teaching on Component 3 (teaching practices). This is an expected score range.

Discussion

In making early programmatic decisions about our PD design, these findings allow us to make some initial points related to the PD design (Penuel et al., 2007). Our initial findings with the first cohort of MTFs demonstrates significant relationships even with a small sample size less than 15 teachers. However, by going even deeper into the item-level analyses, we see the importance of a "conceptual teaching focus" with the TF item 6 on teaching practice construct outcomes (need >2.0 mean on item 6), as well as the SE item 13 on differentiation of instruction to engage students with a high-quality classroom culture (need ultimately a 3.0 mean) to warrant accomplished teaching for the differentiation construct. That is not to say that the other Spearman significant correlations are not worth considering. Items 4&7 are also important. Item 4 focuses on both TF and SE factors, pointing to teachers who facilitate lessons with a focus on students critically examining mathematical strategies. Item 7, modeling, is also important as we found a relationship to content knowledge with a post-hoc exploration. We were not initially examining the Component 1 content knowledge relationship to MCOP^2 observations, it was secondary that we found some significance with the modeling item 7 though a similar finding exists (Yenmez et al., 2017).

We believe this finding begins to shed potential that modeling may be more present in teachers' classrooms with greater content knowledge. We will be interested to see if this finding holds up with the second cohort when data collection is available May 2024. Holistically, there were also TF, SE, and total MCOP^2 scores that were important. We view this holistically that most items are important on each factor, but certainly as our data set grows, we expect to see more granular findings in future research analyses. We would need further analyses, interviews, and a larger data set to fully begin to draw some conclusions.

Acknowledgments

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EXPLORING THE INTERPLAY OF PROOF VALIDATION AND CONSTRUCTION: A STUDY OF STUDENTS' RATIONAL BEHAVIOR

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This study explored the transformability of student rational behavior between proof validation and construction in a university-level online transition-to-proof course. Through a combined "proof validation-construction" activity, this study investigated how students recontextualized their rational behavior from proof validation to subsequent proof construction activities according to Habermas' three components of rationality. The results emphasized the need for additional support and practice in proof construction, extending beyond proof validation, to adequately fulfill the requirements of epistemic and communicative rationality. The study also demonstrated the positive impact of proof validation on proof construction, emphasizing the interconnectedness between two proof activities.

Keywords: Reasoning and Proof, Research Methods, Instructional Activities and Practices, Design Experiments.

Validating arguments and constructing mathematical proofs are essential skills for all mathematicians, and they constitute vital components of advanced mathematics courses in higher education. Proof construction is the process of developing correct proofs and proof validation involves evaluating these proofs to assess their correctness (Selden & Selden, 2017). Research has shown that most university students have experienced substantial difficulties with proof validation and construction (Alcock & Weber, 2005; Selden & Selden, 2003; Sommerhoff & Ufer, 2019; Weber, 2010). Although proof validation and construction are distinct activities, with construction involving the identification and application of theorems and definitions and validation dealing with the understanding and verification of these elements (Selden & Selden, 2017), research indicates that the two activities are closely connected. For instance, Kirsten and Greefrath (2023) investigated different kinds of validation activities (e.g., reviewing, rating, etc.) used by 11 undergraduates in constructing a proof. They found that students' difficulties in constructing proofs may stem from insufficient validation strategies and a lack of suitable criteria for accepting their own proof constructions. Powers et al. (2010) found that students learn written proofs better by providing them with opportunities to validate proofs in an abstract algebra course. Therefore, some researchers have proposed that proof validation and construction should be taught together for better learning (Kirsten & Greefrath, 2023; Powers et al., 2010; Stylianides & Stylianides, 2009) because "constructing or producing proofs appears to be inextricably linked to the ability to validate them reliably, and a 'proof ' that could not be validated would not provide much of a warrant." (Selden & Selden, 2015, p. 9).

Although it is suggested that proof validation can influence students' performance on proof construction, limited studies have explored the specific knowledge or skills acquired through proof validation that mediates this effect. The present study implemented a combined "proof validation-construction" activity to explore how students' rational behavior (cf. Habermas, 1998) was recontextualized from proof validation to subsequent proof construction activities according to Habermas' three components of rationality.

Theoretical Framework

Habermas' (1998) construct of rational behavior has been adopted by a few researchers (e.g., Boero, 2006; Boero & Planas, 2014; Morselli & Boero, 2011; Zhuang, 2020, 2023; Zhuang & Conner, 2018, 2020, 2022) into mathematics education as a theoretical framework for studying a variety of mathematical activities, including argumentation, modeling, proving, and problem-solving process. In these studies, Habermas' construct was used to analyze students' or teachers' rational behavior according to the three rationality components: epistemic rationality (focuses on the knowledge that use and play with); teleological rationality (focuses on how strategically choose tools or means to achieve the goal of the activity); and communicative rationality (focuses on the use of language toward reach understanding in the given community). The proof is a specific kind of mathematical activity in which students are expected to strategically choose specific tools to achieve the validity claims (teleological rationality) on the basis of mathematical rules, theorems, axioms, and principles (epistemic rationality), and communicate in a precise way with the aim of being understood by the given community (communicative rationality), which corresponds to Habermas' elaboration about rationality requirements in discursive practices (Boero et al., 2010; Boero & Planas, 2014). In order to analyze students' rational behavior in applying proof methods within set theory, as examined in this study, the requirements of the three components of rationality were defined as follows, adapted from Habermas' (1998) construct of rational behavior:

Epistemic rationality. Accurately validate or apply set theory definitions or operations (e.g., set difference, set inequality); ensure the sufficient and correct use of reasons and warrants in developing mathematical arguments.

Teleological rationality. Accurately validate or apply proof methods (e.g., proof by contradiction and contrapositive); ensure the use of proof methods is strategic in achieving the goal of the proof.

Communicative rationality. Accurately validate or write proof statements using correct mathematical language, visual representations, or symbolic notations; ensure a clear and concise proof structure without any irrelevant or distracting elements.

Methodology

This critical case study (Yin, 2018) focused on a cohort of mathematics graduate students from an online transition-to-proof course. Students' proof skills were exercised through a review of proof methods and their applications to set theory. The main objective of this course was to equip students who were pursuing a master's degree in mathematics with the proof skills needed before they proceed to advanced graduate-level mathematics courses.

In the initial weeks of the course, the instructor introduced the Proof Validation Framework (PVF; for more details, see Zhuang, 2023) and guided students in using it to validate proofs based on the three rationality requirements. Following this, students were expected to use the PVF to complete the weekly Proof Writer's Workshop (PWW) assignment individually, which required them to validate at least one purported proof common to the new proof methods covered in the course. Then, students were assigned to complete a weekly Proof Written (PW) assignment, which contains proof problems similar to the ones in their PWW assignment. The data of this study consisted of the written responses of 16 participating students to one of their weekly PWW assignments and PW assignments (see Figure 1), focusing on the week when they were introduced to proof by contradiction and proof by contrapositive methods. This study employed a microanalysis approach (Corbin & Strauss, 2015) to examine the written responses provided by each participating student.

Proof Writer's Workshop

> **Theorem 1:** Suppose $A \subseteq B$ and C is any set. If $x \in A - C$ then $x \in B - C$.

Proof. We will give a proof by contradiction. Suppose for contradiction that $x \in A \setminus C$ but $x \notin B \setminus C$. From the definition of set difference, this tells us that $x \notin B$ and $x \in C$. But $x \in A \setminus C$ tells us that $x \in A$ and $x \notin C$. It is impossible to have $x \in C$ and $x \notin C$, so we have a contradiction. Therefore the theorem statement is true. □

Proof Written Assignment

1.1 Suppose $A \subseteq B$, $a \in A$, and $a \notin B - C$. Prove by contradiction that $a \in C$.

1.2 Prove that if $A - B = A$, then $A \cap B = \emptyset$.

Figure 1: Proof Writer's Workshop (PWW) and Proof Written (PW) Assignment

Results

Proof Writer's Workshop

The results showed that out of the 16 students, 14 students were assigned either a revised or a failed grade to the purported proof presented. Among 14 students, 11 correctly identified errors about the definition of set difference (epistemic rationality), with some offering corrections, "using the definition of set difference and DeMorgan's [law] it should be $x \notin B$ or $x \in C$" or by offering a counterexample. Three students mistakenly believed that the writer accurately stated the definition of set difference. Of eight students who focused on teleological rationality, half correctly assessed the use of the proof by contradiction, while the others had an incorrect understanding of assumptions used for proof by contradiction. One student suggested that a direct proof method could potentially reduce the length of the proof. A consensus was reached on the clarity in writing of the purported proof for eight students who attended the communicative rationality component.

Proof Written Assignment

Four students struggled with epistemic rationality in exercise 1.1, failing to justify their arguments using existential quantifiers. One student wrote, "$a \in B$ contradicting $a \notin B$" but did not provide adequate warrants for where the assertion "$a \in B$" came from. All students successfully applied the proof by contradiction method with correct assumptions (teleological rationality). Several students included unrelated or redundant points (e.g., restating the definition in a general sense) in their proof writing (communicative rationality). For exercise 1.2, it was observed that 8 students failed to meet the requirements of epistemic rationality. Many students struggled to interpret the set inequality "$A-B \neq A$" (epistemic rationality) when using proof by contrapositive method. One student wrote, "We know that $x \notin A-B$, but since $x \in A$, we can conclude that $A-B \neq A$" without explaining why the exclusion "$x \notin A-B$" held, showing a gap in their logical reasoning. Regarding the teleological rationality, 14 students established the correct assumptions and worked logically toward the final claim based on the proof method they had chosen. One student failed to employ any proof method, and another student started with proof by contrapositive but lost track of what needed to be proven in the end. A few students presented unclear statements and constructed their proofs using lengthy sentences which made the proof harder to follow (communicative rationality). Table 1 presents students' rational behavior within PWW and PW assignments. The table provides the number and percentage of students who satisfied the requirements of each rationality component, taking into account the varying total number of students attending to specific rationality component in the PWW assignment.

Table 1: Students' Rational Behavior within Proof Writer's Workshop (PWW) and Proof Written (PW) Assignment

Components of Habermas' rationality	Number and percentage of students who met the requirements of rationality					
	PWW assignment		Exercise 1.1 (PW assignment)		Exercise 1.2 (PW assignment)	
	n	%	n	%	n	%
Epistemic Rationality	11	79	10	63	8	50
Teleological Rationality	4	50	16	100	14	88
Communicative Rationality	8	100	9	56	8	50

Note. The number of students counted in the PWW assignment may vary depending on how many students attend the specific rationality component.

Discussion and Conclusion

Table 1 illustrates a decline in the percentage of students meeting the requirements for epistemic and communicative rationality from proof validation to construction. It showed that merely knowing definitions or theorems (epistemic rationality) and checking languages or notations (communicative rationality) in proof validation did not guarantee students meet the requirements of these rationality components in subsequent proof construction exercises. The results reveal that proof construction is more challenging than validation, supporting the argument made by Selden and Selden (2017). On the other hand, the results of this study suggest that students' rational behavior in the three rationality components can be partially transferable from proof validation to proof construction. While some students did not fully satisfy the requirements of rationality in proof construction, they demonstrated the ability to bring accurate definitions of set difference (epistemic rationality) and to apply the proof by contradiction method (teleological rationality) learned from proof validation. These results support the previous studies (e.g., Kirsten & Greefrath, 2023; Powers et al., 2010; Stylianides & Stylianides, 2009; Yee et al., 2018) in underscoring the interconnectedness between proof validation and construction and the benefits of fostering them together. Further research is needed to continually explore the interrelationships between two proof activities and the advantages of fostering them together.

Drawing on Habermas' (1998) construct of rational behavior, this study examined the transformability of students' rational behavior between proof validation and construction. The adaption of Habermas' construct enables a thorough examination of students' attention, interpretation, and response to the three components of rationality, which help us better understand students' cognitive processes and abilities in these proof activities. Sometimes, students may arrive at a correct judgment while providing incorrect interpretations in terms of rationality components, as found in this study. Examining students' rational behavior in proof activities can offer valuable information for gaining insights into their understanding of mathematical concepts, strategies, and language.

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Chapter 15:

Technology and Learning Environment Design

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

AFFORDANCES AND LIMITATIONS OF PRISMS MATH VR SIMULATIONS FOR STUDENTS' MATHEMATICAL REASONING ABOUT RATIO AND SLOPE

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Virtual Reality (VR) provides unique learning experiences that allow students to immerse themselves in three-dimensional environments where they can explore mathematical ideas. Prisms VR is a suite of VR simulations for mathematics that allows students to have simulated real-world modeling experiences while working through topics such as ratios and proportionality. We examine affordances and constraints of VR for mathematics learning, focusing on manipulating and switching attention between different dynamic representations of mathematical principles, some of which involve immersion. Findings showed that some students enjoyed the immersive experiences and were able to be successful moving between the representational forms in VR, while others struggled with the overwhelming environment.

Keywords: Cognition, Learning Theory, Technology, rational numbers, proportional reasoning

Introduction

Virtual Reality (VR) in math education provides learning opportunities for students through a multimodal approach, using body movements, hand gestures, and perspective-taking to interact with objects in virtual worlds. VR is when a user is transported to a fully virtually-rendered world, often by wearing a headset that completely overlays their visual field with dynamic digital graphics. There is currently a lot of enthusiasm about the possibilities of VR for mathematics education, with many meta-analyses highlighting VR's positive results on learning (Cao, 2023; Villena-Taranilla et al., 2022; Yu, 2023). Prisms Math (Prisms of Reality, 2020) is the first widely-available and relatively-affordable suite of VR mathematics simulations for middle and high school, intended for use at scale in schools. But research on this program has again focused on its positive effect on pre- and post-scores (WestEd, 2023), without an accounting of what happens as students engage in mathematical reasoning in these environments. Our research question is: What are the affordances and constraints of immersive VR simulations for middle school students' mathematics learning?

Theoretical Framework

Contemporary VR systems can allow students to interact with virtual objects using their bodies to manipulate and move around the objects. This capitalizes on embodied views of the nature of cognition (Lakoff & Núñez, 2000; Nathan, 2021), which posit that all conceptual knowledge is understood and experienced through the body and is action-based in nature. Embodied views of mathematical cognition often give rise to learning pedagogies where students engage in perceptual, sensorial, and motor activities to deepen their understanding of mathematical ideas (e.g., Abrahamson & Sánchez-García, 2016; Dimmel & Bock, 2019; Smith et al., 2014), including through gestures.

Here we examine three affordances of VR highlighted by Dimmel et al., (2021) – these include that spatial inscriptions in VR can be: (1) viewed from different visual perspectives, (2) explored at different scales, and (3) interacted with in three spatial dimensions. When students interact with 3D objects in VR, they do not only move their hands but their entire bodies as part of their perceiving processes (Bock & Dimmel, 2021; Dimmel et al., 2021).

In terms of constraints of VR, Walkington et al. (2021) found that mathematics teachers highlight their technical and logistic limitations, as well as issues with behavior management and content coverage. In another work, Ke and Carafano (2016) conducted a mixed-methods study where the authors explored an immersive, flight-simulation learning game for high school students. The found that the more immersive simulation generated distractions and sensory overload, potentially hindering learning, with additional challenges in physical setup.

Methods of inquiry

Participants and Procedures

Participants included 9 middle school students who were participating in a district summer school. The students consented to participate in a VR summer tutoring program from a larger study. The students self-identified three as male and six as female; six identified as Hispanic and two as African American. Students wore the Oculus Quest VR headset for 1-4 afterschool tutoring sessions that lasted 30 minutes – 1 hour. They played the VR math game “Prisms Math” while being supported by a trained math tutor.

Within Prisms Math, students were given two simulations to work through. Each simulation started with an interactive immersion activity (Table 1: Immersive Satellite, Train). In the first simulation of ratios, students attempted to hit the “send” button to send location and time data back to Earth from a satellite. In the second simulation of rate of change, students were stopping the train but pulling down the brake lever to calculate “thinking distance.” In both simulations, after the initial immersive experience was complete students were brought to the “lab” to figure out if the satellite was in its orbit in the ratios simulation and the rate of change train simulation to calculate various types of weather conditions (Table 1: tasks Lab Satellite, Lab Train).

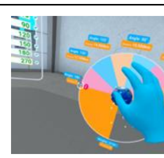
Description	Image	# of Ss Reached
Immersive Satellite: Students are in a control center and manipulate where a satellite goes with their hands at different angles, learning about ratios.	 	9
Lab Satellite: Students are moving the satellite to find the distance traveled for multiple measures such as ½ and ¾ of the orbit. They use this information to calculate the unit rate.	 	9
Immersive Train: Students are in a moving train roleplaying being a train conductor. The goal is to stop the train with minimal reaction time.	 	5
Lab Train: Students compared the thinking distance of a train conductor through three types of weather to see if the weather had an impact on the time it takes to stop the train.	 	5

Table 1: Descriptions of the Simulations Used in Prisms Math

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

Data collection and analysis

Participants took a pre-survey. They used the Meta Oculus 2 VR goggles through four sessions working through the Prisms simulations of *Ratios* and *Rate of Change*. After each session, students were given post-surveys regarding their experience with Prisms and the VR headset. Participants were also video-recorded and their videos coded into three categories. The first category was *manipulation of the satellite simulation*, or the number of times students moved the satellite to show the satellite's distance traveled and fill in the table with the correct distance. While doing this a few times showed engagement and immersion in the VR context, we saw that doing this many times indicated struggle. For the second (*switching representations in satellite simulation*) and third (*switching representations in train simulation)* categories, in both simulations, students had to move within the visual field to four sections: a data table, an interactive simulation of the satellite or train, a hint button with a hint displayed, and an answer tab where they fill in their answer. This movement to complete the task was counted, as again we saw that having to switch rapidly between these 4 different complex and immersive representations signaled difficulty.

Next, videos were coded using thematic analysis (Clark & Braun, 2017) where we looked broadly at affordances and constraints of the VR. For this coding, we focused in on elements of what was occurring that were *unique* to the environment being VR – that would that have been as likely to happen if the same simulation was on a flat screen. Themes were determined inductively, and multimodal analysis (Walkington et al., 2023) was then conducted for each theme to look at students' functional actions, speech, and gestures in the VR environment.

Results

In our results, we give one emergent theme for the affordance of VR for mathematics learning and two emergent themes for the constraint of VR for math learning. Our theme for the affordances of VR was that the simulations allowed some students to successfully *immerse themselves in a real-world scenario using 3D visualizations*. In Figure 1, the student manipulates the satellite by moving the handle to calculate distance. As they moved the handle the Virtual Assistant in the simulation filled in the table automatically. The transcript shows that this student, an eighth grade, male Hispanic, EL learner, was able to move fluidly between mathematical representations that were situated in real-world activities (i.e., the satellite moving) and more formal mathematical representations (i.e., the table and calculating the numerical answer) in the immersive environment (Lines 1-2 Figure 1). This affordance is supported by the data showing students making fewer satellite moves and left to right movements in both simulations. The students' positive response to the environment is further highlighted in their post-survey response, where they said with respect to the VR simulation: ("It is intesesing and is not boreag" (It is interesting and not boring)).

1. **Virtual Assistance (VA):** [Explains the task given to the student], *(Student moves satellite one click at a time and each time turn head to the left to check the graph being filled in with the angle. P1-P2)*
2. **Student (S):** *(student does not have 1/2 of the orbit and attempts to press on the graph. Then goes back to the satellite model and manipulates it clockwise to the ½ of the orbit which then completes the graph. P2)*

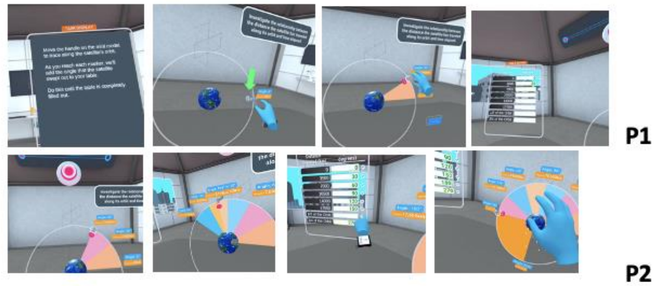


Figure 1: Using Ratios Simulation in Prisms VR

Our first theme for constraints of VR is that students *struggled to move between different representations* presented in the VR environment. The immersive simulation, along with different math representations and hints, overwhelmed some students. Those who didn't understand the satellite's role tended to manipulate it excessively. In both simulations, we observed students frequently scanning the visual field from left to right. In the post-survey answers to the question, "What didn't you know about VR that you do know?", an eighth-grade, male Hispanic student said, "That math is more hard in VR the in real life". Another post-survey question was "What did you dislike about VR?" An eighth-grade Hispanic male, responded, "I need more hints than just one."

Our second theme was that the facilitator *struggled to understand the issues students were experiencing* due to not being able to see from their immersed first-person perspective (Figure 2). Figure 2 shows the researcher communicated with the student multiple times to try to assist. However, the student struggled to explain where they were in the visual field (Line 4, P2). They also say in Figure 2, "I don't know what to divide it by" and "Maybe I can times it by 3?" showing that the facilitator was not able to provide assistance.

1. **Student (S):** OK OK so 3100 all of these are unit travel. 60 minutes so right there and I'm like right here so 12345678 9 so 9 minutes your travels we don't know. Oh, I need help. *(Student toggles around the space not sure where to start. -P1)*
2. **Researcher (R):** What part are you on? Where are you at? *(Researcher waits for student to explain)*
3. **(S):** I think the rate the amount of drivers in a minute is 3500 and I'm trying to and I don't know what to divide it by. *(Student attempts to use calculator and is not sure what exactly needs to be done so solve it. -P2)*
4. **(R):** What do you see? What does it say in the hint? *(Student doesn't share what they see, student reads the hint. -P2)*

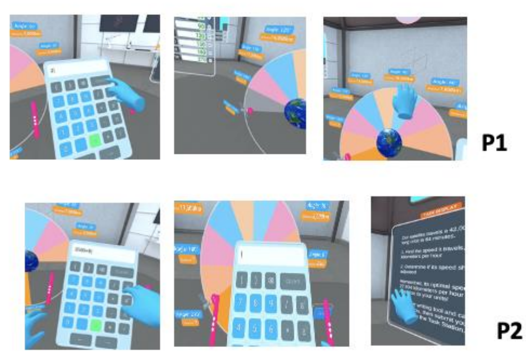


Figure 2: Communication Issue Using Ratios Simulation in Prisms VR

Discussion and Conclusion

This study analyzed rich, in-the-moment interaction data as students used a commercial VR simulation suite for secondary mathematics – Prisms Math. We found that the immersive nature of the simulations of real-world phenomena (driving a train, controlling a satellite in orbit) was a huge affordance of VR, as were the positive affective reactions students had to this environment. However, we also found that students struggled to make sense of the multiple, immersive representations of mathematical ideas in a very crowded and sensory visual field, and the facilitator struggled to provide meaningful assistance to students without access to their first-person view in the VR. One implication of this research is that additional and new considerations of how to scaffold students' mathematical learning need to be brought in VR environments. Scaffolds are needed that focus students' attention, reduce visual and sensorial complexity, and allow teachers to deeply understand what they are experiencing in their immersive environment. A second implication of this research is that future VR designs should follow Prisms Math in leveraging immersive, real-world, even fantastical situations (like controlling a satellite from a space station or driving a train), which students would be unlikely to ever experience in real life. This kind of immersion can be powerful for student engagement.

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THE BEST START: A BLOOMSIAN PERSPECTIVE ON AI-POWERED INNOVATIONS IN MATHEMATICS EDUCATION

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The field of mathematics education faces multifaceted challenges in uncertain times, and preparing learners for the future of work and participation in society requires fostering essential skills and knowledge, including quantitative literacy and mathematical reasoning. Aligning mathematics education with emerging societal trends necessitates a strategic approach, leveraging technologies like AI to enhance instruction while mitigating associated risks. This call to action uses Bloom's (1984) Four Objects of Change as a framework for discussing recommendations for AI supported designs and solutions for learners, educators, curriculum content and materials, and families.

Keywords: Early Childhood Education, Learning Theory, Policy, Systemic change, Artificial Intelligence, Technology

Introduction

The field of mathematics education presently faces a myriad of multifaceted challenges that demand a re-evaluation of our current practices and a renewed commitment to addressing persistent issues. The conference theme which asks us to consider the "*future of mathematics education in uncertain times*" urges us to confront the complexities of the present moment while charting a course towards a more equitable and effective future.

At the heart of any discussion about the future of mathematics education is an acknowledgment of present challenges. Consider that despite decades of effort and substantial investment of time, energy, and capital, we have not substantially increased the mathematics achievement of nearly two-thirds of students in the United States to grade-level proficiency (deBrey et al, 2019; NAEP, 2022). Additionally, the concept of "uncertain times" encompasses a range of challenges, including a worldwide pandemic that saw an explosion of edtech solutions and alternative schooling options, raising concerns about quality and equity, and exacerbating mathematics achievement challenges (Dorn et al., 2021; Patrinos, Vegas, & Carter-Rau, 2022). Furthermore, recent developments in AI have increased its integration into education spaces underscoring the need for more strategic foresight and critical examination of its impacts (U.S. Department of Education, 2023).

Preparing Learners for the Changing Nature of Work & Society

Preparing learners for their future involves two things: (1) understanding the future students will inhabit, and (2) facilitating educational experiences that empower learners with the requisite skills, attitudes, and knowledge to succeed in that future. Though the purpose of education has been defined in different ways over time, more recent conceptions include its framing as an economic good for both society and individuals (Locatelli, 2018; Murray, 2023). Meaning, an educated citizenry not only promotes the successful functioning of a society through economic prosperity as a whole, but also through the economic prosperity of the individual.

The future of work and society is already here. Technological advancements are changing the nature of work and society at a pace unseen in human history. Current projections show that "by

2030, activities that account for up to 30 percent of hours currently worked across the US economy could be automated—a trend accelerated by generative AI" (Ellingrud et al., 2023, p. iv). These changes predict increased occupations in the field of healthcare, STEM, and development (i.e., building and construction)--all of which require strong mathematical knowledge and reasoning. Analyses conversely estimate dramatic reductions in jobs earning the lowest wages with the least educational requirements: "Automation's biggest effects are likely to hit other job categories. Office support, customer service, and food service employment could continue to decline" (Ellingrud et al., 2023, p. iv). These changes are coming fast, with workers in the lowest wage quintiles (i.e., pay ranges of $30K to $38K) 10 to 14 times more likely to need to change occupations by 2030 (Ellingrud et al., 2023).

Work is not the only aspect of adult life that is changing; the past few years have seen unprecedented change in society as well. Students today swim in a sea of data, bombarded by a proliferation of unsubstantiated information, biased interpretations, "alternative facts," "fake news," and more. Successful navigation of adult life today and for the foreseeable future requires acute quantitative literacy that empowers individuals to evaluate, interpret, and develop insights from data in new ways (Elrod, 2014; Yeom, 2021). Calls "for educational approaches that can promote an understanding of how mathematics and statistics are used to serve social power structures or manipulate and shape public opinion" (Gal & Geiger, 2022, p.7) are growing in response to the changing social, civic, and political landscapes.

The implications of these changes in the future of work and society are monumental. They require us to create mathematics education systems that empower learners with deeper mathematics knowledge, stronger mathematical competencies, more creative problem solving, and more rigorous quantitative reasoning. In a world where computational and procedural aspects of work are done by machines, tasks that can't be automated or synthesized, or predicted by large language models, will need to be done by humans who have the ability to think critically, engage in high level problem posing and creative problem solving, and to reason logically. As Pink (2006) said decades ago, "we are moving from an economy and society built on the logical, linear, computer like capabilities of the Information Age to an economy and a society built on the inventive, empathic, big-picture capabilities of what's rising in its place, the Conceptual Age" (p. 2). Learners with strong math knowledge and skills who reason quantitatively, think critically, and creatively problem-solve, will have a distinct advantage in the future they face.

Given this, important questions to ask as stakeholders in mathematics education include:

(1) What is being done to develop learner competence in these areas (e.g., mathematical knowledge & skills, problem-solving, quantitative reasoning, etc.)?
(2) How are we leveraging emerging technologies (such as generative AI) to facilitate learner competence in these areas at scale?
(3) How are we applying the learning sciences to increase learner mathematics competence across the PreK-12 learning cycle?

Perhaps a further question we might pose is, *how will we know our efforts are succeeding?* Presently, we lack the proper instruments to truly answer this question. Consider that even though some progress has been made in moving from assessments that measure procedural knowledge to better measure conceptual knowledge and higher-order reasoning (Thompson, 2008), state and national measures of achievement only have limited means of assessing student critical thinking, creative problem solving, and real-world applications of their mathematical knowledge (Manouchehri et al., 2016). Moreover, in the broader field of education, resistance (at best) or denial (at worst) permeates efforts to change. Tripses (2019) points out that "a lack of deep motivation exists, whether individually or on a societal level, to understand how innovative education differs from past practice. At most, innovations are tolerated as long as they lead to adequate performance on traditional measures. Assessments are almost all geared for classical subject matter and rarely offer the means to assess the flexible, cooperative thinking required for interdisciplinary thought" (p. 25). Conversely, another challenge includes a rush to adopt and implement new technologies (such as generative AI), which can be viewed as problem-solving panaceas, without critical thought about how best to do so (U.S. Department of Education, 2023; NCTM, 2024). That being said, new paradigms for thinking about the future of mathematics education are vital if we hope to prepare learners for the additional challenges the future will bring.

AI's Potential Impact

The rapid evolution and adoption of artificial intelligence (AI) has the potential to contribute to the development of these new instructional paradigms. However, the speed of development and adoption has far outpaced the adoption of frameworks and policies that guide the use of AI. This discrepancy leaves room for misuses that can undermine educational objectives or compromise student privacy. As professional organizations and government organizations alike begin to grapple with AI's implications for education, two themes emerge. First, AI has the promise to enhance mathematics instruction, in part by "creat[ing] positive pressure to avoid the 'shallow assessment' trap and create assignments and assessments that blend the fundamentals and creative thinking." (NCTM, 2024). Second, AI has the potential to limit opportunities, perpetuate biases, and spread misinformation (OSTP, 2024), so a cautious and judicious approach is warranted.

Theoretical Framework

In considering how to obtain maximum benefit from AI while reducing its risks, it is helpful to consider a learning sciences approach as a framework for investigating AI's potential impact on mathematics education. Bloom's (1984) research showed that learners working one-to-one with an expert tutor using a mastery-based approach were able to increase their achievement two standard deviations over that of their peers in a business-as-usual classroom condition. Much of Bloom's subsequent work centered on how to adjust the conditions of the whole class environment such that the achievement of these students could begin to approach the desired achievement of those working with an expert tutor. As part of this work, Bloom identified four "objects of change" which could be acted upon to produce better outcomes for student learning which included the *(1) child, (2) materials, (3) teacher,* and the *(4) environment,* which for Bloom largely meant family and peers.

Often interventions or initiatives designed to improve mathematics performance may only focus on one or two of these four objects. For example, programs may focus on improving learning materials only, or perhaps they might add in "implementation" training for the teacher. However, Bloom viewed these objects as the focus of critical, sustained efforts to improve quality. In other words, serious and ongoing professional development for the teachers to increase their content area and pedagogical expertise in the subject area (e.g., mathematics), not just providing training for program implementation. Bloom further explained that working only through one of the objects was less likely to produce the desired outcomes as working through multiple objects, essentially adopting an ecosystems approach.

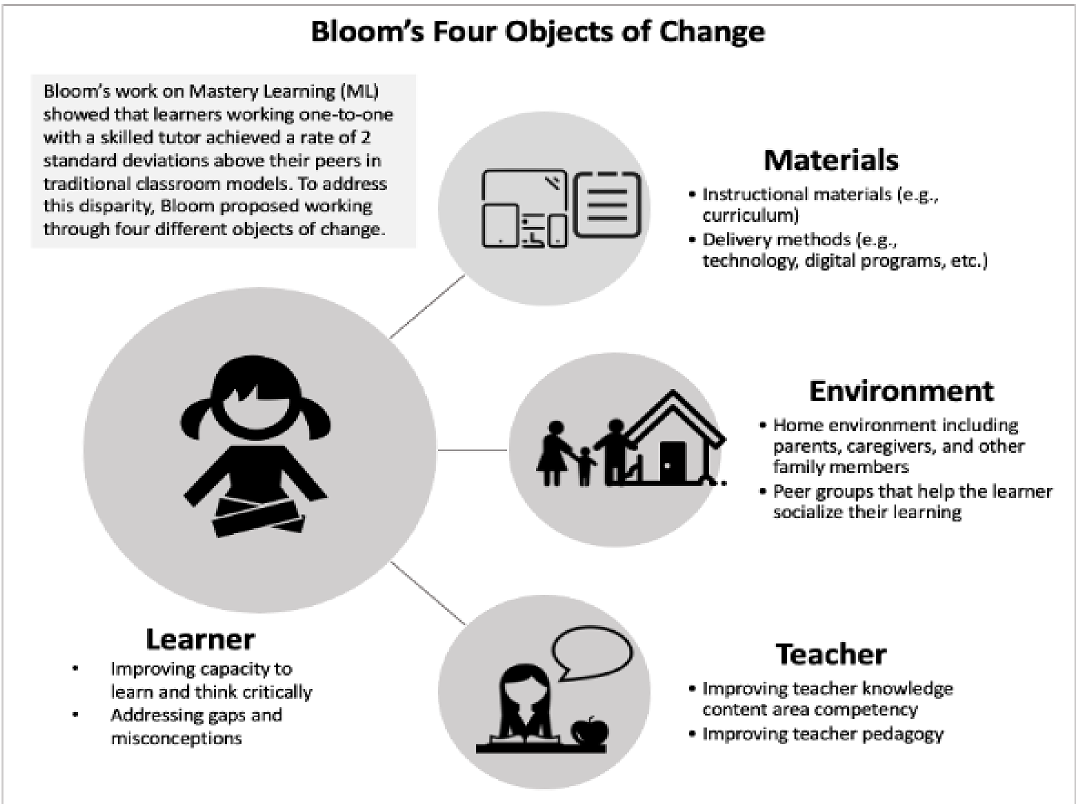

Figure 3: Bloom's Four Objects of Change (from Authors, Date)

Bloom's recommendations that efforts to increase student learning and achievement by leveraging these four objects of change seem, on the one hand, perfectly self-evident. Of course initiatives targeting to improve student learning should seek to work through all avenues possible – and yet so often, efforts do not give equal attention to each of these objects. More often than not, programs and initiatives may overly focus on one object (e.g., the materials and content) and only tangentially focus on the others (if at all). However, in recent years, adopting an ecosystems approach to learning, which involves giving equal attention to increasing the quality and performance of all parts of the system (e.g., child, materials, teacher, families, etc.), has received more concentrated attention (Authors, Date). This has included international calls from the Office of Economic Cooperation and Development (OECD), stating "to attain the desired outcomes of a curriculum reform (i.e., students' development and application of knowledge, skills, values, and attitudes) necessitates more than changing teaching and learning outputs. It involves making coordinated, multifaceted changes at the classroom, school, and policy level." (Taguma & Barrera, 2019, p.3).

A Bloomsian Perspective on the Future of Mathematics Education

Bloom's four objects of change, combined with a future learning landscape disrupted by evolving applications of artificial intelligence, provides a useful theoretical grounding for an examination of the future of mathematics education. With Bloom's framework and AI in mind, we can begin to think about the child who sits at the heart of the mathematics learning ecosystem, how their mathematics learning journey begins in early childhood, and the ways AI can be leveraged to give every child the very best start (NAEYC & NCTM, 2010).

The Future Early Mathematics Learner

Early childhood is a pivotal time for shaping a child's trajectory in mathematics. During these formative years, children are inherently equipped with a remarkable capacity for learning, characterized by extensive neuroplasticity that allows for the rapid absorption and integration of new knowledge (Muthukrishnan et al., 2019). This period is critical for embedding fundamental mathematical concepts such as number sense, pattern recognition, and spatial understanding, which serve as the building blocks for all future mathematical learning (Clements & Sarama, 2020).

Early childhood is the best time to prevent gaps in foundational mathematics knowledge before they form (Claessens & Engel, 2013; Duncan et al., 2007). However, even as early as pre-kindergarten, learners are beginning formal schooling with a patchwork of "math readiness" knowledge and skills (Authors, Date). This uneven foundation leads to variability in learners' ability to learn, and the pace at which they can learn the required skills in kindergarten. Though teachers may wish to attend to the personalized needs of every student, the constraints of traditional classroom environments often prevent such personalization.

The potential of artificial intelligence to enhance personalization through the identification and amelioration of student errors and misconceptions, even in early childhood, represents a significant area of promise (Authors, Date). A model, when trained with the learning experiences of a vast number of students, ranging from hundreds of thousands to millions, could be capable of discerning the typical mistakes and underlying misconceptions students possess (Lin, Luo, & Qian, 2023). Once these errors and misconceptions are recognized, a model that has been educated on examples of skillful instructional methods can generate corrective feedback. This feedback, specifically designed to address the misconception directly, could then guide a student through the necessary steps to comprehend the correct solution. This approach ensures that students receive the precise assistance they require in a timely manner, thus minimizing time spent on ineffective strategies and maximizing engagement with the most effective pathways to successful learning outcomes. Examples of this exist and are already deployed with thousands of Prekindergarten through Second Grades students, and more importantly, are producing learning outcomes at scale (Authors, Date; Bang, Li, & Flynn, 2023; Thai, Bang, & Li, 2022).

It is important to acknowledge that the strategies mentioned herein are not inherently innovative; the effectiveness of personalized learning has been understood for decades. The transformative potential of AI lies in its capacity to emulate these effective classroom practices within digital platforms, enabling dramatic amplification and scaling.

The Future Early Mathematics Curriculum Content and Materials

Among Bloom's Four Objects of Change, instructional materials stand out in the current discourse as significantly poised for disruption and evolution by artificial intelligence. The prevalent discussion focuses on AI's tendency towards inaccuracies, colloquially termed as "hallucinations," and the subpar performance of current models in basic mathematical tasks. Concerns have been raised about the potential for such inaccuracies to permeate AI-generated instructional materials. However, these issues are likely to be transient. Substantial progress is being made in the detection and mitigation of hallucinations through strategies that include the careful selection of training data, enhanced detection techniques, and self-correction methods (Huang et al., 2023). In parallel, innovations such as NVIDIA's OpenMathInstruct-1 problem/solution database are poised to significantly improve the mathematical reasoning abilities of models (Shah, 2024).

Assuming that these problems will be solved, it seems best to turn our attention toward applying these models effectively. While large language models in particular are improving at a rapid clip, they are still being trained on general language, and lack the specific expertise for sound, developmentally appropriate mathematics learning pedagogy. In a future where mathematics instructional materials are custom-generated for each student, it is paramount that the AI-enabled generation is trained to emulate best practices in early mathematics instruction.

Even as we innovate in the creation of instructional materials, it becomes crucial to critically evaluate the curricula these materials support. In an era where AI augments human capabilities, the value of memorization and computational skill diminishes in comparison to an individual's aptitude for discerning which problems merit attention. Essential to this competence is the capacity to pose relevant questions and to judiciously evaluate the mathematical soundness of AI-provided solutions. Transitioning from a focus on procedural fluency to fostering critical quantitative reasoning requires an ongoing shift in the educational paradigm. It underscores the importance of nurturing analytical prowess, critical thinking, and the judgment required to effectively navigate an environment increasingly characterized by automated processes and AI-driven outcomes.

The Future Early Mathematics Educator

One of the primary obstacles to delivering quality early childhood mathematics education is the increasing the mathematical content and pedagogical expertise among teachers in this field—especially early childhood educators. Often, educators in early childhood settings are under-trained in mathematics, lacking the depth of knowledge required to effectively teach the subject (Clements & Sarama, 2020). This deficiency is compounded by a widespread phenomenon known as math anxiety—a condition that affects a considerable number of educators, who may then inadvertently transmit this apprehension to their students (Hertz, Beilock, & Levine, 2019). The combination of insufficient training and personal math anxiety among teachers leads to suboptimal instruction in early childhood mathematics. This not only hampers the delivery of high-quality mathematical education but also potentially instills a similar sense of anxiety and aversion to mathematics in young learners, thereby affecting their long-term relationship with the subject (Hertz, Beilock, & Levine, 2019).

It follows, then, that if we expect the teachers of the future to provide students with higher-order math abilities, teachers themselves require further education. The existing literature points to the positive impact that redeveloping teachers' math expertise can have on their ability to teach effectively (Reid & Reid, 2017; Stoddart, Connel, Stofflet, & Peck, 1993) as well as their attitudes toward the subject (Hertz, Beilock, & Levine, 2019). Just as AI can be put to the task of providing instruction and correcting misconceptions in children, it could do the same for their teachers. Such intervention would ensure that teachers are fully equipped to develop their students' mathematical reasoning at the required level of sophistication.

The Future of Family Support for Early Mathematics Learning

In line with recent calls to focus on mathematics teaching and learning in early childhood, the National Council for Teachers of Mathematics issued a call to "strengthen partnerships with families and communities to create a shared vision of deep mathematics learning that is meaningful and relevant to children's lives" (Huinker et al., 2020, p.126). Further to this call, it is important to note that approximately 60% of children in the United States are not enrolled in any kind of pre-primary school (NCES, 2022). For these children, their primary early learning environment is the home, where parents and caregivers—who may or may not have the knowledge needed to effectively support their children's early math learning—are their first teachers. Without more equitable beginnings, it is difficult to ensure that *every* child will have access to the best start in mathematics.

Children who begin kindergarten with the needed prior mathematics knowledge are more ready to learn, more likely to master key early math competencies, and as a result go on to later success in mathematics in school (Claessens & Engel, 2013; Duncan et al., 2007). In addition, children who experience enriching home mathematics environments, filled with warm and nurturing parent-child shared math activities, are more likely to begin school ready to learn math (Blevins-Knabe, 2016). However, many parents report not knowing how to support their children's early math development, or for those children enrolled in preschool or pre-K, parents may prefer to rely on the teacher to foster the math learning of the child (Author, Date; Clements & Sarama, 2020).

For the most part, parents and families want to be involved and want to help their children to prepare to successfully learn math, but they need support (Author, Date). Adults often have deficiencies in their own math understanding, which can lead to math anxiety that they can easily transmit to their children (Hertz, Beilock, & Levine, 2019). As with teachers, AI can be a tool for addressing these deficiencies, providing personalized instruction on both math concepts themselves and the ways in which children develop understanding of those concepts. AI can also serve as a thought partner, helping caregivers to devise games and activities that ambiently incorporate math into family time in a way that is engaging and productive for all involved. AI-enabled chatbots can be designed to simply answer parent questions about what, how, and why to expose children to key math competencies, using carefully vetted databases of information that can replace parents' current "go to"--You Tube (Author, Date). When involving the family, the objective is less to turn caregivers into math teachers, but rather to support them in the raising of confident and capable math learners.

Conclusion

The future of mathematics education in uncertain times requires new paradigms, fresh perspectives, and the ability (and willingness) to leverage new tools and methods to tackle long-standing challenges. This includes, among other measures, a proactive focus on increasing the quality and accessibility of early childhood mathematics education as a means of ensuring that disparities do not have a chance to develop in the first place. An ecosystems approach that seeks to address the child, the instructional materials, the educator and the family has been shown in the research to be greater than the sum of its parts (Bloom, 1984; Authors, Date). Ecosystems intentionally designed to leverage artificial intelligence may be an effective means of impacting each of these objects of change, fostering personalization and building capacity to prepare students to reason mathematically. Envisioning the future of mathematics education in uncertain times is a call to action—an invitation to reimagine our approaches, rethink our priorities, and redouble our efforts to ensure that every student has the opportunity to succeed in mathematics.

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DEVELOPING A MACHINE LEARNING RUBRIC FOR PROPORTIONAL REASONING WITHIN DIGITAL CURRICULUM MATERIALS

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Keywords: Technology, Curriculum, Rational Numbers & Proportional Reasoning

Machine learning techniques focus on how computers acquire new information. Typically, artificial intelligence used in mathematics education focuses on intelligent tutoring systems, profiling and prediction, and adaptive systems and personalization (Hwang & Tu, 2021). We propose an alternative use of machine learning techniques to train and optimize a model that gives feedback on student strategies, deploying it as a real-time, interactive feature of the digital curriculum platform. In this poster, we report on the development of the proportional reasoning rubric using students' inscription data—log files and any digital student work saved in the digital platform. The set of curriculum materials embedded in the digital collaborative platform we discuss is the middle grades problem-based curriculum, *Connected Mathematics4* (CMP; The Connected Mathematics Project, 2023; Phillips et al., in production). Given the emphasis of the CMP curriculum on fostering student thinking (Choppin et al., 2015), the rubric is thoughtfully designed to adapt to the diverse array of proportional reasoning strategies that may emerge from students' log files.

To develop the rubric, we examined the research literature on proportional reasoning strategies (e.g., Ben-Chaim, Fey, Fitzgerald, Benedetto, & Miller, 1998) and investigated student work already in the digital collaborative platform. The development of the rubric was based on two critical aspects. First, the rubric could be used across the different units in which proportional reasoning appears throughout the year. Second, the information for each criterion generated from machine learning is useful for teachers, such as aiding teachers in structuring whole-class discussions based on students' strategies. Our final rubric focuses on three criteria: (a) student approach (e.g., part-to-part, or part-to-whole), (b) mathematical representation (e.g., text, drawing, equation, or table), and (c) solution strategy (e.g., unit rates, common parts/wholes, or building up).

Our next steps include applying the rubric to code students’ digital work to train the machine learning model. We also created a new annotation feature to elicit more information on students’ strategies. We are testing a new digital interface that allows students and teachers to sort student digital work based on human and machine learning applications of the rubric criterion.

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USING AI TO DEVELOP PRE-SERVICE TEACHERS' LESSON PLANNING SKILLS

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We propose a study that aims to utilize ChatGPT to foster and develop the lesson planning skills of pre-service elementary teachers. The goal of the study is to show that pre-service teachers can interact with ChatGPT in a way that takes their initial drafts of mathematics lesson plans and, through working with peers and ChatGPT feedback loops, design improved lesson plans that are more inquiry-based and student-centered.

Keywords: Technology, Preservice Teacher Education, Elementary School Education

Introduction

The use of technology in math education has evolved significantly, from simple calculators to sophisticated online tools. When a new tool is introduced, it is often initially met with doubt, but eventually recognized for its value in teaching and learning. For example, scientific calculators and computer algebra systems were initially questioned but have since proven their worth in classrooms, supported by research that endorses their positive impact on mathematical understanding and student attitudes toward learning (see e.g., Chorney, 2021; Yohannes & Chen, 2021). Major stakeholders such as The National Council of Teachers of Mathematics (NCTM) have asserted that "It is essential that teachers and students have regular access to technologies that support and advance mathematical sense making, reasoning, problem solving, and communication" (NCTM, 2015, p.1). Indeed, a considerable body of research supports the idea that advanced digital technologies can support learning in general (Borokhovski et al., 2020) and mathematics concepts in particular (Hoyles, 2018; Yohannes & Chen, 2021) as well as having positive effects on students' attitude to learning (Fabian et al., 2018).

Artificial Intelligence (AI), particularly through tools like ChatGPT, represents the latest advancement in this continuum. This study aims to use Large Language Models (LLMs) to enhance the lesson planning abilities of pre-service PreK-5 math teachers. By utilizing LLMs to improve existing lessons or create new ones, our research seeks to achieve several objectives: enhance teachers' proficiency in creating effective lesson prompts (prompt engineering), foster a critical approach to evaluating AI-generated teaching materials, and enhance overall lesson planning skills through critical analyses of teaching materials.

Recognizing the inevitability of AI tools in educational settings, we aim to equip pre-service teachers with the confidence and know-how to harness these technologies for the benefit of students in their future classrooms. We see LLMs not just as tools, but as collaborative partners with unique strengths and limitations, aiming to prepare teachers for a future where technology and pedagogy intersect more seamlessly.

Literature review and relationship to research

Lesson planning is pivotal in teaching, with research highlighting challenges encountered by teachers, especially novices, in crafting effective plans. Cevikbas, König, & Rothland (2023) emphasize that while initial teacher education and professional development can enhance lesson planning competence, teachers often require support in areas such as understanding student thinking and leveraging innovative pedagogies. Collaborative methods such as communities of practice (Wenger, 1999) and lesson study groups (Fernandez, 2002) offer valuable frameworks for enriching lesson planning through shared knowledge and a research-based approach.

The advent of ChatGPT (OpenAI, 2023) introduces a novel resource for lesson planning, though empirical studies on its efficacy are still emerging. Rogeaux & Sharp (2023) note that with appropriate prompts, ChatGPT's output can rival the lesson plans of pre-service teachers, suggesting AI's potential to streamline the lesson planning process. Additionally, practitioner accounts (Schulten, 2023; Mok, 2023) and emerging platforms like Eduaide advocate for ChatGPT's role in enhancing productivity and offering personalized support in education.

This study aims to explore the utilization of ChatGPT and similar LLMs in lesson plan development among pre-service PreK-5 mathematics teachers, with a focus on improving candidates' prompt engineering skills, critical usage of LLMs, and overall lesson planning proficiency. Specifically, we investigate the following research question: *What is the impact on pre-service teachers' understanding of the role of AI in designing, revising, and evaluating lesson plans for mathematics instruction of deploying ChatGPT in the design of lesson plans?*

Methods and Methodologies

Setting and Participants

The study encompasses prospective elementary school teachers enrolled in a mathematics teaching methods course (ED 3XX), focusing on the integration of LLMs for planning (and implementing) mathematics lessons. In this study, we leverage AI to enhance collaboration, synthesize ideas, and communicate concepts in a natural language.

The ED 3XX course is divided into three modules, each designed to deepen the participants' lesson planning skills with an emphasis on inclusion, student engagement, and the effective use of technology. Each module spans three weeks, incorporating activities that guide pre-service teachers from initial lesson plan analysis to AI-assisted revision and critique. The modules are structured as follows:

Module 1: Introduction to lesson plan analysis and revision through a blend of classroom activities and homework assignments. This includes analyzing a sample lesson enacted by the instructor, with candidates critiquing it and engaging with ChatGPT for improvements.

Module 2: Similar in structure to Module 1 but focuses on commercially-produced curricula enacted by small groups of candidates, encouraging collaborative lesson planning and AI-assisted revision.

Module 3: Builds on the previous modules with an initial lesson generated using ChatGPT from initial ideas gleaned during students' field work, challenging students to refine the AI-generated lesson further.

Data Collection and Evaluation

The following course assignments have been constructed, in part, to help answer our research question. Students complete versions of these assignments in each of the three modules:

Pre-Teaching Analysis (PreTA): This tool is used before teaching a lesson, where candidates assess a lesson plan based on eight criteria such as learning objectives alignment, lesson detail, and engagement strategies. It includes a rubric and prompts for candidates to anticipate the lesson's strengths and areas for improvement.

Post-Teaching Analysis (PostTA): Conducted after teaching, this reflective tool prompts candidates to evaluate the lesson's execution against their initial analysis and expectations. It focuses on what went well, surprises encountered, and how the lesson could be improved based on the actual teaching experience.

Prompt Effect Worksheet (PEW): This worksheet documents the interaction with AI tools during the lesson revision process. Candidates note down specific prompts used, the AI-generated responses, and provide a critical analysis of how these interactions influenced the lesson plan's revision. It serves as a record of candidates' engagement with AI in refining educational materials.

In addition, candidates complete questionnaires at the beginning and end of the study.

Lesson Planning Questionnaire: Administered at the course's start and end, this questionnaire gauges candidates' perspectives on effective lesson planning, their experience with LLMs, and reflections on AI's role in lesson planning. It aims to capture shifts in candidates' understanding and attitudes towards AI-assisted lesson planning.

Data Analysis

Our analysis employs a multi-tiered approach. Initially, we compare the initial and final lesson plans in each module, focusing on qualitative differences highlighted by candidates regarding lesson strengths and weaknesses. This involves a direct assessment of the improvements attributed to AI assistance. We also look for differences in plans across the three modules. For instance, *do candidates focus more attention on lesson differentiation in later modules? As they gain experience in PreK-5 classrooms during the field component of the course, do they incorporate these experiences and understandings into prompts they provide to LLMs?*

Adding to this, we propose an innovative qualitative analysis by loading all Prompt Effect Worksheets (PEWs) into a ChatGPT interface. This will allow us to systematically evaluate how closely the AI-guided discussions center around eight key quality indicators of a lesson plan identified in the PreTA. Such an analysis not only quantifies AI's focus on critical lesson plan components but also highlights areas for further pedagogical development in AI interactions.

Further, open coding and the constant comparative method (Glaser & Strauss, 1967) will be used to analyze improvements suggested across all modules, developing a codebook that reflects emerging themes such as standards alignment and effective AI prompts. This codebook will then guide the analysis of final reflections and follow-up interviews, providing a comprehensive view of the AI's impact on pre-service teachers' lesson planning skills and their understanding of AI's role in education.

This layered analysis strategy aims to answer our central research question, exploring the depth of ChatGPT's influence on the pedagogical process and its potential to enrich lesson planning in mathematics education.

Preliminary Results

The initial phase of our study has revealed insightful trends regarding pre-service teachers' lesson planning approaches before and after interaction with AI tools. The initial phase of our study has illuminated the nuanced journey of pre-service teachers integrating AI into lesson planning. Five key themes emerged from their reflections:

Specificity in AI Interaction: Pre-service teachers learned the value of specificity when interacting with AI, noting that detailed prompts yielded more relevant and useful responses. A student observed, "I became more and more specific with my requests...the AI does not respond very well to vague or large overarching questions."

AI as a Resource for Inclusive and Engaging Activities: Many appreciated AI for generating diverse and engaging activities, acknowledging, "It created different activities...inclusive and engaging for all my students, that I may not have thought of on my own." Notably, one student remarked, "ChatGPT helped me see beyond the standard curriculum, offering fresh angles on engagement." This echoes Cevikbas et al.'s (2023) findings on the potential of training to elevate teacher competence in lesson planning, particularly through technological integration. Feedback underscored AI's role in surmounting common planning challenges, particularly in diversifying engagement tactics and addressing varied learning styles. Notably, one participant reflected, "AI opened new avenues for differentiation I hadn't considered, making my plans more inclusive."

Overcoming Initial Hesitance: Despite initial hesitance, the practical utility of AI in revising lesson plans was acknowledged, with reflections such as, "I was hesitant about using AI...but when I started the process...it truly wasn't that bad." Engagement with AI tools marked a pivotal shift, resulting in more innovative approaches and a broader perspective on inclusivity.

AI's Role in Enhancing Lesson Content: Teachers found AI particularly helpful in refining lesson content and structure, making lessons more interesting and accessible. "For almost every prompt that I gave AI, I was able to find...suggestions to be extremely helpful."

Learning to Leverage AI for Detailed Revisions: The process revealed that AI's effectiveness increased with detailed, focused prompts, leading to significant improvements in lesson plans. "The AI's responses gave me details and content that was much needed...I learned to include specifics on what I specifically wanted revised," a teacher reflected.

Anticipations for Modules 2 and 3

As we progress, we anticipate a deeper exploration of AI's capacity to enrich lesson planning in Modules 2 and 3. In Module 2, with candidates leading lessons from the Investigations curriculum (TERC, 1998), we expect an enhanced application of AI in real-world teaching scenarios, reflecting on Rogeaux & Sharp's (2023) insights on AI-generated plans related to real-world experiences. Module 3 promises further exploration into AI-initiated lesson plans, where we foresee candidates critiquing and refining ChatGPT-generated content, potentially redefining the boundaries of traditional and AI-assisted teaching methodologies, resonating with Schulten (2023) and Mok (2023)'s advocacy for AI as a transformative tool in lesson planning.

Relationship to Conference Theme

This study directly addresses PMENA 2024's theme by exploring how AI, specifically ChatGPT, can be integrated into mathematics education to navigate and adapt to the changing educational landscape. Through examining pre-service teachers' engagement with AI in lesson planning, our research contributes to envisioning the future of mathematics education amidst uncertainties, highlighting the potential of AI to support innovative teaching strategies and enhance educational outcomes.

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DEVELOPING AN AUGMENTED REALITY SYSTEM FOR EMBODIED LEARNING OF VOLUME MEASUREMENT

DESARROLLANDO UN SISTEMA DE REALIDAD AUMENTADA PARA EL APRENDIZAJE INCORPORADO EN MATEMÁTICAS

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Volume measurement requires children to develop skills in the coordination of three dimensions, the mental structuring of space, and flexible volumetric reasoning. However, students often face difficulty in these areas. Augmented reality (AR) technology holds promise as a solution to this problem as it can create an environment that introduces a computer-generated layer into the visual environment of the user. In this paper, we introduce an AR smartphone application, MeoGeo, that we developed. This application enables students to use their perceptuomotor actions as pedagogical resources in forming a concept of volume. By employing a multimodal analysis of an elementary student's embodied interaction with this app, we revealed the moments that a student enacts volume measurement of a 3D object. This study suggests that coordinating perception and action in an AR-enabled environment can facilitate mathematical cognition.

Keywords: Technology and Learning Environment Design, Geometry and Measurement, Student Learning and Related Factors

Purpose of the Study

This Brief Research Report aims to demonstrate the potential of an augmented reality (AR) system as an embodied learning technology. AR allows users to augment their view of the real world with computer-generated information. In this paper, we illustrate the embodied repertoire of behavior of an elementary student in learning volume measurement, guided by AR smartphone application, *MeoGeo*, which was developed by the authors of this paper.

Children often struggle with volume measurement, as it requires coordination of three dimensions (3D; Battista & Clements, 1996). Previous research revealed four mental schemes of volume identified by young children: packing, building, filling, and comparing (see Van Dine, 2014; Curry & Outred, 2015). Each scheme can bring particular misconceptions to children. For example, children who use the packing approach (i.e., quantifying the space within a 3D container by iterating unit cubes) tend to miscount only visible faces (Ben-Haim et al., 1985; Panorkou, 2019). In general, mentally organizing the space and imposing structure for volume measurement can be challenging for children, and a carefully designed manipulative can be helpful (Ferrara & Mammana, 2014).

In this study, we introduce *MeoGeo.* This smartphone application was designed for children to *bodily* coordinate 3D models using virtual arrays of unit cubes in their everyday environment in real-time. For example, children can move their body (pre-symbolic register) in alignment with the virtual three axes of object. This experience constitutes situated action where mathematics meanings can arise (Varela, Thompson & Rosch, 1991; Morgan & Abrahamson, 2016). We extend literature on the positive impact of dynamic virtual manipulatives on volume learning (Panorkou, 2019; Rupnow, 2022) to the AR-incorporated embodied learning environment.

Given the increasing popularity of immersive technology across every sector of life, this study aims to report design principles of AR-enabled mathematics learning environments that emphasize pre-symbolic experience for children. We hypothesize that a student's coordination of action with *MeoGeo* can mobilize reasoning with respect to their concept of volume. Our research question asks, "What forms of volumetric reasoning can develop as a result of a child's guided engagement in/with an AR-enabled learning environment?"

Theoretical Framework

Our work is rooted in the theory of embodied cognition and sociocultural theory. Embodied cognition posits that cognitive processes are rooted in the body's interaction with the world, including perceptuomotor coordination and the spatial system (Wilson, 2002). With this framework, this study can analyze how students utilize their body with AR technology (Abrahamson, 2020; Walkington, 2023). Additionally, we also utilize recent scholarship that marries embodied cognition with sociocultural theory to study human-technology interaction (Danish, 2020; Kaptelinin & Nardi, 2017). This approach enables an analysis of the mutually constitutive role of a student within an AR system-applied learning environment (*MeoGeo*).

The Design Study as Mode of Inquiry

We conducted a design-based study with eight students in grades 2-6 to examine how the forms of volumetric reasonings emerge within the context of our AR system (Cobb et al., 2003). The students participated in a seven-week-long mathematics program in New England, with weekly sessions featuring a one-hour-long AR activity session. For this pilot study, we focused on Greg (pseudonym), as he demonstrated motor skills in operating an iPad. Greg showed limited understanding of volume before using the app. To address our research question, Greg was asked to explore how many virtual cubes make up real-objects of his interest. He chose to measure the air-conditioner (AC) (See Figure 1). Two data sources were used: screen recordings of the app and video recordings from an iPad placed in the classroom. These recordings captured Greg's verbal and non-verbal actions. We employed a microanalytic investigation to capture the evidence of how and why *MeoGeo* may help learning of volume (Erikson, 1992). We reviewed the video recordings and identified the forms of interactions linked to the features of the app (e.g, half-transparent virtual unit cubes) using the MAET framework focusing on a user's body position, gaze, and body movement (Walkington, 2023). Below, we illuminate three excerpts that show the moments of interaction between Greg and *MeoGeo*.

Results

Volume as Coordination of 3D

What if children could structure 3D models using virtual unit cubes that visualize hidden portions of shapes and navigate around to experience them in a real world (3D)? Simply observing a physical cube does not enhance children's mental coordination of 3D objects (Battista & Clements, 1996). The utilization of dynamic geometric software, such as *GeoGebra*, can assist students in coordinating 3D representations of volume, but it relies on a fixed computer screen (Rupnow, 2022). *MeoGeo* synchronizes the dimension of the object studied (3D) with the environment in which a child explores it (3D). Excerpt 1 demonstrates a student actively coordinating the 3D of an object, aided by *MeoGeo*. Greg, to measure the height of AC, adjusted his body positioning up and down (Excerpt 1.01). To ensure that the height of the unit cubes was the same as AC, he sat down to align his eye position with AC (Excerpt 1.02). He then moved on to measure the width (Excerpt 1.03), followed by examining the length of AC from another angle (Excerpt 1.04). He moved his body as if there were *x*, *y*, and *z* axes in space, aligning his body with these axes. When children employ volume as a packing or building method, they limit focus to visible faces (Van Dine, 2014). However, Greg's bodily maneuver, enhanced with the transparency of virtual unit cubes projected onto the AC, served to demonstrate the pedagogical utility of an AR environment in reasoning 3D properties.

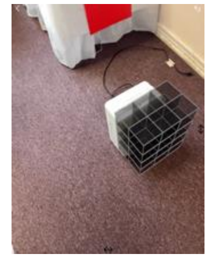
Table 1: Excerpt 1. Greg's spatial structuring of virtual unit cubes

01	Greg:	Hmm. And then, stack it! ***(Fig. 1**: Greg stands up and places the iPad face down)*	 **Fig. 1**
02	Greg:	So, this. *(**Fig. 2**: Greg sits down, moves the iPad to change the camera angles, and then stacks up one more cube on the first column*)	 **Fig. 2**
03	Greg:	Yep. Let's go. Yeah! It's perfect. *(Greg taps iPad to add another column of virtual unit cubes.*)	 **Fig. 3**
04	Greg:	I think there is another [column to stack up for width.] I can't quite decide. *(**Fig. 3**: Greg stands up and moves beside AC)*	

Volume as Unit Structuring Scheme

Children's flexible coordination of units and composite units (such as rows) can aid in volume reasoning, which later helps in the construction of volumetric calculation algorithms (Clements & Sarama, 2021; Rupnow, 2022). To emphasize the unit structuring scheme, we designed an app where children can build virtual unit cubes without gaps or overlaps, a mistake often made by children (Curry & Outhred, 2005). Drawing on gestalt psychology of continuity, we assumed children are more likely to create a row, column, or layer with cubes rather than randomly placing them. This design choice aimed to encourage students to flexibly manipulate units, units of units, and units of units of units for volumetric reasoning. Excerpt 2 shows evidence that children intuitively and bodily explored these unit construction schemes. Greg stacked up three layers with three cubes per row (Excerpt 2.01). Subsequently, he walked around the object to measure the length. He added three more columns (Excerpts 2.02-2.03).

Table 2: Excerpt 2. Greg's interaction with *MeoGeo* in real time

01 Greg: I think there is another [column to stack up for width.] I can't quite decide. *(Greg stands up and moves beside the AC)* 02 Greg: Ugh. Is this… Hoo.. Uhm. *(Greg* stacks up another column. ***Fig. 4****: Greg stands up and walks far away from the AC*) 03 Greg: How is it doing that? Alright. This one [column]. I figured it out. *(Greg* walks back closer to the AC. ***Fig. 5****: he stacks up three more columns, steps back, pauses, and observes his virtual cubes*)	 **Fig. 4**	 **Fig. 5**

Flexible use of multiple reasoning approaches to volume

A previous study found that children were likely to choose one volumetric reasoning strategy (e.g., packing) for a volumetric measurement problem (Vasilyeva et al., 2013). In contrast, our data shows Greg employed three interrelated reasoning strategies for volume. Greg stacks up virtual cubes *next* to the AC, which shows the volume as building approach (see Fig. 6). Then, he compared the size (comparing) using an allocentric frame of reference (Iachini et al., 2023). Then, Greg tried to 'fill' the virtual unit cubes, with the AC (Fig. 7). Although the filling strategy is often observed when measuring the volume of fluid (Van Dine, 2014), the affordances of AR, combining the digital and physical world, made the filling strategy available. It was also interesting that Greg used the AC (the object being measured) as the measuring unit itself. This finding shows that the use of AR can bring flexible volumetric reasoning strategies.

Table 3: Excerpt 3. Greg engaging in various reasoning strategies with *MeoGeo*

01 Greg: (***Fig. 6****: Greg stacks up another virtual layer. He inserts his foot into the cubes. He circles around the AC and walks farther from it and come back where he stood.)* 02 Greg: (***Fig 7.*** *Greg* grabs the AC and then put inside virtual cubes.) 03 Greg: I am done.	 **Fig. 6**	 **Fig. 7**

Discussion and Conclusions

This study examined video recordings of a fifth grader's embodied interaction with *MeoGeo*. This coincides with the increasing popularity of immersive technology featured in the 2024 PME-NA conference theme. We tested our hypothesis that a student's coordination of action (body position, gaze, body movement) mobilized reasoning about volume by (a) coordinating 3D, (b) exploring unit structuring, and (c) flexibly employing three approaches on volume. As a pilot study, this research is limited to the interaction of only one student. However, it highlights the spontaneous and non-deterministic meaning-making process of a student performing volume measurements within an AR-enabled immersive setting. The virtual arrays of unit cubes, structured with no gaps and overlaps, seemed beneficial for students to arrive at reasonings of volume. We believe our study would be beneficial for educators and designers to attend to such design principles that value students' pre-symbolic register of the body in teaching abstract math concepts.

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Pre-service Teachers' Decision-Making Process to Use Technology For Early Algebraic Thinking

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Technology is becoming increasingly integrated into many K-12 education classrooms. This study documents pre-service teachers' decision making in utilizing technology to support algebraic thinking for elementary students. A design research study was conducted where pre-service students designed and taught lessons to elementary students. The findings revealed that pre-service teachers intentionally considered the student perspective of their technology selections. Findings were categorized by purpose in for mathematical thinking and role of technology using the PICRAT Framework (Kimmons, 2018) However, selecting the right tool to specifically support early algebra content was more challenging. Implications for integrating technology into a mathematics methods courses to support pre-service teachers' TPACK knowledge are discussed.

Keywords: Pre-service Teacher Education, Technology,

Introduction

Technology has become a normative instructional practice (Horne & Staker, 2011; US Department of Education 2004, 2010). It has been used flexibly as a tool to support mathematical thinking (Romberg & Kaput, 1999; Warren et al., 2016). This requires pre-service teachers to develop technological, pedagogical, and content knowledge (TPACK) (Mishra & Koehler, 2006). While the development of TPACK knowledge for pre-service teachers has been established (Graham et al., 2012). There is little research on pre-service teachers' rationale, anticipated use, or attitudes about technology (Hughes et al., 2020). This paper presents findings from a whole class teaching experiment conducted in a pre-service methods course, specifically focusing on pre-service teachers' technology decision-making processes as they developed and taught lessons on early algebraic thinking to individual elementary students.

Review of Literature

To effectively incorporate technology to support student learning of algebraic thinking, pre-service teachers must develop knowledge in their content (early algebraic thinking), instructional practices, and technology tools. Early algebraic thinking has been well-defined (Blanton, 2008; Carraher & Schliemann, 2018; Kieran, 2018) and emphasizes numeracy, characteristics, and properties of numbers and their arithmetic operations with or without the formal use of symbolic algebraic notation.

Selecting the right technology for teaching math content is challenging. When working with educational math applications for teaching, there is a need for pre-service teachers to be aware of the learning theories within the apps to better align the app use with specific learning objectives or outcomes (Larkin, 2013; Dubé et al., 2020; Kay & Kwak, 2018a, 2018 b). Overall, pre-service teachers primarily select and use technology from a teacher-centered and directed approach (Hughes et al., 2020; Hu & Yelland, 2017). Therefore, it is important for pre-service teachers not to use technology as a quick add-on for their classrooms, but rather to use it in conjunction with research-based practices that support technology for student learning (Bai, 2019; Lai, 2018) for the teaching and learning of mathematics. There is a need to distinguish between the purpose in per-service teachers' use of technology and technology use by the student in order to support early algebraic thinking. This gap in knowledge leaves pre-service teachers with little guidance on how to select or sequence technology tools within their algebraic lessons. Therefore, we investigated: *What do elementary pre-service teachers consider when integrating technology to support early algebraic thinking?*

Methodology

A whole class teaching experiment was conducted in a math methods course in a Western state. Twenty-two K-8 pre-service teachers (N=22) participated in the study for nine weeks. The course integrated content knowledge and pedagogical content knowledge and intentionally integrated instructional technologies. To provide an authentic lesson planning experience, the pre-service teachers were provided a practicum experience where they had to tutor individual students within the methods course. A design research approach (Barab & Squire, 2016; Collins et al., 2016) was used to design and deliver the lessons. The course duration was 9 weeks and was designed to incorporate technology throughout the course using TPACK Knowledge Progression (Chai et al. 2010). The purpose of technology selections was framed using the PICRAT Framework (Kimmons, 2018). The goal was to support pre-service teachers in developing lesson plans to support the learning of algebraic thinking by integrating technology. Additionally, the course incorporated early algebraic thinking (Early Algebra Progression (LEAP) Blanton et al. (2020) and a lesson planning framework (Lamberg, 2019)

The data collected included field notes, audio transcripts, and two lesson plan assignments that required pre-service teachers to support elementary students' algebraic thinking by integrating technology into the lesson. The first assignment was given in Week 5 as an online discussion. The pre-service teachers were given a lesson plan introducing multiplication in this assignment. They were asked to find or create a technology that they could use to support students' early algebraic thinking. The second assignment was given in Week 8, reflecting on the pre-service teachers' lesson sequence with their elementary students. In their reflection, they were asked to describe a specific aspect of students' algebraic thinking they noticed and justify the technology that could be used in their instructional sequence to support their learning objectives. The data was analyzed using (Corbin & Strauss, 1990). Constant comparative method for whole class discussions. Deductive coding was used to distinguish between arithmetic thinking (Radford, 2018) and algebraic thinking (Kaput, 2008).

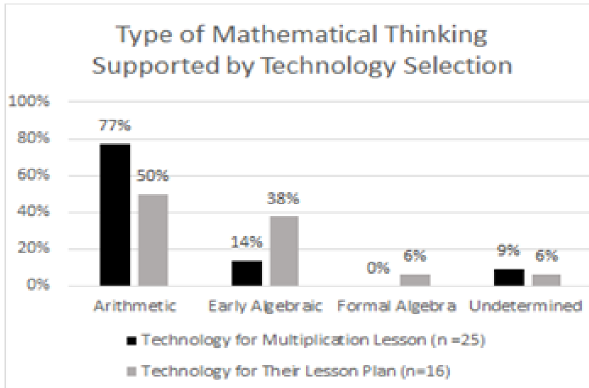
The technology use was also coded for themes using the PICRAT framework by Kimmons (2018) which addresses the student relationship with technology and the teacher's use of technology. To determine the purpose for technology and integration within lesson planning, codes were created inductively using descriptive coding (Saldaña & Omasta, 2018; Saldaña, 2021) .

Results

Selected Technologies Primarily Support Arithmetic Thinking

Pre-service teachers initially selected technologies related to arithmetic concepts, focusing on the operations being done. In the first task of selecting mathematics technology for a lesson plan on multiplication, 100% of participants identified a technology that aligned with the mathematical learning objective in both assignments. The pre-service teachers primarily selected arithmetic technologies to support students' arithmetic (operational thinking). Figure 1 summarizes the types of mathematical thinking supported by the technology selected in each lesson task.

Figure 1: Type of Mathematical Thinking Supported by Technology Selection

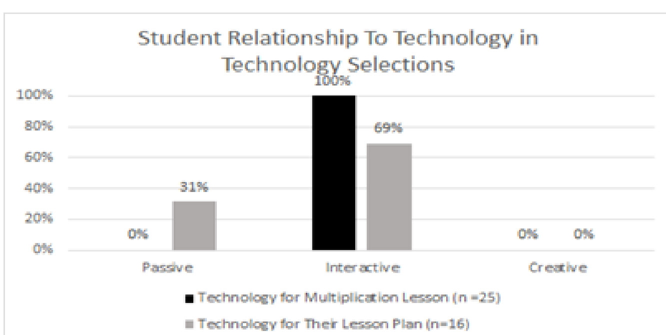


However, the second lesson planning task increased early algebraic thinking and formal algebra. Pre-service teachers identified technology uses that supported generalization, relational thinking, or additive or multiplicative properties. Algebraic thinking could be further distinguished as supporting early algebraic thinking or formal algebraic thinking.

Selected Technologies for Math Lesson Plans Were Passive & Interactive

Coding schemes were created according to PICRAT Framework (Kimmons, 2018), technologies were coded according to student relationship to technology and teacher's use of technology. In both lesson planning tasks 100% of technology selections were chosen based on the student's relationship with technology. Within those tasks, they were further distinguished by passive, interactive and creative technologies. Overall, the pre-service teachers selected three types of technology for both tasks: games or digital manipulatives (interactive) and videos (passive). For both assignments creative technologies were not selected. Figure 2 summarizes the percentage of pre-service teachers who selected each type.

Figure 2: Student Relationship to Technology in Technology Selections



Creative & Teachers' Use of Technologies Emerge

In small grade level groups were asked to create a poster about how they think of technology during lesson planning. While interactive and passive technologies still appeared, creative and a teachers' perspective of technology emerged. Not only do the pre-service teachers describe technologies for supporting math instruction, but they make connections to the lesson planning sequence and provide justifications for technology selection.

Discussion

Initially, pre-service teachers only primarily selected interactive activities to support specific mathematics objectives. However, collectively, in a group discussion, it was revealed that pre-service teachers are aware of and can differentiate between the purposes of passive, interactive, and creative activities for non-content-specific generalized instructional activities. This study differs from Hughes et al. (2020) in showing that pre-service teachers primarily selected technologies that were from the students' perspective but were also aware of the difference in technology for students and teachers' use. Overall, this supports the PICRAT framework (Kimmons, 2018) and continues to validate Kimmons et al. (2020) in utilizing PICRAT within pre-service methods courses to distinguish between student and teacher use of technology. Furthermore, incorporating technology into the lesson planning framework (Lamberg, 2019) goes a step further by linking technology as an intentional decision embedded within the instructional sequence. As technology becomes an instructional norm for teaching, methods courses must support content and pedagogical knowledge and technological pedagogical content (TPACK) knowledge. This study highlights how the pre-service teachers' technology decision-making does not occur within a single lesson but how TPACK knowledge is developed in smaller lessons throughout the course. As technology cannot simply be an add-on to K-12 education, it cannot be a one-time supplement to pre-established learning goals in mathematics methods courses.

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TEACHER BELIEFS ABOUT COMPUTER SCIENCE INTEGRATION IN SECONDARY MATHEMATICS CLASSROOMS

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Keywords: Computing and Coding, Integrated STEM / STEAM, Teacher Beliefs, Technology

This study explores high school mathematics teachers' beliefs about integrating computer science (CS) content into mathematics curricula. Beliefs are defined as teachers' convictions about the value and efficacy of CS integration. The objectives are to examine (1) teachers' perceptions of the benefits and challenges of CS integration in mathematics classrooms and (2) teachers' beliefs regarding the impact of CS integration on student engagement, problem-solving skills, agency, identity, and access to future opportunities. With thirty US state legislatures mandating the inclusion of at least one CS course (e.g., Missouri General Assembly, 2022) and only 41% of small high schools offering standalone CS courses (Code.org et al., 2023), there is a pressing need to explore innovative alternatives for CS integration within existing core classes (e.g., Missouri Department of Elementary and Secondary Education, 2023).

This study draws on Expectancy-Value Theory (Eccles, 1983) and constructionism (Papert, 1980) to explore integrating CS content into high school mathematics curricula. Expectancy-Value Theory provides a lens to examine teachers' motivations and perceived barriers to incorporating CS content, while constructionism grounds this study in the view that students learn most effectively through active engagement in creating and designing tangible projects. The principles of constructionism advocate for a mathematics education that fosters ownership, agency, and relevance through personally meaningful projects that resonate with students' cultures, values, and societal contexts (Papert, 1980). CS integration aligns with these principles, positioning learners as active creators and designers who use technology to explore, experiment, and construct, thereby deepening their mathematical understanding.

The methodological approach for this study centers on qualitative inquiry through semi-structured interviews with participants identified through a targeted data request to the Missouri Department of Elementary and Secondary Education (DESE) and subsequently emailed for recruitment. The sample included five educators meeting the dual certification criteria for teaching high school mathematics and computer science. Interviews were recorded via Zoom, with thematic analysis applied to the transcribed content. Initial codes were developed from the research questions and theoretical frameworks. Coauthors met to discuss and reconcile coding variances to ensure the coding process's reliability and validity.

Teachers expressed enthusiasm for integrating CS into mathematics curricula, citing benefits such as enhanced student engagement and its potential to foster a positive identity in students as learners of mathematics. However, they also highlighted barriers like limited resources and insufficient training. Teachers emphasized the value of hands-on, project-based learning, noting that students engage more deeply with mathematics when seeing practical applications through CS projects. They believed this integration could empower students from underrepresented demographics by exposing them to CS within a core subject, thus broadening their career aspirations and reducing the gender gap in computing fields. Implications include targeted support for smaller schools through grants and professional development programs and advocating for innovative and resource-aware curriculum approaches.

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ILLUSTRATED STORY PROBLEMS: PROSPECTIVE TEACHERS' MATHEMATICALLY PEDAGOGICAL USAGE OF TEXT-TO-IMAGE GENERATIVE ARTIFICIAL INTELLIGENCE

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Keywords: Preservice Teacher Education, Technology, Artificial Intelligence, Adobe Firefly.

Artificial intelligence (AI) has quickly become a prominent discussion topic in academia, including within the field of mathematics education (Kaplan-Rakowski et al., 2023). In broad strokes, AI refers to machines that simulate aspects of human thinking, and generative AI refers to AI capable of producing new content (e.g., text, images, music, etc.) based on particular user-specified input (Pavlik, 2023). Professional organizations have begun to enter the AI conversation in the form of recommendations, position statements, or calls for manuscripts on the topic; for example, a recent NCTM President's Message challenged mathematics educators "to learn how to integrate [AI] into [their] instruction and into our profession" (Dykema, 2023). At this point, exactly how mathematics educators should productively do this remains a largely open question, especially with respect to AI that generates non-text output.

In this poster, we consider the ways in which prospective elementary and middle-grades teachers (PTs) intentionally used Adobe Firefly—a text-to-image generative AI—toward mathematically pedagogical ends. In a semester-long mathematics content course on number and operation, 50 PTs were introduced to Adobe Firefly and were directed to use it for selected assignments. Here, we report on PTs' responses to one assignment, which involved: (a) writing a story problem that could be used to support elementary students' understanding of addition and subtraction, (b) using Adobe Firefly to produce an image with countable items matching those in their story problem, and (c) writing 4–8 sentences to explain why they designed their image the way they did, specifically focusing on features they deemed important for supporting students' mathematical thinking. On this assignment, PTs were also encouraged to attempt multiple prompts to find a satisfactory image; additionally, PTs were encouraged to use Firefly's generative editing features to further refine their image (e.g., by using additional rounds of generation to insert or remove objects from a previously generated image).

In the poster, we share examples of the problems, prompts, and images that PTs submitted for this assignment. We present a thematic analysis (Braun & Clarke, 2006) of the features PTs considered in designing their images to support students' mathematical thinking as well as generative edits they reported making to Firefly's initial image outputs. Additionally, we offer a brief analysis of PTs' story problems in terms the Cognitively Guided Instruction problem types (Carpenter et al., 2014) and consider the features PTs considered in conjunction with problem types. We close the poster by (a) considering the affordances and challenges (from PT and instructor perspectives) in using Adobe Firefly for mathematically pedagogical purposes, (b) offering other examples of ways we have used text-to-image AIs in our courses for prospective mathematics teachers, and (c) providing some questions and directions that might be explored in future research endeavors.

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USING ASSISTMENTS FOR COLLEGE MATH: EVALUATING THE EFFECTIVENESS OF SUPPORTS AND TRANSFERABILITY OF FINDINGS

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Online homework systems are widely used in the instruction of college algebra at the post-secondary level (e.g., Boyce & O'Halloran, 2020; Hauk & Segall, 2005). The immediate feedback and variety of tutorial supports in these systems provide an opportunity for students' autonomy in their learning (Boyce & O'Halloran, 2020), encourage students to correct their mistakes (Affouf & Walsh, 2007), and participate in a cycle of learning of attempt-feedback-reattempt (Brewer, 2009). Studies from middle to post-secondary school indicate that students prefer to complete online assignments, which provide a source of motivation (Hauk & Segall, 2005; Ostrom & Heffernan, 2014). One no-cost system, ASSISTments (www.assistments.org) has been used in large quasi-experimental studies at the middle and high school level (Murphy et al., 2020; Singh et al, 2011). Research on the use thereof to support student learning and the use of formative feedback in instruction in middle-school and secondary classrooms is very promising (e.g., Feng et al., 2014; Kehrer et al., 2013; Kelly et al., 2013; Koedinger et al., 2010; Mendicino et al., 2009). Three of these studies provided insight into the effectiveness of "Best so far" supports, namely video hints vs. text hints (Ostrow et al., 2014), worked examples vs. hints (Shrestha et al., 2009), and single vs. multiple template questions (Jiang et al., 2020) and showed positive learning gains for students with these supports. Given this promise, we proposed adapting ASSISTments for use at the undergraduate level.

This research study conducted at two southeastern universities over two years aims to replicate the three aforementioned studies using ASSISTments in college algebra courses. The goal of this study is to determine whether the use of ASSISTments at the college level leads to similar gains in mathematics learning and analogous changes in pedagogical practices as reported in younger grades. Using the E-TRIALS platform, action-level data will be collected from individual students at the universities and combined with administrative data. Preliminary results on the use and effect of support within ASSISTments will be reported.

An understanding of the efficacy of different support can provide insights into the impacts of the design, implementation of education materials, and pedagogical practices. This speaks to research in the areas of learning sciences and technology, and mathematics education. Further, examining how college students use within-problem supports compared to younger students could help understand how mathematics learning changes chronologically. Finally, we can report average effect sizes on performance disaggregated by student characteristics to ensure findings do not inadvertently harm underrepresented groups in college mathematics courses, which was a component missing in many of the prior studies at the K-12 level.

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HIGH SCHOOL STUDENTS LEARNING DYNAMIC GEOMETRY ON AN IPAD VERSUS IN AUGMENTED REALITY

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Keywords: Technology, Geometry and Spatial Reasoning, Reasoning and Proof

Augmented Reality (AR) technologies allow for holograms to be layered over the real-world, "augmenting" human vision by overlaying illustrations onto 3D space. Contemporary AR systems allow students to interact with virtual objects using their bodies. This capitalizes on embodied views of the nature of cognition (Lakoff & Núñez, 2000; Nathan, 2021), which posit that all conceptual knowledge is understood and experienced through the body and is action-based in nature. Meta-analyses are numerous and estimate the effects of AR in educational interventions to be anywhere from d=0.64 (Garzón & Acevedo, 2019), to g=0.65-0.75 (Chang et al., 2022). However, less is known about the impact of AR holograms in math education specifically. Here we explore the question: What affordances does AR in a collaborative environment have for the future of mathematics education in uncertain times? We conducted a study where 120 high school students were randomly assigned in pairs to the AR treatment condition which used the Hololens2 or to the control condition of the iPad. Using the same Dynamic Geometry Software (DGS; Hollebrands, 2007), subjects were asked to collaboratively explore, evaluate, and justify their reasoning about six geometry conjectures (e.g., alternate interior angles are congruent; Figure 1). Half of the conjectures were about two-dimensional objects (e.g., triangle), and half were about three-dimensional objects (e.g., pyramid).

Figure 1. iPad Condition (left) versus HoloLens condition (right)

Data were analyzed in R using a multilevel cross classified logistic regression. Subjects with the iPad evaluating 2D conjectures performed better than both subjects with iPads evaluating 3D conjectures and subjects with HoloLens evaluating 2D and 3D conjectures (log-odds $\beta = 2.04$, SE = .062, $p < .001$). This effect was moderated by scores on a spatial reasoning test, which assessed ability to mentally rotate 2D and 3D objects (log-odds $\beta = 1.3$, SE = .45, $p = .004$). A simple linear mediation analysis suggests that students manipulating the geometric objects while evaluating the conjecture mediated 22% of the effect subject performance in iPad 2D vs. HoloLens 3D (mediation $\beta = 0.027$, SE = 0.012, $p = .024$, total effect $\beta = 0.118$, SE = 0.045, p =

.008). These results show important factors that impact AR's effectiveness for mathematics learning – including dimensionality, spatial ability, and types of interactions.

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RE-IMAGINING CLASSROOM TOOLS: MODELING MATHEMATICS IN MICROPROGRAMMING ENVIRONMENTS

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This paper presents an initiative aimed at integrating tailor-made micro-programming environments (MPEs) into middle school mathematics education to foster student learning and enhance computational thinking skills. We examine the effectiveness of MPEs in engaging students in computational thinking, aligning with mathematical practice standards, and usability in promoting interdisciplinary connections between mathematics and computing in the middle school setting. Findings suggest MPEs can advance computational thinking skills and enrich lesson alignment with educational standards. Most participants indicated approval of integrating MPEs into middle school classrooms.

Keywords: Mathematical Representations, Computing and Coding, Computational Thinking

In pedagogy-centered mathematics classrooms, educators encourage students to articulate and demonstrate their mathematical thinking through various representations, including diagrams, graphs, and verbal explanations. Incorporating MPE tools for mathematical modeling can enhance modeling effectiveness while exposing students to computer science. The idea of using programming to help students learn mathematics and science has a long history (Papert, 1980; Mayer, 2004; Hickmott et al., 2018), but few studies (Benton et al., 2017; Calao et al., 2015) explore learning key mathematical ideas through computing (Schanzer et al., 2015; Bråting & Kilhamn, 2021). Programming languages are excellent for externalizing and manipulating thought processes (diSessa, 2001; Schaffer & Kaput, 1998). While paper and pencil modeling or physical manipulatives are typically used to externalize thoughts, programming requires students to use explicit commands for each step in the problem-solving process. This approach can aid students in correcting their thinking and help teachers recognize misconceptions. General-purpose programming languages, or even tools such as spreadsheets, may be too unconfined to be effective for students and require prerequisite knowledge. Our design allows teachers to limit features, tailoring the MPE to the specific purpose of the activity. Through survey analysis, we assess MPEs' ability to engage learners in computational thinking, align with math practice standards, and overall usefulness in the middle school classroom. Findings suggest MPEs advance computational skills, align lessons to standards, garner general approval for classroom integration, and feature intuitive elements requiring minimal learning time, necessitating no expertise from teachers to implement.

Theoretical Framework

This study is grounded in Constructivism, which posits that learners construct their understanding and knowledge through experiences and reflection (Piaget, 1972). Constructivism highlights the importance of learners being active in their learning process, engaging in meaningful tasks (Vygotsky, 1978). The MPE embodies the constructivist approach by providing hands-on, experiential learning in computing. The MPE allows students to visualize and model their step-by-step problem-solving processes, making their understanding and misunderstandings visible. When the program does not perform as intended, students quickly realize mistakes, enabling immediate feedback and correction through debugging, a key aspect of learning in programming (Papert, 1980).

With the intentionality to bridge the gap between computer science and mathematics, the aim of the MPEs in this study is to create a constructivist environment that develops computational thinking and enhances student engagement in the Standards for Mathematical Practice (SMP):

- SMP1 “Make sense of Problems and Persevere in Solving them.”
- SMP2 “Reason Abstractly and Quantitatively.”
- SMP3 “Construct Arguments and Critique the Reasoning of Others,”
- SMP4 “Model with Mathematics.”
- SMP5 “Use appropriate tools strategically.”
- SMP6 “Attend to Precision.”
- SMP7 “Find and make use of structure.”
- SMP8 “Find and make use of repeated reasoning.”

MPE Modeling

Our MPE offers diverse strategies for solving mathematical problems. Consider the example: "A seventh-grade class needs 5 leaves daily to feed 2 caterpillars. How many leaves would the students need daily for 12 caterpillars?" Various mathematical strategies can be represented through operations performed on ratio tables. Teachers can limit available blocks to encourage specific strategies or provide options for students. Figure 1 illustrates four typical strategies, showcasing features of the MPE and its evolution of reasoning from concrete to abstract.

Explicit Copies: Manual Additive reasoning	**Repeat Block:** Automated Repeated Addition	**Avoiding Magic Numbers:** Multiplication by a scale factor, using only starting values	**Abstract Reasoning:** Generalized to calculate with any starting values.
Make a table make column caterpillars with 2 make column leaves with 5 copy row 1 copy row 1 copy row 1 copy row 1 copy row 1 add the last 6 rows together show the table	Make a table make column caterpillars with 2 make column leaves with 5 repeat 5 times do add row 1 to the last row show the table	Make a table make column caterpillars with 2 goal for caterpillars is 12 make column leaves with 5 multiply the last row by 12/2 show the table	Make a table make column caterpillars with 2 goal for caterpillars is 12 make column leaves with 5 divide the last row by caterpillars multiply the last row by goal show the table
row caterpillars leaves 1 2 5 2 2 5 3 2 5 4 2 5 5 2 5 6 2 5 7 12 30	row caterpillars leaves 1 2 5 2 4 10 3 6 15 4 8 20 5 10 25 6 12 30	row caterpillars leaves 1 2 5 2 12 30	row caterpillars leaves 1 2 5 2 1 2.5 3 12 30

Figure 1: MPE block-code is shown above, with the corresponding output table below.

Research Methodology and Design

This study addresses the following research questions:

- RQ1. Do micro-programming environments (MPEs) facilitate and enhance participant engagement in computational thinking skills?
- RQ2. Does integration of MPEs support the alignment with and application of mathematical practice standards in the context of middle school mathematics?
- RQ3. What are the perceptions of pre-service teachers on the usefulness and challenges of integrating MPEs into mathematics education?
- RQ4. Is the MPE tool intuitive enough for participants to make sense of the tool without teacher expertise?

This study employed a mixed-methods research design, combining qualitative and quantitative approaches to explore the effectiveness of integrating micro-programming environments (MPEs) into middle school mathematics education. The qualitative approach allowed for an in-depth understanding of participants' experiences and perceptions regarding the use of MPEs, while quantitative data were collected through surveys to supplement the qualitative findings.

As a preliminary to the in-class implementation detailed in publications to come, this trial was run on college students enrolled in a pre-service teacher program. The participants in this mixed-methods study were eleven pre-service teachers who engaged in a self-guided activity using the MPE. Among the participants, 2 (18%) were males and 9 (82%) were females. In terms of racial demographics, 7 (64%) identified as White, 3 (27%) as Black/African, and 1 (9%) as Asian/Vietnamese. The participants had diverse academic backgrounds, with 8 participants majoring in Biology, and one each studying Math, Physics, and Chemistry.

Data were collected through self-reported surveys administered after the MPE activity. The surveys assessed participants' educational backgrounds, experiences with the MPE tool, perceptions of computational thinking, and the usefulness of the MPE in middle school classrooms. Qualitative data analysis was conducted to identify themes and patterns in participants' responses. Thematic analysis was employed to categorize and interpret qualitative data related to participants' experiences, perceptions, and feedback on the integration of MPEs. Quantitative data from the surveys were analyzed using descriptive statistics to provide additional insights into participants' demographics and perceptions.

Analytical Results

RQ1: Computational Thinking

Survey responses revealed significant insights into participants' perceptions of computational thinking. Participants reported an increase in engagement with computational thinking skills during the MPE activity to 82%, compared to 49% in their previous education experiences. Comments included: “This activity developed my computational thinking skills well because I had to think about what I was doing and it helped me develop skills for computer thinking and problem-solving,” and “It really forced me to think about the meaning behind simple arithmetic.”

RQ2: Alignment to Standards for Mathematical Practice

Participants rated the MPE lessons' alignment with the standards for mathematical practice (SMP) on a scale of 0 (no alignment) to 5 (full alignment). The results showed that participants reported full alignment (5/5) with SMP1 "Make sense of Problems and Persevere in Solving them" and SMP3 "Construct Arguments and Critique the Reasoning of Others." The average alignment rating for the other standards was 4 out of 5, indicating near full alignment. Researchers observed that participants engaged more in discussions and debates when using the MPE (SMP4, SMP5), which required precision commands (SMP6, SMP7), than when modeling their thinking on paper. Quotes from the lesson include: "We can't get there from here," and "Yours is more efficient, but mine more clearly shows my thinking. In this format, the table can be used repeatedly for different numbers of days and different numbers of caterpillars."

RQ3: Usefulness of the MPE

Participants rated the usefulness of the MPE on a scale of 0 to 5. Analysis showed that 73% of participants considered the tool generally useful (rating of 3 or higher) in a middle school setting. Comments included: "I think that tools like this would be useful for introducing coding concepts for use in problem solving," and "For students that struggle to solve stepwise problems, this could be a useful tool to outline their thinking. Middle school is a good age to develop these skills." Thematic analysis revealed that participants appreciated the hands-on and experimental nature of the MPE activity, which made their thought processes visible. Some initially struggled with block coding but found it easier once they understood the process. Comments included: "The restriction of having to show every single step forced us to figure out how to solve every step using the program." Participants who rated the MPE "low" (0-1) reported 80% less PBL in their middle school experiences than those who rated it "high" (4-5), and half as much PBL as those who rated it "moderate" (2-3). This suggests that participants with less exposure to similar challenges in their own education found the tool less suitable for middle school settings, while those with more exposure found the tool more suitable. One participant commented: "I don't think I've done project based learning in my entire K-12 schooling, especially in math."

RQ4: Intuitive Design

Although students initially engaged in a productive struggle, all 11 participants were able to quickly deduce the mechanics of the tool without prior knowledge, teacher guidance, or intervention. One participant commented: "It took a while to model my step-by-step process on the block coding, but once I did, it was easy to see the process visually."

Conclusion

This study underscores the potential of Micro Programming Environments (MPEs) to transform middle school mathematics education by fostering computational thinking and aligning with mathematical practice standards. The integration of MPEs facilitates an active, constructivist learning environment where students can model and visualize their problem-solving processes, receiving immediate feedback and opportunities for correction through debugging. Participants reported significant engagement in computational thinking skills and observed near-full alignment with the Standards for Mathematical Practice.

The findings suggest that MPEs are effective tools for enhancing mathematical modeling and computational thinking skills among middle school students. Most participants indicated a positive reception towards integrating MPEs into classrooms, highlighting the tool's usefulness and ease of adoption without requiring extensive teacher expertise. This study contributes to the growing body of research advocating for the integration of computational tools in education, suggesting that MPEs can enrich the learning experience and bridge the gap between mathematics and computer science.

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ENHANCING SECONDARY PROSPECTIVE TEACHER'S EQUITABLE PRACTICE THROUGH TECHNOLOGY INTEGRATION

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Keywords: Technology, Pre-Service Teacher Education, Mathematics Teaching, Assessment, TPACK,

Research efforts have demonstrated that pre-service teachers (PSTs) often encounter challenges in interpreting students' mathematical thinking and conceptual understanding when technology-enhanced math tasks are involved (McCulloch et al., 2023; Smith et al., 2017; Yeo & Webel, 2019). The practice of teachers noticing students' mathematical thinking often involves three key components: attending to students' strategies, interpreting their conceptual understandings, and determining appropriate responses (McCulloch et al., 2023). PSTs can describe students' interactions with technology but they have challenges in finding sufficient evidence to connect these interactions with students' mathematical thinking (McCulloch et al., 2023; Smith et al., 2017; Yeo & Webel, 2019) which influences how they interpret students' conceptual understanding of the mathematical ideas embedded in the technology-enhanced task (McCulloch et al., 2023). The Association of Mathematics Teacher Educators (2017) Standards for the Preparation of Teachers of Mathematics notes the importance of teachers being proficient with tools and technology designed to support mathematical reasoning and sense-making. The primary question explore here involves the ways in which the integration of technology enhances future mathematics teachers' ability to draw insights about students' learning processes.

In an undergraduate Methods in Secondary School Mathematics course, we identified this common challenge among pre-service teachers. To address this challenge, we adopted a research-based approach to develop curriculum materials, integrating two theoretical frameworks: the TPACK framework (Technology Pedagogical Content Knowledge) and TQE process (Tasks, Questions, and Evidence). In each module, initially, pre-service teachers, based on the TPACK framework, learn about ways to recognize different types of technology in serving different purposes by considering technology choice, curriculum goals, and pedagogical content knowledge (Mishra & Koehler, 2006). Then, we asked them to engage with a technology-enhanced math task as learners by using technology tools: Desmos and GeoGebra. In this step, based on the TQE process, they involved responding to questions and interacting with peers in small group discussions through using technology. This active involvement allows PSTs to anticipate student thinking and understand the learning process more comprehensively. Following this approach, PSTs observed the ways that using technology and implementing the TQE process generate various evidence for formative assessment (Nolan et al., 2016), while also integrating individual learning and small group discussions to explore mathematical concepts deeply. Finally, we asked the teacher candidates to analyze other groups' work and engage in discourse in whole class discussion. As PSTs continue using the TPACK framework and TQE process within technology-based learning during the course, they gather more evidence to connect students' engagement with technology to the students' mathematical understanding. This knowledge enables them to provide personalized instruction and consistent feedback to address the needs of all learners through the use of technology (Black & Wiliam, 2009; Black & Wiliam, 2018; Neumann et al., 2021).

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THE IMPACT OF VIRTUAL REALITY TO SUPPORT STUDENT UNDERSTANDING OF POWERS OF TEN

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Keywords: Number Concepts and Proportional Reasoning, Teaching Practice and Classroom Activity, Technology and Learning Environment Design, Algebraic Thinking

This study examined how virtual reality (VR) supports students' understanding and visualization of the nanoscale. A 90-minute VR session allowed students to explore the human hand and zoom inside the hand to see cells, blood vessels, and even individual molecules. Students were in control of the simulation and could look around and explore what was happening

Methods

This study had eleven participants: students aged eleven to fourteen, seven identifying as female, and three as male. This camp was conducted at a mid-sized Western university. The camp focused on nanotechnology. The culminating design project asked students to create the largest soda geyser possible based on what was learned throughout the camp. Even after multiple activities students still struggled to visualize the nanoscale. Therefore, a design decision was made to utilize virtual reality to allow students to explore the nanoscale.

Findings

The VR gave students a concrete understanding of the nanoscale and allowed students to bridge the gap between macroscopic and nanoscale scales. After the VR session, students could discuss and understand how materials react at the nanoscale. In the post-activity discussion, students understood just how small nano is and the scale at which the nanoworld exists.

Discussion

Virtual reality allowed students to develop a concrete understanding of powers of 10. Students were able to use proportional reasoning to scale down to the nanoscale. This aligns with the work of Lamon (2020), where students were able to reason up and down to the appropriate scale. This is evident through how students talked and thought about the nanoscale and how they could use this knowledge to ultimately solve a design problem (creating the tallest soda geyser they could). The VR session provided students with an embodied cognitive perspective. Students engaged in graspable math and playful learning in a virtual learning environment (Abraham et al., 2020). The virtual reality simulation provided a real-world context inside the hand. It was an open-ended environment where students could flexibly explore things they found interesting to make sense of size and scale. Further research is needed to explore the use of virtual reality to support math learning.

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UTILIZING CONJECTURE MAPPING TO DESIGN A DIGITAL TASK FOR DEVELOPING PRODUCTIVE GRAPHING MEANINGS

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In this preliminary theoretical paper, we describe our use of conjecture mapping (Sandoval, 2014) to guide the design of a digital task sequence to support 6th graders' meanings for points as a simultaneous representation of the amount-ness of two quantities. The conjecture map ultimately serves as a theoretical framework with testable conjectures about how the design of the digital task sequence might promote the intended learning outcomes.

Keywords: Design Experiments, Technology, Mathematical Representations

Graph construction and interpretation are critical skills for advanced mathematics coursework and consumption of popular media (Glazer, 2011). Despite the importance of graphs in K-12 mathematics and science curricula (CCSS; NGSS), research indicates that students struggle with graph construction and interpretation well into their post-secondary studies (e.g., Carlson et al., 2002; Glazer, 2011). One explanation for students' challenges with graphing is that they develop meanings for graphs that are useful initially but are limited as they advance through the mathematics curriculum (e.g., understanding points as a set of directions for how far to move over and up from the origin; Frank, 2016). We posit that one way to address this challenge is to support students in developing more productive meanings for graphs when they first encounter graphs in the curriculum. A promising approach to understanding graphs is emergent graphical shape thinking (EGST; Moore & Thompson, 2015). In this theoretical report, we describe our effort to design a task sequence that supports 6th grade students (11-12 years old) in developing EGST. This task design effort was undertaken between rounds of a design-based research study (Cobb et al., 2003), and we utilized conjecture mapping (Sandoval, 2014) to guide our design toward the dual goals of developing theory about how the design of the learning environment functions and about how the development of productive graphing meanings occurs.

Background

Within a multi-year design-based research study (Cobb et al., 2003), we have been working to develop an instructional sequence that supports 6th grade students in developing EGST. We conducted multiple rounds of small group teaching experiments (Steffe & Thompson, 2000) in which pairs of students worked through our task sequences. The ultimate learning goal we intended to support was students' development of EGST.

EGST entails conceiving of a graph as a record of covarying quantities (Moore & Thompson, 2015) which can be created by imagining the trace of a point moving through the coordinate plane such that the motion of the point is constrained by the relationship between situational quantities. As such, developing EGST requires attention to students' quantitative reasoning within situations and graphs. Quantities are conceptual entities grounded in an individual's conception of a situation. "A person is thinking of a quantity when he or she conceives a quality of an object in such a way that this conception entails the quality's measurability" (Thompson, 1994, p. 7). An attribute of a situation is measurable to an individual if they can conceive of a process for measurement that results in an amount and a unit (or the anticipation of a unit). The situation in which a person constructs quantities can be a graph (e.g., constructing vertical distance above the horizontal axis as an attribute of a point) or some experientially real context (Gravemeijer & Doorman, 1999; e.g., constructing weight and habitat temperature for animals at the zoo). Although engaging in EGST entails conceiving of varying quantities, in this report we detail an effort to support students in what we conjecture is a prerequisite meaning for graphs as representing relationships between static quantities.

Informed by Paoletti et al.'s (2023) LIT for developing EGST with advanced 8th graders, our high-level conjecture is that repeated occasions to draw explicit connections between meanings for situational quantities (situational quantitative reasoning; SQR) and meanings for graphical quantities (graphical quantitative reasoning; GQR) is critical for students' developing EGST. Given space constraints in this report, we focus on the first level of SQR and GQR. Denoted as SQR1 below, students must first construct quantities in a situation and conceive of the quantities as being able to take particular amounts in the situation. GQR1 entails considering the length of a magnitude bar as representing a static amount (i.e., constructing the quantity of length). Bridging SQR1 and GQR1 (SQR1↔GQR1) entails considering a magnitude bar as representing the static amount-ness (Stevens & Moore, 2017) of a situational quantity.

We developed the Zoo Task sequence to provide students with opportunities to reason about points as a simultaneous representation of the amount-ness of two situational quantities. However, retrospective analysis from our first two rounds of teaching experiments that used the task indicated that students needed more (or different) opportunities to bridge SQR and GQR. We decided to redesign the Zoo Task sequence to meet this need and took up conjecture mapping (Sandoval, 2014) as a strategy for redesigning the task in ways that would enable us to test and refine our conjectures about the task design and process of developing SQR and GQR. Conjecture mapping attends to the dual goals of design-based research by differentiating between theories about the design of the learning environment and theories about the process of learning. Conjecture maps depict the ways researchers anticipate the design of the learning environment supporting learners in engaging in observable processes as well as conjectures about how engagement with those observable processes results in the desired learning outcomes. Our initial conjecture map is in Figure 1.

The Zoo Task

We describe the opening sequence (screens 2-5) of the Zoo Task to ground descriptions of the conjecture map in the next section. Due to space constraints, we only report on the opening sequence of the task which we designed to support students' SQR1, GQR1, and SQR1↔GQR1. To support the reader, we provide a link to the opening activity sequence so that the digital interactions we describe here can be experienced as we designed them (https://bit.ly/ZooTaskPMENA). On Screen 2 (Figure 1a), students are prompted to weigh three mystery animals at the zoo and record their weight. Students can weigh each animal by dragging it to a scale and pressing the 'Weigh It' button. In response to those actions, the scale depresses with a bounce (imagine a heavy object being placed on a spring-loaded plate) and the weight of the animal is represented with numbers and a magnitude bar. Students then record the animal's weight by dragging a point to construct a vertical magnitude bar with a numeric readout. On Screen 3 (Figure 1b), students are prompted to order five animals from lightest to heaviest given five vertical magnitude bars. Three of the magnitude bars are copied over from their construction on Screen 2 and include a numeric readout, but the other two magnitude bars represent the weight of new animals and do not have numeric readouts. When students select an order and press the ‘Check It’ button, the vertical magnitude bars are dynamically rearranged to reflect the student’s selection and evaluative feedback (a green checkmark) appears next to the animal names in the list if they are in the correct location within the list. Screens 4 and 5 follow a similar design but with opportunities to measure habitat temperature by dragging a temperature probe into each enclosure and then recording values on horizontal magnitude bars.

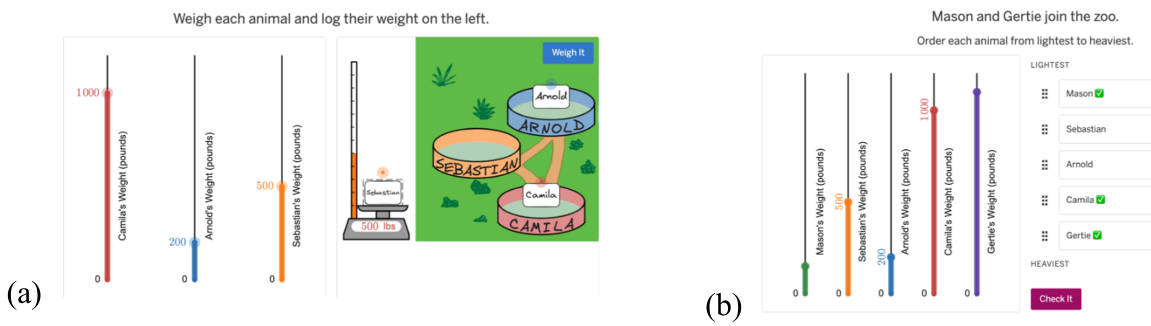

Figure 1: (a) Screen 2, and (b) Screen 3 of the redesigned Zoo Task.

An Initial Conjecture Map for the Redesigned Zoo Task

Recall, our high-level conjecture is that repeated occasions to draw explicit connections between meanings for situational quantities (SQR) and meanings for graphical quantities (GQR) is critical for development toward EGST. To test that conjecture, we need students to (1) develop SQR, (2) develop GQR, and (3) bridge SQR and GQR meanings. Due to space constraints, we report on the opening sequence of the zoo task that we intend to support students in developing the first level of SQR, the first level of GQR, and bridging between those two meanings.

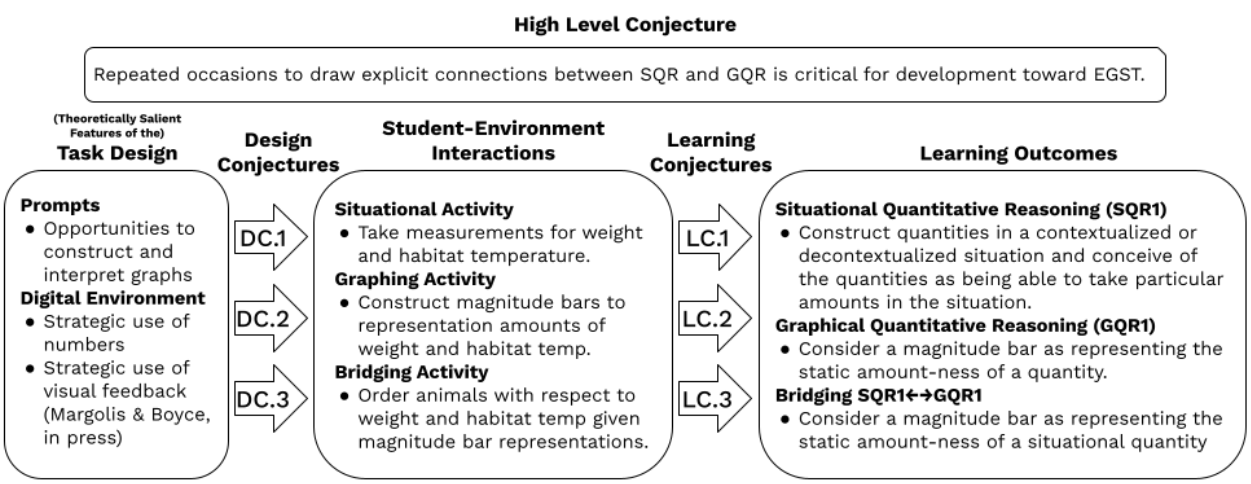

Figure 2: An initial conjecture map to guide the redesign of the Zoo Task.

To support SQR1, we theorized that directly measuring weight and habitat temperature for several animals supports students in conceiving of an attribute of the animals (i.e., heaviness;

hotness) as being measurable (i.e., having an amount and a unit) (LC1; see Figure 2). To support GQR1, we theorized that constructing magnitude bar representations to record particular values would support students in understanding that the length of a magnitude bar represents an amount-ness (e.g., longer bar = larger amount) (LC2). Furthermore, using the magnitude bars to represent particular amounts of weight and habitat temperature derived through direct measurement can help students bridge SQR1⟷GQR1 (LC2). Lastly, we theorized that ordering animals with respect to a situational quantity (weight or habitat temperature) when provided with information about those quantities via magnitude bar representations would support SQR1⟷GQR1 because students would have to set a goal related to the situation (e.g., determine which animal weighs the least) and then use information from the graphical representation to achieve that goal (e.g., which magnitude bar is the shortest) (LC3).

The observable interactions between student and digital environment that are necessary for testing these learning conjectures are listed in the middle column of the conjecture map. Students need to (1) directly measure the weight and habitat temperature of several animals (toward LC1), (2) construct magnitude bar representations of particular weight and habitat temperature amounts (toward LC2), and (3) interpret magnitude bar representations to order animals with respect to weight and habitat temperature (toward LC3).

Next, we developed design conjectures that link the theoretically salient aspects of the task design to the production of desired student-environment interactions. Our goal was for this activity to be a stand-alone digital activity, so we wanted students to be able to directly measure the weight and habitat temperature of zoo animals *within* the digital environment. We theorized that we could design student-environment interactions that emulate direct measurement by coordinating available actions (i.e., drag to the scale and click 'Weigh It') and visual feedback (i.e., "bouncing" on the scale) (DC1). To support students in constructing magnitude bar representations of particular amounts of weight and habitat temperature, we theorized that strategic use of numbers could support students in linking the result of direct measurement in the situation with their understanding of magnitude bars as representing amounts (DC2). The reification of this design conjecture can be seen in Figure 1a where the 500 pounds can be seen as the dynamic label on the magnitude bar representing Sebastian's weight as well as the result of measuring Sebastian's weight on the scale. Our final design conjecture is that strategic use of numbers and strategic use of visual feedback (Margolis & Boyce, in press) can support students in utilizing magnitude bar representations to order animals with respect to weight and habitat temperature (D3). Specifically, when we prompt students to order animals (Screens 3 and 5; Figure 1b), they can view the magnitude bars with numeric readouts for the three animals that they measured on Screens 2 and 4 but are not provided with numbers for the two new animals. We anticipate that this strategic use of numbers will result in students' reasoning about the bars' lengths rather than reasoning about the relationship between values. After students select an order and press the 'Check It' button, the magnitude bars dynamically rearrange to reflect the order of their list. When a student has the animals out of order (as in Figure 1b), we anticipate that the reordered magnitude bars will be useful for reasoning about the necessary adjustments.

Discussion & Future Work

We posit that conjecture mapping is a useful tool for studying the complex links between task design and mathematics learning. Our initial conjecture map serves as a preliminary theoretical framework with testable conjectures about how to design digital tasks that support 6th grade students in developing SQR1, GQR1, and SQR1↔GQR1. Future work can focus on empirically verifying the design and learning conjectures from this conjecture map. Additional work can focus on whether and how the development of SQR, GQR, and SQR↔GQR support the development of EGST. Such research could lead to the development of curricular materials that alleviate student struggles with graph construction and interpretation important for their future in advanced coursework and as critical citizens.

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COMPARING CONSTRUCTS FOR ASSESSING TEACHERS' PREPARATION TO TEACH MATHEMATICS WITH TECHNOLOGY

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This study examines the preparation of prospective teachers (PSTs) in teaching secondary mathematics with technology. It compares the assessment of PSTs' preparedness using two constructs: Vision of High-Quality Mathematics Instruction with Technology (VHQMIw/T) and Technological Pedagogical Content Knowledge (TPACK). To unpack this, we explore the journey of Avery, a prospective secondary teacher, within the context of Teaching Secondary Mathematics with Technology course. The study finds that while TPACK focuses on technological integration, VHQMIw/T may offer a more comprehensive understanding of PSTs' preparedness, especially in envisioning instructional practices with technology. The authors recommend using both constructs to assess PSTs' preparedness effectively.

Keywords: Instructional Vision, Preservice Teacher Education, Technology

The field of mathematics education has long agreed on the importance of secondary mathematics teachers being prepared to support students' learning of mathematics using technology that supports students' mathematical reasoning and sense making (ISTE, 2000, 2017; AMTE, 2017, 2022; NCTM, 2014, 2023). However, assessing the development of prospective mathematics teachers (PSTs) toward this goal is difficult (e.g., Abbitt, 2011). Researchers have called for the use of PSTs' instructional vision (Hammerness, 2001) to assess the development of their pedagogical practices during teacher preparation programs (Fieman-Nemser, 2001; Arbaugh et al., 2021). Munter (2014) described instructional vision as "ways of seeing the world that encompass horizons not yet reached" (p. 587). While instructional vision has been shown to be a helpful construct to assess PSTs' preparedness to teach mathematics (e.g., Arbaugh et al., 2021; Walkowiak, et al., 2015), to date there is scant research on the use of instructional vision to assess PSTs preparedness to teach mathematics with technology. Rather, the most common way of assessing PSTs' preparedness to teach with technology is through attending to the development of their specialized knowledge for teaching mathematics with technology referred to as technological pedagogical content knowledge (TPACK; Mishra & Koehler, 2006). The purpose of this paper is to compare and contrast what we can learn about PSTs' preparation to teach secondary mathematics with technology through attending to these two constructs, vision of high-quality mathematics instruction with technology (VHQMIw/T) and TPACK.

Theoretical Frameworks

Our approach to preparing PSTs to teach mathematics with math action technologies (MAT: Dick & Hollebrands, 2011) suggests two different ways of framing, and assessing, their development, 1) attending to the development of their VHQMIw/T or 2) attending to the development of their TPACK. We describe both theoretical approaches in the sections that follow.

VHQMIw/Tech

Instructional vision is a discourse that teachers (including PSTs) "employ to characterize the kind of ideal classroom practice to which they aspire but have not yet necessarily mastered" (Munter & Wilhelm, 2021, p. 343). As such, one's instructional vision is an expression of their appropriation of the principles, frameworks, and ideas about teaching and learning that they have encountered through their personal and professional learning experiences (Munter & Wilhelm, 2021). Munter (2014) described a specific vision of mathematics instruction deemed "high-quality" that is aligned with the literature on effective mathematics instruction, guiding frameworks in mathematics education (e.g., NCTM, 2014) and data collected from the Middle School Mathematics and the Institutional Setting of Teaching project (Cobb & Smith, 2008). As our work is in the context of using MATs, we refer to discourse about high-quality mathematics instruction that incorporates MATs in ways that are aligned with the literature on effective teaching with technology one's VHQMIw/T.

Like Munter's VHQMI, researchers have long characterized successful technology integration with a specific vision toward "constructivist, student-centered technology use" that includes "active and collaborative learning through authentic problem solving and knowledge construction" (Kopcha et al., 2020 p. 730). In fact, Kopcha et al. (2020) point out that many of the frameworks used to describe technology integration characterize student-centered approaches as high-quality. Thus, when considering a VHQMIw/T in the context of using MATs, the only real difference from Munter's VHQMI should be in the type of task used during instruction.

To operationalize how closely one's vision is aligned with the specific VHQMI described in the literature, Munter (2014) created three interrelated rubrics: *role of teacher, classroom discourse,* and *mathematical tasks*. Each dimension has its own 5-point rubric indicating a trajectory of VHQMI with 4 as the highest and 0 as the lowest. We extensively adapted the *mathematics tasks* rubric since the curriculum materials used in our work focused specifically on the use of tasks that include MATs. On the adapted VHQMIw/T *technology-enhanced mathematics task* rubric the descriptions are parallel to Munter's. A score of 0 or 1 indicates that the PST does not envision using a MAT and either "does not view tasks as a manipulatable features of classroom instruction" (0) or "emphasizes tech tasks that provide students with an opportunity to practice a procedure before applying it conceptually to a problem" (1). To score a 2 or higher it must be clear that a MAT is being used in the task. A score of 2 emphasizes "'reform'-oriented aspects of MAT tasks [e.g., "explore," "higher-order"] without elaborating on their function in terms of learning mathematics—often more about motivation/engagement", 3 emphasizes "MAT tasks with multiple solution paths, potential for complex thinking/problem-solving, but no emphasis on generalization, connections btw strategies/representations, etc.", and a score of 4 is characterized by an emphasis on MAT use in ways "that have the potential to engage students in 'doing mathematics' Munter (2014, p. 633)".

TPACK

Researchers in teacher education have built upon Shulman's (1986) notions of teachers' pedagogical content knowledge (PCK). Grossman (1989, 1990), Even (1990), and Hill et al. (2008) drew upon ideas of PCK and further delineated specific constructs for mathematics within PCK. However, none of this work considered the knowledge that comes with teaching mathematics with technology. In 2005, Niess adapted Grossman's (1989, 1990) components of PCK to take technology into consideration and referred to technology-enhanced PCK. In 2006, Mishra and Koehler identified such knowledge as TPACK. The TPACK framework describes the type of knowledge teachers need to understand how to use technology effectively to teach specific subject matter. In 2013, Neiss identified four components of TPACK with detailed descriptors and claimed they provide insight for "developing a transformed knowledge [of] (TPACK)" (p. 196). These four components: "(1) an overarching conception of what it means to teach a particular subject integrating technology in the learning; (2) knowledge of instructional strategies and representations for teaching particular topics with technology; (3) knowledge of students' understandings, thinking, and learning with technology in a particular subject; and (4) knowledge of curriculum and curriculum materials that integrate technology with learning in the subject area" (Neiss, 2005, p. 511) capture the skills teachers need to develop TPACK.

Many strategies have been used to assess TPACK including teacher interviews, team planning and classroom observations, self-reported surveys, open-ended questionnaires, and performance-assessment instruments (Mouza et al., 2014). Studies that use validated instruments to assess performance-assessments or artifacts of teaching have proven quite useful (e.g., Harris et al., 2010; Hofer et al., 2011; Lyublinskaya & Tournaki, 2012) in providing a glimpse into teachers' TPACK applied in their classrooms. Lyublinskaya and colleagues used the four levels as a guide to develop a validated TPACK Levels Rubric (Lyublinskaya & Tournak, 2012; Lyublinskaya & Kaplon-Schilis, 2022) used to code teachers' technology-enhanced teaching artifacts (e.g., lesson plans) for evidence of their TPACK.

Lyublinskaya and Tournak's (2012) rubrics include five levels of TPACK development (1-Recognizing, 2-Accepting, 3-Adapting, 4-Exploring and 5-Advancing) which are applied to each of Neiss' (2009, 2013) four components of TPACK. At the lowest level, 1-Recognizing, teachers use technology as a motivational tool, not to support students' mathematical thinking and are focused on rote practice. At level 2-Accepting, the use of technology is instructor led and focused on teacher delivery of information often mirroring traditional textbook material. For the middle level 3-Adapting, teachers begin to use technology as a source of student inquiry to support students' mathematical thinking under direct teacher guidance and without opportunity for student reflection; math action technologies may or may not be used at this level. At level 4-Exploring, students become the primary driver of explorative technology making full use of math action tools within the technology, however the teacher "still guides the students to see the meaningful consequences of those actions" (Lyublinskaya & Tournak, 2012). At the highest level, 5-Advancing, students are provided opportunities to explore, make conjectures, reflect and develop their own conceptual understanding of mathematical concepts.

Methods

This is an instrumental case study (Stake, 1995) of a single PST, Avery, who participated in a course titled *Teaching Secondary Mathematics with Technology*. Avery was a mathematics major and secondary mathematics education minor. This was the first mathematics specific methods course. The course was designed using principles of practice-based teacher education (Grossman et al., 2009), used curriculum materials from the Preparing to Teach Mathematics with Technology – Examining Student Pracices project [PTMT-ESP], and the course text was *Exploring Math with Technology: Practices for Secondary Teachers* (McCulloch & Lovett, 2024). Throughout the course Avery had many opportunities to engage in high-quality technology-enhanced math tasks as a learner, to analyze the design of these tasks, to analyze video of students and teachers engaged with these tasks, to create technology-enhanced math tasks, and to plan lessons that incorporate technology-enhanced math tasks. Through these experiences, Avery was developing both their VHQMIw/T and their TPACK. Avery was selected as the case for this study because they scored high on both VHQMIw/T and TPACK at the end of the semester, we hoped this similarity would provide insight to what we can learn about a PSTs' preparedness from these two different perspectives. As such, we aim to answer the following research question: What are the similarities and differences in what we can learn about PSTs' preparation to teach mathematics with technology-enhanced mathematics tasks by attending to their VHQMIw/T and their TPACK?

For the purposes of this study, we are focusing on artifacts from the end of the semester to understand Avery's preparation to teach secondary mathematics with technology at that point in time. This includes his description of his VHQMIw/T and a technology-enhanced math task that he created along with its accompanying lesson plan. The data sources and our analysis of them are described in the sections that follow.

TPACK: Data Source and Data Analysis

All PSTs enrolled in the *Teaching Secondary Mathematics with Technology* course along with Avery, were asked to design a technology task and an accompanying lesson plan. In a prior lesson PSTs had engaged in a sequence of approximations of practice related to a Desmos Activity designed to introduce amplitude, midline, and period of the sine function (tinyurl.com/IntroSine). This included anticipating student thinking, noticing student thinking, scripting whole class discussion, and analyzing video of the classroom teacher as she monitored small groups and facilitated a whole class discussion. At the end of this sequence the PSTs were assigned to design a follow up lesson. They had the option to create a task and lesson that either a) provides an opportunity for the students *to further develop* their understanding of amplitude, midline, and period related to sine functions and their graphs, or b) provides an opportunity for her students *to apply* their knowledge of amplitude, midline, and period to a real context through modeling, or c) an investigative task intended *to introduce* phase shift to go along with amplitude, midline, and period. For this assignment, Avery chose option c.

To analyze the tasks and accompanying lesson plans, we used the TPACK Levels Rubric (Lyublinskaya & Tournak, 2012; Lyublinskaya & Kaplon-Schilis, 2022) to capture the PSTs' TPACK levels across the four dimensions: overarching conception, knowledge of student understanding, knowledge of curriculum, instructional strategies. Each dimension has its own 5-point rubric indicating a growth trajectory of TPACK with 5 as the highest and 1 as the lowest. All tasks and accompanying implementation plans were coded by four researchers and then discrepancies were discussed until consensus was reached across the four coders. A composite score was computed (i.e., the sum of the four dimensions).

VHQMIw/T: Data Sources and Data Analysis

Similarly, all PSTs enrolled in the course with Avery responded to a vision prompt adapted from Munter (2014): If you were asked to observe a technology-using math teacher's classroom for one or more lessons, what would you look for to decide whether the mathematics instruction (including the use of technology) was high quality? In your response make sure you describe what you would expect to see/hear from the teacher, students, and mathematical tasks during your observations.

To analyze the PSTs' vision statements, we used our adapted version of Munter's VHQMI rubrics. These rubrics include 3 interrelated dimensions: role of teacher, discourse, and technology-enhanced math tasks. Each rubric indicates alignment with a research informed VHQMIw/T with 4 being the highest, 0 the lowest, and N/A indicating the dimension was not included in the PSTs' vision statement. Like the TPACK analysis, all vision statements were coded by four researchers and then discrepancies were discussed until consensus was reached across the four coders.

Findings

Avery's scores for both VHQMIw/T and TPACK are found in Table 1. Based on the analysis of their vision statement and lesson plan artifacts, Avery would be described as having a VHQMIw/T that is aligned with the literature on effective teaching and learning with technology and TPACK that is aligned with the advancing level of technology integration. In what follows we unpack what we learn from these rubric scores about Avery's preparedness to teach mathematics with technology-enhanced tasks. We begin with VHQMIw/T, then TPACK, and finally we compare and contrast the two.

VHQMIw/T Rubric Scores (max: 4)		TPACK Rubric Scores (max: 5)	
Role of Teacher	4	Overarching conception	5
Discourse	4	Knowledge of student understanding	5
Tech-enhanced math task	4	Instructional strategies	4
		Knowledge of curriculum	5
Composite Score	4	Composite Score	4.75

Table 1: Avery's TPACK and VHQMIw/T Rubric Scores

Avery's VHQMIw/T

When responding to the VHQMIw/T prompt, Avery began by clearly stating their overarching VHQMIw/T and then went on to describe how to achieve that vision. Avery articulated their VHQMIw/T as,

> A high quality, equitable math instruction with technology would include the following: students exploring the mathematics with dynamic math technology, each student/group working with the technology equally, and a lesson designed to incorporate the student's understandings and work into a discussion that furthers the whole-class understanding around the mathematics topic.

In terms of the *role of the teacher*, Avery expanded on this statement by describing that when students are working on a task, a teacher's role is to "ask students assessing questions that help the teacher see students' understanding then ask advancing questions that lead students to a more developed understanding." Avery envisions that an instructor "guides and facilitates, occasionally may need to step in to reach the necessary daily goals, but should almost never directly explain a topic in full without student input." Avery scored a 4 on the *role of the teacher* rubric since they described the teacher's role as more than just a facilitator. By describing that the teacher "should almost never directly explain a topic in full without student input" it is clear that Avery's vision is the teacher as a more knowledgeable other.

With respect to *discourse* Avery also wrote that their VHQMIw/T includes a "full-class discussion around understanding the mathematics." Avery expanded on this stating that "an instructor should facilitate a discussion on the topics using the students' work. This is done through carefully cultivating the responses and ordering them for discussion." Avery went on to explain that a teacher "should lead the discussion by having open-ended questions posed to them that allow them to identify, compare, contrast, and critique the responses." Avery provides concrete images of students learning from each other. Therefore, Avery scored a 4 on the *classroom discourse* rubric.

Finally, with respect to t*echnology-enhanced tasks*, Avery noted that students should explore mathematics and expanded this idea noting that a high-quality technology-enhanced math task "must be dynamic, it must allow students to explore and notice facts and relationships about the topics being presented." Here Avery focused on connections between the mathematical ideas and described a task that would align with "doing mathematics" (Smith & Stein, 1998), therefore Avery scored a 4 on this rubric.

Avery's TPACK

For three of the four components of TPACK – *overarching conception, knowledge of student understanding,* and *instructional strategies* – Avery scored at the advancing level (5). With regard to their *overarching conception*, Avery's lesson plan included a high-cognitive demand (Smith & Stein, 1998), technology-enhanced task built in Desmos Activity Builder. The task focused on developing students' understanding of phase shift and provided opportunities for inquiry and reflection. For example, Avery's task asked students to explore the relationship between the parameters of the sine function when a new parameter was added. Students were then asked to write down what they notice when examining the slider for *h*. This example demonstrates the connection to conceptual understanding through both inquiry and reflection that was seen throughout the task.

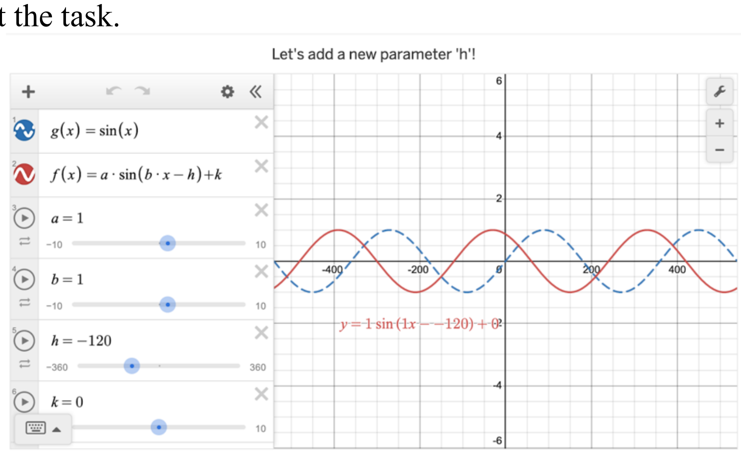

Figure 1: Students could explore the phase shift of the function using sliders

Within *knowledge of student understanding,* Avery's advanced rating of 5 was evident in Avery's task that relied on students taking mathematical actions (using sliders in Desmos) to examine the impact of *h* and *b*, see the consequences of their mathematical actions, and draw conclusions about phase shift (i.e., phase shift = *h/b*) based on these consequences. The lesson plan included how Avery planned to use the resulting student thinking to facilitate discussions and make connections across multiple representations, allowing student thinking to drive the direction of the lesson. For example, Avery included,

> After students have completed the activity, I plan to facilitate a whole class discussion on phase shift. I will ask students to share their noticing looking for…
> - An informal description of how the h slider alters the graph (horizontally shifts)
> - A description of the term phase shift
> - An informal description of the relationship between the slider and phase shift
> - A precisely described mathematical relationship between h, b, and phase shift (direct with h-h/b)

I want to make sure students connect the horizontal shift with phase shift and understand their slight differences…

Avery's advanced rating (i.e., score of 5) for *instructional strategies* was due to the use of sliders within the mathematical task that provided students with an inductive strategy that effectively supported students' exploration of phase shift and included prompts to promote reflection and sense making/reasoning.

With respect to the remaining component, *knowledge of curriculum*, Avery demonstrated an exploring level related to TPACK (i.e., score of 4). This was evidenced in how effectively the task was aligned to the learning and performance goals included in the lesson plan, and how the task provided students an alternative way to explore the mathematical topic (i.e, alternative to using paper and pencil methods) and expand on the mathematical ideas they build with respect to amplitude, midline, and period in the prior lesson through their exploration of the function. Avery's lesson did not score a 5 on the *knowledge of curriculum* rubric because the task did not make connections outside of the curriculum or challenge the traditional curriculum to have students learn different topics.

Comparing and Contrasting Avery's VHQMIw/T and TPACK

Both the VHQMIw/T and TPACK rubric scores suggest that Avery is well-prepared to teach mathematics with technology. What we are curious about is what we learn about Avery's preparation from each of these measures. We begin with the more commonly used construct, TPACK.

Since Avery was in a class focused on teaching mathematics with technology and the lesson planning assignment at the end of the semester required the inclusion of a technology-enhanced task that included a MAT, as long as they created a lesson that met the requirements of the assignment his TPACK rubric scores were going to all be 3, 4 or 5, leaving little room for variability. The TPACK rubrics highlight that Avery is well-prepared to design a high-cognitive demand technology-enhanced task, aligned with learning goals, that provides ways for students to interact with the mathematical objects and prompts them to both explore and reflect. However, we know less about how Avery plans for students and teachers to interact with each other when engaging with the task. The *knowledge of student understanding* and *instructional strategies* rubrics do indicate that the teacher plans to facilitate students' use of the technology in ways that lead to deep understanding of mathematics, and that they will use both deductive and inductive strategies to do so. So there is a sense from the rubric scores that Avery will carefully facilitate the implementation of the technology-enhanced task and resulting whole-class discussion, but this does not provide insight to how Avery's hypothetical students will interact with each other's ideas.

In contrast, the VHQMIw/T rubrics provide insight to not only what Avery envisions a high-quality technology-enhanced math task to be, but also provides further details for how Avery envisions the teacher and students' interacting around such a task. The *technology-enhanced mathematics task* rubric indicates that Avery designed a task that uses a MAT and is of high-cognitive demand, to score a 4 on this rubric they also had to explain how such a task would support student learning (i.e., function view). This additional explanation is not captured in the TPACK rubric. The *role of the teacher* rubric score indicates that Avery envisions the teachers' role as a more knowledgeable other who is proactively supporting students' learning through anticipating student thinking as part of the lesson planning process and then during the lesson, using student work to drive whole class discussions around the important mathematical concepts and connections. Thus, Avery is not only planning to use deductive and inductive strategies as indicated in the TPACK rubrics, but is going to leverage the students' thinking to drive the deepening of their understanding. Finally, the *discourse* rubric indicates that Avery envisions students' learning from each other, with the mathematical discourse often being student initiated and students talking to each other, not solely through the teacher. None of the TPACK rubrics capture the nature of the planned discourse.

Discussion and Conclusion

Comparing and contrasting VHQMIw/T and TPACK using Avery's work at the end of a course focused on preparing PSTs to teach secondary mathematics with technology does reveal some differences in what we can learn about PSTs' preparedness using these two constructs. The most striking finding is that the TPACK rubrics do not capture the nature of planned discourse, including how one envisions the role of the teacher during small group and whole class instruction. The VHQMIw/T *role of teacher* and *discourse* rubrics do capture these important aspects of mathematics instruction. In their review of technology-enhanced pedagogy in teacher learning, Zinger et al. (2017) called for less attention to PSTs' use of technological tools and more attention on the role of the teacher in using those tools to address problems of practice. Our findings suggest that VHQMIw/T might be a helpful framing for researchers taking on that work.

PSTs often do not have field experiences in courses in which they are learning to teach mathematics with technology (McCulloch et al., 2021), making assessing TPACK based on their practice difficult. With this in mind, rather than attending to their enacted instruction, researchers have called for attending to PSTs' instructional vision as an indication of their developmental progress during teacher preparation programs (e.g., Feiman-Nemser, 2001), noting that changes in instructional vision often occur before changes in practice (e.g., Munter, 2014). Our findings suggest that the VHQMIw/T rubrics do provide insight into how PSTs envision the design of technology-enhanced mathematics tasks – the main focus of the TPACK rubrics – while also providing insight to how they hope to one day facilitate students' working on such a task.

Based on the findings in this study, we would ultimately recommend using both the VHQMIw/T and TPACK constructs to understand PSTs' preparedness to teach mathematics with technology. However, assessing TPACK as a pre-test is difficult to do when PSTs have yet to be introduced to teaching mathematics with technology (i.e., it is unfair to ask them to design a technology-enhanced mathematics task and accompanying lesson plan when they have not yet been taught how to do so), yet VHQMIw/T can provide insight to what they aspire. It is not uncommon for researchers to use self-reported TPACK measures like self-efficacy or beliefs as pre/post measures alongside TPACK as post measures (e.g., Akapame et al., 2019). We recommend attending to instructional vision over other self-reported measures as the latter "suggest a relatively static set of decontextualized ontological commitments" and "vision is intended to communicate a more dynamic view of the future" (p. 587). To further compare these two constructs, future work following PSTs into the field to study how their VHQMIw/T informs their practice and whether or not their practice aligns with their VHQMIw/T and enacted TPACK would be useful.

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INSIGHTS FROM A CASE STUDY OF AI-INTEGRATION IN UNDERGRADUATE MATHEMATICS

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AI's influence in mathematics education is growing rapidly with the expansion of new technologies and AI tools. Researchers argue that a deeper study of AI's role in mathematics classrooms is needed to unlock the full potential that technology offers (e.g., Aleven et al., 2023; Gattupalli et al., 2023; Aljarrah & Towers, 2022). In this paper, we share preliminary findings from a study exploring the impact of the AI-powered guide, Khanmigo, on undergraduate students' mathematical learning. Participants completed four tasks drawn from the History of Mathematics (Katz, 1988) and four focused on mathematical modeling (Erbas et al., 2016).

Methods

We chose a case study methodology (Stake, 2000), investigating individual cases (undergraduate students) to address the overall research goal. In this poster, we present findings related to one student's (second author, Li) interactions with Khanmigo specific to this task: *At their last concert, Oscar & the O's sold 2200 tickets for $32 each. The promoter believes that a $1 decrease in the price of a ticket will attract 50 more listeners. Oscar points out that charging $1 more per ticket is just as likely to drive away 50 people. The promoter wants your help to determine the ticket price that will bring in the most money.* Data collected and analyzed include Li's individual response to the task, solution based on her interaction with Khanmigo, and reflections on the task engagement with the AI-tool. Li used the tool for brainstorming ideas, generating suggestions, and seeking clarification on challenging concepts.

Li's engagement with the AI tool is characterized by a progressive exploration of generalized topics related to optimization. Her prompts to the tool evolve gradually, becoming more refined based on the responses received. Khanmigo plays a pivotal role in shaping the conversation's direction by posing questions that prompt Li's thinking and assess her current understanding. An example of this is when the tool inquires, "Do you know what a 'constraint' is in this context?" and follows up with, "Can you think of a situation where you might have a constraint?" The subsequent clarification and the query, "Does that make sense?" demonstrates the tool's role in gauging Li's mathematical understanding. This interactive dynamic underscore the AI tool's function not only as a source of information but also as a supportive tool facilitating hands-on problem-solving and conceptual understanding. However, there was a specific instance when Khanmigo failed to catch the Li's mathematical mistake until she pointed it out. In a classroom setting, this oversight can mislead the learner into thinking that their answer (and thought process) is correct, when in reality, it is incorrect. This hinders the learning process and is counterproductive rather than facilitating. Khanmigo's inability to present graphs and other drawings presents another challenge.

In summary, the above findings demonstrate how an AI tool can be effectively used as a collaborator in the learning process. While Khanmigo has the potential to serve as both an amplifier and reorganizer (Pea, 1985), using it cautiously as a reorganizer proves to be the most effective approach to leverage its AI capabilities, considering the potential modification of Li's thinking processes in this context.

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EXPLORING LEARNER-AI ENGAGEMENT IN MATHEMATICS EDUCATION USING A MATEHMATICS MODELING TASK

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Recent recommendations from national organizations on advancing mathematics education underscore the necessity of integrating artificial intelligence (AI) in the classroom for empowering students. It is necessary to move past the question of "Why use AI" and focus on "How to use AI", "What tools are conducive for AI learning", and "When to use these tools". This shift is necessary for developing a broader and a deeper understanding of the use of AI in mathematics classrooms. In this paper, we feature key findings from a longitudinal case study aimed to explore, understand, and analyze the interactions unfolding between undergraduate teacher education students and an AI tool (ChatGPT) during their engagement with mathematics modeling tasks. We highlight specific findings related to a specific modeling task and describe key aspects of learner-AI engagement.

Keywords: Undergraduate Education, Computational Thinking, Technology, Modeling, Geometry and Spatial Reasoning.

With the rise of new technologies and AI tools, the influence of AI in mathematics education is growing, leading to a current debate within the educational community about if and how AI should be integrated within mathematics education (Kovács et al., 2022, p. 23). AI tools have the potential to transform how learners engage with mathematical tasks (Hwang et al., 2020); however, this requires a comprehensive understanding of AI's nuances and its effective integration within existing curricular and pedagogical frameworks and finding ways to support teachers and learners in the effective and responsible use of AI tools (Celik et al., 2022; Hwang & Tu, 2021). The authors are engaged in a longitudinal research project, aimed to explore the role of AI use in mathematical classrooms, specifically focused on how prospective mathematics teachers (PMTs) engage with AI as 'learners' of mathematics. The primary objective of this research is to explore, understand, and analyze the interactions unfolding between learners and AI tools as they engage with cognitively demanding mathematical tasks. This paper particularly discusses the interactions between learners (PMTs) and Chat Generative Pretrained Transformer (chatGPT) during their engagement with a mathematics modeling task.

Our working definition of AI is that it is a network of "computational systems that simulate human intelligence in machines to reason, learn, and act on complex tasks" (Copur-Gencturk et al., 2023, p.4). ChatGPT, a natural language processing model, has been found to have the potential to assist with both the teaching and learning of mathematics. Teachers use it for various purposes such as lesson planning, student assessments, and providing feedback (Crust, 2023; Firat, 2023; Wardat et al., 2023). Moreover, the chatbot can support learners in generating explanations, answering queries, and engaging in interactive problem-solving related to mathematics (Guo et al., 2023).

Proponents of activity theory (e.g., Vygotsky 1978; Leontiev, 1978) emphasize the central role tools play in mediating human activity. The theory highlights the notion that the relationship between the subject and object is mediated through a tool. Using Activity Theory to investigate learner engagement with AI involves analyzing how the tool-mediated relationship between the subject and the object influences the learning process. In the present context, the subject is the learner, the object is the AI technology, and the mediating tool is the specific AI application (e.g., a chatbot) used for learning. The term "tool-mediated" suggests that tools, whether physical or intellectual, act as facilitators in the relationship between the learner, mathematics, and technology highlighting the importance of these tools in mathematics classrooms.

Methodology

We use a case study methodology (Stake, 2000; Yin, 1994), specifically, an instrumental case study approach to address the research goal. We intend to shed light on a particular phenomenon: the nature of learners' interactions with chatGPT as they engage with a mathematical modeling task. By investigating individual cases (learners), we seek to show how the phenomenon itself could be described, and hence an instrumental case study is appropriate. In this paper, we focus on a subset of data collected for the longitudinal case study.

The geographical context is a large south-western university in the United States. The study setting is a mathematics course for middle school and secondary PMTs - Advanced mathematics from a secondary perspective. Fifteen PMTs engaged with middle school and high school-level topics, including algebra, geometry, and calculus. Within this course, the first two authors developed a course project to explore AI's role in mathematics learning. PMTs completed various activities, beginning with a survey aimed at gathering their initial impressions on the topic. Following this, they researched AI tools applicable to mathematical contexts and completed three mathematics modeling tasks outside of regular class hours. In this paper, we present data specific to the following task (Figure 1).

A company that produces canned food needs short-term storage to store the cylinder-shaped can it produces. The company wants to do this with the least possible cost. Each of the right circular cylinders can be kept 10 cm in radius and 30 cm in height. The company plans to store 175 cans for 2 months. There are 3 different sizes of storage cabinets that the company can store. The rental costs are shown in Table 1 according to the dimensions of base of the storage cabinets, each of which is 100 cm high.

Table 1. Size of the storage cabinets and costs

Width (cm)	Length (cm)	Rental Cost per Month ($)
110	110	100
110	220	150
110	330	200

a. If you were the company owner, in what way(s) would you use which storage cabinet to minimize the cost?
b. The company may need to store different numbers of cans in future production. For this, would it be appropriate for the company to always use the same type of storage cabinets? What do you suggest? Why?
Please note that keeping the cans in an upright position is important for the safety of the storage.

Figure 1: The Mathematical Task (Erbas et al., 2016)

Data collected and analyzed include PMTs' responses to the task, documented interactions with the AI tool, and their reflections on task engagement. Emphasizing inclusivity and ease of use, we allowed learners to self-select the AI tool. During their AI interactions, PMTs were advised to a) use prompts for brainstorming instead of directly seeking answers and b) encouraged to independently solve problems first before collaborating with the AI tool for alternate solutions. The first and the second authors independently read and coded PMTs' interactions with the AI tool, cross-checking their findings. Elsewhere, we have presented a framework that represent levels of learner engagement with AI from foundational recall to creative exploration of mathematical ideas (Naresh et al., 2024). In this paper, we provide a brief synopsis of one case Manu (a pseudonym) – as an illustrative example that highlights a creative level of engagement with AI.

Results

Manu took a distinctive approach by prompting AI to create a recommended table of variables essential for solving the task. After transcribing the AI-generated variables onto the board, Manu analyzed the information, generating initial ideas for solving the task (Figure 2).

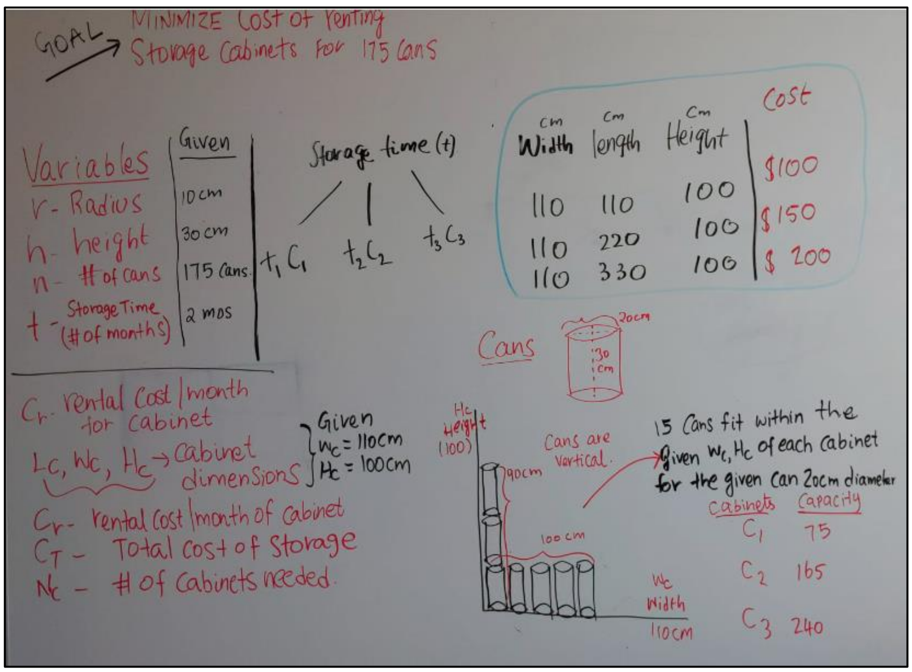

Figure 2: An Overview of Manu's Solution Process

Manu used multiple representations in the solution strategy, illustrating how each can would be positioned and stacked in the storage unit. Using the table, Manu calculated the number of cans each storage unit could accommodate. Next, Manu inquired about an alternative way to model the situation for determining the most cost-effective storage unit, specifically prompting the chatbot to "*act like the college professor and to provide guidance rather than a straightforward answer*". The bot responded with suggestions, including formulating equations to express the volume of a single can and the volume of the storage unit. Manu, recognizing the rigid dimension of the cans, opted for a "Guess and Check method". She calculated the cost associated with using a single storage unit multiple times (e.g., using unit 1 three times costing $300/month) or a combination of multiple units (such as units 1 and 2 together costing $250/month), ultimately choosing storage unit 3 as the most cost-effective option.

Transitioning to modeling for varying number of cans, Manu recognized the problem as an optimization challenge and proposed a simple function to the chatbot. Manu received a set of instructions and identified the following guidance as the most helpful: "*Ensure Minimization: Confirm that the objective is framed as a minimization problem. The goal is to typically minimize the total cost, so the objective function should reflect this"*. Further advancing the inquiry, Manu prompted the chatbot to write code in MATLAB (a programming and numeric computing platform) using the optimization toolbox to mathematically model the situation. After an interactive conversation with chatGPT, debugging the generated code, and testing it in MATLAB, Manu successfully developed a code that recommends the least costly option for storing any number of cans. Reflecting on the practical implications, Manu highlighted the potential real-life application of the code, envisioning a program that allows customers to input the number of cans and receive cost-effective storage recommendations.

Discussion

Manu's engagement surpassed immediate problem-solving by extending into the generalization of mathematical concepts and processes, yielding additional outputs. To further discuss the dynamics between the subject, object and the mediating tool, we turn to Activity Theory. Manu's thoughtful consideration of various solution approaches indicates an activity where understanding the task and exploring potential solution pathways take precedence over immediate task completion. Manu's engagement involves a broader exploration of the task's context before narrowing down to specific mathematical ideas. The progression from a general understanding of the task to a focused engagement with mathematical content suggests a dynamic interplay between the learner, the AI tool, and the evolving understanding of the task. These nuanced engagements highlight the varied depth and creativity in interactions with the AI, offering insights into the potential for more thorough and constructive engagements.

Conclusion

Our findings have significant implications for teaching and research, indicating potential lines of inquiry for further exploration. For teaching, we noted that the AI tool enhances dynamic learning but requires proactive engagement from learners, and incorrect answers or self-corrections necessitate caution for those with varying mathematical understanding. Additionally, the use of AI outside the classroom limits the instructor's ability to monitor adherence to guidelines, with some learners treating the AI as a tutor rather than a collaborator. This highlights the need to further explore AI integration methods, classroom dynamics, pre-AI engagement, and the instructor's role (NCTM, 2024). In our research, we recognize the need to examine how learners interact with AI across various content domains and problem types. Investigating the instructor's role can add new dimensions to our study, providing further insights. Conducting semi-structured interviews will help us understand learners' thought processes and metacognitive factors. Additionally, examining the experiences of prospective teachers with AI tools can offer insights into AI's role in future classrooms.

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HOW ARTIFICAL INTELLIGENT TOOLS USE IN MATHEMATICS EDUCATION MEET OR FALL SHORT OF ESTABLISHED EQUITY (IDENTITY) DEFINTIONS: THE CASE OF PHOTOMATH

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Research (Eubanks, 2018; Holstein & Doroudi, 2022) offers insights into how the use of AI tools can exacerbate or ameliorate historical issues such as inequities experienced by non-dominant groups, particularly in Artificial Intelligence Education (AIED). Despite the National Council of Teachers of Mathematics [NCTM] (2024) call for such research in Mathematics Education, few studies exist. Findings from a deductive analysis of video transcripts (Powell et al., 2003) from three comparative case studies conducted over three months in rural, multilingual, technologically constrained, low socio-economic community and school settings in Ghana contribute to this research. As shown in Figure 1, these findings offer insights into how AI tools used for teaching and learning mathematics either meet or fall short of established definitions of equity, particularly concerning students' identity (Gutierrez, 2007). As AI tools in AIEd often exacerbates existing inequities concerning language identity by being trained primarily on mainstream dialects (Holstein & Doroudi, 2022), so are they in mathematics education. Contrary to the lack of flexible customization in AI tools in AIEd, which worsens existing one-size-fits-all pedagogies (Nye, 2014), AI tools in mathematics education explainability's mitigate this. These findings involve five high school students, five parents with two being francophone speakers, a mathematics and AI researcher who integrated the free mobile version of PhotoMath for differential calculus.

Figure 1: AI Tools in Mathematics Education Meeting or Falling Short of Established Equity (Identity) Definitions

Established equity (Identity) definitions in Mathematics Education	AI Tools Meeting Established equity (Identity) definitions in Mathematics Education	AI Tools Falling short of Established equity (Identity) definitions in Mathematics Education
1. Students engage in mathematical tasks according to their preferences, including algorithms and strategies to solve problems (Gutierrez, 2007; Myers, 2014). 2. Students maintain and draw upon cultural and linguistic capacity during mathematics	1. Students had the autonomy to select their preferred Photomath explanability such as step-by-step solution, promoting inclusivity and accommodating diverse learning styles. 2. In the out of school context, Francophone parents actively engaged with teachers and students, utilizing Photomath's French	Photomath support is limited to mainstream dialects such as Spanish, French, and English, commonly taught in Ghanaian schools. This restriction excludes Ghanaian dialects, restricting students' ability to connect with mathematics through their linguistic repertoire and hindering teachers' capacity to adapt to local contexts.

learning (Gutierrez, 2007).	interface to support students learning.	

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ENHANCING TRANSPARENCY AND INTEGRATION: EXPLORING PHOTOMATH'S EXPLAINABLE FEATURES IN MATHEMATICS EDUCATION

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The integration of Artificial Intelligence (AI) tools in mathematics education is growing, but many tools lack transparency in their outputs. Updated versions of AI tools such as Photomath have incorporated explainable features to improve user comprehension. This paper offers insights into Photomath's explainable features under two areas: technique and practice related. Technique-related comprises the descriptions of available explainable while practice-related covers how they impact users. Findings from the video transcript of a three-month project integrating Photomath into a differential calculus curriculum show that its explainable, such as the embedded dictionary, can impact teachers' mathematics curriculum decisions, amplify equitable mathematics practices, develop students' mathematics communication skills, and provide insights into algorithmic bias, depending on the teacher's instrumental orchestration.

Keywords: Technology, Instructional Vision, Instructional Activities and Practices, Instructional Leadership

Introduction

Fey (1989) identified key areas, including connections to Computer Science and Mathematics Curricula, Artificial Intelligence, and Machine Tutors, as focal points for research into integrating technological tools within mathematics education. In 21st-century Mathematics Education, there is a significant shift towards leveraging Artificial Intelligence (AI) tools (Engelbrecht & Borba, 2023; Lagrange et al., 2023; Richard et al., 2023). These are the newest additions to the digital and analogue tools for teaching and learning mathematics, designed to learn and function based on data that simulates human thought processes. The book titled "Mathematics Education in the Age of Artificial Intelligence: How Artificial Intelligence can Serve Mathematical Human Learning” by Richard et al., (2023) outlines three key areas of focus within this transition: encompassing the process of creating AI in mathematics, examining the integration of AI in learning and investigating the role of AI in optimizing current and future learning based on insights from empirical research.

This paper focuses on the second and the third focal point: the integration of photomath explainable features in the teaching and learning of mathematics and how they can enhance certain teaching practices. Photomath is an AI tool designed to recognize, extract, and solve handwritten as well as printed text mathematical problems using computer vision technology. Photomath's explainable features mainly consist of alterable and non-alterable interfaces, including step-by-step solutions, embedded mathematical dictionaries, variants of mathematical formulas, and approaches to tailor a specific generated solution to offer users an explanation for its output. Among the array of AI tools in mathematics education, Photomath is emphasized for two reasons. Firstly, due to its widespread adoption, with over 220 million downloads worldwide, addressing 2.2 billion problems monthly, and being utilized by over 1 million teachers (Photomath, 2023). Secondly, due to its updated features such as explainability’s aimed at overcoming some of its documented challenges.

In the second section of the paper, a snapshot of the literature documenting challenges with the integration of AI tools such as Photomath for teaching and learning is presented. The third section continues with an insight into the updated features in Photomath aimed at mitigating some of these challenges. The fourth section explores how mathematics educators can instrumentally orchestrate some of these updated features to inform important mathematical practices.

Challenges with the Integration of AI Tools for Teaching and Learning

An artificially intelligent (AI) tool is a software or hardware application that simulates and performs tasks or functions typically associated with human intelligence, such as learning, problem-solving, perception, and decision-making (Zawacki-Richter et al., 2019). Frequently, users remain unaware of the explainability (the extent of understanding regarding how the AI-based system produces specific outcomes) (ISO, 2020). Consequently, scholars argue that this lack of transparency in AI tools renders complex mathematical concepts seemingly irrelevant for students (Newman, 2014). Despite ChatGPT's advanced capabilities as a chatbot model trained on the large language model Generative Pretrained Transformer (GPT), enabling it to handle a broad variety of text-based requests through vast volumes of data (Azaria et al., 2023; Lund, 2022), Bliss (2023) observed its lack of transparency. She stated that that while AI assisted tools such as the large language model Generative Pretrained Transformer (ChatGPT) can aid in delivering information to learners, it cannot think for them or facilitate genuine learning. Bliss expressed concern that students often rely on AI as a quick solution for instant answers, rather than engaging in critical thinking or retaining knowledge, even as educators strive to foster interest and skill acquisition in various subjects. Her concerns are rooted in the premise that human intelligence is innate, developing naturally as we interact with others, contemplate the world, and strive to enhance our connection to it.

In mathematics education, Webel and Otten (2015) discuss concerns regarding the utilization of the Photomath app, noting that "In conceptual problems, Photomath can assist only with computations; it cannot generalize the patterns" (p. 370). They propose potential responses, including banning access, restricting access, or considering a different division of labor. Capinding (2023), citing Muslimah et al. (2023) and Latham (2020), suggests that students may struggle to grasp essential concepts when learning mathematics via Photomath since it provides opaque generated answers instead of the pen-and-paper approach where students engage in manual and transparent calculations. This could lead to an insufficient foundation for future learning, as some may opt for convenience by relying solely on Photomath (Capinding, 2023, p.3). In their 2024 position statement on AI, the National Council of Teachers in Mathematics [NCTM] highlighted concerns about bias in AI tools' training data, which sometimes can lead to amplifying existing inequities faced by non-dominant groups. For instance, in our use of Photomath for differential calculus, we found that Photomath support is limited to mainstream dialects such as Spanish, French, and English, commonly taught in Ghanaian schools (Nti-Asante, 2024). This exclusion of Ghanaian dialects reinforced existing challenges in the study context, where students are restricted from connecting with mathematics through their linguistic repertoire, perpetuating known challenges faced by multilingual students.

Mitigating Challenges with AI Tools for Teaching and Learning

To address challenges associated with AI integration, researchers, policymakers and AI tool making companies have outlined possible approaches. Cox (2020) advocates for educators to cease adversarial engagement with available technology, urging instead an acceptance of its presence and the exploration of ways to leverage it for students' benefit. In line with this, the NCTM advises educators to develop assessments that diminish reliance on computation while fostering problem-solving skills. They stated this approach positions students to critically engage with AI-generated outputs, aiding in the identification of potential biases.

Recognizing the significance of these challenges, AI tool developers such as Photomath have taken proactive steps to ameliorate them. They have introduced updated features, including alterable and inalterable explainable features, aimed at providing users with comprehensive explanations for generated outputs. Due to the profound insights offered by explainability's in tackling challenges faced by users of AI tools, there has been extensive research on the techniques and practices associated with them (Wang et al., 2023). According to Wang and colleagues, technique-related research provides users with an understanding of the various explainability's integrated into a specific AI system and its applications, while practice-related research primarily focuses on evaluating the impact of AI-powered tools' explanations on users. In the subsequent sections, I explore both the techniques and practice aspects of Photomath's explainability's to inform research in the mathematics education community.

Technique Aspect of Photomaths Explainability

The integration of explainability in AI corresponds to teachers' authentic provision of explanations regarding students' task performance, improvement suggestions, self-monitoring prompts, and affect-level comments (Hattie & Timperley, 2007). Khosravi et al., 2022 categorized AI technique related explainability into two main criteria: global approaches vs. local approaches and self-explanatory models vs. post-hoc explanations.

These explanations, generally fall under User-Alterable and Inalterable. User-alterable explainability features empower users to customize and interact with explanations based on their specific needs, while inalterable explainability delivers fixed, predefined explanations by the AI tool that users cannot modify. Photomath incorporates various types of inalterable explainability to enhance user understanding and trust in its system. Global explanations provide users with an overall approach and methodology used by Photomath when solving mathematical problems, contributing to a more comprehensive understanding of its workings. This is evident in Photomath's step-by-step solutions and math concepts tutorials. Self-explainable features offer clear and understandable explanations for their decisions and actions without requiring additional external explanations. These features are exemplified by Photomath's AI-generated explanations and hints for each problem, as well as its proprietary Animated Tutorials, which utilize AI audio voiceover paired with video explanations. Post hoc explanations are provided after the Photomath system has made a decision or taken an action, aiming to clarify and justify its behavior. Users can see this in Photomath's aftermath problem-specific explanations and error analysis, which offer opportunities to delve deeper into the solution process with additional "how" and "why" tips provided. Photomath's alterable explainability focuses on local explanations, which explain how the system works with different inputs and explores alternative solutions to demonstrate how changes affect specific decisions or predictions for inputs or instances. This includes integrating interactive elements, such as buttons and settings, allowing users to select their preferred type of representation for a solution, choose the languages to present their solutions, and explore animated, step-by-step instructions to solve equations—a visually engaging method for learning. The smart calculator feature enables users to input or modify scanned equations via an intuitive math keyboard to experiment with alterations, gaining a more profound comprehension of mathematical problems. Photomath also utilizes graphs to illustrate equations, enabling exploration of graph details like the root, domain, and interpretation of equation solutions. Moreover, Photomath employs Knowledge Graph Embeddings (KGE) models to map and relate mathematical concepts to real-world applications, such as students' home languages. Users can access the embedded Math Dictionary to learn the correct terminology, meaning, and translation by tapping on highlighted vocabulary terms. Photomath also detects mathematical word problems and translates them into equations and other mathematical representations.

Practice Aspect of Photomaths Explainability

Existing practice-related research on AI tools' explainability primarily focuses on their potential to offer explanations for their outputs and how that impacts users' trust in these outputs (Brdnik et al., 2023; Wang et al., 2023). Beyond fostering trust in AI tools, explainability offers opportunities for teachers to orchestrate these tools to address or enhance existing challenges and opportunities in the teaching and learning process. Instrumental orchestration (Drijvers et al., 2010; Trouche, 2004), as used here, refers to the teacher's intentional and systematic organization and use of the various explainable features of an AI tool in a given mathematical task situation, enabling students to develop intertwined technical knowledge about the AI explainability and their mathematical affordances.

Drijvers and colleagues (2010) describe three ways in which teachers' instrumental orchestration can be observed: didactical configuration, exploitation mode, and didactical performance. In terms of AI explainability, didactical configuration includes teachers' decisions on how to set up the classroom environment to integrate AI explainability into mathematics teaching and learning. This can involve various setups, such as whether to use computers, cameras, mobile phones, or the free or paid version of the software. Exploitation mode refers to the hypothetical decisions a teacher makes regarding the potential ways to utilize AI explainability and the related mathematical knowledge and skills to be developed by students. Lastly, didactical performance encompasses the teacher's operationalization of the planned teaching, including the questions posed to achieve the intended use of the AI explainability, handling unexpected inputs from students or the explainability that diverge from the intended purposes of teaching, and navigating these challenges effectively. In the following sections, I use the transcript of a teacher's orchestration of the embedded mathematical dictionary of Photomath as part of our three-month Photomath-infused differential calculus project to demonstrate which mathematical practices were amplified for each type of orchestration.

Methods

Research Design

This study is a revelatory case study (Yin, 2009), revealing the mathematical practices that get amplified through teachers' orchestration of Photomath's explainability features, such as the embedded mathematical dictionary.

Study Context and Participants

The transcript is from one teaching episode purposefully selected from twelve episodes involving five high school students, five parents (two of whom are francophone speakers), a mathematics educator, and an AI researcher. They integrated the free mobile version of

Photomath for teaching and learning differential calculus and geometry. This project was conducted over three months in rural, multilingual, technologically constrained, low socio-economic settings in Ghana. An episode comprises the intended purpose of teaching. The selected transcript was chosen because its purpose was to plan and teach differential calculus by integrating Photomath's embedded dictionary.

Data Collection

Data were collected through video, audio, field notes, and student artifacts, including screenshots from Photomath emailed to the teacher and researcher. The teaching and learning activities for each episode in the project followed a three-part lesson plan: before, during, and after. In the "before" stage, the teacher, accompanied by the researcher, describes his objectives and the supporting resources in place to achieve these objectives. In the "during" stage, students join the teacher and researcher, and the teacher executes his teaching plan. In the "after" stage, the teacher and students meet with the researcher in groups and individually to describe their experiences, including the opportunities and challenges they faced with Photomath and how the teacher intends to address these in subsequent teaching sessions.

Data Analysis

The transcript of the selected episode was generated using an AI tool and reread to correct errors. The generated transcript was then segmented according to the three-part lesson plan: before, during, and after teaching. The three forms of teacher orchestration were used as an analytical lens to analyze the respective transcript segments and describe what transpired. Inductive reading of this description provides insights into which existing practice-related frameworks in mathematics education are amplified by the teacher's orchestration. Specifically, didactical configuration and exploitation mode were used as deductive analytical lenses for the "before teaching" segments of the transcripts. Didactical performance was used as a deductive analytical lens for both the "during" and "after" segments of the transcripts. Components of the transcript are presented based on Powell et al. (2003) description of the composition of a transcript: names, quotes, activities, and time.

Findings

Table 1: Transcript of the Before Teaching

Time on Transcript	Quotes	Deductive code	Inductive Codes/Amplified Mathematical Practice
00:00:30	**Researcher:** Mr. Austin, can you explain what the objective is for class today? **Mr. Austin:** I will integrate the photomath mathematics register into the teaching and learning process today. **Researcher:** How will you do that? **Mr. Austin:** As you know, we currently have the free	**Deductive Configuration**	

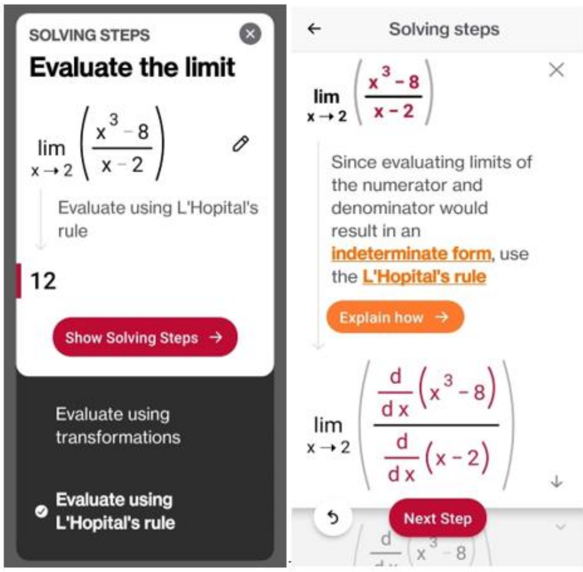
00:05:00	version so we will not have a complete use of it. If we subscribe to it, Photomath will highlight the mathematical terminologies in the sentence for students to click on to have the meaning. Currently students can only see the highlighted text but will not be able to click on for the meaning to show. **Researcher:** Oh! Then what is the worth of use? **Mr.: Austin:** I have a plan. We have mathematics dictionary in the library though I have not added it to my lesson plan before. Because of this inconvenience from our current use of photomaths, I have brought some for each students use. **Researcher:** How will they use photomaths along with mathematics dictionary?		
	Mr. Austin: The goal is for students to create mathematics diary which will be their mathematics vocabulary bank. They will each have a book where they will have four columns; highlighted math word from photomath, the related symbol, the sentence, and meaning the highlighted word they will generate from the mathematics dictionary.	**Exploitation mode**	**Equitable Access to Technology (Guitierrez, 2007; Meyers, 2012)**
00:10:00	**Researcher:** This seems rich. Yet how and when will they use this mathematics diary? **Mr. Austin:** Since we use the photomath as a guide, but each student thoroughly discusses their output for justification, I will ensure that they use the words, its meaning and symbols in the right context during their explanation. Also, I will do a weekly collection of students diary for verification and assessment. Lastly, I will allow students to construct their own word problems in differential Calculus and		**Development of Students Mathematical communication skills (**Brenner, 1998; Herbel-Eisenmann et al. 2015) **Teaching activity design for students to become aware of**

	highlight the vocabularies, to compare with what photomath output. **Researcher:** Okay, let us roll it to see the outcome.	**algorithmic bias (NCTM, 2024)**

As shown in Table 1, before the teaching, Mr. Austin explained that his goal is to see how the students will use the photomaths embedded mathematics dictionary during the differential calculus session. However, the challenge is that students can only see the highlighted mathematics register in the text but will not be able to click on it to see the meaning. This is because as of this time on the project we were using the free version. We needed to subscribe to the paid version in order to access the meaning of the highlighted mathematical register. To allow students to have this benefit, he brought to the class mathematics dictionaries from the school library. He stated that the students will develop a mathematics diary where they will develop a mathematics vocabulary bank. To do this, they will pick the mathematical registers photomaths highlights from the text, find its meaning from the mathematics dictionary, create four columned table with headings: differential calculus register, meaning, symbol and embedded mathematics word problem to record. He expects them to do this for the project until we get sponsors to help us access the paid version. When asked where how he will see the students use these mathematical diary, Mr. Austin stated that he will ensure that the students provide explanations to photomaths generated outputs while using these mathematical registers and also provide verbal and symbolic explanations to these registers. Mr. Austin also suggested that he will have students generate their own word problems for differential calculus and underline the mathematical registers in them. Then, they will submit these problems to Photomath to see which mathematical registers it highlights and check if they align with the students' identifications. Table 2 presents just a snapshot from the during the teaching transcript, which was deducted as the Didactical performance. Due to space the after the teaching transcript is not presented.

These efforts by Mr. Austins show how his instrumental orchestration of the photomath explainable offer opportunities for amplifying existing mathematical practices such as equity, mathematics literacy, algorithmic verifications, and mathematical communication skills. Although Mr. Austin had not previously incorporated a mathematics dictionary into his teaching, his desire to provide students with access to the latest educational technology despite constraints motivated him to leverage the mathematics dictionary from the library. This highlights how a teacher's orchestration of Photomath’s explainable features can influence curriculum decisions, including what resources to integrate or exclude. As shown in Table 2, Mr. Austin's approach to engaging students in justifying their Photomath-generated outputs by explaining the vocabulary and symbolic representations facilitates their comprehension and application of mathematical language and symbolism. This demonstrates how Photomath’s explainable features can enhance existing mathematical practices, such as students’ communication skills in mathematics (Brenner, 1998). Furthermore, by orchestrating Photomath’s explainable features to enhance students’ mathematical communication, Mr. Austin reinforces the teaching practice of developing mathematical literacy as an integral part of the learning process. Regarding the amplification of equity in mathematics education, Mr. Austin's use of up-to-date AI (Myers, 2014), support for learning outside of class hours, and the creation of a conducive classroom environment and participation (Gutierrez, 2007) characterize equitable access. Additionally, facilitating communication with mathematics empowers students. Lastly, his efforts to have students verify mathematical registers before using Photomath demonstrate his commitment to educating students about algorithmic bias, a practice advocated by the NCTM for incorporation into teaching and assessment practices.

Table 2: Transcript of During the Teaching Deducted as the Didactical Performance

Time	Quotes	Students screenshot
00:30:00	**Mr. Austin:** Before you justify the methods for your solution, what are some of the vocabularies you generated? Show it on the photomath. **Elvis:** For this problem, because I wanted my solution to be in the L'hopital's method, photomath highlighted terms such as indeterminate limits and L'Hopital's rule. So, I documented them and also used the mathematics dictionary to find their definitions. L'hopital's rule, which is a method of evaluating limits of indeterminate form using differentiation of one variable approach. **Mr. Austin:** That is good to know. However, let me see on photomath. **Elvis:** Here is the screenshot I took. **Mr. Austin:** Faustina, what are some of the Mathematical registers you also generated. **Mr. Austin:** Coming back to you Elvis, what is the symbolic representation you had for those two registers?	**Elvis's Screenshot** 
00:45:00	**Elvis:** The indeterminate limits are represented by two different symbols 0/0 or ∞/∞. There is no one specific symbol for the L'Hopital's rule but it involves the connected processes of Lim (x → c) [f(x) / g(x)] = Lim (x → c) [f'(x) / g'(x)] where f'(x) and g'(x) represent the derivatives of f(x) and g(x) respectively. **Mr. Austin:** How about you Faustina.	

Conclusion

The evolving landscape of digital and analogue technologies within mathematics education, notably with the integration of artificially intelligent systems, presents challenges concerning the assurance of transparency in their utilization. While developers continuously enhance these tools with explainable features, a notable lacuna exists in scholarly inquiry providing mathematics educators with comprehensive insights into these updates and their pedagogical implications. This paper examines Photomath's explainable features and their pedagogical impact, urging thoughtful orchestration by educators for optimal educational benefits.

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TEACHERS' COLLABORATIVE REFLECTIONS AND PLANNING ON USING DIGITAL AND NON-DIGITAL RESOURCES

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This study investigates how two middle school teachers of mathematics reflected on and planned for the use of a digital collaborative platform embedded with a problem-based curriculum. As digital resources can help teachers enact mathematics problems that are responsive to the needs of their students, more empirical work is needed to understand and inform relevant teaching practices that leverage evidence of student thinking (Pepin et al., 2017). Drawing on a documentational approach to didactics (Gueudet & Trouche, 2009), we examine the influences of collaborative reflections on teachers' decisions about the use of digital resources. Our preliminary findings show that based on their reflective conversation, teachers considered the affordances and constraints of both digital and non-digital resources in their planning. Our findings suggest collaborative reflections help teachers critically examine digital resources.

Keywords: Technology, Instructional Activities and Practices, Curriculum, Problem-Based Learning

Rationale and Purpose

A critical teaching practice in student-centered, inquiry-oriented classrooms is for teachers to elicit evidence of student thinking and connect with prior understanding (Jacobs et al., 2010; Kazemi & Franke, 2004; NCTM, 2014). Given the power and potential of digital technologies, a growing number of studies are examining the important relationship between teachers and their use of digital resources (Gueudet et al., 2012; Remillard et al., 2009; Geiger et al., 2023). Yet, more empirical research is needed to understand the factors influencing teachers' decisions to integrate technology into mathematics classrooms (McCulloch et al., 2018). Further, the potentialities of using digital resources are amplified when students and teachers use digital curriculum materials or digital platform systems that are designed and developed around student thinking in problem-based classrooms (Edson & Phillips, 2021). This study examines how teachers reflect on their individual use of digital or non-digital resources on the same problems.

Theoretical Perspectives

The notion of the teacher has shifted from teachers as curriculum implementers to teachers as enactors, and more recently, to teachers as instructional designers who make decisions on how to use curriculum materials (e.g., Jones & Pepin, 2016; Remillard et al., 2009). This study takes the perspective of teachers as instructional designers who interact with resources to achieve their instructional goals (Brown, 2009). As the scope of curriculum resources expands to include digital materials such as digital textbooks or interactive online platforms (Pepin et al., 2017; Remillard, 2016), how teachers appropriate and transform resources plays a critical role in their teaching work (Adler, 2000).

In our study, we draw on the documentational approach to didactics (Gueudet & Trouche, 2009), which centers the selection, planning, and enactment of resources (i.e., digital, non-digital) "at the core of teachers' professional activity and professional development" (p. 199).

The documentational approach to didactics emphasizes the dual nature of teachers' interactions with resources. When teachers shape a given artifact to achieve goals (*instrumentalization*), the affordances and constraints of the artifact influence teachers' usage of the artifact (*instrumentation*). As teachers develop schemes of how to use resources in their classroom context, their decisions are influenced by both explicit learning goals and implicit beliefs and knowledge from their teaching experiences (*operational invariants*). Over time, teachers develop documents and resource systems that entail both a set of resources and the utilization schemes of how to enact the resources (Gueudet, 2019; Ruthven, 2019).

We also draw on the perspective that discussions with other teachers are part of resources, highlighting the importance of collective documentation work among teaching colleagues (e.g., Gueudet, 2019; Gueudet & Trouche, 2009, 2012; Gueudet et al., 2016). Drawing from the perspective of understanding individual teacher learning within the context of collective practices (e.g., Lave & Wenger, 1991; Wenger,1998), the documentational approach to didactics provides insights into the influence of collaborative reflections and planning on individual teachers' development of resource system. Given the limited evidence of how teachers navigate the affordances and constraints of resources and how teachers orchestrate them (Pepin et al., 2017; Rezat et al., 2021), this study seeks to understand how teachers extend their perceptions of digital curriculum resources and incorporate them into their teaching practices.

Methods

This study is guided by the following question: How do individual teachers select, adapt, enact, and reflect on the use of resources in classrooms where there is readily access to both digital and non-digital resources? We situate the study within a larger design-based research project that focuses on developing a digital collaborative platform embedded with a problem-based middle school mathematics curriculum (Edson & Phillips, 2021). This platform contains various tools for digital inscriptions, such as texts, graphs, tables, drawings, and images. Also, students and teachers can access and co-opt others' digital work in real-time.

Our study focuses on two teachers, Ms. Evans and Ms. Foster, who teach seventh-grade mathematics at the same school in the Midwestern suburban area. Both teachers have more than ten years of teaching experience with a problem-based curriculum, *Connected Mathematics* (CMP; The Connected Mathematics Project, Phillips et al., in press). They have been using the developed digital platform for several years. During their daily planning meetings, they used to select problems together to utilize the digital platform. In this study, they were asked to enact the curriculum in two different learning environments—digital or paper-and-pencil. This paper focuses on two problems from one geometry unit. For Problem 1.3, Ms. Evans made use of the digital platform, and Ms. Foster did the print curriculum. For Problem 3.1, the type of resources is reversed. Ms. Foster used the digital resources, and Ms. Evans did the non-digital resources.

Teachers' collective planning meetings and their individual reflection interviews after each problem were video recorded and transcribed. Secondary data sources included classroom video recordings, researchers' field notes during classroom observations, teachers' weekly reflection survey responses, and classroom documents generated by the teachers and their students (digital and non-digital). The data analysis was guided by the reflective investigation to study teachers' documentation work (Gueudet et al., 2012). First, we identified how teachers perceived the affordances and constraints of digital and non-digital resources and how they would incorporate those resources into their teaching (*documentation work*). Then, we coded what influenced the teachers to develop their utilization schemes of resources (*operational invariants*). After applying descriptive codes to each teacher's data, we looked for patterns, similarities, and differences in the codes between the two teachers' cases.

Preliminary Findings

Our analysis revealed that after their conversations about teaching the same problem, (a) teachers considered the benefits and limitations of using both digital and non-digital resources in association with students' development of mathematical ideas, and (b) teachers changed their planning on future problems in order to integrate digital affordances.

Seeing Both Sides: Drawing From Each Other's Reflections

When asked to select between digital and non-digital resources, Ms. Foster indicated it would make no difference in student learning experience. It was because she believed that "seventh graders are visual learners" and hands-on activities help visualization. For Problem 1.3, she had her students use patty papers to trace shapes and compare them with other shapes to explore the similarities. After having a reflective conversation with Ms. Evans, however, she recognized the differences in student engagement and understanding in different learning environments. While she appreciated how the hands-on manipulatives' color format helped her students see the corresponding sides, Ms. Foster was "rethinking" the limitations of patty papers compared to the affordances of digital manipulatives. In other words, while maintaining her belief in visualization (*operational invariant*), she could develop a scheme of using digital resources for the same problem. Her rethinking was influenced by Ms. Evan's reflection. Ms. Evans shared how her students explored various approaches using digital tools (Figure 1), whereas most of Ms. Foster's students took the same approach by which they measured the lengths of hypotenuses. Ms. Foster claimed that using patty papers was more challenging to connect with mathematical ideas during the whole-class discussions compared to using digital manipulatives:

> The connections that I heard, Ms. Evans talked about today, was when they could actually go into the platform, make copies of the original, and play with the copies in the new image to where they could actually formulate a conclusion about the corresponding sides, corresponding angles, and even the area. [It is] because my kids didn't, we didn't even talk about the area, whereas Ms. Evans said some of her kids actually picked up on the area. […] Now that I'm talking about this and reflecting, I think, timewise, the digital platform allows us a shorter amount of time but a deeper sense of questioning and understanding.

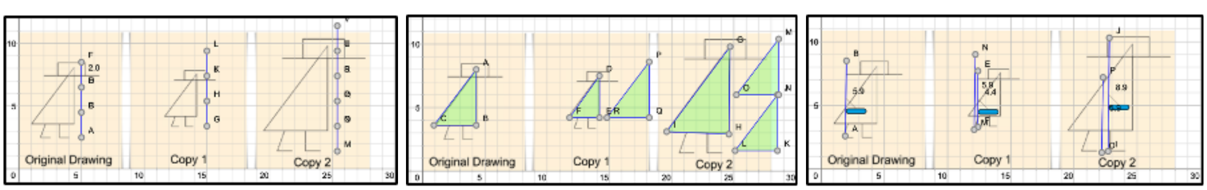

Figure 1: Students' Different Strategies Used on the Digital Platform

Although teachers used one format for each problem, either digital or non-digital resources, they drew on each other's different experiences and considered both resource types to better help students "actually see" embedded mathematical ideas and their conversations "go deeper." After teaching Problem 3.1 on the digital platform, Ms. Foster imagined the possible limitations of

Kosko, K. W., Caniglia, J., Courtney, S., Zolfaghari, M., & Morris, G. A., (2024). *Proceedings of the forty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Kent State University.

using patty paper. She continued to say, “It’s an aha moment for me because I used to be the queen of paper-and-pencil” who always used patty paper for these two problems.

Looking Ahead: Adapting Plans for Future Problems

Teachers adapted their plans for the same problem to integrate digital affordances. This means that teachers revised their documents as they perceived that digital manipulatives could help students focus on exploring mathematical ideas (*operational invariant*). For example, after teaching Problem 3.1 using patty paper, Ms. Evans claimed it took longer for students to work on the problem because her class did not have sufficient time to facilitate as deep conversations about scale factors as Ms. Foster’s class. Ms. Evans further pointed out students’ tracing was not accurate enough to examine scale factors, because she learned that from the digital representations, Ms. Foster’s students saw underlying mathematics “right away.” After this conversation, Ms. Evans planned to discard patty papers during her next class, and instead, she provided each group with a set of plastic shapes. She hoped students would struggle less with drawing and focus more on the relationships with scale factors. At the end of the day, however, Ms. Evans said that using plastic shapes was no better than patty paper for seeing the patterns. She noticed that most students stopped drawing after making a shape twice as big. With digital tools, she imagined students would easily have duplicated shapes to stack them into larger shapes more than twice. Thus, Ms. Evans wanted to revisit the same problem using the digital platform:

> I’ll just make some different shapes […] and then have them try and rep-tile it [because] some of them are still not sure about the scale factor. [In the platform,] they’re still looking at how many shapes versus side lengths, so I think that will help with connecting that a little bit better for them. […] Getting to do it again in a slightly different format will help solidify it.

For both problems, teachers compared the benefits and limitations of digital and non-digital resources and concluded that the digital platform would be more beneficial for student learning. Interestingly, they mentioned that if they had to teach these problems in a paper-and-pencil environment, they would prepare multiple pre-cut shapes so students could skip the drawing step and focus on manipulating shapes to explore mathematical ideas. That is, teachers changed the scheme of using hands-on manipulatives to remove unnecessary distractions and help focus on deepening mathematical understanding, inspired by the digital affordances they observed.

Discussion and Conclusions

This study provides empirical evidence of how teachers shifted from using non-digital, hands-on manipulatives toward using digital resources. This shift was based on collaborative reflections about student understandings of the key mathematical ideas. The teachers developed operational invariants that digital resources help students focus on visualizing their thinking more efficiently than hands-on manipulatives. Such belief will influence the ways teachers consider digital affordances and represent their resource systems (Ruthven, 2019).

Findings from the study pose important implications as teachers select, adapt, and use digital and non-digital resources in their mathematics classrooms. First, we provide evidence that teachers can utilize co-planning meetings to share the different affordances of resources for the same problems that inform how to use technology in supporting student learning (McCulloch et al., 2018). As our preliminary findings focused on the two teachers’ cases, further research should investigate the impact of collaborative reflections compared to individual reflections. Second, we provide evidence that digital features in the platform prompted teachers to re-examine the benefits and challenges of using digital resources (Geiger et al., 2023). As our preliminary findings were specific to geometry problems involving patty paper, further research is needed, e.g., How do teachers develop resource systems regarding their use of digital/non-digital resources in problem-based classrooms?

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A SYNTHESIS ON THE INTEGRATION OF COMPUTER TECHNOLOGY IN GEOMETRY CLASSES

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This synthesis examines recent research from 2016-2022 on integrating computer technology (CT), specifically as exploratory environments, into geometry education across grade levels. Searches were conducted using Google Scholar and ERIC with relevant keywords. Inclusion criteria focused on peer-reviewed empirical studies combining geometry education and exploratory CT environments. Out of 338 initial search results, 64 articles met the criteria. Key findings show the number of studies increased substantially in recent years, with a peak of 14 studies in 2022. The predominant computer technology used was GeoGebra, followed by Geometer's Sketchpad and emerging Augmented Reality/Virtual Reality. Studies overall found positive effects of CT integrations on test scores, problem solving, spatial skills, conceptual understanding, and attitudes. Furthermore, CT facilitated students' exploration, visualization, conjecture generation, justification, and comprehension.

Keywords: Technology, Geometry and Spatial Reasoning.

Historically, geometry has not been valued as highly as algebra (Nirode, 2013), and research shows that many students struggle with learning geometry concepts and developing proficiency in geometric thinking (Sinclair et al., 2017). The use of computer technology (CT), especially dynamic geometry software (DGS), has been proposed as an effective approach to enhance geometry teaching and learning since it offers interactive diagrams and visualization features that provide students with the opportunity to dynamically construct, manipulate, and analyze geometric shapes and measurements (e.g., Hollebrands & Okumuş, 2018; Sherman & Cayton, 2015; Sinclair & Moss, 2012).

Sinclair et al. (2017) provided a systematic overview of research on geometry education, including studies on the use of technology in geometry classrooms up to 2016. They discussed many ways that computer technology has impacted the teaching and learning of geometry. The use of technology in geometry education has accelerated rapidly in recent years due to the COVID-19 pandemic (Bellamy, 2021; Borba, 2021). This study aims to expand the work done by Sinclair et al. (2017) with the following specific questions: 1) What are the trends of peer-reviewed journal articles focused on the use of computer technology in exploratory environments with a focus on geometry education during the last seven years (2016-2022)? 2) What kinds of computer technology are they using in these studies, and what are the main findings?

In this study, we restrict computer technology (CT) to software rather than hardware. Exploratory environment is one of the four uses of CT in classroom (Li & Ma, 2010; Lou et al. 2001; Means, 1994). This focus on exploratory environments aligns with constructivist learning theories that emphasize students constructing their own knowledge through active exploration and discovery-based learning (Lou et al., 2001). By examining the current role and impact of technology, this research will inform our vision for the future of geometry education.

Methodology

This synthesis examines peer-reviewed journal articles on the use of computer technology in geometry teaching and learning that were published between 2016-2022. The search was conducted using academic databases including Google Scholar and ERIC keywords related to technology, dynamic geometry software, geometry education, learning, and teaching. The initial pool from the search was screened based on criteria in Table 1, resulting in 64 out of 338 articles (18.95%) meeting the inclusion criteria. These articles are indicated with an asterisk (*) in the reference list, or you can click this link for the complete reference list of these 64 studies.

Data analysis involved categorizing the selected articles based on year published, type of computer technology (CT), geometry topics, participants' grade level, findings, and methodology (qualitative or quantitative for most CTs). A spreadsheet program was used to determine frequencies and percentages for each category.

Table 1: Inclusion and Exclusion Criteria

Criterion	Inclusion	Exclusion
Type	Full peer-reviewed journals	Non-peer-reviewed journals, book chapters, books, preliminary reports, short papers, brief reports
Topics	Studies focused geometry and computer technology (CT) integration	Studies focused on non-geometry topics, geometry without technology, technology integration in non-geometry topics
Type of CT	Exploratory environments	CT other than exploratory environments (e.g., tutors or tutorials, communication media, tools)
Populations	Research on technology integration in geometry classes at all levels, including stakeholders (e.g., in-service teachers)	Non-human subject research on geometry and technology even if in an educational context (e.g., meta-analyses, bibliographic studies, syntheses)
Language	Studies written in English	Studies in other languages

Results

The trends of peer-reviewed journal articles

Our analysis shows that the number of peer-reviewed articles meeting the criteria increased consistently from 2016 to 2020, peaking at 14 articles in 2022. In terms of grade levels, the most common participant groups across the research on technology integration in geometry education were at the K-12 levels (38 studies, 59.4%), with the most at grades 6-8 (19 studies, 29.7%). About the same number of studies focused on high school (11 studies, 17.2%) and elementary school (9 studies, 12.5%) levels. Approximately a third of the studies focused on participants in teacher education, with most of them (17 studies, 26.6%) on preservice teachers (PSTs) and a handful on in-service teachers (6 studies, 9.4%). Only 3 studies focused on higher education participants (non-teacher education).

In terms of the trend of geometrical topics, this study found that 70.3% of the reviewed studies focused on 2D geometry only, 18.8% focused on 3D geometry only, while 7 studies focused on both 2D and 3D geometry. Using the primary geometry context as the basis for coding, the majority of the studies (34 studies, 53.1%) focused on the classifications, properties, and relationships of a variety of 2D or 3D shapes, for example, studies that required participants to construct or identify the properties of 2D shapes, focus on parallel and perpendicular lines and angles, and on congruence and similarities of triangles. Eleven studies (17.2%) focused on measurement concepts such as area, perimeter, surface area, and volumes. Nine studies (14.1%) focused on a variety of transformation concepts. Three studies focused on proof, and seven studies (10.9%) focused on a variety of other concepts such as limits or analytic geometry.

The types of computer technology and what are the main findings

This study found that the predominant computer technology (CT) used from 2016 to 2022 that met our search criteria was GeoGebra, with 42 out of the 64 studies (65.63%) either using it as a standalone CT or along with other CTs such as GSP, Desmos, and tools for video, animation, and games. GeoGebra was also incorporated into web-based learning platforms, such as Virtual Math Team, or as an app in AR headsets, such as GeoGebra AR.

The quantitative studies that incorporated GeoGebra found that participants who received GeoGebra interventions demonstrated positive improvements compared to control groups on test scores, problem-solving skills, self-efficacy and self-regulated learning, conceptual understanding of geometry, and spatial geometry skills. Some studies showed that teaching with GeoGebra statistically improved preservice teachers' and in-service teachers' creative thinking skills, spatial visualization skills, creative thinking skills, and in-service teachers' beliefs in using GeoGebra in their teaching instructions. Whereas the qualitative studies found that the dragging feature in GeoGebra helped participants see static figures as movable ones, helping their understanding by enabling exploration of geometrical objects and seeing/justifying relations. Additionally, some studies found that using dynamic geometry activities with GeoGebra helped students progress in their van Hiele levels from basic visual recognition of shapes to a more advanced understanding of geometrical properties and relationships between shapes. Furthermore, some studies found that GeoGebra helped preservice teachers (PSTs) understand geometry proofs and expanded their technological pedagogical knowledge, inspiring the integration of similar activities into their teaching. A study on student opinions of technology-integrated lessons (Dikkartın Övez & Kıyıcı, 2018) found positive attitudes about math being more attractive, entertaining, allowing them to discover without memorizing, making information memorable, and increasing participation.

The second most used CT was Geometer's Sketchpad (GSP), with 10 out of the 64 selected articles using this tool, either standalone or along with other CTs like GeoGebra. We found that the experimental groups whose participants received GSP interventions obtained significantly better test scores or achievements compared to control groups. Some studies found that GSP integration improved students' attitudes toward math, such as self-confidence and curiosity to actively explore geometry concepts. In-service teachers also had positive views on GSP's usefulness for pedagogy (Huang et al., 2020) and its potential as a part of the learning process rather than merely as a tool (Zambak & Tyminski, 2017).

Studies incorporating Augmented Reality (AR) and/or Virtual Reality (VR) have grown in recent years, particularly from 2020 to 2022. The CTs or software used in these studies were Unity software, GeoGebra, Geometry and Quiver, Zappar - HP Reveal, and Neotrie VR. Out of the 64 selected articles, there were five studies that used AR and/or VR. Like the other CTs, quantitative studies using AR/VR found that experimental groups receiving AR/VR interventions obtained significantly better test scores or achievements compared to control groups. Some studies (Elsayed & Al-Najrani, 2021; Flores-Bascuñana et al., 2020; Moral-Sánchez et al., 2022) found that students learning with AR were more motivated in mathematics, although Sarkar et al. (2020) found no significant motivation difference between individual and collaborative AR.

Four of the 64 selected articles focused on Cabri. Similar to other CTs, a quantitative study found that the experimental group receiving Cabri interventions had significantly better scores than the control group (Chang et al., 2016). Meanwhile, qualitative studies on incorporating Cabri showed that students actively explored geometry concepts through guided discovery (Selman & Tapan-Broutin, 2018) and improved their conceptual understanding and confidence with geometry after using Cabri (Gülburnu, 2022). Additionally, Cabri Geometry's dragging feature allowed students to test and refine constructions (Kepceoglu, 2018). Cabri also provided an enjoyable, exploratory learning environment (Gülburnu, 2022).

Other CTs such as Google SketchUp, Scratch, Minecraft, Desmos, and Microworld saw more limited implementation. Google SketchUp was found to improve students' van Hiele levels (MdYunus et al., 2019) and spatial visualization skills relating to rotating, viewing, transforming, and mentally manipulating 3D objects and shapes (Abdullah et al., 2022). Setyawan et al. (2018) found that while Desmos helped students and teachers understand concepts, Scratch helped students understand concepts like coordinates, movement, rotation, and angles (Calder, 2019). A Scratch-based game improved student achievement and computational thinking (Acar & DiKkartin Övez, 2022). Minecraft activities promoted pedagogical content knowledge and experience with game-based math instruction (Kim & Park, 2018).

Discussion

Our analysis yielded 64 studies meeting our strict criteria from an initial pool of 338 search results. This substantial number aligns with the findings of Çavuş and Deniz's (2022) meta-analysis, which identified a similar rise in research examining technology's impact on math and geometry achievement (2000-2016). The trend suggests a growing interest in integrating technology into geometry education which might be partially influenced by the COVID-19 pandemic, which forced many teachers to adopt online learning methods (Borba, 2021; Bellamy, 2021). Further supporting this trend, the most common computer technology used was GeoGebra, followed by GSP, AR/VR, and Cabri which is consistent with the results in Abidin et al.'s (2018) study that the commonly used CT during 2009 to 2017 research articles was GeoGebra and GSP, further suggests continued stability in the use of GeoGebra and GSP for geometry instruction.

Studies consistently demonstrated that experimental groups using CT interventions outperformed control groups on students' scores, problem-solving, spatial skills, conceptual understanding, etc. Like this result, other studies related to article syntheses on technology related to mathematics classes (e.g., Abidin et al., 2018; Çavuş & DeniZ, 2022; Ondes, 2021) also found that the participants who received technology-assisted interventions outperformed the participants who did not receive the technology. Similar to Abidin et al. (2018), this study also found that CTs improved participants' attitudes, confidence, motivation, curiosity, and technology integration skills, possibly due to the dragging feature that enables the participants to visualize static geometry figures more dynamically, which facilitates exploration of geometrical objects and relation noticing (Sinclair et al. (2017).

Abidin et al.'s (2018) found that the incorporation of AR/VR not only improved students' achievement but also their attitudes towards using the technology. However, our study only identified AR/VR use in middle grades and for pre-service teachers (PSTs). This highlights a gap in research, particularly at the elementary level, where the exploratory features of AR/VR could significantly benefit spatial understanding in 3D geometry. Additionally, there is an emergence of

AI tools like ChatGPT or PhotoMath that are changing how students learn math. The future of math education may involve more knowledge construction using immersive exploratory technologies and productively leveraging AI tools.

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COLLABORATING DIGITALLY: DESIGNING DIGITAL FEATURES TO SUPPORT SMALL GROUP AND WHOLE-CLASS DISCUSSION

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Technology use in mathematics classrooms can be isolating, dehumanizing, and detrimental to collaboration by encouraging students to withdraw from other people and from society, causing a breakdown in the student-teacher relationship, jeopardizing human values, and over standardizing education (Nissenbaum & Walker, 1998). In these uncertain times, using technology should support the future of mathematics education, which includes *collaboration*, "a joint production of ideas, where students offer their thoughts, attend and respond to each other's ideas and generate shared meaning or understanding through their joint efforts" (Staples, 2007, p. 162). The use of technology in teaching and learning mathematics continues to expand, as it can provide more accessibility, instantaneous feedback, and multiple representations (e.g., Clark-Wilson et al., 2020; Edson & Phillips, 2021; Powell et al., 2018). In this poster, drawing from larger design-based research, we report on the design characteristics of a digital curriculum platform that supports small-group and whole-class collaborative discussions.

When working in small groups, it is essential that students make sense of mathematics, building on each other's thinking. Our digital platform allows students to share work with their groupmates in real time through a "four-up" view that displays all group members' workspaces at once. Sparrows (student proportional reasoning arrows) allow students to explicitly connect and annotate relationships among different representations. Digital messages can be sent from a teacher to individuals and groups to advance student thinking. Teachers can collaborate with colleagues around student work as they plan and use student work for whole-class discussions.

During whole-class discussions, students' various strategies to solve problems are shared to unpack embedded mathematics. In the digital platform, teachers can quickly display different student strategies using a group view or the gallery walk feature. Also, teachers can create a digital document to collect student work and record class discussions, then make it available to all students. The learning logs allow students to reflect on the problem and accumulate their work and ideas over time. Moving forward, we continue to elicit teacher feedback on the features and explore professional development to both support teachers' implementation and discuss what features can be added or changed to enhance small-group and whole-class discussions.

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IT IS ART: TEACHER SCAFFOLDING AND STUDENT PROBLEM POSING DURING MATH WALKS AT AN ART MUSEUM

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This study investigates the interactions between informal educators and adolescents during math walk activities at an art museum. "Math walks" are activities where students notice and wonder about mathematics in the world around them, often creating their own "math walk stops" where they ask and answer mathematical questions. Drawing upon theories of informal math learning, scaffolding, and problem-posing, our research aims to enhance understanding of math walk implementation. Through video content, interaction analysis and artifact analysis of participants' iPad photos, we explore students' mathematical learning processes and the role of adult facilitators in guiding these activities. Results from a three-day summer camp are given, and findings offer implications for designing effective informal math education programs and fostering meaningful student engagement with mathematics in real-world contexts.

Keywords: Informal Education, Integrated STEM / STEAM, Middle School Education

Research Purpose and Question

Investigating interactions between informal educators and adolescents in informal learning settings provides valuable insights into students' perceptions of mathematics. "Math walks," or "math trails," a method linking mathematics to real-world occurrences, can foster meaningful dialogues in community-based settings (English et al., 2010; Fesakis et al., 2018; Wang & Walkington., 2023). During math walks, learners critically assess their surroundings with their "math lenses," observing both mathematical and non-mathematical elements, generating and addressing their own questions (Wang & Walkington, 2023). However, the role of informal educators in guiding learners through this process remains underexplored (Sager et al., 2023).

This study explores how informal educators facilitate connections between school math and real-world math during math walk activities at a downtown art museum. Focusing on scaffolding techniques and student-created math walk stops, we aim to address gaps in the literature on informal math learning, where research has documented the challenges that learners have while making connections between school math and real-world math (e.g., Inoue, 2005; Lave & Wenger, 1991; Masingila et al., 1996). Insights from our case study of six students shed light on student-educator interactions and problem-posing processes, with implications for informal math education. Our research questions are: (1) How do student and facilitator interactions unfold during math walk activities as educators employ scaffolding techniques? (2) What are the characteristics of, and problem-posing processes leading to, student-created math walk stops? Next, we present our theoretical framework, methodology, findings, and conclude with a discussion on future research implications.

Theoretical Framework

Students' ability to describe mathematics can be explicit or implicit (Kaur et al., 2013). Explicit forms often involve the use of standard mathematical terminology. Whereas implicit forms involve understanding and using mathematics without explicitly describing it as such, often embedded within real-world contexts, like those found in informal settings, and are influenced by factors such as prior knowledge and cultural background (Kaur et al., 2013). To this end, we draw upon *informal math learning*, *scaffolding*, and *problem-posing* in our study.

Firstly, *informal math learning* research has examined how people use math in their everyday lives and in careers (e.g., Nunes et al., 1993; Walkington et al., 2014). More recent empirical studies on learning math within informal settings has helped to understand program effects on achievement outcomes (e.g., grades, GPA, test scores) in school mathematics (Lauer, et al., 2006; Lynch et al., 2023). Although place-based mathematics education in informal learning environments is gaining increasing interest (Mokros, 2006), research on this topic remains limited (Pattison et al., 2017). They go on to explain that visitors in place-based settings are often unaware of their engagement with mathematics, and that promising mathematical thinking and social interactions around mathematics can emerge in informal spaces (Pattison et al., 2017).

Secondly, *scaffolding* describes the guidance and support a teacher (or knowledgeable adult) provides a student during problem solving activity in a particular learning context (Dingman et al., 2019), namely in the context of adult-child interactions (Stone, 1998). This is to center the students' learning and reasoning through a process of "the adult 'controlling' those elements of the task that are initially beyond the learner's capacity, thus permitting him to concentrate upon and complete only those elements that are within his range of competence" (Wood et al., 1976, p. 90). From their work, we employ four of the six primary scaffolding strategies in our methodology and in our findings: (1) *Recruitment:* the instructor elicits the student's interest in the problem and highlights the requirements of the task. (2) *Direction maintenance:* the instructor keeps the student in pursuit of a specific objective. (3) *Marking critical features:* the instructor highlights or emphasizes the relevancy of certain features of the task. (4) *Frustration control:* the instructor reduces stress from working the problem (Wood et al., 1976, p. 98).

Thirdly*, problem posing* in mathematics education involves teachers and students (re)formulating or expressing new mathematics problems within a specific context, as described by Cai et al. (2023). These tasks require students to generate or shape new problems based on real-life mathematical situations, which include both contextual situations and prompts (Cai, 2022; Cai & Hwang, 2023). Contextual situations provide problem posers with necessary data to craft their problems, while prompts guide students in problem posing tasks (Cai et al., 2023). Creating math walk stops is a problem-posing task with the potential to enhance students' interest in and understanding of mathematics. Studies by Walkington and Bernacki (2014) and Wang and Walkington (2023) highlight the challenges students face in problem posing due to the need for prior math knowledge and familiarity with "school math" norms. Problem posing research offers opportunities to enrich the informal math literature by transcending the constraints of formal "school math." Next, we present methods for data collection and analysis.

Methods

Background Context

This study highlights findings from the second year of the MathExplorer project, a research practice partnership (RPP) connecting a university in the southwestern United States, a STEM-oriented nonprofit, and nine informal learning sites. We partnered with informal educators at a local art museum ("Art Museum") to conduct a three-day summer camp. Students from the local community participated, alongside five researchers and supervising professor from the university and the director of the STEM-based nonprofit. The Museum Teacher who led the camp was trained on the app, scaffolding strategies, and problem-posing techniques by the research team.

Student participants used the app to explore real-world objects that they encountered during their math walks at the Art Museum. Accompanied by a facilitator, they gained an understanding of and proficiency in applying mathematical concepts while engaging with objects. During the camp, participants explored selected art pieces each day with the facilitators. They watched previously recorded videos embedded in the App, discussing math concepts related to the informal learning space. Afterward, they freely explored the museum to create their own math walk stops about things they noticed and wondered about in their environment (Sager et al., 2023). At the end of the day, they convened for whole group discussions, sharing their photos and math question(s). On the final day, students presented their math walk stops to the group.

Research Participants

The student participant group was diverse, and relevant demographic information including anonymized pseudonyms is summarized in Table 1.

Table 1: Camp Participants

Name (Pseudonym)	Grade	Race/Ethnicity	Gender
Astrophel Seven (505)	8th	Hispanic/African American	Male
Hamal Slope (201)	3rd	White/African American	Male
Apollo Osmium (202)	4th	White	Male
Zania Copper (203)	6th	African American	Female
Zenith Bit (204)	5th	African American	Female
Daniah Roentgenium (506)	5th	African American and Other (not specified)	Female

Data Collection

We collected video and artifact data while observing students and teachers during the Art Museum's three-day summer camp. Students were divided into partner groups, each paired with at least one adult from either the research team or the Art Museum. Each group was provided with a tablet containing the MathExplorer app. Researchers recorded video footage of each small group using handheld recording devices, resulting in twelve videos totaling 340 minutes of footage. Additionally, we retrieved photos of various artifacts from each participant's iPad, some of which were annotated with markings and included posed questions and answers. All collected data was stored in a shared Box folder, organized by data source type (video, image), group number, and date to facilitate the subsequent analysis.

Data Analysis

This study focuses on three main forms of collected data: demographic surveys, small group video recordings, and walk stop screenshots taken from students' iPads. Transcripts and content logs of videos were manually created by the researchers (Jordan & Henderson, 1995). Eighteen math walk stops were created and shared by camp participants over the three days. Following an iterative process informed by the data analysis spiral (Creswell & Poth, 2018), the authors engaged in systematic reading, viewing, and memoing of data, followed by collaborative meetings to categorize and recategorize codes (Saldana, 2021). Further, the authors rewatched videos, read content logs, and revisited transcripts. We used an inductive process to identify various types of interactions between the adults and students. After two rounds of inductive coding, we categorized interactions by the four scaffolding strategies (Wood et al., 1976). Codes were labeled with a schema (theme-interaction category-type), seen in Table 2 under Findings.

Artifact analysis involved a comprehensive review of student-created walk stops from iPad photos to address the second research question. Authors employed an inductive process to identify walk stop types, categorizing them into explicit and implicit mathematics themes using a "problem-posing-type" schema. Subcodes were generated where necessary to denote specific aspects of the walk stops. A decision was made collectively to classify codes as "explicit school mathematics," "implicit mathematics," or "unrelated" discussed later in detail.

Triangulation involved comparing student-selected walk stops with facilitator-student interactions to understand engagement leading up to each walk stop creation. The process included systematic comparison and integration of data sources to ensure coherence and reliability in the analysis. By using triangulation strategies, our data analysis methods offer a transparent framework for analyzing collected data and generating meaningful insights.

Findings

We present our findings for the qualitative case study by looking at each research question separately. The number of instances we observed for each code is provided in Table 2.

Table 2: Codebook from Interaction Analysis and Artifact Analysis

Code	Definition	Coded As	Example	Count
Scaffolding: Recruitment	Instructor elicits student's interest in problem and highlights requirements of task	Probing (prior knowledge)	"You mean Zeus?"	24
		Probing (connection)	"You think it's interesting? What makes it interesting?"	52

Scaffolding: Direction Maintenance	Instructor keeps student in pursuit of a specific objective	Probing (directive)	"So, remember…at each stop there's gonna be a little video for you to watch…and then there'll be some questions that they ask…"	32
Scaffolding: Marking Critical Features	Instructor highlights or emphasizes relevancy of certain features of the task	Probing (mathematical)	"How can we find the diameter of the eye?	48
Scaffolding: Frustration Control	Instructor reduces stress from working the problem	Probing (redirect)	"It's art."	2
Problem Posing: Explicit School Mathematics	Student explicitly uses school mathematical terms	Measurement, Count, Patterns, and Shapes	"…it was about finding a symmetrical ah a little symmetrical with the white and black arrows."	31
Problem Posing: Implicit Mathematics	Student poses questions about aesthetic elements without using explicit school math terms	Design, Functionality, Artist Motivation	"Q: My question is why is it so colorful and how was it made??"	9
Problem Posing: Unrelated	Student poses questions of situational interest, but not of mathematical interest	Unrelated	"Q: If she sad."	1

RQ1. How do student and facilitator interactions unfold during math walk activities in informal learning settings as educators employ scaffolding techniques?

In addressing Research Question 1, we observed facilitators employing four of the six traditional scaffolding methods (Wood et al., 1976).

Recruitment. Facilitators elicited students' interests as learners observed artworks, sometimes probing students by accessing their *prior knowledge* or making *personal connections*. For instance, Figure 1 illustrates a facilitator-student exchange employing both recruitment strategies. Green highlights indicate scaffolding coded as "probing-prior knowledge" and yellow highlights indicate scaffolding coded as "probing-connection." In this example, and others, the facilitator is eliciting student's interest in the art (situational context) to foster the prompting portion of the problem posing task, as summarized next.

11:37	202	"Can I take more than one picture?"
11:39	Researcher A	"Mmm, hmm."
11:41	202	"Just in the orange?"
11:43	Researcher A	"Would you like to walk around just a little bit more? (pauses for 8 sec) Oh, that's a good picture!"
11:55	202	"What's this?" (taking picture with iPad)
12:10	Researcher A	"202, what do you like about this?"
12:12	202	"It looks like a magician's hat."
12:15	Researcher A	"Okay. What else does it remind you of?"
12:17	202	"Umm, a magician's hat!"
12:22	Researcher A	"What do you notice about it?"
12:25	202	"It looks like a mouth and two eyes."
12:27	Researcher A	"Yeah, okay. (pauses for 3 sec) Do you think it's beautiful?"
12:34	202	"I think it's interesting."
12:36	Researcher A	"You think it's interesting. What makes it interesting?"
12:38	202	"Umm, the thing-a-ma-bob at the top and the thing-a-ma-bob at the bottom. (points to different parts of the object)

Figure 1: Photo-transcription of Museum Teacher Scaffolding by Recruitment

Direction Maintenance. Our analysis also shows facilitators employing direction maintenance techniques (coded as "probing-directive"), to keep students focused on the task of posing mathematical questions. This involved providing explicit directives related to the task, ranging from simple reminders to more descriptive instructions, as exemplified in a transcript excerpt where facilitators guide students through the task step-by-step beginning with the Museum Teacher:

> We can also spread out so we're not like all clustered together to make listening to it a little more easily…Please don't leave the orange area over here…So, you're gonna watch the video, and then you're going to formulate questions and answer the questions. Okay?...

Researcher A, then builds with further instructions to assist a nearby student with using the App's embedded voice recorder:

> You're gonna click on 'record answer' to answer the question that it asks you in the box. (student listens to question) So, what are some math questions you could ask here? (student records herself but has trouble hearing her recording) Can you hear yourself?...Do you want to try again? (student tries again)…Do you have any other questions that you need to answer? Think about another question. You can go look at it if you need to so that you can notice some things. They can be any math questions that you're thinking.

Researcher C, overhearing the adult-student exchange, interjects, "Or any…It doesn't have to be math, just any question you have about this place." This dialogue example is rich with scaffolding techniques that moved students along in the problem posing task including self and group management, technical support, and clarifying the task.

Marking Critical Features. At times, facilitators marked critical features of the artwork, highlighting or emphasizing emphasized the relevancy of certain features of the task. For example, Student 202 is drawn to a large indigenous artifact (shown in Figure 2) stating, “The eyes are really big (30:43).”

Figure 2: Photo Example of Marking Critical Features Taken from Video

Researcher A acknowledges, and then draws attention to the eyes to highlight and emphasize their relevance to the problem posing task, “Okay. Do you see anything on the eyes?” This scaffolding made way for Student 202 to formulate and then pose his question, “I've got a question…What is the diameter of the eye?” It is important to note that scaffolding is also used to mitigate student frustration when seeking mathematical connections with diverse artforms.

Frustration Control. Additionally, facilitators employed frustration control techniques to mitigate student frustrations and maintain focus on mathematical learning. During one session, Student 202 deemed a particular exhibit as inappropriate for kids because it contained nudity. Then, Student 201 agreed how he hoped “nobody makes a walk stop about the naked people”. Researcher A replied, “It’s art,” and Researcher H reinforced, “It is art.” However, the students carried on about “it’d be creepy,” and if there was one it wouldn’t be “for kids.” Researcher A steered the conversation back to mathematical inquiry with a definitive, “All right.”

In addition to facilitator strategies, students described specific characteristics of and the problem-posing process that led to their math walk stops, further enriching our understanding of student-facilitator interactions during math walk activities.

RQ2. What are the characteristics of, and problem-posing processes that lead to, student-created math walk stops?

Students described mathematics in both explicit (coded as “problem-posing-explicit school mathematics-type”) and in implicit ways (coded as “problem-posing-implicit mathematics-type”); “type” refers to more specific characteristics or problem posing processes observed in our analysis. Figures 3a-d illustrate select mathematical examples captured during artifact analysis, categorized by subcodes – *measurement*, *count*, *patterns,* or *shapes*.

Explicit School Mathematics. These codes corresponded closely with the questions students posed during the camp and with their accompanying iPad photos. Firstly, *measurement* questions often pertained to length, such as “how long” (see Figure 3a). Secondly, students inquired about quantities, or *count,* exemplified by questions like, “How many carvings in this photo?” (see Figure 3b). Next, *patterns* revealed students’ describing identified patterns within artworks, posing questions like “How many patterns are there?” (see Figure 3c). Lastly, s*hapes* refer to named or drawn geometric shapes, mostly circles and triangles, as seen in Figure 3d. These explicit school mathematics connections were pronounced during our analysis; however, we also found some implicit mathematical connections as well.

Implicit Mathematics. In contrast, some students described mathematics implicitly, focusing on artistic aspects, namely the object's *design* or *functionality* and the *artist's motivation*. Such descriptions reflect students' ability to see applied mathematical principles in art (Figures 3e-g). For instance, Student 505 questioned the materials used to *design* a clock (Figure 3e), while Student 203 expressed curiosity about an object's usage (Figure 3f), exemplifying *functionality*. Also, Student 204 inquired about an *artist's motivation* for color choices and sewing techniques (Figure 3g). Nine photos fell under this theme, illustrating students' intriguing observations of mathematical applications in art through aesthetic.

Unrelated. Lastly, a student's inquiry about a painting's emotional content (Figure 5d), without any explicit or implicit mathematical connections, highlights the diversity of student responses and interests during the math walk activities.

Our findings present a plethora of observations that underscore the multifaceted nature of student problem posing during math walks. Next, we discuss their significance related to the literature and implications for informal math education.

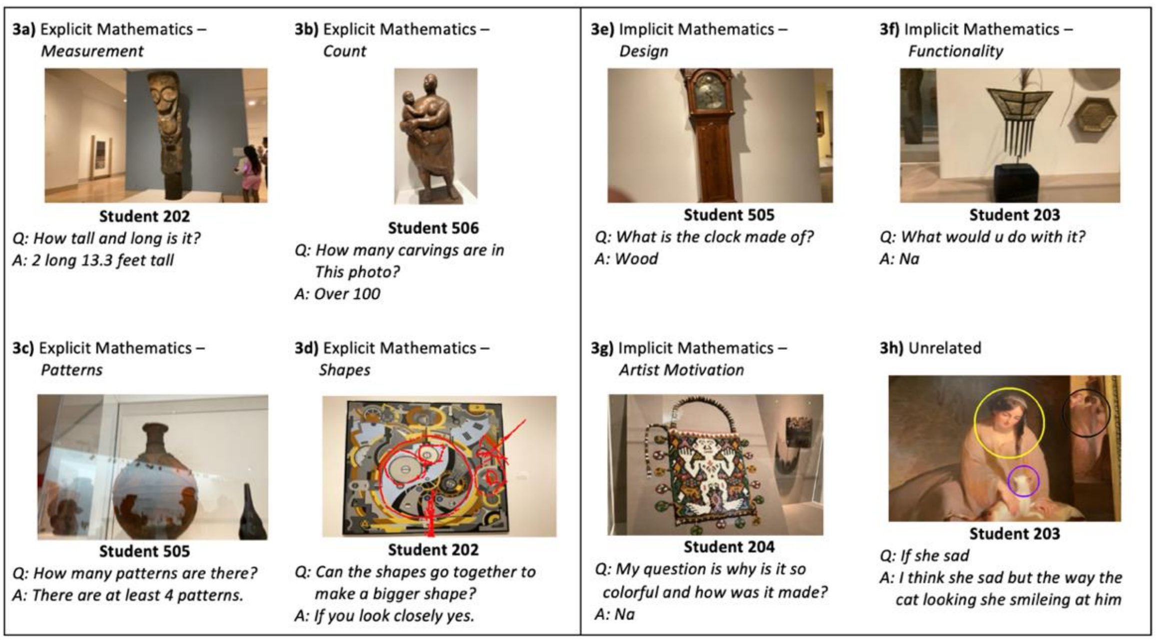

Figure 3: Problem Posing Student Examples (by Codes)

Discussion and Conclusion

While math walks have been identified as an important informal mathematics learning activity, little research has examined how student-facilitator interactions unfold during math walks. Having students generate their own noticings and wonderings from their surroundings is a challenging process, as it involves creativity and the ability to see mathematics as an expansive and situated domain for looking at the world. Here, we show how facilitators can use scaffolding processes like recruitment, direction maintenance, marking critical features, and applying frustration control to make these activities more feasible and rewarding for students, while still allowing students to maintain their independent voice. This offers important guidance for how informal educators can be best trained or prepared to implement math walks – by rehearsing, watching videos of, and discussing these scaffolding strategies.

We also show the kinds of math walk stops students created at an art museum, highlighting the explicit and implicit mathematics they noticed. One striking finding from this study was that the students' walk stops in Figure 3 was quite simple and un-nuanced compared to the rich conversations students had ***while creating*** these math walk stops. Thus, the math walk stops themselves are not the most important demonstration of or product of students' learning from math walks – instead, it is the mathematical discussions that students and facilitators have leading up to the submission of the formal walk stop that best show students' transformations.

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AN ESTIMATION GAME TO PROMOTE SECONDARY STUDENTS' CLIMATE CHANGE UNDERSTANDING USING DATA AND VISUALIZATIONS

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Data and data visualizations have the potential to shift learners' attitudes and conceptions about controversial science topics. However, many people, particularly secondary students, struggle to make scientific meaning from data. This design-based research project aimed to support data literacy and science learning by developing an online estimation game to support secondary students' understanding of climate change with data and data visualizations. Over the course of three design iterations, we interviewed 12 racially diverse secondary students and documented the design of a climate change number estimation game. Inductive coding analysis illustrated dimensions of students' (a) estimation strategies employed (e.g., drawing from prior knowledge, mental computation, wildly guessing), and (b) emotions experienced while estimating climate change numbers (e.g., emotions about climate change vs. about performance).

Keywords: data literacy; design-based research; numerical estimation; secondary education

Scientific data visualizations—such as maps, charts, and graphs—can communicate critical socioscientific information to the public (Allen, 2018; Harold et al., 2016). However, many people lack the quantitative reasoning skills needed to interpret these visualizations (Börner, et al., 2016; Peters et al., 2006; Thacker & Sinatra, 2019). Secondary students, in particular, struggle with number magnitudes using conventional number line representations (Doyle, 2015; Vamvakoussi & Vosniadou, 2004; 2007; 2010; Wilensky, 1991). This magnitude knowledge predicts math and science achievement (Booth & Siegler, 2006; Sasanguie et al., 2012; Siegler & Booth, 2004; Siegler et al., 2012), and students' inability to use visual representations to compare rational numbers can lead to misinterpretations of science topics (Siegler, 2016).

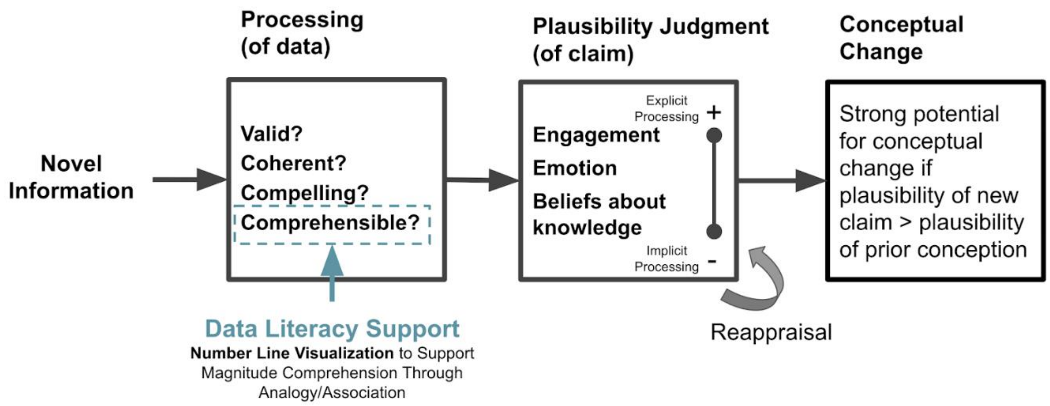
A relevant topic relying on quantitative evidence is climate change. National science standards require students to understand quantities, tables, and graphs related to human-induced climate change (NGSS, 2013). However, students have serious misconceptions about climate change (Dawson & Carson, 2013; McNeil & Vaughn, 2012). There is a need for learning contexts that support climate change understanding and data literacy. Several approaches, such as micro-interventions with surprising numbers about climate change, support learning (Ranney & Clark, 2016; Thacker & Sinatra, 2022), though evidence thus far uses undergraduate samples.

The current project developed an intervention to shift secondary students' climate change misconceptions by leveraging data visualization skills and documenting design choices for integrated STEM learning. The intervention used number line visualizations to support data literacy and climate change learning and investigated estimation strategies students used therein.

Theoretical Framework

To explain how novel data and data visualizations might help shift scientific misconceptions, we integrate theories of conceptual change and numerical development. *Conceptual Change* involves restructuring conceptions to align with scientific consensus (Dole & Sinatra, 1998). The Plausibility Judgments for Conceptual Change (PJCC) model (Lombardi et al., 2016) posits that novel information can shift conceptions if it is *comprehensible*, *coherent*, *compelling*, and *valid*. More explicit plausibility judgments—which are influenced by motivation, engagement, and emotion—increase the likelihood of conceptual change (see Figure 1 for a summary).

Figure 1.
Conceptual Change Process Model



Empirical research shows that novel data can inspire conceptual change about climate change. Estimating climate change numbers before presenting the true values can reduce undergraduates' misconceptions (Thacker & Sinatra, 2022; Thacker, 2023). Further, instruction on data-literacy skills can enhance knowledge gains (Thacker, 2023; Thacker et al., 2024). However, no research thus far has tested this approach with secondary students nor assessed the benefits of supplementing the experience with data visualizations.

Data visualizations can support understanding of scientific quantities and developing of numerical knowledge. Siegler's (2016) *Integrated Theory of Numerical Development* posits that people develop an accurate understanding of number magnitudes and their relationships as they connect numbers (e.g., representing rising global temperatures) to the things that those numbers refer to (e.g., global climate change). Such connections between numbers and their referents happen through processes of analogy and association, as facilitated by conventional representations and visualizations. The linear number line is central to representing real numbers and helps students compare magnitudes and understand abstract concepts (Van De Walle et al., 2013). This can support math and science learning, retention, and engagement (Gunderson et al., 2012; Schwartz & Heiser, 2006; Siegler, 2016; Saxe et al., 2013; Stevens & Hall, 1998).

The current research concentrates on the design of an online intervention presenting secondary students with novel climate change data. The study investigates using number line visualizations to enhance student comprehension of climate change data, math and science learning, and identifies strategies students employ that support this learning. Namely, we ask:

1. *How can a learning intervention be developed to leverage number line estimation skills for the learning of climate change science among secondary students?*
2. *What numerical estimation strategies do students employ when estimating climate change numbers? And how do they respond when presented with the true value?*

Methods

To answer our research questions, we used a design-based research (DBR) methodology (Anderson & Shattuck, 2012; Hoadley & Campos, 2022) to guide the design of an online intervention that we call the "Estimation Game." Typical of DBR, the design, implementation, and revision occurred over several iterations (Bakker, 2019; Cobb et al., 2003).

We revised an existing online estimation game (Thacker et al., 2024), where undergraduate students estimated 12 climate change numbers before being shown the scientifically accepted number. Half of the prompts included given benchmark values (e.g., "hints") that learners could arithmetically manipulate to estimate unknown values. After estimating each value, a pop-up window would appear displaying the true number, accuracy feedback (one to five "gold stars"), an explanation of the climate change number to help students contextualize the quantity in terms of students' prior knowledge, and links to sources of the information to improve credibility.

For this study, we aimed to modify this design to improve data comprehensibility and engagement. Modifications included (a) personalized feedback via a number line visualization to help students compare their estimates with the true value, (b) revised text that is more suitable for secondary students, and (c) a revised look-and-feel to be more game-like. This work resulted in an online, open-source estimation game for secondary students with number line visualization feedback (ianthacker.com/design.html). For a related quantitative study, see Thacker (in press).

Participants and Procedure. The intervention design, implementation, and revision occurred over three iterations. We conducted 12 one-on-one "think-aloud" interviews (Desimone & Le Floch, 2004) via Zoom with a diverse sample of secondary students (grades 7-12) from a southern U.S. metropolitan area. Students identified as female (50%), male (50%), Hispanic (60%), White (58%), two or more races (33%), and Black (8%). Each iteration included pretests, engagement with the game, posttests, and a demographics questionnaire.

Survey Materials. Students completed measures of climate change knowledge and plausibility at pretest and posttest. The knowledge measure assessed the scientific consensus on climate change using a five-point agreement scale (Lombardi et al., 2013). The plausibility perceptions measure assessed endorsements related to human-induced climate change using a seven-point scale (Lombardi et al., 2012).

Analysis. Qualitative analysis occurred in four waves: three after each design iteration and a fourth after all data was collected. Interviews were transcribed and analyzed. Interviewers wrote analytical memos based on open analyses of each transcript, and conclusions informed modifications to the Estimation Game. After three iterations, recordings were open-coded for student thinking dimensions (Corbin & Strauss, 2004; Saldaña, 2021). Themes centered around students' quantitative reasoning strategies and emotions experienced during the game.

Findings

Survey results showed growth from pretest to posttest. At pretest, students had an average knowledge score of 2.4 of 5 and plausibility perception score of 4.87 of 7. Posttest scores improved to 2.91 of 5 for knowledge and 5.08 of 7 for plausibility.

RQ1: Design of an Open-Access Data Estimation Game with Number Line Visualizations

Three design iterations informed several game modifications. The first design iteration redeployed the central design features of the original intervention created by Thacker et al. (2024). We asked participants to estimate 12 climate change numbers. After making each estimate, a pop-up window would display the actual value along with accuracy feedback (one to five gold stars), an explanation of the true value, and links to sources of the information. At the end, a summary page provided an overview of all estimated items and accuracy ratings.

Four key improvements were introduced on top of the original design prior to the first round of interviews. First, we modified the accuracy feedback to also present students with a linear number line visualization illustrating each estimate alongside the true value to facilitate comparisons between the two and boost comprehension. Second, we adjusted the procedure for calculating accuracy feedback to present students with gold stars indicating order of magnitude error rather than absolute error, which is a more generous assessment of estimation accuracy and has been used in similar research (Bröder et al., 2022). Third, to make the intervention more "game-like" we modified the progress bar to include an earth icon, the summary page to provide students with a "final score" indicating the sum of their accuracy ratings, and we updated the "look and feel" of the web app to include more color and animation.

RQ2: How Students Responded to the Intervention

Inductive coding revealed themes in students' estimation strategies and emotional responses. Estimation strategies included drawing from prior knowledge, mental computation, and wild guesses. Emotional responses varied from relief, sadness, and surprise about climate change information to excitement or disappointment about their performance. A summary of related results are presented in Table 1.

Table 1. *Summary of Themes Related to Student Reactions to the Estimation Game*

Dimension	Sub-Dimension	# of Students	Example Excerpts from Student Interviews
Theme 1: Strategies for Estimating Climate Change Quantities			
Prior Knowledge	Educational	1	I was in geography and I saw a picture from 2009 to 2020, and it raised by a lot, so I'll say, say 65 inch increase.
	Personal	7	It's really cold right now… so [I'll estimate] maybe like 7
	Prior item	6	I'm going to use the same answer that I used in my first question.
	Vague	11	I remember hearing somewhere… it was 30.
Mental Computati-on Applied to a Given Value	Extrapolation	10	[Reading hint] Global sea levels rose by 1 inch between [1900 and 1920] that's not a lot…. But [now] it's more… like 5 or 6.
	Arithmetic	9	I would just say 12. I'm just going to multiply by four.
	Unspecified	4	I think it's gone down [compared to what was given in the hint]
	Rounding	1	So we'll round the 53 to 50…
Wild Guess	Wild Guess	12	I'd say four. Wild guess
Theme 2: Emotional Response to New Information			
Emotions (about climate change)	Relief	5	I'm glad that more countries are [committing to climate action] than I thought.
	Sadness	8	Geez [glaciers are melting], well that's sad.
	Surprise	8	I am very surprised and happy about it too. Huh. That is a nice surprise to know that I am very wrong.
Emotions (about	Excitement	6	Amazing job. I was so close. *claps* Look at that. It was 151%. Yay, I got five stars!

performance)	Disappointment	5	Seriously? 5 billion times, I got one star. That's my lowest score. That's really sad.

Contributions

We sought to design an Estimation Game and explore the strategies that diverse secondary students use when estimating climate change data. The intervention, guided by theories of conceptual change and numerical development, reduced climate change misconceptions and increased climate change plausibility perceptions. We found that, as students estimated numbers, they tended to draw from their prior knowledge and/or employ mental computation strategies, supporting the idea that estimating real-world quantities may provide opportunities for students to coordinate their magnitude knowledge and prior knowledge in such a way that is mutually beneficial (Siegler, 2016). Future research might explore relationships between estimation strategies and emotional responses and their impact on STEM-integrated learning.

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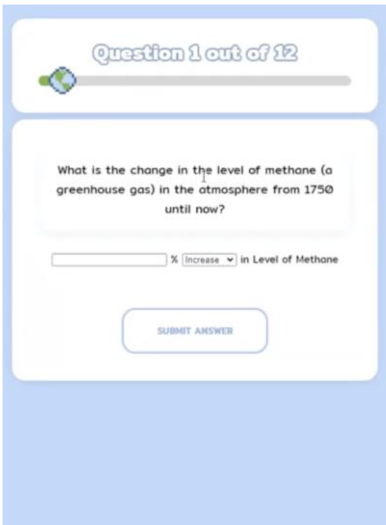
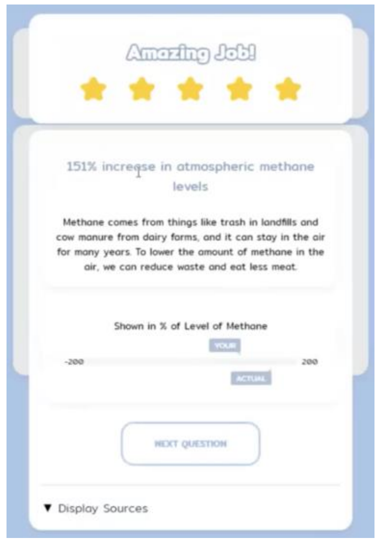
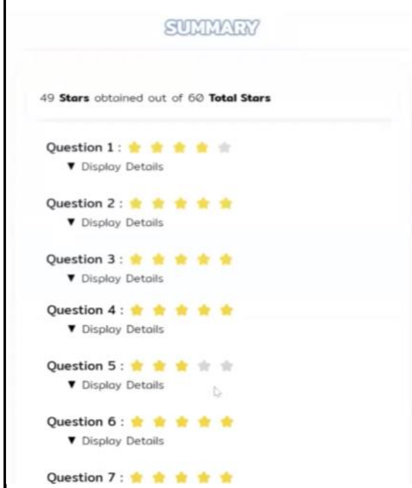
Appendix A: Screenshots of the Estimation Game Developed for this Study

<table>
<tr><td>Students are prompted to estimate a climate change number</td><td>True value pops up with accuracy feedback & number line & sources</td><td>Summary screen after Estimating 12 Quantities</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
</table>

Note. See ianthacker.com/design.html for details.

MATHEMATICAL REASONING AND GESTURES IN GEOMETRY WITH AUGMENTED REALITY

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Recent developments in Augmented Reality (AR) headset technologies have allowed for users to see and manipulate the same 3D hologram at the same time, allowing for collaborative, embodied interactions using gestures and actions. Three-dimensional geometry has traditionally been taught largely through 2D projections of 3D figures on pages or screens, thus dynamic holograms open new opportunities for understanding and extending mathematical cognition. In the present study, we examine multimodal interactions for high school students with either joint 3D holograms or without holograms while exploring geometric conjectures. We examine cases in which changing modalities is associated with changes in reasoning and conclude that holograms have important tradeoffs for embodied mathematical reasoning.

Keywords: Mathematical Processes and Practices, Technology and Learning Environment Design, Geometry and Measurement.

Objectives and Purpose

For many secondary students, formal geometric concepts can be hard to grasp. The abstract way in which geometric figures and concepts are presented in "school math" can create challenges for obtaining deep understanding (Ubi et al., 2018). Grounded representations like gestures and manipulatives can help make more concepts more accessible through connections to prior knowledge and real-world experiences (Cook & Goldin-Meadow, 2006; Demitriadou et al., 2020; Goldin-Meadow, 2005; Nathan et al., 2014; Pier et al., 2019; Walkington et al., 2019). In this inherently *embodied* perspective, learning and practicing mathematics by using the body grounds the concepts through perceptual and motor systems resulting from simulated and performed actions (Abrahamson et al, 2020; Wilson, 2002). The framework of embodied cognition has major affordances for improving the accessibility of these concepts that are presented like amodal mathematical operations that have little connection to the real world, social systems, and physical bodies that sparked the ideas and principles behind them (e.g., Lakoff & Núñez, 2000). Previous literature in the field suggests that embodied learning technologies like augmented reality (AR) can have a positive impact on motivation, learning experience, and understanding (Jabar et al., 2022).

In the present study, we sought to investigate the effect of AR goggles on students' geometric reasoning within the context of mathematical learning. Recent advances in AR goggles have allowed three-dimensional (3D) holograms to be projected in the real-world scene in front of students wearing the goggles, with these holograms controlled through intuitive hand gestures (Walkington et al., 2023). Even more significantly, students can engage with these holograms collaboratively, seeing their partners and the room around them, with each able to manipulate the holograms and see the effects of others' manipulations (Washington & Walkington, Accepted). Here we investigate 28 high school students using AR goggles in pairs to explore geometric conjectures and describe the possibilities this modality offers for sense-making, reasoning, and access to mathematical ideas through bodily and perceptual actions. In this study we sought to answer the following research question: What is the effect of Augmented Reality (AR) on high school students' geometric understanding and reasoning compared to without the use of AR?

Theoretical Framework: Collaborative Embodiment and Gesture

The integration of AR technologies in mathematics education holds significant potential, particularly when viewed through the lens of embodied cognition and gestures. Research shows the critical role of AR in fostering embodied cognition, emphasizing the integration of motion, gesture, and meaningful movement to enhance comprehension (Walkington et al., 2021; Bujak et al., 2013). Embodied cognition, rooted in motor behavior, posits that cognitive processes involve active engagement of the physical body within the learning environment (Schneegans & Schöner, 2008). AR, with its interactive capabilities, immerses learners in virtual surroundings, requiring active physical engagement through gestures and manipulation. Gestures, defined as hand or arm movements used to communicate or explore mathematical ideas (McNeill, 1992), play a crucial role. Research (Alibali, 2005; Goldin-Meadow, 2010) highlights the significance of bodily action in learning (Washington & Walkington, Accepted), suggesting that gestures enhance knowledge structure creation in long-term memory compared to verbalization alone (Cook et al., 2008).

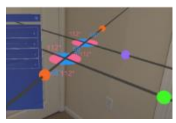
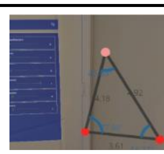
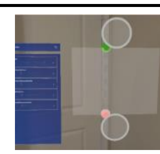
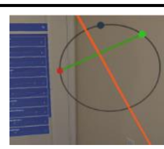
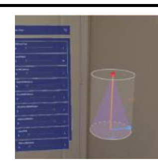
Embodied learning, within an enactivist framework, considers cognition as a result of dynamic interaction with the environment (Gallagher & Lindgren, 2015). AR, as noted by Bujak et al. (2013), encourages the creation of embodied representations, facilitating spatiotemporal alignment and a more meaningful learning environment. The ability to dynamically interact with virtual geometric shapes, observe real-time changes, and manipulate objects in AR provides profound affordances for deeper understanding (Washington & Walkington, Accepted). Studies (Price et al., 2020) suggest that immersive AR environments fill gaps in comprehension for mathematical concepts by allowing learners to use their bodies as tangible resources (Washington & Walkington, Accepted). This aligns with the idea that sensorimotor experiences and meaningful movement enhance geometry and spatial learning (Leitão et al., 2014; Özçakır & Çakıroğlu, 2022). Incorporating technology into education, especially AR, scaffolds reasoning and student capabilities, fostering a better assessment of mathematical understanding and knowledge growth. The literature also emphasizes the importance of interactive AR technologies that go beyond passive engagement (Walkington et al., 2023; Washington et al., Accepted;). Theoretical frameworks centered on gestures and embodied learning highlight how AR enhances sense-making, problem-solving, and collaborative learning. Dynamic gestures in AR, coupled with its immersive nature, make mathematical concepts more concrete and visually comprehensible (Bujak et al., 2013; Dunleavy et al., 2009; Tomaschko & Hohenwarter, 2019).

However, while extant literature provides compelling evidence for the benefits of AR in education, there is a gap in research on the possibilities offered by AR holograms for embodiment and gesture. A comparative study of students exploring geometry conjectures with and without AR goggles projecting 3D holograms seeks to uncover the affordances and limitations of AR technologies for embodied collaboration. This investigation delves into the changes in reasoning, encompassing speech, gestures, actions on objects, and whole-body movements in a distributed cognitive system with multiple learners. Through this analysis, the study aims to contribute insights into the potential of AR holograms in enhancing embodied learning and collaborative problem-solving in mathematics education.

Methods of Inquiry

Participants included 28 high school students enrolled in an enrichment program at a local university intended to support high school students likely to become first-generation college students. Twenty-three of the participants were female and 5 were male; 12 identified as African American, 14 as Hispanic, 1 as White, and 1 as Other Race/Ethnicity. They ranged from freshman to seniors, with an average age of 16.04 years. Only 2 students had not yet taken high school Geometry. Participants were asked to reason about 6 of 8 geometric conjectures about different shapes (see Table 1) in pairs, without AR goggles on. Then they put on the Microsoft HoloLens 2 AR goggles and justified 3-4 of the conjectures a second time. They used an app on the Microsoft HoloLens 2 AR goggles developed by GeoGebra (Hohenwarter & Fuchs, 2004), which rendered the shapes as 3D holograms, with the ability for the researcher to toggle different features (e.g., add a plane intersecting the circle or add a cylinder inscribing a cone). All simulations can be viewed at: https://sites.google.com/view/flatlandxr/home.

Table 1: Sample Conjectures Students Were Asked to Reason About and Explain

2D Simulation	Example Conjecture	Simulation Photo	3D Simulation	Example Conjecture	Simulation Photo
Parallel Lines	If two parallel lines are cut by a third line, the pairs of corresponding angles are congruent.		Sphere	A plane can only intersect a sphere at zero, one, or infinite points.	
Triangle	The sum of the lengths of any two sides of a triangle must be greater than the length of the remaining side.		Cylinder	Given a cylinder with radius r and height h, the cylinder can be unrolled to include a rectangle with length h and width 2**π**r.	
Parallelogram	Consecutive angles in a parallelogram add up to 180 degrees.		Prism	If the length, width, and height of a cube are each doubled, then the volume increases by a factor of 8.	
Circle	The perpendicular bisector of any chord always goes through the center of the circle.		Cone	The volume of a cone is one-third the volume of a cylinder with the same base and height.	

The feed from each students' goggles was recorded, and camcorders also recorded the room. Videos were transcribed and entered into NVivo. A coding scheme was determined that focused on the nature of the mathematical reasoning, including gestures, collaborative talk moves, actions on virtual objects (i.e., holograms), and whole-body movements (e.g., walking around a triangle). Reasoning was coded as students showing multimodal evidence they understood key mathematical *insights* related to the geometry conjecture (Nathan et al., 2021; Zhang et al., 2016). For example, for the triangle conjecture, a key insight the students can have is that if the sum of the lengths was greater, the triangle will not be able to close. The key criterion for an insight is that it must extend what was explicitly given in the conjecture text, and that it must be consistent with the actual properties and interactions of Euclidean shapes. Insights can occur across modality, and insights communicated by speech were not privileged. Responses were coded as to whether they included mathematical insights when solving each conjecture without the goggles versus on a subsequent attempt with the goggles. Cases where the students did have key mathematical insights both without the goggles and with the goggles were coded as "Insight Stay," and cases where students did not have key mathematical insights without the goggles and then with the goggles were coded as "No Insight Stay." More interestingly, cases where students did not have key insights without the goggles but had key insights with the goggles were coded as "New Insight" and cases where the students had key insights without the goggles that they discarded with the goggles were coded as "Discarded Insight." We first counted all instances of these 4 categories, and then used multimodal analysis (Walkington et al., 2023; McNeill, 1992; Alibali & Nathan, 2012) to analyze one representative instance of each case.

Results

The dynamic nature of the holograms and their 3D nature seemed to have some powerful affordances for mathematical reasoning and embodied collaboration by either changing a student's mind when they were able to see the shape in front of them as a hologram or reaffirming their initial reasoning in both negative and positive ways. See Table 2 for the reasoning counts of changes or stays in reasoning broken down by conjecture.

Table 2: Reasoning Insight Changes or Lack of Changes by Conjecture

	Insight Stay	No Insight Stay	New Insight	Discarded Insight	N/A	Total
Triangle	6	9	4	1	0	20
Circle	3	0	1	1	3	8
Parallelogram	3	4	2	1	0	10
Parallel Lines	6	2	10	3	1	22
Sphere	7	3	4	7	1	22
Cube	0	2	1	1	0	4
Cylinder	2	3	5	1	1	12
Cone	4	1	3	2	0	10
Total	32	24	29	17	6	108

Some students were able to use the holograms and their associated embodied resources and motions to reason more effectively or reaffirm prior reasoning. Figure 1 demonstrates such instances of "Insight Stay" and "New Insight." Figure 1 shows students discussing the parallel lines conjecture from Table 1 using collaborative speech, actions on objects, and gesture. In Figure 1, we see S107 and S108 use dynamic gestures (P1-P3 – moving their hands diagonally to show the angle and vertically parallel to show the parallel lines) without the goggles to explain reasoning. We coded this as **Insight** for S107 as he talked about the intersecting line not effecting the corresponding angles as they would be congruent for both parallel lines (Line 4, P1). We coded S108 as **No Insight** because she references the length of the parallel lines being the reason the corresponding angles are congruent (Line 5, P2-P3). They then moved to the AR goggles portion where having access to the 3D shape seemed to give the students novel insights, and we see them using the shape manipulation to change the perspective they have on the parallel lines to help with sense-making. The dynamic nature of the hologram, combined with its scale (i.e., students were immersed in large-scale parallel lines together) seemed to allow for effective forms of collaborative multimodal communication to occur. We also see S107 using dynamic gestures to reference the path of the parallel lines and the intersecting line (Line 8, P6) and collaborative actions to dynamically modify their parallel lines (P4 and P7). We coded this as **Insight** for S107 and S108, making this an **Insight Stay** for S107 and **New Insight** for S108. As seen in Table 2, this conjecture had the second highest number of Insight Stays (6) and the highest number of New Insights (10) by over double compared to the next conjecture (4). Thus, the immersive, dynamic parallel lines seemed to be powerful as a hologram, leading to reaffirmed and new mathematical insights.

(S107 and S108 start in Verbal discussion of the parallel lines conjecture)

1. 107: I would say it's true.
2. 108: Yeah true.
3. 107: Because if it were to be a parallel line, it doesn't matter what angle or what type of pass the third line takes, it is always going be congruent to your…or, well the corresponding angle is gonna always be congruent because they're always hitting the same type of angle, the line is going to be the same type of angle for both of the lines.

(S107 gestures hand up and down diagonally to show the line angle of intersection to the parallel lines – P1)

4. 108: Same as him, and then since they're parallel, they are never going to stop, they are just going to keep on going straight, no matter the angle.

(S108 gestures hands vertically parallel to show the parallel lines and then moves them upwards to show them as continually parallel – P2 and P3)

(S107 and S108 move to the HoloLens discussion of the parallel lines conjecture)

(S107 moves the angles of the parallel lines – P4)

5. 107: Wait can you show the image again? *(P5)* Okay. Yeah, they're both still similar. Because they still go through the same trajectories and angles that the third one has which leads to it still having the same effect on both.

(S107 gestures his hand back and forth to show the trajectory of the parallel lines – P6)

6. 108: I was about to say, right now I was trying to move the parallel lines and they stayed the same even if I move the middle line.

(S108 moves all three manipulatives, changing the distance and angles between the parallel lines, and the angle of the intersecting line– P7)

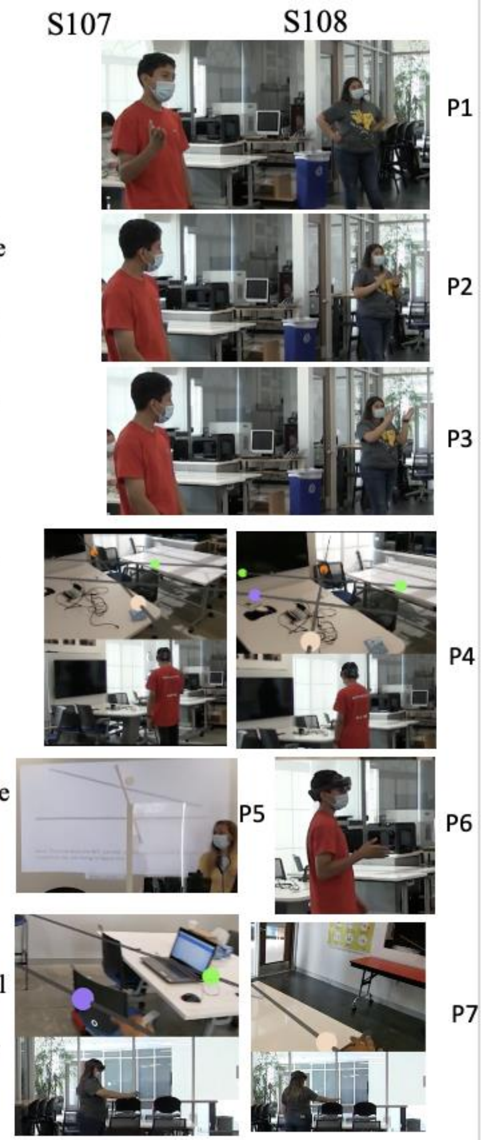


Figure 1: Insight Stay and New Insight: Reasoning about Parallel Lines Conjecture

In Figure 2, we see a case of "No Insight Stay." In this transcript without the goggles, S123 uses representative gesture (P1and P4) that represents two vertically parallel lines, and in P2-3 one arm vertical to show a 90-degree angle and one arm horizontal to show a 180-degree angle. They also use dynamic gestures (P5 – moving his hands back and forth in parallel lines). We coded this as **No Insight** as he talked about the angle and direction to which the sets of parallel lines in the parallelogram were having an effect (Line 2-4 and 9). They then moved to the AR goggles portion where seeing the 3D shape did not seem to give the students novel insights, despite them using the shape manipulation to change the perspective they have on the parallelogram. Despite S124 making a relevant observation in Line 12 (P7), the students cannot see how the angles can equal 180-degrees. In Lines 15-18, S123 and S124 add two sets of angles and explain that neither add to 180-degrees and use pointing gestures (P8-P10 – pointing at the angles) and collaborative actions to dynamically modify their parallelogram (P6) to show their thinking. We coded this as **No Insight Stay**. As seen in Table 2, No Insight Stays were the most common (4) within this conjecture.

(S123 and S124 start in Verbal discussion of the parallelogram conjecture)
1. 123: If parallel lines go straight, they don't touch.
(S123 gestures to represent vertical parallel lines with his arms – P1)
2. 123: The line depends where they really are. Cuz I mean if it goes up it's 90 degrees. But regular straight line is 180 degrees.
(S123 gestures to show a vertical line representing a 90• angle, then gestures a flat horizontal line to represent a line with a 180• angle – P2)
3. 123: So, it kinda depends where the lines are pretty much going.
(S123 gestures both arms vertically parallel and then horizontally parallel – P3)
4. 124: So false?
5. 123: Yeah, I'm guessing false here.
6. 123: So, if they both go up, then maybe they can add up to 180. Depending, cause they're going go up so they're pretty much going to add up to 90•.
(S123 gestures vertical parallel lines three times, once moving his arms up and down two times, then gestures one hand horizontal and one vertical to create a 90• angle – P4)
7. 123: But if they goes sideways you want to slide down the bottom with 180 degrees. So pretty much if you add those up, they will equal 360 degrees, not 180.
(S123 gestures hands left to right horizontally parallel crossing each other to represent the parallel lines, then gestures with both forearms to create a long horizontal line – P5)

(S123 and S124 move to the HoloLens discussion of the parallelogram conjecture)
(S123 and S124 move the shape changing the size and creating different angles – P6)
8. 123: Consecutive angles are angles inside the shape that are next to each other. So next as in parallel or like right next to each other as in...?
9. 124: Yeah, as in top and bottom or left and right, I think that's what it means.
(S124 gestures to the left corner and right corner to show these angles are top and bottom or equivalently left and right that are consecutive angles – P7)
10. 123: So, it said it's always equals to 180. Okay, then it says consecutive angles so its two angles so I'm guessing its 121 plus 121, that doesn't equal 180.
(S123 points at the bottom right angle that is 121• and the top left angle that is 121• – P8)
11. 124: Because it already passed 180.
12. 123: Yeah, 58 and 58, but it doesn't equal to 180. Do we make it smaller to see if it's still to the point where it's about to be...
(S123 points at the bottom left angle that is 58• and the top right angle that is 58• – P9)
13. 123: Make it smaller. If I make it smaller the angles get bigger depending on which side. This one is one... There, it's there. It's smaller but the angle is 126 and this one is 126 and these two angles top and bottom...wait is that left or right?
(S123 gestures to point at two angles and then point to the remaining two angles – P10)
14. 123: False.
15. 124: False.
16. 123: Since it depends on really the angles but even the angles are beyond the point where they can make it to 180 together.
(S123 gestures to point at the angles)

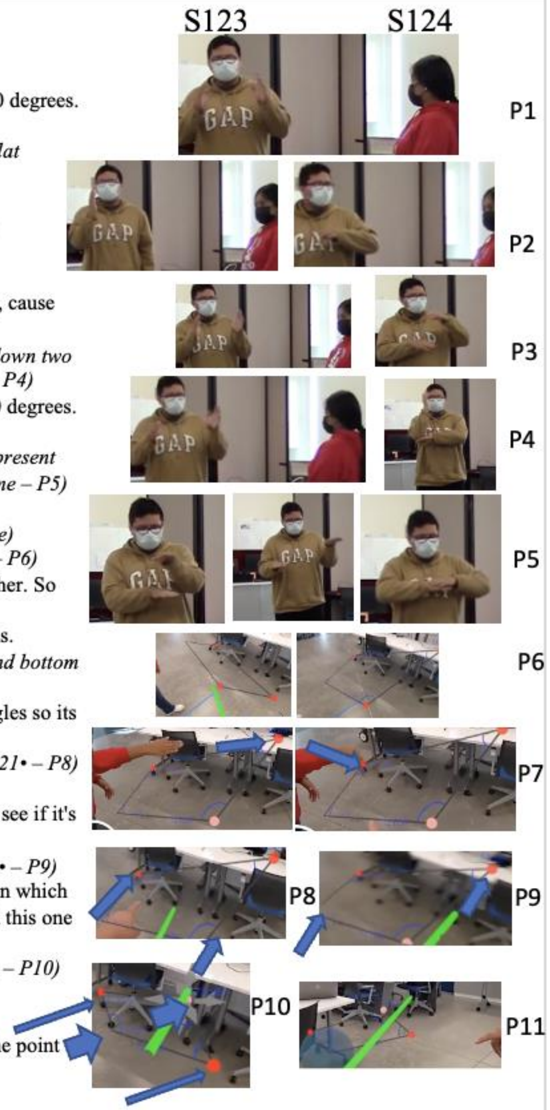


Figure 2: No Insight Stay: Reasoning About the Parallelogram Conjecture

In Figure 3, we see the "Discarded Insight" case where a pair attempts to prove the sphere conjecture. Without the goggles, we see S111 use representational gestures (P1 and P3 – creating an intersection and a sphere with her hands), dynamic gestures (P4 – moving her hand forward to show a cut or moving intersection) and pointing gestures (P3 – pointing at a single point on the imaginary sphere) to explain her reasoning. We coded this as **Insight** as she communicated having an infinite piece of paper and the places that would intersect the sphere (Line 6-7). They then moved to the AR goggles portion where having access to the 3D shape seemed to give the students novel insights, and in the video, we see them using the shape manipulation to change the perspective they have on the sphere to help with sense-making. We see them using pointing gestures to reference places on the sphere and collaborative actions to dynamically modify their sphere, all while engaging in collaborative talk moves. However, despite this strong start (Lines 11-14, P6), they did not see how zero points of intersection was possible with the hologram now present (Lines 16-20). We coded this as **Discarded Insight**. As seen in Table 2, the sphere conjecture had the highest number of Discarded Insights (7) amongst conjectures; thus, the hologram was not functioning for students' math reasoning in the way we had intended.

(S111 and S112 start in Verbal discussion of the sphere conjecture)

1. 111: So, hmm it may be true. Well, I don't know about it being zero. Think it's true?

(S111 gestures one hand flat and another on top gesturing "intersection," then creates a representation of a sphere with her hands – P1)

2. 112: Yeah. I think it's true.
3. 111: Yeah, I think it's true.
4. 111: Umm, I don't know, if you get an actual sphere and a piece of paper but think of it as an infinite piece of paper, and you place the sphere on the piece of paper, it could intersect it that one point.

(S111 gestures her hands outwards, then points to the bottom "one point" she referenced, then creates a sphere with her hands – P2 and P3)

5. But if you take the piece of paper and intersect it through the whole sphere, that is more than one point. That is basically like half the circle or the diameter of it. I am probably thinking too much but that is how it infinite points, so, that's why I said it's true.

(S111 gestures one hand flat and moves it forward to represent the paper "intersecting" through the sphere, then points her finger and moves it in the air drawing a circle – P4)

(S111 and S112 move to the HoloLens discussion of the sphere conjecture)

(S111 and S112 make the sphere bigger and smaller – P5)

6. 111: So, this is the plane and the circle, okay maybe not zero points. I mean it is intersecting it so...okay wait where is the point?
7. 112: I think it's true because it can literally intersect anywhere, so I think it's true.
8. 111: Well, it does say at zero, one, or infinite points so it's including zero, oh crap wait. Well, it's not really intersecting at zero points, so to say. Because it's intersecting it right here and right here. Ummm...

(S111 point to the left side and the right side of the sphere – P6)

9. 111: Okay. Yeah, I mean it still intersects at at least one point. I wouldn't say zero points.
10. CW: Alright so you don't think zero points is possible.
11. 111: No, well if it's intersecting it, it's intersecting at one point, yeah so, I think that's false.
12. 112: Well now that she said zero points, it's kind of making me think about it. But I don't know because I get the one but I also kind of don't get the zero. So, I'll go with false.

(S112 makes the sphere bigger and smaller – P7)

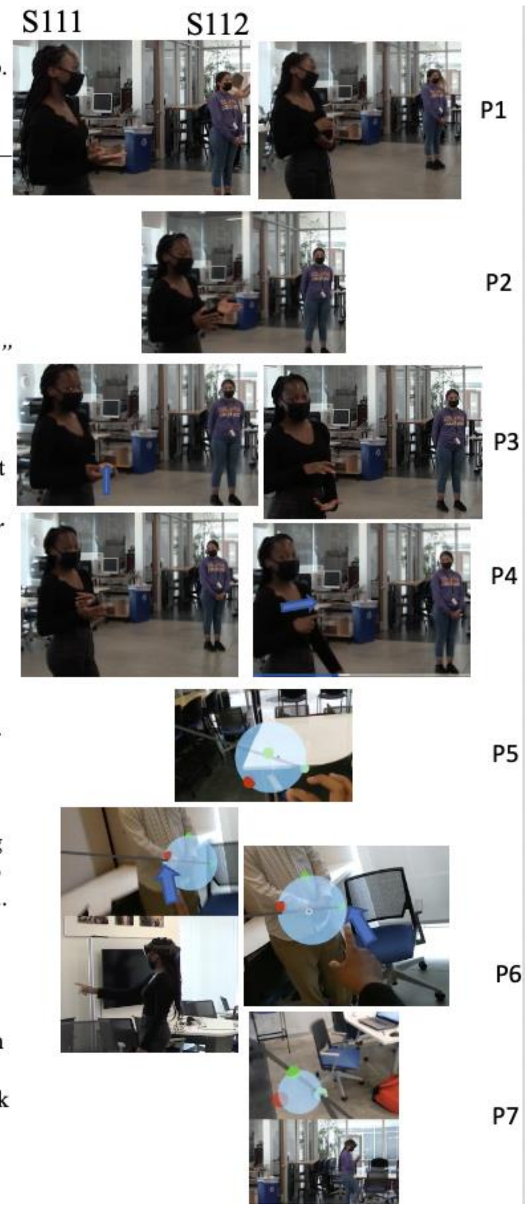


Figure 3: Discarded Insight: Reasoning About the Sphere Conjecture

Discussion

This study has several implications. The first is that AR environments can change, support, or reinforce students understanding, beliefs, or thought processes in their sense-making of geometric objects, in some situations. Students can use these technologies effectively to support in creating meaning to illustrate students' "knowledge" (Bujak et al., 2013; Jabar et al., 2022; Tomaschko & Hohenwarter, 2019; Washington & Walkington, Accepted). AR can also be an effective tool that helps clear up misconceptions or errors in mathematical reasoning with 'physical' manipulatives and visuals to aid in this process (Price et al., 2020; Jabar et al., 2022). However, some students who use the technologies could have trouble with altering their reasoning to be consistent with mathematical insights about how figures and shapes work in Euclidean space. This could come from not using the technology effectively, forgetting to use the technology, or trouble seeing the mathematics in the hologram, which could reinforce students' original reasoning. Another idea is that AR holograms may be more impactful and appropriate for concepts that are inherently three-dimensional, as these objects and our simulations leverage profound affordances; however, our data does support their effectiveness for some 2D objects like parallel lines cut by a transversal. Finally, AR allows students to interact with and explore objects in novel and nontypical ways (Bujak et al., 2013; Dunleavy et al., 2009; Walkington et al., 2023; Washington et al., Accepted; Washington & Walkington, Accepted). However, if you bring more traditional "school" tasks into an AR environment, students' reasoning and attempts at meaning-making could be restricted. In our context, each simulation could use more time for exploration of properties of a figure, rather than confronting a specific conjecture immediately. Future study alternatives we are exploring is to scaffold students by having them work up to making and proving or disproving conjectures after familiarizing themselves with the movements of the figures. Tasks in an AR environment become particularly meaningful if they come from students and their embodied, gestured, collaborative experiences (Walkington et al., 2019; Washington & Walkington, Accepted).

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CONCEPTUALIZATIONS AND REPRESENTATIONS OF FRACTION DIVISION IN ONLINE OPEN ACCESS GEOGEBRA RESOURCES

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This study aimed to examine the quality of the GeoGebra applets for fraction division by attending to the conceptualizations, representations, and cognitive actions prevalent in these digital resources. The results reveal that a majority of the existing applets conceptualize fraction division as measurement while other conceptualizations of fraction division are underrepresented. Among the various representations used, the length model in the form of fraction bar and area model are the predominant choices, with the number line representation being notably less prominent. The results from this study also show that although most applets attempt to visualize the process of fraction division, there are limited opportunities in most applets for users to enact mental actions associated with fraction division, such as partitioning, unitizing, iterating, and disembedding. These results not only increase our understanding of the affordances and limitations of existing applets for fraction division so that we can become more intentional in our choice of them but also inform the design of new applets that support students' development of a rich and robust understanding of fraction division.

Keywords: Rational Numbers, Technology, Mathematical Representations

A robust understanding of fractional concepts not only extends students' understanding of numbers but also is foundational for learning more advanced mathematical content areas and everyday quantitative reasoning. Research has shown that understanding fractions is predictive of students' long-term success in mathematics (NMAP, 2008; Booth & Newton, 2012; Siegler et al., 2012). Despite its importance, researchers have documented challenges in understanding fraction concepts among students in both elementary and secondary schools (e.g., Pitkethly & Hunting, 1996; Siegler & Pyke, 2013). This is especially the case for fraction division, where the literature indicates that students and even prospective mathematics teachers often struggle to create and use representations appropriately, overgeneralize properties of operations with natural numbers to fractions, interpret division primarily using a primitive partitive model of division, and tend to absorb only mechanical procedure like the invert-and-multiply algorithm and thus create "bugs" in computing division expressions (e.g., Tirosh, 2000; Lo & Luo, 2012; Adu-Gyamfi et al., 2019). In other words, the lack of conceptual understanding and representation fluency of fraction division has been identified as a major issue for learning fraction division.

The past few decades have seen the rise of a large repertoire of mathematics-specific technological tools, which can support teachers and students in visualizing, displaying, acting upon, observing, and validating mathematics relationships (Heid & Blume, 2008). By including dynamic and/or interactive representations, these digital tools have the potential to provide students with new ways to conceive and represent mathematics ideas, which are likely to support their development of conceptual understanding and representation fluency of mathematical concepts. In the domain of fractions, evidence suggests that the use of digital technology can provide opportunities for students to work with/on fractions in interactive and dynamic ways, which are likely to support the development of robust understandings of fractional concepts and their operations (e.g., Steffe & Olive, 2002; Poon, 2018; Anat et al., 2020; Yeo & Webel, 2024). Meanwhile, researchers have pointed out that the design of tool-based tasks can embody different conceptualizations and representations of a mathematical idea (Leung & Bolite-Frant, 2015). The use of different conceptualizations and representations in the design of tool-based tasks might impact not only what mathematical ideas are learned but also how they are learned. Therefore, it is important to analyze how mathematical ideas are conceptualized and represented in different tool-based tasks and resources. This is particularly true for concepts like fraction division, which are not only difficult to teach and learn conceptually but also open to multiple conceptualizations and representations.

As an interactive geometry, algebra, statistics, and calculus application that can run on multiple platforms (e.g., desktops, tablets, and online), GeoGebra is developed for learning and teaching mathematics from primary school to university level. The GeoGebra website hosts more than one million free activities, simulations, exercises, lessons, and games for mathematics and science. A significant number of these GeoGebra applets are created for students to learn fraction division. Given that GeoGebra is a community of millions of students and teachers who are potential users of these applets, it is important to examine the conceptualizations and representations of fraction division in these applets.

Literature Review and Conceptual Grounding

Conceptualizations of Fraction Division

The literature has described diverse conceptualizations of fraction division, including , division as partition, division as the determination of a unit rate, division as measurement and division as the inverse operation of multiplication (Sinicrope et al., 2002; Gregg & Gregg, 2007; Lamon, 2012; Adu-Gyamfi et al., 2019). When interpreting *fraction division as partition*, fraction division is perceived as the process of equally sharing a given quantity (dividend) between a given number (divisor) of groups in order to determine the amount in each group. Note that the conceptualization of fraction division as partition is efficient only in situations where the divisor is a whole number. When interpreting *fraction division as the determination of a unit rate*, the focus is not on the action of equally sharing but rather on the size of one unit. Under this conceptualization, $\frac{a}{b} \div \frac{c}{d}$ is interpreted as $\frac{a}{b}$ of one unit of quantity A corresponds to $\frac{c}{d}$ of one unit of quantity B, and the result of $\frac{a}{b} \div \frac{c}{d}$ represents the amount of quantity A in relation to one unit of quantity B. Therefore, $\frac{a}{b} \div \frac{c}{d} = \left(\frac{a}{b} \div c\right) \div \frac{1}{d} = \left(\frac{a}{b} \times \frac{1}{c}\right) \times d = \frac{a}{b} \times \frac{d}{c}$, in which $\frac{a}{b} \div c$ represents only $\frac{1}{d}$ of a unit and multiplying the quantity by d will get the whole unit. It is worth noting that this conceptualization of fraction division makes use of the partition interpretation of fraction division (i.e., $\frac{a}{b} \div c = \frac{a}{b} \times \frac{1}{c}$) and proportional reasoning. When interpreting *fraction division as measurement*, $\frac{a}{b} \div \frac{c}{d}$ is conceptualized as the number of times that the quantity $\frac{c}{d}$ can go into $\frac{a}{b}$. In other words, $\frac{c}{d}$ is considered as a unit of measure for $\frac{a}{b}$. This conceptualization of fraction division can lead to the common denominator algorithm for fraction division: $\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times d} \div \frac{b \times c}{b \times d} = ad \div bc$. In this algorithm, the divisor and the dividend are first expressed as fractions with like denominators, and the numerators are then divided to compare the fractions multiplicatively. As *the inverse operation of fraction multiplication*, a fraction division acts as an inverse function to that of fraction multiplication, mapping some set onto another set. More specifically, the original amount (product in multiplication, dividend in division) is multiplied by the denominator and divided by the numerator. This conceptualization of fraction division revolves around actions such as shrinking and enlarging, compressing and expanding, or simply multiplying and dividing.

Mental Actions Associated with Fraction Division

Mental actions constitute the key component of mental schemes. Mental actions, such as unitizing, partitioning, disembedding, iterating, and splitting are essential when working with fractions and their operations (Steffe & Olive, 2009). *Unitizing* is a mental action that treats an object or collection of objects as a unit, or a whole. Unitizing is essential for understanding fraction division because understanding fraction division demands the ability to visualize reference units, to move between the original unit and the intermediate unit, and to interpret a symbol or operation in terms of those units. *Partitioning* is the mental action of dividing a unit, or a whole, into equal parts. Equal-sized parts are fundamental to partitioning and to constructing the part-whole conception of fractions. *Disembedding* is the mental action of imaginatively pulling out a fraction from a whole while keeping the whole intact. The part-whole conception of fraction relies on mental actions of partitioning and disembedding, with which students can project n equal parts in a continuous whole (partitioning) and pull out m of those parts without losing track of their containment within the whole (disembedding), resulting in the fraction $\frac{m}{n}$. *Iterating* is the mental action of repeating a part to produce identical copies of it. The part can be a unit fraction or a non-unit fraction. Iteration helps children conceive of a whole as a multiple of a same-size unit, which draws their attention to the number of times a unit fraction fits within the whole. *Splitting* is the simultaneous composition of partitioning and iterating. Students who can perform the splitting operation recognize that partitioning and iterating are inverse to each other. They can split an unpartitioned piece of a larger or smaller whole to re-create the whole. These mental actions are vital for developing a robust understanding of fraction concepts and their operations, including fraction division.

Dick (2008) proposed a method of evaluating digital tools and resources for their pedagogical, mathematical, and cognitive fidelity, claiming that high levels of fidelity in these areas are necessary for a significant impact on student learning. The notion of cognitive fidelity indicates that digital tools and resources for mathematics teaching and learning should reflect students' cognitive actions with an emphasis on illuminating mathematical thinking processes rather than simply arriving at the final results. A GeoGebra applet with high cognitive fidelity for fraction division should not only demonstrate but also enable its users to enact the mental actions associated with fraction division. Given the importance of mental actions such as unitizing, partitioning, iterating, and disembedding in different conceptualizations of fraction division, it is important to assess how well GeoGebra applets support these mental actions.

Representations of Fraction Division

It has been agreed that mathematical objects such as fractions cannot be directly perceived or observed with instruments, and access to mathematical objects is bound to the use of a system of semiotic representations that allow them to be designated (Goldin, 1998; Duval, 2006). In the mathematical domain of fraction division, representations that model the process of division may include but are not limited to set models (i.e., the whole is understood to be a set of objects and subsets of the whole consist of fractional parts, such as pictures of familiar objects and dots), area models (i.e., fractions are represented as parts of an area or region of a shape, such as circles, rectangles, and grids), length models (i.e., a fraction is identified as being a particular distance from the "start" of the whole, such as fraction bars and number lines), numerals, and algebraic symbols. These representations have been used to model the different conceptualizations of fraction and fraction division. Research shows that specific representations and models of fraction concepts and operations do have their strengths and limitations. For instance, concrete set models may be well-suited to illustrate part-whole concepts but may not be appropriate for helping children understand fractions as numbers with specific magnitudes (Cramer & Wyberg, 2009). Circle representations can illustrate both the part-whole relationship and the meaning of the relative sizes of fractions (Cramer & Wyberg, 2009), but they are less helpful in supporting the conceptualizations of fractions beyond equal partitioning and tend to support additive thinking rather than multiplicative thinking needed for understanding fractions (Moss, 2005). Number lines help students see fractions as not only parts of a whole or parts of a set but also a part of distance. Many studies have argued the superiority of the number line over other representations of fractions (e.g., Sidney et al., 2019; Hamdan, & Gunderson, 2017). However, compared with other visual representations, the number line is more abstract and cognitively demanding because students have to coordinate symbolic information and visual cues in order to bring meaning to this model since the number line uses symbols to convey part of its meaning. Therefore, it is important to use multiple representations in teaching and learning fractions and their operations. To support students in developing a rich and robust understanding of fraction division, it is important to provide with them the opportunity to work with fraction division constructs (e.g., partition unit, referent units, and whole) via mental actions on representations (e.g., unitizing, partitioning, disembedding, iterating, and splitting) and reflective abstraction on those actions, to utilize multiple representations of the same fraction division conceptualization to observe isomorphic transformations between representations and to utilize multiple representations of the different conceptualizations of fraction division.

Research Questions

Informed by the literature on fraction division, this study was guided by the following questions:

1. What conceptualizations of fraction division are used in the GeoGebra applets for fraction division?
2. What representations are used in the GeoGebra applets for fraction division?
3. What cognitive actions are supported by the GeoGebra applets for fraction division?
4.

Methodology

Data Collection

GeoGebra applets for fraction division were collected from two different sources. First, on August 21, 2023, the keywords "fraction division" and "dividing fractions" were typed into the built-in search engine provided by the GeoGebra community resources cloud service (https://www.geogebra.org/materials), which resulted in 156 and 44 outputs, respectively. Data Miner (https://dataminer.io), which is a Google Chrome Extension and Edge Browser Extension that helps its users crawl and scrape data from web pages and export the data into a CSV file or

Excel spreadsheet, was used to collect the names and URLs of the GeoGebra applets. After combining the results from the two searches, 11 files were first removed due to identical URLs. Second, a set of GeoGebra applets for fraction division was identified from a larger raw data set of GeoGebra applets for fractions that aimed to be used to investigate the conceptualizations and representations of fraction concepts and their operations in the existing resources on the GeoGebra website. The GeoGebra fraction applets in the larger data set were collected in two different ways. First, the keyword "fraction" was typed into the built-in search engine provided by the GeoGebra community resources cloud service. Second, fraction applets were also identified through the organization chart on the GeoGebra (https://www.geogebra.org/t/fraction). 1054 GeoGebra applets were left in the larger data set after initial data cleaning, of which 84 GeoGebra applets were likely about fraction division as their names included the word "division". After combining the results from the two sources, 57 files were first removed due to identical URLs. Among the remaining 216 files with unique URLs, 20 applets were in languages other than English, 75 applets were not on fraction division (e.g., modeling division $a \div b$ as a fraction, where a and b are both integers), 61 applets were duplicates of an existing applet with a different URL, and 4 applets were incomplete. This data cleaning process resulted in 56 unique GeoGebra applets for fraction division, of which 43 applets demonstrate fraction division and 13 applets generate practicing problems on fraction division. This study focused on the 43 applets.

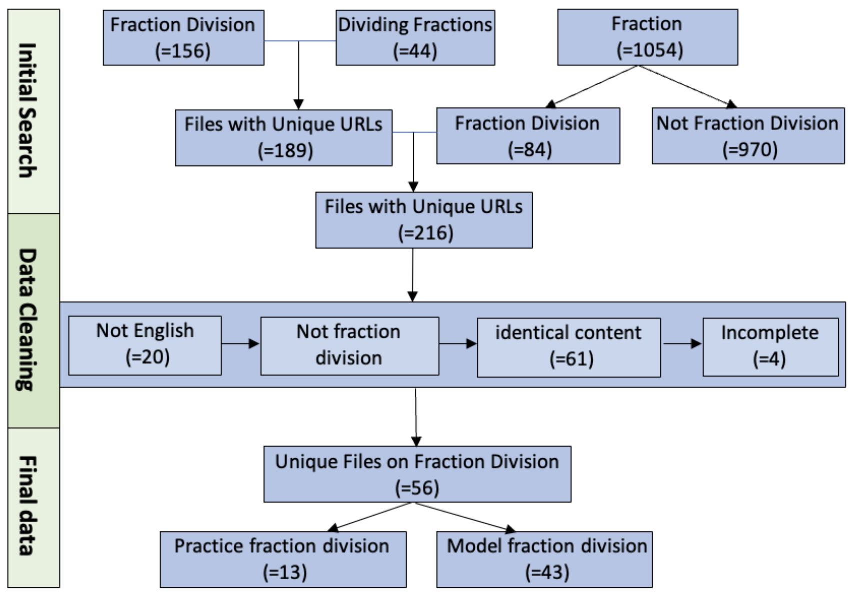


Figure 1: Data Collection and Cleaning Process

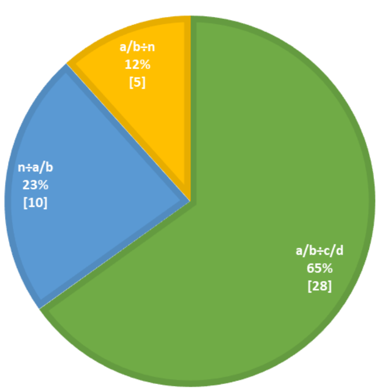
Data Analysis

When analyzing each applet, we considered the conceptualization of fraction division that the applet was based upon, the representation that was used in the applets, and the cognitive actions associated with fraction division that the applet could support. We also considered the type of fraction division being modeled, as it can influence the conceptualization used. The types of fraction division include a fraction divided by a whole number $\left(\frac{a}{b} \div n\right)$, a whole number divided by a fraction $\left(n \div \frac{a}{b}\right)$, and a fraction divided by a fraction $\left(\frac{a}{b} \div \frac{c}{d}\right)$. A fraction in an applet could be either a proper or an improper fraction. The conceptualizations of fraction division include fraction division as measurement, fraction division as partitioning, fraction division as the determination of unit rate, and fraction division as operator. The types of representations that could be used in an applet include fraction bar and number line in the length model, rectangle and circle in the area model, numerical representation, and algebraic representation. When coding cognitive actions supported by an applet, we considered the affordance of the applet in supporting the following actions, namely, changing the dividend/divisor fraction, unitizing, partitioning, iterating, and disembedding. We differentiated between the affordance of demonstrating the actions and the affordance of enacting the actions. We believe that such distinction is important because an applet might use a particular representation to visualize the process of fraction division without allowing its users to enact the action in the applet. Each applet was coded by the two authors. There was a high inter-rater reliability between the two authors. Disagreements in codes were resolved through discussion.

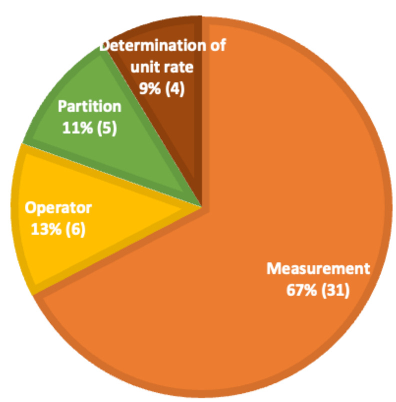
Results

Conceptualizations of Fraction Division in the GeoGebra Applets

As shown in Figure 2a, among the 43 applets, 28 were found to model "a fraction divided by a fraction" $\left(\frac{a}{b} \div \frac{c}{d}\right)$, 10 modeled "a whole number divided by a fraction" $\left(n \div \frac{a}{b}\right)$, and the remaining 5 applets modeled "a fraction divided by a whole number" $\left(\frac{a}{b} \div n\right)$. As shown in Figure 2b, among the 43 GeoGebra applets designed to model fraction division, 67% (31) of applets adopt the conceptualization of fraction division as measurement. The remaining conceptualizations are distributed fairly evenly, with 9% (4) applets framing fraction division as a determination of unit rate, 11% (5) as partitioning, and 13% (6) as the inverse operation of multiplication. There are a small number of applets (n=3) that integrate two conceptualizations.

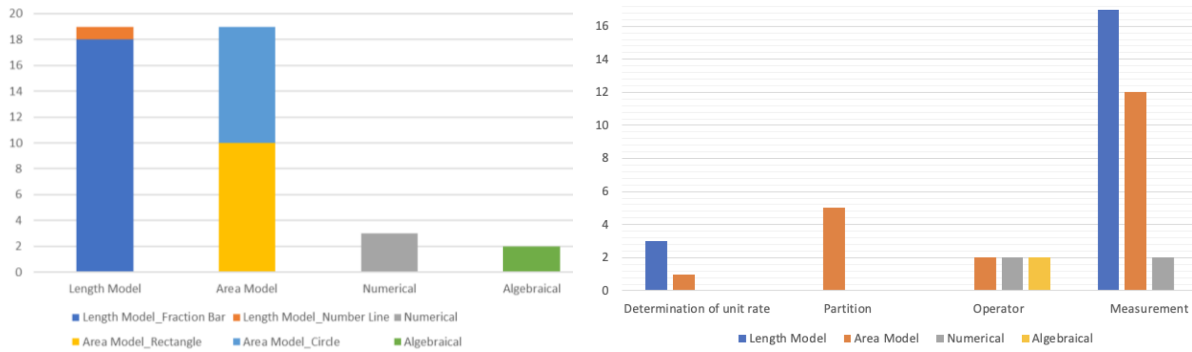


a. Types of fraction division situations



b. Conceptualizations of fraction division

Figure 2: Types and Conceptualizations of Fraction Division in the GeoGebra Applets



a. Representations used for fraction division b. Representations used by conceptualizations

Figure 3: Representations Used in the GeoGebra Applets for Fraction Division

Representations of Fraction Division in the GeoGebra Applets

Regarding the representations employed, a notable number of applets use the length model (n=19) and the area model (n=19) to convey their respective conceptualizations of fraction division. Among the 19 applets using the length model, only one utilizes the number line model, while the remaining 18 applets use a fraction bar model. In the case of the area model, 10 applets use a rectangular area model, while the remaining 9 applets opt for a circular area model. 5 out of the 43 applets provide numerical or algebraic procedural explanation without incorporating any visual representations.

A cross-analysis of the use of conceptualizations and representations reveals several noteworthy patterns. The use of the length model is preferred in the conceptualizations of fraction division as measurement and fraction division as the determination of unit rate. Specifically, among the 31 applets that use the measurement interpretation, the length model is used in 17 instances (16 use fraction bar, and 1 uses number line), followed by 12 instances of area models (6 rectangular, 6 circular). However, in contrast to that, all 5 applets that conceptualize fraction division as partition exclusively use the area model, with 3 instances of the rectangular area model and 2 instances of the circular area model. When fraction division is interpreted as an operator, although the length model is not used in any of the 6 applets, there is no distinct preference identified in the use of the other three representations.

Cognitive Actions Supported by the GeoGebra Applets

The analysis of cognitive actions supported by the GeoGebra applets for fraction division reveals a predominant reliance on passive demonstration rather than interactive user engagement. As shown in Figure 4, among the analyzed applets, 70% (30) applets showcase the action of partitioning, providing a visual representation of the fraction division process without allowing users to execute the partitioning action. Similarly, 42% (18) applets demonstrate the action of unitizing, while 32% (14) applets and 5% (2) applets illustrate the process of iterating and disembedding, respectively. Out of the 43 applets, 74% (32) applets allow users to change the values of the divisor and/or dividend. However, this prominence of fraction modification is counterbalanced by the limited incorporation of interactive features that support students' enactment of cognitive actions associated with fraction operations. More specifically, only 11.6% (5) applets allow their users to iterate a fraction, only 2.3% (1) applet allows its users to partition a fraction, and only 2.3% (1) applet allows its users to enact the action of disembedding. None of the applets allow the users to enact the action of unitizing.

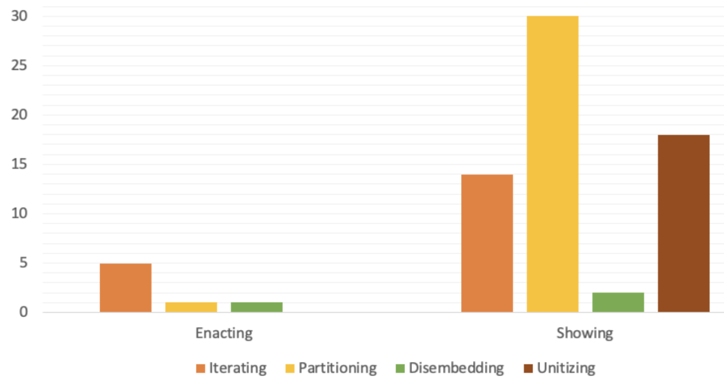


Figure 4: Cognitive Actions Supported by the GeoGebra Applets for Fraction Division

Discussion and Conclusion

Although dynamic mathematical software programs such as *GeoGebra* make it easy for their users to create and share digital applets with interactive and dynamic features, the quality of these applets is less clear. This study aimed to examine the quality of the *GeoGebra* applets for fraction division by attending to the conceptualizations, representations, and cognitive actions prevalent in these digital resources. The results offer valuable insights into the features of the GeoGebra applets for fraction division, which could inform the choices and design of GeoGebra applets for fraction division. This type of content-specific analysis of GeoGebra applets is important because it moves beyond the dynamic and interactive features of digital applets and focuses on what they actually can or cannot afford and how they can be improved in terms of supporting the teaching and learning of fraction division.

One noteworthy observation is the underutilization of the number line in the GeoGebra applets for fraction division, despite its advocated superiority in various studies (Sidney, Thompson, & Rivera, 2019; Hamdan & Gunderson, 2017). While the number line is argued to align well with the measurement conceptualization of fraction division, its scarcity in the analyzed applets suggests a gap in its integration. This might be due to the contradictory facts that the number line is more abstract and cognitively demanding as compared to other visual representations, yet fraction division is usually taught to young students in Grade 5. This also indicates that educators and applet designers may still need to fully explore how the number line can be used to support different conceptualizations of fraction division. Future design of GeoGebra applets for fraction division may consider utilizing the number line to model different conceptualizations of fraction division.

Furthermore, the findings illustrate the popularity of the use of length model (i.e., fraction bar) to represent the measurement interpretation of fraction division and the use of area model (both circular and rectangular representations) to represent partition interpretation of fraction division. This may be partially attributed to the common representations that are typically associated with different conceptions of fraction. While area models such as circles and rectangles are often used to highlight the action of equal partitioning, length models such as fraction bar and the number line are often used to underline the action of iterating. Since the partition interpretation does not necessitate the cognitive action of “iterating”, there is no specific need to use a length model, as an area model is more than enough to enable the cognitive actions of “partitioning” and “unitizing”.

It is evident from the findings that the conceptualization of fraction division as measurement is more prominent than the other conceptualizations. This might be partially attributed to the promotion of measurement conception of fraction by the mathematics education research community and curriculum standards. There is no doubt that the conceptualization of fraction division as measurement can support students in developing a conceptual understanding of fraction division and the common denominator algorithm for fraction division, which might make more sense for many students than the invert-and-multiply algorithm for fraction division (Van de Walle, Karp, & Bay-Williams, 2022). However, the dominance of measurement interpretation of fraction division in the analyzed applets also raises concerns about the underrepresentation of other conceptualizations of fraction division. Working with different conceptualizations of fraction division enables students to not only develop a rich and robust understanding of fraction division but also connect fraction division with other mathematical ideas. For instance, developing an understanding of fraction division as the determination of unit rate not only allows students to see how fraction division as partition can be extended to a fraction divided by a fraction but also provides them with an opportunity to use and deepen their understanding of ratio and proportional reasoning. Therefore, the design and choice of GeoGebra applets for fraction division should provide students with opportunities to learn multiple conceptualizations of fraction division to foster a rich and robust understanding of it.

Despite GeoGebra's functionality and affordability to support the design of high cognitive fidelity applets, the results from this study show that most existing open-access GeoGebra applets for fraction division fail to leverage this potential. That is, many provide their users the opportunity to visualize the process of fraction division but not the opportunity to enact the mental actions associated with fraction division. This might be attributed to the emphasis on visualization and teacher-centered use of technology, in which students are positioned to listen and watch digital information rather than actively act on mathematical objects on the screen. Future design might consider enhancing the cognitive fidelity of the GeoGebra applets for fraction division by enabling students to enact essential mental actions associated with fraction division.

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PRE-SERVICE TEACHERS' PERCEPTIONS ON EXPLORING NUMBER THEORY CONCEPTS USING KHANMIGO: BENEFITS AND CHALLENGES

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This study reports on a part of a broader research project on enhancing preservice elementary mathematics teachers' understanding of number theory concepts in AI-integrated independent learning sessions. Exit interviews were conducted with participants to understand their perceptions of using an AI-powered guide (Khanmigo) in their mathematics learning experiences. Findings suggest that preservice teachers found AI to be beneficial in helping them understand mathematical problems better, providing personalized learning opportunities, and promoting open sharing of questions and challenges. They also faced challenges with AI such as confusing responses and AI's limitations in getting to know individual learners.

Keywords: Preservice Teacher Education, Artificial Intelligence, Technology

Research on the role of AI in mathematics teacher preparation is in its nascent stage. Emerging research has demonstrated that the integration of AI in mathematics education can provide opportunities for cultivating a deeper understanding of mathematical concepts and situating mathematics in meaningful contexts (Aleven et al., 2023; Gattupalli et al., 2023). AI-powered guides are recognized for their potential to boost students' mathematical learning and thinking skills (Hwang & Tu, 2021). Despite the potential benefits, little progress has been made in exploring ways to support teachers and learners in the effective and responsible use of AI tools (Celik et al., 2022; Hwang et al., 2020; Hwang & Tu, 2021). Moving forward, there's a critical need to delve into various aspects of AI's role in mathematics education, including creating innovative and personalized learning environments, understanding the impact of AI on learning experiences, and evaluating the effectiveness of AI through theoretical perspectives (Balacheff, 2023; Hwang et al., 2020; NCTM, 2024).

In our commitment to fostering meaningful student engagement with AI in mathematics classrooms, we recognize the pivotal role of teacher education. Future teachers must grasp the significance of AI and engage with it meaningfully and productively (Hwang et al., 2020; McGrath et al., 2023; NCTM 2024). Consequently, it is the responsibility of mathematics teacher educators to devise innovative instructional approaches into teacher education courses, to model the integration of AI into productive mathematics learning environments. In this paper, we present an initiative where an AI-powered guide, Khanmigo (https://www.khanmigo.ai), was integrated into five teaching modules in content courses for preservice elementary mathematics teachers (PEMTs). As part of a broader research project investigating the impact of teaching modules on PEMTs' understanding of basic number theory concepts, this paper is focused on findings related to this research question: How do PEMTS describe their experiences with an AI-powered guide during independent learning sessions (ILS) focusing on number theory concepts?

Methods

Interviews were used to gain an in-depth understanding (Brenner, 2012) of PEMT's perspectives on their experiences as they interacted with Khanmigo in AI-integrated content course. An ILS refers to a period within class time in which the PEMTs worked collaboratively in small groups, utilizing Khanmigo to explore number-theory concepts and solve open-ended problems. Students engaged in discussions without direct instruction from the mathematics teacher educator who played a supportive role when needed.

We selected Khanmigo to enhance mathematical discourse through personalized tutoring and interactive learning (Kshetri, 2023) during each ILS. Utilizing GPT-4, Khanmigo's chatbot simulates a tutor who facilitates conversations ad provides constructive feedback to the learner rather than providing direct answers (Ofgang, 2023).

Research Context

The study was conducted at a university in the midwestern United States during Fall 2022 in a Fundamental Mathematical Concepts for Elementary Teachers course. Participants were 26 PEMTs who engaged in five 75-minute AI-integrated ILS. The sessions were centered around the following concepts: prime and composite numbers, divisibility rules, prime factorization, greatest common denominator (GCD), and least common multiple (LCM). The instructor (third author) created a set of open-ended problems for each concept and modeled how PEMTs could use Khanmigo (used interchangeably with AI in this paper) to engage with the problems. During each session, the instructor guided and provided support to the PEMTs when they faced difficulties while utilizing AI to solve problems and/or comprehend number theory concepts. The PEMTs used Khanmigo's tutor mode to solve the problems and explore number theory concepts while also participating in discussions with their peers.

Data Collection and Analysis

Data collected for the broader study included observations and screen recordings, content-based pre and post-tests, and end-of-semester exit interviews. In this paper, we will focus on the analysis of the exit interviews conducted with three volunteer PEMTs from different small groups. A 30 to 45 minutes individual interview was conducted with each PEMT to gain an in-depth understanding of the PEMTs' perspectives on the perceived benefits and challenges of the use of AI to solve number theory problems. Interview data were analyzed in three phases. First, interviews were transcribed to identify the text segments (DiCicco-Bloom & Crabtree, 2006) in which the PEMTs' shared benefits and challenges during their use of Khanmigo. We engaged in an open-coding process to identify the patterns in the data to develop themes and categories. The first author coded all text segments, and the second author independently coded a third of randomly selected text segments. Then, these researchers discussed the disagreements in coding until they reached a consensus. The agreement rate was found to be 86%.

Findings

We classified our interpretations of PEMTs' perspectives on Khanmigo using two themes, benefits and challenges. Within the classification of benefits of AI use, we identified three benefits and two challenges as described in the sections below.

An AI Tutor Supports the Problem-solving Process

PEMTs shared that the AI tool supported their engagement with the problems by a) helping them to understand the problems by breaking the problem into steps (n = 2), b) allowing them to elicit meaning of the mathematical concepts and words used (n = 3) and c) exploring multiple solution approaches (n = 2). For instance, PEMT1 shared how Khanmigo helped her understand story problems and elicit the meaning of mathematical concepts: "I really liked about AI is that you can input story problems. And some problems can be hard to dissect, kind of what it's asking, especially when you are unfamiliar with those terms."

Two PEMTs emphasized how AI supported them to elicit and explore multiple solutions for the problem assigned in the course. For instance, PEMT2 stated:

> …to find the least common multiple of, say, like nine and six, and …sometimes it would already give you two ways to do it. But then if you asked, is there another way to find this out? Is there a simpler way? What does this mean? Then it would explain that to you more.

An AI Tutor Supports Individualized Learning

Participant describe the benefits of Khanmigo in structuring self-paced learning and attending to individual learning needs. All PEMTs highlighted the challenges a teacher faces in larger size classrooms of moving beyond direct instruction to support diverse learning needs. They emphasized how AI could foster more differentiated instruction. PEMT3 describe the how she could work at her own pace.

> In a normal classroom, you usually are given those strategies right off the bat, for me, I was able to ask for one strategy at a time from the AI. It was a slower process. And it was really nice for me to be able to learn and figure out each strategy before moving on to the next.

PEMT2 extended this idea about self-paced learning as she reflected on the increased opportunities to learn because the interactions focused on her level of knowledge and curiosity.

> In the AI class, It wasn't me teaching myself, but it was definitely more me centered in a classroom of 20-30 students. It is challenging for it to really be geared towards one student specifically. With the AI, I was able to breeze past what was simple, what came easy, what I already knew, and then spend more time on what was confusing. And what I didn't know.

An AI Tutor Responds to Student Struggles and Questions

All PEMTs indicated that they felt comfortable sharing their struggles and questions with the AI tutor. For instance, PEMT3 stated:

> We felt a little more on our own, at least I did [using AI]. But I liked that aspect of it because I have some social anxiety and it's hard for me to ask questions in front of a whole class sometimes. So being able to just ask a computer and it tells me step by step that was something that was great for me at least.

PEMT1 pointed out that the individualized interactions with AI significantly improved her comfort in asking questions: "Whereas with the AI, it's between you and the computer. I feel like you get more out of it, because you can ask questions on every problem and not feel ashamed and not be worried." This benefit supported PEMTs in engaging in positive and productive struggle with mathematics concepts

An AI Tutor Produces Confusing Responses

Although the conversational nature of AI was a benefit to PEMTs in fostering productive struggle, the PEMTs felt that the occasional lack of clarity in the responses created a challenge in using AI. Two PEMTs' stated that sometimes AI produced mathematically confusing responses. For instance, PEMT3 stated:

> …And so we finished and we got a different answer. And then, we saw that AI had gotten something very off. And so all we did was we typed in what we thought was wrong. And we did it. I'm not quite sure how AI ended up getting the right answer. So that was something *interesting*, but also a struggle almost.

PEMT3 noted how their group reacted to this discrepancy between their solutions and the AI tutor's solution. They questioned the AI tutor's solution instead of accepting it as correct, but they also felt the need to seek out the instructor's support in trying to resolve the discrepancy.

The AI Tutor Doesn't Know Students' Learning Preferences

Although all PEMTs shared how AI could support individualized learning experiences, PEMT1 and PEMT3 highlighted the challenge of interacting with an AI tutor as a non-human. Unlike a classroom teacher, the AI-tutor is not able to respond to individual students' learning preferences and building relationships with students in a mathematics classroom. For instance, PEMT3 stated:

> There's definitely the human interaction portion where you can build that relationship with your professor, they know your learning style. So, it's a lot easier for them to explain things right away. And then with AI, there's still a learning process. They don't know you. So, it's harder for them to help out, get it, get through the process of figuring out the problem.

Discussion and Conclusion

The results suggest that the PEMTs perceived both benefits and challenges for their mathematical thinking in AI-integrated ILS. PEMTs highlighted the benefits of using an AI tutor to assist in the problem-solving process. They helped students grasp the meaning of mathematical concepts and words used in the problems while encouraging students to explore multiple problem-solving strategies. PEMTs noted that AI acted as a supportive and non-judgmental resource, providing a more comfortable environment for seeking assistance compared to asking questions in a whole class setting.

The results also highlight the challenges that PEMTs experienced during engagement with the AI tutor. The benefits that they described suggest that PEMTs valued a humanistic responsiveness in their AI expert tutor that could enhance their learning and their mathematical confidence. These perceived benefits became challenges because at times the PEMTs felt the need to question the reliability of the AI's answers. They struggled to accept that they might have a better solution and sought reassurance or explanation from their instructor. They also felt challenged by the lack of humanistic responsiveness in an AI tutor who could not understand their unique learning preferences build sustained relationships with them. The PEMTs compensated for these challenges by actively seeking support from their instructors and collaborating with peers. The PEMT's perceived benefits should motivate continued exploration of the individualized content learning opportunities that AI integration can create, while the PEMT's perceived challenges highlights the need for students and teachers to understand that the

AI tutor is neither perfect or human. As NCTM (2024) position statement on AI and mathematics education, the need for high-quality pedagogy to support AI integration is an important implication of this study and an area for further research.

Optimizing the use of AI in developing PEMT's mathematical knowledge requires collaborative efforts between educators and AI developers. Such partnerships can ensure that the tools and platforms align with the needs of PEMTs and provide meaningful support in their learning. In our further work, we plan to design AI-integrated sessions that focus on other mathematical topics and investigate how they impact both PEMTs perception of learning and instructor decision-making.

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Chapter 16:
Working Groups and Colloquia

PRODUCTIVE STRUGGLE IN MATHEMATICS EDUCATION RESEARCH, TEACHING, AND LEARNING WORKING GROUP

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Facilitating productive struggle is an essential aspect of teaching that is responsive to students' thinking and develops deep and meaningful understandings of mathematics. Researchers, teacher educators, and teachers would all benefit from being able to draw on a common definition for the term "productive struggle" and a common understanding of the teaching practices needed to facilitate it. This working group will bring researchers and educators together to investigate what we know about facilitating productive struggle and draw on the expertise in the room to move our individual and collective understanding and practice forward.

Keywords: Research Methods, Instructional Activities and Practices, Preservice Teacher Education, Teacher Educators

Over the past 15 years, the term *productive struggle* has become commonplace in mathematics teaching and teacher education. Both the National Council of Teachers of Mathematics (NCTM, 2014) and the Association of Mathematics Teacher Educators (AMTE, 2017) identify providing students with opportunities to engage in productive struggle as a key component of effective mathematics teaching and learning. However, a preliminary systematic review of how researchers define the term productive struggle leaves many unanswered questions about what productive struggle actually entails (Kamlue & Van Zoest, 2024). This new working group seeks to develop a shared, consensus definition of productive struggle and a related conceptual framework that can be utilized to support research and opportunities for students to experience productive struggle in learning mathematics in the classroom.

Theoretical Background

Several seminal writings influence how we perceive and use the term productive struggle in our different contexts. Here, we highlight three seminal works. First, Hiebert and Grouws (2007) define *struggle* as follows, "students expend effort in order to make sense of mathematics, to figure something out that is not immediately apparent" (p. 387). Several researchers (e.g., DiNapoli & Miller, 2022; Warshauer, 2015) use Hiebert and Grouws' (2007) definition of struggle as their definition of productive struggle in their studies. Second, NCTM (2014) frames struggle as involving, "opportunities for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions" (p. 48). This framing continues to inform educators', teacher educators', and researchers' understandings of productive struggle (e.g., Edwards, 2018; Warshauer et al., 2021). Third, practitioners and researchers often invoke the first Standard for

Mathematical Practice: *Make sense of problems and persevere in solving them* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) when speaking of struggle.

As we seek to illustrate here, there is great variety regarding how the field has described and conceived of productive struggle (e.g., expend effort, delve deeply, perseverance). This lack of consensus and common language presents a challenge to those who research productive struggle, those who teach how to facilitate productive struggle, and those seeking to create opportunities for their students to experience productive struggle.

Structure of the Sessions

The purpose of this working group is to bring together researchers and teacher educators to examine existing and new conceptualizations of productive struggle. This work is crucial for the mathematics education community, as the current lack of consensus regarding the use of the term productive struggle presents a host of issues, including the inability to: (a) compare findings across studies and build on each other's work precisely, (b) determine when productive struggle occurs, and (c) effectively model how teachers can support productive struggle. Each session will be focused on a different aspect of productive struggle research and classroom practice.

Session 1: Developing a Consensus Understanding of Productive Struggle

Our goal for this session is to co-develop a shared definition of *productive struggle* and a theoretical underpinning of the term. We will first present how productive struggle has been defined by researchers, teacher educators, in-service teachers, and preservice teachers, and what theoretical frameworks they attended to in their contexts (Hiebert & Grouws, 2007; Jarry-Shore & Anantharajan, 2024; Warshauer, 2015). We will surface unanswered questions from working group participants and ask them to situate and relate their use and understandings of the term to existing literature in order to move our understanding of the term forward.

Session 2: Analyzing Classroom Videos

For this session, working group participants will be asked to apply the conceptualization of productive struggle developed in Session 1 to attend to and interpret instances of struggle in classroom video (Jacobs et al., 2010). Specifically, participants will view a collection of three 5-minute video-clips that display struggle of different forms. Video-clips will portray students solving problems individually and in collaborative groups so that we can assess the usability and applicability of our collective conceptualization of productive struggle to varied problem-solving contexts. The work during this session will be iterative, with participants revisiting and refining ideas between each video viewing, with the goal of drafting a more comprehensive and refined conceptualization of productive struggle.

Session 3: Facilitating Productive Struggle

In the third session we will shift to thinking about how to operationalize the co-constructed conceptual framework developed in sessions 1 and 2 for research, PK-16 classroom practice, and teacher education/professional development. We will attend to two questions. First, how can we use the framework(s) we have developed in our practice to (a) more effectively engage students in productive struggle and (b) teach educators to facilitate productive struggle? Second, how can we use the framework(s) we have developed in our research to more effectively study productive struggle and its facilitation. For both questions, we will focus on assessing the usability of the framework and the benefits and challenges of having such a framework.

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USE OF CHILDREN'S LITERATURE IN MATHEMATICS TEACHING AND LEARNING

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In this new initiative working group, our goals are to explore and share our experiences of teaching and learning mathematics using children's mathematics and build a community for future collaborations. Over three days, drawing from the mathematical exploration lens, we will engage the participants in open notice and wonder reads, focused math lens reads, and idea investigations with different children's literature, discuss the challenges and benefits of mathematizing children's literature with elementary students and elementary prospective teachers, and initiate the conversation for follow-up activities at future PME-NA conferences.

Keywords: Elementary Childhood Education, Instructional Activities and Practices

Using children's literature to teach and learn mathematics provides a context for problem-solving, creates mathematical investigation and reasoning opportunities, and supports deeper conceptual understanding (e.g., Young et al., 2018). The research has supported the benefits of using children's literature to teach mathematics to significantly increase mathematical learning, interest, and engagement in mathematical discourse. Within this body of research, the focus has been primarily on early childhood education (e.g., Elia et al., 2010; Monroe & Young, 2018; Hassinger-Das et al., 2015; Hong, 1996; Jennings et., 1992; van den Heuvel-Panhuizen et al., 2016; van den Heuvel-Panhuizen & van den Boogard, 2008). While envisioning the future of elementary mathematics education as mathematics teacher educators and researchers, it is essential to facilitate the teaching and learning of mathematics by moving beyond the boundaries of mathematics education. At the boundaries of mathematics and literacy education, our goals of this new initiative working group are to (a) share our experiences of using children's literature in mathematics teaching and learning with elementary students and elementary preservice teachers, (b) engage in mathematical explorations using children's literature, and (c) build a community of scholars and educators for future research collaboration to develop research-based resources for preservice and in-service teachers.

Theoretical Background

One of the key recommendations for mathematics instruction described in *Catalyzing Change* is that "Each and every child should develop deep mathematical understanding as confident and capable learners; understand and critique the world through mathematics; and experience the wonder, joy, and beauty of mathematics" (Huinker et al., 2020, p. 11). Using children's literature in mathematics provides opportunities to follow this recommendation. Hintz and Smith (2022) provide a framework for thinking about three types of explorations with children's literature (whether the literature has an explicit or implicit mathematical focus). The *open notice and wonder reads* are an initial exploration that focus on promoting wonder, joy, and beauty in mathematics. First, in an open notice and wonder read, teachers invite students to notice and wonder about what they are hearing and seeing in a story. This process helps students develop a common background of the story's content and begin to think about how the story could be enhanced by viewing it through a mathematical lens. Second, the *focused math lens reads* involve revisiting the notice and wonders, considering which of the students' ideas might be helpful to pursue mathematically, and then inviting students to think about the story from a mathematical lens (Hintz & Smith, 2022). Third, the *idea investigations* involve extending mathematical explorations beyond the book, helping them understand their broader world through similar mathematics. These mathematical explorations with children's literature provide opportunities for students to engage in mathematical problem-posing and open-ended problem-solving (Cai et al., 2015; Warden, 2022).

Several factors play a role in how mathematical explorations with children's literature unfold. One factor is the type of book used: mathematical-focused, general fiction, or nonfiction. A second factor is the people involved: teacher, preservice teacher, and students. A third factor is the mode of sharing the book: electronic (or multimedia) or physical. A better understanding of how mathematical explorations emerge from situations with different combinations of these factors is important for understanding mathematics instruction using children's literature.

Focus and Organization of the Working Group

This working group focuses on understanding the benefits and practices of using children's literature in mathematics teaching and learning. We organize the working group by sharing our experiences of using children's literature in mathematics teaching and learning; learning about different approaches, challenges, and benefits of mathematizing children's literature; and forming a community to build upon our learnings throughout the conference.

Plans for Participants' Active Engagement

Day 1. We first will share our plans for the three days, introduce the theoretical background around mathematical explorations with children's literature (Hintz & Smith, 2022), and provide research evidence supporting the benefits of using children's literature in mathematics teaching and learning. Second, we will ask the participants to share their interests in this working group and their experiences using children's literature in teaching and learning mathematics. Third, by drawing from the theoretical background, we will engage the participants in an open notice and wonder read of two books, one with an explicit mathematical focus (e.g., *Lion's Share* by McElligott, 2009) and one with an implicit mathematical focus (e.g., *Two of Everything* by Hong, 1993). Fourth, the participants will brainstorm ways of using and doing research around this mathematical exploration.

Day 2. We will start the second day by reviewing the first day. Then, in small groups, we will invite the participants to mathematically explore and categorize books (e.g., concept, explicit or implicit mathematical focus, grade level). Groups will select one book and problem-pose around one page to develop a problem-solving situation (Cai et al., 2015; Warden, 2022) and connect these to the types of mathematical explorations from our theoretical background (Hintz & Smith, 2022). Finally, we will share future opportunities for research and teaching in using children's literature to teach and learn mathematics within teacher education programs.

Day 3. Reflecting and building on the last two days, we will brainstorm about the future directions of this working group for follow-up activities at future PME-NA conferences. We will aim to focus on the use of children's literature in mathematics content and method courses for prospective elementary teachers. We will invite participants to share their insights and expertise around challenges prospective elementary teachers and teacher educators might encounter. We will wrap up the last day by taking a survey of participants to document their learning throughout this working group and their interests in future collaboration.

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TEACHING AND LEARNING WITH DATA INVESTIGATION: WORKING GROUP REPORT FROM 2023

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During and since the 2023 PME-NA Annual Conference the working group has carried out many ongoing activities with the goals of creating community and producing scholarship in relation to our group's interests in statistics and data science education. During the conference, issues of environmental justice/sustainability and social justice were two major common interests of participants. Though both themes are clearly linked, discussions of social justice do not always include conversations about environmental justice. We then broke into two groups based on these interests to discuss current work we might have related to the topic or ideas for future work. These groups have continued to meet and have discussions throughout the year and will continue into the 2024 conference. We additionally had a group of interested participants publish an article in Mathematics Teacher: Learning and Teaching PK-12 based on a gap in the literature identified by the group.

Keywords: Data Analysis and Statistics, Sustainability, Social Justice

The 2023 conference marks the fifth PME-NA conference that our working group has met. Each year has been more productive than the last, with growing momentum and membership over the years. The working group began at the 2019 conference where we explored a recently published framework for meaningful statistical learning environments (Ben-Zvi et al., 2018), trying to start conversations about different aspects of design. Then, in the 2020 conference, we focused on considering statistics education across boundaries with collaboration between scholars in the US and Mexico with the sessions held in Spanish and English. In 2021, we began to consider how data science education might play a role in statistics education and the mathematics curriculum. In 2022, we took seriously the growing interest and movement toward incorporating data science education into K-12 schools (Education Development Center, 2015; Lee et al., 2022) and focused on data investigations as a vehicle for addressing the broad data literacy needs of people today rather than considering statistics and data science education as fields. The working group has grown in size year after year and has increasingly taken on an explicit focus on using data investigations to explore issues of justice.

Overview of the Working Group Activities

Coming into the 2023 PME-NA conference, we had two main goals for the working groups 1) sustaining the ongoing cross-institutional collaboration to develop frameworks and resources for supporting mathematics educators in facilitating data investigations and 2) advancing our discussions on designing data investigations with an eye towards taking action to promote socially and environmentally just outcomes. We recognize context is at the core of data investigations, which are driven by a problem. We view data investigation as a vehicle to make sense of and act upon issues of social and environmental justice around the world. Based on the role of context in teaching and learning with data, we organized this working group into three themes: the context of mathematics teacher preparation, the context of cross-disciplinary work, and the context of research and collaboration. Across the three days of the working group sessions, we moved from making connections between interested participants to making explicit plans for action after the conference. In the sections that follow, we discuss in further detail the activities during each day of the conference followed by a discussion of our activities since the conference.

Day 1: Overview of Statistics and Data Science Education

Following a round of introductions, the initial day started with a reexamination of pivotal questions central to our working group discussions. Six questions were placed onto large Post-it papers and a digital Jamboard, facilitating remote participants to collaboratively brainstorm and share their insights in smaller groups. Subsequently, a whole group discussion ensued to collectively analyze the generated ideas. This conversation aimed to discern overarching themes of interest, resulting in the identification of three prominent themes: (1) the use of social and environmental justice as key contexts for data investigations, (2) the exploration of interdisciplinary collaborations within K-12 teacher education, and (3) the integration of statistics education and data science within the realm of mathematics education.

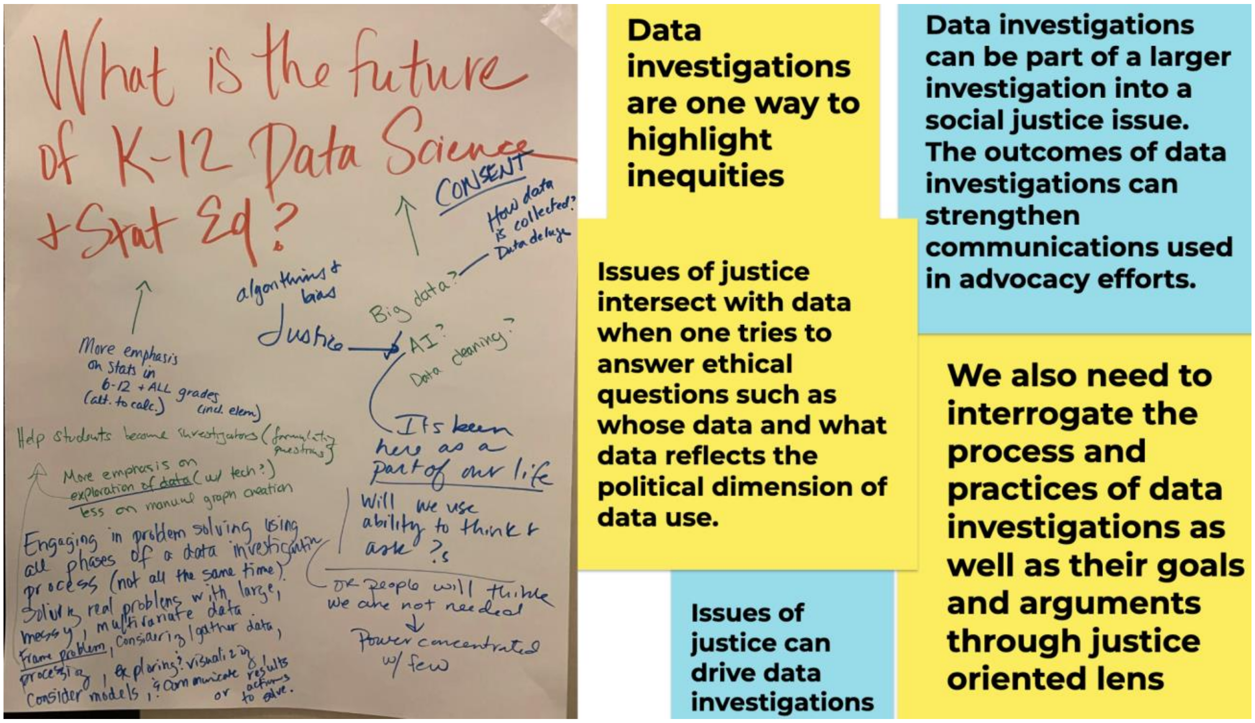

Figure 1: Sample Responses by Participants

In our conversations, we noted the increasing significance of K-12 statistics and data science education and a notable shift towards the recent development of computing and artificial intelligence (AI), leading to an anticipated larger role in K-12 education. AI integration in education, exemplified by tools like ChatGPT, prompts a need for thoughtful adaptation in teaching and evaluating processes. Anticipated challenges include the need to strike a balance between data literacy and traditional learning pathways within the K-12 mathematics curriculum, considerations of data ethics, and addressing algorithmic biases. Additionally, we acknowledged the varied interpretations of the term “data science,” which at times was perceived more as a "buzzword" associated solely with working on large datasets (Donoho, 2017). It is crucial to recognize that applying computing and programming to data investigation is a significant aspect of data science.

All participants expressed a heightened interest in investigating social and environmental justice as a promising approach to teaching and learning with data investigations. Data investigations, which can play a pivotal role in illuminating social inequities and issues of justice, can be a means to critically examine the ethical considerations inherent in the use of data science and AI through a justice-oriented lens. The intersection between data investigations and justice-oriented inquiries can amplify the significance of each other and contribute to deepening participation in civic engagement that attends to societal and environmental challenges. These types of inquiry gain particular relevance in the context of data-driven decision-making, where considerations of justice play a key role.

Day 2: Lessons from Prior Collaborations and Designing New Collaborations

The second working group session began with a report from Smucker on a publication that was initiated during the previous working group in 2022. During that working group, a group of participants wanted to pursue connecting the 5 Practices Framework (Smith & Stein, 2018) to statistical investigations. This sub-group met throughout the first half of 2023, which led to a manuscript submitted to *Mathematics Teacher: Learning and Teaching PK-12*. During the conference, the manuscript was returned with minor revision requests. The manuscript has now been published as a Front and Center article in the journal (Smucker et al., 2024). The group discussed the benefits of this collaboration and the potential for future publications coming out of this working group. Weiland then reported on possible pathways of pursuing a special issue on statistics or data science education in multiple journals in the field. Working group members could participate in different ways, such as guest editors and manuscript authors. Multiple participants indicated an interest in this, with some participants willing to take on editorial responsibilities. This work could begin in 2024-2025.

The rest of the session was spent considering the themes that had come up during the gallery walk on Day 1 to determine which area(s) the group might want to focus on further. The working group organizers had grouped the ideas from Day 1 into the following categories: cross-discipline data investigations, social justice and data science/statistics education (specifically issues of ethics and power), environmental justice in data science/statistics education, and supporting practicing and prospective K-12 teachers to prepare students to make sense of data in today's world. Based on the interests of those in attendance, the larger group was divided into two subgroups to continue discussing the next steps, with one group focused on social justice and the other on environmental justice in statistics and data science education. It was determined that these two subgroups would continue their conversations on Day 3.

Day 3: Planning for Year-Long Collaborations

First, the social justice subgroup planned for a year-long collaboration, building on the diversity of experiences working with social justice topics within the subgroup. The initial part of the discussion was too broad, leaving the group undecided on whether the focus should be on working with students or teachers and at which grade levels. The group also noted that the relevance and appropriateness of certain social issues varied across different grade levels, although some topics remained pertinent across all ages. Issues such as immigration, gender differences, diversity, and climate change were among those that captured our attention.

Subsequently, the discussion shifted towards the characteristics of datasets. Questions such as the following emerged: What variables could facilitate discussions on such social issues (e.g., school discipline)? What types of data could be collected? While existing databases could be valuable, we pondered if there were ways for students to gather relevant data related to social justice. Fernandes, who was working on a research project related to social justice, also described his work with a database of car stops and school discipline ("Statistical Investigations of Systemic Racism", 2023). More questions than answers arose during the discussion, indicating the need to extend our work beyond the conference. Consequently, we decided to continue our conversations throughout the year.

Second, the discussion of the environmental justice group began by sharing their past and ongoing projects or research studies related to environmental data and classroom connections. Subsequently, the group engaged in a discussion about integrating environmental justice and science into various content areas, emphasizing the incorporation of mathematics, science, and social studies. The primary challenge identified was providing students with opportunities to delve into authentic research in environmental justice and science. In addressing topics like climate change, for example, there are challenges with going beyond having students merely feel hopeless; it requires balancing negative with positive affective developments without overwhelming students with large-scale climate data.

Our discussion then shifted to questions regarding the means to enhance K-12 students' data literacy skills. How can working with environmental data be integrated in elementary and pre-service elementary education across disciplines? This integration could manifest through citizen science projects that are locally relevant to students' communities. What are useful tools for data exploration, analysis, and visualization? An example we discussed was the free Common Online Data Analysis Platform (CODAP; Concord Consortium, 2023) and its incorporation of NOAA data. Members of this group planned to continue these discussions beyond the scope of the conference and are committed to identifying goals for future collaboration regarding the integration of authentic environmental data into schools.

Post-Conference Working Group Activities

Social Investigation Group

The group focused on connections between data science education and social justice and has met several times since the conference to refine our focus and consider potential projects for collaboration. At our first meeting in November, we discussed some of the ideas that came up during our sessions at the conference, including how we might connect to existing projects in K-12 data science education and teacher preparation like Data Science 4 Everyone ("Data Science 4 Everyone", 2023), Introduction to Data Science materials from UCLA ("Introduction to Data Science", 2022), ProCivicStat materials, ("ProCivicStat", n.d.) and the ESTEEM project (ESTEEM, 2024), along with Fernandes and Weiland's work on creating data science modules around the contexts of traffic stops and school discipline using real data which was previously mentioned. In particular, projects that might support teachers in states with standards and pathways specific to data science (e.g. Georgia, Oregon, North Carolina, Virginia) seemed like one place where there might be space for a contribution from this group. There was also a discussion on how the group might serve different purposes for various individuals, with some looking for collaboration partners while others may be interested in learning more about what others are currently doing in their research and teaching.

At the January meeting, there was a shift towards thinking about how to encourage mathematics and social studies teachers to collaborate at the intersection of data science and social justice topics. Those who have been working to implement data investigations with prospective and practicing teachers involving issues of social justice noted the challenge in finding mathematics teachers and teacher educators who feel comfortable both facilitating a data investigation and having a productive discussion in these contexts. A cross-curricular collaboration with social studies educators might help resolve this issue. Mathematics teachers can provide expertise in facilitating data investigations, and social studies teachers can serve as experts in the contexts and navigating conversations around sensitive topics. The group built a consensus that this would be a productive idea to pursue further, but they also agreed that additional meetings were needed to determine what exactly this project might look like. Ideas for future discussion included applying for a grant to create professional development that could bring mathematics and social studies teachers together, along with using and adapting existing resources mentioned above. The importance of connecting data science and statistics to more than mathematics classrooms was highlighted in a recent *Passport to Potential: Exploring Data Literacy Worldwide* webcast that included the President of the National Council of Social Studies Education along with members of the Global Partnership for Sustainable Development Data (Data Science 4 Everyone, 2023). We plan to continue meeting on a monthly basis to refine this focus further.

Environmental Investigation Group

The group met once soon after the conference to further develop the collaboration on examining teaching and learning with environmental data. The first meeting was organized as a "data session," in which members brought data (either research data gathered from their own research projects or datasets that are relevant to environmental justice) to formulate collaborative research ideas to contribute to the common environmental justice goal. Three members showcased data from their individual projects or professional development programs (e.g., MODULE(S2), n.d.). One member showed data collected from an interview study, in which participants were asked to estimate climate-related numerical values. Another member shared student work, which led to a discussion on how the task could be modified for the next iteration. Additionally, we highlighted potential data sources, such as the USDA website featuring a Food Access Research Atlas (USDA, 2024) and the CDC website featuring a National Environmental Public Health Tracking resource (CDC, 2024) for future consideration. The aim was to build on existing research projects and collectively explore collaborative studies that the group could undertake. The group is determining the next steps for the collaboration.

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TEACHING AND LEARNING WITH DATA INVESTIGATION: SUPPORTING INTERDISCIPLINARY COLLABORATION

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In going into our sixth year as a working group, our aims are to build off of previous work of the group and look to new productive avenues of scholarship. In particular, based on ongoing conversations of people who met during the 2023 PMENA conference, we seek to find ways of forming interdisciplinary collaborations to support the teaching and learning of statistics and data science through data investigations in meaningful contexts drawing from the contextual knowledge of experts from other fields such as social studies education and science education. We hope to begin work on a grant proposal to pursue funding for ideas developed and refined during the conference to support the ongoing work of the working group. We also invite new members who are interested in exploring issues around teaching and learning with data investigations.

Keywords: statistics education, data science education, data investigation

As our working group continued the effort to identify ways to bring existing recommendations for teaching statistics and data science, such as GAISE II, and NCTM Principles and Standards (Bargagliotti et al., 2020; NCTM, 2000) to the variety of K-12 education settings, we built a consensus that one of the key considerations is facilitating and supporting interdisciplinary teaching and learning. Mathematics teachers are often bound to teach mathematical aspects of statistical thinking, which often overlook students' meaningful engagement with the context while teaching and learning statistics and data science. This issue is exacerbated by a confluence of complex factors including standardized testing, limited teacher learning opportunities, and resources. Whereas, science, social studies, and career and technical education bring abundant opportunities to explore authentic issues with real data that are situated in disciplinary contexts and closely aligned with students' civic lives and professional trajectories. Therefore, we suggest that statistics and data science education researchers need to look beyond K-12 mathematics classrooms and investigate feasible and sustainable approaches to position statistics and data science education as a critical interdisciplinary link between mathematics and other multiple subject areas. Initiating these interdisciplinary efforts and designing mechanisms for collaborations are the primary goals of this year's working group.

Theoretical Background

We draw from two key discussions in the field to inform this year's working group: the interdisciplinarity of statistics and data science and the known challenges with supporting teachers in teaching them. First, as Cobb and Moore (1997) described, statistics is a methodological discipline in service of other fields, such as medicine, science, and sociology. Data science, by building on its foundation of computing, mathematics, and statistics, offers powerful modeling tools in a variety of disciplinary contexts. Though there are multiple views on what data science specifically entails, its fluid boundaries allow us to look beyond what has been considered as computing, statistics, and mathematics when we think of teaching and learning with data investigation (e.g., digital humanities). The utility of data science is well reflected in numerous data science programs and initiatives in higher education institutions, drawing students from across colleges and majors who want to investigate their topics of interest with data.

Second, the literature also presents known challenges associated with supporting teachers in fully leveraging these emerging understandings of data investigation in the classroom. These challenges include supporting teachers in facilitating data investigation (Bargagliotti et al., 2020), their development of statistical knowledge for teaching (Groth, 2013; Lovett & Lee, 2017) and the knowledge needed to adapt the use of technology to teach with data (Lee & Hollebrands, 2008). Some scholars reported promising approaches to professional development for teachers to address these challenges (e.g., Suazo et al., 2015), but the professional development that did not work as intended often remains unreported, which obscures existing challenges in preparing and supporting teachers to teach with data investigation.

Informed by these discussions in the field, our working group plans to explore key challenges that are faced when teacher educators work to bring data investigations into mathematics classrooms and other subject areas, such as science and social studies. We aim to develop collaborative projects and funding proposals that can address some of the identified challenges based on the diverse expertise and experiences represented in the group.

Organization of Working Group Activities

The working group activities are organized in a way that the participants can briefly recap the ongoing progress in collaboration. This would allow the newly participating members to see the history of the group and also welcome them to existing collaborative efforts. Also through our collaboration, we learned that securing grant funding would substantially propel the ongoing collaborations. Based on the rich experience of grant writing by multiple members in our working group, we will progress by identifying key challenges, organizing resources and expertise, and developing a plan for proposal development.

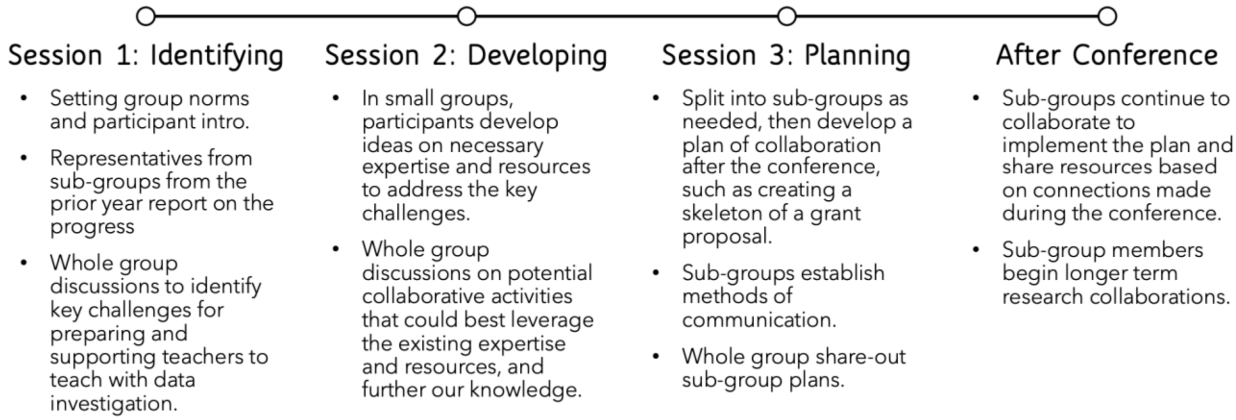

Figure 1: Session Plans and Goals

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WORKING GROUP REPORT: CONCEPTUALIZING THE ROLE OF TECHNOLOGY IN EQUITABLE MATHEMATICS CLASSROOMS (MATH TECHQUITY)

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At PME-NA 2023 in Reno, NV, the Conceptualizing the Role of Technology in Equitable Mathematics Classrooms (now referred to as TechQuity) working group met for the first time. While in Reno, we invited scholars of all backgrounds to contribute their voices to our new working group with the goal being to engage in conversation and plan research agendas centered on exploring how technology can promote equitable classrooms. The group has been able to sustain a dedicated group of members that have been meeting regularly online since the end of PME-NA 2023. In this report, we summarize the history of the TechQuity group, some insights gained through our work, and share some plans for the future of the working group.

Keywords: Technology; Equity, Inclusion, and Diversity

Our PME-NA working group focuses on the importance of using technology in equitable ways to advance our field of teaching and learning mathematics and preparing mathematics teachers. Two recent position statements from the National Council of Teachers of Mathematics (NCTM, 2023) and the Association of Mathematics Teacher Educators (AMTE, 2022) emphasize the importance of equitable integration of technology for mathematics learning. An urgent challenge in our time is the need to advance equity in mathematics education, particularly for those who have been historically marginalized. Technology is ubiquitous in our society and innovative technologies are showing promise in the teaching and learning of mathematics.

Building on research in mathematics education focused on equity and technology, our TechQuity working group draws on several theoretical frameworks including Gutiérrez's (2009) dominant and critical axis of equity. We are working on evaluating the role technology has on the dominant axis in promoting and increasing the mathematics achievement and participation of each and every student as well as the role technology has on the critical axis in promoting identity and power. In addition, as we work to bridge these two areas of interest (i.e., technology and equity), we build on transformative ways technology: offers student-centered learning experiences for exploration, discovery, collaboration and facilitating discourse (McCulloch & Lovett, 2024); advances equitable teaching practices (Suh et al., 2022); provides access to rigorous mathematics that connect to society (Byun et al., 2023); and allows for ease of differentiation and useful formative assessment to support teaching and learning (Barlow et al., 2020).

History of the Working Group

This group began through conversations at the PME-NA 2022 conference after a presentation on leveraging mathematical action technology to develop a sociopolitical disposition. During this conversation, some of the founding members of the group discussed the idea of engaging in more equity-centered work in conjunction with technology use in mathematics education. Given PME-NA's history of successful working groups, we felt starting a new working group was the best avenue to help us connect with others who have similar interests and move this work forward. Thus, at the beginning of 2023, we met to determine the goals, possible outcomes, and structure of the meeting times for the working group. Initially, our goals were (a) "[To] lay the groundwork for a research program dedicated to better understanding how technology can be used as a tool to support equity in mathematics education"; and (b) "bolster existing technology-centered frameworks with considerations of equity from various dimensions" (Witt et al., 2023). With these goals in mind, we planned to engage the participants in our sessions as shown in Table 1.

Table 1: PME-NA 2023 Plan for Engagement

	Activities	Next Steps
Session One	1. Introduction and goals 2. Our work around technology and equity 3. Questions and small group discussions about technology and equity frameworks	Come to session two prepared to share how/what other frameworks may inform your work
Session Two	1. Small group discussions about which frameworks we should consider 2. Group debrief to share current stances on how technology can support equity	Think about what subgroups would be useful to examine various perspectives on equity
Session Three	1. Form subgroups and plan 2. Subgroup share-out 3. Reflections and future work plan	Implement subgroup plan and prepare to share subgroup work next year

Overall, we felt these three sessions were a success in reaching a broad range of mathematics education researchers. We were pleased with the turnout to the group during our first session in Reno, NV (approximately 20-25 people), and we were encouraged to see many returning faces on days two and three. Below we share more about what transpired during the PME-NA 2023 conference, the progress we have made since the close of the conference, and what we plan to do next.

2023 Conference Activities

Before the Conference

Prior to the PME-NA 2023 conference, we asked the conference organizers to share an interest form with potential conference attendees so that we could gather information about who might attend. In this interest form, we asked them what goals they have in relation to the group, which sessions they plan to attend, and to share frameworks or constructs related to technology and equity in mathematics education that they were familiar with, including a place for them to share relevant files (i.e., screenshots of diagrams or frameworks, articles, etc.). While only two people completed the interest form before attending the conference, we used the information they shared as we designed the facilitation plan for each session.

Day One

We began our time together at PME-NA 2023 with the founding members briefly sharing about our work and how we came together to form the working group. We also asked participants to introduce themselves and share why they were interested in this working group. From there, we transitioned into six groups to do a card sort activity examining 17 technology-centered and nine equity-centered frameworks that are used in mathematics education research. The frameworks were all mixed together and participants were not informed from which category each framework originated. We asked the participants to read each framework card and consider how the framework might support mathematics teacher educators (MTEs) and/or prospective mathematics teachers (PMTs). In groups, participants were asked to organize the frameworks into categories of their choice and provide a title for each grouping explaining their sorting criteria. During this time, participants worked together to find connections between and common themes across frameworks to identify and describe their groupings. Figure 1 shows two groups in action. At the end of the card sort activity, we asked each group to share their process for sorting the frameworks, their final categories, what they noticed, and how they felt this could help us articulate where we are within mathematics education in relation to these ideas.

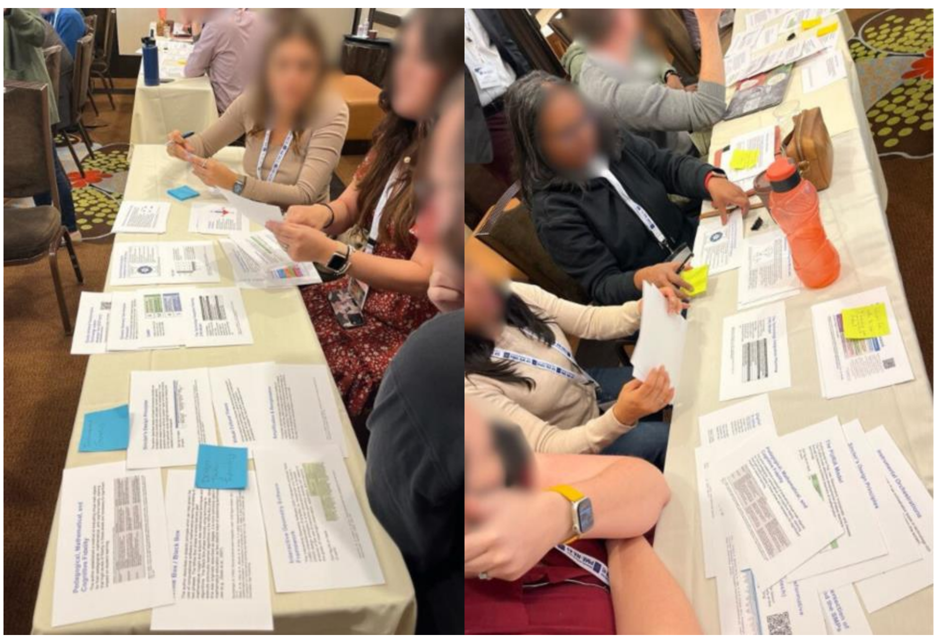

Figure 1: Participants working in groups on framework card sort

Most groups seemed to initially separate the frameworks into two broad categories of equity and technology. This result was not surprising given our working group's focus on finding ways these areas of work may intersect. In other words, it seemed from the card sort activity that the participants, much like us, struggled to make connections across the two sets of frameworks.

And, within the technology category, some groups created subcategories similar to the categories McCulloch et al. (2021) identified when examining technology frameworks MTEs use in preparing secondary PMTs. For example, some groups included a subcategory related to the McCulloch et al.'s (2021) design and evaluation of technology tools and tasks with groupings titled "evaluation," "design tasks & technology," and "task design and evaluation".

While the results of the sorting activity described above were what one might expect, other categories emerged as conversations continued within groups about how we might bridge technology-centered and equity-centered frameworks. These discussions included considering if the intersection of these two ideas should be about the teacher's use of technology in equitable ways, the teacher and student interactions with technology, or how the students are perceiving the use of technology. One group furthered the discussion by considering if we needed to distinguish between macro and micro theories of equitable technology use. Still, some other groups had collections of frameworks labeled "don't know yet" or "IDK", indicating there are more conversations needed to work out a description of this category or determine where these frameworks fit within other existing groupings. Most notably, however, we found that there seemed to be a collection of categories across the groups that might all fall under the same theme of "pedagogy". For example, four groups had categories named "teacher and teaching focus", "pedagogy", and "teaching practices" (Figure 2 shows three of these groupings). It was interesting to see that although the ways that the groups sorted the cards were not identical, there were some common ways in which the participants were thinking about these frameworks as they engaged in conversation. We were encouraged when we noticed that within some of these groupings, both technology-centered and equity-centered frameworks appeared, as this captured the essence of an important goal and theme of the group: finding intersections between teaching mathematics with an equity focus and teaching mathematics with technology.

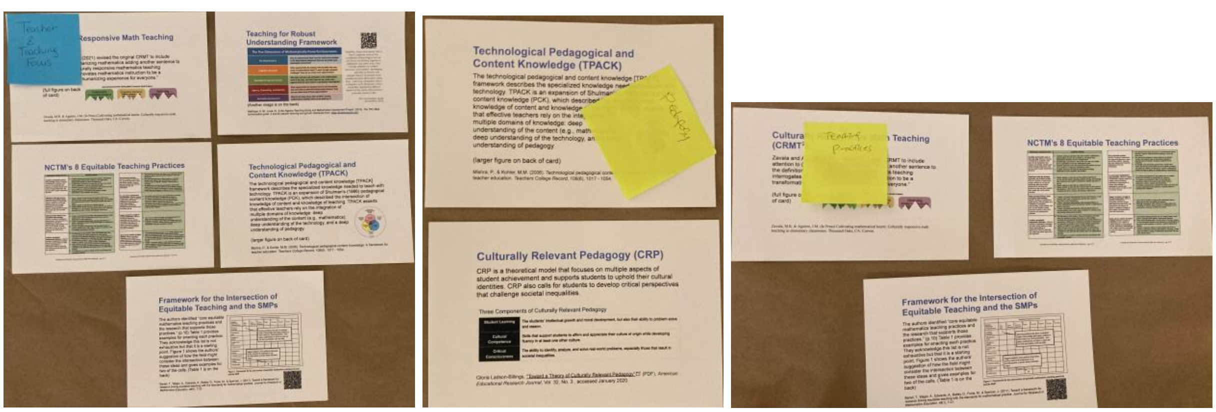

Figure 2: Pedagogy-related categories

We ended our first session asking participants to complete the interest survey that we had distributed prior to the conference so we could better gauge what aspects of technology use and equity participants were familiar with and collect their contact information. We also asked them to complete some homework by reading through position statements related to technology and equity from two national organizations in mathematics education, NCTM (2023) and AMTE

(2023). We asked that they come to the second session prepared to share what resonated with them from these statements. Finally, we asked all participants to share their reflections from the first session via a Padlet. The Padlet contained the following prompts: “What are you hoping will come out of this working group?” and “Do you have any particularly illustrative examples of how you or how you’ve seen technology used to support equity?”.

Day one team debrief. Shortly after the first session, the founding members of the working group met to debrief and adjust our plans for the next working session. Here, we briefly discussed how the day one session went and what the participants shared in the interest survey and Padlet. We used this information to guide the small group discussions for day two.

At this debrief, we reviewed the 16 new responses to the interest survey. Across both the technology-centered frameworks and equity-centered frameworks questions, there was a lot of variety in what the participants shared. For example, three participants indicated that they did not know any frameworks or constructs related to teaching and learning mathematics with technology, while at least one participant from the survey indicated that they were familiar with the following technology-related frameworks and constructs: constructionism, microworlds, situated abstractions, instrumental genesis, instrumental orchestration, semiotic mediation, theory of semiotic register, TPACK, SAMR model, embodied cognition, and new materialism. From this list, two important things stood out to us: 1) some of the frameworks and constructs are technology-specific (e.g., TPACK, SAMR model), while others are not technology-specific (e.g., constructionism, semiotic mediation), and 2) some of the frameworks and constructs they shared are more general, while others are more specific in that they target particular aspects of technology use in mathematics teaching and learning. We found these ideas particularly interesting as they indicated there is much variability in the way the participants are thinking about the use of technology in mathematics education and that there may be multiple areas of interest that emerge from the group depending on one’s view of technology use in mathematics teaching and learning. In terms of equity-related frameworks and constructs in mathematics teaching and learning, half of the respondents indicated that they did not know of any, while others included frameworks such as critical theory, teaching mathematics for social justice, critical race pedagogy, equity pedagogy, black critical theory, culturally responsive teaching, funds of knowledge, Gutiérrez’s four dimensions of equity, rehumanizing mathematics, Watson’s care framework, and neurodiversity. Finally, our analysis of the survey responses about the participants’ goals for attending the working group fell across three main ideas: 1) integrating equity focused goals and impacts into research on teaching and learning with technology, 2) connecting existing technology- and equity-related frameworks on mathematics learning and learning, and 3) looking for future research collaboration on issues related to equity and technology use in mathematics teaching and learning.

We also took a few minutes to review the responses to the Padlet reflection questions. From the six responses to the question: “What are you hoping will come out of this working group?”, two stated they would like to see a framework bridging equity and technology in math education that can be used for research and teacher preparation, two indicated they want to broaden their knowledge in understanding of technology and equity, one expressed interest in seeing a balance between technology and equity in instruction, and one stated they wanted to find research partners to explore topics of equity and technology use. The second Padlet question about providing examples of technology used to support equity only elicited two responses, both of which were general descriptions. Yet, we agreed that the ideas shared within these responses would be a nice addition to the day two discussions, and hopefully by asking the question within the session, we might learn more about how others are using technology in equitable ways.

Day Two

At the start of our second working group session, we asked participants to share what resonated with them from the NCTM (2023) and AMTE (2022) position statements using Jamboard slides for each statement or declaration. We intentionally built in time to allow participants to review the statements if needed and discuss in small groups what they noticed and wondered about. This led to a whole group discussion about each statement or declaration where we asked participants to share to what extent their work aligned with the statement and any other ideas that stood out from the Jamboard. During this time, participants also pointed out that one important idea not captured in these statements is guidance for how mathematics teachers or teacher educators might design tasks that promote these ideas. Conversations like these about possible ways to equip educators led us into a nice transition to share what the founding team learned in their debrief of the day one activities.

We used participants' responses about their reasons for joining the working group from verbal communications on day one and from the interest survey to generate a word cloud to help us visualize overall themes in why we as a group came together (Figure 3). From this word cloud, it was evident that participants aimed to learn about, collaborate on, bridge (i.e., make connections across technology and equity), design around, and conduct research to better understand how we can support equitable technology use in mathematics teaching and learning.

Figure 3: Word cloud illustrating participants' reasons for joining the working group

Then, building from the responses to the second question from the day one Padlet (i.e., provide examples of technology used to support equity), we felt it was important to allow time for participants to share more about their own work and interests. Following these conversations, we then brainstormed possible subgroups that would be of interest to participants. We left day two with the charge of identifying which subgroup you might want to be part of and any relevant research questions or ideas you may have.

Day two team debrief. After the day two working group session, the lead team met to debrief and determine how to structure the conversation for day three. We were encouraged by the amount of participants that returned on day two from day one and how willing everyone was to share about their own work and ideas. We agreed that there are many avenues that we can pursue together to move this work forward and that the conversation on day three needed to center around determining subgroups and possible products that may result from our work.

Day Three

On day three, we began by revisiting the list of possible subgroups identified on day two. We then engaged in conversations about what we are excited about exploring to help us narrow down where to start as many of us had numerous interests from the original subgroup list. Our discussions led to the creation of two subgroups, the Frameworks subgroup and the Task Design and Implementation subgroup. We then spent the remainder of the session determining potential products from each subgroup as well as establishing how we would remain in contact and continue working between conferences. We ended this last working group session with a plan for each group to meet within the next month and establish a regular meeting schedule. We also agreed to maintain a working group website where participants could find the most up to date information about the working group.

After the Conference

In the three months following the close of PME-NA 2023, the subgroups met separately multiple times, for a total of nine times across both groups. A few organizers led these sessions and recorded ideas in a running meeting notes document. At each first post-conference meeting, each subgroup initiated a process of having participants share literature through a Google form that then populated a folder with the relevant literature. This collection of literature helped inform how each group began to conceptualize their next steps. Additionally, one of the organizers, who is a member of both subgroups, has been maintaining our website (https://bit.ly/TechQuity) and distributing email updates to each group about the other group's work and progress.

Subgroup Activities and Artifacts as of February 2024

Frameworks subgroup. The Frameworks subgroup aims to examine a variety of frameworks on equity and on technology and develop a framework on equitable technology integration for mathematics education. At the first meeting, the group created a Google form to enrich the pool of frameworks that were used in the card sort. Through this form, we aim to gather more equity frameworks, technology frameworks, and search for any frameworks that bridge the two. At the same time, we initiated discussions on what a framework on equitable technology integration would focus on, what we refer to as the TechQuity Framework. At our next meeting, we began to conceptualize the TechQuity Framework. Figure 4 illustrates these initial conversations.

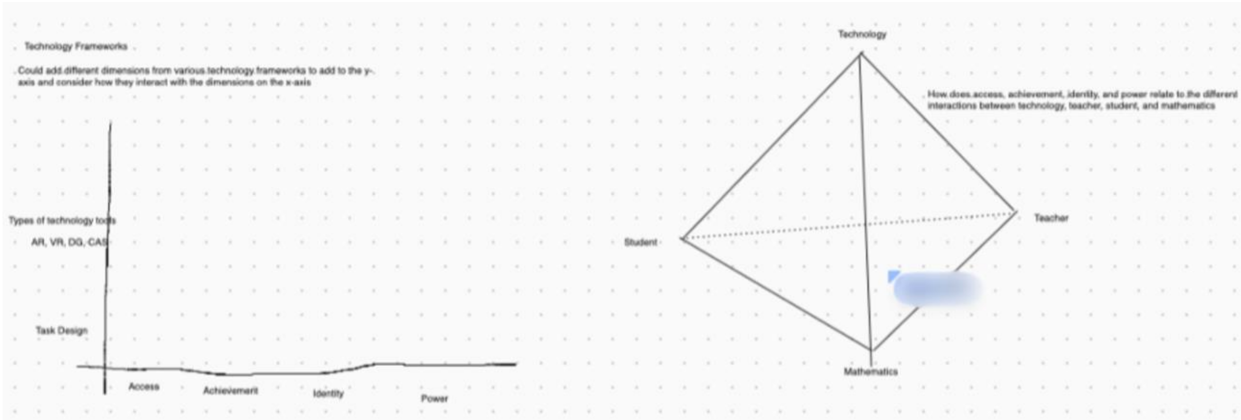

Figure 4: Diagrams of our conceptualizations of the TechQuity Framework

The diagram on the left shows our initial attempt to visualize the intersection of technology and equity as a graph. On the y-axis, we placed task design principles and the types of technology tools one might use while the x-axis focused on Gutiérrez's (2009) four dimensions of framing equity, namely Access, Achievement, Identity, and Power. Based on this diagram, we asked questions such as "How can technology help build student identity? How can it support the student voice? How can technology support access to resources and mathematics?" We also discussed what we mean by "access": access to high quality tasks with technology, a tool's accessibility, or access to diverse populations of learners.

These conversations led to the creation of the diagram on the right. In this conceptualization as a didactic tetrahedron (Hollebrands, 2016), the framework would model interactions between the students, the teacher, the technology, and the mathematics. Our current endeavor is to examine how each of these interactions may be framed and described in terms of Gutiérrez's (2009) four dimensions. We plan to expand upon these ideas between now and the next conference; our goal is to share the first draft of the framework with the 2024 participants.

Task design and implementation subgroup. The Task Design and Implementation subgroup began their work by also using a Google form to gather documents. Specifically, participants used this form to upload tasks that they felt were either (a) centered technology but had potential to be reframed from an equity lens, or (b) centered on some component of equitable teaching and had potential for being productively used with technology. As this subgroup met to discuss how technology can enhance mathematics curriculum and bring equity to the forefront, we began to highlight the many ways MTEs were engaged in integrating Artificial Intelligence (AI) into their curricular tasks and lesson development. Thus, it quickly became apparent after a few meetings, that many participants in this group had a budding interest in exploring the role of AI in mathematics education. As such, the subgroup decided to move forward by shifting our focus to the role of AI for creating equitable mathematics tasks. Subsequent conversations have also considered the role of AI for building equitable mathematics classrooms more broadly, including the ways AI can be used for assessment and as tools for instruction. Our discussions led to the realization that we need funding to bring various stakeholders together. We have since been in communication with NSF program officers about where our ideas might best fit.

Between now and the 2024 conference, we plan to continue these conversations. At the upcoming conference, we will share what we have learned and determine how to proceed.

Next Steps

Expanding on our prior efforts, the working group sessions in 2024 will delve into the following five agendas. First, we will focus on the development of the TechQuity framework, a framework that bridges theories in technology and equity and describes the relationships between mathematics, technology, students, and teachers (Figure 4). Second, we will delve deeper into an examination of the role of AI in promoting equitable mathematics instruction and the ways that various stakeholders might contribute to this. Third, we will continue our conversations around task design by exploring existing literature on mathematical tasks using technology for students in grades K-12 and seeking new mathematical tasks that are designed to enhance equity in teaching and learning mathematics. Fourth, we will grow our dissemination and networking efforts by expanding our website to serve as a hub for collecting resources for the equitable use of technology in mathematics for a larger audience and identifying ways to broaden participation in our professional community (e.g., newsletter). Lastly, we will continue to seek grant opportunities for technology use in mathematics education, particularly in the realm of artificial intelligence, such as applying for an AI workshop conference grant.

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INTEGRATING MATHEMATICS AND LITERACY WORKING GROUP: USING LITERACY AND TRANSLANGUAGING TO SUPPORT EMERGENT BILINGUAL STUDENTS

INTEGRANDO MATEMÁTICAS Y ALFABETIZACIÓN: UTILIZANDO LA ALFABETIZACIÓN Y TRANSLENGUAJE PARA MEJORAR EL APRENDIZAJE DE ESTUDIANTES BILINGÜES

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This working group focuses on integrating mathematics with translanguaging literacy to support emergent bilingual students and strives to gain a better understanding of how translanguaging practices and critical literacy can support the mathematical learning of emergent bilingual students. Participants will be encouraged to share current work supporting emergent bilingual students, network with other mathematics education researchers interested in integrating mathematics and literacy and establish new collaborations and research partnerships.

Keywords: Elementary School Education; Equity, Inclusion, and Diversity; Culturally Relevant Pedagogy; Curriculum

With the recent calls to consider how mathematics and literacy can coalesce to provide connected learning in both domains (NCTE & NCTM, 2024), this working group would be a new initiative taking up that call to consider how mathematics and literacy can be integrated, and in particular to support teaching and learning for emergent bilingual (EB) students. In the initiation of this group, we hope to be able to provide space and networking for scholars interested in connecting mathematics and literacy, drawing on translanguaging literacy and translanguaging practices. Our overarching aim is to explore opportunities for collaboration and exchange of ideas, to ultimately support the mathematical learning of EB students.

Theoretical Background

Decades of research have been dedicated to understanding mathematics teaching and learning of EB students. Many studies emphasize the importance of maintaining robust mathematical learning opportunities alongside language development and suggest that engaging in rich tasks can actually enhance language proficiency (de Araujo et al., 2018; Poza, 2019). Yet, for EB students in dual-language education (DLE) programs the learning opportunities often depend on what language mathematics is taught and assessed in (Author, 2023), given that DLE programs often lack clear directives as both federal and state levels regarding the distribution of instructional time between language and content areas (Boyle et al., 2015). As these DLE programs continue to expand, understanding how the separation of language shapes EB students' mathematical learning becomes crucial. Maldonado Rodriguez & Krause (2020) caution against strict separation of language as a means to improve language proficiency, and instead advocate for embracing translanguaging as an "empowering and critical act" that encourages EB students to convey their mathematical thinking utilizing both languages (p. 21).

Other researchers have additionally advocated for leveraging EB students' cultural backgrounds as valuable resources (Moschkovich, 2022 de Araujo et al., 2018). In his framework for teaching mathematics for social justice, Gutstein (2006) pushes us to create learning opportunities for students to "[read] and [write] the world with mathematics" (p. 23). Within the landscape of literacy, Critical Literacy scholars suggest that "any issues that capture learners' interest, based on their experiences…can and should be used as a text to build a curriculum that has significance in their lives" (Vasquez et al., 2019, p. 301). At the core of critical literacy lies the importance of centering students' lived experiences to build on meaningful texts that address power, inequality and social justice, emphasizing the students' role as agents of change. Critical literacy fosters personal connections where students can interrogate the status quo, encouraging them to look beyond themselves. This is critical to EB students as they gain voice and make meaningful personal connections to mathematical texts.

Structure of Workshop

Expanding on these theoretical foundations across research in mathematics, translanguaging, and critical literacy to support EB students, this working group will focus on sharing current research across these topics to cultivate further development in this burgeoning area of study. In the following sections, we outline our approach to engage participants throughout the three days of the working group sessions.

Day 1: We will focus on introductions and providing opportunities for participants to share their interests in the group, their current/ongoing research, and what they would like to be able to accomplish throughout the three days. Participants will be asked to share research experience (or interest) in supporting emergent bilingual students' mathematical learning. Participants will be encouraged to begin identifying intersections for possible collaborations. current work integrating mathematics with translanguaging literacy. Contributing researchers will each help guide discussion at different groups/tables and document common themes to support establishing smaller working groups for the remaining two days.

Day 2: Contributing researchers (Author1, Author2, and Author4) will lead a short presentation (20 minutes) on rationale for the new working group. Presentation will focus on theoretical underpinnings of research and methodologies that have focused on supporting translanguaging practices of EB students in both mathematical and literacy contexts. Participants will then establish smaller groups interested in either collaborating on a manuscript or proposing a new project. Groups will begin developing ideas for their proposed research projects and outlining their proposed manuscripts.

Day 3: Participants will continue developing ideas for proposed projects and manuscripts. Groups will also develop a working plan to move continue moving forward their projects or manuscripts after the conference. We will also design and schedule follow-up activities that will include setting optional virtual monthly meetings for all participants to support their ongoing work.

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Third Report: Working Group on Mathematics Curriculum Recommendations for Elementary Teacher Preparation

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This report summarizes the collaborative work and discussions from the third meeting of the working group on Mathematics Curriculum Recommendations for Elementary Teacher Preparation at PME-NA 45. Specifically, we share details from our meetings at the PME-NA conference in Reno, NV, as well as updates about all working group endeavors after the conference. We also share specific accomplishments and updates from each of the eight research subgroups of the working group, and our overall plans for the future of the working group. Across this report, we emphasize how we will continue to welcome and actively involve new members to our working group across North America.

Keywords: Preservice Teacher Education, Teacher Educators, Mathematical Knowledge for Teaching, Elementary School Education

The Mathematics Curriculum Recommendations for Elementary Teacher Preparation (MCRETP) working group entered Phase III of its long-term plan during the PME-NA 45 conference in Reno, NV in 2023. This phase centers on the research subgroups of MCRETP, specifically designing and supporting each other in conducting small-scale research studies that could address questions relevant to producing a research-based set of recommendations for elementary teacher preparation programs. The organizers of the working group created structures during PME-NA 45 for the MCRETP research subgroups to meet, brainstorm, collaborate, and share information about these small-scale research studies. Furthermore, the organizers welcomed several new members to the MCRETP working group at PME-NA 45, and meaningfully involved them in the work of the research subgroups.

In this report, we will summarize the outcome of our meetings at PME-NA 45 in Reno, NV. Next, we will describe post-conference activities that have helped our MCRETP working group continue to make progress. Then, we will review the eight research subgroups and describe each subgroup's work and accomplishments at PME-NA 45 and beyond. We will end this report with our revised long-term plan for the working group and discuss what questions we hope to address in the future.

We continue to emphasize that the MCRETP working group welcomes new participants to join at any time. We are especially interested in new members from North American countries outside of the United States, in an effort to better represent the diverse nationalities and perspectives of the whole PME-NA community. New members of our working group can expect to be involved immediately, either in being supported in conducting their own original research in a subgroup(s), joining an existing research team in a subgroup(s), and/or by being a non-researcher in a subgroup(s) and finding ways to support the endeavors of the group. We encourage anyone interested in joining the MCRETP working group to reach out to the working group organizers and/or research subgroup leader(s) to learn how they can get involved.

Our Meeting at PME-NA 45

Day 1

On Day 1 of the PME-NA 45 conference, MCRETP organizers aimed to orient everyone to the working group and to the research subgroups of the working group. We welcomed several new members to the working group, so we began Day 1 by restating our overarching goals, reviewing what we've accomplished so far, and previewing our long-term plan.

We explained that the overarching goal of the working group is to develop research-based, specific, and usable recommendations for the curriculum of elementary mathematics teacher preparation. We reviewed the past work of the MCRETP working group, which included understanding the concerns about elementary teacher preparation (Phase I in 2021) and designing research studies that could inform curricular recommendations (Phase II in 2022). Then, we previewed the present and future work of the working group, which included conducting these research studies and disseminating and synthesizing the findings (Phase III in 2023-2024), drafting research-based curricular recommendations (Phase IV in 2025), and drafting and submitting a grant proposal for a larger-scale, multi-site research study on the curricular recommendations (Phase V in 2026).

We also introduced and explained each research subgroup of the MCRETP working group. These subgroups included (in no particular order): (1) Literature Review and Critique of Current Recommendations, (2) Survey of Current Program Structures and Local Requirements, (3) Preparation Program Alumni Survey and Interviews, (4) Selection of Topics in Content Courses, (5) Pedagogical Practices, Tasks, and Thinking-Oriented Approaches in Content Courses, (6) Pedagogical Practices in Methods Courses, (7) Integration of Content and Methods Courses, and (8) Field Experiences and Clinical Work. Each subgroup leader(s) described their subgroup and its goals, research, and frameworks that guide their vision, and any progress that has been made toward achieving these goals. These descriptions helped reorient working group members to these eight areas of research, as well as introduce new members of the working group to potential subgroups they could join. Details about these research subgroups, their goals, and their up-to-date progress can be found in the Updates About Our Research Subgroups section below.

Day 1 ended with time for new members of the working group to visit with different subgroups and decide on the research subgroups they would like to join. Also, MCRETP organizers used this time to collect updated contact information from all working group members, as well as to preview the goals for Days 2 and 3. In all, Day 1 was a productive meeting because it reconnected members of the working group to the MCRETP goals and progress, and also meaningfully involved new members of the working group with research subgroup(s) of their interest.

Days 2 and 3

On Day 2 of the PME-NA 45 conference, each research subgroup convened and engaged in discussion. Subgroup members were prompted to share issues related to the mission of the subgroup that were salient in their current institutional context. Subgroup members were also invited to share details about a small-scale study they were planning to conduct or had already started conducting. The main purpose of these subgroup convenings was for subgroup members to gather feedback about these in-progress, small-scale research studies. Moreover, as more subgroup members shared about their in-progress research studies, subgroups could start to hypothesize about how findings from these studies might complement one another and to work together to test some hypotheses important to the subgroups' research agendas. Near the end of Day 2, about half of the subgroups shared summaries of their conversations with the entire working group and sought feedback. Members of the working group asked questions of each presenting subgroup and provided suggestions to support the development or progress of the study presented.

On Day 3 of the PME-NA 45 conference, subgroups continued to convene and engage in discussions, similarly to Day 2. Subgroups that had not yet shared about their conversations with the entire working group were given time to do so and collect more feedback. Each subgroup was also encouraged to start to make plans about how to keep their subgroup active and connected post-conference. The goal of the MCRETP organizers was for each member of the working group to leave PME-NA 45 with concrete steps for conducting research (or collaborating in such research) that will support our eventual curricular recommendations. Subgroup leaders used this time to start planning follow-up meetings and check-ins with their subgroup members. Days 2 and 3 proved to be valuable meetings because not only did all working group members have an opportunity to share their voices and visions, each research subgroup was able to make practical plans to stay connected with their collaborators after the conference. Next, we describe some of our post-conference activities that have helped our MCRETP working group continue to make progress after PME-NA 45.

Post-Conference Activities

Between November 2023 and February 2024, we have done the following:

1. Each research subgroup convened (some several times), via Zoom, to provide updates about research endeavors and to continue to get support and feedback from each other.
2. The leader/co-leaders of each research subgroup contributed information about their subgroup's progress to inform this report.
3. The MCRETP working group organizers met to discuss the overall progress of research subgroups. Using these updates, this team met to plan how to make PME-NA 46 as productive as possible.

Next, we review the eight research subgroups and describe each subgroup's work and accomplishments to date. See the author list of this report for contact information for each group leader/co-leader.

Updates About Our Research Subgroups

Literature Review and Critique of Current Recommendations

Group co-leaders: Nicole Wessman-Enzinger, Dana Olanoff, Kim Johnson, Jennifer Tobias, and Neet Priya Bajwa

This subgroup has two main aims. First, we are compiling and analyzing existing recommendations for the mathematical preparation of teachers. Using established criteria (e.g., the number of mathematical content/methods courses, specific content/methods recommendations, course delivery recommendations), this group is analyzing recommendations from groups including the Conference Board of Mathematics Sciences (CBMS), the Association of Mathematics Teacher Educators (AMTE), the National Council of Teachers of Mathematics (NCTM), and the National Council on Teacher Quality (NCTQ). We plan to articulate the ways that these recommendations align (and do not align) with each other. Second, we will use this analysis and synthesis to create a literature review of work related to these recommendations. We know from research (e.g., Garner et al., 2023; Masingila & Olanoff, 2022) that the majority of teacher preparation programs are not meeting the current recommendations. Our hope is that the work from this subgroup will help us to create a set of practical recommendations related to the existing work that programs would more likely be able to implement.

Survey of Current Program Structures and Local Requirements

Group co-leaders: Tuyin An and Dan Clark

The purpose of this research subgroup is to understand how elementary teacher preparation programs are structured across the nation, focusing on models of mathematical content and pedagogy integration, as well as how programs address social-justice-related topics. Since we presented our project idea to explore the structure of elementary teacher preparation programs at the 2023 PME-NA working group meeting, three colleagues have joined our sub-group. We shared similar interests in the policy, design, and structure of teacher preparation programs. We established our goal as a cross-institutional team: applying for an NSF IUSE grant to fund the development of our project. Our working group worked diligently throughout Fall and Winter 2023 and early Spring 2024. We set up weekly and bi-weekly working objectives and supported each other in conquering these challenging tasks: project design, application material preparation, external evaluator collaboration, and cross-institutional budgeting. We successfully submitted our grant proposal in January 2024. The title of our project is "Fostering Cross-Institutional Communication and Partnerships in Mathematics Teacher Preparation Programs." The total request amount is $396,828 over two project years.

The objectives of the proposed project are two-fold. First, we seek to understand the various programmatic structures that preservice elementary teachers (PT) preparation programs across the nation utilize in the mathematical preparation of their teacher candidates, as well as their perceived needs. Second, we will build an online, cross-institutional platform to foster communication and partnerships among PTs preparation programs regarding the mathematical preparation of their teacher candidates. Through the interactive platform, PT preparation programs would be able to easily find other programs (like or unlike themselves) in order to share resources and collectively optimize their courses and program structures. Having this catalog of resources available in one place would facilitate communication across programs of all types, as well as lessen the financial burden of programs looking to improve themselves. In addition, as more and more participants are involved, the online platform will regularly publish newsletters to keep the community informed. Faculty from all over the country can share their program changes, course redesigns, collaborative projects, related publication news, and other resources through the newsletters. This will all lead to PTs entering the workforce with stronger mathematical preparation.

Preparation Program Alumni Survey and Interviews

Group co-leaders: Dana Olanoff and Nicole Wessman-Enzinger

One of the best ways to determine what should be included in the mathematical preparation of teachers is to ask teachers which aspects of their preparation they find useful in their teaching practice. This subgroup has created a survey which we intend to distribute to recent alumni (within 5 years) of our teacher preparation programs. The survey asks respondents how prepared they felt to teach mathematics as well as to identify aspects of their teacher preparation programs that they felt were impactful to their current practice. We also ask for suggestions for aspects of teacher preparation programs that were missing that respondents wish had been included in their teacher preparation. We plan to distribute the surveys in the Spring of 2024 and to analyze the data in the Summer and Fall of 2024 in the hope of sharing preliminary data analysis with the whole Working Group at PME-NA 46.

Selection of Topics in Content Courses

Group co-leaders: Joseph DiNapoli, Valerie Long, and Jennifer Tobias

This research area investigates the affordances and constraints related to the instructional time spent (or not spent) on specific mathematics topics included in content courses for elementary PTs. Our subgroup is largely motivated by a recent longitudinal research study (Corven et al., 2022) that shows that survey courses, or courses that devote small amounts of high-quality instructional time to many mathematical topics, may have little or no impact on a future teacher's retention or use of knowledge related to those topics. Conversely, this study showed that teachers could more easily and effectively access knowledge related to mathematical topics for which ample high-quality instructional time was spent, years earlier, in their content coursework. This suggests that challenging decisions must be made about which topics to teach in content courses, and which to omit.

At PME-NA 45 in Reno, NV, this subgroup met to continue to support each other in designing and conducting original research to test our hypotheses about instructional time in content courses. We made ample progress during these meetings, including designing studies that we could conduct at our home institutions. For example, one study that is currently underway seeks to investigate the relationship between instructional time and the development of mathematical practices like perseverance in problem-solving: do elementary PTs learn to persevere more with challenging tasks about topics for which more instructional time was spent? Another study that is currently in process concerns the relationship between teaching fewer topics and students' summative assessment outcomes. After PME-NA 45, this subgroup has continued to convene virtually to offer support to those who are interested in designing and conducting their own original research relative to our subgroup's goals. Just recently, an idea for a new study to survey North American programs about their selection process for topics in content courses was born in one of these virtual meetings. In this subgroup, several studies are underway and some are even under review as potential presentations for PME-NA 46 about in-process research.

Pedagogical Practices, Tasks, and Thinking-Oriented Approaches in Content Courses

Group co-leaders: Julien Corven, Jennifer Tobias, and Neet Priya Bajwa

The purpose of this research subgroup is to consider recommendations for how to teach mathematics content within elementary content courses. Although recommendations for what to teach are important, without parallel recommendations for how to teach the content, the recommendations may end up ineffective. Li and Howe (2021) assert that it is critical for teacher preparation programs to focus not just on PTs' knowledge of the mathematical content (a knowledge-oriented approach), but more so on how to apply this knowledge to the craft of teaching by, for example, using such knowledge to understand and base instruction on students' thinking (a thinking-oriented approach). However, Li and Howe (2021) acknowledge that this assertion is their opinion, and they echo a call from Hiebert and Berk (2020) for more research that would support a professional knowledge base for MTEs on thinking-oriented approaches.

Subgroup members are currently planning specific research projects in the content courses they teach to assess the effectiveness of particular pedagogical approaches and tasks. For example, one study underway focuses on how PTs analyze student thinking on student solutions to decimal division tasks through the use of base-10 blocks. Another study evaluates tasks and the effect their modifications have on PTs' understanding of number concepts and operations.

We see this research subgroup's work as supporting our future recommendations by giving examples of how to design, modify, and/or implement tasks to support PTs' development of the mathematics they are to teach in their future careers. An underlying aspect of this is pedagogical practice recommendations for MTEs who can then modify and iterate on these tasks and activities (Hiebert & Morris, 2009). We also anticipate results from our studies will help us develop a repository that includes rationales for instructional decisions and information about how PTs may respond to the activities as a way to support novice MTEs (e.g., Superfine & Pitvorec, 2021; Suppa et al., 2020).

Pedagogical Practices in Methods Courses

Group co-leaders: Valerie Long, Richard Velasco, and Hartono Tjoe

This subgroup has two research aims: (1) to investigate the learning experiences that mathematics teacher educators utilize to help elementary PTs develop a rich understanding of equitable pedagogical practices in mathematics; and (2) to examine elementary PTs' implementations of these practices in their field placements as far as advancing their own students' mathematical thinking and learning. These learning experiences might include integrating culturally sustaining pedagogies, selecting and analyzing mathematics tasks, writing lesson plans, conducting teaching experiments, analyzing videos, examining misconceptions for a topic, etc.

At PME-NA 45 in Reno, NV, subgroup members discussed next steps for the current research project pertaining to PTs' analysis of tasks. These next steps are writing a practicum-based article about how researchers implemented the semester-long project in elementary math methods classes, analyzing data from PTs' evaluations using a class rubric of a low-level place value task, and comparing these results with those from PTs' evaluations using a class rubric for a high-level place value task. Additionally, members discussed applying for seed grants for research projects, starting a literature review of access and equity practices within elementary mathematics methods courses, creating pedagogical tasks grounded in the literature that promote access and equity in mathematics, making recommendations for teacher preparation programs associated with mathematics methods courses, beginning preliminary/pilot work within our own respective elementary mathematics methods courses, and using findings to initiate a NSF grant application. Some recent accomplishments from our group include: (1) a manuscript proposal submission to *Review of Research in Education* (a systematic literature review for culturally sustaining pedagogies in mathematics teacher educator programs, inclusive of elementary teacher programs), and (2) a potential article publication to the *School Science and Mathematics* journal for PTs' analysis of tasks.

Integration of Content and Methods Courses

Group co-leaders: Kristy Litster and Bona Kang

This research area seeks to develop recommendations for integrating pedagogical methods and mathematical content that support and assess PTs' learning. Specifically, it is looking at what MTEs can do in courses to support elementary PTs' ability to develop specialized mathematical content knowledge and pedagogical methods to teach any mathematical standard or grade level. AMTE (2017) noted this is an important area for MTEs as they provide guidance in designing and implementing course structures, assignments, and fundamental methodologies when preparing high-quality elementary mathematics teachers. To support the learning of a diverse student population, PTs should learn to evaluate and reflect on both the content and methods they use in the classroom. Research shows many elementary teacher preparation programs have limited time available due to course and program structures, which constrains MTEs' efforts to help PTs develop deep knowledge of children's mathematical thinking as well as ambitious teaching practices (Berry, 2004; Bertolone-Smith et al., 2023; Cochran-Smith et al., 2015; Saclarides et al., 2022). Integrating content and methodologies through a thinking-oriented approach within mathematical content and methods courses helps mitigate limitations of allotted time in programs to support PTs to build cohesive understandings of both specialized content knowledge and pedagogical practices (Li & Howe, 2021).

During the 2023 calendar year, this research group met monthly to review goals and plan potential collaborative research projects based on goals set at the 2023 AMTE conference in New Orleans, LA. We chose to focus on one of our project ideas: To try out specific strategies to integrate content and methods and assess PTs' content and pedagogical knowledge through qualitative and quantitative outcomes. Specifically, we were able to develop a project to assess these outcomes using a diagnostic interview assignment in content and methods courses across multiple universities. We developed a rubric using the teacher noticing (Jacobs et al., 2010), domains of Mathematical Knowledge for Teaching (Ball et al., 2008), and NCTM's effective mathematical practices (NCTM, 2014), which will be used to evaluate specialized content knowledge and pedagogical knowledge of PTs. We finalized the project idea at the PME-NA 45 conference in Reno, NV, and we will be gathering data during Spring 2024.

Field Experiences and Clinical Work

Group leader: Kim Johnson

At PME-NA 45, this subgroup narrowed our focus on paired field experiences and embarked on research on the benefits of providing this opportunity for PTs during their math methods courses. It is the aim of the methods courses that the PTs should observe mentors that use and implement effective mathematical teaching practices (NCTM, 2014). However, choosing the best mentor teachers for PTs to work with is often problematic given the large number of students enrolled in many programs. As noted in Bullough et al. (2002), using paired placements allows the reduction in the number of mentor teachers required and enables the concentration of effort and energy needed to produce the simultaneous renewal of effective placements. According to previous research (Baker & Milner, 2006, Bullough et al., 2002), paired placements also bring other benefits for PTs such as an increase in confidence and deeper reflection on lessons.

Our subgroup has met monthly since Reno, NV, and is pursuing a research study at University of Missouri to look closely at paired field placements versus individual field placements. This is an opportunity to do a comparison study, as they are piloting both types of placements. Additionally, group members are doing some work on developing planning tools for both in-service teachers and PTs to conduct more meaningful number routines in the classroom as part of professional development for mentor teachers. These lesson planning tools can be used by the PTs as part of the field experiences to help PTs focus on student thinking.

Our Long-Term Plan

The long-term plan for our MCRETP working group consists of five phases. We are currently in Phase III. Our working group, and consequently Phase I, began in 2021 at PME-NA 43 in Philadelphia, PA, with a primary focus on understanding the issues surrounding the preparation of elementary teachers. We entered into Phase II in 2022, at PME-NA 44 in Nashville, TN, with an emphasis on formulating research subgroups and brainstorming research studies aimed at providing insights that could inform our eventual curricular recommendations. As detailed in this report, we entered into Phase III in 2023, at PME-NA 45 in Reno, NV, and have continued this work since the conference. Phase III is designed to encompass two years, 2023 and 2024, as members of the MCRETP working group continue to design, conduct, and synthesize the findings from small-scale research studies that address questions relevant to producing a research-based set of recommendations for elementary teacher preparation programs. We aim to continue our work in Phase III at PME-NA 46 in Cleveland, OH, and in the months following the conference. Each research subgroup will continue to meet periodically, outside of our formal working group meetings at PME-NA conferences, to offer research support and feedback. The full MCRETP working group will also convene periodically, providing a platform for subgroups to share their progress and solicit additional feedback. We anticipate several members of the working group to be able to start disseminating findings from their small-scale studies near the culmination of Phase III. We will encourage working group members to write proposals to present research findings at PME-NA or other relevant professional conferences and to publish reports in other appropriate outlets.

Looking beyond our current work, Phases IV and V are scheduled for 2025 and 2026, respectively, with a primary focus on synthesis and the eventual production of research-based recommendations. In Phase IV, we plan to propose a PME-NA 47 research colloquium, where the research findings from each subgroup of the MCRETP working group will be cohesively presented. This colloquium will serve as a formal platform for synthesizing collective findings and addressing questions about the implications of our work. Simultaneously, we anticipate organizing a special issue of a journal, featuring a synthesis article from each research subgroup and a concluding article discussing the policy implications of the research findings. Moving into Phase V, our initial goal is to publish the special issue prepared during Phase IV. Furthermore, we intend to pursue a grant to host a 2026 conference with the objective of crafting and debating a research-supported set of recommendations for elementary teacher preparation.

In conclusion, our MCRETP working group has formulated a distinct vision focused on creating a research-based set of recommendations for elementary teacher preparation programs. Our time at PME-NA 45 in Reno, NV was exciting and productive, fueling our enthusiasm as we embark on the subsequent phases of our long-term plan. We emphasize one last time our openness to new participants, inviting them to join the MCRETP working group during its current phase, contribute to a research subgroup, and collaborate with us in achieving our long-term goals. Your participation is welcomed as we continue this impactful journey.

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OUR INACTION IS ACTION: COUNTERING ANTI-BLACKNESS IN SECONDARY MATHEMATICS TEACHER PREPARATION

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This group is dedicated to dismantling anti-Blackness in secondary mathematics teacher preparation through the development of critical, research-informed modules for Mathematics Teacher Educators. Our mission is to equip Pre-Service Teachers with knowledge, skills, and dispositions necessary to foster critical consciousness and counter systemic inequities in mathematics education. Leveraging improvement science methods, we aim to create, refine, and share six modules addressing instructor reflexivity, PST beliefs, identity and agency, political and historical context, systemic anti-Blackness, and actionable plans for student liberation. These modules are poised to transform mathematics teaching practices, promoting equity and stimulating interest in STEM among historically marginalized students.

Keywords: Preservice Teacher Education, Equity, Inclusion, and Diversity, & Social Justice

This newly established working group will contribute to ongoing efforts to increase access (Males et. al, 2020) to much needed research-based resources for Mathematics Teacher Educators (MTEs) to engage their pre-service teachers (PSTs) in learning to better serve the students in their diverse classrooms. The primary artifacts of discussion, development, and distribution will focus on six modules. These six modules are rooted in critical pedagogies (Duncan-Andrade & Morrell, 2008; Shor, 2014) to support PSTs in understanding how a) anti-Blackness functions in mathematics education and b) to counter cultural hegemonic practices. In line with Whitehead (2021), we define anti-Blackness as “whiteness and anti-Blackness are discrete, entangled ideologies through which Black people are stripped of their humanity. … [and the] relationship between whiteness and anti-Blackness operates both systemically and interpersonally” (p. 310). Programmatic structures must allow students to commit to a praxis Freire, 1996) reflecting on how anti-Blackness (white supremacy culture) impacts mathematics education and focus on the actions they may have already taken and will continue to take in the future. Through an awareness of anti-Blackness and the agency to challenge white supremacy, PSTs will develop critical consciousness. In mathematics education critical consciousness is related to the context of learning (connections to the real world) and how we see/treat students. Critical consciousness in mathematics education harkens on the importance of connecting mathematics to the socio-political (real) world (Kokka, 2019; Seda & Brown, 2021). However, our prior work shows that MTEs lack resources for doing this work (Males et al., 2020). Therefore, we aim to use improvement science methods (Bryk et al., 2015; Martin et al., 2020) to design and refine a series of modules for use in undergraduate 6-12 mathematics methods courses that address PSTs knowledge, skills, and dispositions to counteract anti-Blackness. The goal of this working group is to create a space for MTEs to collaborate on research and reflect on the harm caused by inaction, in order to sustain action towards dismantling systemic anti-Blackness within mathematics education, fostering a community of praxis that actively engages in the development and implementation of transformative pedagogies and curricula grounded in social justice and equity.

Modules & Session Structure

Recognizing the need for resources that empower MTEs to confront anti-Blackness in educational settings, our working group moves from theory to action. The six modules are designed to deepen PSTs understanding of anti-Blackness in mathematics education and equip them with the practical skills and reflective practices necessary for transformative teaching. PSTs will be encouraged to reflect on their own positions, challenge systemic issues, and develop actionable strategies for change.

The first module, **Instructor Positionality & Reflexivity (Module 0)**, prompts educators to reflect on their beliefs and practices related to anti-Blackness. **Understanding PSTs' Perspectives on Mathematics (Module 1)** encourages PSTs to examine and reflect on their own beliefs about mathematics teaching and learning. **Identity & Agency (Module 2)** explores the influence of personal identities in the learning environment. The fourth module, **Political & Historical Context (Module 3)**, links the history and politics of race to teaching practices. **Anti-Blackness and the System (Module 4)** delve into systemic issues perpetuating inequities and the impact of the culture of white supremacy, while the final module, **Developing a First Year Teacher Action Plan for Student Liberation (Module 5)**, focuses on actionable strategies for PSTs to foster an inclusive and liberating classroom.

Session 1 – Overview and Reflexivity. We will begin by providing an overview of the project and the work completed thus far, including sample data from the first rounds of implementation (30 minutes). We will engage participants in aspects of Module 0 to reflect on our own beliefs and practices as MTEs (30 minutes). The session will close with a whole group discussion on the necessity of developing critical consciousness (30 minutes).

Session 2 - Developing PSTs Critical Consciousness. Day 2 will engage participants in an activity from Module 2 called "Discarding Identities" (30 minutes). Participants will analyze the activity and further explore the modules in small groups (45 minutes). We will close with a whole group discussion on implementation of modules within participant contexts (15 minutes).

Session 3 – Moving Towards Action. Based on day 2 participation, we will dive deeper into the modules and existing data in smaller interest groups (45 minutes). Groups will discuss next steps and movement towards action to inform future PMENA working group meetings (45 minutes).

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EQUITY-ORIENTED CASES FOR MATHEMATICS TEACHING: CREATING RESOURCES AND EXPANDING CRITICAL CONVERSATIONS

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This new working group will reflect on intersectional equity issues in today's mathematics learning environments and create a resource to expand critical conversations through equity-oriented cases. We plan to engage participants in discussions regarding current uses of case-based instruction to reframe and dismantle violent and ideological views (e.g., racism, sexism, ableism) that diminish and disparage individuals from participating in mathematics settings. As a collective, participants will develop equity-oriented cases specific to mathematics education and establish collaborations to use the newly developed cases in mathematics teacher preparation. This working group aims to provide the community with space to generate a book of cases that will be shared with the mathematics education community.

Keywords: Preservice teacher education; instructional activities and practices; equity, inclusion, and diversity

Standards documents recommend teachers have a repertoire of equitable teaching strategies to be ethical advocates for every student (AMTE, 2017, Standards C.2 and C.4). Teacher education can provide opportunities for preservice teachers (PTs) to learn how social, historical, and institutional contexts influence mathematics teaching and learning, particularly with respect to equity and the various ways to challenge inequities that affect marginalized students' learning experiences. Additionally, PME-NA's Equity Statement (2020) calls for re-conceptualizing pedagogical practice away from deficit ways of knowing and learning to promote a humanized perspective of education that positions all students as capable contributors to mathematics education. In response to these initiatives, mathematics teacher educators (MTEs) use various research-based practices to support PTs' understanding of power and privilege in mathematics education, how to cultivate positive mathematical identities, draw on students' mathematical strengths, and enact ethical practice for advocacy. One such practice is using case-based instruction, which uses stories to explore the complexities of mathematics classrooms to develop a critical lens for teaching (Gorski & Pothini, 2018; Kavanagh, 2020). MTEs continue to develop, use, and share their cases at conferences and through personal connections. However, the sharing is not easily accessed by all in the mathematics education field.

This new working group will create a resource to expand critical conversations through equity-oriented mathematics cases that can be shared widely. The cases will illuminate the mathematical brilliance of all students, focusing on the intersectional identities of oppressed and marginalized students. While this work is not new, as there have been many equity-oriented cases shared at PME-NA, AMTE, and AERA where MTEs report on how PTs responded, including some of our presentations (Gonzalez et al., 2022; Moldavan et al., 2023), the goal of this working group is to leverage the productive critical conversations that have foregrounded the collective work to meet the following goals (1) assist participants in becoming familiar with recent research and resources addressing the benefits of case-based instruction and how cases can be used in mathematics teacher education to help PTs challenge equity issues in K-12 mathematics settings and (2) develop equity-oriented cases specific to mathematics education that can then be implemented in their settings and submitted to a case book.

Theoretical Background

Case-based instruction is widely used as a pedagogical approach to examine authentic, real-life scenarios that can engage teachers in analyzing dilemmas of practice (White et al., 2016). It can be used as a tool for uncovering teachers' deficit noticings and perspectives that can lead to differential student treatment, such as over-punishing Black students, which is a form of systemic violence in mathematics education (Childs & Wooten, 2023; Martin, 2019). While cases can be used to present a complex classroom dilemma situated within discussion questions for individual and group reflections (Gorski & Pothini, 2018; Redman & Redman, 2011), the cases often need to be modified to address the specific needs of mathematics teacher educators using the cases to bridge theory to practice in the teaching and learning of mathematics (Gonzalez et al., 2022). Furthermore, cases are typically organized around specific issues, such as race, gender, or socioeconomic status, and do little to explore the intersectionality of oppressed and marginalized identities that create compounding systems of disadvantage. The field of mathematics education needs a resource of equity-oriented cases that represent mathematics learning and teaching while also highlighting the importance of attending to multiple students' identities.

To frame the equity-oriented cases, we consider Cochran-Smith and Keefe's (2022) definition of strong equity, considering redistribution, recognition, representation, and reframing. MTEs can use equity-oriented cases to explore the power of one's story, which provides entry points to self-reflect and learn from other's experiences. The cases can encourage PTs to critically reflect on particular scenarios to reframe, integrate, and unlearn deficit views of students and transform equitable instructional practices that position all students as capable mathematics learners and doers. Recognizing the need to reframe and dismantle violent and ideological views (e.g., racism, sexism, ableism) that diminish and disparage individuals from participating in mathematics settings can guide the purpose and use of cases, especially cases that present counternarratives to challenge deficit perspectives.

Organization and Presentation Plan

The working group will be organized into three sessions where participants can actively contribute their experiences and expertise to satisfy a deliverable. The first session will welcome participants with an introduction to case-based instruction, an exploration of recent resources, and a discussion about the desired product that the working group will produce. As a collective, we will collaboratively critique existing cases focusing on equity and mathematics spaces. After sharing an example template for writing equity-oriented cases, participants will be tasked with drafting their own case to bring to the next session.

In the second session, participants will work in small groups to provide feedback on their drafted cases. They will collaboratively unpack the intersectional equity issues the cases explore and critically reflect on how the cases were written so as not to perpetuate biases and deficit perspectives. They will then be tasked with coming to the next session with a revised draft with discussion questions that will engage PTs in unpacking the inequities alongside the mathematics content. The last session will provide a space to refine the discussion questions, coordinate implementation plans to use the cases, and note recruitment plans to contribute to the case book.

By the end of the conference, the group will have produced multiple equity-oriented cases that are ready to be used with PTs. Additionally, we intend to continue this work with broader participation from the mathematics education community by including the voices of mathematics teachers, coaches, and school leaders.

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GENDER AND SEXUALITY WORKING GROUP: EXPLORING INTERSECTIONALITY WITH A FOCUS ON CONCEALABLE STIGMATIZED IDENTITIES

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Gender research has a long history in mathematics education, whereas sexuality research is a newer research focus in the field. Much of this research has been conducted using a singular lens, that is, strictly focusing on the influence of participants' genders (or sexualities) in their mathematics experiences. However, in recent years, increased focus has been placed on the importance of conducting research with an intersectional lens, with the acknowledgement that experiences within the same identity group (e.g., boys, lesbians) differ due to multiple intersecting identities. Although research has been conducted about the intersections of gender and race in mathematics education, far less attention has been paid to identity aspects that are less (visually) obvious than race, such as socioeconomic status, first-generation college student status, and certain disabilities. In this year's working group, we will explore how to make gender/sexuality research more intersectional, addressing both methodological and theoretical considerations, with special attention to concealable stigmatized identities, which have rarely been explored in gender/sexuality research.

Keywords: Gender; LGBTQIA+; Equity, Inclusion, and Diversity

Mathematics education and mathematics education research are subject to influence from broader societal shifts and government policies. In recent years, there have been numerous bills introduced and passed in North America (French, 2024; Trans Legislation Tracker, n.d.) that have negatively impacted the lived experiences, including the schooling experiences, of LGBTQIA+ students and teachers (Kosciw et al., 2021). Hence, for these groups of individuals, their learning and teaching of mathematics are indeed taking place during uncertain times.

Given these uncertainties, as well as recent societal shifts regarding perceptions of gender and sexuality, it is crucial that we, as mathematics education researchers, acknowledge and honor the complexities of participants' lived experiences through our work. One such topic that has increasing relevance is intersectionality. Coined by Crenshaw (1989), the term *intersectionality* refers to "the intersection of people's identity categories but also the intersection of individual and institutional factors" (Pugach et al., 2019, p. 207). Although some mathematics education research is intersectional, intersectional research is still uncommon in the field and typically only pertains to intersections of gender and race (e.g., Battey et al., 2022; Hsieh et al., 2021). Hence, scholars (e.g., Bullock, 2018; Hall et al., 2024) have called for an increased focus on intersectionality, as well as for intersectional analyses beyond gender and race.

Social (or membership) identities are based on belonging to groups that share those identities, such as gender, sexuality, race, and language (Langer-Osuna & Esmonde, 2017). These identities, especially as they intersect with each other, influence students' experiences in and out of school (Eddy et al., 2015; Ferguson, 2017). Social identities can be conspicuous or concealable (Quinn, 2006). Concealable stigmatized identities (CSIs) are those "that can be hidden from others and that are socially devalued and negatively stereotyped" (Quinn & Earnshaw, 2013, p. 40). Examples include minoritized sexualities, certain health conditions (e.g., mental illness), and low socioeconomic status (Busch et al., 2023). Both hiding and disclosing CSIs can have consequences for people's lives in school and beyond (Quinn & Earnshaw, 2013; Weisz et al., 2016). This relatively neglected research topic is thus a significant area of exploration for mathematics education researchers because mathematics identity is formed by interactions between mathematics learning and social identities (Gresalfi et al., 2019).

History and Goals of the Working Group

Since the Gender and Sexuality Working Group began in 2018, various topics have been explored, including methodologies and theories for conducting inclusive gender and sexuality research. Over the years, the working group has grown, indicating expanding interest in this research area. In last year's working group sessions, participants shared their experiences conducting gender and sexuality research, including discussing how they initially became involved in the field. They expressed that they felt that the working group was a safe space to discuss research challenges that they faced, due to the similar perspectives, goals, and experiences of those in attendance. Participants also shared examples of research successes and pragmatic ideas to support each other's research.

One topic that arose in last year's working group discussions was intersectionality. Participants expressed interest in exploring this topic in more depth in future working group sessions, which is why we selected the topic as this year's focus. Specifically, we plan to use the working group sessions to provide participants with an opportunity to share their experiences and suggestions for conducting intersectional gender and sexuality research, with a focus on CSIs.

This Year's Working Group Sessions

Each session will have a different, but related, focus. Specifically, **Session 1** will have an introductory focus so that participants can become better acquainted with the topics of intersectionality and CSIs, their histories, and their places in gender and sexuality research, via two 10-minute presentations: one about intersectionality and one about CSIs. In the remainder of Session 1, participants will share their experiences conducting intersectional research, particularly research involving CSIs.

Session 2 will begin with a 15-minute presentation in which examples of intersectional research involving CSIs (conducted by other researchers in the field) will be shared. The remainder of Session 2 will be a working session in which participants will be provided with case studies involving sample research questions and will be invited to conceptualize practical ideas for conducting the research using an intersectional lens. Participants will consider the case studies in small groups, and the responses to the case study questions will be shared in a jigsaw format.

In **Session 3**, participants will be asked to form groups based on common areas of interest and will work together to brainstorm potential study designs around these topics. We anticipate that these discussions will lead to future research collaborations among the working group participants.

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100-Word Description of the Working Group Sessions

The focus of this year's Gender and Sexuality Working Group is intersectionality. Although some intersectional gender/sexuality research has been conducted, these studies typically only involve gender and race. Hence, in the sessions, we plan to specifically focus on concealable stigmatized identities (e.g., mental health conditions, socioeconomic status), as they are rarely addressed in gender/sexuality research. Participants will share their experiences conducting intersectional gender/sexuality research, particularly research involving concealable stigmatized identities. Participants will also be able to put their understandings into practice, through considering case studies, and will be able to brainstorm future research collaborations with other group members.

COMPLEX CONNECTIONS: REIMAGINING PURE AND APPLIED RESEARCH IN UNITS CONSTRUCTION AND COORDINATION

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The theory of units construction and coordination originally posited a rationale for students' mathematical reasoning with whole numbers; for instance, explaining that students may conceive of 12 as 12 units of one, 1 unit of 12, or 1 unit of 3 groups of 4, among other possibilities. Research has subsequently identified connections between students' construction and coordination of whole number units to fractional units, proportional reasoning, and other mathematical domains. This working group aims to engage mathematics educators in discussion to deepen the field's understanding of units construction and coordination, including pure theoretical research, and research applied in classrooms.

Keywords: Learning Theory, Learning Trajectories and Progressions, Number Concepts and Operations

Background

For the past several years, the Complex Connections working group has met to broaden the research, understanding, and application of units coordination in mathematics education. This work has included discussions around algebraic reasoning, covariational reasoning, combinatorial reasoning, mathematics knowledge for teaching and many other mathematical concepts. The goal each year is to provide an opportunity for researchers to discuss the links between the mathematical concepts and the types of of units constructions and coordinations needed for reasoning. This year, the working group is hoping to move beyond these discussions and look ahead at research we may develop to broaden the understanding of units construction and coordination.

Units coordination is related to student reasoning, particularly multiplicative reasoning, fractional reasoning, algebraic reasoning, covariational reasoning, and combinatorial reasoning (Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009; Norton et al., 2015; Antonides & Battista, 2022; Tillema, 2013, 2018). Units coordination refers to one's ability to construct and operate on composite units made up of sub-units and switch between the multiple units (Norton et al., 2015). For example, in a multiplicative task (e.g., You have 5 towers of cubes, each with 3 cubes, how many cubes do you have altogether?), an individual reasoning multiplicatively could coordinate three different units within the task (composite units – towers, single units – cubes in each tower, and total compilation – total cubes across the 5 towers) (Tzur et al., 2017).

Day 1: The Future of Units Construction and Coordination

Day 1 will have the working group transitioning from what has already been studied to what the future holds in units construction and coordination research. The session will begin with an introduction to provide a framework for the working group's goals and allow for a discussion on the distinctions between applied research versus pure theory research. This will give members the opportunity to consider which direction their future research may take based on their own aspirations and research goals. From there, members will be introduced to a scenario to provide context for further discussion: If we had unlimited funding to study units construction and coordination, what would we, as a community of researchers, want to study, what questions would we want to answer, and what problems would we want to explore? The goal is to provide a possible roadmap of where we want our research to go based on the progress we have already made and what we have left to explore. This discussion will provide members with an opportunity to collaborate with others interested in similar research goals and organize times to meet throughout the conference days and plan future research.

Day 2: Nuances of Units Construction and Coordination

Over the last several years, the working group has come to realize that the nuances of units construction and coordination research vary across researchers and their work in the field. Therefore, the group will spend the second session exploring these nuances and coming to some agreements and commonalities for our research field. These nuances include how we assess units construction and coordination, the stages of units coordination we observe and analyze, and the terminology we use in our research dissemination (i.e., unit vs units; structure; coordination vs. construction, etc.). We will culminate the session by analyzing discrepancies within the units coordination literature through guided prompts provided by the leadership team. The goal is to find a common foundation on which units coordination research can further build.

Day 3:

Day 3 will be a session bookmarked for members to report to the group what they are currently studying in units coordination and providing an opportunity to ask questions and discuss the research currently going on. The session will also include time to plan for the 2025 working group, as well as expectations of working group members over the next year.

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IDENTIFYING AND STUDYING BLACK HOLES OF MATHEMATICS EDUCATION RESEARCH ON INSTRUCTIONAL PRACTICE

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Matney et al.'s (2020) Black Hole of mathematics educational research involves an instructional practice that has attained critical gravity in the mathematics teaching field. While practitioner anecdotes and related research suggest the practice's efficacy, there is a lack of rigorous research into the practice itself. Launching from Matney et al.'s example of Number Talks, this new working group will describe and elaborate on the Black Hole metaphor, identify additional possible Black Hole research domains, define parameters for researching and ultimately illuminating Black Holes, and establish future research collaborations.

Keywords: Instructional activities and practices, research methods

In mathematics education, practitioners often take up an instructional practice that intuition tells them will be impactful. This intuition is guided by experience and research related to the practice, but at times school systems use this practice widely without objectively understanding the efficacy of this practice. Furthermore, the practice often evolves into practices bearing the same name while being somewhat different in purpose and substance. This working group seeks to promote a "practice-to-research" focus on scholarship examining the 'Black Holes' surrounding these instructional practices in mathematics classrooms. The authors of this proposal are from various institutions and have connected at various conferences, including PME-NA, over their shared interest in widely used classroom practices that lack a solid foundation of empirical research. This working group seeks to formalize these connections and bring others into the conversation. In doing so, we have the following goals:

- Introduce and discuss the metaphor of Black Holes of research and identify examples.
- Describe constraints to working on research to illuminate a Black Hole.
- Discuss what it means to contribute to research on practices represented as Black Holes of research.
- Develop future work across institutions for systematic investigation of the practices.

Theoretical Background

Matney et al. (2020) labeled the phenomenon described above as a Black Hole of research on instructional practice, meaning an instructional practice that has attained 'critical gravity' in the teaching field, but little to no blind-peer-reviewed research has investigated its efficacy. When a practice resides in a research Black Hole, the Black Hole does not represent empty space; rather, there is 'something' there attracting practitioners, but research has yet to systematically guide or illuminate it. Despite this lack of direct investigation, the Black Hole of research is surrounded by an accretion disk made up of connected knowledge from the results of rigorous research that suggests the practice's validity (Matney et al., 2020).

Matney et al.'s (2020) impetus for developing the metaphor of the Black Hole for instructional practices that lacked a rigorous research base was the practice of Number Talks (e.g., Humphreys & Parker, 2015; Parrish, 2011). Number Talks are said to promote sensemaking and computational fluency (Humphreys & Parker; Parrish). However, while established research on several relevant topics—such as classroom environment, discourse, mental math, number sense, and teacher questioning—suggest the practice's efficacy, Matney et al.'s systematic literature review identified only one study that examined the impacts of the practice itself with K-12 students. Matney et al. asserted that investigation of the practice itself is necessary in order to bridge the gap between research and practice and identify the features of the practice and its implementation that contribute to the practice's impact.

Structure of the Sessions

On **Day 1** of the working group, we will begin with introductions of the participants of the working group and review the goals of the working group (15 minutes). Then, based on his original literature review (Matney et al., 2020), Matney will introduce the Black Hole metaphor and explain its origin (10 minutes). Two research teams will present an overview of their research into the practice that spurred the creation of the Black Hole metaphor, Number Talks (10 minutes each for a total of 20 minutes). One research group investigates in-service teachers' curriculum assemblages with Number Talk resources. The second research group explores when and how preservice teachers use teacher authority in their enacted Number Talks and their related reflections. The remainder of the session (45 minutes) will involve small and whole group brainstorming and discussion of other potential instructional practices that are surrounded by a Black Hole of research. Discussion points will include parameters for what 'counts' as a Black Hole, what sources should be reviewed in order to identify a Black Hole, and what constitutes a lack of systematic research.

On **Day 2**, we will review the items discussed on Day 1 (15 minutes). Three research projects will present overviews of their research on practices they have identified as surrounded by a Black Hole of research (10 minutes each for a total of 30 minutes). One researcher studies pre-service elementary teachers' engagement in open mathematics tasks such as Which One Doesn't Belong? (Danielson, 2016), How Many? (Danielson, 2018), and Notice and Wonder (Fetter, 2021; Ray-Riek, 2013). The second research group investigates in-service teachers' purposes for using Notice and Wonder. A third researcher studies Complex Instruction (Cohen & Lotan, 2014) from its theoretical underpinnings of expectation states theory. In the remaining 45 minutes, in small groups and as a whole group, discussion will address issues such as research questions, populations, data collection, data analysis. We will also discuss what it means to provide evidence of a practice's 'efficacy.' We will discuss challenges inherent in researching in Black Holes such as establishing research in K-12 settings with K-12 students.

Day 3 will focus on establishing joint goals and initiating research collaborations. We plan to have a website to keep a record of our work between and across conference meetings.

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For conference program:

Matney et al.'s (2020) Black Hole of mathematics educational research involves an instructional practice that has attained critical gravity in the mathematics teaching field. While practitioner anecdotes and related research suggest the practice's efficacy, there is a lack of rigorous research into the practice itself. Launching from Matney et al.'s example of Number Talks, this new working group will describe and elaborate on the Black Hole metaphor, identify additional possible Black Hole research domains, define parameters for researching and ultimately illuminating Black Holes, and establish future research collaborations.

WORKING GROUP REPORT: CRITICAL DISABILITY STUDIES IN MATHEMATICS EDUCATION

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In continuing with past working groups on research at the intersection of disability studies and mathematics education, this working group met at PME-NA 45 to share new developments from group members, to make new connections, and work toward shared goals. Over the course of the three days, we articulated topics and ideas in which to invest our energies, specifically: (a) the development of the Mathematics Education Conference Access Committee, (b) a commitment to supporting the leadership of PME-NA as they work toward increasing accessibility to and at the annual conferences, (c) a plan for a Disability Justice and Mathematics book proposal, and (d) a plan for pursuing grant funding for a Disability Studies in Mathematics Education conference.

Keywords: students with disabilities, special education, equity, inclusion, and diversity

History

Kai Rands and James Sheldon conceptualized and convened this working group, formally from 2016–2018, although other informal meetings occurred before and after these dates. During these early meetings, group members discussed employing different theoretical perspectives within their work, described current projects, and raised critical issues. This core group consisted of faculty, graduate students, disability activists, and classroom teachers and consisted of: Amber Candela, Jessica Hunt, Rachel Lambert, Katie Lewis, Paulo Tan, and Cathery Yeh. The working group convened again in 2022 and 2023 at PME-NA. In between formal meetings, many working group members collaborated to conduct research, write grants, produce manuscripts and books, and engage with the mathematics education, special education, and disability communities.

Work at PME-NA 45

Across the three sessions at PME-NA 45, 27 people participated in our hybrid working group. This hybrid approach allowed access for people to participate and engage in the group who would have otherwise been excluded from an in-person-only working group. The need for a hybrid working group surfaced when group members expressed concerns about COVID-19, travel, in-town transportation, and the difficulty many caregivers face when being away from home for multiple days. Our group discussed the various ways hybrid access allowed for more inclusivity, specifically for disabled, immunocompromised, and otherwise marginalized scholars.

Hybrid access allowed many working group organizers to meet and form collaborations across space before, during, and after the conference. Over the three days we built community, worked in small groups, and planned for next steps beyond the bounds of conference meetings. We created a set of shared documents to capture meeting notes, conversations, and ideas.

Day 1: Community, Connections, and Generating Small Group Topics

Our group started with introductions and sharing access needs. The importance of the invitation to share access needs is discussed by activists and scholars (e.g., Reinholz & Ridgway, 2021; Sins Invalid, 2019). For example, sharing one's access needs might sound like, "I am [name], and I am at [university]. My pronouns are she, her, hers, and it is important that I can get up and stretch; I need to take breaks periodically," or, "My access needs are being met through having the virtual environment." Through these introductions, our group built camaraderie as people identified similar needs and ways to collectively meet various needs. These introductions served as opportunities for group members to identify people with whom they would like to connect based on, for example, their professional role or interests. These introductions were also an inroad for folks to learn about ableism and how professional norms often (unintentionally) impede access. During these introductions, we recorded names and contact information on a shared electronic document so group members could reach out to each other during or after the conference. One access need shared by many participants was having a virtual modality for the working group. Throughout the session, organizers regularly ensured that access needs were met and that communication throughout both modalities was successful and comfortable. This helped create an environment in which those participating virtually were viewed as full and contributing members of the community and spurred a discussion about implicit ableism in the way that virtual participation is often viewed as less legitimate. Several members expressed disappointment that they, and other marginalized scholars, were unable to engage in the remainder of the conference.

We generated a list of topics for members to pursue in a small group; ideas were documented on the shared electronic document. As we generated this list, it became evident that attendees had a range of experiences, background knowledge, and interests. It was clear that attendees, both old and new, were interested in a space to feel intellectually, professionally, and personally seen and heard. Thus, the first meeting organically developed into a space for community and connection.

Day 2: Generating and Planning

Our next meeting afforded the opportunity to enact the value that access is a collective responsibility, thereby moving toward a collaborative actualization of Disability Justice (Piepzna-Samarasinha, 2019). One small concrete example of this occurred as people arrived at the session. Group members immediately began collaboratively rearranging chairs and tables, with members joining in as they arrived, to honor multiple access needs simultaneously, even if those needs were not shared by those doing the rearranging. They did this because increasing access meant we all had a greater understanding of each other's access needs and were thus all able to honor those needs and engage in deeper work.

We had multiple new attendees and so took care to (re)introduce ourselves and conduct an access check-in. We continued generating ideas and raising issues to address in small groups. We asked for volunteers to lead different small groups and then split into those groups to pursue different topics or projects. The groups were: (a) Disability Justice and teacher education, (b) Disability Justice and Mathematics book proposal, (c) addressing and/or responding to the

"Science of Math" website, including international perspectives, (d) accessibility and ableism within mathematics education research spaces, specifically at professional conferences (This group formalized themselves into the Mathematics Education Conference Access Committee.), and (e) conference grant proposal writing to convene educators and disability-led organizations and coalitions.

Day 3: Updates, Strategizing, and Articulating Next Steps

On our final day we shared updates on our work in relation to the "Science of Math" website and heard from group members who were on the PME-NA steering committee. We discussed ideas to push PME-NA to make its conference hybrid and to examine biases and presumptions regarding different approaches to conference engagement (e.g., virtual v. in-person). We also worked towards establishing how the working group community could continue to make progress throughout the year. We assigned leaders to the logistical and intellectual work we wanted to pursue. Then, we planned to (a) update the group's email listserv, (b) publicize speaking engagements and papers generated by group members, (c) we summarized what we had accomplished during this working group and (d) articulated next steps.

Progress Since PME-NA 45

This group maintains regular communication through an established listserv and has continued to work on several collaborative projects since our last meeting. In July 2023, four members of the working group submitted a paper that analyzed the language used on the "Science of Math" website. Since PME-NA 45, this paper has been rejected, revised, and is currently under review.

In October 2023, following the conference, one group member reported that they and other members of the Mathematics Education Conference Access Committee had engaged in ongoing conversations with PME-NA conference organizers, specifically about increasing conference accessibility via a hybrid option. The group member elicited feedback from the working group about priority accessibility concerns and took those concerns to others in conference decision-making roles. This small group continues to meet and advance our position about increased conference accessibility.

In November 2023, three members of the working group elicited ideas for chapters and art works for a proposal to Teachers College Press' Disability, Equity, & Culture series. The editors anticipate submitting a revised prospectus to the publisher in spring 2024.

Actions to Include Participants

To include as many people as possible, we employed several intentional strategies. First, by offering a hybrid option for our working group we signaled to our colleagues and in-person conference attendees that we prioritized enacting our values of inclusivity and access. Further, when people joined the working group on the second or third days of the conference, we took care to pause the work and properly introduce newcomers and orient them to the ongoing work. Beyond the conference, we maintain two different email groups: a PME-NA working group list and a broader listserv for anyone interested in staying connected to information about mathematics education and disability. This provides community, connection, and a recruitment avenue as our group positions PME-NA as a focal site for folks interested in taking up critical questions about mathematics education and disability.

Questions to Address in the Future

Critical questions that our group will continue to address include:

- What do we mean by "critical disability studies in mathematics education"?
- How, if at all, does "critical disability studies" relate to "critical special education"?
- As mathematics educators, how might we continue to engage folks in other disciplines including special education?
- How could this group promote activism and advocacy within and beyond mathematics education?
- What kinds of conversations about disability and mathematics education are happening in international contexts and how can those inform the work of this group?
- What else is needed with respect to addressing disability and other intersectional identities in mathematics education?
- What is the ethos of this group?
- How can we meaningfully engage and mentor doctoral students within mathematics education and between mathematics education and other fields?
- How can we support novice and experienced researchers to work and learn with disabled students, educators, and scholars?
- What role can this working group play in advocating for and actualizing increased conference accessibility, specifically through organizations with an explicit commitment to equity and inclusivity?

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CRITICAL DISABILITY STUDIES IN MATHEMATICS EDUCATION: ENVISIONING THE FUTURE

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Keywords: students with disabilities, special education, equity, inclusion, and diversity

Goals and Effort Toward These Goals

"Taking up space as a disabled person is always revolutionary" (Ho, 2020, p. 115).

As we *Envision the Future of Mathematics Education*, the theme of PME-NA 46, that future necessarily includes the voices, perspectives, and experiences of disabled students (and their families), disabled teachers, disabled activists, and disabled scholars. Yet, disability is persistently omitted from research in mathematics education (Lambert & Tan, 2020). Because mathematics education classrooms and content are often used as tools for marginalization, we draw upon critical theories such as Disability Studies in Education, Critical Race Theory, and Disability Critical Race Studies (DisCrit) to offer a justice-oriented vision of mathematics education. PME-NA 46 would represent the group's sixth meeting; therefore, our goals necessarily encompass the continuation of past work as well as the development of new work:

- Create an inclusive and accessible space that centers disability and topics related to mathematics education;
- Anchor our work in the 10 principles of Disability Justice (Sins Invalid, 2019);
- Generate interdisciplinary and cross-institutional collaborations; and
- Leverage the group's collective wisdom to (re)imagine solutions to persistent problems.

Our strategies to meet these goals include:

- Provide a hybrid option for the working group;
- Unpack the 10 principles of Disability Justice (Sins Invalid, 2019);
- Design spaces that prioritize focused, small-group work; and
- Identify small-group leaders who are equipped to facilitate the cross-pollination of ideas.

Beyond this meeting, we anticipate proposing future PME-NA working groups to ensure the sustainability of this community and the work that this community generates.

Theoretical Background

We theorize ableism through Disability Studies in Education (Connor et al., 2008). We seek to disrupt ideologies of "ability" and "normal" (e.g., Annamma et al., 2013; Siebers, 2008) to study the pervasiveness of ableism in education. We apply scholarship on ableism to investigate how this operates in mathematics education (Dolmage, 2017; Price, 2021; Westby, 2021). We draw on the Disability Justice framework, which was developed by disabled queer activists and disabled people of color (Sins Invalid, 2019). We specifically draw upon the 10 principles for disability justice created by Patty Berne and others from Sins Invalid (Sins Invalid, 2019): intersectionality, leadership of the most impacted, anti-capitalist politics, cross-movement solidarity, recognizing wholeness, sustainability, cross-disability solidarity, interdependence, collective access, and collective liberation. Given the harm done to disabled communities, Disability Justice allows researchers to recognize the intersecting legacies of white supremacy, colonial capitalism, gender oppression, and ableism to understand how people are labeled "deviant", "unproductive" and/or "invalid" (Sins Invalid, 2019). Acknowledging that disabled students and communities of color have gifts (Annamma & Morrison, 2018), such as unique mathematical ideas, perspectives, and solutions, challenges the deficit narrative and structures of ableism (Lewis & Lynn, 2018; Yeh, 2023).

Plan for Active Engagement and Working Group Organization

In specific response to the growing size and diversity of our working group members, we have designed a three-session sequence that is both structured and flexible. Group members will have choice in how and with which topics they engage. Small-groups will have the autonomy to define what "meaningful outcomes" mean for small-group members and a timeline by which those outcomes will be achieved; these strategies are designed to promote engagement.

Session 1: Community, Connections, and Co-Designing

Most of this session will be to introduce and orient people to the working group. We will start with introductions and access needs (20min), followed by an update about the group's accomplishments, sharing the goals of the group, and introducing the principles of Disability Justice (30min). We will introduce small-group topics and their leaders and identify any additional small groups that may arise (10min). Small-group topics include, but are not limited to: (1) teacher education (ongoing from 2023); (2) intersectional identities, specifically LGBTQIA+; (3) interdisciplinary work with special education and addressing the "Science of Math" website (ongoing from 2022); (4) advocacy / activism; (5) conference accessibility (ongoing from 2022); (6) international contexts; (7) conference proposal development (e.g., Mathematics Education and Society; ongoing from 2023); (8) theoretical dives into Disability Justice, DisCrit, and Critical Disability Studies; and (9) articulating the working group's ethos. This session will conclude with small-group meetings (30min).

Session 2: Generating and Planning

Most of this session will be working in small groups. We will (re)introduce people and share access needs (15min). We will transition into the small groups identified in Session 1 (55min). We will conclude by coming back to the whole group and sharing out (20min).

Session 3: Articulating Next Steps

Most of this session will be working in small groups. We will begin in small groups (60min). We will come back to the whole group and share out (20min). Finally, we will wrap-up the working group by articulating a plan for ongoing communication and support beyond the conference (10min).

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A NEW WORKING GROUP: SUPPORTING TEACHERS AND TEACHER EDUCATORS WITH IMPLEMENTING BUILDING THINKING CLASSROOMS

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This new working group seeks to produce innovative ways to support the implementation of and research about 'Building Thinking Classrooms' at the university and K-12 levels.

Keywords: Professional Development, Preservice Teacher Education, Classroom Discourse.

Introduction

Since it was first introduced to the mathematics education community, *Building Thinking Classrooms* (BTC) (Liljedahl, 2021) continues to grow in relevance among teachers and teacher educators across North America. We see this growth in regional conferences, professional learning programs, and learning communities across social media platforms. The relevance of this initiative has grown so much that the first annual International Building Thinking Classrooms Conference was held in Indianapolis in 2023.While BTC is a research-based initiative, the resources for teachers are mainly limited to the book and online forums for sharing ideas. For teachers to successfully implement, critique, and supplement these practices, guidance is needed to support their work. Our goal as mathematics teachers and mathematics teacher educators is to study how the practices and their implementation are supporting effective teaching and learning. Our aim is to support teachers to draw on the practices, along with other research-based initiatives that align with increasing student thinking in the classroom, as they make professional decisions in their classroom.

The authors of this working group have engaged in various initiatives centered around BTC. One author has created an Instructional Circle (Author, 2022) to provide personalized and deprivatized feedback to teachers as they implement BTC. There are three stages to implementation of BTC that become more difficult to implement in a classroom (Liljedahl, 2016). The instructional circle is a form of professional learning where teachers can find support from experienced teachers and knowledgeable others about ways they can improve their instructional practices. Another author developed an initiative with a local school district to provide instruction and materials for teachers wanting to implement BTC but lacking the support or materials to be successful. Lastly, one Author's team is engaging in an initiative looking at specific aspects of BTC including task writing and “thin slicing” and how teachers are utilizing them in their implementation of BTC (Frazee & Scharfenberger, 2023).

As mathematics teachers and mathematics teacher educators we feel it is worthwhile to support the implementation of BTCs in a variety of contexts. This instructional practice supports ambitious teaching as defined in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014). The creators of BTC also understand professional learning principles that state incremental changes to teacher practices are more sustainable long-term than sudden transformational professional developments (Otten et al., 2022).

Goals of the Working Group

The goals of this working group are two-fold. First, we want to serve as a forum for authors and participants to share a variety of research projects centered around *Building Thinking Classrooms* (Liljedahl, 2021). This will inform the conversations about how to create a more effective implementation and support structure for teachers using this instructional practice in both K-12 and university levels. Second, instructional circles (Melville, 2022) will be created for the participants of the working group to form professional learning communities for support around their specific interest with BTC beyond this conference. Specifically, these contexts may include using BTC in math content courses, math methods courses, in-service professional learning, and ways in which research projects can be designed to study various aspects of these settings. We believe that these instructional circles will be a valuable resource as we implement and study the effects of BTC on student learning and teacher education and the growth of mathematics teacher educators.

We expect this working group to continue for multiple years; however, this initial setting will be for the introduction of BTC to some, and reports of what the authors and participants are currently doing. For future iterations of this working group, we hope to develop more concise ideas about the needs, struggles, and benefits of implementing BTC in content courses, method courses, and other professional learning settings.

Organization and Presentation Plan

The first session will start with presentations from the authors about what they are currently doing regarding coaching, implementing, and teaching BTC in methods courses and professional learning contexts. The authors alone have a variety of initiatives around BTC. One example is that BTC is being studied as a vehicle for task assessment and professional learning of teachers around planning (Frazee & Scharfenberger, 2023). has created an Instructional Circle (Melville, 2022) to provide personalized and deprivatized feedback to teachers as they implement BTC. The last initiative studies how BTC is being used in a more traditional professional learning course for teachers who are or would like to implement these instructional strategies.

The second session will include reports from non-author participants about their current initiatives that utilize BTC or that could benefit from using BTC. These participants will volunteer during the first session for this activity. The second session will end with forming break out groups according to participants’ interest in utilizing BTC (i.e., math content classrooms, math methods classrooms, professional learning settings, or researching the effects of BTC in these other contexts).

The third session will continue the breakout groups for the different contexts that participants are interested in utilizing BTC. During these breakout sessions, we plan to help organize the participants into instructional circles to provide support and fresh ideas to implement or study BTC in different settings. This session will also include each group sharing their current ideas and goals with the whole group.

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EMBODIED MATHEMATICAL IMAGINATION AND COGNITION (EMIC) RESEARCH COLLOQUIUM

COLOQUIO DE INVESTIGACIÓN: COGNICIÓN E IMAGINACIÓN MATEMATICA INCORPORADA

COLLOQUE DE RECHERCHE: IMAGINATION ET COGNITION MATHÉMATIQUES INTÉGRÉES

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The Embodied Mathematical Imagination and Cognition (EMIC) Research Colloquium offers hands-on activities; technological, curricular, and pedagogical demonstrations; and open spaces for exploration and discussion regarding the embodied nature of mathematics education. We use the PME-NA 46 conference theme "Envisioning the Future of Mathematics Education in Uncertain Times" to invite the community to experience perspectives across multiple institutions and expand notions of mathematical activity, effective teaching, learning, assessment, and learning technology design. We focus on current uses, future prospects, and challenges of virtual and augmented reality (VR/AR) to foster inclusive and effective educational experiences.

Keywords: Cognition; Learning Theory; Teaching; Social Justice; Systemic Change; Technology

Theoretical Background

Virtual and augmented reality (VR/AR) serve an increasingly central role in mathematics education and teacher training (Bock & Dimmel, 2021). As with any such advancement, the research community must contribute to this process of technology transfer and translation based on rigorous methodology and sound theoretically-principled designs for curriculum, instruction, and assessment. However, few studies in AR/VR are informed by a clear theoretical framework (Mikropoulos & Natsis, 2011). VR/AR offers learners unprecedented access to mathematical objects and tools through sensorimotor and collaborative processes that may generate new forms of mathematical inscriptions, even entirely new branches of mathematics (Nathan, 2024). EMIC contributes a compelling framework for the empirical study of embodied interactions (collaborative gesture; multimodal analytics) using VR/AR, the design of future platforms and activities (affordances and constraints), and formative and summative assessments of nonverbal ways of knowing (Abrahamson, 2014; Alibali & Nathan, 2012; Arzarello et al., 2009; De Freitas & Sinclair, 2014; Edwards et al., 2014; Radford, 2009; Walkington et al., 2021) that address the *whole learner*, facilitate the development of diverse learners, and foster inclusive and effective educational experiences (McKinney de Royston, et al., 2020).

A Brief History and Motivation of the EMIC Working Group

The EMIC working group officially started in East Lansing, MI during PME-NA 2015. We have convened annually at PME-NA ever since, organizing events that investigate the embodied nature of mathematics through physical and digital games, arts and crafts, individual and collaborative dance and movement, language, perception, immersive experiences, and more. EMIC members continue to explore new technologies (e.g., Harrison et al., 2018; Walkington et al., 2021) and research methods (e.g., Abrahamson et al., 2021; Closser et al., 2021) that acknowledge and leverage the inherently embodied nature of mathematical cognition. They go beyond scholarly publications and presentations to offer: novel embodied technologies, including some that have gone to scale for classroom use, special education, and emerging bilinguals; teacher learning for instructional uses of gestures; and webinars for teachers and parents. In addition to the authors, the organizers/institutions will include Dr. Teruni Lamberg (UNR), David Kirkland (Clark County School District), and Dr. Candace Walkington (SMU).

VR/AR: Envisioning the Future of Mathematics Education in Uncertain Times

VR/AR technology is rapidly altering the landscape of mathematics education and STEM professional work through immersion and direct, body-based interactions with (holographic) mathematical objects. In 2024, we will organize hands-on activities and group discussions around the proliferation of VR/AR technologies in classroom instruction and research. On **Day 1**, we will do introductions and then give an overview of grounded and embodied mathematical thinking and teaching. We will then talk about published findings on the educational value of VR/AR technology for supporting math learning and teaching and its connection with embodied learning (e.g., Garzón et al., 2020). We will invite participants to engage and observe the use of an example VR system for teaching about powers of ten. We will explain why students often struggle to understand orders of magnitude and reason about very small and very large numbers in domains such as nanotechnology, and demonstrate how VR provides perceptual experiences that can improve their mathematical reasoning. Day 1 will conclude with an open-ended Q&A on the nature of embodiment theory and embodied design principles for VR/AR.

On **Day 2**, organizers will facilitate exploring EMIC accounts of VR/AR in collaborative, embodied math activities. We address research methods for analyzing collaborative embodiment in immersive learning environments and some of the benefits and challenges for implementing these interventions in authentic learning contexts. We will then break into small design groups where participants consider ways VR/AR could be used to explore other topics in mathematics education, and ways of supporting diverse learners with varying physical abilities and linguistic and cultural experiences. We will report these ideas to the whole group, theorizing about the nature of learning in ways that inform research and the design of valid knowledge assessments for a broad range of learners and math topics, which will be chronicled in a shared document.

Day 3 will start with reflections on the design ideas and learning principles generated during Days 1 & 2. We explore ways that VR/AR enables novel forms of mathematical activity by directly interacting with and transforming geometric objects using body movements and perceptual processes. We then consider how VR/AR can support expanded notions of mathematical representations, such as inscribing operations through person- and group-centered actions and perceptions, in contrast with disembodied, formalism-centered, symbolic notations and diagrams. We plan to have a facilitated discussion on why embodiment offers a unique set of theoretical and methodological resources for designing meaningful learning experiences.

Conclusions and Looking Ahead

Embodied learning theory and design principles provide key insights into how perceptually concrete, bodily experiences and collaboration in VR/AR can engage learners across a wide range of topics and grade levels. Movement- and perception-based mathematical activities can reach students who may otherwise feel left out of highly discursive and symbol-based activities. Through EMIC, we aim to foster an engaged community committed to transformative education.

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100-word description

VR/AR technology is rapidly altering the landscape of math education and STEM professional work. Embodied approaches to mathematics learning, instruction, design, and assessment offer natural inroads for understanding the challenges and benefits of using VR/AR to engage all learners. Since 2015, the Embodied Mathematical Imagination and Cognition (EMIC) Research Colloquium has organized hands-on, collaborative, and generative activities for experiencing the contributions that embodied mathematics has to offer. During this 3-part research colloquium, participants are invited to experience, reflect on, and design activities for immersive learning experiences using VR/AR across a range of topics to suit all learners across learning settings.

MATHEMATICS CURRICULUM RECOMMENDATIONS FOR ELEMENTARY TEACHER PREPARATION WORKING GROUP: RESEARCH IN PROGRESS

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In our third year of meeting at PME-NA, we brainstormed and solidified plans for conducting research studies that would support our future recommendations for how to prepare future elementary teachers to teach mathematics. Our goals for this year are to continue to support such research conducted by working group members, adding new members and studies as appropriate, and to prepare to synthesize and disseminate our research findings.

Keywords: Preservice Teacher Education, Teacher Educators, Mathematical Knowledge for Teaching, Elementary School Education

Elementary teacher preparation in North America is vitally important to ensure a robust mathematics education for elementary students. However, preparation programs for elementary teachers contain large variations in terms of what, how, and how much is taught. Various organizations (e.g., AMTE, 2017; CBMS, 2012; NCEE, 2016) have made recommendations about teacher preparation including the number of credit hours that prospective teachers (PTs) should take as well as what content should be taught, and what types of experiences PTs should have. However, research (e.g., Garner et al., 2023; Masingila et al., 2012) has shown that the majority of teacher preparation programs are not meeting these recommendations. In many programs, it seems that the current recommendations are not feasible, given the other requirements for preparing PTs. The purpose of this working group is to create a set of research-based, specific recommendations for elementary teacher preparation that can be feasibly implemented by teacher education programs. We will do this by providing research-based evidence to stakeholders to show that our claims for the amount of instruction time and specific topics and instructional practices are needed to prepare PTs for their future jobs.

Progress of Current Work

According to Garner et al. (2023), the overwhelming majority of elementary teacher preparation programs in the United States fall short of the number of credit hours in elementary mathematics content and methods recommended by the *Standards for Preparing Teachers of Mathematics* (AMTE, 2017). More specific research-based recommendations of what topics to teach, how much time to spend on each topic, and what pedagogical strategies are effective in teaching these topics will help programs advocate for change at their institutions. Corven et al. (2022) indicate that elementary prospective teachers require 6-8 hours of instructional time to acquire lasting and deep specialized content knowledge (Ball et al., 2008) for key mathematics topics from the elementary curriculum. Whereas Corven et al. (2022) claim this finding underscores the need to limit the number of topics taught in elementary teacher preparation (cf. Walsh et al., 2022), this result further supports the conclusion of Garner et al. (2023) that programs with fewer credit hours than AMTE's (2017) recommendations may not be serving future elementary teachers well. Our working group continues to conduct research that will be useful in providing more specific recommendations for elementary teacher preparation programs.

Organization and Presentation Plan

This will be the fourth year that this working group has met at PME-NA. We have already created a number of subgroups that are undertaking research projects as detailed in our report. At PME-NA 46, we plan to have each research subgroup present their current research plans and research studies in progress during the first two sessions, with time allocated in the first session to orient new members to the working group. After each presentation, the audience will have an opportunity to ask general questions and give feedback. Each day, after the presentations, the subgroups that presented will meet to discuss the feedback and revise their research plans. Members of the working group who are not members of any of the subgroups that presented that day will be asked to serve as "outside reviewers" to a subgroup of their choice to provide additional specific feedback on research plans. This structure will allow new members of the working group time to figure out which subgroup(s) they want to join and be able to contribute meaningfully to subgroup work right away. Additionally, this structure will permit members who are part of two subgroups to meet with both subgroups during the conference.

During the third session, all subgroups will meet for the first part of the session to discuss the concrete next steps for their research plans. We will provide a calendar template to subgroups to help with this planning. In the second half of the third session, we will give each subgroup five minutes to present their plan to the whole group. We will ask subgroups to talk specifically about the potential results they might present at a PME-NA Colloquium in 2025.

Opportunities for New Members to Contribute and Future Plans

New members of the working group should select one or two research subgroups that interest them. They will have opportunities to join these subgroups and contribute meaningfully to the in-progress research plans as a collaborator. We encourage anyone interested in joining the working group to reach out to a subgroup leader before the conference (contact information for the subgroup leaders is in our report) to get added to the email notifications list.

We expect the working group to transition to a Colloquium for PME-NA 47 (2025) to allow the subgroups to present research results. We also encourage researchers who are not members of the working group but who have conducted research related to one of the subgroups to reach out for inclusion in the Colloquium. After the Colloquium, we intend to organize a special issue of a journal or a book to disseminate the results of the working group. We expect to have an introductory article, one article for each subgroup that summarizes and synthesizes their work, and a concluding article that contains policy recommendations and implications of each article. After the journal/book is published, we intend to hold a conference to discuss and debate a draft of the curricular recommendations.

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GENDER AND SEXUALITY WORKING GROUP REPORT: SUPPORTING LEARNERS AND SCHOLARS THROUGH OUR STORIES

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Gender and sexuality research in mathematics education remains a significant focal point for discussions among researchers, educators, and students. During the 2023 Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA), the working group met to explore our journeys as gender and sexuality researchers, the challenges and supports encountered throughout our endeavors, and the stories told through our research. In this report, we discuss the key activities conducted during the working group sessions at PME-NA 2023, including a presentation about a recent literature in gender and sexuality, as well as two focus group discussions. Furthermore, we delve into how the discussions held during the 2023 working group sessions propelled us toward embracing intersectionality as a central focus for the 2024 working group initiatives.

Keywords: Gender; LGBTQIA+; Equity, Inclusion, and Diversity; Social Justice

History of the Working Group

The Gender and Sexuality Working Group originated from discussions on equity in mathematics education at PME-NA 2017 in Indianapolis and first convened at PME-NA 2018 in Greenville. Since its inception, the working group has met annually at PME-NA to facilitate crucial discussions on gender and sexuality in mathematics education. At these gatherings, we have focused on various research approaches, including theory and data (Przybyla-Kuchek et al., 2022), conceptual and methodological frameworks (Jackson et al., 2021), and data collection (Ataide Pinheiro et al., 2023). The group has been a welcoming and supportive venue for early-career faculty and graduate students, and has provided opportunities for them to share their research, assume leadership roles, and engage with researchers from institutions across the United States, Canada, Brazil, and Australia.

A significant aspect of the working group has been its efforts to broaden participants' perspectives on gender and sexuality research in mathematics education and to support individuals in developing and advancing their work for publication. For instance, the group was pivotal in supporting contributions to the special issue on gender and sexuality in *Mathematics Education Research Journal*, co-edited by Jennifer Hall and Eva Norén (2021). Notably, during PME-NA 2021, the working group organized discussions featuring authors of articles from this special issue, providing a platform for in-depth exploration and debate, facilitated by a designated discussant (Jackson et al., 2021). The Gender and Sexuality Working Group continues to serve as a vital forum for advancing research, fostering collaboration, and promoting inclusivity within gender and sexuality research in mathematics education.

2023 Working Group Sessions and Efforts Made to Include Participants

After several years of engaging in crucial discussions on theory and methodologies in gender and sexuality research, the 2023 Gender and Sexuality Working Group shifted its focus towards exploring the personal narratives and motivations that shape us as gender and sexuality researchers. Specifically, we sought to understand the life stories of researchers in this field, including what led them to develop an interest in this research topic, their experiences navigating research across various academic venues (e.g., faculties of education, departments of mathematics), and how these lived experiences may or may not have influenced their research agendas and the stories that they have told through their research. This exploration encompassed the challenges encountered and the support received in conducting gender and sexuality research.

Next, we provide a detailed overview of the activities that transpired during the PME-NA 2023 Gender and Sexuality Working Group sessions.

Day 1

We began the working group session on Day 1 with introductions of attendees and an open discussion to learn about the types of research that attendees were interested in pursuing in connection to gender and sexuality in mathematics education. After these introductions, we outlined the plans for the working group sessions.

On Day 1, we also had the opportunity to delve deeper into a literature review prepared and presented by one of our 2023 leadership team members, Ana Dias of Central Michigan University. Ana systematically reviewed research methods utilized in previous research articles focusing on gender, sexuality, and mathematics education from 2020 to 2024, using the databases Google Scholar, EBSCO, ERIC, Web of Science, and ProQuest. In her review, Ana drew upon and expanded the dates covered in the review done by Becker and Hall (2023). English, French, Spanish, and Portuguese were the languages used in the review. For this systematic literature review, the types of work that were excluded were thesis and dissertations, conference proceedings, books, opinion pieces, theoretical papers, and reports on lessons and activities; studies in which STEM was treated as a whole (as opposed to those that had separate data about mathematics students, preservice teachers, or in-service teachers); and research articles in which gender or sexuality were used as a variable (as opposed to as a category of analysis). After the exclusion criteria were applied, 31 articles remained for review. Only four studies involved quantitative methodologies (Copur-Gencturk et al., 2021; Teague Tsopgny et al., 2020; Voigt, 2022; Wolff, 2021). The rest were qualitative studies.

Methods used in the qualitative studies were discourse analysis, content analysis, and interviews with drawing solicitation (Gjøvik et al., 2022; Guichot-Reina & De la Torre-Sierra, 2023; Lafay, 2022; Neto & Ataide Pinheiro, 2021); interaction analysis for analyzing video data (Kolovou et al., 2023); semi-structured interviews and coding (Jaremus, 2021; Jaremus et al., 2020); task analysis (Rubel et al., 2022), interventional studies (de Souza Ortolan et al., 2020; Soares et al., 2023); a combination of interviews, focus groups, photo solicitation, and discussion (Hall & Robinson, 2020); narrative inquiry and grounded theory (Kersey & Voigt, 2021); longitudinal mixed methods and national surveys (Barbosa et al., 2021; Hsieh et al., 2021); document analysis and ideology critique (Martins et al., 2021); questionnaires analyzed qualitatively (Guse et al., 2020); feminist post-structural discourse analysis (Przybyla-Kuchek, 2021); doctrinal methodology (Sharma, 2021); and comparative methodologies (Sharma, 2021; Jao et al., 2023). Some of the perspectives from which the studies were conducted were post-structural feminism (Jaremus, 2021; Jaremus et al., 2020; Przybyla-Kuchek, 2021), and decolonization (Barbosa et al., 2021; Soares et al., 2023). This introductory discussion laid the groundwork for subsequent conversations during Days 2 and 3 of the Working Group.

Day 2

Day 2 discussions were focused on stories of how the working group participants became gender and sexuality researchers, and what led us to be interested in researching gender and sexuality in mathematics education. We also discussed the challenges and supports that we have encountered and continue to encounter as gender and sexuality researchers. In our discussions, we aimed to support graduate students and early-career faculty embarking on research in gender and sexuality who may benefit from learning about and learning from obstacles that we have faced, and that they may themselves face. Furthermore, such discussions are beneficial for all scholars as we shared knowledge, skills, and resources that are useful for successfully navigating conducting research in this field.

The discussions on Day 2 were organized as focus group discussions. Therefore, we divided the leadership team and the participants into two groups. The questions discussed on this day by both leadership team members and participants included, but were not limited to:

(a) How did you become interested in gender and sexuality research?
(b) What challenges have you faced while doing gender and sexuality research?
(c) What supports have you encountered while doing gender and sexuality research?
(d) Is there anything else you would like to share about your story?

Some of the themes that arose during these focus group discussions included challenges of conducting this kind of research, including having our voices not be heard in university departments, difficulty obtaining IRB approval, and differences in norms between countries.

Day 3

On the last day of the working group sessions, we engaged participants in focus groups in a similar manner to the ones that we employed on Day 2. The specific discussion questions we used to guide the discussion on Day 3 were:

(a) What stories have you told through your gender and sexuality research? (For those who haven’t had the chance to conduct research in this area yet, what would you like to know before engaging in gender and sexuality research?)
(b) How have your years of experience affected the stories that you have told through gender and sexuality research?
(c) What ways do you envision this working group moving forward as a space to engage in discussions of gender and sexuality in mathematics education?

Some of the topics that emerged from this focus group discussion were the need for better pre-service teacher education regarding these topics, the challenges of teasing out intersectional identities (especially hidden ones), the need to move beyond binaries and normativities, and the increased expectation of researchers sharing their identities with research participants and in publications.

We concluded Day 3 by consolidating our plans for future research discussions using a Jamboard, as shown in Figure 1. During the conversation that followed, we noted that many of the topics suggested on the Jamboard pertained to intersectionality and thus determined that for the 2024 working group sessions, we would focus on a theme related to intersectional identities.

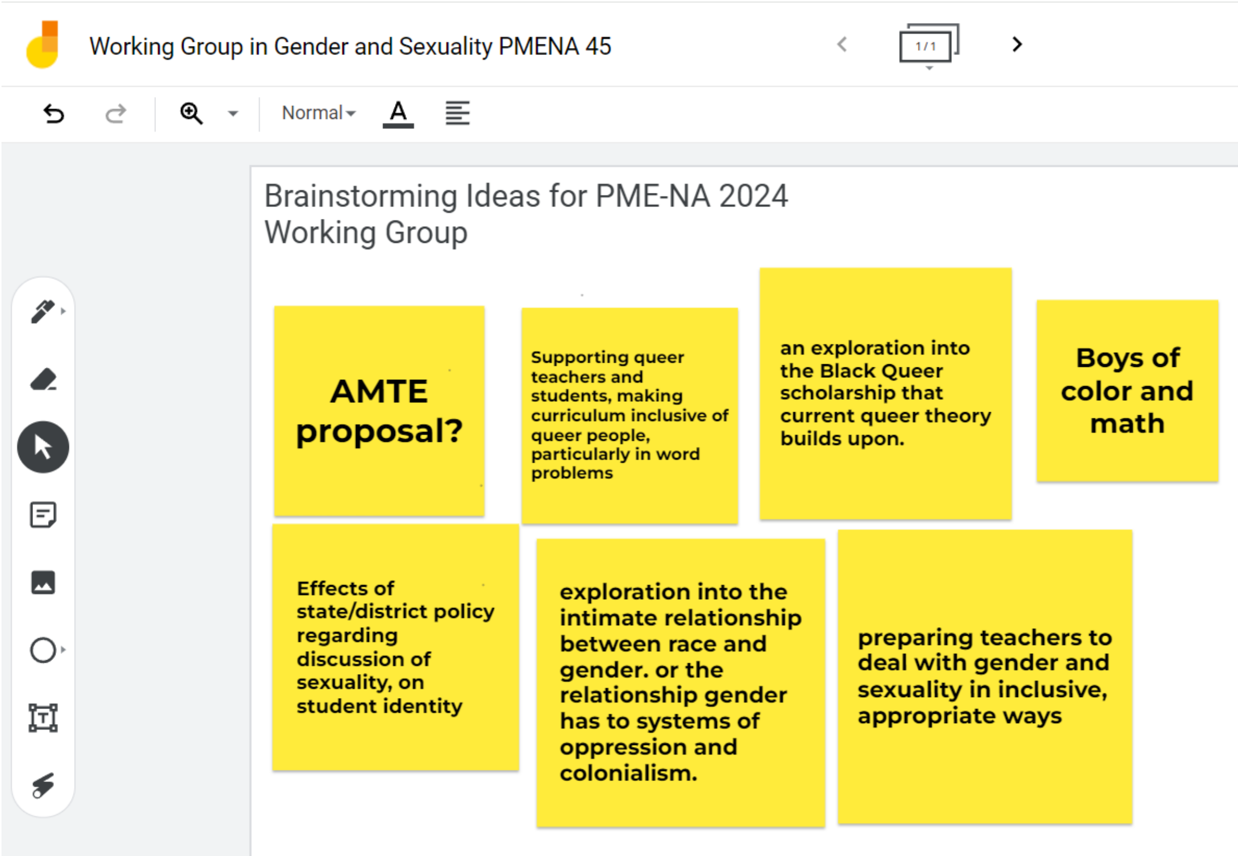

Figure 1: Jamboard Used by Participants During the Day 3 Session

Progress Made From the 2023 Conference to the 2024 Conference

As discussed, the initial design of the PME-NA 2023 Working Group on Gender and Sexuality involved building a repository of why and how individuals become gender and sexuality researchers, the challenges and supports that they have encountered when conducting research in this field, and the stories they have conveyed through their research. We are planning to follow up from the focus of the 2023 working group sessions and investigate possible outlets in which our stories could be shared more publicly (e.g., a special issue of a journal). We are eager to better understand: What motivates scholars to become gender and sexuality researchers? What challenges and supports do gender and sexuality researchers experience along the way? What stories have gender and sexuality researchers told through their research? How do their lived experiences affect the research questions that they choose to pose?

Questions to Address in the Future

As mentioned, one area of interest raised by participants of the 2023 working group was the topic of intersectional identities. Specifically, participants discussed how the multitude of individuals' social identities, in addition to gender and sexuality, could be used to control, oppress, and exclude individuals from mathematics education. Therefore, discussing gender and sexuality without considering race, socioeconomic status, disability, and other identities makes it difficult to have a more holistic discussion about how gender and sexuality impact the teaching and learning of mathematics. Participants demonstrated an interest in further discussions by the working group that bring to the forefront the intersections of race, gender, and sexuality, as well as socioeconomic status, disability, and other identity factors that may be less 'visible' than gender or race. Participants also raised questions regarding (a) the effects of national/state/district policy on discussions of sexuality on student identity, (b) supporting queer teachers and students by making the curriculum inclusive of queer people, and (c) preparing teachers to address gender and sexuality in inclusive, appropriate ways. These questions may be considered and explored in working group sessions at future PME-NA conferences.

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AESTHETIC AND AFFECTIVE DIMENSIONS OF MATHEMATICS LEARNING

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Students' aesthetic and affective responses are intertwined and are both central to mathematics learning. This working group will continue the conversation begun in 2022 to investigate the connection between the affective and aesthetic dimensions of mathematics education, and how connecting these dimensions can help us to better understand students' experience with mathematics. The goals of this third meeting of the working group are to explore useful tools and methodologies for studying aesthetics and affect, as related to participants' sharing their research interests.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Curriculum; Instructional Activities and Practices.

Theoretical Background

The relationships between aesthetics and affect (beliefs, engagement, motivation, etc.) for their centrality in understanding students' mathematical experiences and learning is clear (e.g., Cheeseman & Mornane, 2014; DeBellis & Goldin, 2006; Malmivuori, 2001). The purpose of this working group, from Riling et al. (2023), is to investigate "the potential enabled by identifying the shared interests in aesthetics and affect. Our aim is to enable new solutions to a persistent problem [poor student experience in mathematics] that can become possible by bringing together these two domains" (p. 596).

We describe the theoretical foundations for our working group, from our first proposal (Satyam et al., 2022): *Affect* has been defined as all aspects of experience that involve feeling (McLeod, 1988). This ranges from deeply held, long duration constructs such as beliefs, attitudes, math anxiety, and motivation, to shorter-term and in-the-moment feelings such as emotions and engagement (Grootenboer & Marshman, 2016; McLeod, 1992; Middleton et al., 2017). In mathematics, where success versus failure is often visible, students' affective responses can be quite strong (Boaler, 2015) and impactful (Grootenboer & Marshman, 2016; Op 't Eynde et al., 2006).

Yet, similar to how we would also examine a piece of art for explanations for an individual's gasp in a museum, we argue that researchers also need to attend to the nature of the mathematical experience for explanations of how it potentially impacted students (e.g., inspiring a question, enabling predictions and eliciting surprise). We refer to the way a lesson supports the felt impulses that compel (or impede) a student to continue to progress (or not) through an experience as its *aesthetic dimensions* (Dietiker, 2015). Some researchers have recently begun to study the aesthetic potential of mathematical learning environments (e.g., Dietiker, 2016; Sinclair, 2001), learning for example how the design of technological tools can offer surprise and appeal (Sinclair et al., 2009). While still emerging, the field is learning how to design and enact what Sinclair (2001) calls “aesthetically-rich” mathematical experiences, which she describes as those that “enable children to wonder, to notice, to imagine alternatives, to appreciate contingencies and to experience pleasure and pride” (p. 26).

This working group will continue to explore the linkage between affect and aesthetic to tackle multiple dimensions of one of the most significant dilemmas facing mathematics classroom practice today: poor student experiences with mathematics that fuel negative student dispositions. Through our working group design, we plan to bring together graduate students, early career and senior faculty together, to solidify the current state of the field and help form informal and formal connections for future research endeavors.

History of Working Group at PME-NA 2023

The goals of the second meeting of this working group were to bring researchers together to build community through structured but also informal activities, learn about each other’s work through participants sharing, and explore potential writing projects. We met over three days and a mixture of university faculty and graduate students attended.

Day 1: Task as Anchoring Experience

We began with an overview of aesthetics and affect, the purpose of our working group, and a recap of what occurred at the previous (first) meeting of the working group. This was important as we had some returning participants and some new participants joining. We then all engaged in an online task, Parable of the Polygons, as a shared experience through which participants could then have grounded reflection and discussion about their aesthetic and affective reactions. This was successful in new and returning members meeting each other, equalizing their contributions, yet bringing them together on the same page. Participants discussed which aspects of the activity most “drew them in” to explore more and how that related to the aesthetic and affective aspects of their experiences.

The session ended with us collecting their information: research interests, population of interest (students or teachers, grade band, etc.) and what they would like from the working group. We also provided a sign-up for participants to share their work with the whole group the next day.

Day 2: Participants Share Work

On Day 2, we facilitated extended introductions, in which participants shared their research and/or teaching interests that fell under the umbrella of aesthetics and/or affect. We intentionally planned for this to occur after our participants had an experience together first (the previous day), so that participants would have shared language and experiences when learning about each other’s work and interests. The main activity of this session was for participants to share their work for 5-10 minutes. There were seven presenters (including the organizers):

Brady Tyburski: Undergraduate STEM students’ art and artist statements
Megan Selbach-Allen: Caring instruction
Matt Melville: Teacher reactions to deprivitization of practice
Tracy Dobie: Student perceptions of math usefulness
Sandra Hall: Frameworks of aesthetic and affective domains
Leslie Dietiker: Mathematically Captivating Lesson Experiences
Meghan Riling: Types of aesthetic experiences and creativity

Participants shared digital copies of their posters from the current PME-NA and other conferences, as well as artifacts from their research (see Figure 1).

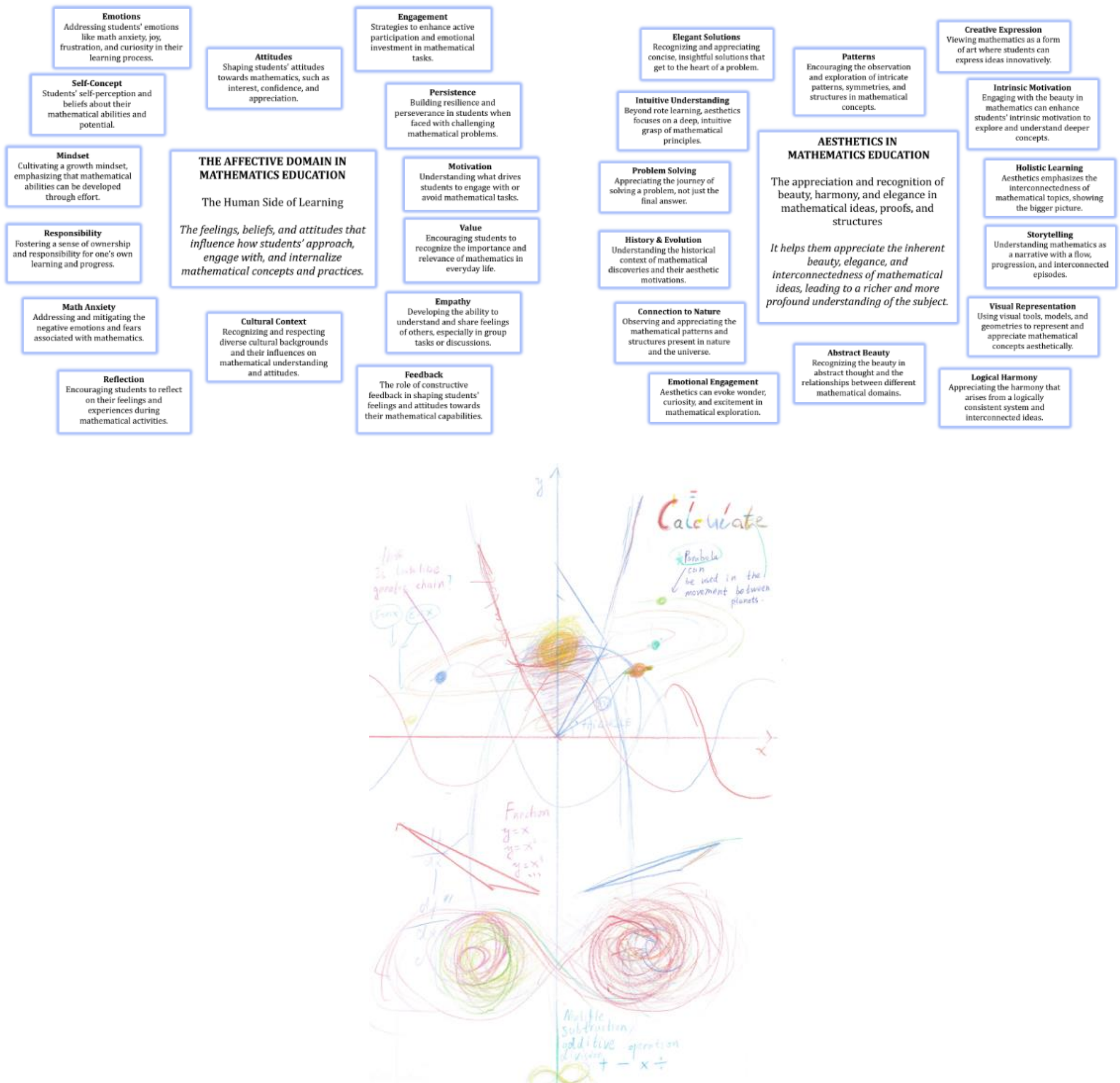

Figure 1. Samples of participants sharing conceptual diagrams and student work. Above: Sandra Hall. Below: Brady Tyburski

At the end, participants reflected with a partner on what they found interesting across the presentations. This served as a wrap-up for Day 2 and as preparation for Day 3.

Day 3: Comparing and Contrasting Conceptualizations

On day three, we worked to bring the group together and set directions for future research collaboration. We finished presentations from Day 2. Then, we did a video analysis: we together watched a classroom video from a 9th grade geometry classroom about midpoint quadrilaterals and posed the following questions. *How can you describe the aesthetic elements of the* *interaction? How can you describe the affective elements of the interaction?* Participants were given a transcript of the video and in groups, recorded their observations and preliminary analysis electronically in google docs. Participants noted constructs such as engagement, affective reactions, relations to Dewian aesthetic, and the overall benefits for how aesthetic and affective aspects created a space in which students could share ideas. The group also pondered the relationships between affect and aesthetic. For example: "Affective as part of the aesthetic - if aesthetic is felt reaction to a stimulus, then the affect is just the felt part. Aesthetic tries to tell the story of what happened and how student felt. Affect zeroes in on the student feelings."

The video analysis served as an example of future work together and a segue into an open discussion about next steps. We provided participants these prompts:

Opportunities for Collaboration: What ideas or projects might you want to work on with a colleague? (Casual/informal)
Group Interaction: What kinds of communication would you be interested in receiving from the group between now and next year?
Connections: What other constructs are you most interested in connecting to aesthetics/affect, and how do you currently see them as being connected?
Questions: What questions related to aesthetics or affect are you wondering about now?
Anything Else: Is there anything else you'd like to share with any of us?

We also shared an outline for a theoretical paper Dr. Dietiker had started, about aesthetics and affect, including possible journal venue and potential inclusion of video analyses.

Next Steps for PME-NA 2024: Session Organization

The goal of this third iteration of the working group will be on exploring useful tools and methodologies for aesthetics and/or affect, by way of participants sharing and talking about their own research and identifying resources to explore together. Given the success of the prior meetings, we will follow a similar structure as before. In particular, we have found that doing a task together on Day 1 to provide a collective, grounded experience provides a solid foundation that is memorable and useful.

Prior to the working group start, we will send out an interest survey for participants' research and/or teaching interests, population, and if they'd be interested in sharing any of their work (published work, data, instruments, etc.).

Day 1: We will start by having participants introduce themselves, share their research and/or teaching interests and experience in aesthetics and affect so far (15 min). We will all engage in an anchoring mathematical activity from an existing research study, aimed toward producing affective and aesthetic reaction (30 min). This will elicit informal conversation among participants, returning and new, centered around a task and focus. We will pair this experience with some research data, such as video, that we will all watch together (30 min). This picks up where Day 3 of our last working group left off: selecting a task from a study and engaging with its data. We'll end with an overview of the goals of this working group and will offer opportunities for participants to share a piece of their work for the next session on Day 2 (15 min).

Day 2: We will provide a short recap of Day 1 for any new participants (10 min). The main activity will be for a few participants who are currently working in the domains of aesthetics and/or affect to share their research. To elicit conversation and interaction, we will split the entire working group into 2-3 groups, so people can both share research within the group (20 min) and pick something else to explore together as a subgroup (40 min). Topics for further exploration may be other constructs, methodologies, or instruments related to the research shared, if applicable. We will then share out collectively what each group discussed and pursued (20 min), as a record of the working group's thinking.

Day 3: We will provide time and space for participants to discuss next steps. Options include: writing short pieces for future conferences or journals such as *For the learning of mathematics*, a conference proposal grant, and/or adapting powerful activities for students based on participants' teaching interests. We will maintain a Google Drive folder, where Google Docs with work that we generate together will be kept as a record after the conference. There will also be a folder for participants to share their relevant published work with the group to facilitate spread of ideas, especially from new scholars.

We plan for future iterations of the working group to be a larger, shared authorship team with participants. Our report would be a compilation of reflections from the participants, and those that agree to share for the report would be listed in the working group as authors.

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YEAR 2 OF TECHQUITY WORKING GROUP

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This continuing working group, Conceptualizing the Role of Technology in Equitable Mathematics Classrooms (TechQuity), plans to extend the ongoing conversation regarding the role of technology in building equitable mathematics classrooms. During PME-NA 2023, the group met for the first time to better understand the intersections between technology- and equity-centered mathematics teaching and learning. While our initial meeting was successful, we found the complexity of the role of technology in equitable mathematics classrooms to be far-reaching. This year we plan to grow our community and enact ideas to build sustainable programs of research dedicated to TechQuity in mathematics education.

Keywords: Technology; Equity, Inclusion, and Diversity

In this second year of meeting, the TechQuity working group will continue to develop opportunities for collaboration through the sharing of our current work in progress and facilitating opportunities to grow this network of mathematics educators dedicated to exploring TechQuity.

While attending to precisely what *equity* means can be a challenge, our current conceptualization of TechQuity centers around examining potential intersections between, Gutiérrez's (2009) four dimensions of equity, namely Access, Achievement, Identity, and Power, and the didactic tetrahedron (Hollebrands, 2017) that models interactions between students, the teacher, technology, and mathematics. This year we plan to continue unpacking what these intersections might look like in research and practice, especially in light of new advancements in technology (e.g., AI). Prior work on this topic (e.g., Suh et al., 2022; Witt, 2022) offers promising starts to promoting TechQuity. For example, Suh et al (2022) present,

> "...dimensions for technology that have transformative potential to enhance access to inquiry based learning, promote positive math identity through authorship and agency, provide formative assessment and differentiation, encourage collective thinking, and amplify mathematical thinking processes" (p. 1564).

Additionally, Witt (2022) argues for the leveraging of mathematical action technology– that is, dynamic interactive digital technology to support student-centered exploration (Dick &

Hollebrands, 2011) to support the development of a sociopolitical disposition in students. While promising, we acknowledge much more work and many more voices are needed to pursue TechQuity.

Throughout the three-day working group sessions at the 2023 PME-NA conference, we set the stage for sustaining two interconnected subgroups. The *Frameworks Subgroup* focuses on creating a framework for equitable technology integration, what we refer to as the TechQuity Framework. The *Task Design and Implementation Subgroup* focuses on the design and implementation of mathematical tasks using technology that promote equitable teaching practices. Additionally, the *Task Design and Implementation Subgroup* decided to pursue further grant opportunities for examining the role of AI use in mathematics education. Our recent discussions in this subgroup have supported calls (e.g., NCTM, 2024) to bring more mathematics educators into the discussion on researching and developing AI for equitable access and outcomes in mathematics education. Ultimately, the most promising outcome of our work together thus far has been the building of a community of mathematics educators who are committed to pursuing more work related to TechQuity.

With these two subgroups meeting regularly, we see this year's working group as a great opportunity to share with the PME-NA community what our group has learned and plans to pursue while also welcoming new members interested in exploring these ideas with us further. Thus, we organize this year's working group around the goals of these two subgroups: 1) *sharing progress and gathering feedback on developing a TechQuity Framework,* 2) *compiling research on the implementation of and development of tasks that support TechQuity*, and 3) *examining ways that AI can support equitable opportunities to learn in mathematics classrooms*. With these three emergent areas of interest, we intend to broaden participation by welcoming current and new members to join us and contribute to these conversations and collaborations.

Table 1 shows our tentative plan for engagement with working group participants; we plan to adjust these plans to accommodate participant interests.

Table 1: 2024 Plan for Engagement

Session	Activities
One	1. Introductions (20 mins)
	2. Updates and discussion of goals (70 mins)
	a. Developing a TechQuity Framework
	b. TechQuity Task Design and Implementation
	c. Equitable Use of AI in Mathematics Classroom
	d. Reflection: "What is TechQuity?" and "What are we missing?"
Two	Identifying emerging subgroups (15 mins)
	Collaborating in subgroups (60 mins)
	Sharing out ideas from the subgroups (15 mins)
Three	1. Planning for future work in subgroups (45 mins)
	2. Sharing of subgroup plans (30 mins)
	3. Planning for dissemination, including website expansion (15 mins)

Our anticipated follow-up activities include: (a) collaborative research and task development, (b) grant writing, (c) continuing to develop a TechQuity Framework, (d) expanding our website to disseminate our work, and (e) growing our community.

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PRODUCTIVE INTEGRATION OF COMPUTATIONAL THINKING INTO MATHEMATICS EDUCATION

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The implicit connections between the practices of computational thinking and mathematics have the powerful potential to strengthen students' reasoning about quantitative and spatial relationships. Working group participants will build upon their PME-NA 45 exploration of the synergies between CT and mathematics education to refine directions for collaborative research. We will continue to work toward our overarching goal of providing more equitable access to authentic mathematical problem solving through computing. Discussions will relate participant experiences with computationally-integrated mathematics tools and curriculum to the construction of productive learning environments and problem-solving identities.

Keywords: Computational Thinking, Mathematical Thinking, Teacher Identity, Student Identity

The integration of computational thinking (CT) and mathematics learning (Shute et al., 2017; Weintrop et al., 2016) has the powerful potential to deepen K-16 students' mathematical skills and practices as they seek patterns, create visualizations, make strategic guesses, and experiment systematically (Pei et al., 2018). The natural connections across disciplines can foster students' active engagement in the CT practices of abstraction, decomposition, pattern recognition, and algorithm design as they solve authentic mathematical problems (Gadanidis, 2017, Ng et al., 2023). Ye and colleagues (2023) further describe this disciplinary relationship as an interactive and cyclical process of reasoning mathematically and computationally to generate knowledge. By extending upon constructionist perspectives of computational artifacts as "objects to think with" (Papert, 1980), mathematics educators integrate computational artifacts as dynamic visual representations of mathematical thinking (Dahshan & Galanti, 2024; Cui et al., 2023).

Several directions for potential research were discussed at the inaugural meeting of this working group at PME-NA 45. While all participants were interested in exploring the synergies between CT, mathematics education, and data science education, there was a consensus that an actionable research agenda demanded a narrower focus. This year's working group will draw on the expertise of our current and new participants to focus specifically on the use of CT to enrich K-16 mathematics teaching and learning. We will continue to work toward our overarching goal of more equitable access to authentic mathematical problem solving through computing.

Session Plans and Focus

Session 1 – Are we using mathematics to teach CT or CT to teach mathematics?

In their systematic review on CT-based mathematics instruction and student learning, Ye et al. (2023) highlighted the need for research on curriculum resources and instructional designs aimed at integrating CT into school mathematics beginning from early grades. However, existing research is still in process to unpack what productive integration of CT and mathematics might look like and how the synergies and differences between these disciplines might influence integration. Studies suggest there is a reciprocal relationship between CT and mathematical thinking (Wu & Yang, 2022). Thus, we will explore whether we are using mathematics to teach CT or are we using CT to teach mathematics. To kick-start our inquiry, Ms. Strickland will share her extensive experience in elementary computer science and mathematics integration with a focus on culturally responsive pedagogy and curriculum. Then, small groups based on grade-level interest discuss the overarching questions of session 1.

Session 2 – How have we experienced CT integration in mathematics teaching?

This session will focus on an international, multilevel exploration of existing curricular resources centered on the integration of CT into mathematics education in both the United States and Europe. Participants will contextualize their concepts of interdisciplinary teaching from session 1 as they engage with two presentations. The first presentation focuses on how many European countries have incorporated CT into their mandatory education systems. While some countries, such as England and Poland, have integrated CT skills as part of a separate subject (Bocconi, 2022), the Nordic countries Finland, Sweden, and Norway, have integrated CT and programming into existing subjects, notably in mathematics (Vinnervik & Bungum, 2022). We will discuss a selection of one or two tasks that exemplify aspects of the integration of CT into primary and lower secondary education mathematics. We also will discuss illustrative examples of the opportunities and challenges encountered by teachers when incorporating CT into mathematics. The examples are drawn from a longitudinal evaluation study (Burner et al., 2022).

Unlike the Nordic coordinated integration into school curriculum, there is no unified approach for integrating CT into US postsecondary mathematics classrooms. The second presentation will discuss research on curricular materials integrating computational modeling and linear algebra (Castle, 2022). Participants will critically consider the affordances and constraints of CT integration as described by students, focusing on the power that computation has to potentially disrupt previously held mathematical notions and confront mathematical assumptions. Small groups based on grade-level interest will then discuss the overarching question of session 2 along with potential structural barriers to classroom implementation.

Session 3 – How do we foster learner and teacher identities at the intersection of CT and mathematics education?

The integration of CT into mathematics education necessitates a critical exploration of the potential intertwining of computational identities (e.g., Kafai & Proctor, 2022; Kong & Lai, 2022) and mathematical identities (e.g., Boaler & Greeno, 2000; Graven & Heyd-Metsuyamin, 2019). After primer presentations on both student and teacher identities in CT and mathematics, participants will develop potential theoretical framings for the interaction of computational and mathematical identity. We will consider how identity development can be supported and how it can be interpreted so it becomes an outcome of the learning processes in both K-16 classrooms and teacher education contexts. Specific attention will be given to topics such as avoiding the perpetuation of stereotypes of who belongs in these disciplines and focus on how the integration can bring out positive self and world views for students. We will also explore ways in which mathematics teachers (and mathematics teacher educators) can shape classroom discourse and power dynamics to strengthen CT and mathematics learner identities (Perez, 2018).

Acknowledgments

The first and second authors contributed equally to writing this proposal.

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WORKING GROUP REPORT: THE POWER OF COMPUTATIONAL THINKING IN MATHEMATICS AND DATA SCIENCE EDUCATION

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The "Power of Computational Thinking in Mathematics and Data Science Education" working group held its inaugural meeting at PME-NA 45 in Reno, Nevada. The skills and practices of CT can empower teachers to emphasize abstraction, automation, modeling, and simulations as their students investigate relationships in mathematics and data science. The focus of the three sessions was to advance conversations about the integration of CT in mathematics and DS education with aims to launch new collaborations. Our overarching goal of providing more equitable access to authentic mathematical problem solving through guided the design and facilitation of the working group sessions. Participants experienced three CT-integrated data science tasks on Day 1, created working visuals of the synergies across the disciplines on Day 2, and proposed directions for future research on Day 3.

Keywords: Computational Thinking, Computing and Coding, Data Analysis and Statistics, Modeling

A Brief History and Motivation of the Working Group

This working group journey started at the PME-NA 44 conference in Nashville in 2022. During a session focused on computational thinking (CT) in mathematics education, the founding members (Yilmaz et al., 2023a) embarked on a journey, laying the groundwork for this thematic group. Mr. Alegre and Dr. Yilmaz presented on the transformative integration of mathematical thinking (MT) and computational thinking (CT) within the "Cultural Quilts" coding project. They shared how the coding efforts of high school students not only offered valuable insights into mathematical strategies (e.g., defining functions, using geometric transformations) within their code but also served as a means for students to express their cultural identities and values through uniquely personalized quilt designs (Alegre et al., 2022). Dr. Galanti (2022) presented on how elementary teachers in an online graduate-level CT course engaged in mathematical sensemaking using block-based programming. She shared how teachers modified parameters in a Scratch project to explore the dynamic relationship between changing height and changing volume to create a box with maximum volume from a sheet of paper. Teacher's coding artifacts modeled the reciprocal relationship between CT (algorithmic thinking, abstraction, and automation) and MT (pattern seeking and generalization). These two presentations contribute to the growing knowledge base on the integration of CT and mathematics and its potential to deepen learning in both disciplines (e.g., Brady et al., 2021; Goldenberg et al., 2021; Kallia et al., 2021; Weintrop et al. 2016).

Our shared passion for exploring the integration of CT and mathematics extended informal conversations at the PME-NA 44 conference to more format collaborations. Dr. Lawler invited the presenters from the CT session at PME-NA 44 to co-host a mini-symposium at Kennesaw State University supported by the National Science Foundation (https://research.kennesaw.edu/cistemer/culturally-relevant-integration-cs-mathematics.php) This two-day event was attended by over 30 computer science and mathematics education scholars from diverse backgrounds and contexts. They considered ways in which culturally relevant pedagogy might enhance integrated approaches to learning computation and mathematics.

Furthermore, our discussions extended to wonderings about the role of CT in data science education. We draw on several definitions of data science as an overlap of the skills of multiple disciplines including mathematics, business, statistics, and computer science (Lee et al., 2022). Acknowledging that data science requires not only CT but also MT, we were excited to expand our exploration to the synergies among these three disciplinary domains. Dr. Yilmaz envisioned extending these efforts beyond our one-time symposium event. She facilitated our submission of a thematic group proposal for PME-NA 45 by its founding members. This proposal created a "community focused on advanced conversations about synergies between CT in mathematics and data science education with the aim to launch new collaborations" (Yilmaz et al., 2023a, p. 674). Recognizing the importance of CT as a crucial yet underemphasized aspect of K-16 education in an increasingly technological world, this working group aims to challenge the mathematics education community to advance the teaching and learning of CT within both mathematics and data science.

Progress Made Throughout the PME-NA 45 Working Group

Day 1 Progress and Outcomes

Workshop participants collaboratively engaged in three integrated problem-solving tasks. These tasks, all focused on statistical concepts (measures of central tendency), were selected to provide multiple entry points to our discussion of the synergies among CT, MT, and data science without an assumption of prior knowledge. The discussion questions were designed to stimulate conversations about the ways in which the concepts and practices of CT could be integrated in mathematics and data science teaching.

Task 1 was adapted from a set of freely available elementary CT-integrated mathematics modules (Education Development Center, 2021). Participants used an unplugged CT approach (without a computer) to write algorithms to move two bots to calculate the mean and median of a data set (See Figure 1). They considered the following synergistic questions:

- How does CT contribute to the development of the mathematical concepts of mean, median, and spread in this task?
- How does the context of a mathematics task create opportunities to develop CT skills and practices?
- How do the algorithms developed by the students in this task relate to concepts and techniques commonly used in data science?

Task 2 was an adaptation of a plugged YouCubed K-12 Data Science activity (2020). Participants accessed an open-source Common Data Analysis Platform (CODAP) to explore mean, median, and mode for a mammals data set See Figure 2). They considered the following synergistic questions:

- How does data science contribute to the development of the mathematical concepts of mean, median, and spread in this task?
- How does the context of a data science task create opportunities to develop CT skills and practices?
- What shifts do you see in CT skills and practices used when students transition from Task 1 (algorithmic design as problem-solving) to Task 2 (data exploration and visualization as problem-solving)?

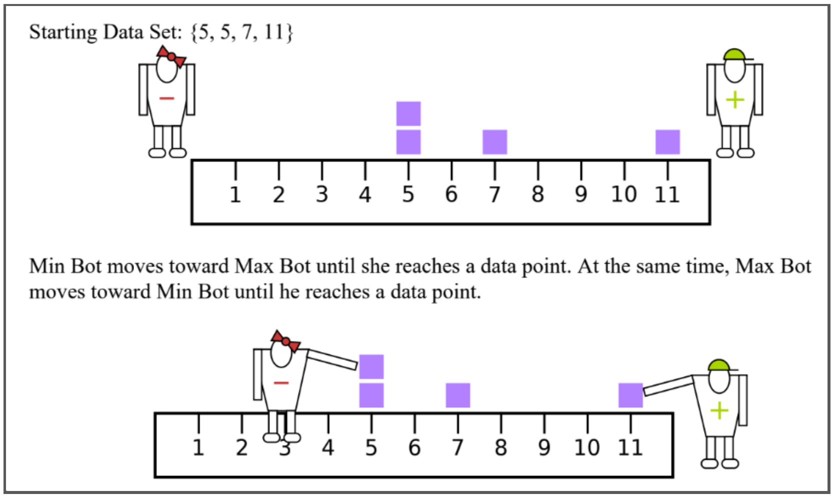

Figure 1: Excerpt from Task 1 - Algorithm Writing to Compute Mean and Median) (Education Development Center, 2021)

Mammals

Mammals (27 cases)

index	Mammal	Order	LifeSpan (years)	Height (meters)	Mass (kg)	Sleep (hours)	Speed (km/h)	Habitat	Diet
1	Africa...	Probosc...	70	4	6400	3	40	land	plants
2	Asian...	Probosc...	70	3	5000	4	40	land	plants
3	Big B...	Chiropt...	19	0.1	0.02	20	40	land	meat
4	Bottl...	Cetacea	25	3.5	635	5	37	water	meat
5	Chee...	Carnivora	14	1.5	50	12	110	land	meat
6	Chim...	Primate	40	1.5	68	10		land	both
7	Dom...	Carnivora	16	0.8	4.5	12	50	land	meat
8	Donk...	Perisso...	40	1.2	187	3	50	land	plants

Figure 2: Excerpt from Task 2 - Using CODAP to Explore Attributes of Mammal Data Set

Task 3 used a simple computer program (Figure 3) to explore the interactions of CT and MT in block-based programming approaches to calculating the mean of a set of numbers. Participants had the opportunity to modify code to compare and contrast two approaches to a statistical calculation: 1) the use of a single variable to accumulate the sum of numbers as they are entered by the user and divided by the number of entries; and 2) the use of a list to first store all entered values before calculating the mean by iterating over a list of numerical entries.

All three of these tasks were intended to highlight abstraction, decomposition, pattern recognition, algorithmic thinking, logical thinking, modeling, and automation as productive aspects of CT within mathematics education (Kallia et al., 2021). The participants were able to draw upon these different problem-solving experiences within a data science context to think about how dynamic computer models and the underlying algorithmic thinking can support generalization and evaluation in problem solving.

Figure 3: Excerpt from Task 3 (Using a Block-based Scratch Computer Program to Calculate Mean)

Day 2 Progress and Outcomes

The three tasks on Day 1 provided a shared experience from which to elicit participants' own teaching and learning experiences with CT, MT, and data science. After examining a series of published visual conceptualizations of relationships between the three disciplines (e.g., Lee et al., 2022; Sneider et al., 2014), the participants worked in small groups to create their own visuals (see Figure 4 for examples). To contextualize these visualizations of disciplinary relationships, Mr. Alegre and Dr. Lawler presented research that challenges us as mathematics education researchers to reflect on whose ways of knowing are valued in CS and mathematics education. Mr. Alegre described a research practice partnership between Louisiana State University and secondary teachers to integrate CS in mathematics. Dr. Lawler shared work in after school programs designed to engage middle grades students in imagination, creativity, reasoning, and discourse through integrated coding and mathematics experiences. These presenters encouraged conversations about the need to draw upon teachers' and students' cultural assets in considering future directions for disciplinary integration.

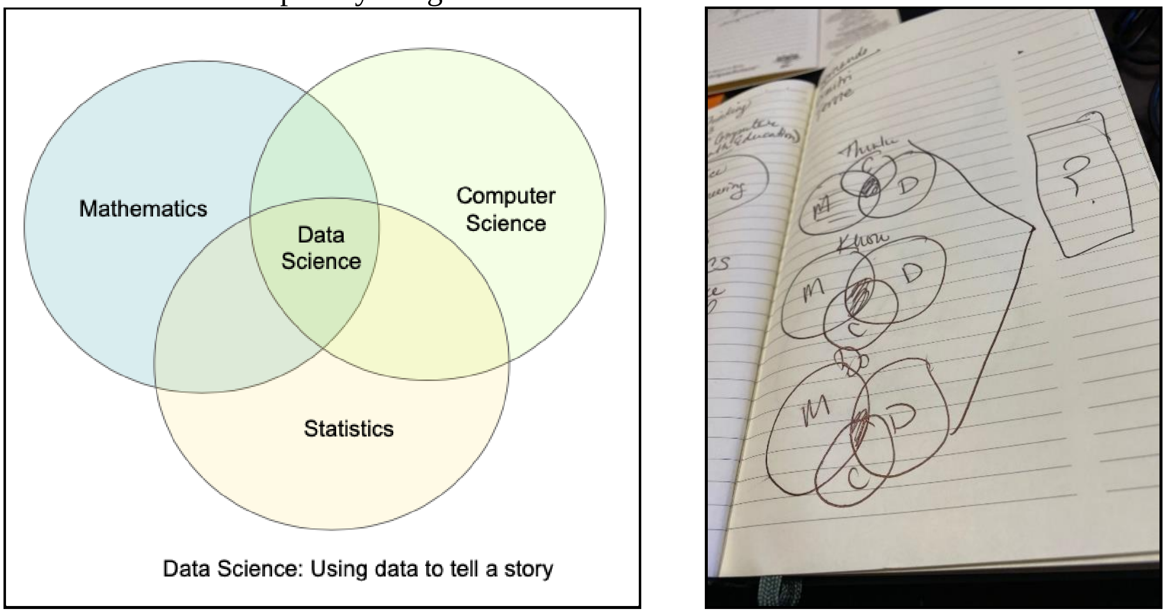

Figure 3: Participant-created visualizations of CT, MT, and Data Science

Day 3 Progress and Outcomes

The final session started with participants' reflection on the visuals created on Day 2 and organic conversation across two main areas: 1) Computational Thinking: Knowing, Doing, Thinking and 2) Equity and Access: Practical Applications of Computation and Mathematics in Schools.

Computational Thinking: Knowing, Doing, Thinking.

The participants delved one of the visuals created on Day 3 (See Figure 3 above). This visual emphasized the three aspects of computational integration as knowing, doing and thinking. They suggest computational knowing could include understanding algorithms and data structures and computational doing could include coding. They argued that both knowing and doing are both essential for CT. They also discussed concrete examples on the misconception that learning to code with an application such as Scratch (a visual block-based programming language) equates to CT. They shared the need for a broader conceptual understanding of what CT encompasses. The participants generated two questions for inquiry and reflection.

- Is computational doing limited to coding, or are there other ways to enact CT?
- What constitutes computational knowing, thinking and how can it be effectively taught and assessed in mathematics and data science education?

Examination of various visuals from Day 1 and Day 2 motivated participants to consider the benefit of creating a unified visual showing the synergies between CT, MT, and data science. They suggested such a visual could support educators, policymakers, and practitioners in conceptualizing the varied approaches to integrating computation in K-16 mathematics and data science education and identifying areas for cross-disciplinary collaborations.

The participants acknowledged the challenges in creating a unified visual. One significant challenge is the establishment of common language and frameworks across disciplines. Each discipline may have its own terminology, methodologies, and epistemologies, making it challenging to develop a shared understanding. For instance, while mathematicians may approach problem-solving through abstraction and proof, computer scientists may focus on algorithms and computational complexity. Bridging these diverse perspectives requires careful negotiation and collaboration among experts from different disciplines. Additionally, the rapid evolution of technology and research methods further complicates efforts to create a unified representation that remains relevant over time.

Equity and Access: Practical Applications of Computation and Mathematics in Schools

The participants discussed how mathematics could serve as a pathway to increase access to computer science and related fields, potentially addressing issues of equity. Unlike elective courses like computer science, which may have prerequisites or limited availability, mathematics is a core subject taught to all students. One of the participants described how some students, particularly those underperforming in traditional subjects like mathematics and English, may be denied access to elective courses like computer science. This lack of access denies them the opportunity to explore practical applications in which they can use both CT and MT. Concrete applications of CT and MT skills, such as pattern making and measurement in fields like interior or fashion design, could improve students' understanding of mathematical concepts. Similarly, engaging students in computational activities rooted in mathematics might not only enhance their computational skills but also deepen their mathematical understanding. By incorporating CT into mathematics, schools can provide students with opportunities to explore real-world problem-solving and creative expression, potentially empowering students who might otherwise struggle with traditional math instruction.

Consideration of future research on equity and access to CT within K-12 mathematics and data science education settings yielded two questions for further inquiry. The first question was “Can students succeed in computer science courses without first meeting traditional grade-level math standards?” This question delves into providing alternative pathways or interventions to support more diversity in engagement with computer science in K-12 settings. The second question was “How can we as mathematics educators promote equitable access to CT in K-12 settings, particularly for students from marginalized backgrounds?” This question underscores the need to address systemic barriers and biases that may hinder students participation in CT programs.

Progress Made Following the October 2023 Working Group

Our working group convened virtually in November 2023 and in January 2024 to continue to discuss directions for research and collaboration at the intersection of CT, MT, and data science. The participants collapsed our working list of 13 researchable topics generated on Day 3 of the October meeting to a set of seven topics for collaborative research (See Table 1). The attendees also considered methodologies to support exploration of these topics. A literature review seemed well aligned with Topic 1, while an investigation of existing curricula seemed to be a productive entry point for Topic 4. Discussions of potential research centered on the stakeholders in disciplinary integration, specifically students and classroom educators. The ways in which students and educators experience the integration of CT and MT was of continuing interest, with multiple theoretical frameworks (e.g. identity, content knowledge and pedagogical content knowledge) and mathematical domains (e.g., geometry, algebra, and number) guiding the conversations.

Our working group has continued its effort to define a near-term collaborative project by reading a 2023 systematic literature review on the integration of CT in K-12 mathematics education with a focus on instruction and learning (Ye et al., 2023). The central question of this literature review was, "How do students' CT and mathematics learning interact when they are involved in CT-based mathematics instruction, and what are the consequences of such interactions?" (p. 13). Participants met virtually in January 2024 to share their key takeaways and their key wonderings based upon their reading. The following questions emerged:

- Ng & Cui (2021) offered the descriptor of "computationally-enhanced mathematics education". Would this description attract more interest in innovating traditional school curriculum with mathematics as a standalone subject?.
- There was no mention of the role that artificial intelligence or machine learning could play in the teaching and learning of CT and MT. Should we expand our focus to these potential integrations?
- Dick and Hollebrands (2011) define a mathematical action technology that can "perform mathematical tasks and/or respond to the user's actions in mathematically defined ways" (p. xii). How does computer programming fit within this definition?
- Dynamic geometry software applications are mathematical action technologies. What CT skills and practices are teachers engaging in with students using dynamic algebra and geometry software applications?

Table 1: Directions for Collaborative MT-CT- Data Science Research

Direction for Collaborative Research	Number of Interested Researchers
Synergies between MT, CT, and Data Science	9
Integration of MT, CT, and Data Science in K-12 Education: Curriculum, Implementation, and Challenges	6
Integration of MT, CT, and DS: Effect on Underserved Students	3
Affordances of Data-Driven Math Curricula	2
Teaching Mathematics, Data Science, and Computer Science Simultaneously	4
STEM Teachers as "Integrators" of computational thinking	3
STEM Parenting Identity	1

Ye et al. (2023) stated that "future research on teacher professional development concerning the emergent competency of CT-based mathematical thinking is indispensable" (p. 24), but research on preservice and practicing teachers was an exclusion criterion for their literature review. We are tentatively working toward a collaborative literature review on preparing pre-service and practicing teachers to integrate CT in mathematics instruction.

Conclusions and Looking Ahead

Based on existing literature that explores the connections between CT and MT (e.g. Brating & Kilham, 2021; Hickmott et al., 2021) , as well as the integration of CT into mathematics as a school subject (e.g., Chan et al., 2023; Rich et al., 2020), our group is motivated to delve deeper into identified gaps, needs and challenges in these studies. Limited empirical studies have explored the synergies between CT and MT (Hickmott et al., 2018). Only a few studies have incorporated the expertise of mathematics educators to explore the integration of CT in mathematics (Hickmott et al., 2018; Kallia et al., 2021).

Another identified gap in the research is evidence of what resources (e.g., knowledge, curricular materials, tools) teachers need to effectively integrate CT into their mathematics teaching (Wu et al., 2021; Yadav et al., 2016) and how to assess teacher CT learning (Galanti & Baker, 2023). These gaps highlight the need for more professional learning opportunities for teachers. However, only a limited number of PD studies (e.g. Ahamed et al., 2010; Hart et al., 2008; Wu et al., 2021; Yilmaz et al., 2023b) have focused on the integration of CT and MT. This indicates a need for more tailored PD initiatives designed through interdisciplinary collaboration of mathematics and CS educators (Dahshan & Galanti, in press; Menekşe, 2015).

Our group is also interested in how the integration of CT in mathematics education and data science education in fostering teacher and student problem-solving identities. By looking across the literature in CT identity (Kong & Lai, 2022) and use of coding to foster creativity in mathematical thinking (Castle, 2023), we seek to understand how CT integration can build an individual sense of self as a “doer” of mathematics or data science.

As we continue engaging in generative inquiry and collaborative research, our thematic group is committed to seeking ways to address these gaps and needs and to broaden interest in this work in the PME-NA community.

Acknowledgments

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Chapter 17:
Plenary Papers

THE CHALLENGES OF AI IN SHAPING MATHEMATICAL WORK: FROM HUMAN HYBRIDIZATION TO AUTOMATION THROUGH SYNERGIES OF SYMBOLIC AI AND GENERATIVE MODELS

LES DÉFIS DE L'IA DANS LE FAÇONNEMENT DU TRAVAIL MATHÉMATIQUE : DE L'HYBRIDATION HUMAINE À L'AUTOMATISATION GRÂCE AUX SYNERGIES ENTRE L'IA SYMBOLIQUE ET LES MODÈLES GÉNÉRATIFS

LOS RETOS DE LAS IA EN LA CONFIGURACIÓN DEL TRABAJO MATEMÁTICO: DE LA HIBRIDACIÓN HUMANA A LA AUTOMATIZACIÓN GRACIAS A LAS SINERGIAS ENTRE LAS IA SIMBÓLICAS Y LOS MODELOS GENERATIVOS

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This text explores the influence of artificial intelligence (AI) and technology in shaping mathematical work in educational contexts, with a focus on human-machine interaction dynamics. It addresses key aspects such as symbolic and statistical AI, digital artifacts, and the role of hybridization as a technological counterpoint to overcome current limitations. By presenting practical examples, it demonstrates how technology creates new forms of control, necessities, and challenges in mathematics education. Although large language models possess extensive knowledge, their limitations in performing genuine mathematical reasoning remain, highlighting the need for innovative approaches to carry out the new mathematical work.

Introduction

The integration of artificial intelligence (AI) and technology into mathematics education presents both opportunities and challenges in reshaping mathematical work. While the development and application of symbolic and statistical AI, along with digital tools, have become increasingly common in mathematical research, many educators and mathematicians remain hesitant to fully embrace these innovations. This hesitation is partly rooted in a long-standing belief that mathematics is closely intertwined with language—not only as an expression of mathematical thought but as a core component of mathematical activity itself. Is mathematics not a science, and do the mathematical sciences not trace their origins to history, with the expression of writing as a fundamentally human characteristic? Consequently, the notion of delegating this inherently human endeavour to a machine—often perceived as a "black box" with opaque inner workings—sparks concerns about relinquishing control over the mathematical process. Moreover, the concept of reasoning, central to mathematical work, calls for reexamination in this evolving context.

In contrast, the evolving reality of the modern classroom, where students interact daily with powerful digital tools and AI technologies, necessitates a shift in how mathematics is taught. Today's learners, who have grown up in a world increasingly shaped by digital artifacts, are not only familiar with these tools but rely on them as part of their everyday lives. This trend has made it imperative for mathematics education (ME) to adapt to these new technological realities.

However, this adaptation often occurs without a foundation in what might be called instrumented mathematics, or a clear epistemological reference or long-term understanding of the cognitive issues at stake with digital artifacts. This gap makes the theoretical and practical issues surrounding mathematics even more complex.

This paper examines these dynamics by considering the role of AI, particularly through the synergy between symbolic AI and generative models in transforming the nature of mathematical work. It explores how hybridizations, or technological counterpoints—both in the integration of symbolic and statistical AI and in human-AI collaboration—may provide a potential solution to some of the limitations currently faced in the field. By drawing on concepts such as augmented intelligence, idoneity, and the challenges posed by "black box effects," this work seeks to provide a comprehensive understanding of the shifting landscape of the new mathematical work in the context of fast technological advancements.

Our Focus: Traditional and New Mathematical Work

Mathematical Work

Mathematical work refers to the activities, processes, and approaches involved in practising mathematics, whether by students, teachers, or professionals in the field. It encompasses various dimensions, including problem-solving, modelling, demonstrations and reasoning, representations and communication, as well as knowledge, procedures, and attitudes.

In a more formal framework, mathematical work includes reflection on the nature of mathematics itself, on how it evolves, and on how it relates to other disciplines. In the theory of mathematical working space (ThMWS), mathematical work is progressively constructed as a process that bridges epistemological and cognitive aspects through three intertwined genetic developments, identified in the theory as semiotic, instrumental, and discursive genesis (Kuzniak, Montoya, & Richard, 2022). Beyond didactic transposition, the concept of mathematical work stimulates epistemological awareness in educational programs.

New Mathematical Work

The integration of digital artifacts into mathematical activity goes beyond merely facilitating or accelerating tasks; it transforms the nature of mathematical work itself by opening new pathways for problem-solving, exploration, and learning. The human-machine interaction resulting from this integration produces new forms of mathematical activity (Bruillard & Richard, 2024), including the dynamic interplay of representations, automated calculations and reasoning, exploratory conjecture, instant feedback, and the cognitive impact of artifact interactions.

This new mathematical work, shaped by human-digital collaboration, is characterized by automation, dynamic visualization, and interactive feedback. These elements redefine mathematical practices by weaving together semiotic, instrumental, and discursive dimensions—creating a complex "dance of genesis" that brings new depth to learning and problem-solving (Flores Salazar, Gaona, & Richard, 2022).

The effects of this evolving mathematical work are wide-ranging, influencing all aspects of mathematical practice, from problem-solving and modelling to proofs and reasoning. In reasoning, specifically, studies such as Richard, Venant, and Gagnon (2019) show how instrumental proofs and technology-assisted reasoning can already serve as valuable models for future exploration.

Underlying References

Aside from the theory of symbolic mathematical working spaces (ThMWS), our discourse also draws on two main references. First, the book *Mathematics* Education *in the Age of Artificial Intelligence: How Artificial Intelligence Can Serve Mathematical Human Learning* (Richard, Vélez, & Van Vaerenbergh, 2022) underscores the contributions of artificial intelligence to mathematics education, presenting concrete ideas grounded in mathematical work developed through dynamic international collaboration. This book further addresses the evolution of new mathematics in the contemporary world. Its themes and sections explore the creation of AI-enhanced learning environments for mathematics, AI-supported mathematics learning, and the integration of traditional paper-and-pencil techniques with new AI-aided educational working spaces.

The second key reference, the article *Artificial Intelligence and the Didactics of Mathematics: Current Situation and Issues* (Emprin & Richard, 2023), delves into the complex interplay between artificial intelligence (AI) and mathematics education (ME), particularly timely given AI's profound impact on society and the economy. Initially, the article questions the concept of "intelligence" itself, examining its definitions and the biases it may invoke when applied to AI. This foundational exploration paves the way for an analysis of potential links between AI and the didactics of mathematics, illustrated through examples of current projects in the Francophone world that offer insight into actively developed areas. The article then discusses theoretical frameworks in mathematics education and their integration with AI. Finally, it addresses critical questions and challenges that arise from AI usage, presenting promising perspectives for future developments.

Artificial Intelligence (AI)

From Foundations to Collaborative and User-Centric AI Design

The historical and inherent connection between artificial intelligence and the didactics of mathematics is well established. Notably, the foundational references to this relationship were published nearly 30 years ago, with the work *Didactique et intelligence artificielle* (Balacheff, 1994). The previous underlying references by Richard, Vélez, and Van Vaerenbergh (2022), and Emprin and Richard (2023), coincided with the rapid rise of public interest in ChatGPT and generative AI models. Indeed, ChatGPT gained widespread recognition in November 2022, when OpenAI released an accessible version based on the GPT-3.5 model. Its user-friendly interface and advanced conversational capabilities quickly captured the attention of users, media, and businesses alike, generating significant interest in AI applications across various fields. This surge in popularity has often led to a conflation of AI with generative models, overshadowing the foundational contributions of diverse research approaches in the field of AI (as discussed further below). Consequently, these recent publications have become essential references for understanding the challenges of new mathematical work, particularly concerning numerical and symbolic approaches.

Balacheff's (1994) work highlighted advances in AI and their impact on the development of computer environments for human learning. This initial period of enthusiasm was, however, followed by a phase of disillusionment, during which the challenges involved were underestimated. Today, paradoxically, AI appears to be converging once again with didactics by offering approaches focused on solving non-routine problems, even before generative models began attracting significant attention. These approaches integrate phases of learning, modelling, and prediction that evoke both mathematical work and the expertise required to devise solutions. For AI to effectively contribute to academic success and assist teachers in monitoring student progress, any collaboration between a teacher and a tutoring system must involve a well-informed understanding of didactic culture. The tutor must recognize the specific demands of teacher-student knowledge interactions and support these dynamics alongside human reasoning competencies. The system should adapt to human needs, rather than the reverse, to avoid reducing these considerations to mere instrumental issues.

The integration of "machine thinking" and "human thinking" through artificial intelligence requires collaborative input that goes beyond computer scientists alone. Given the fundamental, shared concept that mathematics is pursued through seeking and solving specific problems—and thus continually posing new questions—each problem presents an opportunity to deepen understanding of both mathematics and reality. When students take ownership of this process, problem-solving and modelling provide insights into the nature of learning itself—whether in moments of blockage, overcoming obstacles, or successfully explaining and clarifying complex situations. As AI already proves valuable with traditional techniques (Lagrange, Richard, Vélez, & Van Vaerenbergh, 2023), its potential growth is imaginable through collaborative development involving both didactics and computer engineering, adopting a perspective of new mathematical work. Above all, AI could thrive by incorporating user involvement early in the design process, embracing the principle of "design-in-use" for the learning of mathematics.

Classical Perspectives on AI Research Approaches and Beyond

Artificial intelligence (AI) has long raised fundamental questions about its nature. In his preface titled *AI for the Learning of Mathematics*, Balacheff (2022) explains:

> It would have been good to have a precise and clear definition of Artificial Intelligence (AI) unanimously accepted. Unfortunately, it is not the case today as it was not the case formerly. The common criterion, AI is a property of machines "exhibiting certain behaviours which strikes as intelligent," reminds us that this is a judgement underpinned by a kind of human empathy. Looking closer, it appears that such a judgement assesses both the task which has been achieved by the machine and the way it has been achieved: a behaviour is striking because the task is acknowledged complex and/or the way in which it has been achieved looks smart.

Is it simply a "machine story," a modelled extension of human capabilities, or a human-machine collaboration that can take the form of both partner and adversary? While all three perspectives are possible, it is certain that the idea of collaborative interaction aligns more closely with the concept of mathematical work. Drawing on the classical perspective—and a bit beyond—we will initially explore contemporary issues related to AI in mathematics education.

AI is traditionally divided into two dichotomous approaches, as illustrated by the OECD report (2019) and studies by the Massachusetts Institute of Technology (MIT) (Fig. 1). Generally, symbolic approaches are based on rules and symbols, striving to model human thought through logical processes that generate truth from truth. While powerful for certain tasks, symbolic AI is limited by its inability to handle uncertainty and the complexity of real-world situations. It stands out for the transparency of its decisions due to explicit rules, its precision in logical reasoning, its formal demonstrations, and its resolution of symbolic equations. It is well suited to geometric constructions for exploring properties and performing visual proofs. It is effective in symbolic analysis to solve integrals, derivatives, and limits formally, without numerical approximation. Finally, it is useful in the symbolic treatment of statistical data to model distributions or establish theoretical correlations. However, it remains rigid when faced with new data, requiring manual updates or adaptations. It is difficult to adapt to uncertain data or unexpected variations, and generally, it is less effective at learning from large datasets.

Statistical approaches to AI are based on data analysis to establish correlations and make predictions. They use techniques like machine learning and neural networks, allowing AI to recognize patterns and learn from large datasets. Unlike symbolic AI, statistical AI effectively handles uncertainty and unexpected variations, adapting its models autonomously as new data becomes available. It is particularly effective for predictive tasks, time-series analysis, and pattern recognition in complex data. In mathematics, it excels in the approximate resolution of numerical problems, such as statistical simulations and probabilistic modelling. However, it lacks transparency in its decisions (the "black box" effect) and does not provide formal demonstrations. Its results are often approximations, and despite its flexibility, it may lack precision in contexts requiring rigorous logic or explicit proofs.

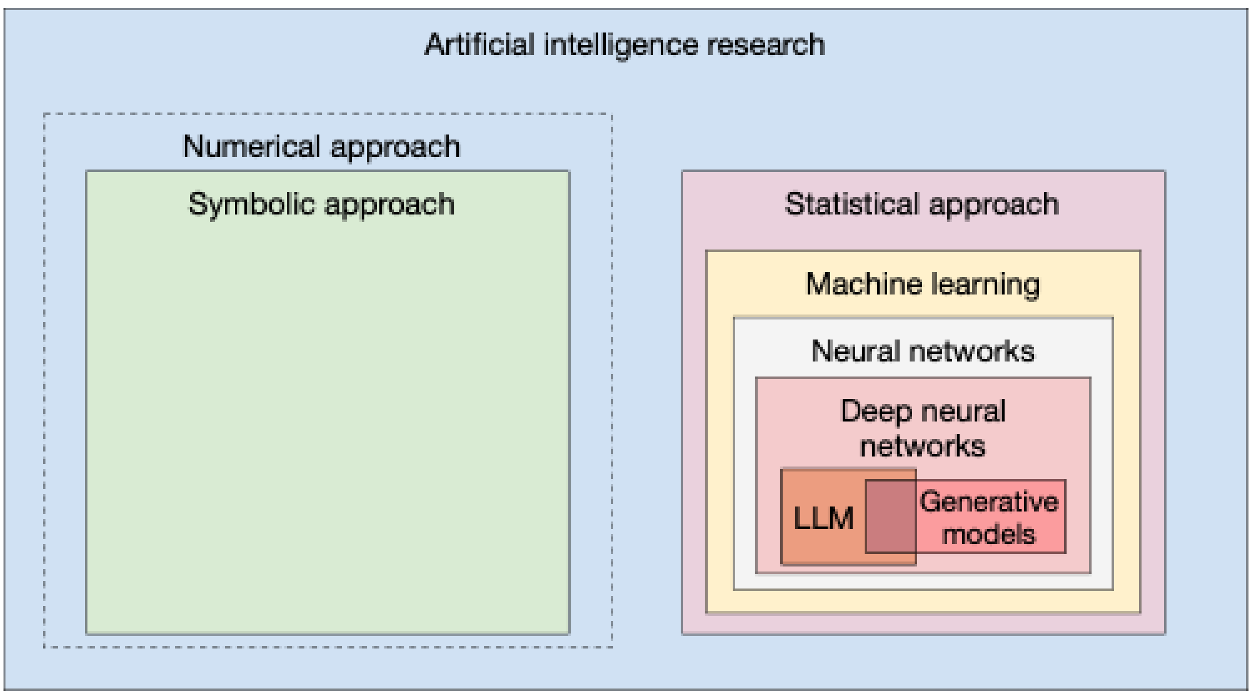


Figure 1: Adaptation of the original diagram in OECD, in which we added the "numerical approach" in dotted lines around the symbolic approach, and the "LLMs" and "generative models" in the deep neural networks.

As shown in Fig. 1, while there is no doubt that “LLMs” and “generative models” fall under the category of deep neural networks, it is less certain whether numerical approaches belong to AI for mathematical work. The answer may be yes, as numerical approaches seem essential for AI in mathematics. Indeed, statistical AI and large language models often struggle to achieve the precision needed for rigorous mathematical tasks, making them less suitable for high-level mathematical applications. For example, while symbolic approaches are effective for derivable functions, they fall short when integrating functions without primitives, which limits their scope. Additionally, numerical methods are crucial for solving complex and nonlinear differential equations, expanding the potential applications of AI in mathematics. Thus, without numerical models, AI’s capacity in mathematics is constrained. With them, AI gains a broader, more robust range of applications, making numerical approaches integral to mathematical AI.

Technological Counterpoint

In the title of our presentation, we used the term *hybridization* due to its familiarity. However, we need a term that operates on two levels: first, it should convey the integration of symbolic and statistical AI, representing a novel technological synthesis; second, it should capture the emerging interaction between humans and AI as a complex, cohesive system, like an instrumented human or augmented intelligence. While *hybridization* commonly denotes the crossbreeding of species—as seen in the historical interbreeding between humans and Neanderthals—it might not fully represent our intent in the context of human-AI interaction. Viewing the learner in mathematics as a “cyberborg” similarly implies an identity, we prefer to avoid.

The term *technological synergy* could also be considered, as it suggests a harmonious collaboration where distinct elements combine to create a greater force. However, *synergy* often implies a fusion or productive integration between humans and technology, which does not accurately reflect the nuanced, non-fusional interaction we aim to describe.

Instead, we propose the term *technological counterpoint*. Borrowed from music, counterpoint evokes a contrast or constructive opposition, presenting technology and human intelligence as distinct, parallel perspectives that mutually critique and enrich one another, creating a subtle dynamic tension. For describing human-AI interaction, *counterpoint* captures the idea of two entities acting in concert while preserving their individuality. Similarly, in the case of symbolic and statistical approaches, counterpoint aptly reflects two distinct currents that coexist and enrich the field of AI in a complementary, yet non-blending, manner.

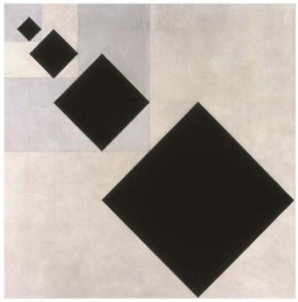
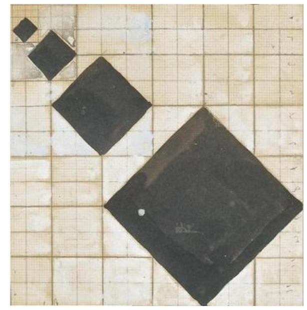
 

Figure 2: The pinnacle of concrete art is exemplified in *Arithmetic Composition* by Theo van Doesburg (1929–1930). Black, white, and gray squares, repeated with varying dimensions and intervals according to a geometric progression, merge space and time. The artwork is on the left, and its study, on 12 cm graph paper, is on the right.

Human "Hybridization"

To assess how AI influences learners' engagement in problem-solving, we developed an activity that fosters mathematical exploration by incorporating critical thinking and argumentative skills. The study is framed by a historical context: in the late 1920s, Dutch painter, architect, and art theorist Theo van Doesburg created his *Arithmetic Composition* (Fig. 2), at a time when mathematical character evidently took precedence over impressionism. Drawing from the artist's approach, we designed a structured activity focused on exploring mathematical necessity within the artwork. This activity employs a guiding questionnaire and involves trainer interventions in sessions for teachers and researchers in mathematics education. It emphasizes critical reflection, in-depth analysis, and creative problem-solving, while allowing participants freedom in their use of technological tools.

For the design of the activity, we incorporated findings and expertise gained from previous research in teacher training. The first area involves task definition by teachers and the establishment of hypothetical learning progressions within the context of *Arithmetic Composition* (García-Honrado, Clemente, Vanegas, Badillo, & Fortuny, 2018). The second area introduces an instrumented approach using GeoGebra's automated reasoning tools to support the discovery, derivation, and demonstration of mathematical properties (Kovács, Recio, Vélez, 2021). Initial analysis results are presented in García-Honrado, Fortuny-Aymemi, Recio, & Richard (2024), and the complete questionnaire is included in the appendix.

The questionnaire is structured around two exercises, each designed to immerse students in training in distinct aspects of AI-supported mathematical exploration and problem-solving. The first exercise emphasizes qualitative analysis of Theo van Doesburg's *Arithmetic Composition* (Figure 2) by engaging primarily with statistical AI, including advanced generative language models such as ChatGPT (developed by OpenAI) and Gemini (developed by Google DeepMind), alongside conventional tools like Google and Bing search engines. This exercise invites participants to examine the visible regularities and recursive patterns within the artwork, fostering critical observation skills that AI tools can supplement. For instance, students in training are guided to use ChatGPT to generate hypotheses about the mathematical properties embedded in the artwork, then compare these with factual data sourced from Google. This blend of AI tools allows participants to evaluate the distinct contributions of statistical AI and search engines, providing insight into how generative models can support exploratory reasoning within a mathematical and artistic context.

The second exercise shifts focus toward symbolic AI and traditional paper-pencil methods, encouraging participants to analyze the geometric structures in *Arithmetic Composition* through the automated reasoning capabilities of GeoGebra Discovery. As outlined in Figure 1, this exercise exemplifies the symbolic AI approach within the broader AI landscape, enabling the exploration of recursive geometric sequences and Thales' configurations (Exercise 2) using GeoGebra's interactive geometry modules. Students in training are tasked with verifying the alignment of square vertices and investigating proportional relationships, such as the scaling ratios and the positioning of elements in Thales' configurations. These tasks highlight how symbolic AI, unlike statistical AI, provides precise calculations and direct verification of mathematical properties.

The questionnaire embodies a dual "hybrid" approach through a structured *technological counterpoint*, bridging human-AI and symbolic-statistical AI paradigms to enhance mathematical exploration and validation. This approach aligns with the concept of *new mathematical work*, where digital artifacts transform traditional mathematical practices by enabling new forms of engagement. As illustrated in Figure 1, which depicts symbolic, statistical, and generative AI models, students in training are immersed in both the generative, language-based reasoning of models like ChatGPT and Gemini, and precision-oriented symbolic tools such as GeoGebra Discovery. This interplay fosters *augmented intelligence*, whereby human cognitive capacities are amplified through interaction with digital tools, supporting deeper exploration of mathematical concepts and enhanced problem-solving processes. Within this framework, the concept of *idoneity* becomes central (Emprin & Richard, 2023): through iterative questioning, students engage in a recursive adjustment process between their problem-solving intentions and the digital responses provided by AI tools, leading to an evolving alignment between human initiative and the technological affordances of AI. In their written answers, students also reveal the outcomes of their decisions, indicating whether the AI tool served as a partner, an adversary, or introduced new and surprising challenges in the process. This approach not only diversifies problem-solving methods but also cultivates critical thinking and argumentative skills, equipping students with a nuanced understanding of AI's role in mathematical inquiry and fostering an adaptable, optimal integration of AI into pedagogical practices.

Technological "Hybridization"

When examining the intentions of AI system designers, the concept of technological counterpoint highlights the automated integration between symbolic and statistical AI. A notable example is the neurosymbolic approach implemented by OpenAI and Wolfram in ChatGPT. In the following, we aim to compare ChatGPT's standalone performance with that of this type of "hybridized" system.

Let's be clear: when doing mathematics with ChatGPT, it can be misleading. It gives the illusion of performing genuine mathematical work, but this is not truly the case. For example, if we pose the following problem to the system:

Prove that a quadrilateral with three right angles is a rectangle, ChatGPT provides a solution (Fig. 3, left) derived from its training data. When we repeat the question immediately afterward (Fig. 3, right), its response changes because it now uses the prior context (our first question and its solution) as additional contextual input. This creates a form of learning by the system, where contextual inferences help maintain continuity in the conversation.

To prove that a quadrilateral with three right angles is a rectangle, let's use the properties of quadrilaterals and their internal angles.

Step 1: Recall the properties of a quadrilateral's angles

A quadrilateral is a closed figure with four sides, and the sum of its internal angles is always 360°.

Step 2: Hypothesis

Let's assume that quadrilateral $ABCD$ has three right angles:

- $\angle A = 90^\circ$
- $\angle B = 90^\circ$
- $\angle C = 90^\circ$

Step 3: Calculate the missing angle

Since the sum of the angles in a quadrilateral is 360°, the angle $\angle D$ can be calculated as follows:

$$\angle D = 360^\circ - (\angle A + \angle B + \angle C) = 360^\circ - (90^\circ + 90^\circ + 90^\circ) = 90^\circ$$

Thus, $\angle D = 90^\circ$.

Step 4: Conclusion

We have shown that all four angles of the quadrilateral $ABCD$ are 90°. By definition, a quadrilateral with four right angles is a **rectangle**.

Final Conclusion

A quadrilateral with three right angles must be a rectangle because the fourth angle also has to be a right angle to make the sum of the angles 360°.

To demonstrate that a quadrilateral with three right angles is a rectangle, we will proceed with the following steps:

Statement

Consider a quadrilateral $ABCD$ such that angles $\angle A$, $\angle B$, and $\angle C$ are right angles. We need to prove that $ABCD$ is a rectangle.

Proof

1. **Properties of the angles:**
   - By assumption, we know that:
   $$\angle A = 90^\circ, \quad \angle B = 90^\circ, \quad \angle C = 90^\circ.$$
2. **Calculating angle D:**
   - The sum of the angles in a quadrilateral is 360°. Therefore, we can write:
   $$\angle A + \angle B + \angle C + \angle D = 360^\circ.$$
   - Substituting the known values, we get:
   $$90^\circ + 90^\circ + 90^\circ + \angle D = 360^\circ.$$
   - This implies:
   $$270^\circ + \angle D = 360^\circ \quad \Rightarrow \quad \angle D = 360^\circ - 270^\circ = 90^\circ.$$
3. **Conclusion about the angles:**
   - We have shown that:
   $$\angle D = 90^\circ.$$
   - Thus, all angles of the quadrilateral $ABCD$ are right angles.
4. **Properties of the sides:**
   - In a quadrilateral with all right angles, opposite sides are parallel and equal in length. Indeed, by definition, a quadrilateral is a rectangle if and only if it has four right angles and opposite sides that are parallel.
   - Since we have already proven that $ABCD$ has four right angles, we can conclude that $ABCD$ is a rectangle.

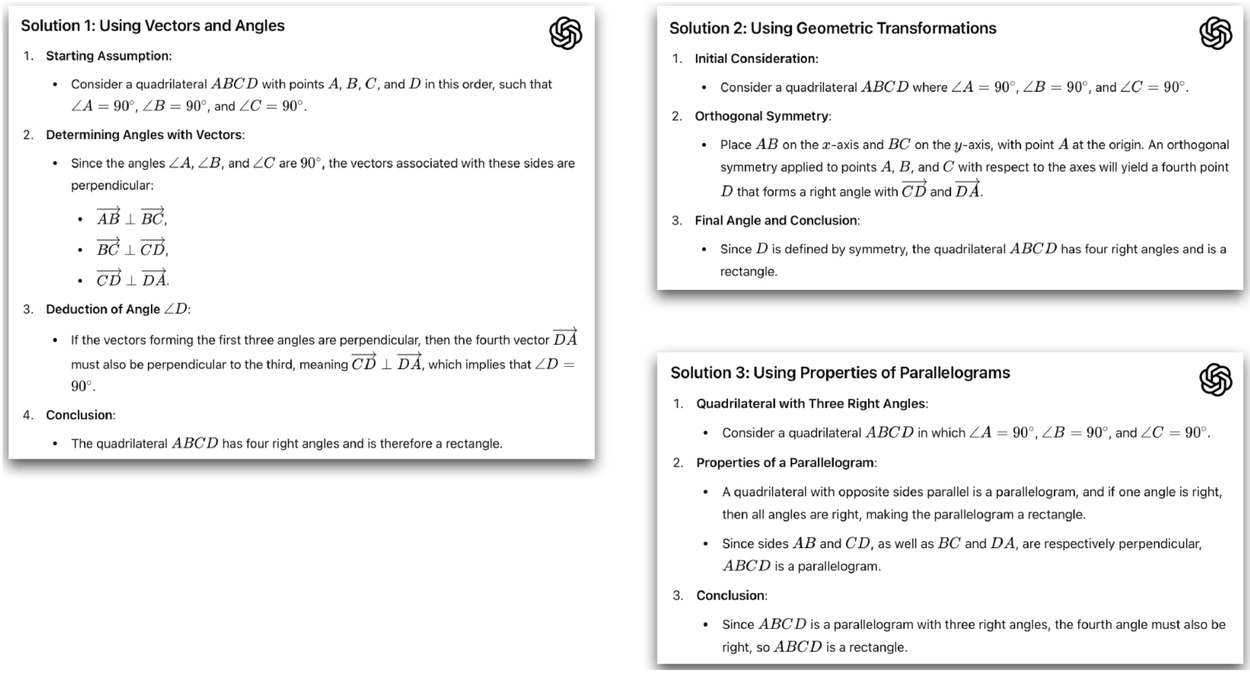
Conclusion

The quadrilateral $ABCD$ with three right angles (and thus four right angles) is a rectangle.

Therefore, we have shown that any quadrilateral with three right angles is necessarily a rectangle.

Figure 3: First and second requests to ChatGPT for the rectangle problem.

Although the problem itself is relatively simple, ChatGPT's responses reveal two key aspects. In the first instance (first request), the solution appears well structured, almost textbook-like, with mathematical elements (angles, the property of the sum of a quadrilateral's angles, the equation to calculate the missing angle, the definition of a rectangle, etc.) and metamathematics elements (initial hypothesis, recall of geometric properties, logical steps in reasoning, "final conclusion" as a general reasoning) that seem appropriate for the problem. However, when the question is repeated (second request), instead of confirming its initial solution or exploring alternatives, ChatGPT adds contextual arguments that, while seems true, are unnecessary. This behaviour resembles the *comportement heuristique* observed in mathematics education research, where students employ heuristic strategies while attempting to solve proof problems (Richard, 2004) or when they "say everything they know" on an exam to gain points. When explicitly asked for three different solutions (Figure 4), ChatGPT generates responses that use vectors and angles, geometric transformations, and parallelogram properties—an impressive variety, but how reliable are these?



Solution 1: Using Vectors and Angles

1. **Starting Assumption:**
   - Consider a quadrilateral $ABCD$ with points A, B, C, and D in this order, such that $\angle A = 90^\circ$, $\angle B = 90^\circ$, and $\angle C = 90^\circ$.
2. **Determining Angles with Vectors:**
   - Since the angles $\angle A$, $\angle B$, and $\angle C$ are 90°, the vectors associated with these sides are perpendicular:
     - $\overrightarrow{AB} \perp \overrightarrow{BC}$,
     - $\overrightarrow{BC} \perp \overrightarrow{CD}$,
     - $\overrightarrow{CD} \perp \overrightarrow{DA}$.
3. **Deduction of Angle $\angle D$:**
   - If the vectors forming the first three angles are perpendicular, then the fourth vector $\overrightarrow{DA}$ must also be perpendicular to the third, meaning $\overrightarrow{CD} \perp \overrightarrow{DA}$, which implies that $\angle D = 90^\circ$.
4. **Conclusion:**
   - The quadrilateral $ABCD$ has four right angles and is therefore a rectangle.

Solution 2: Using Geometric Transformations

1. **Initial Consideration:**
   - Consider a quadrilateral $ABCD$ where $\angle A = 90^\circ$, $\angle B = 90^\circ$, and $\angle C = 90^\circ$.
2. **Orthogonal Symmetry:**
   - Place AB on the x-axis and BC on the y-axis, with point A at the origin. An orthogonal symmetry applied to points A, B, and C with respect to the axes will yield a fourth point D that forms a right angle with $\overrightarrow{CD}$ and $\overrightarrow{DA}$.
3. **Final Angle and Conclusion:**
   - Since D is defined by symmetry, the quadrilateral $ABCD$ has four right angles and is a rectangle.

Solution 3: Using Properties of Parallelograms

1. **Quadrilateral with Three Right Angles:**
   - Consider a quadrilateral $ABCD$ in which $\angle A = 90^\circ$, $\angle B = 90^\circ$, and $\angle C = 90^\circ$.
2. **Properties of a Parallelogram:**
   - A quadrilateral with opposite sides parallel is a parallelogram, and if one angle is right, then all angles are right, making the parallelogram a rectangle.
   - Since sides AB and CD, as well as BC and DA, are respectively perpendicular, $ABCD$ is a parallelogram.
3. **Conclusion:**
   - Since $ABCD$ is a parallelogram with three right angles, the fourth angle must also be right, so $ABCD$ is a rectangle.

Figure 4: Requesting three different solutions from ChatGPT for the rectangle problem.

To understand how ChatGPT "reasons" in mathematical contexts, it becomes evident that it generates solutions based on identifying patterns and utilizing knowledge derived from text data within its training set. Rather than performing calculations or deductive reasoning, ChatGPT combines mathematical rules and examples it has "learned" from text exposure, but it does not employ symbolic engines like like Wolfram Mathematica or Maple for validation or calculation.

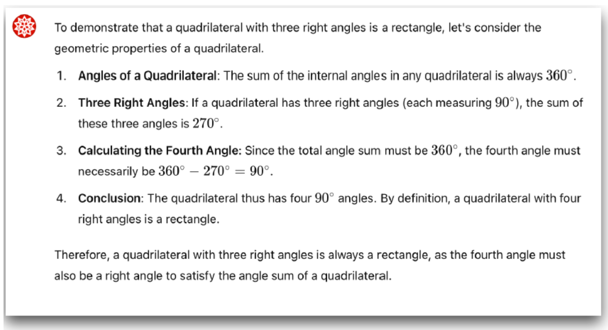
This approach has significant limitations. First, ChatGPT lacks exact calculations and real-time validation, meaning it does not verify each step as a symbolic tool would, which can lead to errors, particularly in complex calculations or advanced logical steps. As a result, ChatGPT's solutions are often plausible but remain unverified, offering logical coherence but risking inaccuracies, especially in complex or ambiguous cases.

A closer look at its inferential processes shows that ChatGPT's "deductive reasoning" is closer to inductive inference, as it primarily relies on pattern recognition to produce responses that are relevant yet unverified. This contrasts with tools like WolframAlpha, GeoGebra Discovery, and other Computer Algebra Systems (CAS) that offer validation mechanisms. WolframAlpha combines CAS functionality with numerical and statistical tools, while GeoGebra Discovery uniquely integrates CAS with automated reasoning for symbolic AI, particularly in geometry. Unlike these tools, ChatGPT alone does not provide a mathematical validation, depending instead on unverified inductive patterns rather than formal, symbolic deduction.

This tendency to make non-deductive deductions or to combine deductive fragments derived from training data in problems that have already been solved poses several problems. By blending different types of rationality, such as those found in experimental sciences versus mathematics, ChatGPT creates a confusion that can lead to significant challenges. For example, induction is useful for identifying invariants, models or conceptualizing categories, but its validation is an experimental matter and generally requires further research or long-term studies to stand the test of time. In other words, induction is not designed to produce necessary truths from established truths, as deductive reasoning does, nor to articulate knowledge that necessarily follows from other knowledge, as is expected in mathematics.

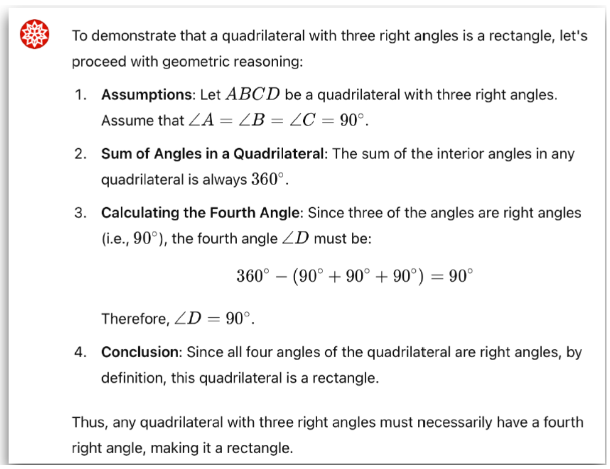
Moreover, when ChatGPT uses abduction, it can generate creative responses and explore potential solutions based on partial information, fostering interaction and opening up avenues of inquiry, while also mitigating the “blank page” effect. However, these abductive responses can be misleading by giving an impression of certainty, when in fact they are merely unverified hypotheses that require additional validation to ensure their accuracy.

Paradoxically, a deeper understanding of how this AI functions allows us to tackle real problems and engage in an authentic modelling process, thereby moving away from often artificial school exercises. This approach recentres exploration and scientific inquiry—essential elements of learning that align with the natural curiosity of students, especially given the new possibilities offered by digital tools. However, this openness may unsettle inexperienced teachers, who might feel a loss of control over their students’ learning due to the element of uncertainty introduced. Should we then prioritize the student’s curiosity or the teacher’s need for control?


To demonstrate that a quadrilateral with three right angles is a rectangle, let's consider the geometric properties of a quadrilateral.

1. **Angles of a Quadrilateral**: The sum of the internal angles in any quadrilateral is always 360°.
2. **Three Right Angles**: If a quadrilateral has three right angles (each measuring 90°), the sum of these three angles is 270°.
3. **Calculating the Fourth Angle**: Since the total angle sum must be 360°, the fourth angle must necessarily be $360^\circ - 270^\circ = 90^\circ$.
4. **Conclusion**: The quadrilateral thus has four 90° angles. By definition, a quadrilateral with four right angles is a rectangle.

Therefore, a quadrilateral with three right angles is always a rectangle, as the fourth angle must also be a right angle to satisfy the angle sum of a quadrilateral.


To demonstrate that a quadrilateral with three right angles is a rectangle, let's proceed with geometric reasoning:

1. **Assumptions**: Let $ABCD$ be a quadrilateral with three right angles. Assume that $\angle A = \angle B = \angle C = 90^\circ$.
2. **Sum of Angles in a Quadrilateral**: The sum of the interior angles in any quadrilateral is always 360°.
3. **Calculating the Fourth Angle**: Since three of the angles are right angles (i.e., 90°), the fourth angle $\angle D$ must be:

$$360^\circ - (90^\circ + 90^\circ + 90^\circ) = 90^\circ$$

Therefore, $\angle D = 90^\circ$.

4. **Conclusion**: Since all four angles of the quadrilateral are right angles, by definition, this quadrilateral is a rectangle.

Thus, any quadrilateral with three right angles must necessarily have a fourth right angle, making it a rectangle.

Figure 5: Rectangle problem, first and second requests with ChatGPT + Wolfram.

With the neurosymbolic technological counterpoint provided by the integration of ChatGPT and Wolfram, new dynamics emerge in mathematical reasoning and problem-solving (see Fig. 5). This “hybrid” model combines the precision of symbolic AI (Wolfram’s computational engine) with the generative flexibility of ChatGPT’s large language model. However, achieving coherence between these two approaches introduces distinct challenges, especially around communication, interpretation, and validity, where human input plays a central role.

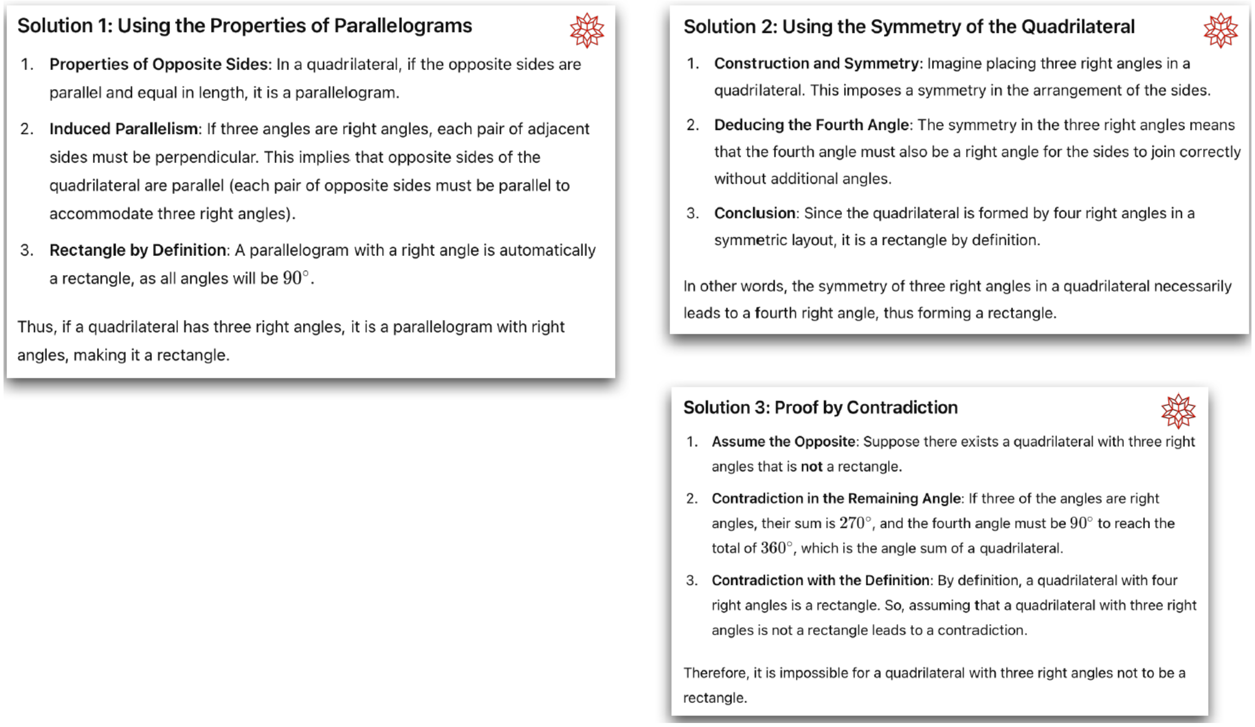
Firstly, effective integration relies heavily on clarity in questioning, where the user’s role is crucial. The phrasing, precision, and structure of the user’s question directly influence ChatGPT’s interpretation and the subsequent instructions it sends to Wolfram. For instance, if a user ambiguously asks for multiple proofs of a geometric property, ChatGPT may interpret this in ways that don’t align with the structured, symbolic expectations of Wolfram’s system. The user, therefore, must provide carefully framed questions, guiding ChatGPT to relay mathematically precise requests to Wolfram, minimizing ambiguity and maximizing alignment between the systems. This is a typical case of the quest for idoneity in which scientific questioning is at the heart of the approach. Secondly, interpretation by ChatGPT introduces another layer where human oversight is beneficial. ChatGPT's pattern recognition generates responses based on linguistic data and probabilistic associations from its training, yet it lacks formal verification for each logical step. Human users must be prepared to guide ChatGPT toward interpretations that are contextually relevant for Wolfram's symbolic processing. For example, while ChatGPT might generate additional contextual information that is linguistically coherent, human oversight can help filter out unnecessary content that doesn't contribute to the mathematical precision required by Wolfram. This interaction requires users to understand potential interpretive gaps and adjust their input for reliable outputs.

Ensuring validity within this neurosymbolic integration hinges on the verification methods designed by developers, but also on human interaction during validation. Wolfram's computational engine provides verified mathematical outputs, yet when combined with ChatGPT's generative responses, unverified elements may arise. Human users play a vital role here, reviewing and assessing the outputs for coherence and accuracy, especially in complex or ambiguous queries. This oversight helps bridge the flexibility of generative responses with the strict validation of symbolic AI, guiding the combined system toward reliable and rigorous outcomes. The integration of ChatGPT and Wolfram offers a powerful tool for mathematical inquiry, blending exploratory reasoning with computational rigour. However, the human role in questioning, interpreting, and validating remains essential to navigate the system's interpretive boundaries and to ensure coherent and accurate outputs.

ChatGPT alone provides a step-by-step solution, including a recall of geometric properties and a progressive reasoning process that makes the solution more "pedagogical." This approach includes metamathematical elements such as the initial hypothesis, the reasoning logic, and a general conclusion, making the reasoning relatively accessible to learners. In fact, this is not necessarily a drawback in itself, especially considering that, in a mathematics classroom, both human argumentative reasoning and deductive reasoning with calculations are typically present. However, as ChatGPT relies solely on probabilistic models, the solution lacks formal validation, introducing a latent uncertainty or a low epistemic value.

With the integration of Wolfram, the solution becomes more concise and benefits from rigorous mathematical validation through symbolic computation. Wolfram enhances the accuracy and reliability of calculations, but the presentation adopts a more demonstrative style, focusing on results rather than intermediate "pedagogical" explanations. Consequently, while this approach is mathematically more precise, it may be less clear for beginner learners who wish to understand each step of the reasoning process.

In summary, ChatGPT alone provides a step-by-step reasoning process that can help learners visualize the logical progression in a mathematical solution. However, in the absence of formal validation, this approach carries the risk of errors and inaccuracies, particularly for more complex problems. ChatGPT with Wolfram, on the other hand, incorporates formal mathematical verification that ensures the rigour of calculations and conclusions, though it may be less detailed in explaining intermediate steps. For situations where mathematical precision is essential, integration with Wolfram offers a more reliable and rigorous solution, and it can serve as a model for correct mathematical reasoning, though it may sometimes require additional work to elaborate the complete reasoning pathway for learners.


Solution 1: Using the Properties of Parallelograms

1. **Properties of Opposite Sides**: In a quadrilateral, if the opposite sides are parallel and equal in length, it is a parallelogram.
2. **Induced Parallelism**: If three angles are right angles, each pair of adjacent sides must be perpendicular. This implies that opposite sides of the quadrilateral are parallel (each pair of opposite sides must be parallel to accommodate three right angles).
3. **Rectangle by Definition**: A parallelogram with a right angle is automatically a rectangle, as all angles will be 90°.

Thus, if a quadrilateral has three right angles, it is a parallelogram with right angles, making it a rectangle.

Solution 2: Using the Symmetry of the Quadrilateral

1. **Construction and Symmetry**: Imagine placing three right angles in a quadrilateral. This imposes a symmetry in the arrangement of the sides.
2. **Deducing the Fourth Angle**: The symmetry in the three right angles means that the fourth angle must also be a right angle for the sides to join correctly without additional angles.
3. **Conclusion**: Since the quadrilateral is formed by four right angles in a symmetric layout, it is a rectangle by definition.

In other words, the symmetry of three right angles in a quadrilateral necessarily leads to a fourth right angle, thus forming a rectangle.

Solution 3: Proof by Contradiction

1. **Assume the Opposite**: Suppose there exists a quadrilateral with three right angles that is **not** a rectangle.
2. **Contradiction in the Remaining Angle**: If three of the angles are right angles, their sum is 270°, and the fourth angle must be 90° to reach the total of 360°, which is the angle sum of a quadrilateral.
3. **Contradiction with the Definition**: By definition, a quadrilateral with four right angles is a rectangle. So, assuming that a quadrilateral with three right angles is not a rectangle leads to a contradiction.

Therefore, it is impossible for a quadrilateral with three right angles not to be a rectangle.

Figure 6: Three solution requests to ChatGPT + Wolfram for the rectangle problem.

Comparing the three solutions proposed by ChatGPT alone (Fig. 4) and by ChatGPT with Wolfram (Fig. 6), we observe notable differences in terms of reasoning depth and diversity of geometric approaches. ChatGPT alone employs several methods, such as the use of angles and the properties of parallelograms, thereby offering solutions that evoke direct methods. In contrast, ChatGPT with Wolfram demonstrates increased rigour by adopting various geometric frameworks, notably moving from synthetic geometry to vector geometry, and by incorporating reductio ad absurdum arguments similar to those found abundantly in the original Elements of Euclid, prior to the introduction of algebraic calculation in subsequent versions. This flexibility in approach and the ability to rigorously validate each step through symbolic calculations confer a higher epistemic value and greater adaptability to the solutions provided by ChatGPT with Wolfram, allowing for a deeper exploration of mathematical concepts and processes while ensuring accuracy.

Inspiring Examples of Technological Counterpoint and AI Classification for ME

AlphaGeometry exemplifies a remarkable case of technological hybridization, showcasing substantial progress in solving geometric theorems. This neurosymbolic system combines a language model trained on synthetic data with a symbolic deduction engine, allowing it to navigate the “infinite space” of possible solution graphs in plane geometry. AlphaGeometry addresses one of the greatest challenges in AI for mathematics: the synthesis of human proof data into machine-verifiable languages, a problem reminiscent of the concept of computational transposition (Balacheff, 1994). With a massive set of 100 million automatically generated theorems, AlphaGeometry produces human-readable proofs and even solves problems at the Olympic level, marking a step toward high-level automated reasoning in a field where rigour is essential (Trinh, Wu, Le, et al., 2024). For comparison, Euclid's *Elements* contains approximately 465 propositions. While these propositions are meaningful condensations derived from human culture, many of AlphaGeometry's properties hold no epistemological value.

Although this example is not directly related to mathematics, it illustrates the resurgence of expert systems fuelled by advancements in generative AI (e.g., Trafton, 2019; Fearn, 2024). Machine translation systems, for instance, exemplify a technological counterpoint where expert systems and generative models work together to interpret and translate content across multiple languages. MIT is currently developing a hybrid architecture that combines symbolic and statistical AI, allowing machines to simulate "reasoning" about complex relationships with less data by integrating perception, language, and reasoning modules for more nuanced interpretations. In truth, the machine only simulates reasoning to some extent; it does not truly reason in the human sense of the term and is incapable of doing so. This hybridization leverages the efficiency of expert models, which provide linguistic precision in specialized fields, alongside the flexibility of generative models, which capture broader contextual nuances. Such architectures apply beyond translation to other areas requiring advanced cognitive functions, where AI can approximate a deeper understanding through a synergy of statistical approaches and symbolic reasoning.

The QED-Tutrix system provides an innovative approach in AI-powered tutoring for mathematical proofs, focusing on enhancing human learning. As described by Font, Gagnon, Leduc, and Richard (2022), QED-Tutrix operates as an intelligent tutoring system, modelling knowledge and reasoning to support learners in understanding mathematical justifications. This kind of AI system plays a vital role in mathematics education by simulating interactive sessions where students can practise proof techniques, gradually mastering the logic and procedural steps necessary for complex problem-solving. Although promising, such systems require ongoing funding, especially in regions like Canada, to reach their full potential and expand their applications in AI-assisted learning.

According to Van Vaerenbergh and Pérez-Suay (2022), AI systems for mathematics education can be classified into four categories: information extractors, reasoning engines, explainers, and data modelling systems. These categories reflect the various ways in which AI supports mathematical learning: information extractors transform raw data into usable mathematical representations, reasoning engines, such as WolframAlpha, solve complex problems, explainer breakdown solutions to facilitate understanding, and data modelling systems personalize learning by analyzing student interactions.

This classification highlights how each type of AI can enhance mathematics education according to specific needs. Reasoning engines provide precise answers, while explainers help make these answers accessible, fostering a deeper understanding. By combining these tools, students strengthen their autonomy and actively engage with mathematical concepts.

This categorization also supports the integration of AI within the technological counterpoint framework, providing teachers with a structured foundation for selecting AI tools that align with learning objectives and student profiles, thereby enriching the educational experience.

Implications and Challenges for the Future

In the era of new mathematical work, it is clear that in-depth studies on technological counterpoint involving human learners are essential. However, the rapid growth of computing and the utilitarian ideology often present in the field of AI developers—who are eager to decide alone on orientations and applications—raise critical questions for mathematics education. This movement risks overlooking the complexity of human learning and the pedagogical dynamics necessary for a thoughtful integration of AI with our students. By prioritizing efficiency and automation over scientific inquiry and the cultivation of critical thinking, we risk creating powerful tools that address learners' educational needs only indirectly, with validity based solely on action itself. The absence of dialogue with researchers in didactics and mathematics education could also lead to AI systems poorly suited to the diversity of learning contexts, thereby limiting their educational impact to narrow usage considerations. Yet, mathematics education plays a central role in AI development, dating back to the rise of cognitive sciences in the 1980s. Enhancing intelligence within the framework of new mathematical work necessarily requires modelling knowledge and reasoning that can be implemented in today's digital tools.

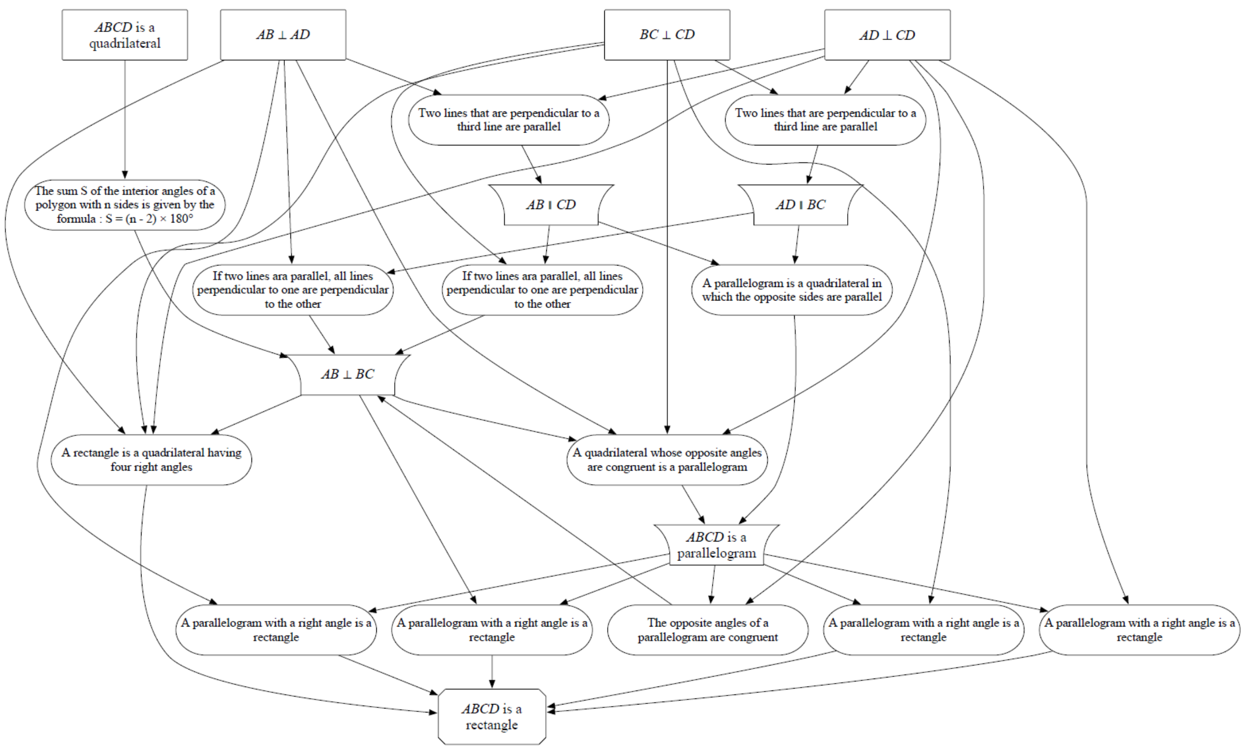


Figure 7: The QED-Tutrix rectangle problem solution graph (Richard, Gagnon, & Fortuny, 2018).

Imagine for a moment that the QED-Tutrix project receives funding and a team of researchers in mathematics education and computer engineering is reassembled. As mentioned in the previous section, QED-Tutrix is an intelligent tutoring system designed to support students in solving complex geometry proof problems. Unlike AlphaGeometry or ChatGPT + Wolfram, QED-Tutrix was created to facilitate students' understanding of geometric reasoning by enabling them to explore, articulate, and validate their problem-solving processes through guided interactions. While reasoning engines can solve mathematical problems and generate correct proofs, they do not necessarily produce results that are accessible to human understanding. Since QED-Tutrix is intended for mathematics education and needs to produce messages when students encounter blocks in the proof-solving process, an approach was required in which each step of the reasoning is known or accessible in some way by the computer system. This is why, at its core, the system models students' thought processes and provides feedback based on an extensive memory of problem-solving steps (Fig. 7), allowing it to guide students out of impasses by suggesting connected problems that enhance understanding without simply providing solutions (Richard, Gagnon, & Fortuny, 2018).

A unique feature of QED-Tutrix is its focus on enhancing mathematical competence by positioning problem-solving as a fundamental skill. Leveraging a comprehensive database of potential solution paths, it aims to simulate a teacher's ability to provide contextually relevant hints and feedback. The system encourages students to analyze and synthesize geometric properties, ultimately empowering them to construct proofs independently. However, QED-Tutrix's development also underscores certain challenges, particularly the need for funding and additional research to expand its capabilities in adaptive learning and AI-supported education.

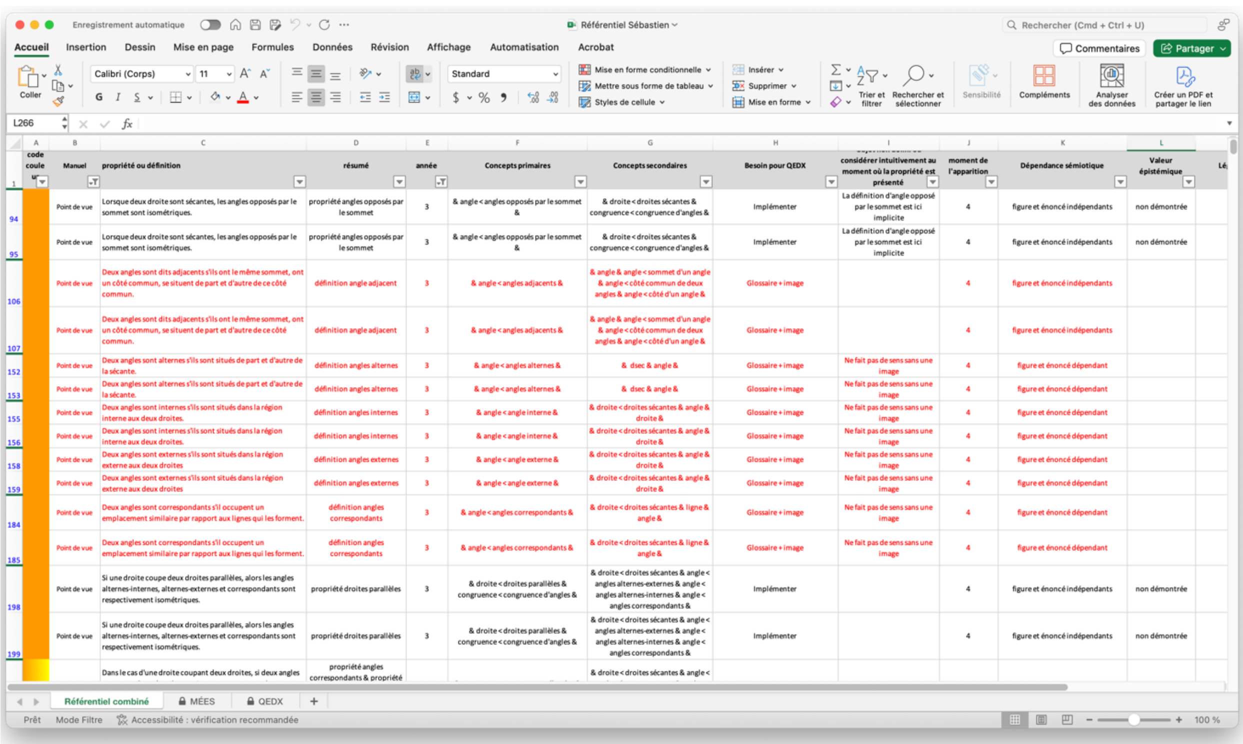


Figure 8: Repository of geometry referential used in Quebec school textbooks and official documents for the creation of a common referential framework for implementation in the QED-Tutrix system (Cyr, 2018).

Looking to the future, QED-Tutrix aims to enrich the pedagogical experience by becoming more adept at identifying student needs and responding with customized prompts and connected problems, thereby fostering an integrated learning environment. Although this approach has not yet been fully implemented, generative models and large language models (LLMs) could provide substantial support. These technologies could not only facilitate human-machine communication but also help the system to accommodate the varied formulations of mathematical properties and definitions encountered in Quebec classrooms and beyond. For example, Cyr (2022) has already documented numerous variations found in official documents and various textbooks, both old and new (Fig. 8). By using AI to identify a representative for each definition or property—acting as a class representative in an internally unified referential—the system can adapt to different learning environments, operating with a streamlined solution graph that respects each classroom's didactic contract. This approach allows QED-Tutrix to preserve unique inferential shortcuts and familiar styles of formulation without compromising rigour.

This methodology is well aligned with the educational objectives of mathematics didactics, aiming to foster a richer, interactive environment where technology and human learning intersect dynamically through projects and purposeful tasks. As a member of the ThEDU community (see below), QED-Tutrix is positioned within a network dedicated to advancing educational technology, particularly in enhancing mathematical reasoning through digital tools, and serves as an intelligent tutoring system for the teaching and learning of mathematical proofs.

The Xena Project, founded by Kevin Buzzard at Imperial College London, introduces mathematicians, particularly undergraduate students, to the use of Lean for formal verification of mathematical proofs. This software is based on the calculus of constructions with inductive types, enabling not only the formalization of deductive reasoning but also the modelling of inductive structures, which are essential in mathematical exploration and discovery. By transforming proofs into a kind of interactive game, it makes each theorem computer-verifiable through the mathlib library, integrating rigorously structured reasoning steps. This approach provides a technological counterpoint to large language models like ChatGPT, which Buzzard compares to students who "know a lot but do not think for themselves." Unlike LLMs, Lean follows a verifiable, step-by-step proof logic, combining the rigour of formal and inductive systems from computer science to enrich mathematical learning and enhance understanding.

ThEDU (Theorem proving and Automated Deduction in Education) is a community dedicated to integrating automated theorem proving and deduction in the field of education (Quaresma, 2022). Its objective is to promote the development and application of systems capable of verifying, proving, and exploring mathematical statements within an educational context. ThEDU supports the creation of educational software that leverages automated proof techniques to enhance learning in mathematics and logic, facilitating students' understanding of proof concepts and rigorous reasoning. The community organizes academic events, such as workshops, to bring together researchers and practitioners around the latest advances in educational AI and proof systems.

In conclusion, while large language models and symbolic AI systems represent significant advancements in computational capability, they are not inherently suited for authentic mathematical work. Despite drawing from an extensive knowledge base, these models lack the structured, autonomous reasoning and control essential for genuine, dialectical mathematical learning—one that actively responds to the learner's questioning and fosters the creation of meaningful knowledge. As we navigate an era of rapid computational evolution, the integration of technological counterpoints, where symbolic and statistical AI work in tandem, offers valuable support but may still fall short of addressing these intrinsic limitations. It is essential that humans act as "chefs d'orchestre" of this technological counterpoint, steering its design and applications to meet specific didactic needs and countering the utilitarian ideologies that often drive AI development. By prioritizing a collaborative, human-centered approach, we can foster a new generation of critical thinkers who not only engage deeply with mathematical concepts but also uphold the primacy of human reasoning in mathematics education.

The illusion of creativity in ChatGPT arises from its ability to establish unexpected connections based on data produced by humans. Novelty, as envisioned in the principle of idoneity, thus becomes an essential reference that influences new mathematical work. It is worth recalling that the concept of idoneity, introduced by Ferdinand Gonseth, proposes an innovative approach in the philosophy of science, particularly for analyzing the relationship between established reference frameworks and the evolution of knowledge. According to Gonseth, these frameworks are not fixed structures; they must be continuously adapted to new experiences and discoveries. Although statistical AI attempts to mimic this adaptive process, it fails to exercise conscious control over these adjustments. Rather than seeking absolute and immutable foundations, we should adopt flexible approaches that allow knowledge to advance in step with ongoing developments. For humans, this implies a "strategy of engagement": an active approach open to uncertainty, in contrast to the often rigid predictive methods, thus valuing adaptability, whether to explore scientific truth or simply to learn how to formulate and solve problems. Understanding the emergence of new ideas, and particularly those with real value, remains complex; yet it is precisely what we must aim for, even as we take into account the contributions and limitations of AI and other technological counterpoints.

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Appendix

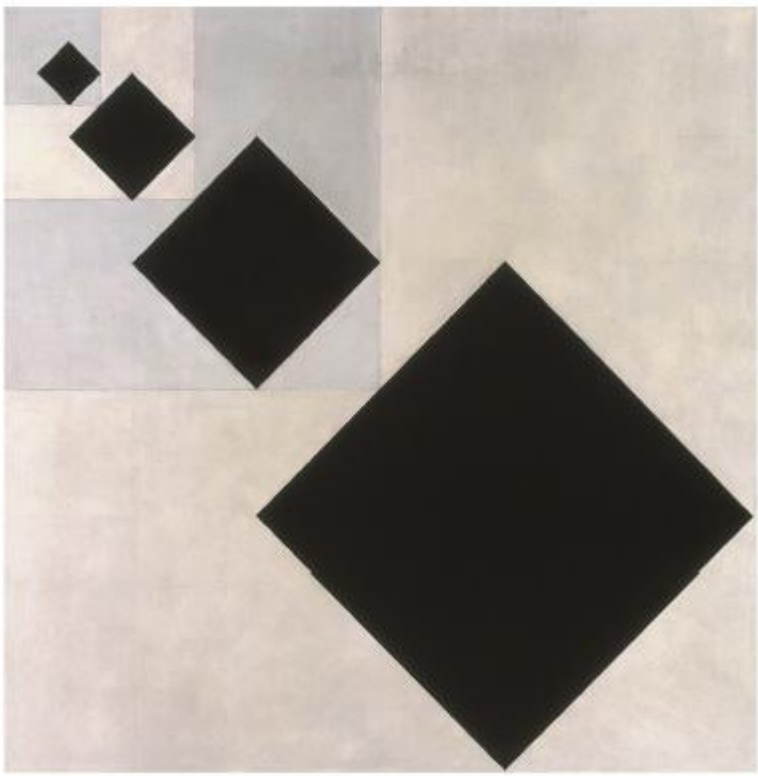
Condensed version of the initial questionnaire.

Mathematical Exploration: Abstract Harmonies in Visual Art

To answer the following questions, you may use the conversational agent ChatGPT or the dynamic mathematics software GéoGébra, at any time, which includes interactive geometry modules and automated reasoning tools.

Exercise 1 *Face-to-Face with Plasticism*

Here is an **image**:



Here is a **problem situation**:

When observing Theo Van Doesburg's Arithmetic Composition (image above), an abstract painting created in 1929-1930 during the neoplasticism period, visible regularities are noticeable. Black, white, and gray squares are arranged with varying dimensions and intervals, evoking a sense of recursion.

Our aim is to understand and explain the mathematical aspects that visually emerge in this artwork before delving into the artist's motivations behind choosing these abstract harmonies.

1. Check if ChatGPT contains information on Theo Van Doesburg's Arithmetic Composition, then compare it with information obtained through a search engine like Google.

2. Describe the visual properties that emerge from Arithmetic Composition. Use ChatGPT for assistance if needed.

3. Share the observed properties of Arithmetic Composition with another team. Compare the methods of direct observation with those using ChatGPT, highlighting the formulation of questions and the responses obtained.

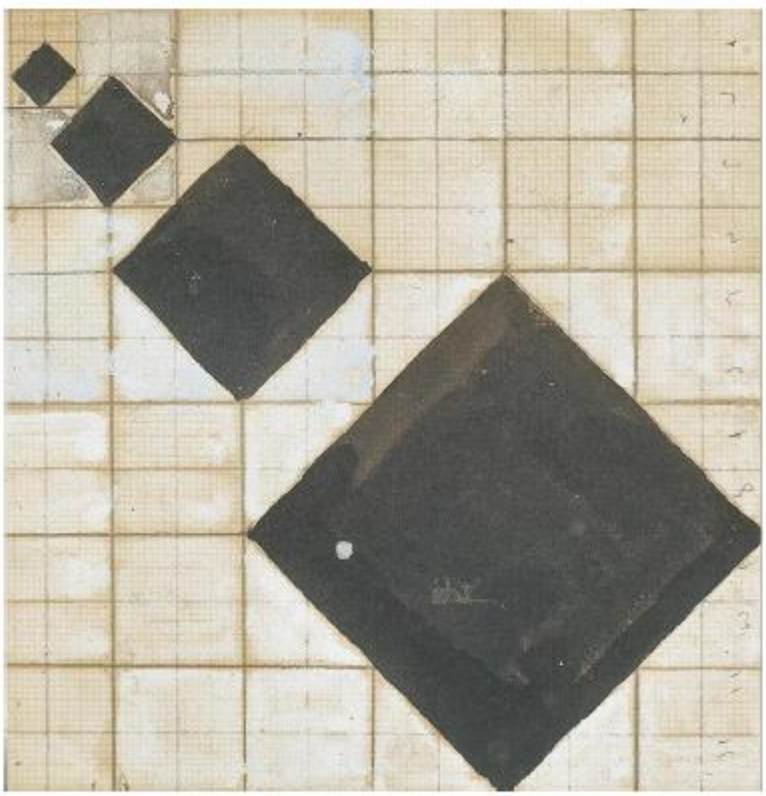
4. Examine if ChatGPT has information from a direct analysis of Theo Van Doesburg's Arithmetic Composition, excluding peripheral data. Explain your approach.

5. Evaluate the usefulness of ChatGPT in its interaction, taking into account its reasoning abilities, its influence on knowledge development, and the advantages and limitations related to its dependence on training data.

6. Explain the significance of the statement, "ChatGPT can make mistakes. Remember to verify important information."

Exercise 2 *Geometric Sequence*

Here is a **sketch**:



Here is the continuation of the **problem situation**:

> It's easy to imagine that creating such a balanced work as Arithmetic Composition required a series of preliminary trials. A sketch, drawn on graph paper and measuring 12 cm per side, has been preserved, though we do not know if it was a draft or a communication tool once the work was completed.

> It's easy to imagine that creating such a balanced work as Arithmetic Composition required a series of preliminary trials. A sketch, drawn on graph paper and measuring 12 cm per side, has been preserved, though we do not know if it was a draft or a communication tool once the work was completed.

1. Are all squares, when taken in pairs, necessarily in a Thales' configuration? Verify this property in general using ChatGPT, and indicate the necessary condition for a Thales' configuration.

2. In the sketch of Arithmetic Composition, it is easy to see that the vertices of the black squares are carefully placed on grid nodes.

- **a)** Why are the "black diamonds" actually squares? Justify your answer.
- **b)** In the file Exercise 2-2b.ggb, points A to H and Z have been fixed on grid nodes. Using the "Relation" tool (on the side), or the command `Relation(a, e)` in the input bar, verify whether the configuration involving Z, [AB], and [EF] constitutes a Thales' configuration, and confirm why this is the case.
- **c)** In the input bar, enter the command `AreParallel(a, e)` and note the result. When compared with previous answers from the "Relation" tool, the software returns "true," "are parallel (evaluated by calculation)," and "are parallel (generally false)." What explains these differences?
- **d)** Calculate the scaling ratios k and ℓ that allow one measurement to be converted into another, specifically "side: AB $\xrightarrow{k}$ EF" and "area: ABCD $\xrightarrow{\ell}$ EFGB".
- **e)** In the previous file, place vertices I(8, 0), J(12, 4), K(8, 8), and L(4, 4) of the large black square, as well as the centers M(4, 8) and N(8, 4) of squares EFGH and IJKL. The lines (MN) and (FL) intersect at point O.
  Calculate the ratio $\frac{OM}{ON}$ and find a significance for point O between the two squares.

3. Looking closely at the paper, we note that the numbers from 1 to 12 are written on the right, suggesting that the author deliberately chose this number. Additionally, using thicker lines, it seems that he grouped the squares on the sheet by "three" to form "┘"shapes, as seen in the artwork itself.

- **a)** Why would a 12 cm grid have been chosen? What purpose might grouping by three serve? Justify your reasoning.
- **b)** Discuss the possibility of extending the construction of black squares indefinitely on the sheet, whether or not this is related to Theo Van Doesburg's grid.

4. We are interested in the largest square that can be inscribed in a triangle.

   a) If 4 points are placed on the sides of a triangle, must at least two points be on the same side? What mathematical property applies in this case?

   b) In Arithmetic Composition, we notice that the large black square is inscribed within half of the grid square. Is this the largest square that can be inscribed in an isosceles right triangle? Demonstrate your reasoning.

Generated by Bing Image Creator.

5. A teacher asserts that for a square inscribed in a triangle with base b and height h, the area of the square is maximal when the side of the square measures $\frac{b \times h}{b + h}$. Is this possible? Prove or refute this assertion using two different methods from the following options:

   i. Discussing the problem with ChatGPT to reach a convergent conclusion;

   ii. Using traditional dynamic geometry through numerical models;

   iii. Utilizing automated reasoning tools or CAS commands;

   iv. A manual approach in the paper and pencil environment, without the use of instrumented mathematical devices (such as ChatGPT, GGB, ART, etc. 🤭) showing through in the written solution 🤔.

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RESEARCH AND THE NEW AI TOOLS MEET FACE TO FACE: TAKING ADVANTAGE OF WHAT WE KNOW ABOUT THE LEARNING OF MATHEMATICS

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This article aims to acknowledge the role played by the findings of mathematics education research in the development of technology learning environments, especially when AI components are incorporated. Reference is made to two examples that illustrate how knowledge of students' mathematical thinking processes enables anticipating critical moments in the understanding of concepts or in the resolution of mathematical tasks. The examples also illustrate how such possibility of anticipation is a fundamental element in didactic designs that include intelligent support for providing pupils with feedback on their performance within the learning environment.

Introduction

After four decades of research in mathematics education, one finds an enormous amount of accumulated knowledge about the nature of the difficulties students face in understanding concepts and developing skills in different areas of mathematics. Access to digital technologies for personal use made some researchers -the undersigned included- consider them as tools to create learning environments. It would be impossible here -in this space- to give a fair accounting of the different programs and devices with which studies and experiments have been carried out for the purpose of delving into the potential of those technologies. Their potential, that is, to help students overcome didactic and epistemological obstacles that make it difficult for them to understand concepts and develop mathematical skills. However, it is worth mentioning that in addition to supporting students in facing such difficulties, programs such as Logo, Scratch, Spreadsheets, Computer Algebra Systems, Dynamic Geometry (in versions of Cabri Géomètre, Geometer Sketchpad and Geogebra) have given rise to didactic designs that encourage exploration, experimentation and different forms of knowledge construction.

In this regard one must acknowledge the role played by the findings of mathematics education research in the development of these didactic designs. Such is the case of studies on generalization processes or on conceptualization by working with different modes of representation of mathematical objects. These studies have led to the use of environments such as Logo and Geogebra (among many others) to realize the approach and verification of didactic hypotheses, as well as to propose their use in the mathematics classroom.

A common denominator of the programs cited is the possibility to provide users with visual and quick feedback on their actions performed on mathematical tasks within the environments. In relatively recent times it has been carried out the development of digital environments"

A research project bearing those characteristics was taken on by a group of academics at the UCL-Knowledge Lab of the University of London, for which the *eXpresser*[1], microworld was developed. In that microworld students are asked to express in different codes the pattern they identify in a given diagrammatic configuration. Through the intelligent system embedded in the environment, natural language prompts are provided to students as they work on a proposed task (Gutiérrez Santos et al., 2012). It is not difficult to imagine how knowledge concerning generalization processes and structure sense, derived from specialized research, could have influenced the design of these interventions with prompts. Similarly, in the project entitled *Intelligent dialogues with tertiary education students*[2], the *Intelligent Dialogues* learning environment was developed with an intelligent support that provides students with feedback in natural language as they perform modeling activities in mathematics and science (Rojano & García-Campos, 2017). In this case, the basis for the design of the feedback/intervention resource was research findings on the difficulties faced by tertiary education students at critical moments in modeling (Molyneux et al., 1999).

Continuing with the theme of intelligent support that includes communication with users in natural language, one cannot ignore the fact that we are now entering a new era arising from progress in development of generative AI chats and expedited access to their use. There is an expectation that with new tools it will be possible to enhance feedback such as those of *eXpresser* and *Intelligent Dialogues.* But in addition to their role in feedback generative AI chats are expected to be used in the creation of increasingly effective assistants or user support guides for learning concepts and solving mathematical tasks and problems. In the design of these assistants, drawing on what is known about the learning of specific mathematics topics is crucial, and it is in this use for design purposes that a meeting between fundamental research in mathematics education and the potential of new AI tools can take place.

The following section uses two examples -taken from the aforementioned projects- to illustrate how knowledge of students' mathematical thinking processes enables (in many cases) anticipating critical moments in the understanding of concepts or in the resolution of mathematical tasks. They also illustrate how such possibility of anticipation is a fundamental component in the development of technology learning environments that include intelligent support for interaction with users, including feedback on their performance within those environments.

Intelligent systems with didactic design based on foundational research

In this section reference is made to two examples of the use of AI systems to encourage students to reflect, analyze and (where necessary) correct or redirect their actions, through

[1] The eXpresser microworld was designed to help students with mathematics generalization processes. Inspired by the robotics and adaptive systems methodology, the MiGen project used a 'layers' approach to develop and assess the intelligent support system for eXpresser.

[2] Intelligent Dialogues with tertiary and university education students is a three-year research project, funded by the National Council of Science and Technology (Conacyt) in Mexico, Reference No. 168620

didactic designs of activities and feedback resources based on findings in mathematics education research.

Example 1. *eXpresser*

Researchers at the UCL-Knowledge Lab, the developers of *eXpresser*, refer to Exploratory Learning Environments (ELEs) as a particular type of computer-based learning environments in which the focus is on learners' exploration of an area of knowledge (Mavrikis et al., 2019). These authors include microworlds in this ELE category, microworlds that are environments developed with a constructionist perspective, characterized by didactic designs based on open-ended tasks aimed mainly at promoting exploration and the construction of conceptual knowledge among students (Healy & Kynigos, 2010). As mentioned in the previous section, the possibility of providing feedback related to learners' actions is a constitutive element of interactive digital environments and ELEs are no exception in that regard. On the contrary, feedback in ELEs and especially in microworlds has distinctive features that distinguish it from the interaction in classical intelligent tutorial systems. Such features are related to the nature and purpose of the tasks designed for the microworlds, which as mentioned above are more about fostering students' exploration, discovery and understanding of concepts, rather than development of operational skills through instructionalist designs (Mavrikis et al., 2019, p.2).

In the *eXpresser* microworld students are asked to build models with mosaics and algebraic rules associated with the models. The suggested strategy is to first construct 'building blocks' (which are perceived as a common piece of the different parts of a given model or diagrammatic configuration) and then repeat these blocks and generate the model with a particular number of constitutive elements and to thus infer/produce a general rule (algebraic formula) that makes it possible to generate a model with *n* elements (Mavrikis et al., 2019). A typical *eXpresser* activity is shown in Figure 1.

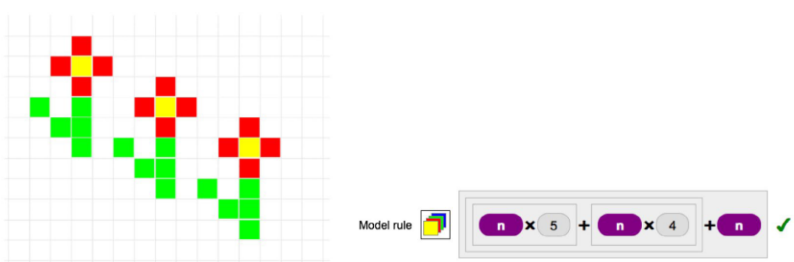

Figure 1: "An example model and rules that students may construct in eXpresser. They can do so by creating "building blocks" to generate the centres of the flowers, the petals and the stalks, which they will then repeat to make the yellow, red, and green patterns. They will be nudged towards deriving general rules for n number of flowers. This is challenging as an "animation" feature applies different random values to the variables used by the student in order to check the generality of their rules." From Mavrikis, et al. (2019, p. 4).

Clearly the construction of such blocks and of the model is a process that involves different stages of generalization, from the perception of the structure of a given pattern and the identification of the common unit that characterizes the structure associated with the pattern as it is perceived, to the steps of expression of that unit and its relation to the diagrammatic configuration in different codes (natural language, diagrammatic, numerical, or algebraic). These processes have been extensively researched, theoretically and empirically, from different perspectives; and the design of the tasks in *eXpresser*, as well as their feedback resources, are based on the results of that research. For example, present in the didactic design of the *eXpresser* tasks are both the well-known fact that the perception of the structure of a pattern is not unique (different individuals may have multiple perceptions of one same configuration), and what Rivera (2010) has pointed out regarding how decisive identification of the unit of repetition is to conceive the structure of the pattern. As already stated, in those tasks students are asked to build blocks that represent the common unit that they identify in the figural representation of the pattern (building blocks strategy).

At the stage of the activity in which the review of and reflection upon the blocks and formulas built by each student or group of students is carried out, by verifying whether their construction (or formula) leads to the reproduction of the given figural pattern, feedback from the environment and/or through the teacher's intervention is usually required. It is at these times of review/reflection that the use of prompts -via questions and suggestions- is crucial for advancing towards the desired generalization. Some of the knowledge concerning the perception and generalization processes of a pattern, underlying the design of the activities that take place in the learning environment, translates into warnings regarding the difficulties that learners may face and they therefore also play an important role in the design of feedback. Geraniou & Mavrikis (2022, p. 134) refer to how thanks to the intelligent support of the MiGen system, of which *eXpresser* is a part, students participating in an experimental study were able to find the minimum number of tiles they could group in a building block and repeat it to form a pattern. This shows the relevance of having supports of this type, of specific and timely intervention, whose design considers both tracking data of student actions while working on a task, as well as knowledge of their learning and comprehension processes in mathematics.

Another recurrent finding in studies on patterns and generalization is the tendency of students to view the relationships in the pattern recursively (and not functionally). This is an obstacle to generalization for case *n* (Stacey, 1989) and therefore it also represents an obstacle to the generation of an algebraic rule associated with the pattern. Hence intervention (of some sort) in this transition to symbolization in order to promote student reflection is also highly relevant. The case of the MiGen project and the intelligent system associated with eXpresser is an example of the usefulness of having knowledge of possible anticipations in student performance in generalization tasks for designing AI resources that allow relevant interventions. But in terms of types of use of such resources as well, the project goes beyond individual adaptive feedback and the article *Intelligent analysis and data visualisation for teacher assistance tools: The case of exploratory learning* (Mavrikis et al. 2019, p.6) describes intelligent support components of the MiGen system that include intelligent Teacher Assistant Tools, and outlines the work that has been done to extend the use of these tools to classroom activity, thus complementing the work of the teacher as facilitator or orchestrator.

Example 2. *Intelligent Dialogues*

The findings of the Anglo-Mexican project *The role of spreadsheets within school-based mathematical practices in science*[3] include the identification of specific moments that were critical for all participating students (tertiary education students) during their performance of modeling activities using spreadsheets. The critical moments in question are: prediction of phenomenon behavior, verification of the prediction and generalization of the model (Molyneux et al., 1999). This finding together with the *Dialogues with Theo* (DiT) program (http://recursostic.education.es/descartes) served as the basis for designing and building the integrated environment *Intelligent Dialogues in Parameterized Modelling Activities*, hereafter referred to as *Intelligent Dialogues.*

Using the DiT program, in the integrated *Intelligent Dialogues* environment it is possible to simultaneously display a dialogue window and a microworld window on the computer screen, both of which are dynamically linked (see Figures 2a and 2b). In that environment students can work on specific modeling tasks in the spreadsheets microworld and dialogue with the system in natural language. Consequently, students receive feedback both in the microworld and in the dialogue window. Figure 2a shows the activity scene *Molecular diffusion in a cell*, by means of a simulation of the phenomenon that consists of considering a simplified cell (in two dimensions) with six compartments. The outer walls are impermeable, although the internal membranes between each two compartments do allow molecules to move from one compartment to another. At time t=0, there are 1,200 molecules in the first compartment. In each time unit the molecules move in the four directions with the same probability. The students are asked to build a spreadsheet model that represents the molecules spreading into the different compartments over time. Figure 2a describes the phenomenon behavior at time t=0 and t=1; Figure 2b shows the questions asked by the system and student answers (dialogue window), as well as the calculations performed by the student (window on the right). The microworld offers students the option of working numerically or with formulae on a spreadsheet (Rojano, 2018, p. 227).

[3] Anglo-Mexican Project developed in collaboration with the Institute of Education of the University of London and the Department of Mathematics Education of the Centre for Research and Advanced Studies (Cinvestav) in Mexico, funded by the Spencer Foundation of Chicago, Ill, Grant No. B-1493.

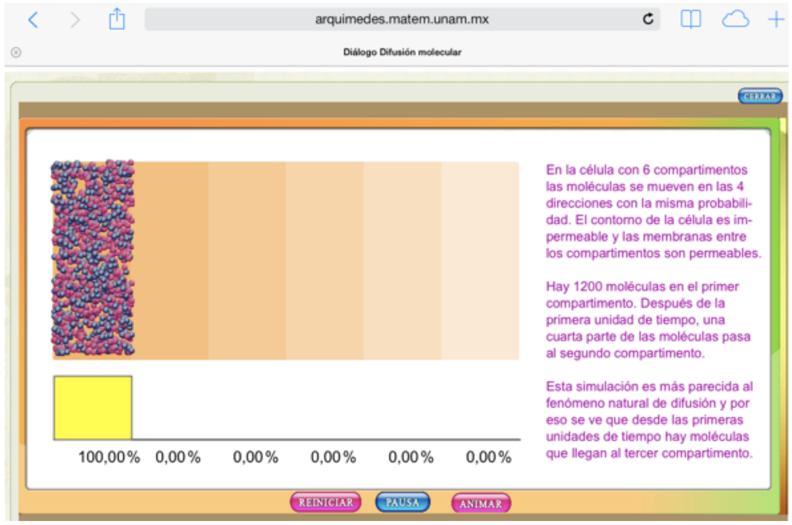

Figure 2a. "Screen with simulation of Molecular diffusion in a cell. From Rojano (2018, p. 227).

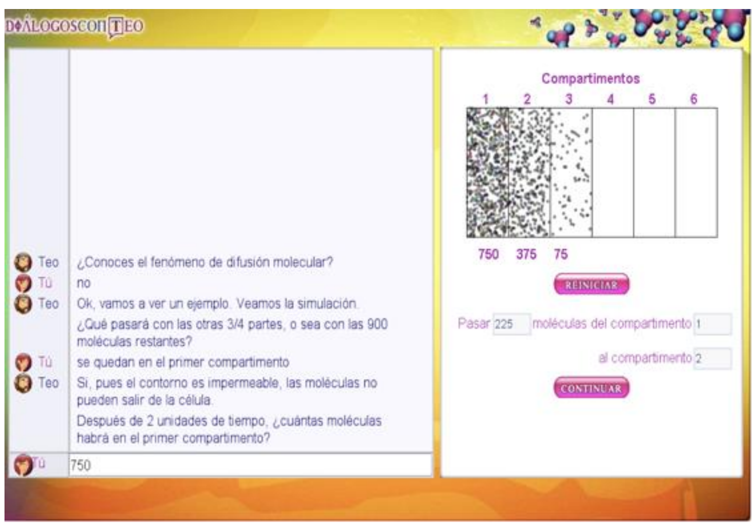

Figure 2b. "Screen with the dialogue (on the left) and the microworld (on the right) windows". From Rojano (2018, p. 227)

For the experimental study associated with this integrated environment, activities were designed considering three stages of the modeling activity: understanding the phenomenon, building the model, and predicting the phenomenon behavior in the long run. In view of the results of the previous study (without the AI layer), the critical moments were anticipated at each of the stages in order to design the feedback in the dialogue window. Figure 3 shows how students are presented with the option of a correct and an incorrect prediction of the long-term behavior of the molecular diffusion to encourage them to reflect on their choice and provide them with hints on how to verify or correct their prediction. Moreover, to support the students in going through the different stages of the modeling activity, open questions, multiple-choice items, suggestions, and prompts were included in the dialogue window, whereas in the microworld window, different mathematical representation systems were used (see Rojano & García-Campos 2017, for the type of activities used and the methodological aspects of the study). The outcomes from the experimental study indicate that the Intelligent Dialogues system responds differently depending on to the type of strategy used by the student, be that spreadsheet-numerical or algebraic (using a formula). It was furthermore observed that the intervention through the intelligent system allowed students to successfully complete the activity. However, in some cases, where students were experiencing great difficulty from the initial stage of the modeling, substantive intervention from the teacher-researcher was necessary (Rojano & García-Campos, 2017, p.29).

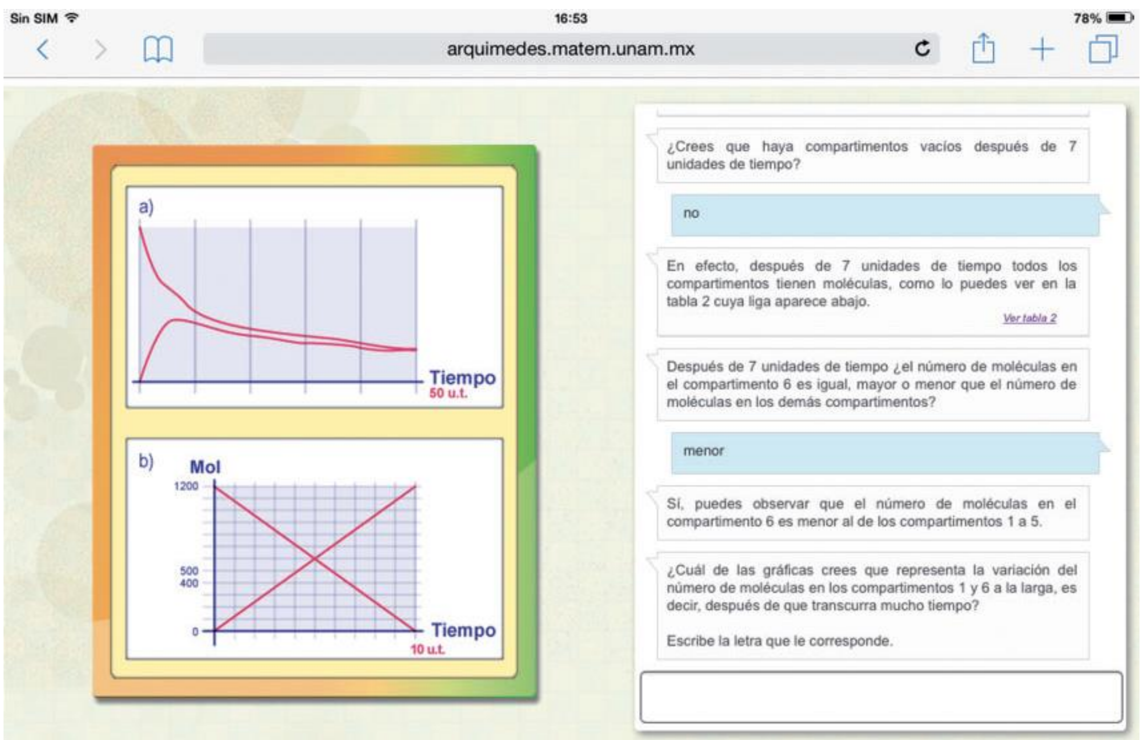

Figure 3: Dialogue and microworld windows. Choosing graphs for compartments 1 and 6. From Rojano, & García (2017, p. 25)

Intelligent Dialogues -inspired by the MiGen project- is another example of integrating an AI component into a microworld, in which natural language is used as a vehicle for interaction with students. Due to the exploratory nature of the activities, both in this environment and in eXpresser, the possibility of dialogue with other students and with the teacher is considered. That is to say, the interaction with natural language goes beyond an individual-intelligent system chat modality. Hence in this type of use of intelligent systems and teamwork, natural language communication plays a key role in the didactic scenario. It is worth noting that in this respect the experiment in empirical studies conducted with both environments confirms the hypothesis formulated by some authors to the effect that interaction through natural language contributes to the improvement of learning achievements, especially when it takes place in a technology environment. That hypothesis is derived from the research conducted by Litman et al (2009), which reports that a higher percentage of content-rich talk is correlated with higher learning gain. To the previous result it should be added that Chi, et al (1994) showed that self-explanation among students significantly improves learning, which reinforces the above-mentioned hypothesis.

From incorporating a feedback system in a microworld to creating AI assistants to foster mathematical thinking: Algebra Structure Sense (ASS) and Expression Machine (*Mex*)

Algebra structure sense (ASS) is the capability to gain awareness of the internal structure of algebraic objects (Rojano, 2022, p. 2). According to an operational definition of that capability, a person has structure sense if he/she: sees an algebraic expression or sentence as an entity, recognizes an algebraic expression or sentence as a previously met structure, divides an entity into sub-structures, recognizes mutual connections between structures, recognizes which manipulation it is possible to perform, and recognize which manipulations it is useful to perform (Hoch & Dreyfus, 2004, page 351). Hoch & Dreyfus used this definition to verify in a study involving 92 16-17 years old pupils whether a subject makes use of ASS when performing an algebraic task. The authors applied the following test:

> Study with 92 16-17 years old pupils. The purpose is to identify students who display structure sense, and if structure sense is affected by the number of sets of brackets (0, 1, 2) and by the placement of the variable (on one side of equation or on both sides of equation). From Hoch & Dreyfus (2004).

A. $1-\frac{1}{n+2}-\left(1-\frac{1}{n+2}\right)=\frac{1}{110}$

B. $\left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{n+1}\right)=\frac{1}{132}$

C. $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$

X. $\frac{1}{4}-\frac{x}{x-1}-x=5+\left(\frac{1}{4}-\frac{x}{x-1}\right)$

Y. $\left(\frac{1}{4}-\frac{x}{x-1}\right)-x=6+\left(\frac{1}{4}-\frac{x}{x-1}\right)$

Z. $\frac{1}{4}-\frac{x}{x-1}-x=7+\frac{1}{4}-\frac{x}{x-1}$

In the data recorded two essentially different types of strategies were observed. One is based on the use of ASS, which allows students to identify identical compound terms that can cancel each other out and thus obtain the solution of the equation immediately. The other strategy consists of developing the operations called for, (including the elimination of parentheses or looking for a common denominator in rational expressions), which led in several cases to long and complicated processes that did not allow the students to arrive at the solution (Hoch & Dreyfus, 2004).

With another approach to the same topic, in the project *Developing algebra structure sense in a digital interactive environment with an adaptive system*[4], *Mex*, an interactive web environment was created for the purpose of helping students from different educational levels to develop their ASS (Muñoz & Xolocotzin, 2022, pp. 79-84). In that environment, interaction is between the user and a multilevel network of algebraic tasks of varying degrees of complexity, and feedback is based on the user's responses to each task, without the use of a chat. The results of a first study with *MEx* (carried out among subjects with varying mathematical experience and knowledge) show that a group of participants make progress in the development and use of structure sense as they solve tasks along paths entailing different levels of complexity. However, the subjects with a tendency to develop algebraic expressions instead of using structure sense found it difficult to distance themselves from that tendency (Solares & Rojano, 2022 pp 91-104). In such cases the need for intervention to guide students to reflect on their responses and procedures was recognized. As seen in a preceding section, in projects such as *eXpresser* and *Intelligent Dialogues*, this need has been addressed through incorporation of prompts and a dialogue window. But in view of the nature of the cognitive ability involved in recognizing the internal order of algebraic objects (Kirshner, 1989), one option to support ASS development is to use generative AI systems applications to develop assistants.

Knowing more about and exploring the potential of the new tools

The features that distinguish generative AI applications from non-generative or conversational AI applications **(**https://mail.google.com/mail/u/0/?tab=rm&ogbl#inbox) make it possible to think of sophisticated and ambitious usage models, both as interlocutors for students and as teacher assistants in the classroom. Unlike conversational AI based in part on predefined answers or rule-based systems, generative AI can be used to create new content with the use of machine learning algorithms and deep learning to generate outputs. This has led to the creation of ChatGPT and Google Gemini, as well as many other applications **(**https://mail.google.com/mail/u/0/?tab=rm&ogbl#inbox). In a short period of time tools such as these have proven to be very powerful in assisting users with activities in different areas; it is at this point when the question arises as to how reliable and effective they are in the field of mathematics. For the moment, what can be said is that in some cases the applications can give mathematically and didactically correct answers or guidance. However, there are cases in which "their knowledge" of the subject matter is poor or scarce and their intervention can be erratic as a result. In other words, their effectiveness depends to a large extent on the mathematical content in question and the AI application’s ability to perform a particular task or solve a particular problem. The latter was evidenced in a preliminary study, in the framework of an ongoing PhD

[4] *Developing algebra structure sense in a digital interactive environment with an adaptive feedback system* is a Frontiers of Science Project, funded by the Council of Science and Technology in Mexico (Conacyt Ref. 2016-01-2347). The main aim of the project is to prove the feasibility to foster the development of high algebraic competencies in heterogeneous groups of students, using a technology environment that promotes autonomous learning. (https://www.teresarojano.com)

project[5], when a selection of items from the Hoch & Dreyfus questionnaire was used to test the ASS of a set of generative AI applications.

In the doctoral research in reference, the author sought to answer the following questions regarding the generative AI applications: 1) Do they apply ASS to solve the Hoch & Dreyfus items; 2) Can they be taught to apply ASS; 3) How can AI be used to foster development of ASS in subjects? When testing the selected applications, it was found that only some of them make use of ASS when asked to solve one of the equations in the questionnaire; while others develop the algebraic expressions present in the equation, just as some of the participants in the study did with *MEx.* The above answers question 1). The following shows the performance of two of these tools when they are asked to solve the equation $1 - \frac{1}{(n+3)} - 1 + \frac{1}{(n+3)} = \frac{1}{72}$, taken from the Hoch & Dreyfus questionnaire (Nebbia, in process).

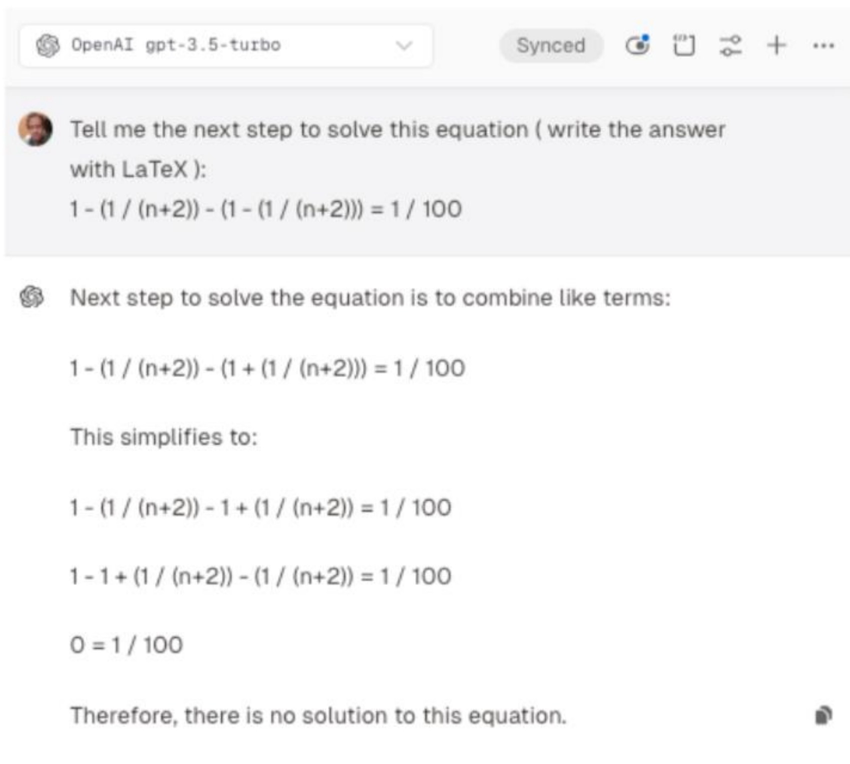

Figure 4b: Example of Open AI GPT 3.5 turbo solving the equation $1 - \frac{1}{(n+3)} - 1 + \frac{1}{(n+3)} = \frac{1}{72}$ (from the Hoch & Dreyfus questionnaire). The AI application used a syntactic rule-based strategy to find the solution (Nebbia, in process).

[5]Nebbia, C. (in process). *Construction of interactive environments for the enhancement of algebra structure sense: Into the world of generative AI tools*. PhD project, Center for Research and Advanced Studies (Cinvestav), IPN, Mexico

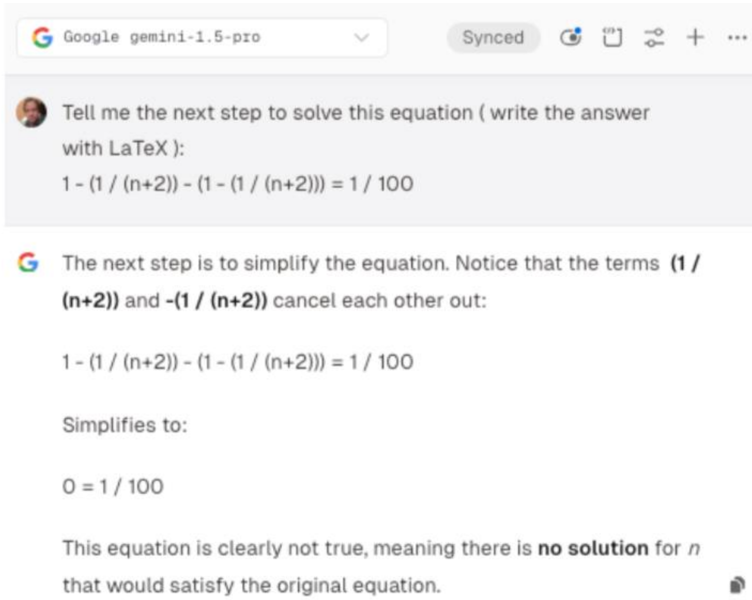

Figure 4b: Example of Google Gemini 1.5 pro solving the equation $1 - \frac{1}{(n+3)} - 1 + \frac{1}{(n+3)} = \frac{1}{72}$ (from the Hoch & Dreyfus questionnaire). The AI application used structure sense to find the solution (Nebbia, in process)

The overall plan of this project entailed choosing generative AI applications that can be used to build an assistant that accompanies students as they solve algebraic tasks designed to promote the use of structure sense. The accompaniment will consist of engaging in a dialogue with the student during the joint resolution (student-assistant) of the tasks, making the student aware of the structural properties that s/he has not perceived and suggesting how to use them to perform the algebraic transformations that lead to the solution.

Final Thoughts

The author aimed to use the examples submitted in this article to emphasize the role of knowledge concerning the mathematical thinking of students in the design and development of technology learning environments. Through empirical studies, the latter environments have shown their potential to foster such thinking among users, both conceptually and in development of mathematical skills and abilities. However, one must acknowledge that matching didactic designs of this type with the dizzying advance of digital technology and AI tools represents an enormous challenge. This is in part because the generation of and access to fundamental research results in mathematics education are slower processes than technological progress. It is also partially due to the fact that widespread access to an expeditious use of the new tools - particularly what we are witnessing today in relation to the use of generative AI applications- favors a great diversity of spontaneous uses in and out of school. The foregoing makes up a very different scenario as compared to what existed in the 90s, when access to use of digital devices and software was very restricted; in many cases the use of technology was barely starting in academia or the workplace. The current context is in many ways very different, and major changes are beginning to be hypothesized both in the ways of teaching and learning, as well as in the manners of conducting educational research in general, and in mathematics education in particular.

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THE IMPLICATIONS OF GENERATIVE ARTIFICIAL INTELLIGENCE FOR MATHEMATICS EDUCATION

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Generative Artificial Intelligence has become prevalent in discussions of educational technology. These AI models can engage in human-like conversation and generate answers to complex questions in real-time, with education reports accentuating their potential to make teachers' work more efficient and improve student learning. In this paper, I provide a review of the current literature on generative AI in mathematics education, focusing on four areas: generative AI for mathematics problem-solving, generative AI for mathematics tutoring and feedback, generative AI to adapt mathematical tasks, and generative AI to assist mathematics teachers in planning. I then discuss ethical and logistical issues that arise with the application of generative AI in mathematics education, and close with some observations, recommendations, and future directions for the field.

Generative Artificial Intelligence has taken the world by storm since the release of ChatGPT in November of 2022. This release marked an important milestone in the development of conversational Artificial Intelligence agents, driven by ChatGPT's ability to engage in human-like conversation and answer complex questions. Stakeholders immediately began imagining how these tools might be applied to education. It has been nearly two years since ChatGPT's release, and research is rapidly emerging on its implications for education. In this paper, I seek to summarize current trends and issues related to GenAI in mathematics education, since the release of ChatGPT.

Artificial Intelligence (AI), according to the U.S. Department of Education, is "automation based on associations" (Cardona et al., 2023, p.1). Generative AI (GenAI) is a class of AI that is capable of generating new data and outputs by learning patterns from training data. Large Language Models (LLMs) are one type of GenAI that "build sophisticated statistical predictors by identifying patterns in a massive set of human-curated training data" (NCTM, 2023, p. 1). What was so revolutionary about GenAI models like ChatGPT when they were released was the ability of a human user to *respond back* to the AI model and ask it to change elements of its response – this practice is known as *prompting*. This ability gave rise to *prompt engineering,* which is the process of constructing inputs for LLMs to elicit precise, coherent, and pertinent responses (Liu et al., 2021). This process allows users to iteratively refine the kinds of output the LLMs provide, customizing the LLM's work to their needs and context.

The U.S. Department of Education gives many possible benefits of AI in education – from increasing the adaptivity of learning materials to students' needs, to providing teachers support via automated teaching assistants, to better customizing learning resources to meet local demands (Cardona et al., 2023). NCTM's (2023) AI Position Statement further expands on these affordances – describing how GenAI can allow for the quick development of multiple problem versions to illustrate a mathematics concept, can efficiently design engaging, personally relevant questions tailored to individual students' experiences, and can generate rich mathematical explanations that adapt to students' current level of expertise. Both of these reports also detail the incredible risks that GenAI poses – including issues of bias towards marginalized groups, concerns related to privacy and surveillance, and GenAI's tendency to hallucinate and provide incorrect information.

I structure this paper by first discussing affordances and use cases of GenAI in mathematics education in four broad areas – focusing on mathematics problem-solving, mathematics tutoring and feedback, adapting mathematical tasks to learner needs, and supporting mathematics teachers in planning. I then move to discussing important ethical, theoretical, and practical issues to consider when implementing GenAI in education. I close by providing some observations and recommendations for the future of GenAI in mathematics education.

Generative AI for Mathematics Problem-Solving

GenAI programs like ChatGPT can have a wide variety of mathematics problems inputted into them and can not only generate an answer to these problems, but also give a detailed explanation of how to get to that answer. The latest version of the LLM GPT-4 (OpenAI, 2023) integrates computer vision, such that the AI can examine mathematical diagrams in addition to the problem's text. GPT-4 scores 700 out of 800 on the mathematics portion of the SAT (OpenAI, 2023), which is in the 89th percentile. This is an improvement on a score of 590 (70th percentile) that was achieved by GPT-3.5, its predecessor. GPT-4 also scores a 4 out of 5 on the AP Calculus BC exam, while its predecessor scored a 1. GPT-4 scores in the 80th percentile on the GRE Quantitative exam, with its predecessor in the 25th percentile. And while GPT-4o scored only 13% on the qualifying exam for the Math Olympiad, the new GPT-o1 model designed for complex reasoning scored an impressive 83% (OpenAI, 2024), although it is slower and more costly than its alternatives. These results paint an impressive picture of the capability of contemporary LLMs for mathematics problem-solving.

LLMs also seem to have a relatively easy time with typical K-12 mathematics word problems used in open-source curricula. For example, GPT-4 solved and generated explanations for middle school mathematics word problems from ASSISTments with only a 4% error rate in its mathematical explanations (Wang et al., 2024a). Interestingly, none of these errors in explanations were associated with incorrect answers, allowing researchers to conclude that the AI was relatively safe for use with middle school students. Other researchers have used GPT-4 to solve more difficult graduate-level mathematics problems, to test whether GenAI can assist mathematicians in their professional activities (Frieder et al., 2023). They found that while GPT-4 could solve undergraduate mathematics problems, it performed poorly on graduate-level work.

They concluded that GPT-4 can best be leveraged to act as a mathematical search engine and query databases of mathematical objects, rather than as a direct solver of advanced problems. Analyses have also been done into the nature of the mistakes GPT makes when solving mathematical tasks. Typical errors made by GPT-3.5 included using incorrect formulas or methods or unclear question definition, along with calculation errors and misinterpretations of the question being asked (Yen & Hsu, 2023). Mathematics problems that have a high number and diversity of mathematical operations needed to solve them tend to be more difficult for LLMs, as are problems that utilize a conversion of a quantity that requires real-world knowledge (Srivasta & Kochmar, 2024). Math word problems with unfeasible solutions (i.e., the analytical solution is not practical with respect to the real-world context), that contain quantities in them that are not needed to solve the problem, or that involve a comparison between quantities are also more difficult for LLMs (Albornoz-De Luise1 et al., 2024). In addition, holding mathematical features constant, longer and more difficult-to-read word problems are harder for LLMs to solve. One reason why understanding the capability of LLMs to solve mathematics problems is important is because these tools are used by students as a form of assistance. Although for primarily symbolic problems computer algebra systems like Maxima and Wolfram Alpha are more accurate, LLMs offer the advantage of communication using natural language, and can explain different steps for problem solving. Thus, LLMs may be preferred by students over alternatives. Integrating computer algebra systems (Matzakos et al., 2023) or other secondary systems that can check the LLM's calculations (Yen & Hsu, 2023) with LLMs will be an important future direction to improve the reliability of these systems.

There is surprisingly little research on how GenAI can be used effectively by *students* to enhance their learning of mathematics. Barana et al. (2023) gave university students combinatorics problems to solve with the help of GPT-3.5. They found that although GPT-3.5 did not consistently achieve correct answers to the problems, the output given by GPT-3.5 could be leveraged by the students. The students used the output to generate ideas for how to approach the problem, to compare their reasoning with GPT-3.5's reasoning, to solve smaller parts of the problem, and to evaluate and then correct GPT-3.5's solution paths. This is an important example of how LLMs can support higher-level thinking in mathematics. Research has also been done with pre-service mathematics teachers using ChatGPT to help them solve mathematical modelling tasks (Naresh et al., 2024). The researchers highlighted that incorrect answers from the AI could be opportunities for student learning and could launch important mathematical conversations. The teachers also recognized that their students could self-explain the different steps shown by ChatGPT as an opportunity for deeper learning. In sum, more research on how students can effectively partner with LLMs to confront challenging mathematical tasks, like mathematical modeling tasks, is needed.

Generative AI for Mathematics Tutoring and Feedback

Several different online learning platforms have launched GenAI chatbot mathematics tutors, which communicate with students through text chat to assist them with solving mathematical tasks. The most well-known is Khan Academy's Khanmigo (Khan Academy, n.d.), which is free for teachers through a partnership with Microsoft, but has a monthly charge for students, families, and districts. Khan Academy describes how "Unlike other AI tools such as ChatGPT, Khanmigo doesn't just give answers. Instead, with limitless patience, it guides learners to find the answer themselves." Notably, Khanmigo is student-facing with no human directly in the loop – children interact directly with the GenAI conversational agent, relying on Khan Academy's safeguards to prevent inappropriate interactions. Khanmigo acknowledges that it will sometimes make mathematical errors, with a disclaimer at the top of the tutoring screen reading “Khanmigo make mistakes sometimes.”

Results are starting to emerge on the effectiveness of GenAI tutoring. Bastani et al. (2024) conducted a pre-registered RCT that involved nearly 1000 high school students and compared students learning mathematics over a semester with either: (1) GPT-4, (2) GPT-4 with knowledge of correct solutions and student mistakes, as well as instructions to not give students answers, or (3) a control condition where students used books and notes with no access to devices. They found that both versions of GPT-4 improved immediate performance, but that the version that lacked the safeguards actually harmed later exam performance by 17%. Additional analyses suggested that GPT-4 was being used as a crutch by students, and that they were often simply asking it for the answer without a substantial conversation. The enhanced version of GPT-4 with the safeguards did not offer any benefit on the exam over simply studying the text and notes without devices, and its effect trended slightly negative compared to the control group.

As little research exists on GenAI tutors, we can look to research on chatbot tutors that were built predating the rise of contemporary LLMs. A study that compared adults learning mathematics with a chatbot to adults learning with Khan Academy videos did not find significant differences in learning (Grossman et al., 2019), suggesting that the chatbot was generally not effective. However, a math tutoring chatbot for a younger population of elementary students showed some evidence of positive results for engagement and learning above a control condition with no support (Ruan et al., 2020). A follow-up study (Ruan et al., 2024) of elementary students found no differences in overall learning compared to a control condition with no support, but some suggestion of increased learning for students with lower pretest scores in the chatbot condition. At the secondary level, a chatbot implemented in ASSISTments was compared to students simply receiving static hints, and researchers found no differences in learning (Cheng et al., 2024). However, students who had interacted with the chatbot actually had *lower* confidence in their problem-solving, due to potentially becoming reliant on the chatbot’s high degree of assistance.

None of these studies compared chatbots to human tutors, and overall, the evidence base for chatbot tutors does not seem particularly promising. However, we should not discount that many marginalized learners may not have access to human tutoring, and that LLMs’ abilities to communicate in different languages may have important affordances. Butgereit and van Staden (2023) report on an implementation in South Africa of adult learners receiving mathematics tutoring through a version of GPT-4 configured for tutoring interactions. The tutoring was delivered in several different languages, including less-resourced African languages that typically perform less well in LLMs.

Given that there is little research on GenAI chatbots, I next look to research on whether LLMs can give actionable feedback to students on their mathematical problem-solving. In online learning platforms, generating text and images for explanations and hints to be administered when students need assistance can be time-consuming for curriculum developers. As a result, many curriculum developers are looking to LLMs to help with this process. Research suggests that GPT-4 has a tendency to over-identify instances of students making mathematical errors

(Karkarla et al., 2024). Other studies have examined the quality of GPT-generated explanations through ratings of explanation quality. Wang et al. (2024a) asked ten undergraduate mathematics majors to evaluate either explanations for middle school mathematics problems written by GPT-4 or explanations for the same problems previously written by educators. They found that the perceived quality of the explanations was higher for GPT-4 than for teacher-written explanations, potentially because the GPT explanations were seen as having a clearer, step-by-step approach. Prior research had shown that GPT-3's explanations for mathematics problems were rated lower than teacher-generated explanations (Prihar et al., 2023). Research comparing pre-service teachers' reactions to educator-generated hints versus hints generated by GPT-4 found that educator-generated hints were preferred in some cases, as they incorporated visuals, while the LLM-generated hints were preferred in other cases, as they often were more thorough and detailed (Gattupalli et al., 2023a). Other research on GPT-4 suggests that the hints it gives may be too procedurally-focused and are not always written appropriately to support students' reading needs (Gattupalli et al., 2023b). The rated quality of GPT-4-written explanations for middle school mathematics problems can be improved if the LLM integrates previous annotations of the student's work from experts into its reasoning (Wang et al., 2024b). This blending of human and GenAI capabilities may be a promising approach.

Research has also examined the learning implications of AI versus human-generated hints, in addition to preference scores. Pardos and Bhandari (2024) compared GPT-3.5-generated hints for mathematics problems in the OpenStax curriculum to human tutor-generated hints. They found that adult learners had higher learning gains in the GPT condition compared to a control condition with no hints, while the difference between human-generated hints and the control condition did not reach significance. However, they found that 32% of the hints generated by GPT were initially disqualified for inaccuracy, and that this percentage was reduced through the use of an LLM hallucination reduction technique. Overall LLMs seem to be improving in their ability to generate hints and feedback but work still needs to be done on ensuring the hints are accurate, conceptually-focused, and do not lead to over-reliance on the LLM.

Generative AI To Adapt Mathematical Tasks to Learner Needs

GenAI can also be used to adapt learning tasks to meet different learner needs. For example, students often struggle to read the text of mathematics word problems (Walkington et al., 2018), and LLMs have the potential to adapt problems to assist emerging readers. Norberg et al. (2024a) showed that having GPT-4 rewrite middle school mathematics problems to improve their readability resulted in similar effects on student performance as having humans rewrite the problems. They also found that compared to the original problems that had not been rewritten, the problems rewritten for improved readability using GPT-4 could in some cases improve students' mastery rates. Using earlier LLMs, like GPT-3, for the same kind of task resulted in less impressive results, where outputs had more error and noise (Patel et al., 2023).

LLMs can also adapt mathematics problems based on students' interests in popular culture areas like sports or music, or career areas like nursing or engineering – this is often called *context personalization* (Walkington, 2013). GPT-3.5 was used in a research study to rewrite probability and statistics problems to correspond to undergraduates' career interests in areas like pharmacology or economics (Einarsson et al., 2023). The study found that the problems sometimes required revisions due to mathematical issues, and that while some students liked the relevant contexts, others did not like the extra length and complexity the personalized contexts added. However, the authors recognize that better results for accuracy with respect to mathematical issues may have been found with GPT-4. Indeed, another small study where GPT-4 was used to personalize secondary mathematics word problems to correspond to students' interests in areas like TikTok, results showed that GPT did not change the difficulty, intent, or values in the problem (Yadav et al., 2023). In a unique approach to personalization, Hwang et al. (2024) used GPT-3.5 to pose mathematical problems based on camera-captured images of real-world geometric objects, personalizing mathematics problems to objects in the learners' environment. They found that 5th grade students using the system outperformed a control group.

While research shows some effects of personalization and readability on student outcomes, it is also important to examine the perspectives of teachers. An interview study with teachers who taught 8th grade math in urban settings asked about the possibilities of using GenAI to personalize mathematics problems (Walkington and Bainbridge, under review). The study found that teachers felt it would be an effective way to draw upon students' real-world knowledge, activate interest, and allow for sense-making around mathematical problems. One teacher described how "If it's talking about a place, thing, or situation that they're actually familiar with, that they've actually had hands-on experience with, or have seen with their own eyes then, of course, it's gonna be a little bit easier for them to comprehend the problem," while another teacher said, "I think them being able to have a little bit of background knowledge makes word problems a little less scary, sometimes, too, where they feel like they understand it better." However, the teachers had concerns that LLM-generated problems would create greater reading burdens for students, that students still lacked important fundamental math knowledge, and that having different problem versions would translate into additional preparation time for teachers and/or create difficulty when going over problems as a class. One teacher described how "But if they're struggling in math, even giving them what they're interested in, it still may pose a challenge." Overall teachers showed some enthusiasm about the approach, mixed with concern that it would not solve the fundamental issues they were experiencing with their instruction.

Research has also examined partnering students with LLMs to engage in mathematical problem-posing activities (Silver, 1994), as a way to create personalized versions of story situations written by students (e.g., Walkington et al., 2024a). Norberg et al. (2024b) engaged middle school students in authoring their own personalized problems relating to probability and ratios and based on their interests, using GPT-4 to assist students. They found that students preferred more control over the personalization system and found slight increases in students' sense of belonging in mathematics. Zhang et al. (2024) conducted a study of 4–8-year-olds writing math stories while partnering with a GPT-4 agent to support their storytelling. They found that compared to a human partner, the LLM's assistance was actually better in helping students comprehend mathematical definitions, and comparable for supporting mathematics language learning and the generation of quality math stories. However, children interacting with the LLM were less likely to provide substantial responses in the conversation and more likely to give uncertain responses, compared to when interacting with a human.

Another study examined middle school girls posing mathematics story problems with the assistance of GPT-3.5, as part of a project-based unit to create a pitch for a mathematics video game (Walkington et al., 2024b). The study found that the girls used prompting to fine-tune the mathematics story problems for their game and their pitch, and typically gave length and style parameters to the LLM. The girls found that using GPT-3.5 created story problems they felt were fun and engaging, was an efficient process, and that the problems had the potential to increase mathematical understanding. However, further analyses found that the quality of the mathematics story problems, in terms of their realism and correspondence to actual real-world objects and events, was relatively low and problems could contain mathematical errors or inconsistencies. Overall, partnering students with GenAI to accomplish complex mathematical tasks that include problem-posing activities may be an important future direction of GenAI research, if students have an appropriate understanding of the issues involved with using GenAI.

Generative AI To Support Mathematics Teachers in Planning

Research suggests that teachers work an average of 50 hours per week, and that only 49% of this time is in direct interaction with students. The rest of this time involves preparing lessons, giving feedback, doing administrative work, and engaging in professional development (Cardona et al., 2023). There has thus been interest in leveraging GenAI to make teachers more efficient during the 51% of time they are not directly interacting with students. Many tools have arisen that use GenAI to help teachers plan their lessons and accomplish logistical classroom tasks. One of the most well-known tools is MagicSchool (powered by GPT-4, among other models), currently advertised to be used by 2 million educators worldwide (MagicSchool, n.d.). The MagicSchool suite has over 70 AI tools for educators that "help you lesson plan, differentiate, write assessments, write IEPs, communicate clearly, and more." The base version of MagicSchool is currently free for teachers, with more advanced plans having recurring charges. However, there are many other GenAI tools for teachers, with Khanmigo, for instance, having a similar suite of free teacher tools (Khan Academy, n.d.). Gemini for Google Workspace (Google for Education, 2024) has also arisen as a major player in the "GenAI for Teachers" field. Gemini has easy integration with Google tools that are widely used in schools already like Docs, Sheets, Slides, and Gmail, as well as integration with Google's Gemini chatbot.

Research on pre-service and in-service mathematics teachers using MagicSchool (Beauchamp & Walkington, 2024) has examined the use of various tools to make mathematics tasks more relevant to students. This study found that teachers felt the tools had the potential to support students' motivation to learn mathematics and that the tools could increase the efficiency with which the teachers could generate tasks. However, the teachers found the tools limited in their support for English Learners and felt that some of the tasks did not accurately or deeply reflect elements of students' real-world experiences or had mathematical issues. Research has also examined pre-service elementary mathematics teachers using Khanmigo as a support for their mathematics learning related to number theory (Yilmaz et al., 2024). This study found that teachers using Khanmigo appreciated the individualized learning and were comfortable sharing questions and struggles with Khanmigo. However, in some cases they found Khanmigo responses confusing or of questionable reliability and missed the human interaction of someone who gets to know them as students.

Research has also examined pre-service mathematics teachers using ChatGPT for lesson planning (Berryhill et al., 2024; Broutin, 2024; Kwom & Ko, 2024; Naresh et al., 2024). One study found that teachers asked ChatGPT for both mathematical and pedagogical knowledge, that they used ChatGPT as an assistant in developing and organizing lessons, they had ChatGPT simulate possible student responses, and they asked ChatGPT to validate or comment on their ideas relating to teaching (Broutin, 2024). The teachers used continuous prompting to adjust the output and ideas that ChatGPT generated, and they extensively modified the output from ChatGPT to meet their needs. Similarly, a study on pre-service mathematics teachers using ChatGPT highlighted that it can be used to anticipate student misconceptions and approaches to problems, and that ChatGPT can simulate being an age-appropriate student to assist teachers in planning (Naresh et al., 2024). Teachers can also use ChatGPT to generate culturally relevant word problems, with research suggesting that the LLM can be a helpful thought partner through the use of iterative prompting and revision (Berryhill et al., 2024). GPT can further be used to generate mathematics assessment items. Secondary mathematics teachers using GPT-3.5 to generate statistics assessment items felt variably in their desire to actually use the GPT-generated problems (Kwon & Ko, 2024). The teachers appreciated the creativity, efficiency, and specificity of GPT, in addition to its ability to produce anticipated student solutions. However, concerns were raised about GPT's mathematical errors, security and copyright issues, GPT's lack of transparency, its inability to know teachers' students and classrooms, as well as issues with item difficulty and discrimination. Overall, LLMs have some functions that will be useful to teachers in lesson planning, as long as the output can be modified and enhanced by the teachers themselves to best fit their needs.

Issues with the Use of Generative AI in Mathematics Education

A myriad of important ethical issues and concerns arise when applying GenAI technologies to education. Bender et al.'s (2021) groundbreaking paper describes some of these issues, highlighting the environmental and financial cost of increasingly complex and accurate GenAI that require more and more computing power (see also Li et al., 2023). In addition, the training data for GenAI is from large internet datasets that overrepresent people in positions of power in society, that show bias towards the inclusion of marginalized groups, and that include derogatory associations and stereotypes towards these groups (Bender et al., 2021).

Issues with training data may be of particular concern in mathematics education, as common textbooks (including open access textbooks) that GenAI is drawing from have been found to be culturally-biased. An analysis of the top 9 textbooks for 8th grade mathematics on EdReports. found that the majority of the problems in these texts were situated in White, middle-class American culture (Pruitt-Britton & Walkington, 2022). Many of the activities described in the story problems in these texts required wealth or transportation to participate in – such as a story problem about renting a jet-ski or vacationing in the Poconos. The analysis also found problems with specialized non-mathematical language that would be challenging for English Learners. If these are the kinds of data that GenAI is trained from, then GenAI is likely to show these same issues and biases when asked to generate problems.

There are also concerns about LLMs having stereotypical negative associations with the subject matter of mathematics itself, given the commonness of mathematics anxiety and negative reactions to mathematics that are prevalent in society and thus in the LLM's training data. Abramski et al. (2023) studied the associations that GPT-4 makes with the academic subject of mathematics, and the degree to which these associations are negative or positive. They found that 10% of sentiments associated with mathematics were negative for GPT-4, compared to a surprising 50% in GPT-3.5. However, the authors still found that in GPT-4, "Math was associated with frustrating, anxiety, fearful, intimidating, confusing, and struggle. These negative associations were not found in the semantic frame of physics, whose negative associates were related to domain knowledge (e.g., chaos, nuclear)" (p. 15).

When discussing limitations of LLMs, Bender et al. (2021) further describe how, "Text generated by an LLM is not grounded in communicative intent, any model of the world, or any model of the reader's state of mind. It can't have been because the training data never included sharing thoughts with a listener, nor does the machine have the ability to do that" (p. 616). Although these models can generate human-like responses, they are not reasoning or "thinking" in the way humans do. Indeed, the development of mathematics concepts themselves and the development of students' mathematics learning is situated in their individual and collective interactions with the physical world (Nathan, 2022). However, it has been argued that AI systems are "fundamentally incapable of understanding people's embodied interactions in the ways that humans understand them" (Nathan, 2023, p.1). These systems cannot account for forms of human reasoning that are non-verbal and non-pictorial, like gestures and actions.

In addition, in education particularly, there are concerns about the protection of users' inputs into the LLM, including privacy and issues of ownership of intellectual property (Gómez Marchant & Hardison, 2024). When an LLM collects data about young students to better adapt learning materials to student needs, issues of who sees the data and how it is deleted are paramount (Cardona et al., 2023). The rise of LLMs integrated into educational settings may also involve increasing possibilities for surveillance of both students and teachers, as the LLM collects data from multiple sources in order to best adapt instruction and assist teachers.

Further, research on using ChatGPT in mathematics teacher education has shown that ChatGPT can create developmentally inappropriate learning activities and materials that include mathematical mistakes. ChatGPT may create inappropriate materials, such as a middle school mathematics scenario about someone losing 5 pounds per month in a weight loss program (Sawyer & Aga, 2024). LLM-generated problems can also involve haphazard, rather than purposeful, choice of numbers, and LLM's lack of authentic connections to learners' lived experiences can "demonstrate a dangerous surface-level approach to culturally relevant pedagogy" (Gómez Marchant & Hardison, 2024, p. 3).

Indeed, Walkington et al.'s (2024b) study of middle school girls using GPT-3.5 to create mathematics story problems, and Beauchamp and Walkington's (2024) study of teachers using MagicSchool to create relevant learning tasks, found that the problems created by these LLMs often involved surface-level connections to students' experiences. For example, one teacher asked MagicSchool to create a lesson relevant to their students' interest in Mexican-American rapper ThatMexicanOT, and it generated the following: "Students will create a Spotify playlist based on songs by ThatMexicanOT. Each song will be represented by a cylinder-shaped object, and students will calculate the volume and surface area of each cylinder. This activity will show students how real-world math concepts are used in creative ways, like organizing playlists based on their favorite music." Obviously, this scenario is nonsensical, as representing songs as cylinders and calculating their volume and surface area makes little sense. Similar issues were found for student-generated math problems in our study of middle school girls – GPT-3.5 generated the problem "Jack, one of the last five remaining humans, is determined to defeat the robot army by factoring the polynomial expression 2x^2 + 5x - 3, representing the robots' central control system. If Jack successfully factors the polynomial into its linear expressions (Ax + B)(Cx + D), where A, B, C, and D are integers, he can exploit the weaknesses in the robots' programming." This again is a shallow connection between the mathematics concepts and the real-world context.

Image-generating GenAI also exhibit significant bias. For example, Figure 1 shows the output that was generated when DALL-E3 was asked to create "An image of a room of mathematics educators attending the Psychology of Mathematics Education - North America conference in Cleveland, Ohio." The lack of diversity in the image is striking. A study of middle school girls using DALL-E2 reported that the girls recognized bias when the GenAI would generate mainly light-skinned images of humans, despite most of the girls being girls of color (Walkington et al., 2024b). One group of girls in this study described how the pictures of the "landlord" character in their game were consistently generated as older, White men. Gómez Marchant and Hardison (2024) further describe how Adobe Firefly's image-generating AI shows negative racial imagery and incorporates an anti-fatness bias. They asked Adobe Firefly to generate images of a mathematics teacher, and all the images were of White adults. In the mathematics textbook analysis mentioned previously (Pruitt-Britton &Walkington, 2022), it was found that the majority of images of humans in mathematics textbooks were of White, able-bodied people. Given that GenAI is largely trained on these kinds of datasets when generating images for mathematical problems, it is not surprising that the generated images lack diversity.

There is also a lack of guidance in schools about how to handle students using GenAI to complete their assignments. This can lead to disciplinary action that may have disproportionate impact marginalized students, specifically special education students (Laird et al., 2023). There is evidence that English Language Learners and neurodivergent students may be disproportionately targeted by AI detection tools (Gegg-Harrison & Quaterman, 2024). Further, there is evidence that Black students are more likely to be false accused of using GenAI tools to cheat (Madden et al., 2024).

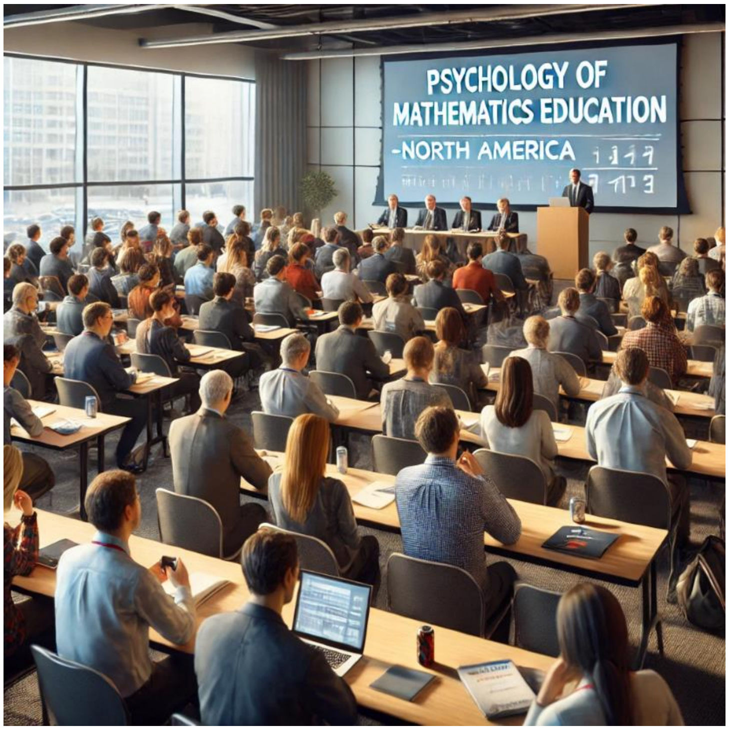

Figure 1: DALL-E3 image of attendees of PME-NA 2024

Discussion

NCTM (2023), in their Position Statement on Generative AI, compares GenAI to advances in technology like calculators, search engines, and image-based solving systems like Photomath. They describe how these tools have the potential to reduce an emphasis on computation in mathematics classrooms and increase focus on creative problem-solving. NCTM (2023) also describes how such tools can create positive pressure for teachers and curriculum developers to pose mathematical tasks that are deeper and involve creative thinking and are thus less prone to being solved with LLMs. NCTM further describes how GenAI tools can shift the focus of mathematics instruction from *solving* tasks to both *solving* and *verifying* – an evolutionary change where students must critically examine outputs from LLM and engage in deeper reasoning.

This is a very optimistic and forward-looking account on how GenAI could be used to deliver on its promise to change education. The research that has emerged before and since this Position Statement, however, paints a different picture. There is certainly some important, emerging research happening that leverages GenAI to engage students in rich and meaningful mathematical problem solving – probably far more than is represented in this review, as results may not be published at this early stage. But much of the effort, funding, and emphasis in GenAI in mathematics education is being directed at creating AI chatbots or personalized feedback systems, and then making small incremental enhancements to these systems to more optimally respond to students' errors or assess students' knowledge. This may sound promising, but these technologies are primarily being developed in contexts where students are solving simple, repetitive, skill-based mathematics tasks individually on a screen. Critiques of GenAI chatbots have been harsh, with Meyer (2024) arguing that Khanmigo "regards math as machine-executable code, a numbered list of steps that a machine can execute one at a time with any error bubbling up the stack and identifying the earliest step that produced it" and "regards students as buggy computers whose errors should be identified and corrected as efficiently as possible." As a result, Meyer describes how "The lie that Khanmigo perpetuates here is that 'math is about a huge number of small ideas.'" This then leads to the important question of, what actual, pressing problem in mathematics education is GenAI suited to solve?

A survey in 2023 of why K-12 teachers are not using GenAI found that the most common reason was "I haven't explored these tools because I have other priorities that are more important" (Klein, 2024). This was echoed by one of the teachers in the Walkington and Bainbridge (under review) interview study. An Algebra 1 teacher with 10 years of experience teaching in a district composed of predominantly marginalized learners, when asked about using GenAI to personalize content to his students' interests, described how:

> It could help, you know - anything is better than nothing, but that's not the issue. The issue is the gaps of what they do and what they don't know based on where they are… We're trying to fill gaps like the city does potholes. If you've ever seen the cities do potholes, man, they just put something over it. But if the car hit that hole, maybe 20-30 minutes later, the pothole gonna be right back there next month. Instead of tearing up the street and starting over. And unfortunately, that's kind of what we're trying to do… we need to be able to kinda almost start over instead of trying to fix their gaps, the gaps are turning into canyons and in doing this, we're kind of wasting a year because we ain't fixing what the real issue is.

These kinds of sentiments relating to teachers having to confront bigger issues than GenAI can solve were also echoed by a first-year mathematics teacher teaching in a district composed primarily of marginalized learners, in the Beauchamp and Walkington (2024) study. During a discussion of using MagicSchool in the classroom, this teacher described how:

> It's so hard, because I feel like coming in, the teaching philosophy was "Oh I want to make sure my kids are well-rounded and critical thinkers." But now, since like the district is like "Why are test scores so bad? Why are test scores so bad?" it's like, my curriculum is going to be test questions basically, to prep them… They still tell us that we need to be stretching our kids thinking, but I'm like, we only have so much time. So I feel like because my district really does want to see test scores higher, I feel like my curriculum really is just test questions.

Discussing the possibilities of GenAI with mathematics teachers can be a reminder that these technologies may not be particularly effective for solving the larger problems they face with mathematics instruction every day. The mixed reviews from teachers we see in the studies of

GenAI we reviewed lends further credence to this point. This leads to the question of whether the hype around GenAI in education, and its transformative potential, is indeed justified.

Conclusion

I close by considering how can GenAI give us opportunities to truly transform the nature of mathematics education, in the way that the advent of calculators and dynamic geometry software was transformative. First, the practice of students engaging critically with the output of LLMs, particularly their function to create an endless amount of worked examples with explanations, could be powerful. This may be especially important for learners who speak diverse languages or learners in low-resourced settings where human tutoring is not possible. This approach may be particularly effective for student learning if the LLM makes mathematical mistakes that learners must grapple with and reason about – but as these LLMs rapidly become more advanced, mistakes happening with regularity seems increasingly unlikely, especially for K-12 mathematics content. Second, students using LLMs as a thought partner for problem-posing or mathematical story-telling activities seems like a promising direction from the existing research – story-telling is one practice that this technology excels at, and mathematics instruction is often missing the integration of rich, compelling stories about quantitative and spatial experiences.

Third, image-generating GenAI still has a long way to go to be ethical and useful. However, a promising way they could be leveraged is to automatically create rich visual representations to accompany mathematical tasks. This could also function to reduce costs associated with the development of high-quality open access materials that are freely available to teachers and districts. Fourth, mathematics teachers using GenAI as a thought partner to help them brainstorm and iterate upon lesson ideas, adapted for their context and needs, certainly has potential, especially if these lessons would be free. However, there are a variety of logistical and structural issues that may prevent this from being possible for individual teachers, and teachers will need to be prepared to modify and adapt the output of LLMs to suit their needs. Depending on the amount of time this modification takes, LLMs may not greatly enhance efficiency, and may instead mainly enhance creativity.

Finally, the power of GenAI to support students with mathematics skill practice, particularly in its ability to adapt to student needs, does have real-world value. These skill-based tasks are ultimately the kinds of mathematical scenarios that students will be held accountable for being able to solve in K-12 settings, and students' mathematical fluency can have high-stakes implications inside of school. However, the field of mathematics education needs to look beyond such applications of GenAI and consider how this technology, coupled with other initiatives, can help us solve the pressing problems teachers actually face with mathematics instruction.

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MATHEMATICS TEACHER NOTICING:
HOW IT STARTED, HOW IT'S GOING, WHAT'S NEXT

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Drawing on a math lesson from a kindergarten class, we reflect on the evolution of the field's thinking about teacher noticing. Changes in the prevailing theoretical trends have influenced how we think about teacher noticing. In addition, evolving interests have led to the expansion of teacher noticing in new directions, pushing at the boundaries of the idea, as it was initially posed. Similarly, changes in recording technologies have profoundly impacted how we study and support teacher noticing. In reflecting on these developments, we first trace some of the origins of research on teacher noticing, then discuss the current state of research in the field, and finally share some considerations of next steps.

Dorothy Fields, an experienced kindergarten teacher, was participating in an online course for early elementary teachers on mathematical argumentation (Lomax et al., 2017). As part of the course, Ms. Fields video-recorded her students as they engaged in weekly math activities. She then selected portions of the recordings to share and discuss with her peers. It was the fourth week of the course, and teachers were continuing to explore what their students understood about equality.

Imagine for a moment, you asked a group of kindergarteners if the following equations were true or false (Figure 1). What might they say?

$$7 = 2 + 5$$
$$2 + 5 = 5 + 2$$
$$2 + 5 = 5 - 2$$

Figure 1: True/False Questions

Ms. Fields' students gathered on the rug in the front of the room, and she set up her computer at the back of the class to record their discussion. Ms. Fields projected each equation on the board, one at a time, asking students if it was "true or false" and stating that "I want to hear your thinking."

Students had mixed views about the first equation, with one student noting that 2 + 5 = 7 would be the "regular equation." When discussing the second equation, Timmy claimed that it was true "because 2 plus 5, it equals 7, *and* 5 plus 2 equals 7" and that "5 plus 2 is just reversed from 2 plus 5." Ms. Field's revoiced Timmy's idea for the class, pointing to each side of the equation “So, this side is just the reverse of this side.” Ellen agreed with Timmy because of how “in the last one…it [2 + 5] was the same thing as the first thing [7].”

Miles then brought up the idea of “the wall” which Ms. Fields interpreted to mean the equals sign. Miles explained that “they’re both 7, but it’s actually kind of wrong because (pointing to the 5 + 2)…the 2 should be there and then the 5 would be there,” suggesting that 5 + 2 should be rewritten as 2 + 5. Rose and Nelson argued that the equation was false because “there’s two pluses” and if the final + 2 “was gone then it wouldn’t be true.” Rose also explained that “the minus is in the middle” instead of “next to the 2.” Ms. Fields asked Rose if she meant that “the *equals sign* should be over here [next to the 2]” and Rose agreed.

Ms. Fields next turned to the third equation. Ellen claimed “I think it’s false because that one’s a minus… and that’s one says plus.” Callie then added that “this one is false because this one (pointing to 5 - 2) is 3 and this one (pointing to 5 + 2) is 7.” At one point, Timmy added that it was “extra, extra, extra false.”

What do you “notice” in the students’ thinking? Is the idea of 2 + 5 = 7 being the “regular equation” something you have heard before? What might Timmy mean when he says the third equation is “extra, extra, extra false?” What do you think the teacher notices? Is Ms. Fields cueing into students’ comments about the order of the numbers in the equations? Does she notice that Callie’s claim is about the *sum* and *difference* of the two equations, rather than the *form* of the equations? What is it about Rose, or about what Rose says, that prompts Ms. Fields to wonder if Rose meant that the “equals sign is in the middle?”

Teacher noticing continues to be of strong interest to the mathematics education community. Recent reviews note the wealth of publications on teacher noticing, the diversity of the work, and key shared assumptions (Amador et al., 2021; König et al., 2022; Weyers et al., 2024). More and more there is consensus that *what* teachers notice, and *how* teachers notice, matter for student learning. In addition, research continues to document that learning to notice is, to some extent, teachable — that teachers can learn to notice through teaching, through professional development, and with the use of tools (e.g., Larison et al., 2024; Sherin et al., 2011; van Es et al., 2017).

Yet we believe this is a good moment to reflect on the evolution of the field’s thinking about teacher noticing. As trends have come and gone in the study of teacher learning, noticing has been impacted as well. Changes in the prevailing theoretical trends have been reflected in how we think about teacher noticing. And evolving interests have led to the expansion of teacher noticing in new directions, thus pushing at the boundaries of the idea, as it was initially posed. Similarly, changes in recording technologies have profoundly impacted how we study and support teacher noticing. In what follows, we reflect on these developments, first tracing some of the origins of research on teacher noticing, then discussing the current state of research in the field, and finally by sharing some considerations of next steps.

Theoretical Perspectives on Teacher Noticing

How it started. The idea that teacher noticing is a component of teaching expertise had multiple origins, largely within cognitive traditions characteristic of the time. One line of thinking had its roots in the literature on expertise, which documented that experts in a variety of domains have the ability to see patterns and structure in domain phenomena that are different from those seen by novices. Furthermore, these patterns tend to be particularly useful for the types of tasks that are characteristic of expertise (e.g., Chase & Simon, 1973). In the 1980s, Berliner and colleagues applied these ideas to teaching, showing across a range of studies that expert teachers perceived meaning and substance in various depictions of classroom interactions, while novice teachers struggled to do so and often focused on features of the classroom that are less useful for the work of teaching (Berliner, 1986; Carter et al., 1988). Returning to our example, consider how Ms. Fields' interaction with Rose's idea about "the minus" being "in the middle" might reflect her teaching expertise. In her response, Ms. Fields pursued how Rose was thinking about the organization of the equation and where certain pieces belong rather than Rose's inaccurate use of the term "minus" to refer to the equals sign. As an experienced teacher, we think it is quite possible that Ms. Fields foregrounds Rose's comments about the organization of the equation because that is an aspect of Rose's reasoning that was more relevant to her learning.

Around the same time, Erickson and colleagues investigated what they referred to as "teachers' practical ways of seeing" (Erickson et al., 1986). Central to the cognitive paradigm was viewing teaching as a way of thinking that involved a particular set of specialized knowledge and cognitive processes. Erickson's claim was that to truly understand teaching, one needed to understand how teachers "observed and made practical sense of what happened in their classrooms daily" (Erickson, 2011, p. 19). Erickson explained further that teacher noticing was selective and multidimensional, and at times opportunistic, tied to areas where teachers believed they might act.

Mason (1982, 1987) also explored the construct of teacher noticing in the early 1980s. Mason's approach emphasized the importance of teacher awareness — of teachers paying attention to what and how they see mathematics and mathematics teaching and learning. Mason (2002) referred to this as the "discipline of noticing" to emphasize that effective teaching involved preparing to notice in particular ways, and required sustained effort and energy to successfully do so. In this way, Mason made clear his position that teachers can develop their noticing capabilities.

My own (M. Sherin's) interest in teacher noticing developed in the early 1990s as I worked with colleagues on a project in which teachers watched and reflected on excerpts of classroom videos (Frederiksen et al., 1998; Gamoran, 1994). Goodwin's idea of "professional vision" (Goodwin, 1994) — that members of any professional group develop specialized ways of noticing — guided my work. A key feature of Goodwin's idea is that expert vision is not simply something that individuals acquire, on their own, through repeated practice and experience. It also is partly shaped from their participation in a profession, one with its own technical foundations. To me, then, the idea of professional vision further elevated teacher noticing and did so at a time when professionalizing teaching was an important goal (Firestone, 1993).

All of these lines of work shared the idea that perception was an active process; that is, that teachers direct their attention to significant features of the classroom environment as they engage in teaching. Additionally, these influences emphasized that teacher decision-making is dynamic and dependent in part on what teachers perceive to be taking place in the moments of instruction.

How it's going. Research on teacher noticing has continued to expand and evolve over the past forty years. Much of this work continues to take a cognitive approach, describing the cognitive processes involved as individual teachers notice classroom events. However, over the last decade, increasing attention has been paid to what König et al. (2022)[6] refer to as a socio-cultural perspective of teacher noticing. This perspective aligns with Goodwin's claim that "the ability to see a meaningful event is not a transparent, psychological process, but instead a socially situated activity" (Goodwin, 1994, p. 606). Furthermore, it foregrounds the idea that noticing is not just a social but a cultural activity, and that what a teacher notices is necessarily situated within the larger systems and discourses in which teaching and learning take place, systems that include, but stretch far beyond, the profession of teaching. In an early example of this approach, Levin et al. (2009) highlighted the challenge a teacher faced in consistently attending to student thinking in her classroom given direction from school administrators to focus on classroom management and content coverage, pressures that the school administrators may have felt more acutely given the recent passing of the No Child Left Behind Act (Klein, 2015). Lefstein and Snell (2011) similarly emphasized the political nature of teacher noticing, suggesting that there are often multiple "professional visions" (p. 505), and that teacher noticing is inherently political given that some ideologies are privileged while others are marginalized.

Expanding on such work, a number of recent studies examine the relationship between teacher noticing and cultural, historical, and ideological perspectives (Chen, 2020; Dominguez, 2021; Louie et al., 2021; Scheiner, 2023; Shah & Coles, 2020). These studies illustrate, for example, that dominant discourses in mathematics education position students of color as less capable mathematicians and, therefore, influence teachers' perceptions of "smartness" in student contributions (Louie, 2018). Returning to our example, a socio-cultural perspective would ask us to consider, among other things, ways in which Ms. Field's noticing was influenced by her perceptions of Timmy, Ellen, Miles, and others more broadly, including their status in the classroom community and their broader societally-relevant identities. In short, this perspective emphasizes that what a teacher does and does not notice is shaped by cultural forces, and that research on teacher noticing must consider the broader framings within which noticing takes place, particularly in our efforts to help teachers develop noticing practices that promote equitable and meaningful mathematics learning (Horn & Garner, 2022).

What's next? The developments discussed in the preceding subsection all push on the notion of teacher noticing and in quite reasonable ways. It is certainly a good idea to think of teacher noticing as socially-situated. It also seems certain that a teacher's noticing will be dependent on aspects of their past histories, such as, generally speaking, their cultural and ideological backgrounds (Hand, 2012; Mendoza et al., 2021).

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[6]König et al. (2022) introduce a total of four perspectives on teacher noticing — cognitive-psychological and socio-cultural, as discussed here, as well as discipline-specific and expertise-related. We would argue that the discipline-specific and expertise-related also take a cognitive approach to the study of teacher noticing. And of course while discussed separately, these perspectives are often interwoven in research.

But it's less clear to us what this should mean for how we think about teacher noticing – how we should theorize about and study it going forward. In the proposals mentioned, there is still something called "noticing" that a teacher does. The proposals add inputs, and they say that what a teacher notices is going to heavily depend on these inputs. The details of the classroom context heavily impact a teacher's noticing. And various inputs throughout a teacher's life impact the noticing that they do in the classroom.

But we should worry whether the developments discussed above require a more fundamental kind of change. Are we doing the equivalent of adding epicycles[7] to a theoretical framework that needs a more thorough rewriting?

Sherin and Star (2011) distinguish multiple possible ways to understand teacher noticing. One of these is noticing as *a cognitive sub-process.* This, loosely speaking, is how noticing was originally understood. A second possibility they discuss is noticing as *an emergent property of the larger cognitive system*. We believe that the above developments suggest the possibility that we should consider a third kind of understanding of teacher noticing: Noticing as *an emergent property of a teacher and a classroom, taken together.*

Let's consider this latter possibility for a moment. If we see noticing as an emergent property of a teacher-classroom system, where do we look to see teacher noticing? How can we say what is "noticed?" One possibility is to simply say that an attribute of the classroom context is noticed if it in some manner impacts a teacher's behavior.

This stance, if adopted, has some intriguing implications for how we might think about teacher noticing going forward. Suppose that Ms. Fields did not react, in any way, to Rose's inaccurate use of the term "minus." If we adopt the emergent perspective on noticing, then we would say that Ms. Fields did not *notice* Rose's use of "minus."

Further, we imagine other teachers may notice differently than Ms. Fields; not all teachers will act the same in a given context. In some manner, a teacher's behavior depends on their past history. A way to accommodate this is to borrow a term from Greeno (1998). We can say that, based on their histories, teachers develop "attunements" - propensities to be impacted by some features of the classroom context.

So, in summary: we might say that teachers develop attunements based on their past histories. In the moment, in part because of these attunements, teachers behave in a manner that depends on some features of the classroom context and not others.

We are not sure we want to argue for this new way of thinking. Our own belief is that it will be productive, going forward, to adopt multiple perspectives. But we do want to make the general point that, given the observations mentioned in the preceding section, we do have to think hard about where to go next. This emerging line of work raises an important question - is it sufficient to add new inputs and dependencies? Certainly, understanding the expansiveness of what captures teachers' attention and why is important. At the same time, it prompts us to ask, how does this work advance our understanding of what constitutes teacher noticing or what

[7]Some early astronomers added epicycles to geocentric models of planetary motion in order to better account for observational data. In these models, the planets move in smaller circles — the epicycles — as they trace their larger orbit around the Earth. The models were complicated, but could make highly accurate predictions.

teacher noticing is? Does it invite us to see noticing in a new light, such as, that noticing is being done and, in the doing, noticing enables new forms of action and social relations? If not, should there still be something called "teacher noticing?"

The Boundaries of Teacher Noticing

How it started. As work in teacher noticing continues to expand and evolve, it is important to carefully consider what we see as the *boundaries* of teacher noticing — that is, what belongs in a theory of teacher noticing. Early work in human perception emphasized the active nature of noticing, which involves both bottom-up and top-down processing (Gibson, 1950). On the one hand, we recognize patterns in available visual information and then apply our knowledge and experiences to make sense of what is seen (think here of walking into an unmarked building, seeing gurneys and people in scrubs, and realizing that you are in a medical center). On the other hand, our knowledge and experiences help us perceive and interpret visual information (think of walking into a medical center and immediately looking for the reception area in order to check in).

In line with such work, initial conceptions of teacher noticing emphasized two key subprocesses: selective attention, or *attending*, reflecting the idea that teaching involves attending to some events while not attending to others, and knowledge-based reasoning, or *interpreting*, reflecting that teaching involves teachers using their knowledge and experiences to make sense of what they observe (Sherin, 2007). Furthermore, attending and interpreting were understood to act in a dynamic manner.

Our decision to include both attending and interpreting in the construct of noticing was not driven by theory alone, as empirical work with teachers illustrated the close connection between these two subprocesses. For example, after viewing the recording of her class, Ms. Fields wrote, "I was not surprised by Timmy's answers because I know he has a solid understanding of the equal sign." In line with Sherin and Star (2011), we believe that Ms. Fields' statement points to something that she noticed about Timmy's comments, but it is not possible to distinguish what she attended to from her interpretation. Nor is it possible to say that these subprocesses operated in some kind of typical chronological fashion (Castro Superfine et al., 2017). It may be that, based on experiences with Timmy during the previous week's activity related to equality, Ms. Fields was primed to attend to Timmy's answers in a certain way. Or it could be that attending to Timmy's comments in class were what convinced her that he had a "solid understanding of the equals sign." Either way, empirical work supports the theoretical claim that "attending and interpreting are inextricably linked" (Castro Superfine et al., 2017, p. 422).

How it's going. Researchers have sought to expand the construct of noticing further by introducing other components. The most widespread of these is related to the introduction of *professional noticing of children's mathematical thinking* by Jacobs et al. (2010). In doing so, Jacobs et al. defined three central processes: (a) attending to children's strategies, (b) interpreting children's mathematical understandings, and (c) deciding how to respond on the basis of children's understandings. They emphasized that "deciding how to respond" is temporally linked to attending and interpreting, and therefore should not be separated. The authors write, "We suggest that… the three component skills of professional noticing of children’s mathematical thinking—attending, interpreting, and deciding how to respond— happen... almost simultaneously, as if constituting a single, integrated teaching move” (Jacobs et al., 2010, p. 173). Blömeke et al. (2015) similarly include decision making as an integrated component of noticing. What might these scholars point to as evidence of Ms. Fields “deciding how to respond”? Perhaps when Ms. Fields followed up with Rose about the placement of the equals sign rather than Rose’s inaccurate use of the term “minus,” Ms. Fields had made a decision based on what she believed about Rose’s thinking — a decision that was not distinct from her attending to and interpreting Rose’s thinking.

Two additional components were introduced in 2021. In their discussion of teacher noticing from a sociopolitical perspective, Louie et al. (2021) positioned *framing* as an important component of noticing. Building on previous studies that explored the relationship between framing and noticing, they chose to “locate framing *within* noticing, as an integral process that both shapes and is shaped by other noticing processes” (p. 97). In doing so, they sought to elevate ways that broader sociopolitical frames, like deficit discourses about students of color or framings of mathematics as “universal, objective, and fixed” (p. 98), are influential in a teacher’s framing in the moment and intimately bound up with processes of attending, interpreting, and (in their framework) responding. How might this idea of framing as connected to noticing be evident in our example? Consider that in the lesson, Ms. Fields probes statements from some of her students, while moving on after the responses of others. Might this indicate that she has a fixed sense of some students’ abilities and is unlikely to notice potential insights in their comments or to believe there is more to what they are thinking that was stated? Thus, what she notices about students’ thinking is connected to her sense of their capabilities as mathematical learners. Alternatively, Ms. Fields may be aware that some of her students seem to have a great deal of authority in the classroom, and so she attends more closely to the thinking of those who do not, in order to ascribe them more agency among their peers. In this case, her noticing about students’ thinking is connected to her beliefs about their positionality among their peers.

Around the same time, van Es and Sherin (2021), revisiting their early work on teacher noticing, introduced a third component to their model of teacher noticing, what they refer to as *shaping* which involves “teachers constructing interactions, in the midst of noticing, to gain access to additional information that further supports their noticing” (p. 23). The idea then, is that teacher noticing involves not only attending to and interpreting what is happening in the classroom, but also actively seeking out, and in some cases creating, information that can be noticed. This component builds from the notion that noticing is active and in addition, that noticing involves interaction with one’s environment. van Es and Sherin discuss shaping specifically in the context of student mathematical thinking. Looking at our example, they might point to Ms. Fields’ question to the class “Does anyone think this is false?” as evidence of shaping. In response to several students having shared their thinking that 7 = 5 + 2 was true, Ms. Fields was creating opportunities to notice students who disagreed. Later in the lesson, Mason responded to the discussion that 2 + 5 = 5 + 2 is true by stating “if you just put the wall on” to which Ms. Fields said "Tell me more about the wall." Again, Ms. Fields was promoting an opportunity to have access to additional information about this idea of "the wall."

What's next? As with our discussion of theoretical perspectives, evolution of the boundaries of a construct like teacher noticing feels natural as the field engages with it. New components can represent progress and refinement. However, new components can also muddy the waters of conceptual and analytical clarity. Here, we consider both sides of the coin in relation to teacher noticing.

Conceptualizing components like *deciding how to respond*, *framing*, and *shaping* as part and parcel of noticing can help to draw our attention to key purposes and facets of the work teachers are doing when they notice classroom events. For instance, as Erickson (2011) discussed, teacher noticing is often tied to aspects of classroom events on which teachers want to act - including *deciding how to respond* as a component of noticing can emphasize this purposeful, selective nature of noticing on the part of teachers. Including *framing* as a subprocess of noticing can intentionally promote the field's attention to the broader contexts in which teachers and classroom interactions operate, especially which culturally dominant frames or counterframes may be in play in acts of noticing. Positioning framing as a component can productively push researchers to consider this in any noticing analysis, not just ones in which relations between framing and noticing are an explicit focus of study.

Yet additional components can also make it challenging to know where teacher noticing starts and stops. Are we headed down a slippery slope in which many acts of teaching fall under the umbrella of noticing (Sherin, 2017)? For instance, engaging in *shaping* in a classroom interaction, in order to create better opportunities for attending to and interpreting student thinking, looks quite similar to revoicing or asking probing questions (Franke et al., 2007). Did we ourselves create this potential slope when we extended noticing beyond attending to interpreting, to also include shaping (van Es & Sherin, 2021)?

In short, we are asking ourselves and wish to ask the field — in what ways and for what purposes is it useful to think of these components as *part of* noticing, versus *entangled with but separate from* noticing? What are the implications for making progress on understanding teachers' practice and thinking writ large? For designing teacher learning environments that cultivate noticing? Can teacher noticing be useful as a construct, if everything is teacher noticing?

Using Video to Support Teacher Noticing

How it started. In the 1960s, video cameras became more portable and less costly, and recording in classrooms as part of teacher education and professional development became increasingly common. One of the earliest uses involved a practice called microteaching in which participants were recorded using a specific teaching skill, then assessed their use of the skill based on the recording. Another approach, interaction analysis, involved the use of an observation instrument that outlined specific student and teacher behaviors to note when viewing a recording of a lesson. Both microteaching and interaction analysis were based on the idea that teaching consisted of a set of observable teacher actions and that there exists a straightforward relationship between those actions and student outcomes (Russ et al., 2016).

By the mid 1980s, the cognitive paradigm had become quite popular in educational research, and research on teaching focused on the mental structures of teachers, including identifying specific categories of knowledge. Recording teachers and classrooms was used as a way for researchers to investigate teachers' thinking, and new approaches to the use of video with teachers were introduced. For instance, in 1994 the National Board for Professional Teaching Standards began to accept applications for certification that included video segments and corresponding teacher reflections (Goldhaber et al., 2004). Around the same time, M. Sherin and colleagues began to work with groups of teachers in *video clubs* in which participants watched and discussed excerpts of videos from their classrooms (Frederiksen et al., 1998).

One of the outstanding questions of the time was what video might help teachers learn. Video cases, for example, were often geared towards the development of teachers' subject matter knowledge or pedagogical content knowledge, though research was mixed on whether such efforts were successful (Sherin, 2004). In contrast, consideration of the affordances of video suggested that video was particularly well-suited to support the development of teacher noticing. In particular, because video provided a permanent record of classroom interactions that could be viewed repeatedly and did not require teachers to respond immediately as is needed when teaching, video came to be viewed as a valuable resource for working with teachers around their noticing capabilities.

How it's going. Over the past 25 years, the use of video to support teacher noticing has received a great deal of attention and innovation. A range of research has documented that video clubs and other video-based professional development can promote the development of teacher noticing as teachers shift in *what* and *how* they notice (Gaudin & Chaliès, 2015). A common focus in this work were efforts designed specifically to help teachers notice students' mathematical thinking (Santagata et al., 2021). Such work demonstrated that reflecting on video of one's teaching enabled both pre-service and in-service teachers to develop productive approaches for attending to and making sense of the ideas that students raise (Kleinknecht & Gröschner, 2016; Seidel et al., 2011; Santagata & Guarino, 2011). Related research looked specifically at how to situate video productively in order to support teachers' learning to notice as well as the role of facilitation in such efforts (González et al, 2016; Seago et al., 2018; van Es et al., 2014). Of interest also was work that explored implications for instruction. Sherin and van Es (2009) and van Es and Sherin (2010), for example, documented that the development of teacher noticing in video clubs did in fact influence teachers' classroom practices, as teachers came to use the noticing strategies developed in the video clubs during instruction.

Research on the use of video to support teacher noticing has also explored the use of various video-based tools in this process. For example, Larison et al. (2024) and Walkoe et al. (2020) describe the use of a video-tagging tool to support teachers' ability to identify and comment on interesting moments of students' mathematical thinking in classroom video excerpts. Others discuss the use of video annotation tools that allow teachers to mark moments of video as associated with particular features of a framework (Suh et al., 2021) or video editing tools that invite teachers to highlight particular moments of a video for themselves and others (Chen & Chan, 2022; Fadde et al., 2009). Another important line of research examined specific features of video clips that promote teacher noticing (Sherin et al., 2009) and in particular, the characterization of “critical events” in which student’s mathematical thinking is particularly salient and therefore can serve as key tools in helping teachers learn to notice (Leathem et al., 2015; Rotem & Ayalon, 2022).

What’s next? For the most part, research on the use of video to support teacher noticing has been based on the idea that teacher learning occurs through review and discussion of video excerpts among teachers. Further, video excerpts for discussion have often been chosen by teacher educators and researchers, rather than teachers themselves. More recently, however, we and others (e.g., Borowiec et al., 2022; Brouwer, 2022; Xiao & Eriksson, 2020) have begun to examine how the acts of self-capturing video in one’s classroom and selecting excerpts to share with others can also be sources of learning to notice for teachers. Sherin and Dyer (2017), for example, document that in preparing to record student mathematical thinking in their classrooms, teachers engage in a kind of *anticipatory noticing* in which they predict in what part of a lesson student mathematical thinking will be visible and what students might say or do at that time. In addition, teachers consider some of the logistics of recording with an eye towards making sure students’ ideas can be seen and heard on the video (Richards et al., 2021). Thus, prior to recording, teachers engage in what Mason (2002) described as preparing to notice. While recording a lesson, teachers also have opportunities to develop their noticing as they identify interesting student mathematical thinking taking place and, in some cases, adjust the lesson in progress in order to provide additional opportunities for student thinking to be visible. Finally, learning to notice can occur as teachers review their videos in an effort to select portions of a lesson to share with others.

Animating some of these opportunities with our example, we focus on Ms. Fields’ initial review of the lesson in an attempt to select a 3-4 minute clip to share with peers in the course. Ms. Fields explained that she was looking for parts “where the thinking [in] the conversation is best,” or what was “interesting or different or surprising.” As she watched the recording on her own, Ms. Fields commented on what she noticed. “That was a good point. It's just reversed.” “So, she's thrown off by the, the set up of the equation.” “Miles changed his thinking here. He wanted it to look like two plus five equals two plus five and then it would be true and then at the end, he realized that it still is the same, which I think is... it's interesting.” In an interview later in the course Ms. Fields explained that sometimes watching the video reminded her of something she had noticed while teaching, while at other times she noticed new aspects of what students shared. In fact at one point she stated “I think this part is better [than] what I remember.”

After viewing the recording Ms. Fields explained that she would go back to “the most interesting moments” and “trim that part” to prepare a 3-4 minute excerpt to share with her peers in the course. As she made the selection for this lesson Ms. Fields explained that she “wanted to capture when she changed it to subtraction.” More generally, she explained that she wanted her recording to “show my colleagues…how [students] approached the numbers andtheir different thinking." Her peers similarly described attending closely to what they considered "more meatier" discussion among students and moments that showed changes in students' thinking.

We believe that this expansion of the use of video for teacher noticing is quite significant. As video equipment becomes smaller and more ubiquitous, teachers can more easily record their classrooms, and we are seeing that doing so can provide novel opportunities for meaningful teacher learning. Further, in our view, these shifts importantly put teachers at the helm of deciding what to elevate and discuss in video as classroom professionals, rather than locating this decision-making primarily or solely with teacher educators or researchers. While we still envision a role for teacher educators and researchers in designing and facilitating video-based programs for teachers, we feel that this represents an important move toward democratizing video-based teacher noticing and learning. Looking forward, we are intrigued by the possibilities of AI for facilitating teachers' selection of video excerpts from their classrooms, while of course cautious of the ethical implications of doing so. Imagine Ms. Fields with a full corpus of video from her kindergarten class asking chatGPT to pull out moments where she said "interesting" in class, or to identify moments where students talked directly to each other. What possibilities might this open up for how the field both theorizes teacher noticing and studies teachers' learning to notice in the future?

Reflecting on the evolution of the construct of teacher noticing and the different ways noticing has contributed to our understanding of teaching, we believe that continuing to question what it means to notice and, therefore, how to study teacher noticing is a worthwhile endeavor. Our snapshot of Dorothy Fields offers a window into the ample and different opportunities for teachers' learning to notice in the context of their recording, selecting, sharing, and reflecting on video of classroom activity. Indeed, we suspect that technological advancements and novel designs for teacher learning will continue to influence research on teacher noticing and how we conceive of and promote teacher noticing.

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RESCUING THE BABY: FORMATIVE DIAGNOSTIC ASSESSMENT USING LEARNING TRAJECTORIES[8] [9]

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The state of assessment in mathematics education is arguably precarious at this time. Many educators question, if not downright reject, the use of high-stakes tests (Au, 2022). In fact, the climate of anti-testing in mathematics education has reached a point where some doubt the value of measurement itself, as a tool for the improvement of learning, (Zhao, 2020). Others are staunch advocates for the use of formative assessment to stimulate powerful reflective teacher noticing of students' approaches but may overlook that formative assessment practices seldom accrue records of student progress over time—either individually or across groups of students. One might say that the field has thrown out the baby (measurement) with the bathwater (high-stakes assessment). My goal is to rescue the baby, placing it carefully in the category of "assessment for learning", rather than "assessment of learning" (Black & Wiliam, 1998).

In this paper, I introduce the concept of a Learning Trajectory-based Diagnostic Assessment system (LTDA)[10] as an alternative genre of assessment, which combines diagnostically valid with low-stakes characteristics and includes longer term record-keeping. I report on the kinds of contributions to instructional improvement made possible by harnessing and making accessible the rich foundation of research in the learning sciences and describe some emergent dynamics that influence student outcomes. This paper does not review related diagnostic systems but rather shares the story of the design and implementation of a prototype with the potential to improve practice at scale.

Section One: Achievement, Student Voice, and Learning Trajectories

The Current Context

Most readers here are probably familiar with the data from The Nation's Report Card or the National Assessment of Educational Progress (NAEP), which I review as a starting point. The NAEP mathematics scores were trending modestly upward until they plateaued in 2013. From 2019 to 2022, during the pandemic years, there was a steep drop on both the fourth grade and eighth grade assessments (NAEP, 2022a; Figure 1). Of the regionally sampled NAEP test takers, the average scores in low-performing percentiles dropped more precipitously than those of the higher ones (NAEP, 2022b; Figure 2), as can be seen by comparing the decreases for students in the 10th percentile (12%) to the 90th percentile (only 3% decline). The declines during 2020-2022

[8] A shorter version of this paper was presented as a talk at the IES Math Summit, Sept. 2024. This version has been modified and expanded considerably.

[9] I wish to thank Alan Maloney, Meetal Shah, and Erin Krupa for their helpful comments on earlier drafts.

[10] At a 2010 conference (Confrey et al, 2011), the term IDAS (Interactive Diagnostic Assessment System) was introduced. Subsequent work discussed here adds in the use of learning trajectories to create a LTDA.

were even more substantial for Black (13 points) and Hispanic (8 points) students than for White students (5 points) (NAEP 2022a; Figure 3). Some scholars have raised concerns that too much focus is placed on the differences in scores (called "gap gazing" (Gutiérrez, 2008)) and that doing so leads to deficit thinking (Davis & Museus, 2019; Valencia, 1997). As valid and as historically justified as these perspectives are, it is still critical to disaggregate achievement scores; these data are essential for demanding action on inequity in instructional opportunities. We should simultaneously keep in mind that 75% of *all* of our 12th graders have been performing below proficiency since 2005. As Bob Moses reminded us, "As a nation, we stand with bated breath, waiting for public schools to reopen, and for a return to normal, while ignoring that for many 'normal' is not only not good enough, it was never really good." (Moses, 2021).

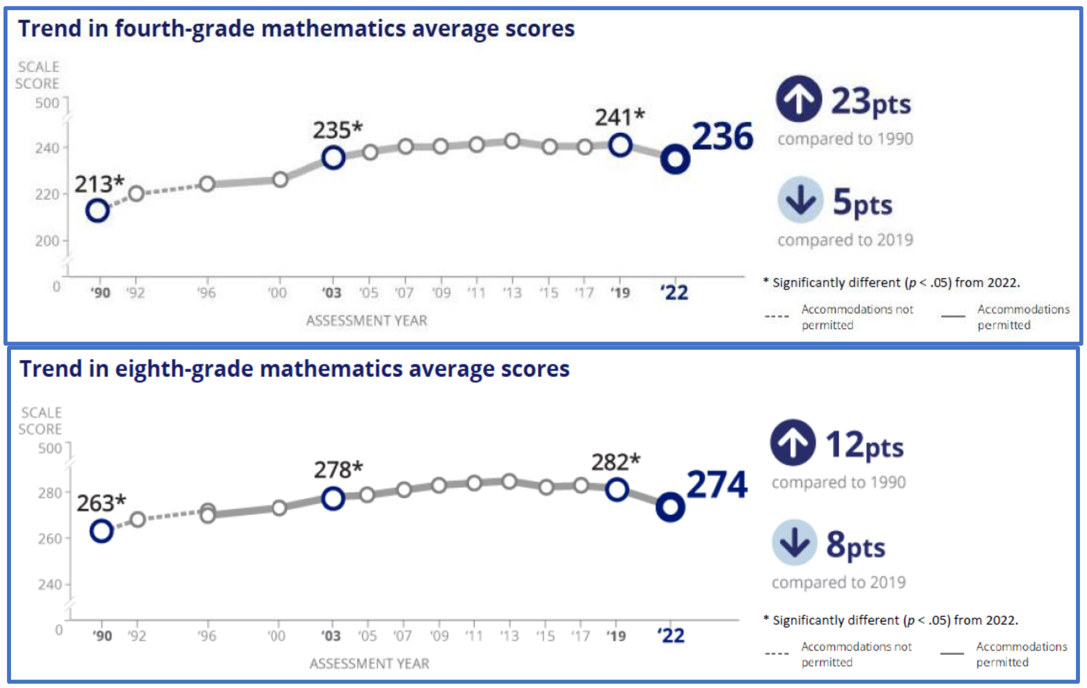

Figure 1: Average math scores trend upward (1990-2013), level off (2014-19), then drop (2019-22) (https://www.nationsreportcard.gov/mathematics/supportive_files/2022_rm_infographic.pdf)

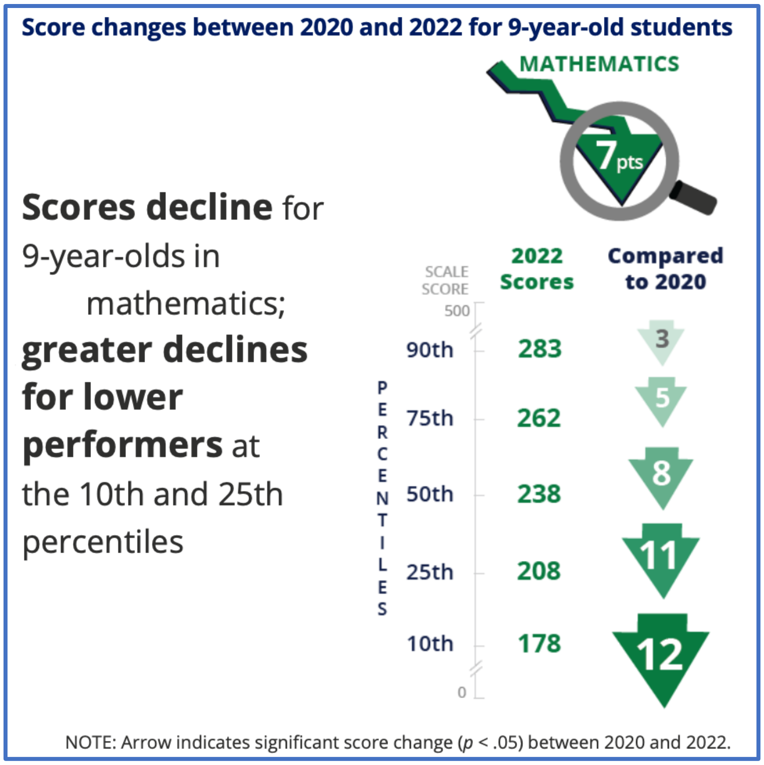

Figure 2: Lower performers' scores dropped more precipitously (https://www.nationsreportcard.gov/highlights/ltt/2022/supporting_files/ltt-2022-age9-infographic.pdf)

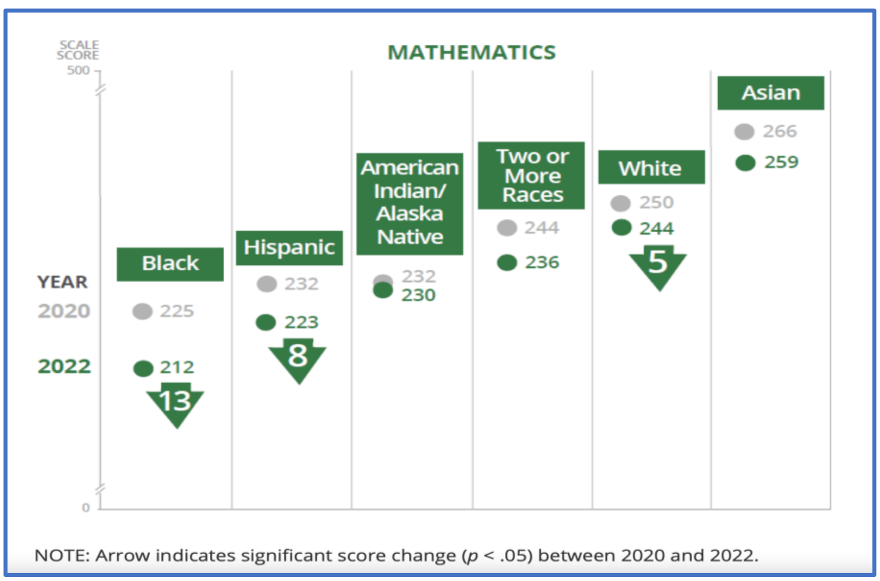

Figure 3: Score reductions are greater for Black and Hispanic students (https://www.nationsreportcard.gov/highlights/ltt/2022/supporting_files/ltt-2022-age9-infographic.pdf)

These data imply massive systemic failure both in policy and in practice. Further, inequitable post-pandemic outcomes were predictable—not as student failure, but as direct evidence of how rapidly learning losses accrue if equitable, adequate resources and supports are withheld (Kane & Reardon, 2023). At this point in our country's history, we should be pursuing a commitment towards solving a nationwide problem; namely, how to achieve quality education as a constitutional right for all citizens (Liu, 2006).

Focusing on Learner's Voice

To generate alternative approaches to improve instruction and outcomes, we need to focus on the learners and their perspectives. I, along with others, have argued repeatedly that students are, ironically, the most underutilized resource in our educational system. I view them as receivers of their education rather than as partners in a journey. In order to describe how to design a more robust approach to assessment and use it to foster instructional improvement, we begin with an admission that, in general, adults are not very good at listening to and seeing learners (Confrey 1991).

To give the reader a sense of what it means to listen to students' mathematical reasoning, I begin with a story. I asked a six-year-old to explain what the digits in the number 314 stood for. Using base 10 blocks, he pulled out 3-100s blocks (flats) and 14 single blocks. His representation (Figure 4a), while correct, was missing the interpretation of the one as a single ten (along with four ones). He saw the digits as representing 300 + 14. As I pondered how to respond to the missing use of ten-stick, he said, "It's like the clock." And he drew two boxes on the paper, as shown in Figure 4b (without adding numbers).

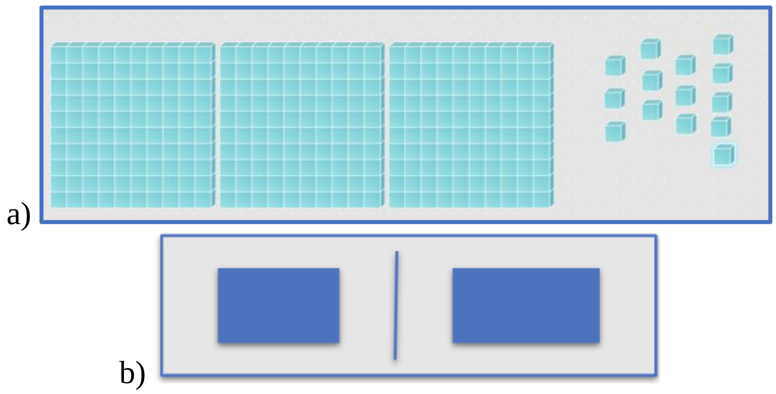

Figure 4: a) Child's representation of the number 314 using Dienes blocks. b) Child's representation of a digital clock display

Now my question is, how would you respond if you were his teacher? Take a second to think about this before reading on.

For the number 314 in our base ten system, each of the digits represents a different unit; hundreds, tens, and ones. This system, associated with counting, operates so that as the number of units reaches ten, the single digit numeral is replaced by a 1 and a zero. The numeral 1 no longer represents a single unit but rather the composite unit 10, consisting of 10 ones, hence it is a system of placement of the digits (hence place) and value of that digit (based on place as ones, tens, hundreds).

Typically, his comment about the clock would either be ignored or corrected, calling his attention only to the need for a ten-stick to represent the fourteen. But if this were how a teacher proceeded, s/he would miss an opportunity to consider how his response sheds light on a particular–and ubiquitous–use of place and value in our culture.

Consider his second representation: two boxes separated by a vertical line. It *does* mirror what one sees on digital clocks. In the case of clocks, the two boxes represent hours and minutes. On a timer (*e.g.,* on a microwave oven), the boxes could represent minutes and seconds. In either case, the numeric values in the boxes or places can exceed 10; they can range to 12 or 24 in the case of the hours, and 1 to 59 for minutes or seconds. The boxes can contain only certain *values*, whose meanings depend on their *placement*. Further, they represent a place value *system*, because there is a relationship between the places that determines the conversion from seconds to minutes or minutes to hours. Thereby one can see that digital timers contain a rudimentary place value system, and one for which his representation of 314 as, for instance, 3 hours and 14 minutes, would be accurate and complete. (And it is also the case that our base ten system of place value is operating *within* each of the places referenced in the clock/timer.)

So, has this child demonstrated an understanding of place value? Note that with his second idea he demonstrates a sound, culturally-situated response to the question of what the digits could represent (time); in doing so, he articulated an understanding of a connection between place and value with his boxes. (If I had asked him about the relative values of the numbers that could be in the two boxes, I could have further assessed the extent of his understanding of a system of exchange or conversion.)

This anecdote and the description of a common response serve several purposes. First, children's inventions should be a basis on which they are made to feel welcome, engaged, and capable in mathematics. However, a common experience of learners is having their ideas under-recognized and undervalued. Children's expressions of ideas, their insights and inventions, frequently and unintentionally, fall on deaf ears, a common occurrence (which I have witnessed in hundreds of classrooms across the country over the course of my career). Alternatively, but no better, children may receive merely superficial encouragement, a kind of pat on the head, when in fact, their idea's validity, vitality, and legitimacy are being overlooked or disregarded. Students' experiences of this kind could even be considered a form of micro-oppression. Over time, for a child to have his or her ideas treated as irrelevant has a deadening effect on a child's eagerness to learn.

Much is asked of teachers. For instance, elementary teachers, who are often weak in or frightened of mathematics, are assigned to teach *all* subjects. What supports are provided for most elementary teachers? Typically, the answer is two meager courses during teacher prep programs around the country. Secondary teacher candidates likewise cannot possibly re-examine all of K-12 mathematics as they complete math and other required coursework. As a result, there is a fair amount of evidence that American teachers are not adequately prepared to recognize children's legitimate alternative competences nor to steer them toward more complete understandings. This requires significant insight and a strong foundation in mathematical reasoning. And few school districts offer professional development with a strong focus on content learning (Scott & Philip 2023).

Ma (2010) conducted a study that compared the ways Chinese and American teachers discuss elementary math content including the topic of place value. Comparing their explanations of place value, she documented that Chinese teachers refer to the relationship among the places in a place value system as “decomposing a unit of higher value” (p. 7). This phrase refers to the fact that each time one switches places, from ones to tens or tens to hundreds and so on, a group of ten of a given unit or value comprises the next unit. In a base ten system, that exchange rate across units is always in the ratio 10:1. In contrast, the digital clock place value system has a variable rate; switching from 60:1 (for seconds to minutes and then minutes to hours) but it switches to 12:1, for an hour to a half day where the clock recycles, or 24:1 if days are recorded. Ma’s study documented that American teachers said little about this underlying mathematical principle, speaking only of the act of “carrying” in addition.

One might hope the Standards will help address these teacher needs. If one examines the relevant standards for place value (Table 1), one sees that they do express what kind of numbers to analyze each year (teens, two-digit, three-digit) and how to represent the digits in terms of bundles of ones, tens, and hundreds. But the Standards provide little insight or guidance into how these composite units are formed, how the key ideas of “place” and “value” are combined and related; nor do they express the idea of a constant or variable “rate of composition/decomposition.”

Table 1: The Common Core Standards for Numbers and Operations in Base Ten for K-2

K.NBT	*1.NBT*	*2.NBT*
Work with numbers 11–19 to gain foundations for place value. *.A.1 Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 +8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones.*	*Understand place value.* *.B.2 Understand that the two digits of a two-digit number represent amounts of tens and ones. Understand the following as special cases:* *.a 10 can be thought of as a bundle of ten ones — called a "ten."* *.b The numbers from 11 to 19 are composed of a ten and one, two, three, four, five, six, seven, eight, or nine ones.* *.c The numbers 10, 20, 30, 40, 50, 60, 70, 80, 90 refer to one, two, three, four, five, six, seven, eight, or nine tens (and 0 ones).* *.B.3 Compare two two-digit numbers based on meanings of the tens and ones digits, recording the results of comparisons with the symbols >, =, and <.*	*Understand place value.* *.A.1 Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases:* *.a 100 can be thought of as a bundle of ten tens — called a "hundred."* *.b The numbers 100, 200, 300, 400, 500, 600, 700, 800, 900 refer to one, two, three, four, five, six, seven, eight, or nine hundreds (and 0 tens and 0 ones).* *.B.2 Count within 1000; skip-count by 5s, 10s, and 100s.* *.B.3 Read and write numbers to 1000 using base-ten numerals, number names, and expanded form.* *.B.4 Compare two three-digit numbers based on meanings of the hundreds, tens, and ones digits.*

Realistically it is unreasonable to expect the Standards alone to support a teacher's readiness to recognize and support the range of students' inventions and nascent reasoning. Standards express goals for accomplishment at particular grade levels to allow coordination of topics across grades and set expectations for levels of achievement.

Other resources could be synthesized to help teachers consider student reasoning about place value, specifically the extensive research on "composite units" (Boyce et. al, 2024; Steffe, 1992; Ulrich, 2015). Built within constructivist paradigms, this research documents how students come to create bundles into larger units and how they use these units in place value, skip counting, and multiplication and division. However, to support learner-centered instruction, this type of research from the learning sciences must be synthesized and made accessible to and actionable by teachers. Seldom does this occur.

In this one small story, I have tried to encapsulate how a child's explanation often needs to be recognized and acknowledged as "correct" in relation to his own experiences and ideas, and that these explanations are potentially productive in a shared journey to more sophisticated reasoning. But it also further represents the need for teachers and researchers to use such student explanations as opportunities to reexamine their own understanding of mathematical ideas and meanings.

I formalized the concepts underpinning this story when I developed the notion of a "voice-perspective dialectic" (Confrey, 1998) and it has served as an underlying theme of much of my subsequent work. *Voice* refers to the *learner's* way of making sense of a situation; *perspective* refers to the way the *more experienced listener* (a researcher, teacher, etc.) makes sense of the situation (Figure 5a). The reason it is a *dialectic* is that a listener cannot directly know what the student is perceiving and thinking. Instead, the listener filters the student voice through their own perspective (Figure 5b). When that perspective is formal mathematics training, understanding a child can require the expert to relax, or momentarily ignore, their prior knowledge. In this case (the student's clock idea), a listener need *not* see 14 indivisible blocks as wrongly ignoring the tens place in place value, but as situated in a place value system where the box on the right can range to sixty before conversion (and hence can have two places in the base ten system). It requires one to break set on the "rigor" expected in mathematics, in order to build a bridge from student ideas and towards the connections and distinctions among concepts. As one begins to understand the student voice, another fundamental opportunity emerges: the voice-perspective dialectic reverses and one can employ student voice as a means to reconceptualize one's own perspective (see Figure 5c). The clock example is an illustration of opportunities to apply this dialectic, and eventually reconsider one's own understanding of place value. Learning to recognize student inventions is a bit like seeing fireflies. The first one is difficult to see, but once you do, then you often realize they are all around you, winking on and off. For me, the interactivity of the voice-perspective dialectic in collaboration with learners comprises one of the most profound and satisfying aspects of teaching mathematics.

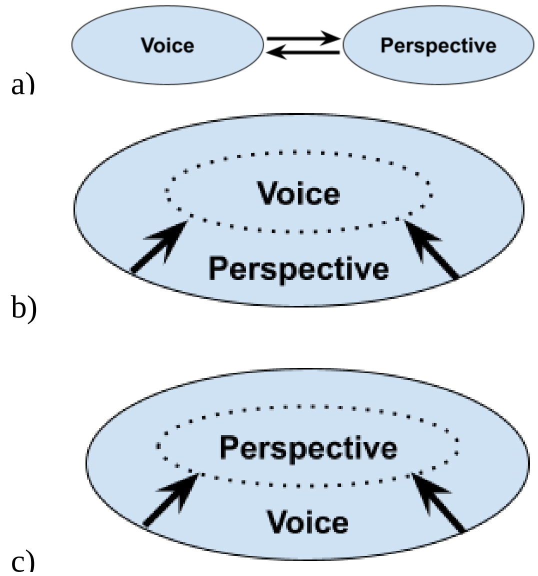

Figure 5: Using the Voice-Perspective Dialectic

The Concept of Learning Trajectories

Developing a learner-centered frame of mind is essential for educators, but it has to be accompanied by informational resources about what is known about patterns in student reasoning. The learning sciences have documented extensive information, concept by concept, about misconceptions, multiple representations, strategies, cases, verbal descriptions, connections to familiar contexts, and predictable approaches of students. Little of this research has been easily available outside of a few systematic efforts such as cognitively guided instruction (CGI) (Carpenter et al., 1996), the Rational Number Project (Behr et al., 1983), and "Turn-On CCMath" (Confrey et al., 2011).

I suggest that learning trajectories offer another resource for organizing relevant research from the learning sciences. In a learner-centered classroom, teachers begin teaching a topic by examining what students already know, often stimulated by posing a provocative task or activity. Depending on the responses, they must figure out how to weave from the students' experiences, conjectures, inventions, or ideas towards an intended learning target. Simon (1995) called these conjectured paths "hypothetical learning trajectories" (HLTs). These HLTs *should* be richly informed by the wealth of research on student thinking, so numerous researchers have proposed actual learning trajectories (LTs) as a systematic way for teachers to access student reasoning patterns as guides to drive instruction (See Confrey, 2019, for one compilation of these.)

The National Research Council (2007) described "learning trajectories" as "successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (p. 214). They typically include a domain-specific target and descriptions of the levels of student reasoning. Some researchers include associated instructional tasks by level in LTs, while others, including myself, provide them for illustrative purposes. In my case, this approach allows me to associate a range of tasks with a level and keep curricular materials distinct from the LTs themselves. For this paper, it is important for the reader to recognize the salient points about what LTs are and are not (Table 2). For more detailed reviews of LTs see Anderson et al., 2012; Confrey, 2019; Daro et al, 2011; Lobato & Walter, 2017).

Table 2: Features of Learning Trajectories (from Confrey, 2019)

Learning Trajectories *Are*:	**Learning Trajectories Are *Not*:**
Domain-specific models	General or universal principles
Expected probabilities	Stage theories
Empirically-based models of student thinking	Logico-mathematical deconstructions
Elicited by rich or novel tasks	Derived from typical exercises
Include strategies, reasons, explanations and cases	Sub-goals of the target
Include exploring misconceptions	A means to avoid errors
Ordered by increasing sophistication	Ordered by difficulty
Evolving	Fixed

Although LTs represent domain-specific models of learning, they rely on aspects of "grand" theories of learning. Learning trajectories draw from sociocultural theory and situated cognition in that they are built on students' ideas, and hence demand unearthing and drawing on learners' ideas and experiences. They also draw from the Piagetian tradition of studying cognitive development using as a key concept "genetic epistemology" (Piaget, 1971). It focuses on the movement in thinking, an action of learning that progresses from an initial idea or set of operations, through their extension to new situations or cases, to reach a generalization, or scheme, which is then available for use in future situations. It requires one to think of mathematical ideas in terms of purpose, asking what a mathematical idea allows one to do, recognize, predict, or explain.

For instance, the concept of percents makes the process of comparing ratios more efficient because it standardizes relative size, expressing it as a single comparable number. Instead of comparing ratios of 2/5 and 1/3, one compares 40% to 33.3% (of the same whole). The driving heuristic in mathematics education is for students to realize the beauty of how mathematics can help them make sense of their world in compelling, efficient ways. How can one apply genetic epistemology? Consider asking if two different-sized concert venues, with different numbers of seats and attendees, are equally full—where "full" can be described as the ratio of attendees to seats. Imagine how students might first argue that the one with the *most* attendees is more full. Or that the one with the *fewest* empty seats is more full. And then might consider when the two venues would be considered equally full or empty. And so on.

Genetic epistemology focuses one's attention on the starting ideas and their transformation from operations to schemes through "reflective abstraction" (Piaget, 1977). In this sense, a LT is not simply a list of behaviors to elicit, but a description of increasingly sophisticated ways of reasoning about a phenomenon, with the key concept being how the levels' descriptions scaffold the movement between levels. That said, LTs are not stage theories nor are they logical mathematical deconstructions built using typical exercises. Learning trajectories are probabilistic; different students may be more or less likely to exhibit the behaviors associated with a level when working on a related task. Learning more about the probabilities will require more research, new methods of data collection, larger samples, and careful attention to the cultural background and experience of a diverse set of participants.

I liken the array of ideas explicated at the level of an LT to the multiple paths on a climbing wall (Confrey, McGowan, Shah, et al., 2019, p. 79) but where the target is clear, obstacles are predictable, as are the handholds and footholds. The levels can include strategies and representations, explanations and cases, and they necessarily include *exploring* misconceptions instead of avoiding them. It is very important to note that grain sizes of LT-levels matter. If they're too fine-grained, they can't be readily used to guide instruction, and if they're too broad or coarse-grained, the level of detail is also insufficient to guide instruction.

In summary, low achievement scores on NAEP continue to remind us that large portions (often minority and poor students) of our student population are still underserved by our mathematics education system, and, at the same time, the overall system is faltering. When students are given opportunities to participate in engaging tasks, they offer their own ideas about mathematical concepts, but these ideas are not typically leveraged to revitalize classroom learning. Learning scientists, however, have studied the patterns in student reasoning and identified fruitful pathways, which can be packaged in the form of learning trajectories. These LTs may be a fruitful source of innovation that can be employed to address the system's weaknesses at scale.

Section Two: Introducing a Learning Trajectory-based Diagnostic System, Math-Mapper Features, Partners, and Validation

Designing a Learning Trajectory-based Diagnostic Assessment (LTDA) System called Math-Mapper

Between 2015 and 2022, my research group built a web-based application in which the LTs, associated assessments, and an overarching measurement validation process were described as a prototype of a "Learning Trajectory-based Diagnostic Assessment" (LTDA) system. We sought to accomplish multiple goals. Those included bringing the rich resources of domain-specific knowledge (LTs) from the learning sciences into the classroom. We sought to provide detailed learning feedback to students and teachers: timely, precise, and relevant descriptions of what students already knew *and* what remained to be learned. We hoped our LT- and feedback-oriented approach would support students in considering their own learning and enrich teachers' understanding of both student responses and the underlying mathematical epistemology. Further, if incorporated as designed into classroom routines, this approach could reduce or avoid the disfiguring pressures and drawbacks of high-stakes testing, and, instead, highlight student voice and put it to work in deepening students' agency as learners.

The assumptions underlying this approach included: 1) students are empowered to be more aware of their own progress, and 2) teachers are able to target subsequent instruction to topics needing more (or different) approaches, and plan more effective and efficient ways to meet the personalized individual or group needs of their students. We had two additional goals: creating such a system in partnership with on-going practice to anticipate issues of scale and establishing measurement bona fides for the system in order to document its effects validly and fairly, and include evidence of growth, as movement along the LTs.

In this section of the paper, I describe a range of features and capabilities of our prototype LTDA called Math-Mapper (MM)[11]. I selected one learning trajectory (LT) to illustrate the use of the LTDA. A report validation process is also provided to illustrate how its measurement model shifted in part due to the low-stakes structure of the LTDA design. In a subsequent section, I report on some insights learned from its implementation.

The model for use of the diagnostic assessment evolved over time and came to be referred to as an "agile curriculum framework" (Confrey et. al. 2018). It placed curricular implementation between the two bookends of standards and policies on the one hand and high-stakes testing on the other. The model for diagnostic assessment was specified to coordinate the administration of diagnostic assessments with the curriculum across a department at a grade level. Diagnostic tests were scheduled approximately 2/3 of the way through a curricular topic's designated time allocations (based on pacing guides). MM returns assessment data in real time, so teachers can review data with their students and make individual instructional adjustments. The model also included periodic professional learning communities (PLCs) where one teacher would present and discuss their data and others would report on their data and discuss explanations and approaches. We referred to this as two-cycle feedback (Figure 6).

[11] It is not possible to acknowledge every contributor to MM–however, I do wish to acknowledge a core group who contributed significantly and consistently across nearly the entire project: Meetal Shah, Youngjae Kim, Emily Toutkoushian, Charlene Marchese, Margaret Hennessey, Pedro Larios, and Alan Maloney.

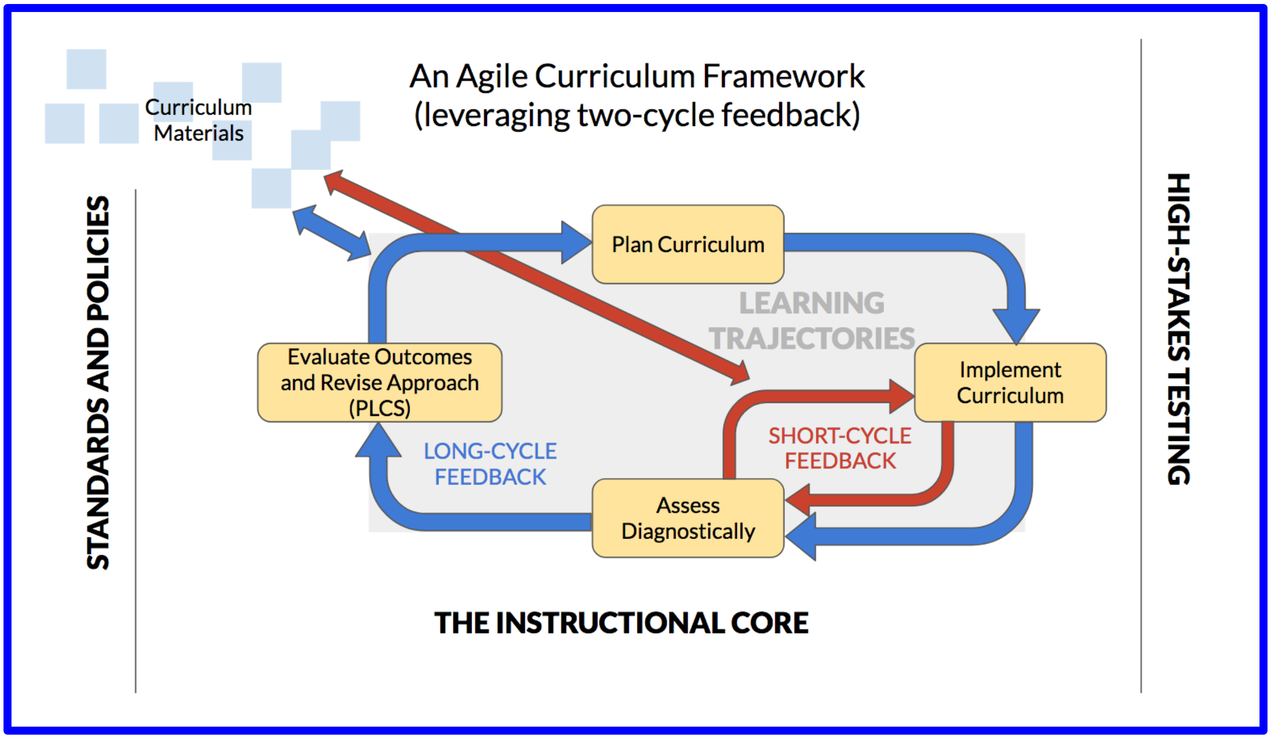

Figure 6: A Two-Cycle Feedback Use in a Diagnostic Formative Assessment System based on LTs (Confrey et al., 2018)

The system consisted of a map of "11 big ideas" to cover all of middle school mathematics including algebra (Figure 7) incorporating 74 individual learning trajectories (called "constructs") organized into a total of 32 "clusters" of topically related constructs. To the degree possible, the LTs were research-based; however, we acknowledge that some topics have a more secure foundation in research than others.

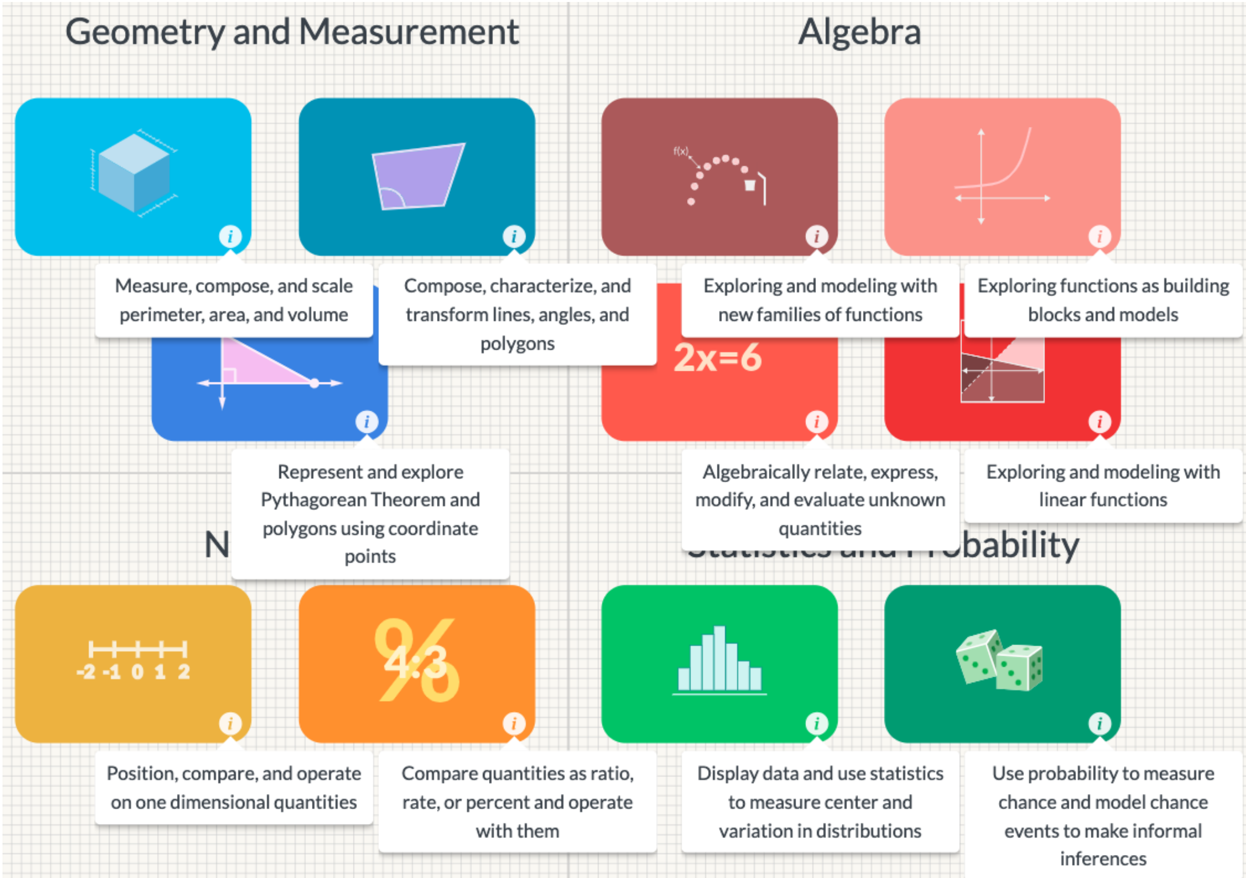

Figure 7: The MM learning map, at top level (i.e. displaying the "big ideas" of the Fields of Geometry and Measurement, Algebra, Number, and Statistics and Probability

In the system, tapping on a construct reveals the associated learning trajectory (Figure 8). The LT levels also include descriptions of related misconceptions (yellow triangles) and their corrected conceptions. Teachers can also access the Common Core Standards related to a LT, or conversely, use a standard to search for its relevant LTs on the map.

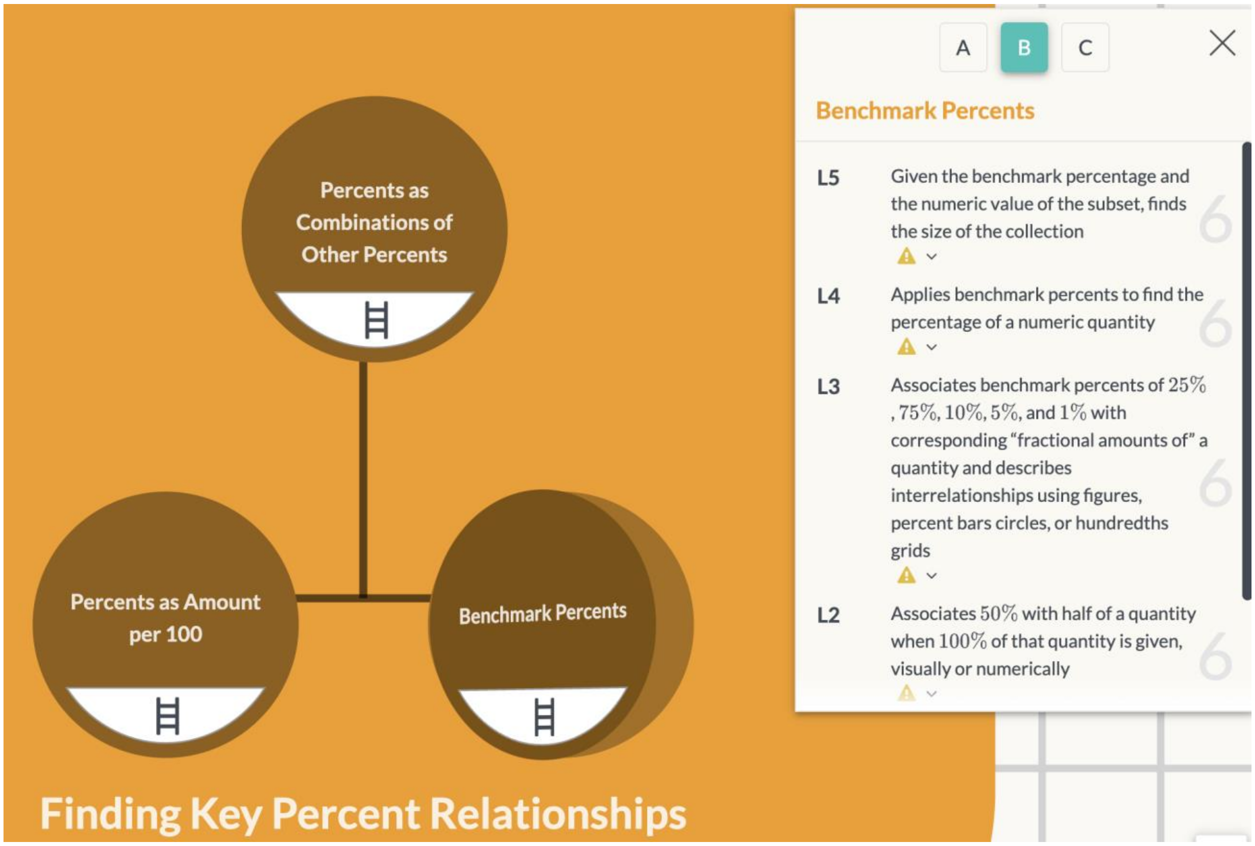

Figure 8: A Close Up of the Cluster "Finding Key Percent Relationships" in the Big Idea "Compare Quantities as Ratio, Rate or Percent and Operate with Them"

To illustrate the system, I will use a single, relatively simple LT, Benchmark Percents (Table 3). We list learning trajectories to mirror the climbing wall from bottom to top. In this LT, the first level children typically can learn to associate 100% with all of a quantity, and 0% with none. At level two, they can associate 50% with half of the quantity when 100% of that quantity is given. And so it goes up the levels, students learning to identify 25%, 75%, 10%, 5%, and 1%, and then applying those to solve problems. An important thing about this trajectory is that it is not based on the teaching of definitions. It builds from the learner's experiences in the world. And they have a lot of experience with percentages as some form of numeric measure. They see percentages on cell phones with the amount of battery left. They see them when they download apps, looking at what percentage has been downloaded. And they see them, of course, in prices and sales in other places out in the world. Therefore, our expectation is that students come to school with some familiarity with the idea, flavored with their own experiences.

Table 3: Learning Trajectory for the Construct "Benchmark Percents"

Level 5	Given the benchmark percentage and the numeric value of the subset, finds the size of the collection
Level 4	Applies benchmark percents to find the percentage of a numeric quantity
Level 3	Associates benchmark percents of 25%, 75%, 10%, 5%, and 1% with corresponding "fractional amounts of" a quantity, and describes interrelationships using figures, percent bars, circles, or hundredths grids
Level 2	Associates 50% with half of a quantity when 100% of that quantity is given, visually or numerically
Level 1	Associates all of a quantity with 100% and none of a quantity with 0% visually and numerically given the value or size of the whole

In the learning map, the learning trajectory for Benchmark Percents is situated in a group of three related LTs which also include Percents as an Amount per 100 and Combinations of Percents (Figure 8). There are also LTs for solving percent problems for the part, the whole and the percentage, and solving multi-step percent problems.

The Context

Two districts were contacted and agreed to work with us on how to meet the individual needs of students more effectively, while striving to improve their scores on state tests. Our district partners differed. One was a rural district with a nearby military base, serving a diverse, high-need, student body (27% African American, 10% Hispanic and Mixed, 53% White, with 57% FRL [free and reduced lunch]). Student mathematics performance at the school, measured on an annual state test, was bimodal. The principal was a strong advocate for the LTDA, and most of the time there was no mathematics supervisor. The other district was suburban, relatively wealthy, and much less diverse (4% AA, 8 % Hispanic and Mixed, 79% White and 10% FRL). Performance measured with the PARCC exam was one of the strongest in the state. The mathematics supervisor was a school leader and a key participant in the design of new affordances in the LTDA.

Over 2,500 original, conceptually-oriented items (16% 1-letter answer such as T/F; 27% multiple-choice; 27% numeric response; 18% multiple-select items; 12% parameterized, algebraic manipulation items), each associated with a single level on a LT, were authored by the research and development team. An assessment of a construct (LT) or a cluster (multiple LTs) comprising 8-12 items, requiring about 20-30 minutes, can be assigned and administered to a class. Once students complete their assessment, the MM system generates individual and class-level data reports. Psychometrically equivalent tests are provided to support repeated assessments in a proficiency-based achievement framework. Students also have access to practice items at any LT level.

High quality, intuitive data reports for students, teachers, and administrators are critical to the diagnostic assessment model. Upon completion of an assessment, each student can access their own data report. Figure 9 shows a student report for the cluster “Finding Key Percent Relationships” (this includes the construct for “benchmark percents” on which the student scored 89%). At the right, the student’s detailed performance on each selected construct is provided using a color-coded ladder that indicates which levels were assessed and the student’s performance on each level (shown: results for construct “Percents as Combination of Other Percents”). The student can also scroll down to access each item on the test, their results, and tools for revising and resubmitting or, alternately, revealing answers for each item (not shown). The green sectors of the circles in Figure 9 indicate that the student has already submitted a correctly revised response.

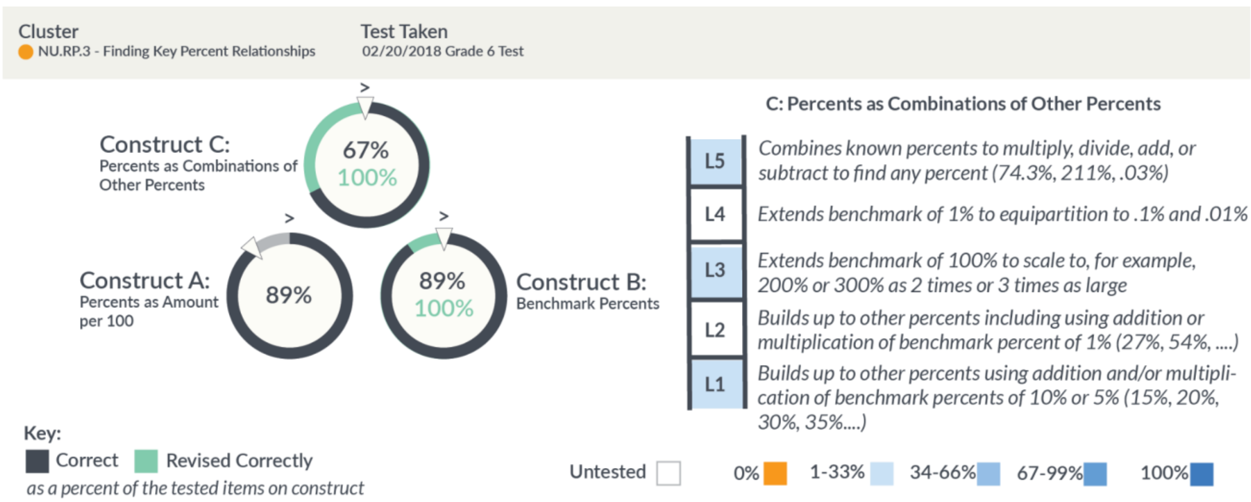

Figure 9: The top half of a student data report on a cluster assessment for “Finding Key Percent Relationships”

As students complete the assessments, teachers have access to all the students’ results in the form of a *heat map* (Figure 10), with a color-coded scale (orange to blue) indicating the percentage correct or designating an untested level (grey) on the particular assessment. Each column (“stack”) represents a student’s score profile for the construct. Students’ score profiles for each construct (LT) are ordered from least to most proficient for that assessment administration.

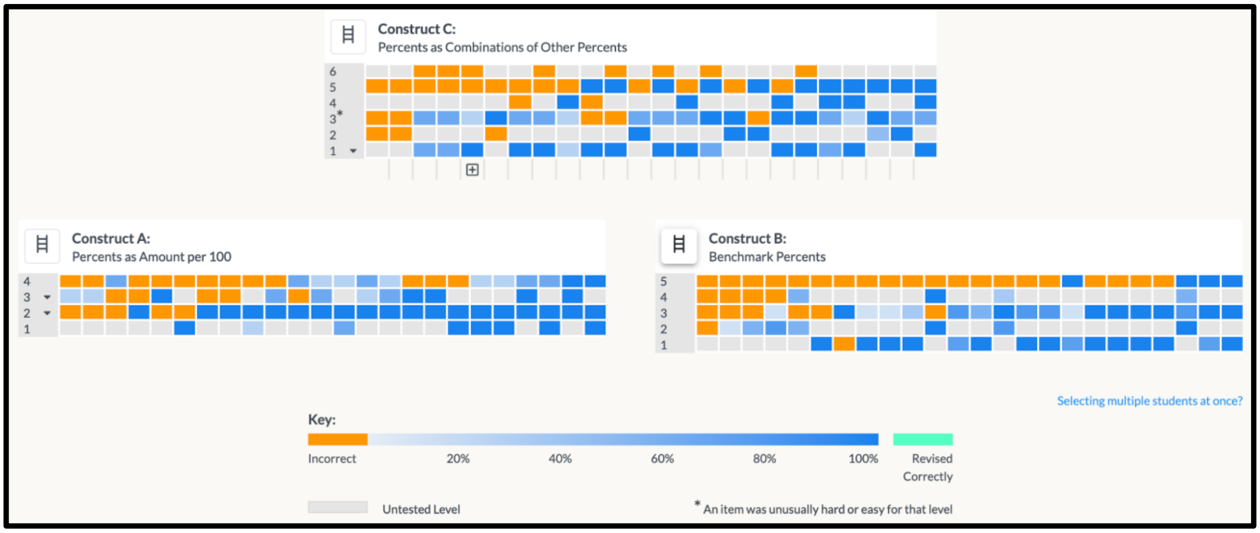

Figure 10: Heat map for assessment for cluster "Finding Key Percent Relationships"

During professional learning sessions, we use Figure 11 to teach teachers how to interpret heat maps (only teachers have access to student names on heat maps; during in-class discussions, they use heat maps without identifiers).

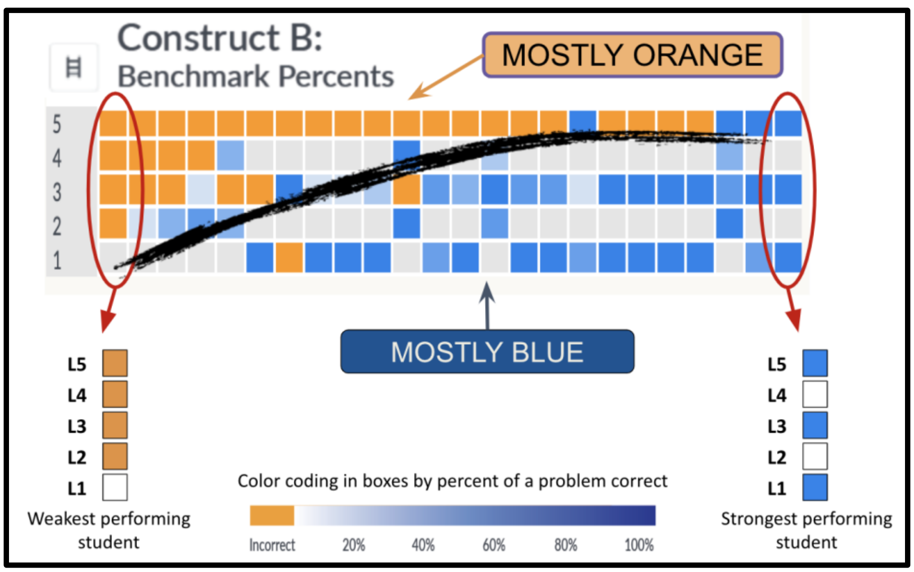

Figure 11: Heat map detail from Figure 10: construct "Benchmark Percents" (explanatory marks and labels added)

Because the items aligned to the LT-levels increase in difficulty, it typically generates a heatmap with more orange towards the upper left (high levels and weaker student performances) and blue towards the lower right (lower levels and strong student performances). Teachers learn to imagine the superimposed curve shown, as an indicator of performance above and below proficiency-level performance. Teachers learn to identify any set of students needing additional instructional help (probably the first two or three students); and any levels that the majority of the class may need to review (levels 4 and 5). To further support classroom discourse, a teacher can display the actual items used in the assessment, reveal the correct answer, display the percentage of students exhibiting misconceptions associated with that item, and examine an item analysis of the students' responses within the heatmap.

Validation of MM

MM is designed as a formative assessment system, to support student- and classroom-level reasoning and discussion. As a system producing scores, it needs to be validated, albeit to a different set of criteria than high stakes tests. The validation of an assessment is "an integrated evaluative judgment of the extent to which empirical evidence and theoretical rationales support appropriate actions in line with the purpose of the test" (Messick, 1990) in this case, feedback on students' progress on LTs that can inform the focus of subsequent instruction. As low-stakes assessments, attention to technical measures of reliability may be de-emphasized. However, fairness, as the absence of bias, must still be examined.

In this low-stakes setting, validation of an LTDA system can be informed by Pellegrino et al.'s, (2016) approach to instructionally valid "classroom assessments". Our validation methodology is described along three dimensions: the validation *argument* (Kane, 2013), the techniques or models selected, and the process of review. Validation arguments help to ensure that the way assessments are designed and are intended to be used are made explicit so that their effectiveness can be knowledgeably and fairly evaluated. We have generated numerous papers about the validation argument (Confrey, Toutkoushian, & Shah, 2019), the validation of particular LTs, and a summary review of the performance of MM based on two rounds of validation for 45 of the LTs.

The validation argument consists of five claims and evidence:

- Construct subscores for a cluster will be highly correlated, reflecting the mutual dependencies implied by the cluster's structure.
- Overall, the empirical difficulty of items will vary positively with level (positive correlation).
- Empirical item difficulties will vary within an LT level in ways closely associated with the meaning of the level; construct-irrelevant variance can be minimized.
- On class (or multi-class)-level reports (such as heat maps or compound bar displays), ordering the levels (and the items by difficulty within the levels) will reveal relative strength of students' performances across items and levels and across the class or classes.
- Based on the data on the students' performance by level, teachers target instruction appropriately.

These five claims tie together the theory of action for MM, moving from the structure of the map into clusters, the relationship between item difficulties and levels, the prediction about the shape of the results and the resulting instructional actions. Further reflection has convinced us that the validation argument needs an additional step that recognizes the importance of the classroom assessment review process.

Following the lead of others, (Lehrer et al., 2014; Wilson, 2005, Graf & van Rijn, 2015), our first step in the validation process was to employ Item Response Theory (IRT) as a psychometric test of the strength of the relationship between the difficulty of the items and the levels of the LTs. Using an iterative regression technique, we identified "potentially non-conforming items" and analyzed the items further. A summary of results from the first two rounds of validation, using 45 of the LT constructs with sufficient data (Confrey et al., 2020) reported a mean correlation of 0.71 between levels and difficulty; 68% of those constructs had strong correlations and 24% had a moderate correlation. Only four constructs had poor correlations that justified major revisions.

Our procedure for deciding what to do with the potentially non-conforming items differed from standard processes. Instead of simply discarding items that did not conform to the expected model, we developed an interdisciplinary review and documentation process conducted by learning scientists, practitioners, and psychometricians. Examining the extent and nature of the data (sample size, school, grade level, and class), we considered seven actions, ranging from discarding the item; revising items; revising, adding, or removing LT levels; collecting more data; re-examining the literature; or completely revising the LT. Building on work by Lehrer (2013) we likened our validation process to a trading zone (Confrey, 2019; Confrey et al., 2021).

Section Three: Five Fundamental Dynamics of the MM LTDA System

I have introduced and described a diagnostic assessment system, based on learning trajectories, whose primary purpose is formative. It is intended to provide learner-centered feedback to students and teachers during instruction, to locate learners' current progress and needs, and to articulate possible paths toward more sophisticated understanding of target concepts. The fine-grained LTs were based on available data from the learning sciences, and the assessments were validated and equated. The just-in-time feedback from the assessments at the teacher-level strengthens reflection and drives instruction and, at the student-level, provides awareness about specific areas that need more work.

The project, situated as "design-based implementation research" (DBIR, LeMahieu, P. et al., 2017; Penuel & Fishman, 2012) resulted in a number of studies, including studies of classroom assessments, validation studies, a lesson study, and a set of think-aloud interviews. We regularly collected data from classroom observations, PLCs, professional development sessions, administrative memos, and software design documents. DBIR involves a complex interplay over time among the theoretical premises of LTs, the software design decisions, measurement choices, student performance data, teacher professional development, curriculum, and classroom implementation.

These rich data sources provided opportunities to consider how underlying design commitments played out in the fray of implementation. Below, I identify five of these design commitments that are critical to successful implementation. I call them "system dynamics" to emphasize that 1) dynamics allow the "study of the effects of forces on the motion of objects in the system” (Wiktionary, https://en.wiktionary.org/wiki/dynamics) and 2) those forces may strengthen *or* weaken the outcomes. Thus, a dynamic here is metaphorically how forces are related to outcomes, and whose intensity and direction varies during implementation. In this context, a dynamic does not function as an independent component of the system but rather as a force that influences components of the system, making them function more or less effectively. Attention to the dynamics has a qualitative effect on the system’s operation. As the importance of these dynamics became increasingly apparent, we added affordances to the LTDA to foster and monitor them in positive ways.

Dynamic 1: Timely, fine-grained LT-situated feedback to teachers supports targeted instruction to increase learning and promote more equitable outcomes.

The central goal of formative assessment is to enact practices that improve instruction and learning over time by providing relevant, timely, and *actionable* feedback to the participants such that *every* student has the opportunity to become proficient in mathematics. To study this dynamic in the MM system, we examine compound bar graphs that describe school or district-wide assessment performance on the LTs.[12] Data from the two partner districts, by grade and class type for the LT “Benchmark Percents” are shown in Figure 12. Bar graph displays mirror the earlier heat map color coding: each bar displays, by LT level, the percentage of students who answered equivalent items aligned to this level incorrectly (orange) and correctly (blue). Note that the rural school only administered this assessment to sixth graders, because they did not group students into classes by performance level. The suburban school had “regular” and “advanced level” classes and administered the assessment in both sixth and seventh grades.

[12] We did not report data by race, gender, ethnicity or SES because we did not negotiate to do so with our partners as we sought to build the system. In our lower SES rural school, we did report evidence of correlations between the increased use of MM and data gains on the state test. We found that assignment by teachers of more of these tests of any topic (just four sections of 25 students) was correlated with the gains of.88 on a five-point scale end of-course test. Conversely, opportunity by individual students to take five more Math-Mapper tests (any topic) per year was associated with gain of.4 on the end-of-course high-stakes test.

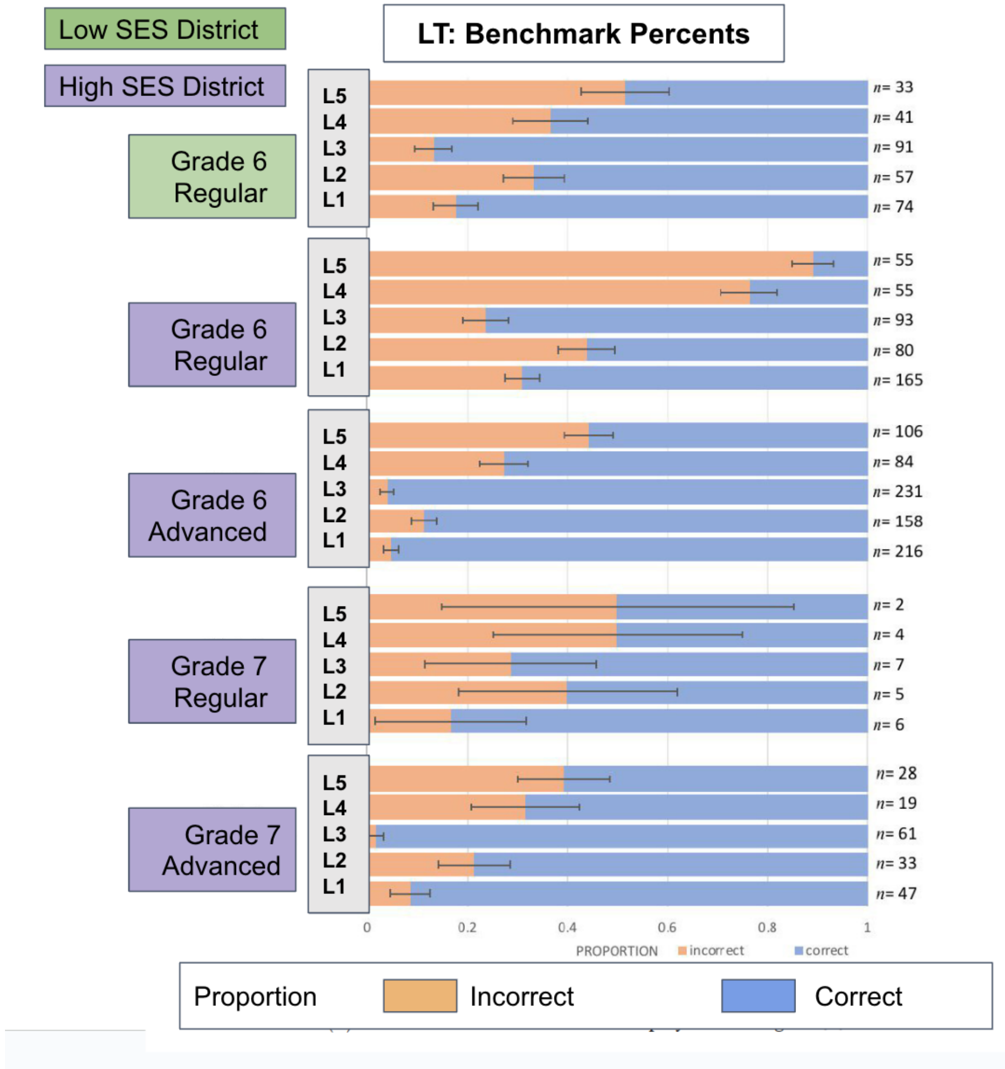

Figure 12: Compound Bar Graphs for the construct Benchmark Percents across the levels, by district, grade, and grouping

These results provide evidence that the LT behaves as expected: for the most part, the higher levels of the LT are more difficult[13].

More importantly, they show that a substantial proportion of students do **not** reach proficiency at the upper levels. Across all the LTs, these proportions of non-proficient students often exceed

[13] Note: Level 3 associates 25%, 75%, 10%, 5%, and 1% with corresponding "fractional amounts", and the items proved easier than expected. Re-examining those items during validation revealed that the presentations of the percentages using pitchers and circles could simply be ordered visually to identify the associated percents without understanding the values. The items were retired.

50%, results which run counter to a proficiency-oriented perspective on learning. During our field studies, teachers had a number of explanations for their students' weak performances at the upper levels, ranging from arguing that the levels were not aligned with standards (they are), that some students do not need to reach the top levels (low expectations), or that students had insufficient time to reach those levels. Based on discussions during professional development, we noted that these upper levels were often less well understood as targets, or that teachers asserted that top levels were more properly assigned to higher grades. We believe this result merits closer attention: teachers may lack adequate time to reach these levels, they may focus more than necessary on lower levels, or may themselves not fully understand the target proficiencies of the LT. Many times, however, we also observed a highly successful teacher explain in PLCs how she/he addressed these levels as a culmination of the LT-related instruction. We see this as an important reminder that schools have their own reservoir of talent that they can draw on to strengthen capacity using their own data from a LTDA.

Across our classroom observations, teachers' review practices varied in the extent to which they drew out student thinking (vs. telling them how to solve the problem), and how they situated the items within the LT. Later, in PLCs, they examined results and shared their interpretations and instructional practices. Re-testing with equated tests could shed some light on student progress and the effectiveness of review and modified instruction. Most teachers we observed did review the data with their classes, but unfortunately, few conducted retesting due to their pacing guide time pressures. We were, however, able to examine the results of repeated testing for a small sample of students on the cluster "Finding Key Percent Relationships" (Figure 13). The distribution of the scores suggests there is a strong potential role for using retesting after a review of the initial diagnostic assessment to support a stronger proficiency-based outcome.

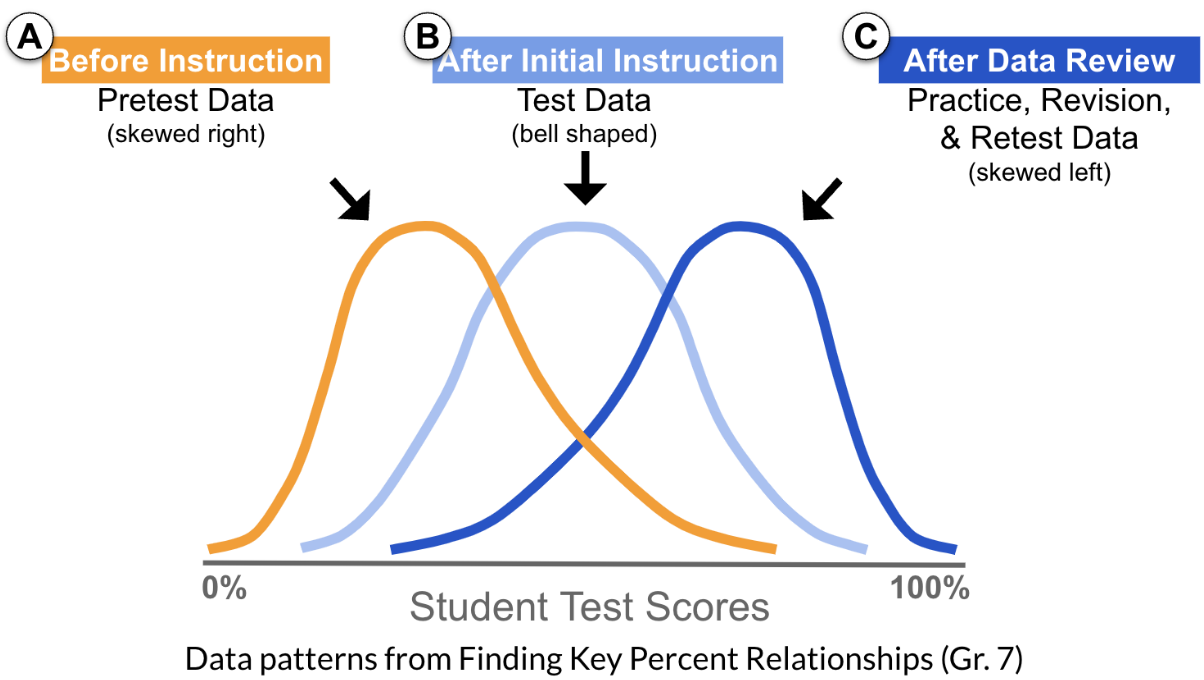

Figure 13: The distribution of student scores across three administrations of equivalent MM assessments

Dynamic 2: Personalized, fine-grained LT-situated data delivered immediately to students, in a context of assessment for learning, fosters growth mindset and student agency.

Learning trajectories place students at the center of instruction. A ground-breaking feature of Math-Mapper is the direct and immediate return of data to students, along with access to the learning trajectories. We argue that this can strengthen student agency, because the students are considered partners in learning, and can evaluate their own strengths and weaknesses based on their own data. The MM system has other affordances to both support teachers in fostering a growth mindset and provide students the opportunities to build their own mathematical strengths. For instance, MM provides students with the option to revise and resubmit answers to any item (with consistent feedback); we repeatedly witnessed students–who at the onset were sure they could not solve a problem–who, through careful reading and persistence, solved the problem.

The student experience of taking MM assessments differed from typical assessments in many ways. Teachers reported that the items were more conceptual. We designed the items, especially at the upper levels, so that students would experience some productive failure/challenge (Kapur, 2009, 2014). Students who were accustomed simply to solving clones of textbook problems found many MM items challenging; on average, overall percent correct scores ranged in the 60-80 percent range. Teachers needed to shift students' expectations of their scores: to understand that the lower range in scores left room for growth (and would not translate directly to the customary grading scale).

In these examples, one can see that teacher expertise is critical to support students in re-orienting themselves to assessment *for* learning in contrast to assessment *of* learning. In our field work, we observed sharp differences in how teachers conducted data reviews. Some treated the items as "hard", and simply explained the solution. In contrast, other teachers asked students to read the problems out loud, queried them about their initial thoughts, and invited sharing and comparison of strategies. The teachers with this second approach often reflected back on the problem with students considering *why* it had been hard and how difficult ways to approach it were warranted. Not surprisingly, when students discussed these challenges as they learned more successful paths to proficiency on the concepts, their growth in learning outcomes accelerated. Students of teachers in the second group learned to approach the assessments with more confidence and determination. These teachers' moves promoted a growth mindset in students and set high expectations.

Supporting students to become change agents for their own learning can (and should) involve groups and not just individual students. We saw evidence of students adopting the language of the LT for themselves to scaffold their own learning. A teacher grouped students with similar response profiles together and asked them to use the practice resource collaboratively on LT levels on which they were weak *before* revising and resubmitting assessment items. Students increasingly used the language of the trajectory levels to explain their reasoning while they taught each other or collaboratively solved the problems.

A recent collaboration with the Young People's Project (directed by Maisha Moses, Bob Moses's daughter and collaborator) involved working with high school students on their way to becoming math literacy (peer) workers. Together we reviewed the LTs on ratio as a means to discuss content **and** pedagogy. After working through the levels, the group somewhat reluctantly took a MM assessment, resulting in a heat map with spotty performances–some stronger, some weaker. However, rather than respond individually to their incorrect results, the group used its own collaborative process to join together and work to “turn the heat map blue”. By doing so, they reappropriated what had too often been an oppressive experience (assessment) into a shared opportunity to assert agency.

Dynamic 3: Data demonstrating variability in learners’ initial success across LTs can challenge beliefs about learning: both student’s self-limiting beliefs and teachers’ stereotyping of students’ learning potentials.

If students feel that they have low chances to succeed in school, they can develop behaviors and perceptions that reinforce failure. Unfortunately, once students are behind the expected pacing of the curriculum, it's increasingly hard to change their negative self-perceptions. Similarly, students who are advanced in their pacing may come to expect that all topics should be easily learned, and feel compelled to hide missed connections and confusion.

To counter such static evaluations, we emphasized the use of flexible grouping using the heat maps for differentiation by LT. Teachers can quickly identify groups of students who need assistance with similar issues. These groups may need instruction to strengthen prerequisite or supplemental topics, to reteach the whole topic (LT), or to focus on specific LT levels. New groupings are formed regularly by need.

MM also allows teachers to explore the consistency of students’ performances across different constructs. In a cluster heat map, for example, selecting a student’s score profile (a stack) in one construct automatically displays that student’s profile in the other constructs of the cluster. It reveals that a score of high (or low) proficiency on one construct does not always correspond to a score of high (or low) proficiency on the others; that is, students frequently do not perform consistently across constructs in a cluster. From this insight, teachers learn that students differ in what they find easier or harder to learn and why one-size-fits-all instruction is unlikely to meet learners’ needs well.

The MM heat maps also allow teachers to select multiple student profiles simultaneously. Using this feature, teachers can investigate patterns of reasoning within groups of learners within a cluster. For example, the profiles of four high-performing students have been selected in “Construct B: Benchmark Percents” (Figure 14). But their performances on the other two constructs varied (Table 4). Multiple conjectures are possible here. Perhaps student O did not understand the new construct about the additive nature of combinations of percent, while students U and E might need more explanations for using certain representations of percents (*e.g.*, using percent bars and extending beyond 100%). That gap in understanding seems to have hampered student E (but not student U) in learning Combinations of Percents. Teachers developed more nuanced understandings of difference and more curiosity about the reasons behind it.

These features of MM can challenge teachers’ and class beliefs about students’ abilities as well as tacit assumptions that mathematical reasoning ability is immutable. By using the MM tools for flexible grouping, teachers can avoid the pejorative and lasting effects of a student always being assigned to the same level or group. In addition, the content-specific conjectures offer rich insights to guide data review discussions and instructional follow-ups. This form of investigation of performance across constructs establishes the value of detailed data on student strengths and needs, using digital affordances to convey this information quickly and accurately. It provides a concrete instance of seeing personalized learning data as a means for all students to experience challenges and success.

Figure 14: Heat map of Assessment Results from Finding Key Percent Relationships. (Color coding added for interpretability)

Table 4: A Summary of Students O, U, E, and D's Proficiency across Constructs

	Student O	Student U	Student E	Student D
Construct B: Benchmark Percents	High	High	High	High
Construct A: Percent as amount per one hundred	High	Moderate	Low	High
Construct C: Combinations of percents	Low	High	Moderate	High

Dynamic 4: Gradual sustained teacher professional learning about the LTs and data use increases the effectiveness of LTDA implementation

Any complex diagnostic technology, from medicine to car repair, requires preparation and ongoing updating for proper implementation. It seems intuitively obvious, then, that making effective use of a LTDA system requires adoption of a continuous improvement approach to building teachers' expertise in implementing the LTDA while gaining their trust and buy-in. A gradual process was needed to learn to use the data effectively in both cycles of the agile curriculum feedback model.

Initially, we documented that during the "short-cycle" feedback process (Figure 6) teachers tended to treat solving each item itself as the direct target of discussion. Gradually, and often with coaching, they learned to shift their focus to 1) treat each item as *representative* of a level, and 2) treat LT levels as *situated within* the trajectory, thereby focusing on movement up the levels as a goal of instruction. Overall, we interpreted this process as the teachers gradually recognizing and coming to trust the LTs as valid frameworks for learning and instruction. During the "long-cycle" feedback process, teachers needed opportunities and coaching to become effective at reading and interpreting data patterns, linking them to potential causes or larger trends, and designing responsive instructional adjustments. Initially, they would jump to propose interventions without careful data examinations. This led us to develop a data-driven PLC framework to guide teachers in the process of 1) closely examining and understanding the student data, led by one teacher, 2) generating conjectures and explanations about connections and possible causes, and 3) planning how to adjust, supplement or change curricular materials or instructional approaches (Figure 15). As an illustration of the value of data use, at our rural school, a sixth-grade teacher shared data showing weak performance for L4: "Builds Up to Percents Larger than 100" within the LT "Percents as Combinations of Other Percents." When other teachers confirmed similarly weak performance in their classes, the teachers as a group realized the topic had simply been overlooked, with the main source of the problem having been the scope and sequence and pacing guides. Some of these problems can be relatively easy to address, with data to guide its identification.

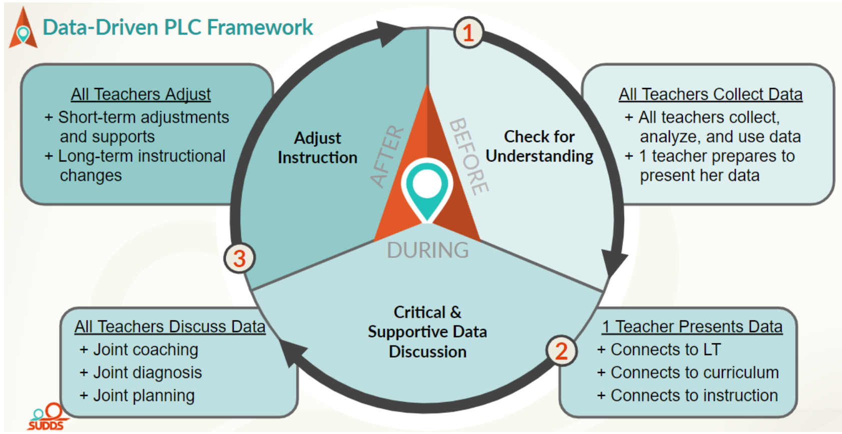

Figure 15: Data-Driven PLC Framework for analyzing and interpreting formative diagnostic assessment data, and making data-grounded short- and long-term instructional adjustments

It is imperative to note that the school or district leaders' commitment, and content- and instructionally focused leadership, are essential for the eventual success of any such adoption. Our schools were more successful from the outset of implementation, when they had the consistent and substantive leadership of the lead district administrator and a mathematics education curriculum specialist, instructional coach, or district supervisor. Success also required school or district commitment to prioritizing the LTDA's use (avoiding competing formative assessment initiatives) and establishing expectations for: 1) content-focused conduct of PLCs, and 2) documented evidence of gradual and sustained improvements in student learning. Supervisors were needed who oversaw the conduct of the PLCs and communicated the district's long-term vision and commitment. To be effective, supervisors should conduct observations of the use of the tool and promote teacher leadership at individual schools. They must co-develop a logistical plan to fully implement MM including choosing focal units to start with after coordinating instruction and common assessments with the pacing guides and the scope and sequence across teachers, and scheduling heatmap review classes and PLCs. The lack of substantive support from those professionals could undercut an implementation.

To enact the high-leverage practices supported by a formative diagnostic assessment system, schools should plan a multi-year ramp to full implementation, even if initial benefits accrue more quickly. Clusters and domains place varying demands on teachers; for most teachers, descriptive statistics and probability and inference are less familiar and more challenging than fractions and decimals. It may take several years to rotate the content focus among the 11 big ideas and 32 clusters: schools should make use of their teachers' varied expertise in different domains. Implementing a LTDA is a systemic enterprise. that dramatically deepens content-based discussions and sharing among teachers. Studies of teachers using MM (Confrey, Shah, Persson, Ciliano, 2019) showed that the most successful teachers exhibited the following patterns:

- emphasis on “assessment for learning,”
- conceptual understanding of, and growing trust in, the LTs, as instructional foundations
- increasingly consistent focus on student reasoning,
- participation in and fostering of peer (teacher) interactions around valid and reliable data,
- growth in the ability to query data in a principled way, and
- commitment to achieve gradual, steady professional growth as a community member.

Dynamic 5: Trading Zone-style collaboration among learning scientists, psychometricians, teachers, *and* students promotes iterative improvement of the design, implementation, and effectiveness of the LTDA.

In general, LTDAs should and can be designed and implemented as dynamic systems to accomplish these dynamic principles (summarized in Table 5). By its nature, an LTDA encompasses all the stakeholders, researchers, and software professionals, as well as the technology product itself. Today’s technology development can be fast and flexible. The map and the LT’s function as a boundary object, that is an object that is meaningful to a variety of communities in different ways, but which also has a common meaning that fosters communication. In this context, changes to LTs and assessment items are expected, including adding, modifying, or removing some, grounded in the validation process. New formats for items can be added. Video exemplars of student reasoning can be shared. As reported, our validation process already alters the interaction patterns among learning scientists, psychometricians, and practitioners to resemble more of a trading zone (in which, the exchange is based on debate and evidence about the relative value, to participants, of various objects at a point in time). A community of stakeholders (now including researchers) with multiple concerns and sets of expertise collaborate, respectful of their different perspectives, with the common goal of promoting growth in all students’ learning.

Changes in MM arose from research, technological advances, and the perspective of practitioners. For instance, integers were originally grouped in the same cluster as fractions and decimals, under the header of “rational numbers”. Teachers pointed out that most curricula do not teach these topics in the same unit, some putting integers in closer proximity to actions on the coordinate plane, so we moved integers to its own cluster. Change can also be initiated based on new research or changes to standards. For example, the constructs encompassing probability were significantly reworked as the topic of early inferential statistical reasoning emerged from research.

A communal process should include critical discussions concerning how to strengthen LTDA design, implementation, and/or practices to equitably support mathematical learning by subgroup, based on race, ethnicity, gender or SES, special education needs, and ESL. The use of the LTDA should be regularly examined for its effects on instructional quality overall and the delivery of quality instruction at all performance levels. Schools should report on the data in easily understandable formats, to support the reduction in learning gaps and improvement in overall performance without shifting its use to a high-stakes entity.

Table 5: Five dynamic principles underlying MM LT-based Diagnostic Assessment System

Dynamic 1: Timely, fine-grained LT-situated feedback to teachers supports targeted instruction to increase learning and promote more equitable outcomes.
Dynamic 2: Personalized, fine-grained LT-situated data delivered immediately to students, in a context of assessment *for* learning, fosters growth mindset and student agency.
Dynamic 3: Data demonstrating variability in learners' initial success across LTs can challenge beliefs about learning: both student's self-limiting beliefs and teachers' stereotyping of students' learning potentials.
Dynamic 4: Gradual sustained teacher professional learning about the LTs and data use increases the effectiveness of LTDA implementation.
Dynamic 5: Trading Zone-style collaboration among learning scientists, psychometricians, teachers, *and* students promotes iterative improvement of the design, implementation, and effectiveness of the LTDA.

Concluding Remarks

By describing Math Mapper's features and identifying dynamics associated with its implementation, I have sought to illustrate a concrete vision of the potential value of diagnostic approaches in a low-stakes feedback data-rich assessment-for-learning environment. MM is both flexible and valid for its purpose as a formative assessment tool. Its validity is enhanced by its proximity and relevance to instruction, and its goal is to stimulate student progress towards more sophisticated reasoning about specific mathematical concepts. Learning trajectories put the learning sciences to work by conveying the richness and diversity of student thinking. In MM, we have synthesized the research on LTs on a full range of the big ideas in middle grades math and accomplished an unprecedented degree of scale in linking those findings to diagnostic assessments. MM's flexibility resides in the way in which it is compatible with a variety of choices of instructional materials whose scopes and sequences vary.

MM provides timely and useful feedback to its users in three notable ways. First, both students *and* teachers immediately and directly receive the data from assessments. Secondly, the data reports are situated within a learning map of the topics, and the underlying learning trajectories provide an overview of the knowledge under study. Thirdly, the data displays include an array of affordances that encourage interactive data review processes. Students are able to revise and resubmit incorrect initial responses. They can practice collaboratively on levels needing more work and then can take equivalent retests supporting the ultimate goal of proficiency at each level. Teachers work with their classes using heat maps to strategically focus instruction at the appropriate levels and to identify students needing additional support to reach proficiency. Affordances of the heat maps allow teachers to project problems that merit further discussion, and to provide access to relevant answers and/or item analysis of the array and percentages of learner responses. Teachers share data with colleagues in search of patterns and alternative approaches or curricular materials.

Through Math-Mapper, we have contributed to the field our methodology for building and refining assessments. Building assessment items specific to the LT levels led us to develop a large base of original items, which were conceptually oriented and nuanced in their cognitive distinctions. Widely accepted psychometric techniques were used to check assumptions about how LTs should behave and identify potentially non-conforming items. Typical practice relies on psychometricians to decide how to modify or remove troubled items; however, our practice was to include learning scientists, psychometricians, and practitioners, all on equal footing. These teams examined the data and decided how to modify items, LT levels, or entire LTs, as we generated explanations based on data characteristics (size of sample, population sampled). With each round of validation studies, we saw the association between LT level and item difficulty improve. Math-Mapper's design made careful use of measurement, and its results supported teachers in making innovative instructional adjustments to meet and record student progress and needs. However, measurement was but one component of this formative diagnostic assessment system. So, I would suggest we have rescued the baby (measurement) and placed it in a new low-stakes context of assessment for learning.

Working at scale within a DBIR project with multiple data sources allowed the identification of five "dynamics." The term "dynamics" was chosen to identify forces that would act on, and interact with, the components of implementation to affect outcomes. Their influence could enhance or diminish the quality of implementation; hence the dynamics are not just additional components. We saw how progress was fostered by a clear understanding of, and focus on, the structure of the LT and on student participation, as evidenced by the methods of review. We observed how fostering a growth mindset and recognizing students as partners and collaborators to each other in learning strengthened student agency. We watched how stereotypes about who can and cannot learn math were diminished as the focus moved to sharing different ideas, challenges, and strategies. And we observed teachers engaging in more specific content-based exchanges about strategies and resources.

The last, perhaps most essential, dynamic was the final one, recognizing the need for transparency and the value of inclusivity[14] in the design and use of formative diagnostic assessment systems in a *low-stakes* context. In settings with strong and energetic leadership, with leaders tuned into their teaching staff, feedback on the tool's use and features accompanied and enhanced the entire endeavor. Absent leadership with actual knowledge of curricular pacing and level of coordination among staff, communication became muddled, and resistance could build. When information flowed in both directions, however, teacher suggestions for changes to MM resulted in notable enhancements. The beauty and power of a low-stakes LTDA is that it can also be improved on a shorter and more flexible timeline because ownership is shared among users and experimentation with techniques can be encouraged.

Bob Moses implored us that it cannot be good enough to return to "normal", because normal has failed to serve such a large proportion of our youth, and has actively disadvantaged specific

[14] Parents are also stakeholders for implementation of an LTDA. They were not included in the current version of MM due to funding and time constraints.

groups. We are suggesting that the next round of LTDA innovation must also be learner-centered with ways to document progress within feedback systems and examine fairness in achieving the system's formative goals. Collaborative contributions by a diverse group of experts and stakeholders must be a central feature. Designed well to produce relevant, timely, and easily used data and implemented in partnership, we can simultaneously make progress on achievement to high levels of mathematical proficiency *and* equity.

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